On the Parity of Ranks of Selmer Groups III

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Abstract. We show that the parity conjecture for Selmer groups is invariant under deformation in p-adic families of self-dual pure Galois representations satisfying Pančiškin's condition at all primes above p.

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0. Introduction

(0.0) Let F, L be number fields contained in a fixed algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$; let M be a motive over F with coefficients in L. The L-function of M (assuming it is well-defined) is a Dirichlet series $\sum_{n\geq 1} a_n n^{-s}$ with coefficients in L. For each embedding $\iota : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, the complex-valued L-function

$$L(\iota M, s) = \sum_{n\geq 1} \iota(a_n) n^{-s}$$

is absolutely convergent for $\mathrm{Re}(s) >> 0$. It is expected to admit a meromorphic continuation to $\mathbf{C}$ and a functional equation of the form

$$(C_{FE}) \qquad (L\cdot L_\infty)(\iota M, s) \stackrel{?}{=} \varepsilon(\iota M, s)\,(L\cdot L_\infty)(\iota M^*(1), -s),$$

where

$$L_\infty(\iota M, s) = \prod_{v\mid\infty} L_v(\iota M, s)$$

is a product of appropriate Γ-factors (independent of ι) and

$$\varepsilon(\iota M, s) = \iota(\varepsilon(M))\,\mathrm{cond}(M)^{-s}, \qquad \varepsilon(M) \in \overline{\mathbf{Q}}^*.$$

(0.1) Let p be a prime number and $\mathfrak{p} \mid p$ a prime of L above p. The $\mathfrak{p}$-adic realization $M_{\mathfrak{p}}$ of M is a finite-dimensional $L_{\mathfrak{p}}$-vector space equipped with a continuous action of the Galois group $G_{F,S} = \mathrm{Gal}(F_S/F)$, where $F_S \subset \overline{\mathbf{Q}}$ is the maximal extension of F unramified outside a suitable finite set $S \supset S_p \cup S_\infty$ of primes of F. According to the conjectures of Bloch and Kato [Bl-Ka] (generalized by Fontaine and Perrin-Riou [Fo-PR]),

$$(C_{BK}) \qquad \begin{aligned} \mathrm{ord}_{s=0} L(\iota M, s) &\stackrel{?}{=} \dim_{L_{\mathfrak{p}}} H^1_f(F, M^*_{\mathfrak{p}}(1)) - \dim_{L_{\mathfrak{p}}} H^0(F, M^*_{\mathfrak{p}}(1)) = \\ &= h^1_f(F, M^*_{\mathfrak{p}}(1)) - h^0(F, M^*_{\mathfrak{p}}(1)), \end{aligned}$$

where $H^1_f(F,V) \subseteq H^1(G_{F,S}, V)$ is the generalized Selmer group defined in [Bl-Ka].

(0.2) Consider the special case when the motive M is SELF-DUAL (i.e., when there exists a skew-symmetric isomorphism $M \xrightarrow{\sim} M^*(1)$) and PURE (necessarily of weight -1). In this case $H^0(F, M_{\mathfrak{p}}) = 0$ and $\mathrm{ord}_{s=0} L_\infty(\iota M, s) = 0$, which means that the global ε-factor $\varepsilon(M)$ determines the parity of $\mathrm{ord}_{s=0} L(\iota M, s)$ (assuming the validity of (C_{FE})):

$$(-1)^{\mathrm{ord}_{s=0} L(\iota M, s)} = \varepsilon(M). \tag{0.2.1}$$

In this article we concentrate on the PARITY CONJECTURE FOR SELMER GROUPS, namely on the conjecture

$$(C_{BK} \pmod 2) \qquad \mathrm{ord}_{s=0} L(\iota M, s) \stackrel{?}{\equiv} h^1_f(F, M_{\mathfrak{p}}) \pmod 2.$$

In view of (0.2.1), this conjecture can be reformulated (assuming (C_{FE})) as follows:

$$(-1)^{h^1_f(F, M_{\mathfrak{p}})} \stackrel{?}{=} \varepsilon(M) \tag{0.2.2}$$

(0.3) The advantage of the formulation (0.2.2) is that the global ε-factor

$$\varepsilon(M) = \prod_v \varepsilon_v(M), \qquad \varepsilon_v(M) = \varepsilon_v(M_{\mathfrak{p}})$$

is a product of local ε-factors, which can be expressed in terms of the Galois representation $M_{\mathfrak{p}}$ alone: for $v \nmid p\infty$ (resp., $v \mid p$), $\varepsilon_v(M)$ is the local ε-factor of the representation of the Weil-Deligne group of F_v attached to the action of $\mathrm{Gal}(\overline{F}_v/F_v)$ on $M_{\mathfrak{p}}$ (resp., attached to the corresponding Fontaine module $D_{pst}(M_{\mathfrak{p}})$ over F_v). For $v \mid \infty$, $\varepsilon_v(M)$ depends on the Hodge numbers of the de Rham realization M_{dR} of M, which can be read off from $D_{dR}(M_{\mathfrak{p}})$ over F_v, for any $v \mid p$.

It makes sense, therefore, to rewrite the conjecture (0.2.2) as

$$(-1)^{h^1_f(F,V)} \stackrel{?}{=} \varepsilon(V) = \prod_v \varepsilon_v(V), \tag{0.3.1}$$

for any symplectically self-dual ($V \xrightarrow{\sim} V^*(1)$) representation of $G_{F,S}$ which is geometric (= potentially semistable at all primes above p) and pure (of weight -1).

In the present article we consider the following question: is the conjecture (0.3.1) invariant under deformation in p-adic families of representations of $G_{F,S}$? In other words, if V, V' are two representations of $G_{F,S}$ (self-dual, geometric and pure) belonging to the same p-adic family (say, in one parameter) of representations of $G_{F,S}$, is it true that

$$(-1)^{h^1_f(F,V)} / \varepsilon(V) \stackrel{?}{=} (-1)^{h^1_f(F,V')} / \varepsilon(V') \qquad ? \tag{0.3.2}$$

The main result of this article (Thm. 5.3.1) implies that (0.3.2) holds for families satisfying the Pančiškin condition at all primes $v \mid p$. The proof follows the strategy employed in [Ne 2, ch. 12] in the context of Hilbert modular forms [(1)] : multiplying both sides of (0.3.1) by a common sign (the contribution of the "trivial zeros"), we rewrite (0.3.1) as

$$(-1)^{\widetilde{h}^1_f(F,V)} \stackrel{?}{=} \widetilde{\varepsilon}(V) = \prod_v \widetilde{\varepsilon}_v(V), \tag{0.3.3}$$

where $\widetilde{h}^1_f(F,V) = \dim_{L_\mathfrak{p}} \widetilde{H}^1_f(F,V)$ is the dimension of the extended Selmer group (defined in 4.2 below) and $\widetilde{\varepsilon}_v(V) = \varepsilon_v(V)$, unless $v \mid p$ and the local Euler factor at v admits a "trivial zero". The goal is to show that both sides of (0.3.3) remain constant in the family [(2)].

The variation of $\widetilde{H}^1_f(F,V)$ in the family is controlled by the torsion submodule of a suitable $\widetilde{H}^2_f$. The generalized Cassels-Tate pairing constructed in [Ne 2, ch. 10] defines a skew-symmetric form on this torsion submodule, which implies that the parity of $\widetilde{h}^1_f(F,V)$ is constant in family:

$$(-1)^{\widetilde{h}^1_f(F,V)} = (-1)^{\widetilde{h}^1_f(F,V')}.$$

The Pančiškin condition allows us to compute explicitly the local terms $\widetilde{\varepsilon}_v(V)$ for all $v \mid p$, which yields

$$\prod_{v \mid p\infty} \widetilde{\varepsilon}_v(V) = \prod_{v \mid p\infty} \widetilde{\varepsilon}_v(V').$$

Finally, it follows from general principles (and the purity assumption) that

$$\forall v \nmid p\infty \qquad \varepsilon_v(V) = \varepsilon_v(V'),$$

hence $\widetilde{\varepsilon}(V) = \widetilde{\varepsilon}(V')$.

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(1) In [loc. cit.] we worked with automorphic ε-factors, but they coincide with the Galois-theoretical ε-factors ([Ne 2], 12.4.3, 12.5.4(iii)).

(2) Morally, $\widetilde{\varepsilon}(V)$ should be the sign in the functional equation of a p-adic L-function attached to the family.

1. Representations of the Weil-Deligne group

(1.1) The general setup ([De 1, §8], [De 2, 3.1], [Fo-PR, I.1.1-2])

(1.1.1) We use the notation of [Fo-PR, ch.I]. For a field L, denote by L^{sep} a separable closure of L and by $G_L = \mathrm{Gal}(L^{sep}/L)$ the absolute Galois group of L. Throughout this article, K will be a complete discrete valuation field of characteristic zero with finite residue field k of cardinality $q = q_k$; denote by $f = f_k \in G_k$ the GEOMETRIC Frobenius element ($f(x) = x^{1/q}$). We identify $G_k \xrightarrow{\sim} \widehat{\mathbf{Z}}$ via $f \mapsto 1$ and denote by $\nu : G_K \xrightarrow{\mathrm{can}} G_k \xrightarrow{\sim} \widehat{\mathbf{Z}}$ the canonical surjection whose kernel $\mathrm{Ker}(\nu) = I_K = I$ is the inertia group of K. The Weil group (of K) $W_K = \nu^{-1}(\mathbf{Z}) = \coprod_{n\in\mathbf{Z}} \widetilde{f}^n I$ ($\widetilde{f} \in \nu^{-1}(1)$) is equipped with the topology of a disjoint union of countably many pro-finite sets. The homomorphism

$$|\cdot| : W_K \longrightarrow q^{\mathbf{Z}}, \qquad |w| = q^{-\nu(w)}$$

corresponds to the normalized valuation $|\cdot| : K^* \longrightarrow q^{\mathbf{Z}}$ via the reciprocity isomorphism $\mathrm{rec}_K : K^* \xrightarrow{\sim} W_K^{ab}$ (normalized using the geometric Frobenius element).

(1.1.2) Let E be a field of characteristic zero.

An object of $\mathrm{Rep}_E(W_K)$ (= a representation of the Weil group of K over E) is a finite-dimensional E-vector space Δ equipped with a continuous homomorphism $\rho = \rho_\Delta : W_K \longrightarrow \mathrm{Aut}_E(\Delta)$ (with respect to the discrete topology on the target). As $\mathrm{Ker}(\rho)$ is open, $\rho(I)$ is finite and $\rho|_I$ is semi-simple.

An object of $\mathrm{Rep}_E('W_K)$ (= a representation of the Weil-Deligne group of K over E) is a pair (ρ, N), where $\rho = \rho_\Delta \in \mathrm{Rep}_E(W_K)$ and $N \in \mathrm{End}_E(\Delta)$ is a nilpotent endomorphism satisfying

$$\forall w \in W_K \qquad \rho(w)N\rho(w)^{-1} = |w|\, N.$$

Morphisms in $\mathrm{Rep}_E(W_K)$ (resp., in $\mathrm{Rep}_E('W_K)$) are E-linear maps commuting with the action of W_K (resp., with the action of W_K and N). We consider $\mathrm{Rep}_E(W_K)$ as a full subcategory of $\mathrm{Rep}_E('W_K)$ via the full embedding $\rho \mapsto (\rho, 0)$. Tensor products and duals in $\mathrm{Rep}_E('W_K)$ are defined in the usual way: $N_{\Delta\otimes\Delta'} = N_\Delta \otimes 1 + 1 \otimes N_{\Delta'}$, $N_{\Delta^*} = -(N_\Delta)^*$. The Tate twist of $\Delta \in \mathrm{Rep}_E('W_K)$ by an integer $m \in \mathbf{Z}$ is defined as $\Delta|\cdot|^m = \Delta \otimes E|\cdot|^m$, where $w \in W_K$ acts on the one-dimensional representation $E|\cdot|^m \in \mathrm{Rep}_E(W_K)$ by $|w|^m$.

The Frobenius semi-simplification

$$\Delta = (\rho, N) \mapsto \Delta^{f-ss} = (\rho^{ss}, N)$$

is an exact tensor functor $\mathrm{Rep}_E('W_K) \longrightarrow \mathrm{Rep}_E('W_K)$. The "forget the monodromy" functor

$$\Delta = (\rho, N) \mapsto \Delta^{N-ss} = (\rho, 0)$$

is an exact tensor functor $\mathrm{Rep}_E('W_K) \longrightarrow \mathrm{Rep}_E(W_K)$.

Following [Fo-PR, I.1.2.1], we put, for each $\Delta \in \mathrm{Rep}_E('W_K)$,

$$\Delta_g = \Delta^{\rho(I)}, \qquad \Delta_f = \mathrm{Ker}(N)^{\rho(I)} \subset \Delta_g, \qquad P_K(\Delta, u) = \det(1 - fu \mid \Delta_f) \in E[u].$$

We also set

$$H^0(\Delta) = \mathrm{Ker}\,(\Delta_f \xrightarrow{f-1} \Delta_f)\,.$$

(1.1.3) In the special case when E is a finite extension of $\mathbf{Q}_p$ ($p \neq \mathrm{char}(k)$) and when $V \in \mathrm{Rep}_E(G_K)$ is a representation of G_K over E (finite-dimensional and continuous with respect to the topology on E defined by the p-adic valuation), then V gives rise to a representation $WD(V) = \Delta = (\rho_\Delta, N) \in \mathrm{Rep}_E('W_K)$ acting on V, which is defined as follows ([De 1, 8.4]): there exists an open subgroup J of I which acts on V unipotently, and through the map $J \hookrightarrow I \twoheadrightarrow I(p)$, where $I(p)$ is the maximal pro-p-quotient of I (isomorphic to $\mathbf{Z}_p$). Fixing a topological generator t of $I(p)$ and an integer $a \geq 1$ such that t^a lies in the image of J, the nilpotent endomorphism

$$N = \frac{1}{a}\,\log \rho_V(t^a) \in \mathrm{End}_E(V)$$

(where $\rho_V : G_K \longrightarrow \mathrm{Aut}_E(V)$ denotes the action of G_K on V) is independent of a. Fix a lift $\widetilde{f} \in \nu^{-1}(1) \subset W_K$ of f and define

$$\rho_\Delta : W_K \longrightarrow \mathrm{Aut}_E(V)$$

by

$$\rho_\Delta(\widetilde{f}^n u) := \rho_V(\widetilde{f}^n u)\exp(-bN) \qquad\qquad (n \in \mathbf{Z},\ u \in I),$$

where $b \in \mathbf{Z}_p$ is such that the image of u in $I(p)$ is equal to t^b. The pair (ρ_Δ, N) defines an object $\Delta = WD(V)$ of $\mathrm{Rep}_E('W_K)$, the isomorphism class of which is independent of the choices of $\widetilde{f}$ and t ([De 1], Lemma 8.4.3), and which satisfies

$$\Delta_f = V^{\rho_V(I)}, \qquad H^0(\Delta) = V^{\rho_V(G_K)}.$$

(1.2) Self-dual representations

(1.2.1) Definition. *Let $\omega : W_K \longrightarrow E^*$ be a one-dimensional object of $\mathrm{Rep}_E(W_K)$. We say that $\Delta \in \mathrm{Rep}_E('W_K)$ is ω-*ORTHOGONAL* (resp., ω-*SYMPLECTIC*) if there exists a morphism in $\mathrm{Rep}_E('W_K)$ $\Delta \otimes \Delta \longrightarrow \omega$ which is non-degenerate (i.e., which induces an isomorphism $\Delta \xrightarrow{\sim} \Delta^* \otimes \omega$) and *SYMMETRIC* (resp., *SKEW-SYMMETRIC*). If $\omega = 1$, we say that Δ is *ORTHOGONAL* (resp., *SYMPLECTIC*).*

(1.2.2) (1) If Δ is ω-orthogonal, then $\det(\Delta)^2 = \omega^{\dim(\Delta)}$.
(2) If Δ is ω-symplectic, then $2 \mid \dim(\Delta)$ and $\det(\Delta) = \omega^{\dim(\Delta)/2}$.

