Koszul Duality and Equivariant Cohomology

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Abstract. Let G be a topological group such that its homology $H(G)$ with coefficients in a principal ideal domain R is an exterior algebra, generated in odd degrees. We show that the singular cochain functor carries the duality between G-spaces and spaces over BG to the Koszul duality between modules up to homotopy over $H(G)$ and $H^*(BG)$. This gives in particular a Cartan-type model for the equivariant cohomology of a G-space with coefficients in R. As another corollary, we obtain a multiplicative quasi-isomorphism $C^*(BG) \to H^*(BG)$. A key step in the proof is to show that a differential Hopf algebra is formal in the category of A_∞ algebras provided that it is free over R and its homology an exterior algebra.

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1. Introduction

Let G be a topological group. A space over the classifying space BG of G is a map $Y \to BG$. There are canonical ways to pass from left G-spaces to spaces over BG and back: The Borel construction $\boldsymbol{t}X = EG \times_G X$ is a functor

$$\boldsymbol{t}\colon G\text{-}\boldsymbol{Space} \to \boldsymbol{Space}\text{-}BG,$$

and pulling back the universal right G-bundle $EG \to BG$ along $Y \to BG$ and passing to a left action gives a functor in the other direction,

$$\boldsymbol{h}\colon \boldsymbol{Space}\text{-}BG \to G\text{-}\boldsymbol{Space}.$$

These functors are essentially inverse to each other in the sense that $\boldsymbol{ht}X$ and $\boldsymbol{th}Y$ are homotopy-equivalent in the category of spaces to X and Y, respectively, cf. [3].

Goresky–Kottwitz–MacPherson [8] have related this to an algebraic phenomenon called Koszul duality (see also Alekseev–Meinrenken [1] and Allday–Puppe [2]). Let $\mathbf{\Lambda}$ be an exterior algebra over some ring R with generators $x_1, \ldots, x_r$ of odd degrees, and $\mathbf{S}^*$ the symmetric R-algebra with generators $\xi_1, \ldots, \xi_r$ dual to the x_i and with degrees shifted by 1. We will denote the categories of bounded below differential graded modules over $\mathbf{\Lambda}$ and $\mathbf{S}^*$ by $\mathbf{\Lambda}$-*Mod* and $\mathbf{S}^*$-*Mod*, respectively. The Koszul functors

$$\mathbf{t}\colon \mathbf{\Lambda}\text{-}\mathit{Mod} \to \mathbf{S}^*\text{-}\mathit{Mod} \quad \text{and} \quad \mathbf{h}\colon \mathbf{S}^*\text{-}\mathit{Mod} \to \mathbf{\Lambda}\text{-}\mathit{Mod}$$

are defined by

$$\text{(1.1)} \qquad \mathbf{t}N = \mathbf{S}^* \otimes N, \qquad d(\sigma \otimes n) = \sigma \otimes dn + \sum_{i=1}^{r} \xi_i \sigma \otimes x_i n$$

and

$$\text{(1.2)} \qquad \mathbf{h}M = \mathbf{\Lambda}^* \otimes M, \qquad d(\alpha \otimes m) = (-1)^{|\alpha|} \alpha \otimes dm - \sum_{i=1}^{r} x_i \cdot \alpha \otimes \xi_i m.$$

Here $\mathbf{\Lambda}$ acts on $\mathbf{\Lambda}^*$ by contraction. Koszul duality refers to the fact that $\mathbf{ht}N$ and $\mathbf{th}M$ are homotopy-equivalent in the category of R-modules to N and M, respectively.

Now let $\mathbf{\Lambda} = H(G)$ be the homology of the compact connected Lie group G (with the Pontryagin product induced from the group multiplication) and $\mathbf{S}^* = H^*(BG)$ the cohomology of its classifying space BG. We take real coefficients, so that $\mathbf{\Lambda}$ and $\mathbf{S}^*$ are of the form described above. Goresky–Kottwitz–MacPherson and Alekseev-Meinrenken have shown that for certain G-spaces X, for instance for G-manifolds, $\mathbf{t}\Omega^*(X)^G$ computes the equivariant cohomology of X as $\mathbf{S}^*$-module, and $\mathbf{h}\Omega^*(\boldsymbol{t}X)$ the ordinary cohomology of X as $\mathbf{\Lambda}$-module. Here $\Omega^*(X)^G$ denotes the G-invariant differential forms on X, and $\Omega^*(\boldsymbol{t}X)$ the (suitably defined) differential forms on the Borel construction of X.

For the case of torus actions, the author has shown in [5] how to generalise this to arbitrary spaces and, more importantly, to an arbitrary coefficient ring R instead of $\mathbf{R}$. Differential forms are thereby replaced by singular cochains. The main problem one has to face is that the action of $\mathbf{S}^*$ on $H^*(Y)$, Y a space over BG, does not lift to an action on $C^*(Y)$ because the cup product of cochains is not commutative – unlike that of differential forms. The solution comes in form of "modules up to homotopy". Although modules up to homotopy – or *weak modules*, as we will call them – have a long history in Differential Homological Algebra (cf. for instance [17] or [18]), they are not familiar to many mathematicians in other areas. They will be defined precisely in Section 2; in the following paragraphs we just explain their main features and why they are useful for us.

A weak $\mathbf{S}^*$-module is a bounded below differential graded module over a differential graded R-algebra A together with elements $a_\pi \in A$, $\emptyset \neq \pi \subset \{1, \ldots, r\}$, such that

$$d(\alpha \otimes m) = (-1)^{|\alpha|}\alpha \otimes dm + \sum_{\pi \neq \emptyset} (-1)^{|x_\pi|} x_\pi \cdot \alpha \otimes a_\pi m \tag{1.3}$$

is a differential on $\mathbf{\Lambda}^* \otimes M$. Here (x_π) denotes the canonical R-basis of $\mathbf{\Lambda}$ consisting of the monomials in the x_i. If $A = \mathbf{S}^*$, one can simply set $a_i = \xi_i$ and all higher elements equal to zero. This shows that any $\mathbf{S}^*$-module is also a weak $\mathbf{S}^*$-module. In general, equation (1.3) puts certain conditions on the elements a_π. For instance, the element a_{12} must satisfy the relation

$$(da_{12})m = (a_1 a_2 - a_2 a_1)m \quad \text{for all } m \in M. \tag{1.4}$$

In other words, it compensates for the lack of commutativity between a_1 and a_2. Gugenheim–May [10] have shown how to construct suitable elements $a_\pi \in A = C^*(BG)$ starting from representatives a_i of the $\xi_i \in \mathbf{S}^*$. As a consequence, the cochains on any space Y over BG admit the structure of a weak $\mathbf{S}^*$-module. One then defines the Koszul dual of the weak $\mathbf{S}^*$-module $C^*(Y)$ to be the $\mathbf{\Lambda}$-module $\mathbf{\Lambda}^* \otimes C^*(Y)$ with differential (1.3), and in [5] it was shown that for tori this computes the cohomology of $\boldsymbol{h}Y$ as $\mathbf{\Lambda}$-module. (That this complex gives the right cohomology as R-module appears already in Gugenheim–May [10].)