(1.2.3) Example: For $m \geq 1$, define $sp(m) \in \mathrm{Rep}_E('W_K)$ by

$$sp(m) = \bigoplus_{i=0}^{m-1} Ee_i, \qquad N(e_i) = e_{i+1}, \qquad \forall w \in W_K \quad w(e_i) = |w|^i e_i.$$

Up to a scalar multiple, there is a unique non-degenerate morphism $sp(m) \otimes sp(m) \longrightarrow E|\cdot|^{m-1}$ in $\mathrm{Rep}_E('W_K)$, namely

$$sp(m) \otimes sp(m) \longrightarrow E|\cdot|^{m-1}, \qquad e_i \otimes e_j \mapsto \begin{cases} (-1)^i, & i+j = m-1 \\ 0, & i+j \neq m-1. \end{cases}$$

This morphism is $|\cdot|^{m-1}$-symplectic (resp., $|\cdot|^{m-1}$-orthogonal) if $2 \mid m$ (resp., if $2 \nmid m$).

(1.2.4) According to [De 2, 3.1.3(ii)], indecomposable f-semi-simple objects of $\mathrm{Rep}_E('W_K)$ are of the form $\rho \otimes sp(m)$, where $\rho \in \mathrm{Rep}_E(W_K)$ is irreducible and $m \geq 1$. This implies that, for each $|\cdot|$-symplectic representation $\Delta \xrightarrow{\sim} \Delta^*|\cdot| \in \mathrm{Rep}_E('W_K)$, the f-semi-simplification Δ^{f-ss} is a direct sum of $|\cdot|$-symplectic representations of the following type:

(1) $X \oplus X^*|\cdot|$ $(X \in \mathrm{Rep}_E('W_K))$ with the standard symplectic form $(x, x^*) \otimes (y, y^*) \mapsto x^*(y) - y^*(x)$;
(2) $\rho \otimes sp(m)$, where $m \geq 1$, $\rho \in \mathrm{Rep}_E(W_K)$ is irreducible and $|\cdot|^{2-m}$-symplectic (resp., $|\cdot|^{2-m}$-orthogonal) if $2 \nmid m$ (resp., if $2 \mid m$).

(1.3) The monodromy filtration

(1.3.1) For each $\Delta = (\rho, N) \in \mathrm{Rep}_E('W_K)$, the monodromy filtration

$$M_n\Delta := \sum_{i-j=n+1} \ker(N^i) \cap \mathrm{Im}(N^j) \qquad (n \in \mathbf{Z})$$

is the unique increasing filtration of Δ by E-vector subspaces satisfying

$$\bigcap_n M_n\Delta = 0, \qquad \bigcup_n M_n\Delta = \Delta, \qquad N(M_n\Delta) \subseteq M_{n-2}\Delta,$$
$$\forall r \geq 0 \quad N^r : \mathrm{gr}_r^M \Delta \xrightarrow{\sim} \mathrm{gr}_{-r}^M \Delta.$$

(1.3.2) Examples: (1) $N = 0 \iff M_{-1}\Delta = 0$, $M_0\Delta = \Delta$.
(2) If $N^r \neq 0 = N^{r+1}$ $(r \geq 0)$, then $M_{-r-1}\Delta = 0$, $M_{-r}\Delta = \mathrm{Im}(N^r) \neq 0$, $M_{r-1}\Delta = \mathrm{Ker}(N^r) \neq \Delta$, $M_r\Delta = \Delta$.

(1.3.3) More precisely, the endomorphism $N \in \mathrm{End}_E(\Delta)$ defines a morphism in $\mathrm{Rep}_E(W_K)$

$$N : \Delta \longrightarrow \Delta|\cdot|^{-1},$$

which implies that each $M_n\Delta$ is a sub-object of Δ^{N-ss} in $\mathrm{Rep}_E(W_K)$,

$$N : M_n\Delta \longrightarrow (M_{n-2}\Delta)|\cdot|^{-1}$$

and, for each $r \geq 0$, the endomorphism N^r induces an isomorphism in $\mathrm{Rep}_E(W_K)$

$$N^r : \mathrm{gr}^M_r \Delta \xrightarrow{\sim} (\mathrm{gr}^M_{-r}\Delta)|\cdot|^{-r}.$$

(1.3.4) The monodromy filtration on the dual representation $\Delta^* = (\rho^*, -N^*)$ satisfies $M_n\Delta^* = (M_{-1-n}\Delta)^\perp$ ($n \in \mathbf{Z}$), which yields canonical isomorphisms in $\mathrm{Rep}_E(W_K)$

$$\forall m \leq n \qquad M_n\Delta^*/M_m\Delta^* \xrightarrow{\sim} (M_{-1-m}\Delta/M_{-1-n}\Delta)^*.$$

(1.3.5) If $\langle\ ,\ \rangle : \Delta \otimes \Delta \longrightarrow E \otimes \omega$ is an ω-symplectic form on Δ, then, for each $r \geq 0$, the formula $\langle x, y\rangle_r = \langle N^r x, y\rangle$ defines an $\omega|\cdot|^{-r}$-symplectic (resp., $\omega|\cdot|^{-r}$-orthogonal) form on $\mathrm{gr}^M_r\Delta \in \mathrm{Rep}_E(W_K)$ if $2 \mid r$ (resp., if $2 \nmid r$).

(1.3.6) Dimensions. The dimensions

$$d_r = d_r(\Delta) = \dim\, \mathrm{gr}^M_r \Delta = d_{-r} \qquad\qquad (r \in \mathbf{Z})$$

can be interpreted as follows. By the Jacobson-Morozov theorem, there exists a (non-unique) representation

$$\rho : sl(2) = sl(2, E) \longrightarrow \mathrm{End}_E(\Delta)$$

such that $\rho(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = N$. Putting $H = \rho(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$ and $\Delta_m = \{x \in \Delta \mid Hx = mx\}$ ($m \in \mathbf{Z}$), then

$$M_n\Delta = \sum_{m \leq n} \Delta_m.$$

Decomposing Δ as a representation of $sl(2)$

$$\Delta \xrightarrow{\sim} \bigoplus_{j \geq 0} \left(S^j E^2\right)^{\oplus m_j(\Delta)},$$

then the multiplicities $m_j = m_j(\Delta)$ are related to other numerical invariants of Δ as follows:

$$\dim(\Delta) = \sum_{j\geq 0}(j+1)m_j, \quad (\forall r \geq 0) \quad d_{-r} = \sum_{i \geq 0} m_{r+2i}, \quad m_r = d_{-r} - d_{-r-2},$$
$$\dim\, \mathrm{Im}(N^r) = d_r + 2\sum_{j > r} d_j, \quad \dim\, \mathrm{Ker}(N^{r+1}) = d_0 + 2\sum_{j=1}^{r} d_j + d_{r+1}. \tag{1.3.6.1}$$

(1.4) Purity

(1.4.1) Definition. *Let E' be a field containing E and $a \in \mathbf{Z}$. We say that $\alpha \in E'$ is a q^a-*Weil number of weight* $n \in \mathbf{Z}$ if α is algebraic over $\mathbf{Q}$, there exists $N \in \mathbf{Z}$ such that $q^N\alpha$ is integral over $\mathbf{Z}$, and for each embedding $\sigma : \mathbf{Q}(\alpha) \hookrightarrow \mathbf{C}$, the usual archimedean absolute value of $\sigma(\alpha)$ is equal to $|\sigma(\alpha)|_\infty = q^{an/2}$.*

(1.4.2) Definition. *We say that $\Delta \in \mathrm{Rep}_E('W_K)$ is* strictly pure of weight *$n \in \mathbf{Z}$ if $\Delta = \rho \in \mathrm{Rep}_E(W_K)$ and if for each $w \in W_K$ all eigenvalues of $\rho(w)$ are $q^{\nu(w)}$-Weil numbers of weight $n \in \mathbf{Z}$.*

(1.4.3) Definition. *We say that $\Delta \in \mathrm{Rep}_E('W_K)$ is* pure of weight *$n \in \mathbf{Z}$ if, for each $r \in \mathbf{Z}$, $\mathrm{gr}^M_r \Delta \in \mathrm{Rep}_E(W_K)$ is strictly pure of weight $n + r$.*

(1.4.4) (1) Each representation $\rho \in \mathrm{Rep}_E(W_K)$ with finite image is strictly pure of weight 0.
(2) If $\Delta, \Delta' \in \mathrm{Rep}_E('W_K)$ are (strictly) pure of weights n and n', respectively, then $\Delta \otimes \Delta'$ is (strictly) pure of weight $n + n'$, and Δ^* is (strictly) pure of weight $-n$.
(3) For each $m \in \mathbf{Z}$, $E|\cdot|^m$ is strictly pure of weight $-2m$.
(4) For each $\rho \in \mathrm{Rep}_E(W_K)$ and $m \geq 1$,

$$\begin{aligned}\Delta = \rho \otimes sp(m) \text{ is pure of weight } n &\iff \rho \text{ is strictly pure of weight } n + m - 1\\ &\Longrightarrow \Delta_f = \rho^I|\cdot|^{m-1} \text{ is strictly pure of weight } n + 1 - m.\end{aligned}$$

(5) If $\Delta \in \mathrm{Rep}_E('W_K)$ is pure of weight $n < 0$, then all eigenvalues of $\rho(\widetilde{f})$ (for any $\widetilde{f} \in \nu^{-1}(1)$) on $\mathrm{Ker}(N) \subseteq M_0\Delta$ are q-Weil numbers of weights $\leq n < 0$, hence $H^0(\Delta) = 0$.
(6) If $\Delta \in \mathrm{Rep}_E('W_K)$ is pure of weight n (but not necessarily f-semi-simple), then $\Delta \xrightarrow{\sim} \bigoplus \rho_j \otimes sp(m_j)$, where each $\rho_j \in \mathrm{Rep}_E(W_K)$ is strictly pure of weight $n + m_j - 1$.

(1.4.5) Definition. *In the situation of 1.1.3, we say that $V \in \mathrm{Rep}_E(G_K)$ is* pure of weight *$n \in \mathbf{Z}$ if $WD(V) \in \mathrm{Rep}_E('W_K)$ is pure of weight $n \in \mathbf{Z}$ in the sense of 1.4.3.*

(1.5) Specialization of representations of the Weil-Deligne group

(1.5.1) Let $\mathcal{O}$ be a discrete valuation ring containing $\mathbf{Q}$; denote by E (resp., $k_\mathcal{O}$) the field of fractions (resp., the residue field) of $\mathcal{O}$.

(1.5.2) An object of $\mathrm{Rep}_\mathcal{O}('W_K)$ (= a representation of the Weil-Deligne group of K over $\mathcal{O}$) consists of a free $\mathcal{O}$-module of finite type T, a continuous homomorphism $\rho = \rho_T : W_K \longrightarrow \mathrm{Aut}_\mathcal{O}(T)$ (with respect to the discrete topology on the target) and a nilpotent endomorphism $N = N_T \in \mathrm{End}_\mathcal{O}(T)$ satisfying

$$\forall w \in W_K \qquad \rho(w) N \rho(w)^{-1} = |w|\, N.$$

The GENERIC FIBRE (resp., the SPECIAL FIBRE) of T is the representation $T_\eta = T \otimes_{\mathcal{O}} E \in \mathrm{Rep}_E('W_K)$ (resp., the representation $T_s = T \otimes_{\mathcal{O}} k_{\mathcal{O}} \in \mathrm{Rep}_{k_{\mathcal{O}}}('W_K)$). We denote by N_η (resp., N_s) the monodromy operator $N_T \otimes 1$ on T_η (resp., on T_s).

(1.5.3) For $T \in \mathrm{Rep}_{\mathcal{O}}('W_K)$, we denote by T^* the representation $T^* = \mathrm{Hom}_{\mathcal{O}}(T, \mathcal{O})$ (equipped with the dual action of W_K and the monodromy operator $N_{T^*} = -(N_T)^*$). Given a representation $\omega : W_K \longrightarrow \mathcal{O}^*$, we say that T is ω-ORTHOGONAL (resp., ω-SYMPLECTIC) if there exists an isomorphism $j : T \xrightarrow{\sim} T^* \otimes \omega$ in $\mathrm{Rep}_{\mathcal{O}}('W_K)$ satisfying $j^* \otimes \omega = j$ (resp., $j^* \otimes \omega = -j$).

(1.5.4) PROPOSITION. *Assume that $T \in \mathrm{Rep}_{\mathcal{O}}('W_K)$ is $|\cdot|$-symplectic (hence so are T_η and T_s) and that $T_s \in \mathrm{Rep}_{k_{\mathcal{O}}}('W_K)$ is pure (necessarily of weight -1). Then:*

(1) $\forall j \geq 0 \qquad m_j(T_\eta) = m_j(T_s)$.

(2) $\forall j \geq 0 \qquad \dim_E \mathrm{Ker}(N_\eta^j) = \dim_{k_{\mathcal{O}}} \mathrm{Ker}(N_s^j)$.

(3) *For each $j \geq 0$, the natural injective map $\big(\mathrm{Ker}(N_\eta^j) \cap T\big) \otimes_{\mathcal{O}} k_{\mathcal{O}} \longrightarrow \mathrm{Ker}(N_s^j)$ is an isomorphism.*

Proof. It is enough to prove (1), since (2) is a consequence of (1) and the formulas (1.3.6.1), and (2) is equivalent to (3) for trivial reasons. We prove (1) by induction on $r = \min\{j \geq 0 \mid N_T^{j+1} = 0\}$. If $r = 0$, then there is nothing to prove. Assume that $r \geq 1$ and that (1) holds whenever $N_T^r = 0$. Recall from 1.3.2(2) and 1.3.5 that

$$\begin{gathered} M_{-r-1}(T_\eta) = 0 \neq M_{-r}(T_\eta) = \mathrm{Im}(N_\eta^r), \quad M_{r-1}(T_\eta) = \mathrm{Ker}(N_\eta^r) \neq T_\eta = M_r(T_\eta), \\ M_{-r-1}(T_s) = 0, \quad M_{-r}(T_s) = \mathrm{Im}(N_s^r), \quad M_{r-1}(T_s) = \mathrm{Ker}(N_s^r), \quad M_r(T_s) = T_s \end{gathered}$$

and that $M_{-r}(T_\eta)$ is $|\cdot|^{r+1}$-symplectic (resp., $|\cdot|^{r+1}$-orthogonal) if $2 \mid r$ (resp., if $2 \nmid r$). The latter property implies that, for any eigenvalue $\alpha \in \overline{k_{\mathcal{O}}}$ of any lift $\widetilde{f} \in \nu^{-1}(1)$ of f acting on $(M_{-r}(T_\eta) \cap T) \otimes_{\mathcal{O}} k_{\mathcal{O}}$ there exists another eigenvalue α' such that $\alpha\alpha' = q^{-r-1}$. On the other hand, $(M_{-r}(T_\eta) \cap T) \otimes_{\mathcal{O}} k_{\mathcal{O}} \in \mathrm{Rep}_{k_{\mathcal{O}}}(W_K)$ is a sub-object of T_s in $\mathrm{Rep}_{k_{\mathcal{O}}}('W_K)$, and all eigenvalues of $\widetilde{f}$ on T_s are q-Weil numbers of weights contained in $\{-r-1, -r, \ldots, r-1\}$; thus both α and α' are q-Weil numbers of weight $-r-1$. In other words, $(\mathrm{Im}(N_\eta^r) \cap T) \otimes_{\mathcal{O}} k_{\mathcal{O}} = (M_{-r}(T_\eta) \cap T) \otimes_{\mathcal{O}} k_{\mathcal{O}}$ is strictly pure of weight $-r-1$, hence is contained in $M_{-r}(T_s) = \mathrm{Im}(N_s^r) = (\mathrm{Im}(N_T^r)) \otimes_{\mathcal{O}} k_{\mathcal{O}}$. The opposite inclusion being trivial, we deduce that $\mathrm{Im}(N_T^r)$ is equal to $\mathrm{Im}(N_\eta^r) \cap T$, hence is a direct summand of T (as an $\mathcal{O}$-module); it follows that

$$m_r(T_s) = \dim_{k_{\mathcal{O}}} \mathrm{Im}(N_s^r) = \dim_E \mathrm{Im}(N_\eta^r) = m_r(T_\eta).$$

The representation $T' = (M_{r-1}(T_\eta) \cap T)/(M_{-r}(T_\eta) \cap T) \in \mathrm{Rep}_{\mathcal{O}}('W_K)$ is also $|\cdot|$-symplectic, satisfies $N_{T'}^r = 0$, and T'_s is pure of weight -1. By induction hypothesis, we have

$$\forall j \geq 0 \qquad m_j(T'_s) = m_j(T'_\eta).$$

The relations

$$m_j(T'_?) = \begin{cases} m_j(T_?), & j \neq r, r-2 \\ m_{r-2}(T_?) + m_r(T_?), & j = r-2 \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad (? = \eta, s)$$

then imply

$$\forall j \geq 0 \qquad m_j(T_s) = m_j(T_\eta).$$

2. Local ε-factors

(2.1) General facts

(2.1.1) Fix an algebraically closed field $E' \supset E$. Let $\psi : K \longrightarrow E'^*$ be a non-trivial continuous homomorphism (with respect to the discrete topology on the target); it always exists. If $\psi' : K \longrightarrow E'^*$ is another non-trivial continuous homomorphism, then there exists unique $a \in K^*$ such that $\psi' = \psi_a$, where $\psi_a(y) = \psi(ay)$. Denote by μ_ψ the unique E'-valued Haar measure on K which is self-dual with respect to ψ; then

$$\forall a \in K^* \qquad \mu_{\psi_a} = |a|^{1/2}\, \mu_\psi, \tag{2.1.1.1}$$

and every non-zero E'-valued Haar measure μ on K is a scalar multiple of μ_ψ: $\mu = b\, \mu_\psi$, for some $b \in E'^*$.

(2.1.2) Deligne [De 1] associated to each triple (Δ, ψ, μ), where $\Delta \in \mathrm{Rep}_E('W_K)$ and ψ, μ are as in 2.1.1, the local ε-factor $\varepsilon(\Delta, \psi, \mu) \in E'^*$ satisfying the following properties.

(2.1.2.1) $\varepsilon(\Delta, \psi, \mu) = \varepsilon(\Delta^{f-ss}, \psi, \mu)$.

(2.1.2.2) If $0 \longrightarrow \rho' \longrightarrow \rho \longrightarrow \rho'' \longrightarrow 0$ is an exact sequence in $\mathrm{Rep}_E(W_K)$, then $\varepsilon(\rho, \psi, \mu) = \varepsilon(\rho', \psi, \mu)\varepsilon(\rho'', \psi, \mu)$.

(2.1.2.3) $\varepsilon_0(\Delta, \psi, \mu) = \varepsilon(\Delta, \psi, \mu) \det(-f \mid \Delta_f)$ depends only on $\Delta^{N-ss} \in \mathrm{Rep}_E(W_K)$. As $(\Delta^{N-ss})_f = \Delta_g$, it follows that

$$\varepsilon(\Delta, \psi, \mu) = \varepsilon(\Delta^{N-ss}, \psi, \mu) \det(-f \mid \Delta_g/\Delta_f).$$

(2.1.2.4) $\forall a \in K^* \qquad \varepsilon(\Delta, \psi_a, \mu) = (\det \Delta)(a)\, |a|^{-\dim(\Delta)}\, \varepsilon(\Delta, \psi, \mu)$.

(2.1.2.5) $\forall b \in E'^* \qquad \varepsilon(\Delta, \psi, b\,\mu) = b^{\dim(\Delta)}\, \varepsilon(\Delta, \psi, \mu)$.

(2.1.2.6) If $\Delta = \rho \in \mathrm{Rep}_E(W_K)$, then $\varepsilon(\rho, \psi, \mu)\, \varepsilon(\rho^* |\cdot|, \psi_{-1}, \mu^*) = 1$ (where μ^* is the measure dual to μ with respect to ψ).

(2.1.2.7) If $\Delta = \rho \in \mathrm{Rep}_E(W_K)$, and if $\chi : W_K/I \longrightarrow E^*$ is an unramified one-dimensional representation, then

$$\varepsilon(\rho \otimes \chi, \psi, \mu) = \varepsilon(\rho, \psi, \mu)\, \chi(\pi)^{a(\rho) + \dim(\rho) n(\psi)},$$

where π is a prime element of $\mathcal{O}_K$ and $a(\rho)$ (resp., $n(\psi)$) is the conductor exponent of ρ (resp., of ψ).

(2.1.2.8) ([Fo-PR, I.1.2.3]) For an exact sequence in $\mathrm{Rep}_E('W_K)$

$$(\beta) \qquad\qquad 0 \longrightarrow \Delta' \longrightarrow \Delta \longrightarrow \Delta'' \longrightarrow 0,$$

set $P_K(\beta) = P_K(\Delta, u)/P_K(\Delta', u)P_K(\Delta'', u)$, $a(\beta) = \dim \Delta'_f + \dim \Delta''_f - \dim \Delta_f$, $\varepsilon(\beta) = \varepsilon(\Delta, \psi, \mu)/\varepsilon(\Delta', \psi, \mu)\,\varepsilon(\Delta'', \psi, \mu)$, and similarly for the dual exact sequence

$$(\beta^*|\cdot|) \qquad 0 \longrightarrow \Delta''^*|\cdot| \longrightarrow \Delta^*|\cdot| \longrightarrow \Delta'^*|\cdot| \longrightarrow 0;$$

then

$$P_K(\beta^*|\cdot|, u^{-1}) = \varepsilon(\beta)\, u^{a(\beta)}\, P_K(\beta, u).$$

(2.1.3) Lemma. *If $\Delta \in \mathrm{Rep}_E('W_K)$, then $\varepsilon(\Delta, \psi, \mu)\,\varepsilon(\Delta^*|\cdot|, \psi_{-1}, \mu^*) = 1$ (where μ^* is the measure dual to μ with respect to ψ).*

Proof. Thanks to (2.1.2.1-2), we can assume that Δ is f-semi-simple and indecomposable: $\Delta = \rho \otimes sp(m)$, $\rho \in \mathrm{Rep}_E(W_K)$, $m \geq 1$. In this case

$$\Delta_g = \bigoplus_{j=0}^{m-1} \rho^I|\cdot|^j, \qquad \Delta_g/\Delta_f = \bigoplus_{j=0}^{m-2} \rho^I|\cdot|^j, \qquad \Delta^*|\cdot| = \rho^* \otimes sp(m)|\cdot|^{2-m}$$

$$(\Delta^*|\cdot|)_g/(\Delta^*|\cdot|)_f = \bigoplus_{j=0}^{m-2} (\rho^*)^I|\cdot|^{2-m+j} = (\Delta_g/\Delta_f)^*$$

(as $\rho(I)$ is finite, we have $(\rho^*)^I = (\rho^I)^*$), hence

$$\det(-f \mid \Delta_g/\Delta_f)\, \det(-f \mid (\Delta^*|\cdot|)_g/(\Delta^*|\cdot|)_f) = 1;$$

we deduce that

$$\varepsilon(\Delta, \psi, \mu)\,\varepsilon(\Delta^*|\cdot|, \psi_{-1}, \mu^*) = \varepsilon(\Delta^{N-ss}, \psi, \mu)\,\varepsilon((\Delta^*|\cdot|)^{N-ss}, \psi_{-1}, \mu^*),$$

which is equal to 1, by (2.1.2.6).

(2.2) $|\cdot|$-symplectic representations

(2.2.1) Proposition. *Let $\Delta \xrightarrow{\sim} \Delta^*|\cdot| \in \mathrm{Rep}_E('W_K)$ be $|\cdot|$-symplectic. Then:*
(1) $\varepsilon(\Delta) := \varepsilon(\Delta, \psi, \mu_\psi)$ *does not depend on* ψ.
(2) $\varepsilon(\Delta) = \pm 1$; *more precisely:*
(3) *If* $\rho \xrightarrow{\sim} \rho^*|\cdot| \in \mathrm{Rep}_E(W_K)$ *is $|\cdot|$-symplectic, then* $\varepsilon(\rho) = \pm 1$.
(4) *If* $\Delta = X \oplus X^*|\cdot|$ *is as in 1.2.4(1), then* $\varepsilon(\Delta) = \varepsilon(\Delta^{N-ss}) = (\det X)(-1)$.
(5) *If* $\Delta = \rho \otimes sp(2n+1)$ ($\rho \in \mathrm{Rep}_E(W_K)$, $n \geq 0$), *then* $\rho|\cdot|^n \in \mathrm{Rep}_E(W_K)$ *is $|\cdot|$-symplectic and* $\varepsilon(\Delta) = \varepsilon(\Delta^{N-ss}) = \varepsilon(\rho|\cdot|^n)$.

(6) *If* $\Delta = \rho \otimes sp(2n)$ $(\rho \in \mathrm{Rep}_E(W_K),\ n \geq 1)$, *then* $\rho|\cdot|^{n-1} \in \mathrm{Rep}_E(W_K)$ *is orthogonal, there is an exact sequence in* $\mathrm{Rep}_E('W_K)$

$$0 \longrightarrow \Delta^+ \longrightarrow \Delta \longrightarrow \Delta^- \longrightarrow 0$$
$$\Delta^+ = \rho \otimes sp(n)|\cdot|^n, \qquad \Delta^- = \rho \otimes sp(n),$$

$H^0(\Delta^-) = H^0(\rho|\cdot|^{n-1})$ *and*

$$\varepsilon(\Delta) = (-1)^{\dim_E H^0(\Delta^-)}(\det \Delta^+)(-1), \qquad \varepsilon(\Delta^{N-ss}) = (\det \Delta^+)(-1).$$

Proof. (1) For each $a \in K^*$,

$$\begin{aligned}
\varepsilon(\Delta, \psi_a, \mu_{\psi_a}) &= \varepsilon(\Delta, \psi_a, |a|^{1/2}\mu_\psi) && \text{(by (2.1.1.1))}\\
&= |a|^{\dim(\Delta)/2}\,\varepsilon(\Delta, \psi_a, \mu_\psi) && \text{(by (2.1.2.5))}\\
&= (\det \Delta)(a)\,|a|^{-\dim(\Delta)/2}\,\varepsilon(\Delta, \psi, \mu_\psi) && \text{(by (2.1.2.4))}\\
&= \varepsilon(\Delta, \psi, \mu_\psi). && \text{(by 1.2.2(2))}
\end{aligned}$$

(2) Writing Δ^{f-ss} as a direct sum of $|\cdot|$-symplectic representations of the form 1.2.4(1) or 1.2.4(2), the statement follows from the explicit formulas (4)-(6) and (3), proved below.
(3) Combining (2.1.2.6), (2.1.2.4) and 1.2.2(2), we obtain

$$\begin{aligned}
\varepsilon(\rho, \psi, \mu_\psi)^2 = \varepsilon(\rho, \psi, \mu_\psi)\,(\det \rho)(-1)\varepsilon(\rho, \psi, \mu_\psi) = \varepsilon(\rho, \psi, \mu_\psi)\,\varepsilon(\rho, \psi_{-1}, \mu_\psi) =\\
= \varepsilon(\rho, \psi, \mu_\psi)\,\varepsilon(\rho^*|\cdot|, \psi_{-1}, \mu_\psi) = 1.
\end{aligned}$$