A fancier way to define a weak $\mathbf{S}^*$-module is to say that it is an A-module as above together with a so-called twisting cochain $u\colon \mathbf{\Lambda}^* \to A$. The elements a_π then are the images under u of the R-basis of $\mathbf{\Lambda}^*$ dual to the basis (x_π). It follows from equation (1.4) that the cohomology of a weak $\mathbf{S}^*$-module admits itself a (strict) $\mathbf{S}^*$-action. Similarly, a weak $\mathbf{\Lambda}$-module is a module N over some algebra A together with a twisting cochain $\mathbf{S} \to A$, where $\mathbf{S}$ denotes the coalgebra dual to $\mathbf{S}^*$. Its cohomology is canonically a $\mathbf{\Lambda}$-module.

For torus actions there is no need to consider weak $\mathbf{\Lambda}$-modules because the $\mathbf{\Lambda}$-action on cohomology can be lifted to an honest action on cochains. In fact, since $C(G)$ is graded commutative in this case, it suffices to choose representatives $c_i \in C(G)$ of the generators $x_i \in \mathbf{\Lambda}$ in order to construct a quasi-isomorphism of algebras $\mathbf{\Lambda} \to C(G)$. In [8, Sec. 12] it is claimed that a lifting is possible for any compact connected Lie group, but the proof given there is wrong. The mistake is that it is not possible in general to find conjugation-invariant representatives of the generators x_i because all singular simplices appearing in a conjugation-invariant chain c_i necessarily map to the centre of G. The example $G = SU(3)$ shows that passing to subanalytic chains (which are also used in [8]) is of no help: apart from the finite centre, all conjugation classes of $SU(3)$ have dimension 4 or 6. Hence, there can be no conjugation-invariant subanalytic set supporting a representative of the 3-dimensional generator.

In the present paper, we extend the approach of [5] to non-commutative topological groups G by constructing a weak $\mathbf{\Lambda}$-structure on the cochain complex of a G-space X. We then show that the normalised singular cochain functor C^* transforms the topological equivalence between G-spaces and spaces over BG, up to quasi-isomorphism, to the Koszul duality between modules up to homotopy over the homology $\mathbf{\Lambda} = H(G)$ and the cohomology $\mathbf{S}^* = H^*(BG)$. The only assumptions are that coefficients are in a principal ideal domain R and that $H(G)$ is an exterior algebra on finitely many generators of odd degrees or, equivalently, that $H^*(BG)$ a symmetric algebra on finitely many generators of even degrees.

A priori, the isomorphism $H(G) \cong \bigwedge(x_1, \dots, x_r)$ must be one of Hopf algebras[1] with primitive generators x_i. But the Samelson–Leray theorem asserts that in our situation any isomorphism of algebras (or coalgebras) can be replaced by one which is Hopf. In characteristic 0 it suffices by Hopf's theorem to check that G is connected and $H(G)$ free of finite rank over R. In particular, the condition is satisfied for $U(n)$, $SU(n)$ and $Sp(n)$ and arbitrary R, and for an arbitrary compact connected Lie group if the order of the Weyl group is invertible in R.

Under the assumptions on $H(G)$ and $H^*(BG)$ mentioned above, we prove the following:

PROPOSITION 1.1. *There are twisting cochains $v\colon \mathbf{S} \to C(G)$ and $u\colon \mathbf{\Lambda}^* \to C^*(BG)$ such that the $\mathbf{\Lambda}$-action on the homology of a $C(G)$-module, viewed as weak $\mathbf{\Lambda}$-module, is the canonical one over $H(G) = \mathbf{\Lambda}$, and analogously for u.*

The cochains on a G-space are canonically a $C(G)$-module and the cochains on a space over BG a $C^*(BG)$-module. Hence we may consider C^* as a functor from G-spaces to weak $\mathbf{\Lambda}$-modules, and from spaces over BG to weak $\mathbf{S}^*$-modules.

We say that two functors to a category of complexes are quasi-isomorphic if they are related by a zig-zag of natural transformations which become isomorphisms after passing to homology.

THEOREM 1.2. *The functors $C^* \circ \boldsymbol{t}$ and $\mathbf{t} \circ C^*$ from G-spaces to weak $\mathbf{S}^*$-modules are quasi-isomorphic, as are the functors $C^* \circ \boldsymbol{h}$ and $\mathbf{h} \circ C^*$ from spaces over BG to weak $\mathbf{\Lambda}$-modules.*

Hence, the equivariant cohomology $H_G^*(X)$ of a G-space X is naturally isomorphic, as $\mathbf{S}^*$-module, to the homology of the "singular Cartan model"

$$\mathbf{t}C^*(X) = \mathbf{S}^* \otimes C^*(X) \tag{1.5a}$$

with differential

$$d(\sigma \otimes \gamma) = \sigma \otimes d\gamma + \sum_{i=1}^{r} \xi_i \sigma \otimes c_i \cdot \gamma + \sum_{i \le j} \xi_i \xi_j \sigma \otimes c_{ij} \cdot \gamma + \cdots , \tag{1.5b}$$

where the ξ_i are generators of the symmetric algebra $\mathbf{S}^*$ and the $c_i \in C(G)$ representatives of the generators $x_i \in \mathbf{\Lambda}$. They are, like the higher order terms c_{ij} etc., encoded in the twisting cochain v. The sum, which runs over all non-constant monomials of $\mathbf{S}^*$, is well-defined for degree reasons.

Similarly, the cohomology of the pull back of EG along $Y \to BG$ is isomorphic to the homology of the $\mathbf{\Lambda}$-module $\mathbf{h}C^*(Y) = \mathbf{\Lambda}^* \otimes C^*(Y)$, again with a twisted differential. (See Section 3 for precise formulas for the differentials.) That the complex $\mathbf{h}C^*(Y)$ gives the right cohomology as R-module is already due to Gugenheim–May [10]. The correctness of the $\mathbf{\Lambda}$-action is new.

[1] Note that $H(G)$ has a well-defined diagonal because it is free over R.