(4) As in the proof of (3), Lemma 2.1.3 together with (2.1.2.4) yield

$$\begin{aligned}
\varepsilon(\Delta) = \varepsilon(X, \psi, \mu_\psi)\,\varepsilon(X^*|\cdot|, \psi, \mu_\psi) = (\det X)(-1)\,\varepsilon(X, \psi_{-1}, \mu_\psi)\,\varepsilon(X^*|\cdot|, \psi, \mu_\psi) =\\
= (\det X)(-1).
\end{aligned}$$

Replacing X by X^{N-ss}, we obtain $\varepsilon(\Delta^{N-ss}) = (\det X^{N-ss})(-1) = (\det X)(-1) = \varepsilon(\Delta)$.
(5) As $\Delta = \rho \otimes sp(2n+1)$ is $|\cdot|$-symplectic, the representation $\rho|\cdot|^n$ is also $|\cdot|$-symplectic, by 1.2.3-4 (in particular, $\det(\rho) = |\cdot|^{(1-2n)\dim(\rho)/2}$). The same calculation as in the proof of Lemma 2.1.3 yields

$$\Delta_g/\Delta_f = \bigoplus_{j=0}^{2n-1} \rho^I|\cdot|^j, \qquad (\rho^I|\cdot|^j)^* = (\rho^*|\cdot|^{-j})^I = \rho^I|\cdot|^{2n-1-j},$$
$$\Delta_g/\Delta_f = \bigoplus_{j=0}^{n-1} \rho^I|\cdot|^j \oplus \left(\rho^I|\cdot|^j\right)^*,$$

which implies that $\det(-f \mid \Delta_g/\Delta_f) = 1$, hence

$$\varepsilon(\Delta) = \varepsilon(\Delta^{N-ss}) = \prod_{j=0}^{2n} \varepsilon(\rho|\cdot|^j, \psi, \mu_\psi) = \varepsilon(\rho|\cdot|^n) \prod_{j=0}^{n-1} \varepsilon(\rho|\cdot|^j \oplus (\rho|\cdot|^j)^* |\cdot|) \overset{(4)}{=} \varepsilon(\rho|\cdot|^n).$$

(6) As $\Delta = \rho \otimes sp(2n)$ is $|\cdot|$-symplectic, the representation $\rho|\cdot|^{n-1}$ is orthogonal, by 1.2.3. The same calculation as in the proof of (5) shows that

$$\varepsilon(\Delta^{N-ss}) = \prod_{j=0}^{n-1} \varepsilon(\rho|\cdot|^j \oplus (\rho|\cdot|^j)^* |\cdot|) \overset{(4)}{=} \prod_{j=0}^{n-1} \det(\rho|\cdot|^j)(-1) = (\det \Delta^+)(-1)$$

and

$$\Delta_g/\Delta_f = \rho^I|\cdot|^{n-1} \oplus \bigoplus_{j=0}^{n-2} \rho^I|\cdot|^j \oplus \left(\rho^I|\cdot|^j\right)^*, \quad \det(-f \mid \Delta_g/\Delta_f) = \left(-f \mid \rho^I|\cdot|^{n-1}\right).$$

As $\rho(I)$ acts semi-simply, the (unramified) representation $V = \rho^I|\cdot|^{n-1} \in \mathrm{Rep}_E(W_K)$ is also orthogonal; applying Lemma 2.2.2 below to $u = f$ acting on V, we obtain

$$\varepsilon(\Delta)/\varepsilon(\Delta^{N-ss}) = \det(-f \mid \Delta_g/\Delta_f) = (-1)^{\dim_E \mathrm{Ker}(f-1: V \longrightarrow V)}.$$

Finally,

$$\mathrm{Ker}(V \xrightarrow{f-1} V) = H^0(\rho|\cdot|^{n-1}) = H^0(\rho \otimes sp(n)) = H^0(\Delta^-).$$

(2.2.2) Lemma. *Let (V, q) be a non-degenerate quadratic space over a field L of characteristic not equal to 2. If $u \in O(V, q)$, then*

$$\det(-u) = (-1)^{\dim_L \mathrm{Ker}(u-1)}, \qquad \det(u) = (-1)^{\dim_L \mathrm{Im}(u-1)}.$$

Proof. The following short argument is due to J. Oesterlé. The two formulas being equivalent, it is enough to prove the second one. Let $a \in V$, $q(a) \neq 0$; denote by $s \in O^-(V, q)$ the reflection with respect to the hyperplane $\mathrm{Ker}(s-1) = a^\perp$. A short calculation shows that

$$\mathrm{Ker}(su-1) = \begin{cases} \mathrm{Ker}(u-1) \oplus Lb, & a = (u-1)b,\ b \in V \\ \mathrm{Ker}(u-1) \cap a^\perp \subsetneq \mathrm{Ker}(u-1), & a \notin \mathrm{Im}(u-1), \end{cases}$$

hence

$$\dim_L \mathrm{Im}(su-1) = \dim_L \mathrm{Im}(u-1) \mp 1. \tag{2.2.2.1}$$

Writing u as a product of $r \geq 1$ reflections, we deduce from (2.2.2.1), by induction, that $\dim_L \mathrm{Im}(u-1) \equiv r \pmod 2$, as claimed.

(2.2.3) Proposition. *Let $\Delta \xrightarrow{\sim} \Delta^*|\cdot| \in \mathrm{Rep}_E('W_K)$ be $|\cdot|$-symplectic and pure (of weight -1). Assume that there exists an exact sequence in $\mathrm{Rep}_E('W_K)$*

$$(\beta) \qquad 0 \longrightarrow \Delta^+ \longrightarrow \Delta \longrightarrow \Delta^- \longrightarrow 0$$

such that the isomorphism $\Delta \xrightarrow{\sim} \Delta^*|\cdot|$ induces isomorphisms $\Delta^\pm \xrightarrow{\sim} (\Delta^\mp)^*|\cdot|$. Assume, in addition, that there exists a direct sum decomposition $\Delta = \Delta_1 \oplus \Delta_2$ in $\mathrm{Rep}_E('W_K)$ compatible with the isomorphism $\Delta \xrightarrow{\sim} \Delta^*|\cdot|$ and the exact sequence (β), and such that $H^0(\Delta_2^-) = 0$, while

$$(\beta_1) \qquad 0 \longrightarrow \Delta_1^+ \longrightarrow \Delta_1 \longrightarrow \Delta_1^- \longrightarrow 0$$

is a direct sum of exact sequences of the type considered in Proposition 2.2.1(6). Then

$$\varepsilon(\Delta) = (-1)^{\dim_E H^0(\Delta^-)}(\det\Delta^+)(-1), \qquad \varepsilon(\Delta^{N-ss}) = (\det\Delta^+)(-1).$$

Proof. It is enough to treat separately Δ_1 and Δ_2. For $\Delta = \Delta_1$, the statement follows from Proposition 2.2.1(6). For $\Delta = \Delta_2$, the assumption $H^0(\Delta^-) = 0$ implies that $P_K(\Delta^-, 1) \neq 0$. As Δ is pure of weight $-1 < 0$, we also have $H^0(\Delta^+) \subseteq H^0(\Delta) = 0$, by 1.4.4(5), hence $P_K(\Delta^+, 1)P_K(\Delta, 1) \neq 0$. Letting $u \longrightarrow 1$ in (2.1.2.8), we obtain $\varepsilon(\beta) = 1$, hence

$$\varepsilon(\Delta) = \varepsilon(\Delta^+, \psi, \mu_\psi)\,\varepsilon(\Delta^-, \psi, \mu_\psi) = \varepsilon(\Delta^+ \oplus (\Delta^+)^*|\cdot|) = (\det\Delta^+)(-1).$$

Finally,

$$\begin{aligned}\varepsilon(\Delta^{N-ss}) &= \varepsilon((\Delta^+)^{N-ss}, \psi, \mu_\psi)\,\varepsilon((\Delta^-)^{N-ss}, \psi, \mu_\psi) = \\ &= \varepsilon((\Delta^+)^{N-ss} \oplus ((\Delta^+)^{N-ss})^*|\cdot|) = (\det(\Delta^+)^{N-ss})(-1) = (\det\Delta^+)(-1).\end{aligned}$$

(2.2.4) Proposition. *In the situation of 1.5.4, $\varepsilon(T_s) = \varepsilon(T_\eta) \in \{\pm 1\}$.*

Proof. For any $\mathcal{O}$-module X, denote by $red : X \longrightarrow X \otimes_{\mathcal{O}} k_{\mathcal{O}}$ the canonical surjection. Proposition 1.5.4 implies that

$$red\left(\frac{T \cap (T_\eta)_g}{T \cap (T_\eta)_f}\right) = (T_s)_g/(T_s)_f,$$

hence

$$\begin{aligned} red\left(\varepsilon(T_\eta)/\varepsilon(T_\eta^{N-ss})\right) &= red\left(\det(-f \mid (T_\eta)_g/(T_\eta)_f)\right) = \\ &= (\det(-f \mid (T_s)_g/(T_s)_f) = \varepsilon(T_s)/\varepsilon(T_s^{N-ss}).\end{aligned}$$

As $\varepsilon(T_\eta), \varepsilon(T_\eta^{N-ss}), \varepsilon(T_s), \varepsilon(T_s^{N-ss}) \in \{\pm 1\}$, we are reduced to showing that

$$red\left(\varepsilon(T_\eta^{N-ss})\right) \stackrel{?}{=} \varepsilon(T_s^{N-ss}).$$

The following argument is based on a suggestion of T. Saito. We can replace (ρ_T, N_T) by $(\rho_T, 0)$ and assume that $N_T = 0$. Furthermore, after replacing E by a finite extension, we can assume (see [De 1, 4.10]) that

$$T_\eta^{f-ss} = \bigoplus_\alpha \rho_\alpha \otimes \omega_\alpha,$$

where $\rho_\alpha \in \mathrm{Rep}_L(W_K)$ for a subfield $L \subset \mathcal{O}$ of finite degree over $\mathbf{Q}$, and $\omega_\alpha : W_K/I \longrightarrow \mathcal{O}^*$ is an unramified representation of rank 1. We have

$$\forall w \in W_K \qquad \mathrm{Tr}(w \mid T_s) = red\left(\mathrm{Tr}(w \mid T_\eta)\right),$$

hence

$$T_s^{f-ss} = \bigoplus_\alpha \rho_\alpha \otimes red(\omega_\alpha).$$

Applying (2.1.2.7) to each direct summand, we obtain

$$red\left(\varepsilon(T_\eta)\right) = \prod_\alpha red\left(\varepsilon(\rho_\alpha \otimes \omega_\alpha, \psi, \mu_\psi)\right) = \prod_\alpha \varepsilon(\rho_\alpha \otimes red(\omega_\alpha), red \circ \psi, red \circ \mu_\psi) = \\ = \varepsilon(T_s).$$

(2.3) The archimedean case

Let $L = \mathbf{R}$ or $\mathbf{C}$. If H is a pure $\mathbf{R}$-Hodge structure over L ([Fo-PR, III.1]) of weight -1, then

$$H = \bigoplus_{r>0} H_r(r)^{\oplus m_r},$$

where H_r is a two-dimensional pure $\mathbf{R}$-Hodge structure over L of Hodge type $(2r-1, 0)$, $(0, 2r-1)$. The standard formulas ([De 3, 5.3], [Fo-PR, III.1.1.10, III.1.2.7]) yield

$$\varepsilon(H_r(r)) = (-1)^{[L:\mathbf{R}]\, r} \times \begin{cases} 1, & L = \mathbf{R} \\ -1, & L = \mathbf{C}. \end{cases}$$

As

$$\forall p < 0 \qquad h^{p,-1-p}(H) = m_{-p},$$

we obtain

$$\varepsilon(H) = (-1)^{[L:\mathbf{R}]\, d^-(H)} \times \begin{cases} 1, & L = \mathbf{R} \\ (-1)^{(\dim_{\mathbf{R}} H)/2}, & L = \mathbf{C}, \end{cases} \quad d^-(H) = \sum_{p<0} p\, h^{p,q}(H). \tag{2.3.1}$$

3. Local p-adic Galois representations

(3.1) General facts

(3.1.1) Notation. Let p be the characteristic of the residue field k of K; then $q = p^h$ and K is a finite extension of $\mathbf{Q}_p$. Denote by $\sigma \in \mathrm{Gal}(\mathbf{Q}_p^{ur}/\mathbf{Q}_p) \xrightarrow{\sim} G_{\mathbf{F}_p}$ the lift of the ARITHMETIC Frobenius element $x \mapsto x^p$. Let L be another finite extension of $\mathbf{Q}_p$.

We use the standard notation

$$\mathrm{Rep}_{cris,L}(G_K) \subset \mathrm{Rep}_{st,L}(G_K) \subset \mathrm{Rep}_{pst,L}(G_K) = \mathrm{Rep}_{dR,L}(G_K) \subset \mathrm{Rep}_L(G_K)$$

for Fontaine's hiearchy of (finite-dimensional, L-linear) representations of G_K ([Fo]), and

$$D_{cris}(V) = (V \otimes_{\mathbf{Q}_p} B_{cris})^{G_K}, \qquad D_{st}(V) = (V \otimes_{\mathbf{Q}_p} B_{st})^{G_K},$$
$$D_{pst}(V) = \varinjlim_{K'} (V \otimes_{\mathbf{Q}_p} B_{st})^{G_{K'}},$$
$$D^i_{dR}(V) = (V \otimes_{\mathbf{Q}_p} t^i B^+_{dR})^{G_K} \subset D_{dR}(V) = (V \otimes_{\mathbf{Q}_p} B_{dR})^{G_K}$$

for various Fontaine's functors (above, $V \in \mathrm{Rep}_L(G_K)$, and K' runs through all finite extensions of K contained in $\overline{K}$). As in [Bl-Ka], put $H^i(K,-) = H^i_{\mathrm{cont}}(G_K,-)$ and, for $* = e, f, st, g$,

$$H^1_*(K,V) = \mathrm{Ker}\left(H^1(K,V) \longrightarrow H^1(K, V \otimes_{\mathbf{Q}_p} B_*)\right),$$
$$B_e = B_{cris}^{\varphi=1}, \qquad B_f = B_{cris}, \qquad B_g = B_{dR}.$$

If K'/K is a finite Galois extension, then

$$H^1_*(K,V) = H^1_*(K',V)^{\mathrm{Gal}(K'/K)}, \qquad (* = \emptyset, e, f, st, g) \qquad (3.1.1.1)$$

(as both $H^1(-,V)$ and $H^1(-, V \otimes_{\mathbf{Q}_p} B_*)$ satisfy Galois descent w.r.t. the extension K'/K, and the functor of $\mathrm{Gal}(K'/K)$-invariants is exact on the category of $\mathbf{Q}[\mathrm{Gal}(K'/K)]$-modules).