Along the way, we obtain the following result, which was previously only known for tori, and for other Eilenberg–Mac Lane spaces if $R = \mathbf{Z}_2$ (Gugenheim–May [10, §4]):

Proposition 1.3. *There exists a quasi-isomorphism of algebras $C^*(BG) \to H^*(BG)$ between the cochains and the cohomology of the simplicial construction of the classifying space of G.*

Any such map has an A_∞ map as homotopy inverse (cf. Lemma 4.1). So we get as another corollary the well-known existence of an A_∞ quasi-isomorphism $H^*(BG) \Rightarrow C^*(BG)$. The original proof (Stasheff–Halperin [22]) uses the homotopy-commutativity of the cup product and the fact that $H^*(BG)$ is free commutative. Here it is based, like most of the paper, on the following result, which is of independent interest and should be considered as dual to the theorem of Stasheff and Halperin.

Theorem 1.4. *Let A be a differential $\mathbf{N}$-graded Hopf algebra, free over R and such that its homology is an exterior algebra on finitely many generators of odd degrees. Then there are A_∞ quasi-isomorphisms $A \Rightarrow H(A)$ and $H(A) \Rightarrow A$.*

It is essentially in order to use Theorem 1.4 (and a similar argument in Section 7) that we assume R to be a principal ideal domain. A look at the proofs will show that once Proposition 1.1, Theorem 1.2 and Proposition 1.3 are established for such an R, they follow by extension of scalars for any commutative R-algebra R' instead of R.
Johannes Huebschmann has informed the author that he has been aware of the singular Cartan model and of Theorem 1.4 since the 1980's, cf. [14]. Instead of adapting arguments from his habilitation thesis [13, Sec. 4.8], we shall base the proof of Theorem 1.4 on an observation due to Stasheff [21].

The paper is organised as follows: Notation and terminology is fixed in Section 2. Section 3 contains a review of Koszul duality between modules up to homotopy over symmetric and exterior algebras. Theorem 1.4 is proved in Section 4. The proofs of the other results stated in the introduction appear in Sections 5 to 7. In Section 8 we discuss equivariantly formal spaces and in Section 9 the relation between the singular Cartan model and other models, in particular the classical Cartan model. In an appendix we prove the versions of the theorems of Samelson–Leray and Hopf mentioned above because they are not readily available in the literature.

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2. Preliminaries

Throughout this paper, the letter R denotes a principal ideal domain. All complexes are over R. Differentials always lower degree, hence cochain complexes and cohomology are negatively graded. All (co)algebras and (co)modules are graded and have differentials (which might be trivial). Let A and B be complexes. The dual $f^* \in \operatorname{Hom}(B^*, A^*)$ of a map $f \in \operatorname{Hom}(A, B)$ is defined by

$$f^*(\beta)(a) = (-1)^{|f||\beta|}\, \beta(f(a)).$$

Algebras will be associative and coalgebras coassociative, and both have units and counits (augmentations). Morphisms of (co)algebras preserve these structures. We denote the augmentation ideal of an algebra A by $\bar{A}$. An $\mathbf{N}$-graded algebra A is called connected if $\bar{A}_0 = 0$, and an $\mathbf{N}$-graded coalgebra C simply connected if $C_0 = R$ and $C_1 = 0$. Hopf algebras are algebras which are also coalgebras with a multiplicative diagonal, cf. [18, Def. 4.39]. (Note that we do not require the existence of an antipode, though there will always be one for our examples.)

Let C be a coalgebra, A an algebra and $t\colon C \to A$ a twisting cochain. For a right C-comodule M and a left A-module N, we define the twisted tensor product $M \otimes_t N$ with differential

$$d_t = d \otimes 1 + 1 \otimes d + (1 \otimes \mu_N)(1 \otimes t \otimes 1)(\Delta_M \otimes 1).$$

Here $\Delta_M\colon M \to M \otimes C$ and $\mu_N\colon A \otimes N \to N$ denote the structure maps of M and N, respectively. Readers unfamiliar with twisting cochains can take the fact that d is a well-defined differential (say, on $C \otimes_t A$) as the definition of a twisting cochain, plus the normalisation conditions $t\iota_C = 0$ and $\varepsilon_A t = 0$, where ι_C is the unit of C and ε_A the augmentation of A. Suppose that C and A are $\mathbf{N}$-graded. We will regularly use the fact that twisting cochains $C \to A$ correspond bijectively to coalgebra maps $C \to BA$ and to algebra maps $\Omega C \to A$. Here BA denotes the normalised bar construction of A and ΩC the normalised cobar construction of C. In particular, the functors Ω and B are adjoint. (See for instance [15, Sec. II] for more about twisting cochains and the (co)bar construction.)

We agree that an exterior algebra is one on finitely many generators of odd positive degrees. Let A be an $\mathbf{N}$-graded algebra such that $\mathbf{\Lambda} = H(A) = \bigwedge(x_1, \dots, x_r)$ is an exterior algebra. Then $H(BA) = H(B\mathbf{\Lambda}) = \mathbf{S}$ is a symmetric coalgebra on finitely many cogenerators y_i of even degrees $|y_i| = |x_i| + 1$, cf. [18, Thm. 7.30]. (The converse is true as well.) We assume that the y_i are chosen such that they can be represented by the cycles $[x_i] \in B\mathbf{\Lambda}$ and $[c_i] \in BA$, where the $c_i \in A$ are any representatives of the generators $x_i \in \mathbf{\Lambda}$. We denote by x_π, $\pi \subset \{1, \dots, r\}$, the canonical R-basis of $\mathbf{\Lambda}$ generated by the x_i, and the dual basis of $\mathbf{\Lambda}^*$ by ξ_π. The R-basis of $\mathbf{S}$ induced by the y_i is written as y_α, $\alpha \in \mathbf{N}^r$. The dual $\mathbf{S}^*$ of $\mathbf{S}$ is a symmetric algebra on generators ξ_i dual to the y_i.

We work in the simplicial category. We denote by $C(X)$ the normalised chain complex of the simplicial set X. (If X comes from a topological space, then $C(X)$ is the complex of normalised singular chains.) The (negatively graded) dual complex of normalised cochains is denoted by $C^*(X)$. If G is a connected (topological or simplicial) group, then the inclusion of the simplicial subgroup consisting of the simplices with all vertices at $1 \in G$ is a quasi-isomorphism. We may therefore assume that G has only one vertex. Then $C(G)$ is a connected Hopf algebra and $C(BG)$ a simply connected coalgebra. In both cases, the diagonal is the Alexander–Whitney map, and the Pontryagin product of $C(G)$ is the composition of the shuffle map $C(G) \otimes C(G) \to C(G \times G)$ with the map $C(G \times G) \to C(G)$ induced by the multiplication of G. Analogously, $C(X)$ is a left $C(G)$-module if X is a left G-space. The left $C(G)$-action on cochains is defined by

$$(a \cdot \gamma)(c) = (-1)^{|a||\gamma|}\,\gamma(\lambda_*(a) \cdot c) \tag{2.1}$$

where $\lambda\colon G \to G$ denotes the group inversion. If $p\colon Y \to BG$ is a space over BG, then $C^*(Y)$ is a left $C^*(BG)$-module by $\beta \cdot \gamma = p^*(\beta) \cup \gamma$.