(3.1.2) For $V \in \mathrm{Rep}_{dR,L}(G_K)$ and $i \in \mathbf{Z}$, define

$$d^i_L(V) := \dim_L\left(D^i_{dR}(V)/D^{i+1}_{dR}(V)\right), \qquad d^-_L(V) := \sum_{i<0} i\, d^i_L(V),$$
$$d_L(V) := \sum_{i \in \mathbf{Z}} i\, d^i_L(V).$$

(3.1.3) If $V \in \mathrm{Rep}_{pst,L}(G_K)$, then $D = D_{pst}(V)$ is a free $(\mathbf{Q}_p^{ur} \otimes_{\mathbf{Q}_p} L)$-module of rank equal to $\dim_L(V)$, which is equipped (among others) with the following structure ([Fo], [Fo-PR, I.2.2]):

(1) An L-linear action $\rho_{sl} : W_K \longrightarrow \mathrm{Aut}_L(D)$, which is $\mathbf{Q}_p^{ur}$-semi-linear in the following sense:

$$\forall w \in W_K \ \forall \lambda \in \mathbf{Q}_p^{ur} \ \forall x \in D \qquad \rho_{sl}(w)(\lambda x) = f_k^{\nu(w)}(\lambda)\, \rho_{sl}(w)(x).$$

(2) An L-linear, σ-semi-linear map $\varphi : D \longrightarrow D$ commuting with $\rho_{sl}(w)$ (for all $w \in W_K$):

$$\forall w \in W_K \ \forall \lambda \in \mathbf{Q}_p^{ur} \ \forall x \in D \qquad \varphi(\lambda x) = \sigma(\lambda)\, \varphi(x).$$

(3) A $(\mathbf{Q}_p^{ur} \otimes_{\mathbf{Q}_p} L)$-linear nilpotent endomorphism $N : D \longrightarrow D$ commuting with $\rho_{sl}(w)$ (for all $w \in W_K$) and satisfying $N\varphi = p\varphi N$.

(4) An isomorphism of $(K \otimes_{\mathbf{Q}_p} L)$-modules

$$(D \otimes_{\mathbf{Q}_p^{ur}} \overline{K})^{G_K} \xrightarrow{\sim} D_{dR}(V).$$

(3.2) Potentially semistable representations and representations of the Weil-Deligne group

We recall how, for each $V \in \mathrm{Rep}_{pst,L}(G_K)$, the structure 3.1.3(1)-(3) can be used to define a representation of the Weil-Deligne group of K ([Fo], [Fo-PR, I.1.3.2]).

(3.2.1) Fix a field $E \supset \mathbf{Q}_p^{ur}$ for which there exists an embedding $\tau : L \hookrightarrow E$, and define

$$WD_\tau(V) := D_{pst}(V) \otimes_{\mathbf{Q}_p^{ur} \otimes_{\mathbf{Q}_p} L, \mathrm{id} \otimes \tau} E,$$

which is an E-vector space of dimension $\dim_E(WD_\tau(V)) = \dim_L(V)$. We define an E-linear action of W_K on $WD_\tau(V)$ by

$$\rho(w) := \rho_{sl}(w) \circ \varphi^{h\nu(w)} \otimes \mathrm{id} \qquad\qquad (w \in W_K)$$

and a monodromy operator $N = N \otimes \mathrm{id} \in \mathrm{End}_E(WD_\tau(V))$. This defines a representation

$$WD_\tau(V) = (\rho, N) \in \mathrm{Rep}_E('W_K),$$

whose isomorphism class does not depend on τ. Furthermore,

$$WD_\tau : \mathrm{Rep}_{pst,L}(G_K) \longrightarrow \mathrm{Rep}_E('W_K)$$

is an exact tensor functor.

(3.2.2) Examples: (1) If V is potentially unramified, then $WD_\tau(V) = V \otimes_{L,\tau} E \in \mathrm{Rep}_E(W_K)$.

(2) If V is semistable, then $WD(V) = D_{st}(V) \otimes_{K_0 \otimes_{\mathbf{Q}_p} L, \mathrm{id} \otimes \tau} E$ ($K_0 = K \cap \mathbf{Q}_p^{ur}$), with $\rho(I)$ acting trivially, $N = N \otimes \mathrm{id}$ and $\rho(f_k) = \varphi^h \otimes \mathrm{id}$. Conversely, if $\rho(I)$ acts trivially, then V is semistable.
(3) If $V = L(n) = L \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(n)$ ($n \in \mathbf{Z}$), then $WD_\tau(V) = E|\cdot|^n = E \otimes |\cdot|^n$.
(4) (Lubin-Tate theory) Fix a prime element $\pi \in \mathcal{O}_K$. The reciprocity map $\mathrm{rec}_K : K^* \longrightarrow G_K^{ab}$ (normalized using the geometric Frobenius element) defines a one-dimensional representation $V_\pi \in \mathrm{Rep}_{cris,K}(G_K)$

$$\chi_\pi : G_K \longrightarrow G_K^{ab} \xrightarrow{\sim} \widehat{K}^* = \pi^{\widehat{\mathbf{Z}}} \times \mathcal{O}_K^* \twoheadrightarrow \mathcal{O}_K^* \hookrightarrow K^*,$$

which arises in the π-adic Tate module of any Lubin-Tate group over K associated to π. In this case

$$D_{pst}(V_\pi) = (\mathbf{Q}_p^{ur} \otimes_{\mathbf{Q}_p} K)u, \qquad \varphi^h(u) = (1 \otimes \pi)^{-1}u, \qquad Nu = 0,$$
$$\forall w \in W_K \quad \rho_{sl}(w)(u) = u.$$

If $E \supset \mathbf{Q}_p^{ur}$ is a field and $\tau : K \hookrightarrow E$ an embedding of fields, then $WD_\tau(V_\pi) \in \mathrm{Rep}_E(W_K)$ is an unramified one-dimensional representation of W_K, on which $f = f_k$ acts by $\tau(\pi)^{-1}$. For $K = \mathbf{Q}_p$ and $\pi = p$ we recover Example (3) for $n = 1$.

(3.2.3) DEFINITION. *We say that $V \in \mathrm{Rep}_{pst,L}(G_K)$ is* PURE OF WEIGHT $n \in \mathbf{Z}$ *if $WD_\tau(V) \in \mathrm{Rep}_E('W_K)$ is pure of weight n, in the sense of 1.4.3.*

(3.2.4) LEMMA. *For each $V \in \mathrm{Rep}_{pst,L}(G_K)$ and each $\tau : L \hookrightarrow E \supset \mathbf{Q}_p^{ur}$,*

$$WD_\tau(V)_g^{f_k=1} = D_{st}(V)^{\varphi=1} \otimes_{L,\tau} E, \qquad H^0(WD_\tau(V)) = D_{cris}(V)^{\varphi=1} \otimes_{L,\tau} E.$$

Proof. As $D_{cris}(V) = D_{st}(V)^{N=0}$, it is enough to prove the first equality. As both sides satisfy Galois descent with respect to finite Galois extensions K'/K, we can assume that V is semistable. In this case, 3.2.2(2) implies that

$$WD_\tau(V)_g^{f_k=1} = WD_\tau(V)^{f_k=1} = D_{st}(V)^{\varphi^h=1} \otimes_{K_0 \otimes_{\mathbf{Q}_p} L, \mathrm{id}\otimes\tau} E \qquad (K_0 = K \cap \mathbf{Q}_p^{ur}).$$

As

$$D_{st}(V)^{\varphi^h=1} = D_{st}(V)^{\varphi=1} \otimes_{\mathbf{Q}_p} K_0$$

(thanks to Hilbert's Theorem 90 for $H^1(K_0/\mathbf{Q}_p, GL_n(K_0))$), it follows that

$$WD_\tau(V)_g^{f_k=1} = D_{st}(V)^{\varphi=1} \otimes_{L,\tau} E.$$

(3.2.5) COROLLARY. *If $V \in \mathrm{Rep}_{pst,L}(G_K)$ is pure of weight $n < 0$, then $D_{cris}(V)^{\varphi=1} = 0$.*

(3.2.6) PROPOSITION. *For each $V \in \mathrm{Rep}_{pst,L}(G_K)$,*

$$(\det_E(WD_\tau(V)))(-1) = (-1)^{d_L(V)} (\det_L V)(-1).$$

Proof. As WD_τ is a tensor functor and $d_L(V) = d_L(\det_L(V))$, we can replace V by $\det_L(V)$, hence assume that $\dim(V) = 1$; denote by $\chi_V : G_K \longrightarrow K^*$ the character by which G_K acts on V. After replacing L by a finite extension, we can assume that L contains the Galois closure of K over $\mathbf{Q}_p$. As V is potentially semistable, there exists a one-dimensional representation

$$\chi : G_K \longrightarrow L^*$$

with finite image and integers n_σ ($\sigma : K \hookrightarrow L$) such that

$$\chi_V = \chi \prod_{\sigma : K \hookrightarrow L} (\sigma \circ \chi_\pi)^{-n_\sigma},$$

where $\chi_\pi : G_K \longrightarrow K^*$ is as in 3.2.2(4). It follows from 3.2.2 that $WD_\tau(V) = (\tau \circ \chi)\alpha$, where $\alpha : W_K/I \longrightarrow E^*$ is the one-dimensional unramified representation satisfying

$$\alpha(f) = \prod_{\sigma : K \hookrightarrow L} \tau(\sigma(\pi))^{n_\sigma}.$$

This implies that

$$(\det_E(WD_\tau(V)))(-1) = \chi(-1), \quad (\det_L V)(-1) = (-1)^n \, \chi(-1), \quad n = \sum_{\sigma : K \hookrightarrow L} n_\sigma.$$

On the other hand,

$$d_L^i(V) = |\{\sigma : K \hookrightarrow L \mid n_\sigma = i\}|,$$

hence $n = d_L(V)$.

(3.3) Representations satisfying Pančiškin's condition

We recall a few basic facts from [Ne 1].

(3.3.1) Definition. *We say that $V \in \mathrm{Rep}_L(G_K)$ satisfies* Pančiškin's condition *if there exists an exact sequence in* $\mathrm{Rep}_L(G_K)$

$$0 \longrightarrow V^+ \longrightarrow V \longrightarrow V^- \longrightarrow 0$$

such that $V^\pm \in \mathrm{Rep}_{pst,L}(G_K)$ and $D^0_{dR}(V^+) = 0 = D_{dR}(V^-)/D^0_{dR}(V^-)$. If this is the case, then $V^\pm$ are uniquely determined ([Ne 1], 6.7), $V \in \mathrm{Rep}_{pst,L}(G_K)$ ([Ne 1], 1.28) and $V^*(1)$ also satisfies Pančiškin's condition (with $(V^*(1))^\pm = (V^\mp)^*(1)$).

(3.3.2) Proposition. *If V satisfies Pančiškin's condition, then:*
(1) $H^0(K, V^-) = D_{cris}(V^-)^{\varphi=1} = D_{st}(V^-)^{\varphi=1}$.
(2) *Assume that there exists a finite Galois extension K'/K over which V becomes semistable and such that $D_{cris}(V|_{G_{K'}})^{\varphi=1} = D_{cris}(V^*(1)|_{G_{K'}})^{\varphi=1} = 0$ (the latter condition holds, e.g., if V is pure of weight -1, by 3.2.5). Then*

$$H^1_e(K, V) = H^1_f(K, V) = H^1_{st}(K, V) = H^1_g(K, V)$$

and there is an exact sequence

$$0 \longrightarrow H^0(K, V^-) \longrightarrow H^1(K, V^+) \longrightarrow H^1_f(K, V) \longrightarrow 0,$$

in which $H^1(K, V^+) = H^1_{st}(K, V^+)$.

Proof. (1) This is proved in [Ne 1, 1.28(3)] under the tacit assumption that V^- is semistable. The general case follows by passing to a finite Galois extension over which V^- becomes semistable and taking Galois invariants.
(2) Over K', the statement is proved in [Ne 1, 1.32]; the general case follows by applying (3.1.1.1).