3. Koszul duality

Koszul duality is most elegantly expressed as a duality between $\mathbf{\Lambda}$-modules and comodules over the symmetric coalgebra $\mathbf{S}$ dual to $\mathbf{S}^*$, see [5, Sec. 2]. It hinges on the fact that the Koszul complex $\mathbf{S} \otimes_w \mathbf{\Lambda}$ is acyclic, where $w\colon \mathbf{S} \to \mathbf{\Lambda}$ is the canonical twisting cochain which sends each y_i to x_i and annihilates all other y_α. In this paper, though, we adopt a cohomological viewpoint. This makes definitions look rather ad hoc, but it is better suited to our discussion of equivariant cohomology in Section 8.

We denote the categories of bounded above weak modules over $\mathbf{\Lambda}$ and $\mathbf{S}^*$ by $\mathbf{\Lambda}$-$\mathcal{M}od$ and $\mathbf{S}^*$-$\mathcal{M}od$, respectively. (Recall that we grade cochain complexes negatively.) Note that any (strict) module over $\mathbf{\Lambda}$ or $\mathbf{S}^*$ is also a weak module because of the canonical twisting cochain w and its dual $w^*\colon \mathbf{\Lambda}^* \to \mathbf{S}^*$. The homology of a weak $\mathbf{\Lambda}$-module (N, v) is a $\mathbf{\Lambda}$-module by setting $x_i \cdot [n] = [v(y_i) \cdot n]$, and $\mathbf{S}^*$ acts on the homology of a weak $\mathbf{S}^*$-module (M, u) by $\xi_i \cdot [m] = [u(\xi_i) \cdot m]$. Before describing morphisms of weak modules, we say how the Koszul functors act on objects.

The Koszul dual of $(N, v) \in \mathbf{\Lambda}$-$\mathcal{M}od$ is defined as the bounded above $\mathbf{S}^*$-module $\mathbf{t}N = \mathbf{S}^* \otimes N$ with differential

$$d(\sigma \otimes n) = \sigma \otimes dn + \sum_{\alpha > 0} \xi^\alpha \sigma \otimes v(y_\alpha) \cdot n. \tag{3.1}$$

(This is well-defined because N is bounded above.)

The Koszul dual of $(M, u) \in \mathbf{S}^*$-$\mathcal{M}od$ is the bounded above $\mathbf{\Lambda}$-module $\mathbf{h}M = \mathbf{\Lambda}^* \otimes M$ with differential

$$d(\alpha \otimes m) = (-1)^{|\alpha|} \alpha \otimes dm + \sum_{\pi \neq \emptyset} (-1)^{|x_\pi|} x_\pi \cdot \alpha \otimes u(\xi_\pi) \cdot m \tag{3.2}$$

and $\mathbf{\Lambda}$-action coming from that on $\mathbf{\Lambda}^*$, which is defined similarly to (2.1),

$$(a \cdot \alpha)(a') = (-1)^{|a|(|\alpha|+1)} \alpha(a \wedge a').$$

A morphism f between two weak $\mathbf{\Lambda}$-modules N and N' is a morphism of (strict) $\mathbf{S}^*$-modules $\mathbf{t}N \to \mathbf{t}N'$. Its "base-component"

$$N = 1 \otimes N \hookrightarrow \mathbf{S}^* \otimes N \xrightarrow{f} \mathbf{S}^* \otimes N' \twoheadrightarrow 1 \otimes N' = N'$$

is a chain map inducing a $\mathbf{\Lambda}$-equivariant map in homology. If the latter is an isomorphism, we say that f is a quasi-isomorphism. The definitions for weak $\mathbf{S}^*$-modules are analogous. The Koszul dual of a morphism of weak modules is what one expects.
The Koszul functors preserve quasi-isomorphisms and are quasi-inverse to each other, cf. [5, Sec. 2.6]. Note that our (left) weak $\mathbf{S}^*$-modules correspond to *left* weak $\mathbf{S}$-comodules and not to right ones as used in [5]. This detail, which is crucial for the present paper, does not affect Koszul duality.

In the rest of this section we generalise results of [8, Sec. 9] to weak modules.

Following [8], we call a weak $\mathbf{S}^*$-module M is called *split and extended* if it is quasi-isomorphic to its homology and if the latter is of the form $\mathbf{S}^* \otimes L$ for some graded R-module L. If M it is quasi-isomorphic to its homology and if the $\mathbf{S}^*$-action on $H(M)$ is trivial, we say that M is *split and trivial*. Similar definitions apply to weak $\mathbf{\Lambda}$-modules. (Note that it does not make a difference whether we require the homology of a split and free $\mathbf{\Lambda}$-module to be isomorphic to $\mathbf{\Lambda} \otimes L$ or to $\mathbf{\Lambda}^* \otimes L$.)

PROPOSITION 3.1. *Under Koszul duality, split and trivial weak modules correspond to split and extended ones.*

Proof. That the Koszul functors carry split and trivial weak modules to split and extended ones is almost a tautology. The other direction follows from the fact that the Koszul functors are quasi-inverse to each other and preserve quasi-isomorphisms because a split and extended weak module is by definition quasi-isomorphic to the Koszul dual of a module with zero differential and trivial action. □

PROPOSITION 3.2. *Let M be in $\mathbf{S}^*$-$\mathcal{M}od$. If $H(M)$ is extended, then M is split and extended.*

Proof. We may assume that M has a strict $\mathbf{S}^*$-action because any weak $\mathbf{S}^*$-module M is quasi-isomorphic to a strict one (for instance, to $\mathbf{th}M$). By assumption, $H(M) \cong \mathbf{S}^* \otimes L$ for some graded R-module L. Since we work over a principal ideal domain, there exists a free resolution

$$0 \longleftarrow L \longleftarrow P^0 \longleftarrow P^1 \longleftarrow 0$$

of L with P^0, P^1 bounded above. Tensoring it with $\mathbf{S}^*$ gives a free resolution of the $\mathbf{S}^*$-module $H(M)$ and therefore the (not uniquely determined) $\mathbf{S}^*$-equivariant vertical maps in the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 & \mathbf{S}^* \otimes L \arrow[l] \arrow[d, "\cong"'] & \mathbf{S}^* \otimes P^0 \arrow[l] \arrow[d] & \mathbf{S}^* \otimes P^1 \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & H(M) \arrow[l] & Z(M) \arrow[l] & M. \arrow[l, "d"'] &
\end{tikzcd}$$

This implies that the total complex $\mathbf{S}^* \otimes P$ is quasi-isomorphic to both $H(M)$ and M. □

4. Proof of Theorem 1.4

In this section all algebras are $\mathbf{N}$-graded and connected unless otherwise stated. Recall that an A_∞ map $f\colon A \Rightarrow A'$ between two algebras is a map of coalgebras $BA \to BA'$, see [18, Sec. 8.1] or [17] for example. It is called strict if it is induced from an algebra map $A \to A'$. If $f\colon A \Rightarrow A'$ is A_∞, then its base component $f_1\colon B_1A \to B_1A'$ between the elements of external degree 1 is a chain map, multiplicative up to homotopy. We denote the induced algebra map in homology by $H(f)\colon H(A) \to H(A')$. If it is an isomorphism, we call f an A_∞ quasi-isomorphism.
In order to prove Theorem 1.4, it is sufficient to construct an A_∞ quasi-isomorphism $A \Rightarrow H(A) = \bigwedge(x_1, \ldots, x_r) = \mathbf{\Lambda}$, due to the following result:

Lemma 4.1. *Let A be an algebra with A and $H(A)$ free over R, and let $f\colon A \Rightarrow H(A)$ be an A_∞ map inducing the identity in homology. Then f has an A_∞ quasi-inverse, i. e., there is an A_∞ map $g\colon H(A) \Rightarrow A$ also inducing the identity in homology.*

(At least over fields one can do better: there any A_∞ quasi-isomorphism between two algebras – even A_∞ algebras – is an A_∞ homotopy equivalence, cf. [20] or [17, Sec. 3.7].)

Proof. According to [19, Prop. 2.2], the claim is true if f is strict. (Here we use that over a principal ideal domain any quasi-isomorphism $A \to H(A)$ of free modules comes from a "trivialised extension" in the sense of [19, §2.1].) To reduce the general case to this, we consider the cobar construction ΩBA of BA. Coalgebra maps $h\colon BA \to BA'$ correspond bijectively to algebra maps $\tilde{h}\colon \Omega BA \to A'$. For h, the identity of A, the map $\tilde{h}$ is a quasi-isomorphism [15, Thm II.4.4] with quasi-inverse (in the category of complexes), the canonical inclusion $A \hookrightarrow \Omega BA$. The composition of this map with $\tilde{f}\colon \Omega BA \to H(A)$ is essentially f_1, which is a quasi-isomorphism by hypothesis. Hence $\tilde{f}$ is so, too. Now compose any A_∞ quasi-inverse of it with the projection $\Omega BA \to A$. □

Recall that for any complex C a cycle in $C^q = \operatorname{Hom}_{-q}(C, R)$ is the same as a chain map $C \to R[-q]$. (Here $R[-q]$ denotes the complex R, shifted to degree q.) The crucial observation, made in a topological context by Stasheff [21, Thm. 5.1], is the following:

Lemma 4.2. *A_∞ maps $A \Rightarrow \bigwedge(x)$, $|x| = q > 0$, correspond bijectively to cocycles in $(BA)^{q+1}$.*

Proof. Note that the augmentation ideal of $\bigwedge(x)$ is $R[-q]$ (with vanishing product). An A_∞ map $f\colon A \Rightarrow \bigwedge(x)$ is given by components $f_p\colon \bar{A}^{\otimes p} \to R[-q]$ of degree $p-1$ such that for all $[a_1, \ldots, a_p] \in B_p(A)$,

$$f_p(d[a_1, \ldots, a_p]) = -f_{p-1}(\delta[a_1, \ldots, a_p]),$$

where $d\colon B_p(A) \to B_p(A)$ denotes the "internal" differential and $\delta\colon B_p(A) \to B_{p-1}(A)$ the "external" one, cf. [18, Thm. 8.18]. In other words, $d(f_p) = -\delta(f_{p-1})$, where δ and d now denote the dual differentials. But this is the condition for a cycle in the double complex $((BA)^*, d, \delta)$ dual to BA. $\square$

By our assumptions, $H^*(BA) = \mathbf{S}^*$ is a (negatively graded) polynomial algebra. Taking representatives of the generators ξ_i gives A_∞ maps $f^{(i)}\colon A \Rightarrow \bigwedge(x_i)$. By [19, Prop. 3.3 & 3.7], they assemble into an A_∞ map

$$f^{(1)} \otimes \cdots \otimes f^{(r)}\colon A^{\otimes r} \Rightarrow \bigwedge(x_1) \otimes \cdots \otimes \bigwedge(x_r) = \mathbf{\Lambda}$$

whose base component is the tensor product of the base components $f_1^{(i)}$. Since A is a Hopf algebra, the r-fold diagonal $\Delta^{(r)}\colon A \to A^{\otimes r}$ is a morphism of algebras. A test on the generators x_i reveals that the composition $(f^{(1)} \otimes \cdots \otimes f^{(r)})\Delta^{(r)}\colon A \Rightarrow \mathbf{\Lambda}$ induces an isomorphism in homology, hence is the A_∞ quasi-isomorphism we are looking for.

REMARK 4.3. Since we have not really used the coassociativity of Δ, Theorem 1.4 holds even for quasi-Hopf algebras in the sense of [15, §IV.5].

5. THE TWISTING COCHAIN $v\colon \mathbf{S} \to C(G)$

This is now easy: Compose the map $\mathbf{S} \to B\mathbf{\Lambda}$ determined by the canonical twisting cochain $w\colon \mathbf{S} \to \mathbf{\Lambda}$ with the map $B\mathbf{\Lambda} \to BC(G)$. This corresponds to a twisting cochain $\mathbf{S} \to C(G)$ mapping each cogenerator $y_i \in \mathbf{S}$ to a representative of $x_i \in \mathbf{\Lambda}$. Since these elements are used to define the $\mathbf{\Lambda}$-action in the homology of a weak $\mathbf{\Lambda}$-module, we get the usual action of $\mathbf{\Lambda} = H(G)$ there.
Note that by dualisation we obtain a quasi-isomorphism of algebras $(BC(G))^* \to \mathbf{S}^*$. This is not exactly the same as the quasi-isomorphism of algebras $C^*(BG) \to \mathbf{S}^*$ from Proposition 1.3, which we are going to construct next.