(3.3.3) Proposition. *Assume that V satisfies Pančiškin's condition, is pure (of weight -1) and that there exists an isomorphism $j : V \xrightarrow{\sim} V^*(1)$ in $\mathrm{Rep}_L(G_K)$ satisfying $j^*(1) = -j$. Then:*
(1) *j induces isomorphisms $V^\pm \xrightarrow{\sim} (V^\mp)^*(1)$.*
(2) *Fix an embedding of fields $\tau : L \hookrightarrow E \supset \mathbf{Q}_p^{ur}$ and put $\Delta = WD_\tau(V)$, $\Delta^\pm = WD_\tau(V^\pm)$. Then $\Delta \in \mathrm{Rep}_E('W_K)$ is $|\cdot|$-symplectic and the exact sequence in $\mathrm{Rep}_E('W_K)$*

$$0 \longrightarrow \Delta^+ \longrightarrow \Delta \longrightarrow \Delta^- \longrightarrow 0$$

satisfies the assumptions of Proposition 2.2.3.
(3) $(\det_E \Delta^+)(-1)/(\det_L V^+)(-1) = (-1)^{d_L(V^+)} = (-1)^{d_L^-(V)}$.
(4) *The ε-factors of Δ and Δ^{N-ss} are equal to*

$$\varepsilon(\Delta) = (-1)^{\dim_L\, H^0(K, V^-)}\,(-1)^{d_L^-(V)}\,(\det_L V^+)(-1),$$
$$\varepsilon(\Delta^{N-ss}) = (-1)^{d_L^-(V)}\,(\det_L V^+)(-1).$$

Proof. (1) This follows from the remarks made in 3.3.1.
(2) Δ is $|\cdot|$-symplectic, since WD_τ is a tensor functor. In order to verify the assumptions of Proposition 2.2.3, we are going to decompose Δ into several components. Firstly, the functor

$$\mathrm{Rep}_E('W_K) \longrightarrow \mathrm{Rep}_E('W_K), \qquad X \mapsto X^{\rho(I)}$$

is exact and commutes with duals. In addition, $X^{\rho(I)}$ is a direct summand of X, with a functorial complement X'. Secondly, for each $\lambda \in \overline{E}$, the minimal polynomial $p_{[\lambda]}(T)$ of λ over E depends only on the G_E-orbit $[\lambda]$ of λ. We define

$$\Delta_1 = \bigoplus_{\lambda \in q^{\mathbf{Z}}} \bigcup_{n \geq 1} \mathrm{Ker}\left((f - \lambda)^n : \Delta^{\rho(I)} \longrightarrow \Delta^{\rho(I)}\right),$$
$$\Delta_2 = \Delta' \oplus \bigoplus_{\lambda \notin q^{\mathbf{Z}}} \bigcup_{n \geq 1} \mathrm{Ker}\left(p_{[\lambda]}(f)^n : \Delta^{\rho(I)} \longrightarrow \Delta^{\rho(I)}\right).$$

The direct sum decomposition $\Delta = \Delta_1 \oplus \Delta_2$ in $\mathrm{Rep}_E('W_K)$ is compatible with the isomorphism $\Delta \xrightarrow{\sim} \Delta^*|\cdot|$ and the exact sequence

$$0 \longrightarrow \Delta^+ \longrightarrow \Delta \longrightarrow \Delta^- \longrightarrow 0.$$

By construction, every subquotient of Δ_2 in $\mathrm{Rep}_E({}'W_K)$ has trivial H^0, hence $H^0(\Delta_2^-) = 0$. As Δ is pure of weight -1, it follows that

$$\Delta_1 = \bigoplus_{m \geq 1} \sigma_m \otimes sp(2m),$$

where each $\sigma_m \in \mathrm{Rep}_E({}'W_K)$ is an unramified representation of W_K on which $q^{1-m} f$ acts unipotently.
As V satisfies the Pančiškin condition, weak admissibility of $V^\pm$ implies that all eigenvalues of f on $\Delta_1^+ = \Delta^+ \cap \Delta_1$ (resp., on $\Delta_1^- = \Delta_1/\Delta_1^+$) are of the form q^n with $n < 0$ (resp., with $n \geq 0$). It follows that

$$\Delta_1^+ = \bigoplus_{m \geq 1} \sigma_m \otimes sp(m)|\cdot|^m, \qquad \Delta_1^- = \bigoplus_{m \geq 1} \sigma_m \otimes sp(m),$$

which proves (2).
(3) This follows from Proposition 3.2.6 applied to V^+.
(4) We combine Proposition 2.2.3 (which applies to Δ, thanks to (2)) with the formula (3) and the fact that

$$\begin{aligned} H^0(\Delta^-) = D_{cris}(V^-)^{\varphi=1} \otimes_{L,\tau} E = (D_{cris}(V^-)^{\varphi=1} \cap D^0_{dR}(V^-)) \otimes_{L,\tau} E = \\ = H^0(K, V^-) \otimes_{L,\tau} E. \end{aligned}$$

4. Global p-adic Galois representations

(4.1) Generalities

(4.1.1) Notation. Let F be a number field. For each prime l of $\mathbf{Q}$, let S_l be the set of primes of F above l. Fix a prime number p, a finite extension $L_\mathfrak{p}$ of $\mathbf{Q}_p$ and a finite set $S \supset S_\infty \cup S_p$ of primes of F. Let F_S be the maximal extension of F (contained in $\overline{F}$) unramified outside S; put $G_{F,S} = \mathrm{Gal}(F_S/F)$. For each prime v of F fix an embedding $\overline{F} \hookrightarrow \overline{F}_v$; this defines a morphism $G_{F_v} \longrightarrow G_F \longrightarrow G_{F,S}$. For each Galois representation $V \in \mathrm{Rep}_{L_\mathfrak{p}}(G_{F,S})$ (continuous and finite-dimensional over $L_\mathfrak{p}$), denote by $V_v \in \mathrm{Rep}_{L_\mathfrak{p}}(G_{F_v})$ the local Galois representation induced by the map $G_{F_v} \longrightarrow G_{F,S}$. For each $v \notin S_\infty \cup S_p$, denote by $WD(V_v) \in \mathrm{Rep}_{L_\mathfrak{p}}({}'W_{F_v})$ the associated representation of the Weil-Deligne group of F_v (see 1.1.3). As in [Bl-Ka], we put

$$\forall v \notin S_\infty \cup S_p \qquad H^1_f(F_v, V) = H^1_{ur}(F_v, V) = \mathrm{Ker}\left(H^1(F_v, V) \longrightarrow H^1(F_v^{ur}, V)\right)$$

$$H^1_f(F, V) = \mathrm{Ker}\left(H^1(G_{F,S}, V) \longrightarrow \bigoplus_{v \in S - S_\infty} H^1(F_v, V)/H^1_f(F_v, V)\right).$$

The $L_{\mathfrak{p}}$-vector space $H^1_f(F, V)$ does not change if we enlarge S.

(4.1.2) Throughout §4, assume that V satisfies the following conditions.

(1) There exists an isomorphism $j : V \xrightarrow{\sim} V^*(1)$ in $\mathrm{Rep}_{L_{\mathfrak{p}}}(G_{F,S})$ satisfying $j^*(1) = -j$.
(2) For each $v \in S_p$, $V_v \in \mathrm{Rep}_{L_{\mathfrak{p}}}(G_{F_v})$ satisfies the Pančiškin condition 3.3.1:

$$0 \longrightarrow V_v^+ \longrightarrow V_v \longrightarrow V_v^- \longrightarrow 0$$

(in particular, $V_v \in \mathrm{Rep}_{pst,L_{\mathfrak{p}}}(G_{F_v})$).
(3) For each $v \notin S_\infty \cup S_p$ (resp., $v \in S_p$), V_v is pure (necessarily of weight -1) in the sense of 1.4.5 (resp., in the sense of 3.2.3).
(4) For each $i \in \mathbf{Z}$, the integer

$$d^i(V) := \dim_{L_{\mathfrak{p}}} \left(D^i_{dR}(V_v)/D^{i+1}_{dR}(V_v)\right)/[F_v : \mathbf{Q}_p]$$

does not depend on $v \in S_p$. This condition is satisfied if $V = M_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic realization of a motive (pure of weight -1) M over F with coefficients in a number field L (of which $L_{\mathfrak{p}}$ is a completion), as

$$d^i(V) = \dim_L \left(F^i M_{dR}/F^{i+1} M_{dR}\right)$$

in this case.

Example: $F = \mathbf{Q}$ and $V = (S^{2m-1}V(f))(mk - m + 1 - k/2)$, where $m \geq 1$ and $V(f)$ is the Galois representation (pure of weight $k-1$) associated to a potentially p-ordinary Hecke eigenform $f \in S_k(\Gamma_0(N))$ of (even) weight k and trivial character.

(4.1.3) ε-factors. We define

$$d^-(V) = \sum_{i<0} i\, d^i(V), \tag{4.1.3.1}$$

$$\forall v \in S_\infty \qquad \varepsilon(V_v) = (-1)^{[F_v:\mathbf{R}]\, d^-(V)} \times \begin{cases} 1, & F_v = \mathbf{R} \\ (-1)^{\dim_{L_{\mathfrak{p}}}(V)/2}, & F_v = \mathbf{C} \end{cases} \tag{4.1.3.2}$$

(in view of (2.3.1), this is the correct archimedean local ε-factor if $V = M_{\mathfrak{p}}$ is as in 4.1.2(4)) and

$$\forall v \notin S_\infty \qquad \varepsilon(V_v) = \varepsilon(WD(V_v)). \tag{4.1.3.3}$$

For any prime v of F, let

$$\widetilde{\varepsilon}(V_v) = \varepsilon(V_v) \times \begin{cases} (-1)^{h^0(F_v, V_v^-)}, & v \in S_p \\ 1, & v \notin S_p, \end{cases} \tag{4.1.3.4}$$

where

$$h^i(F_v, X) = \dim_{L_{\mathfrak{p}}} H^i(F_v, X) \qquad (X \in \mathrm{Rep}_{L_{\mathfrak{p}}}(G_{F_v})).$$

Finally, define

$$\varepsilon(V)=\prod_v \varepsilon(V_v),\qquad \widetilde{\varepsilon}(V)=\prod_v \widetilde{\varepsilon}(V_v) \tag{4.1.3.5}$$

(this makes sense, as $\varepsilon(V_v)=1$ for all but finitely many v). It follows from Proposition 3.3.3 that

$$\forall v\in S_p \qquad \widetilde{\varepsilon}(V_v)=(-1)^{[F_v:\mathbf{Q}_p]\,d^-(V)}\,(\det V_v^+)(-1)=\varepsilon(WD(V_v)^{N-ss}), \tag{4.1.3.6}$$

hence

$$\prod_{v\in S_p}\widetilde{\varepsilon}(V_v)=(-1)^{[F:\mathbf{Q}]\,d^-(V)}\prod_{v\in S_p}(\det V_v^+)(-1).$$

As

$$\prod_{v\in S_\infty}\varepsilon(V_v)=(-1)^{[F:\mathbf{Q}]\,d^-(V)},$$

it follows that

$$\prod_{v\in S_p\cup S_\infty}\widetilde{\varepsilon}(V_v)=\prod_{v\in S_p}(\det V_v^+)(-1). \tag{4.1.3.7}$$

(4.2) Selmer complexes and extended Selmer groups

(4.2.1) For a pro-finite group G and a representation $X\in \mathrm{Rep}_{L_\mathfrak{p}}(G)$ (continuous, finite-dimensional), denote by $C^\bullet(G,X)$ the standard complex of (non-homogeneous) continuous cochains of G with values in X. Fix a set $S_p\subset\Sigma\subset S$ and define, for each $v\in S-S_\infty$, the complex

$$U_v^+(V)=\begin{cases} C^\bullet(G_{F_v},V_v^+), & v\in S_p\\ 0, & v\in\Sigma-S_p\\ C^\bullet_{ur}(G_{F_v},V_v)=C^\bullet(G_{F_v}/I_v,V_v^{I_v}), & v\in S-\Sigma,\end{cases}$$

where $I_v\subset G_{F_v}$ is the inertia group. As in ([Ne 2], 12.5.9.1), define the Selmer complex of V associated to the local conditions $\Delta_\Sigma(V)=(U_v^+(V))_{v\in S-S_\infty}$ as

$$\widetilde{C}^\bullet_f(G_{F,S},V;\Delta_\Sigma(V))=$$
$$=\mathrm{Cone}\left(C^\bullet(G_{F,S},V)\oplus\bigoplus_{v\in S-S_\infty}U_v^+(V)\longrightarrow\bigoplus_{v\in S-S_\infty}C^\bullet(G_{F_v},V)\right)[-1].$$

(4.2.2) Proposition. (1) *For each $v\notin S_\infty\cup S_p$, the complexes $C^\bullet(G_{F_v},V)$ and $C^\bullet_{ur}(G_{F_v},V)$ are acyclic.*
(2) *Up to a canonical isomorphism, the image of $\widetilde{C}^\bullet_f(G_{F,S},V;\Delta_\Sigma(V))$ in the*

derived category $D^b_{ft}(L_{\mathfrak{p}} - \mathrm{Mod})$ does not depend on Σ and S; denote it by $\widetilde{\mathbf{R}\Gamma}_f(F,V)$ and its cohomology by $\widetilde{H}^i_f(F,V)$ (as $L_{\mathfrak{p}}$ is a field, $\widetilde{\mathbf{R}\Gamma}_f(F,V) = \bigoplus_{i\in\mathbf{Z}} \widetilde{H}^i_f(F,V)[-i]$).
(3) *There is an exact sequence*

$$0 \longrightarrow \bigoplus_{v\in S_p} H^0(F_v, V_v^-) \longrightarrow \widetilde{H}^1_f(F,V) \longrightarrow H^1_f(F,V) \longrightarrow 0.$$

(4) *If we put* $h^1_f(F,V) = \dim_{L_{\mathfrak{p}}} H^1_f(F,V)$, $\widetilde{h}^1_f(F,V) = \dim_{L_{\mathfrak{p}}} \widetilde{H}^1_f(F,V)$, *then*

$$(-1)^{h^1_f(F,V)}/\,\varepsilon(V) = (-1)^{\widetilde{h}^1_f(F,V)}/\,\widetilde{\varepsilon}(V).$$

Proof. (cf. [Ne 2, 12.5.9.2]) (1) The cohomology group $H^0(F_v,V) = 0$ vanishes by purity (1.4.4(5)), $H^2(F_v,V) \xrightarrow{\sim} H^0(F_v,V^*(1))^* \xrightarrow{\sim} H^0(F_v,V)^* = 0$ by duality and $H^1(F_v,V) = 0$ by the local Euler characteristic formula $\sum_{i=0}^2(-1)^i\, h^i(F_v,V) = 0$. Finally, $\dim_{L_{\mathfrak{p}}} H^1_{ur}(F_v,V) = h^0(F_v,V) = 0$.
(2) Independence of Σ follows from (1), independence of S is a general fact ([Ne 2], Prop. 7.8.8).
(3) It follows from (1) and [Ne 2, Lemma 9.6.3] that there is an exact sequence

$$0 \longrightarrow \widetilde{H}^0_f(F,V) \longrightarrow H^0(G_{F,S},V) \longrightarrow \bigoplus_{v\in S_p} H^0(F_v,V_v^-) \longrightarrow \widetilde{H}^1_f(F,V) \longrightarrow H \longrightarrow 0,$$

where

$$H = \mathrm{Ker}\left(H^1(G_{F,S},V) \longrightarrow \bigoplus_{v\in S-S_\infty} H^1(F_v,V)/\mathrm{Im}(H^1(U_v^+(V)))\right).$$

As

$$\mathrm{Im}(H^1(U_v^+(V))) = \begin{cases} 0 = H^1_f(F_v,V), & v \notin S_p \\ H^1_f(F_v,V), & v \in S_p \end{cases}$$

by (1) and Proposition 3.3.2(2), respectively, we deduce that $H = H^1_f(F,V)$. Finally, $H^0(G_{F,S},V) = 0$ by purity.
(4) This is a consequence of (3) and (4.1.3.4).