6. PROOF OF THEOREM 1.2 (FIRST PART) AND OF PROPOSITION 1.3

In this section we construct maps

$$\Psi_X\colon \mathbf{S} \otimes_v C(X) \to C(EG \underset{G}{\times} X) = C(\boldsymbol{t}X),$$

natural in $X \in G$-***Space***. We will show that $\psi := \Psi_{\mathrm{pt}}\colon \mathbf{S} \to C(BG)$ is a quasi-isomorphism of coalgebras and that Ψ_X, which maps from an $\mathbf{S}$-comodule to a $C(BG)$–comodule, is a ψ-equivariant quasi-isomorphism. Taking duals then gives Proposition 1.3 and the first half of Theorem 1.2.
Recall that the differential on $\mathbf{S} \otimes_v C(X)$ is

$$d(y_\alpha \otimes c) = y_\alpha \otimes dc + \sum_{\beta<\alpha} y_\beta \otimes c_{\alpha-\beta} \cdot c,$$

where we have abbreviated $v(y_{\alpha-\beta})$ to $c_{\alpha-\beta}$. The summation runs over all β strictly smaller than α in the canonical partial ordering of $\mathbf{N}^r$.

To begin with, we define a map

$$f\colon \mathbf{S}\otimes_v C(G)\to C(EG)$$

by recursively setting

$$f(1\otimes a)=e_0\cdot a,$$
$$f(y_\alpha\otimes a)=\Big(Sf\big(d(y_\alpha\otimes 1)\big)\Big)\cdot a$$

for $\alpha>0$. Here e_0 is the canonical base point of the simplicial construction of the right G-space EG and S its canonical contracting homotopy, cf. [5, Sec. 3.7].

Lemma 6.1. *This f is a quasi-morphism of right $C(G)$-modules.*

Proof. The map is equivariant by construction. By induction, one has for $\alpha>0$

$$d\,f(y_\alpha\otimes 1)=d\,Sf\big(d(y_\alpha\otimes 1)\big)=f\big(d(y_\alpha\otimes 1)\big)-S\,d\,f\big(d(y_\alpha\otimes 1)\big)=f\big(d(y_\alpha\otimes 1)\big),$$

which shows that it is a chain map. That it induces an isomorphism in homology follows from the acyclicity of $\mathbf{S}\otimes_v C(G)$: Filter the complex according to the number of factors ξ_i appearing in an element $\xi^\alpha\otimes a$, i.e., by $\alpha_1+\cdots+\alpha_r$. Then the E^1 term of the corresponding spectral sequence is the Koszul complex $\mathbf{S}\otimes_w \mathbf{\Lambda}$, hence acyclic. □

We will also need the following result:

Lemma 6.2. *The image of $f(y_\alpha\otimes 1)$, $\alpha\in\mathbf{N}^r$, under the diagonal Δ of the coalgebra $C(EG)$ is*

$$\Delta f(y_\alpha\otimes 1)\equiv\sum_{\beta+\gamma=\alpha} f(y_\beta\otimes 1)\otimes f(y_\gamma\otimes 1),$$

up to terms of the form $c\cdot a\otimes c'$ with c, $c'\in C(EG)$ and $a\in C(G)$, $|a|>0$.

Proof. We proceed by induction, the case $\alpha=0$ being trivial. For $\alpha>0$ we have

$$\Delta f(y_\alpha\otimes 1)=\Delta Sf\big(d\,(y_\alpha\otimes 1)\big)=\sum_{\beta<\alpha}\Delta S\big(f(y_\beta\otimes 1)\cdot c_{\alpha-\beta}\big)$$

We now use the identity $\Delta S(c)=Sc\otimes 1+(1\otimes S)AW(c)$ [5, Prop. 3.8] and the $C(G)$-equivariance of the Alexander–Whitney map to get

$$=f(y_\alpha\otimes 1)\otimes 1+(1\otimes S)\sum_{\beta<\alpha}\Delta f(y_\beta\otimes 1)\cdot\Delta c_{\alpha-\beta},$$

where the second diagonal is of course that of $C(G)$. By induction and the fact that $\Delta c_{\alpha-\beta}\equiv 1\otimes c_{\alpha-\beta}$ up to terms $a\otimes a'$ with $|a|>0$, we find

$$\equiv f(y_\alpha\otimes 1)\otimes 1+(1\otimes S)\sum_{\beta+\gamma<\alpha} f(y_\beta\otimes 1)\otimes f(y_\gamma\otimes 1)\cdot c_{\alpha-(\beta+\gamma)}$$
$$=f(y_\alpha\otimes 1)\otimes 1+\sum_{\substack{\beta<\alpha\\ \gamma<\alpha-\beta}} f(y_\beta\otimes 1)\otimes Sf(y_\gamma\otimes c_{(\alpha-\beta)-\gamma}),$$

which simplifies by the definition of f to

$$= f(y_\alpha \otimes 1) \otimes 1 + \sum_{\beta<\alpha} f(y_\beta \otimes 1) \otimes f(y_{\alpha-\beta} \otimes 1),$$

as was to be shown. □

For a G-space X we define the map

$$\Psi_X \colon \mathbf{t}C(X) = \mathbf{S} \otimes_v C(X) \to C(EG \times_G X) = C(\boldsymbol{t}X)$$

as the bottom row of the commutative diagram

$$\begin{tikzcd}
& \mathbf{S} \otimes_v C(G) \otimes C(X) \arrow[r, "f \otimes 1"] \arrow[d] & C(EG) \otimes C(X) \arrow[r, "\nabla"] \arrow[d] & C(EG \times X) \arrow[d] \\
\mathbf{S} \otimes_v C(X) \arrow[r, equal] & \mathbf{S} \otimes_v C(G) \otimes_{C(G)} C(X) \arrow[r] & C(EG) \otimes_{C(G)} C(X) \arrow[r] & C(EG \times_G X),
\end{tikzcd}$$

where ∇ denotes the shuffle map. Ψ_X is obviously natural in X.
It follows from the preceding lemma that $\psi = \Psi_{\mathrm{pt}} \colon \mathbf{S} \to C(BG)$ is a morphism of coalgebras because terms of the form $c \cdot a$ with $|a| > 0$ are annihilated by the projection $C(EG) \to C(BG)$. (We are working with normalised chains!) Using naturality and the commutativity of the diagram

$$\begin{tikzcd}
C(EG) \otimes C(X) \arrow[r, "\nabla"] \arrow[d, "\Delta_{C(EG)} \otimes 1"'] & C(EG \times X) \arrow[d, "\Delta_{C(EG \times X)}"] \\
C(BG) \otimes C(EG) \otimes C(X) \arrow[r, "1 \otimes \nabla"'] & C(BG) \otimes C(EG \times X),
\end{tikzcd}$$

one proves similarly that Ψ_X is a ψ-equivariant morphism of comodules. To see that it induces an isomorphism in homology, consider the diagram

$$\begin{tikzcd}
\mathrm{Tor}^{C(G)}\big(\mathbf{S} \otimes_v C(G), C(X)\big) \arrow[r] \arrow[d, "\mathrm{Tor}^{\mathrm{id}}(f, \mathrm{id})"'] & H\big(\mathbf{S} \otimes_v C(G) \otimes_{C(G)} C(X)\big) \arrow[r, equal] \arrow[d] & H(\mathbf{S} \otimes_v C(X)) \arrow[d, "H(\Psi_X)"] \\
\mathrm{Tor}^{C(G)}\big(C(EG), C(X)\big) \arrow[r] & H\big(C(EG) \otimes_{C(G)} C(X)\big) \arrow[r] & H(EG \times_G X).
\end{tikzcd}$$

The composition along the bottom row is an isomorphism by Moore's theorem [18, Thm. 7.27],[2] and the top row is so because $\mathbf{S} \otimes_v C(G)$ is $C(G)$-flat. Since $\mathrm{Tor}^{\mathrm{id}}(f, \mathrm{id})$ is an isomorphism by Lemma 6.1, $H(\Psi_X)$ is so, too.

[2]In fact, each single arrow is an isomorphism. This follows from the twisted Eilenberg–Zilber theorem, see [9] for example.

7. The twisting cochain $u\colon \mathbf{\Lambda}^* \to C^*(BG)$ and the end of the proof of Theorem 1.2

The map $\psi\colon \mathbf{S} \to C(BG)$ is a quasi-isomorphism of simply connected coalgebras. Similar to the first step in the proof of Lemma 4.1, it comes from a trivialised extension (or "Eilenberg–Zilber data" in the terminology of [11]). By [11, Thm. 4.1*], there is an algebra map $F\colon \Omega C(BG) \to \Omega\mathbf{S}$ whose base component $F_{-1}\colon \Omega_{-1}C(BG) \to \Omega_{-1}\mathbf{S}$ is essentially the chosen homotopy inverse to ψ. Composing such an F with the canonical map $g\colon \Omega\mathbf{S} \to \mathbf{\Lambda}$, we get a twisting cochain $\tilde{u}\colon C(BG) \to \mathbf{\Lambda}$. Write

$$\tilde{u} = \sum_{\emptyset \neq \pi \subset \{1,\dots,r\}} x_\pi \otimes \gamma_\pi \in \mathbf{\Lambda} \otimes C^*(BG) = \operatorname{Hom}(C(BG), \mathbf{\Lambda}). \tag{7.1}$$

Then γ_i is a representative of the generator $\xi_i \in \mathbf{S}^*$ because it is a cocycle (cf. [5, eq. (2.12)]) and

$$\tilde{u}(\psi(y_i)) = g(F([\psi(y_i)])) = g(y_i) = x_i.$$

The dual $u = \tilde{u}^*\colon \mathbf{\Lambda}^* \to C^*(BG)$ is again a cochain, which corresponds under the isomorphism $\operatorname{Hom}(\mathbf{\Lambda}^*, C^*(BG)) = C^*(BG) \otimes \mathbf{\Lambda}$ to the transposition of factors of (7.1). Therefore, the induced action of $\mathbf{S}^*$ on a $C^*(BG)$-module, considered as weak $\mathbf{S}^*$-module, is given by $\xi_i \cdot [m] = [\gamma_i \cdot m]$, as desired.

For a given G-space X, we now look at the map Ψ_X^* as a quasi-isomorphism of $C^*(BG)$-modules, where the module structure of $\mathbf{t}C^*(X)$ is induced by ψ^*. By naturality, it is a morphism of weak $\mathbf{S}^*$-modules. This new weak $\mathbf{S}^*$-action on $\mathbf{t}C^*(X)$ coincides with the (strict) old one because the composition

$$(\Omega\mathbf{S})^* \xrightarrow{F^*} (\Omega C(BG))^* \xrightarrow{\psi^*} (\Omega\mathbf{S})^*$$

is the identity. This proves that Ψ_X^* is a quasi-isomorphism of weak $\mathbf{S}^*$-modules, hence that the functors $C^* \circ \boldsymbol{t}$ and $\mathbf{t} \circ C^*$ are quasi-isomorphic.
The corresponding result for the functors $\boldsymbol{h}$ and $\mathbf{h}$ is a formal consequence of this because they are quasi-inverse to $\boldsymbol{t}$ and $\mathbf{t}$, respectively. This finishes the proof of Theorem 1.2.

Remark 7.1. For $G = (S^1)^r$ a torus (and a reasonable choice of v) one may also take the twisting cochain $\mathbf{\Lambda}^* \to C^*(BG)$ of Gugenheim–May [10, Example 2.2], which is defined using iterated cup_1 products of (any choice of) representatives $\gamma_i \in C^*(BG)$ of the $\xi_i \in \mathbf{S}^*$. (This follows for example from [5, Cor. 4.4].) It would be interesting to know whether this remains true in general if one chooses the γ_i carefully enough.

8. Equivariantly formal spaces

An important class of G-spaces are the equivariantly formal ones. Their equivariant cohomology is particularly simple, which is often exploited in algebraic or symplectic geometry or combinatorics.
We say that X is *R-equivariantly formal* if the following conditions hold.

PROPOSITION 8.1. *For a G-space X, the following are equivalent:*

1. $H^*_G(X)$ *is extended.*
2. $C^*(X_G)$ *is split and extended.*
3. $C^*(X)$ *is split and trivial.*
4. *The canonical map* $H^*_G(X) \to H^*(X)$ *admits a section of graded R-modules.*
5. $H^*_G(X)$ *is isomorphic, as* $\mathbf{S}^*$*-module, to the* E_2 *term* $\mathbf{S}^* \otimes H^*(X)$ *of the Leray–Serre spectral sequence for* X_G *(which therefore degenerates).*

Note that if R is a field, condition (1) means that $H^*_G(X)$ is free over $\mathbf{S}^*$, and condition (4) that $H^*_G(X) \to H^*(X)$ is surjective. A space X with the latter property is traditionally called "totally non-homologous to zero in X_G with respect to R". We stress the fact that for some of the above conditions we really need the assumption that R is a principal ideal domain.
In [6] (see also [7]) it is shown that a compact symplectic manifold X with a Hamiltonian torus action is $\mathbf{Z}$-equivariantly formal if $X^T = X^{T_p}$ for each prime p that kills elements in $H^*(X^T)$. Here $T_p \cong \mathbf{Z}_p^r$ denotes the maximal p-torus contained in the torus T. In particular, a compact Hamiltonian T-manifold is $\mathbf{Z}$-equivariantly formal if the isotropy group of each non-fixed point is contained in a proper subtorus.