5. p-Adic families of global p-adic Galois representations

(5.1) The general setup

(5.1.1) Fix a number field F, a prime number p and a finite set $S \supset S_p \cup S_\infty$ of primes of F.

(5.1.2) Assume that we are given the following data.

(1) A complete local noetherian domain R of dimension $\dim(R) = 2$, whose residue field is a finite extension of $\mathbf{F}_p$ and whose fraction field $\mathscr{L}$ is of characteristic zero.
(2) An R-module of finite type $\mathcal{T}$ equipped with an R-linear continuous action of $G_{F,S}$ (with respect to the pro-finite topology of $\mathcal{T}$). Set $\mathcal{V} = \mathcal{T} \otimes_R \mathscr{L}$.
(3) A skew-symmetric morphism of $R[G_{F,S}]$-modules

$$(\ , \ ) : \mathcal{T} \otimes_R \mathcal{T} \longrightarrow R(1) = R \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1)$$

inducing an isomorphism of $\mathscr{L}[G_{F,S}]$-modules

$$\mathcal{V} \xrightarrow{\sim} \mathcal{V}^*(1) = \mathrm{Hom}_{\mathscr{L}}(\mathcal{V}, \mathscr{L})(1).$$

(4) For each $v \in S_p$ an $R[G_{F_v}]$-submodule $\mathcal{T}_v^+ \subset \mathcal{T}_v$ such that the isomorphism $\mathcal{V} \xrightarrow{\sim} \mathcal{V}^*(1)$ induces isomorphisms of $\mathscr{L}[G_{F_v}]$-modules

$$\mathcal{V}_v^{\pm} \xrightarrow{\sim} (\mathcal{V}_v^{\mp})^*(1) = \mathrm{Hom}_{\mathscr{L}}(\mathcal{V}_v^{\mp}, \mathscr{L})(1),$$

where $\mathcal{V}_v^+ = \mathcal{T}_v^+ \otimes_R \mathscr{L}$, $\mathcal{V}_v^- = \mathcal{V}_v/\mathcal{V}_v^+$.
(5) A prime ideal $P \in \mathrm{Spec}(R)$ of height $ht(P) = 1$, which does not contain p and such that R_P is a discrete valuation ring. Fix a prime element ϖ_P of R_P. The residue field $\kappa(P) = R_P/\varpi_P R_P$ is a finite extension of $\mathbf{Q}_p$. Define

$$\mathcal{T}_P = \mathcal{T} \otimes_R R_P \subset \mathcal{V}, \qquad V = \mathcal{T}_P/\varpi_P \mathcal{T}_P \in \mathrm{Rep}_{\kappa(P)}(G_{F,S})$$

and, for each $v \in S_p$,

$$(\mathcal{T}_P)_v^+ = \mathcal{T}_P \cap \mathcal{V}_v^+, \qquad (\mathcal{T}_P)_v^- = \mathcal{T}_P/(\mathcal{T}_P)_v^+, \qquad V_v^+ = (\mathcal{T}_P)_v^+/\varpi_P(\mathcal{T}_P)_v^+ \subset V_v,$$
$$V_v^- = V_v/V_v^+ \qquad\qquad (V_v^{\pm} \in \mathrm{Rep}_{\kappa(P)}(G_{F_v})).$$

(6) We assume that there exists $u \in \mathscr{L}^*$ such that $u\cdot(\ , \ )$ induces an isomorphism of $R_P[G_{F,S}]$-modules

$$\mathcal{T}_P \xrightarrow{\sim} \mathcal{T}_P^*(1) = \mathrm{Hom}_{R_P}(\mathcal{T}_P, R_P)(1).$$

This implies that, for each $v \in S_p$, $u\cdot(\ , \ )$ induces an isomorphism of $R_P[G_{F_v}]$-modules $(\mathcal{T}_P)_v^{\pm} \xrightarrow{\sim} ((\mathcal{T}_P)_v^{\mp})^*(1)$. Reducing $u\cdot(\ , \ )$ modulo P, we obtain a non-degenerate skew-symmetric morphism of $\kappa(P)[G_{F,S}]$-modules $V \otimes_{\kappa(P)} V \longrightarrow \kappa(P)(1)$ which induces, for each $v \in S_p$, isomorphisms $V_v^{\pm} \xrightarrow{\sim} (V_v^{\mp})^*(1)$ in $\mathrm{Rep}_{\kappa(P)}(G_{F_v})$.
(7) We assume that, for each $v \in S_p$, the exact sequence

$$0 \longrightarrow V_v^+ \longrightarrow V_v \longrightarrow V_v^- \longrightarrow 0$$

satisfies the Pančiškin condition: $V_v^{\pm} \in \mathrm{Rep}_{pst,\kappa(P)}(G_{F_v})$ and $D_{dR}^0(V_v^+) = 0 = D_{dR}(V_v^-)/D_{dR}^0(V_v^-)$.

(8) We assume that, for each $v \notin S_\infty$, V_v is pure of weight -1 (in the sense of 1.4.5 and 3.2.3, respectively).

(9) We assume that, for each $i \in \mathbf{Z}$, the integer

$$d^i(V) := \dim_{\kappa(P)} \left(D^i_{dR}(V_v)/D^{i+1}_{dR}(V_v)\right)/[F_v : \mathbf{Q}_p]$$

does not depend on $v \in S_p$; put

$$d^-(V) = \sum_{i<0} i\, d^i(V).$$

(5.1.3) This implies, in particular, that V satisfies the assumptions 4.1.2(1)-(4).

(5.1.4) Fix $v \notin S_p \cup S_\infty$. As $\mathrm{Aut}_R(\mathcal{T})$ is a pro-finite group containing a pro-p open subgroup, there exists an open subgroup J of the inertia group $I = I_v = \mathrm{Gal}(\overline{F}_v/F_v^{ur})$ such that J acts on $\mathcal{T}$ through the map $J \hookrightarrow I \twoheadrightarrow I(p)$, where $I(p)$ is the maximal pro-p-quotient of I (isomorphic to $\mathbf{Z}_p$). Fixing a topological generator t of $I(p)$ and an integer $a \geq 1$ such that t^a lies in the image of J, then the set of eigenvalues of t^a acting on $\mathcal{V}$ is stable under the map $\lambda \mapsto \lambda^{Nv}$, which implies that there exists an integer $c \geq 1$ divisible by a such that t^c acts unipotently on $\mathcal{V}$. Defining

$$N = \frac{1}{c}\,\log \rho_{\mathcal{T}}(t^c) \in \mathrm{End}_R(\mathcal{T}) \otimes \mathbf{Q}$$

(where $\rho_{\mathcal{T}} : G_K \longrightarrow \mathrm{Aut}_R(\mathcal{T})$ denotes the action of G_{F_v} on $\mathcal{T}$) and (fixing a lift $\widetilde{f} \in \nu^{-1}(1) \subset W_K$ of f)

$$\rho_T(\widetilde{f}^n u) := \rho_{\mathcal{T}}(\widetilde{f}^n u)\exp(-bN) \in \mathrm{Aut}_{R\otimes\mathbf{Q}}(\mathcal{T} \otimes \mathbf{Q}) \subset \mathrm{Aut}_{R_P}(\mathcal{T}_P) \quad (n \in \mathbf{Z},\, u \in I)$$

(where $b \in \mathbf{Z}_p$ is such that the image of u in $I(p)$ is equal to t^b), the pair (ρ_T, N) defines an object $T = (\rho_T, N)$ of $\mathrm{Rep}_{R_P}('W_{F_v})$ in the sense of 1.5.2, the isomorphism class of which is independent of the choice of $\widetilde{f}$ ([De 1], 8.4.3). By construction, the special fibre of T is isomorphic to

$$T_s \xrightarrow{\sim} WD(V_v) \in \mathrm{Rep}_{\kappa(P)}('W_{F_v}).$$

We define

$$\begin{aligned} WD(\mathcal{V}_v) &:= T_\eta = T \otimes_{R_P} \mathscr{L} \in \mathrm{Rep}_{\mathscr{L}}('W_{F_v}) \\ \varepsilon(\mathcal{V}_v) &:= \varepsilon(WD(\mathcal{V}_v)). \end{aligned} \tag{5.1.4.1}$$

If we choose another generator of $I(p)$, then N is multiplied by a scalar $\lambda \in \mathbf{Z}_p^*$, which does not change the isomorphism class of $WD(\mathcal{V}_v)$ ([De 1], 8.4.3).

(5.2) Selmer complexes and extended Selmer groups

(5.2.1) We equip each R-module $Y = \mathcal{T}, \mathcal{T}_v^+, \mathcal{T}_v^{I_v}$ with the pro-finite topology and we denote by $C^\bullet(G, Y)$ the corresponding complex of continuous cochains (for

$G = G_{F,S}, G_{F_v}, G_{F_v}/I_v$, respectively). For $R' = R_P, \mathscr{L}$, define $C^\bullet(G, Y \otimes_R R') = C^\bullet(G, Y) \otimes_R R'$. As in 4.2.1, fix a set $S_p \subset \Sigma \subset S$ and define, for $X = \mathcal{T}_P, \mathcal{V}$, $R_X = R_P, \mathscr{L}$ and each $v \in S - S_\infty$, complexes of R_X-modules

$$U_v^+(X) = \begin{cases} C^\bullet(G_{F_v}, X_v^+), & v \in S_p \\ 0, & v \in \Sigma - S_p \\ C^\bullet_{ur}(G_{F_v}, X) = C^\bullet(G_{F_v}/I_v, X^{I_v}), & v \in S - \Sigma, \end{cases}$$

and

$$\widetilde{C}^\bullet_f(G_{F,S}, X; \Delta_\Sigma(X)) =$$
$$= \mathrm{Cone}\left(C^\bullet(G_{F,S}, X) \oplus \bigoplus_{v \in S - S_\infty} U_v^+(X) \longrightarrow \bigoplus_{v \in S - S_\infty} C^\bullet(G_{F_v}, X) \right)[-1].$$

(5.2.2) Proposition. (1) *For each $X = \mathcal{T}_P, \mathcal{V}$ and each $v \notin S_\infty \cup S_p$, the complexes $C^\bullet(G_{F_v}, X)$ and $C^\bullet_{ur}(G_{F_v}, X)$ are acyclic.*
(2) *Up to a canonical isomorphism, the image of $\widetilde{C}^\bullet_f(G_{F,S}, X; \Delta_\Sigma(X))$ in $D^b_{ft}(R_X - \mathrm{Mod})$ does not depend on Σ and S; denote it by $\widetilde{\mathbf{R}\Gamma}_f(F, X)$ and its cohomology by $\widetilde{H}^i_f(F, X)$ (as $\mathscr{L}$ is a field, $\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{V}) = \bigoplus_{i \in \mathbf{Z}} \widetilde{H}^i_f(F, \mathcal{V})[-i]$).*
(3) *There is an exact triangle in $D^b_{ft}(R_P - \mathrm{Mod})$*

$$\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P) \xrightarrow{\varpi_P} \widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(F, V) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P)[1]$$

giving rise to exact sequences

$$0 \longrightarrow \widetilde{H}^i_f(F, \mathcal{T}_P)/\varpi_P \widetilde{H}^i_f(F, \mathcal{T}_P) \longrightarrow \widetilde{H}^i_f(F, V) \longrightarrow \widetilde{H}^{i+1}_f(F, \mathcal{T}_P)[\varpi_P] \longrightarrow 0,$$

and an isomorphism $\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P) \overset{\mathbf{L}}{\otimes}_{R_P} \mathscr{L} \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{V})$ *in* $D^b_{ft}(\mathscr{L} - \mathrm{Mod})$.
(4) *There exists a skew-symmetric isomorphism in $D^b_{ft}(R_P - \mathrm{Mod})$*

$$\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{R_P}(\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P), R_P)[-3]$$

inducing a skew-symmetric non-degenerate pairing

$$\widetilde{H}^2_f(F, \mathcal{T}_P)_{R_P-\mathrm{tors}} \times \widetilde{H}^2_f(F, \mathcal{T}_P)_{R_P-\mathrm{tors}} \longrightarrow \mathscr{L}/R_P.$$

(5) *There exists an R_P-module Z of finite length such that $\widetilde{H}^2_f(F, \mathcal{T}_P)_{R_P-\mathrm{tors}} \xrightarrow{\sim} Z \oplus Z$.*
(6) *$\widetilde{H}^1_f(F, \mathcal{T}_P)$ is a free R_P-module of rank $\widetilde{h}^1_f(F, \mathcal{V}) := \dim_{\mathscr{L}} \widetilde{H}^1_f(F, \mathcal{V})$.*
(7) *$\widetilde{h}^1_f(F, V) \equiv \widetilde{h}^1_f(F, \mathcal{V}) \pmod 2$.*

Proof. (cf. [Ne 2, 12.7.13.4]) (1) It is enough to prove the statement for $X = \mathcal{T}_P$. By ([Ne 2], Prop. 3.4.2 and 3.4.4), there is an exact sequence of complexes

$$0 \longrightarrow C^\bullet(G_{F_v}, \mathcal{T}_P) \xrightarrow{\varpi_P} C^\bullet(G_{F_v}, \mathcal{T}_P) \longrightarrow C^\bullet(G_{F_v}, V) \longrightarrow 0,$$

which induces injections

$$H^i(G_{F_v}, \mathcal{T}_P)/\varpi_P H^i(G_{F_v}, \mathcal{T}_P) \hookrightarrow H^i(F_v, V).$$

As $H^i(F_v, V) = 0$ by Proposition 4.2.2(1), and $H^i(G_{F_v}, \mathcal{T}_P) = H^i(G_{F_v}, \mathcal{T}) \otimes_R R_P$ is an R_P-module of finite type (by [Ne 2], Prop. 4.2.3), it follows that $H^i(G_{F_v}, \mathcal{T}_P) = 0$. Finally, the unramified cohomology $H^1_{ur} = H^1_{ur}(G_{F_v}, \mathcal{T}_P) = \mathcal{T}_P^{I_v}/(f_v - 1)\mathcal{T}_P^{I_v}$ is an R_P-module of finite type and $H^1_{ur}/\varpi_P H^1_{ur}$ is a subquotient of $V^{I_v}/\varpi_P V^{I_v} = H^1_{ur}(G_{F_v}, V) = 0$; thus $H^1_{ur} = 0$.
(2) This follows from (1), as in the proof of 4.2.2(2).
(3) According to (2), we can take $\Sigma = S$, in which case the exact triangle in question follows from the exact sequences

$$0 \longrightarrow C^\bullet(G, \mathcal{T}_P) \xrightarrow{\varpi_P} C^\bullet(G, \mathcal{T}_P) \longrightarrow C^\bullet(G, V) \longrightarrow 0 \qquad (G = G_{F,S},\, G_{F_v}).$$