Proof. (5) $\Rightarrow$ (1) is trivial. (1) $\Rightarrow$ (2) follows from Proposition 3.2, and (2) $\Rightarrow$ (3) from Proposition 3.1 because $C^*(X)$ and $C^*(X_G)$ are Koszul dual by Theorem 1.2. (4) $\Rightarrow$ (5) is the Leray–Hirsch theorem. (Note that it holds here for arbitrary X because $H^*(BG) = \mathbf{S}^*$ is of finite type.)
(3) $\Rightarrow$ (4): The (in the simplicial setting canonical) map $C^*(X_G) \to C^*(X)$ is the composition of Ψ^*_X with the canonical projection $\mathbf{t}C^*(X) \to C^*(X)$. Since $C^*(X)$ is split, we can pass from $C^*(X)$ to $H^*(X)$ by a sequence of commutative diagrams

$$\begin{array}{ccc} \mathbf{t}N & \longrightarrow & N \\ \downarrow & & \downarrow \\ \mathbf{t}N' & \longrightarrow & N' \end{array}$$

where the vertical arrow on the right is the base component of the quasi-isomorphism of weak $\mathbf{\Lambda}$-modules given on the left. But for the projection $\mathbf{S}^* \otimes H^*(X) \to H^*(X)$ the assertion is obvious because $\mathbf{\Lambda}$ acts trivially on $H^*(X)$, which means that there are no differentials any more. □

9. Relation to the Cartan model

In differential geometry and differential homological algebra many different complexes ("models") are known that compute the equivariant cohomology of a space. We content ourselves with indicating the relation between our construction and the probably best-known one, the so-called Cartan model. We use real or complex coefficients.

Let G be a compact connected Lie group and X a G-manifold. The Cartan model of X is the complex

$$\Big(\mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega(X)\Big)^G \tag{9.1a}$$

of G-invariants with differential

$$d(\sigma \otimes \omega) = \sigma \otimes d\omega + \sum_{j=1}^{s} \zeta_j \sigma \otimes z_j \cdot \omega. \tag{9.1b}$$

Here $\mathrm{Sym}(\mathfrak{g}^*)$ denotes the (evenly graded) polynomial functions on the Lie algebra $\mathfrak{g}$ of G, (z_j) a basis of $\mathfrak{g}$ with dual basis (ζ_j), and $z_j \cdot \omega$ the contraction of the form ω with the generating vector field associated with z_j. The Cartan model computes $H_G^*(X)$ as algebra and as $\mathbf{S}^*$-module, cf. [12].

As mentioned in the introduction, Goresky, Kottwitz and MacPherson [8] have found an even smaller complex giving the $\mathbf{S}^*$-module $H_G^*(X)$, namely $\mathbf{t}\Omega(X)^G$, or explicitly

$$\mathrm{Sym}(\mathfrak{g}^*)^G \otimes \Omega(X)^G, \tag{9.2a}$$

where $\Omega(X)^G$ denotes the complex of G-invariant differential forms on X. The differential

$$d(\sigma \otimes \omega) = \sigma \otimes d\omega + \sum_{i=1}^{r} \xi_i \sigma \otimes x_i \cdot \omega \tag{9.2b}$$

is similar to (9.1b), but the summation now runs over a system of generators of $\mathbf{S}^* = H^*(BG) = \mathrm{Sym}(\mathfrak{g}^*)^G$. (This is of course differential (3.1) for strict $\mathbf{\Lambda}$-modules.) Alekseev and Meinrenken [1] have proved that the complexes (9.1) and (9.2) are quasi-isomorphic as $\mathbf{S}^*$-modules.

For the case of torus actions (where (9.1) and (9.2) coincide), Goresky–Kottwitz–MacPherson [8, Sec. 12] have shown that one may replace $\Omega(X)^T$ by singular cochains together with the "sweep action", which is defined by restricting the action of $C(T)$ along a quasi-isomorphism of algebras $\mathbf{\Lambda} = H(T) \to C(T)$. The latter is easy to construct, as explained in the introduction. Now all ingredients are defined for an arbitrary topological T-space X and an arbitrary coefficient ring R, and the resulting complex does indeed compute $H_T^*(X)$ as algebra and as $\mathbf{S}^*$-module in this generality, see Félix–Halperin–Thomas [4, Sec. 7.3].

Appendix: The theorems of Samelson–Leray and Hopf

All differentials are zero in this section. Recall that an element a of a Hopf algebra A is called primitive if $\Delta a = a \otimes 1 + 1 \otimes a$ or, equivalently, if the projection of Δa to $\bar{A} \otimes \bar{A}$ is zero.

Let A be a Hopf algebra over a field, isomorphic as algebra to an exterior algebra. Then A is primitively generated (Samelson–Leray). If R is a field of characteristic 0 and A a connected commutative Hopf algebra, finite-dimensional over R, then multiplicatively it is an exterior algebra (Hopf), hence also primitively generated. (A good reference for our purposes is [16, §§1, 2].)
We now show that the analogous statements hold over any principal ideal domain. Denote for a Hopf algebra A over R the extension of coefficients to the quotient field of R by $A_{(0)}$.

Proposition 9.1. *Let A be a Hopf algebra, free over R and such that $A_{(0)}$ is a primitively generated exterior algebra. Then A is a primitively generated exterior algebra.*

Proof. Let A' be the sub Hopf algebra generated by the free submodule of primitive elements of A. Then $A'_{(0)} = A_{(0)}$ (Samelson–Leray), hence A' is a primitively generated exterior algebra and A/A' is R-torsion. Take an $a \in A \backslash A'$ of smallest degree. Then $ka \in A'$ for some $0 \neq k \in R$, and the image of Δa in $\bar{A} \otimes \bar{A}$ already lies in $\bar{A}' \otimes \bar{A}'$. Write $ka = a_1 + a_2$ with $a_1 \in A'$ primitive and $a_2 \in \bar{A}' \cdot \bar{A}'$. Note that the image of Δa_2 in $\bar{A}' \otimes \bar{A}'$ is divisible by k. This implies that a_2 is divisible by k in A'. (Look at how the various products of the generators of a primitively generated exterior algebra behave under the diagonal.) Since $a - a_2/k$ is primitive, it lies in A', hence a as well. Therefore, $A = A'$. □

Added in proof. Suppose that G is a compact connected Lie group and let $T \subset G$ be a maximal torus. In their recent preprint "Torsion and abelianization in equivariant cohomology" (math.AT/0607069), T. Holm and R. Sjamaar show that in this situation $H_G^*(X)$ consists of the Weyl group invariants of $H_T^*(X)$. Their assumption on the coefficient ring R is essentially the same as ours. Together with the explicit Cartan model for torus actions [5], this gives another model for $H_G^*(X)$.

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