The isomorphism $\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P) \overset{\mathbf{L}}{\otimes}_{R_P} \mathscr{L} \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{V})$ is a direct consequence of the definitions.
(4) Take again $\Sigma = S$. According to a localized version of ([Ne 2], 7.8.4.4), there exists an exact triangle in $D^b_{ft}(R_P - \mathrm{Mod})$

$$\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P) \xrightarrow{\gamma} \mathbf{R}\mathrm{Hom}_{R_P}(\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P), R_P)[-3] \longrightarrow \bigoplus_{v \in S - S_\infty} \mathrm{Err}_v,$$

in which the error terms Err_v vanish for $v \in S_p$ (as $(\mathcal{T}_P)^\pm \xrightarrow{\sim} ((\mathcal{T}_P)^\mp)^*(1)$), as well as for $v \notin S_p$ (by (1) and [Ne 2], Prop. 6.7.6(iv)). The map γ (which is an isomorphism, by the previous discussion) is skew-symmetric, by ([Ne 2], Prop. 6.6.2 and 7.7.3). The skew-symmetric non-degenerate pairing

$$\widetilde{H}^2_f(F, \mathcal{T}_P)_{R_P-\mathrm{tors}} \times \widetilde{H}^2_f(F, \mathcal{T}_P)_{R_P-\mathrm{tors}} \longrightarrow \mathscr{L}/R_P.$$

is constructed from γ in ([Ne 2], Prop. 10.2.5).
(5) This follows from (4) and the structure theory of symplectic modules of finite length over discrete valuation rings (note that 2 is invertible in R_P).
(6) It is enough to show that $\widetilde{H}^1_f(F, \mathcal{T}_P)$ has no R_P-torsion, which si a consequence of the exact sequence from (3) (for $i = 0$).
(7) In the exact sequence from (3) for $i = 1$, the term on the left (resp., on the right), is a $\kappa(P)$-vector space of dimension $\widetilde{h}^1_f(F, \mathcal{V})$, by (6) (resp., of even dimension, by (5)); thus the dimension of the middle term ($= \widetilde{h}^1_f(F, V)$) has the same parity as $\widetilde{h}^1_f(F, \mathcal{V})$.

(5.3) The parity conjecture in p-adic families

(5.3.1) Theorem. *Under the assumptions 5.1.2(1)-(9), the quantity*

$$(-1)^{h^1_f(F,V)}/\,\varepsilon(V) = (-1)^{\widetilde{h}^1_f(F,V)}/\,\widetilde{\varepsilon}(V) = \\ = (-1)^{\widetilde{h}^1_f(F,\mathcal{V})} \prod_{v\in S_p} (\det \mathcal{V}_v^+)(-1) \prod_{v\notin S_p\cup S_\infty} \varepsilon(\mathcal{V}_v)$$

depends only on $\mathcal{V}$ and $\mathcal{V}_v^+$ $(v \in S_p)$.

Proof. We combine the equalities

$$(-1)^{h^1_f(F,V)}/\,\varepsilon(V) = (-1)^{\widetilde{h}^1_f(F,V)}/\,\widetilde{\varepsilon}(V) \qquad \text{(Prop. 4.2.2(4))}$$

$$(-1)^{\widetilde{h}^1_f(F,V)} = (-1)^{\widetilde{h}^1_f(F,\mathcal{V})} \qquad \text{(Prop. 5.2.2(7))}$$

$$\widetilde{\varepsilon}(V) = \prod_{v\in S_p\cup S_\infty} \widetilde{\varepsilon}(V_v) \prod_{v\notin S_p\cup S_\infty} \varepsilon(V_v) = \prod_{v\in S_p} (\det V_v^+)(-1) \prod_{v\notin S_\infty\cup S_p} \varepsilon(V_v) \qquad \text{(by 4.1.3.7)}$$

$$\forall v \notin S_\infty \cup S_p \qquad \varepsilon(V_v) = \varepsilon(\mathcal{V}_v) \qquad \text{(Prop. 2.2.4)}$$

$$\forall v \in S_p \qquad (\det V_v^+)(-1) = (\det \mathcal{V}_v^+)(-1)$$

(both sides are equal to ± 1, and the L.H.S. is the reduction of the R.H.S. modulo P).

(5.3.2) Corollary. *Under the assumptions 5.1.2(1)-(4), if $P, P' \in \mathrm{Spec}(R)$ are prime ideals satisfying 5.1.2(5)-(9), then the Galois representations $V = \mathcal{T}_P/P\mathcal{T}_P$ and $V' = \mathcal{T}_{P'}/P'\mathcal{T}_{P'}$ satisfy*

$$(-1)^{h^1_f(F,V)}/\,\varepsilon(V) = (-1)^{h^1_f(F,V')}/\,\varepsilon(V').$$

(5.3.3) Open questions. It would be of interest to generalize Corollary 5.3.2 to self-dual families of Galois representations that do not satisfy the Pančiškin condition. Is it true, in general, that

$$(-1)^{[F_v:\mathbf{Q}_p]\, d^-(V)}\, \varepsilon(WD(V_v)^{N-ss}) \qquad (v \in S_p)$$

depends only on $\mathcal{V}_v$, and that

$$(-1)^{h^1_f(F,V)} \prod_{v\in S_p} \frac{\varepsilon(WD(V_v))}{\varepsilon(WD(V_v)^{N-ss})}$$

depends only on $\mathcal{V}$?

(5.3.4) Example (dihedral Iwasawa theory). Assume that $F_0 \subset F_\infty$ are Galois extension of F such that $[F_0 : F] = 2$, $\Gamma = \mathrm{Gal}(F_\infty/F_0) \xrightarrow{\sim} \mathbf{Z}_p$ and $\Gamma^+ = \mathrm{Gal}(F_\infty/F) = \Gamma \rtimes \{1, \tau\}$ is dihedral:

$$\tau \in \Gamma^+ - \Gamma, \qquad \tau^2 = 1, \qquad \forall g \in \Gamma \quad \tau g \tau^{-1} = g^{-1}.$$

Let $V \in \mathrm{Rep}_{L_\mathfrak{p}}(G_{F,S})$ be a Galois representation satisfying 4.1.2(1)-(4); fix a $G_{F,S}$-stable $\mathcal{O}_\mathfrak{p}$-lattice $T \subset V$ ($\mathcal{O}_\mathfrak{p} = \mathcal{O}_{L_\mathfrak{p}}$) such that the pairing $(\ ,\ )_V : V \times V \longrightarrow L_\mathfrak{p}(1)$ induced by j maps $T \times T$ into $\mathcal{O}_\mathfrak{p}(1)$. After enlarging S if necessary, we can assume that S contains all primes that ramify in F_0/F; then $F_\infty \subset F_S$. We define the following data of the type considered in 5.1.2:

(1) Let $R = \mathcal{O}_\mathfrak{p}[[\Gamma]]$ be the Iwasawa algebra of Γ (isomorphic to the power series ring $\mathcal{O}_\mathfrak{p}[[X]]$). The Iwasawa algebra of Γ^+ is a free (both left and right) R-module of rank 2:
$$\mathcal{O}_\mathfrak{p}[[\Gamma^+]] = R \oplus R\tau = R \oplus \tau R.$$
Denote by ι the standard $\mathcal{O}_\mathfrak{p}$-linear involution on $\mathcal{O}_\mathfrak{p}[[\Gamma^+]]$ ($\iota(\sigma) = \sigma^{-1}$ for all $\sigma \in \Gamma^+$).

(2) Let $\mathcal{T} = T \otimes_{\mathcal{O}_\mathfrak{p}} \mathcal{O}_\mathfrak{p}[[\Gamma^+]]$, considered as a left $R[G_{F,S}]$-module with the action given by
$$r(x \otimes a) = x \otimes ra, \quad g(x \otimes a) = g(x) \otimes a(\overline{g})^{-1} \qquad (r \in R,\ x \in T,\ a \in \mathcal{O}_\mathfrak{p}[[\Gamma^+]]),$$
where we have denoted by $\overline{g}$ the image of $g \in G_{F,S}$ in Γ^+ (cf., [Ne 2], 10.3.5.3).

(3) As in ([Ne 2], 10.3.5.10), the formula
$$(x \otimes (a_1 + \tau a_2), y \otimes (b_1 + \tau b_2)) = (x, y)_V (a_1 \iota(b_2) + \iota(a_2) b_1)$$
defines a skew-symmetric R-bilinear pairing $(\ ,\ ) : \mathcal{T} \times \mathcal{T} \longrightarrow R(1)$, which induces an isomorphism
$$\mathcal{T} \otimes \mathbf{Q} \xrightarrow{\sim} \mathrm{Hom}_R(\mathcal{T}, R(1)) \otimes \mathbf{Q}$$
(hence satisfies 5.1.2(3)).

(4) For each $v \in S_p$, define $\mathcal{T}_v^+ = T_v^+ \otimes_{\mathcal{O}_\mathfrak{p}} \mathcal{O}_\mathfrak{p}[[\Gamma^+]]$ (where $T_v^+ = T \cap V_v^+$).

(5) Let $\beta : \Gamma \longrightarrow L_\mathfrak{p}(\beta)^*$ be a homomorphism with finite image (where $L_\mathfrak{p}(\beta)$ is a field generated over $L_\mathfrak{p}$ by the values of β); then $P = \mathrm{Ker}(\beta : R \longrightarrow L_\mathfrak{p}(\beta)) \in \mathrm{Spec}(R)$ is as in 5.1.2(5), with $\kappa(P) = L_\mathfrak{p}(\beta)$. It follows from ([Ne 2], Lemma 10.3.5.4) that
$$\mathcal{T}_P/\varpi_P \mathcal{T}_P = \mathrm{Ind}_{G_{F_0,S}}^{G_{F,S}}(V \otimes \beta),$$
where we have denoted by $V \otimes \beta \in \mathrm{Rep}_{L_\mathfrak{p}(\beta)}(G_{F_0,S})$ the $G_{F_0,S}$-module $V \otimes_{L_\mathfrak{p}} L_\mathfrak{p}(\beta)$ on which $g \in G_{F_0,S}$ acts by $g \otimes \beta(\overline{g})$, where $\overline{g}$ is the image of g in Γ. The discussion in ([Ne 2], 10.3.5.10) implies that 5.1.2(6) holds with $u = 1$. The conditions 5.1.2(7)-(9) for $\mathcal{T}_P/\varpi_P \mathcal{T}_P$ follow from the corresponding conditions 4.1.2(2)-(4) for V.

(5.3.5) In the situation of 5.3.4, putting $F_\beta = F_\infty^{\mathrm{Ker}(\beta)}$ and, for each $L_\mathfrak{p}[\Gamma]$-module M,

$$M^{(\beta)} = \{x \in M \otimes_{L_\mathfrak{p}} L_\mathfrak{p}(\beta) \mid \forall \sigma \in \Gamma \quad \sigma(x) = \beta(\sigma)x\},$$

then we have

$$H^1_f(F, \mathcal{T}_P/\varpi_P\mathcal{T}_P) = H^1_f(F_0, V \otimes \beta) = \left(H^1_f(F_\beta, V) \otimes \beta\right)^{\mathrm{Gal}(F_\beta/F_0)} = \\ = H^1_f(F_\beta, V)^{(\beta^{-1})},$$

and the action of τ induces an isomorphism of $L_\mathfrak{p}(\beta)$-vector spaces

$$\tau : H^1_f(F_\beta, V)^{(\beta^{-1})} \xrightarrow{\sim} H^1_f(F_\beta, V)^{(\beta)}.$$

Applying Corollary 5.3.2, we obtain, for any pair of characters of finite order $\beta, \beta' : \Gamma \longrightarrow \overline{L}_\mathfrak{p}^*$, that

$$(-1)^{h^1_f(F_0, V\otimes\beta)}/\,\varepsilon(F_0, V \otimes \beta) = (-1)^{h^1_f(F_0, V\otimes\beta')}/\,\varepsilon(F_0, V \otimes \beta'). \qquad (5.3.5.1)$$

In this special case one can prove Proposition 2.2.4 directly (at least if $p \neq 2$) by using (2.1.2.7).

It would be of interest to generalize (5.3.5.1) to more general dihedral characters, as in [Ma-Ru].

References

[Bl-Ka] S. Bloch, K. Kato, *L-functions and Tamagawa numbers of motives*, in: The Grothendieck Festschrift I, Progress in Mathematics **86**, Birkhäuser, Boston, Basel, Berlin, 1990, pp. 333–400.

[De 1] P. Deligne, *Les constantes des équations fonctionnelles des fonctions L*, in: Modular functions of one variable II (Antwerp, 1972), Lect. Notes in Math. **349**, Springer, Berlin, 1973, pp. 501–597.

[De 2] P. Deligne, *Formes modulaires et représentations de $GL(2)$*, in: Modular functions of one variable II (Antwerp, 1972), Lect. Notes in Math. **349**, Springer, Berlin, 1973, pp. 55–105.

[De 3] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales*, in: Automorphic forms, representations and L-functions (Corvallis, 1977), Proc. Symposia Pure Math. **33/II**, American Math. Society, Providence, Rhode Island, 1979, pp. 313–346.

[Fo] J.-M. Fontaine, *Représentations ℓ-adiques potentiellement semi-stables*, in: Périodes p-adiques (Bures-sur-Yvette, 1988), Astérisque **223**, Soc. Math. de France, Paris, 1994, pp. 321–347.

[Fo-PR] J.-M. Fontaine, B. Perrin-Riou, *Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L*, in: Motives (Seattle, 1991), Proc. Symposia in Pure Math. **55/I**, American Math. Society, Providence, Rhode Island, 1994, pp. 599–706.

[Ma-Ru] B. Mazur, K. Rubin, *Finding large Selmer rank via an arithmetic theory of local constants*, Ann. of Math., to appear.

[Ne 1] J. Nekovář, *On p-adic height pairings*, in: Séminaire de Théorie des Nombres de Paris 1990/91, Progress in Math. **108**, (S. David, ed.), Birkhäuser, Boston, 1993, pp. 127–202.

[Ne 2] J. Nekovář, *Selmer complexes*, Astérisque **310**, Soc. Math. de France, Paris, 2006.

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