Galois coverings, Morita Equivalence and Smash Extensions of Categories over a Field[1]

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Abstract. Algebras over a field k generalize to categories over k in order to considers Galois coverings. Two theories presenting analogies, namely smash extensions and Galois coverings with respect to a finite group are known to be different. However we prove in this paper that they are Morita equivalent. For this purpose we need to describe explicit processes providing Morita equivalences of categories which we call contraction and expansion. A structure theorem is obtained: composition of these processes provides any Morita equivalence up to equivalence, a result which is related with the karoubianisation (or idempotent completion) and additivisation of a k-category.

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1 Introduction

Let k be a field. The observation that a k-algebra A is a category with one object and endomorphisms given by A leads to Galois coverings given by categories with more than one object, see for instance [3]. In this context the universal cover of the polynomial algebra in one variable is the free category

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over k generated by the infinite quiver with integer vertices and one arrow from i to $i+1$ for each integer i.

A category $\mathcal{C}$ over a field k has a set of objects $\mathcal{C}_0$ and each morphism set ${}_y\mathcal{C}_x$ from an object x to an object y is a k-vector space, the composition of maps of $\mathcal{C}$ is k-bilinear. In particular each endomorphism set ${}_x\mathcal{C}_x$ is an associative k-algebra. Such categories are called k-categories, they have been considered extensively and are considered as algebras with several objects, see [13, 14].

This work has a two-fold main purpose.

In one direction we show that there is a coincidence up to Morita equivalence between Galois coverings of k-categories and smash extensions for a finite group. More precisely we associate to each Galois covering of a k-category with finite group G a smash extension with the same group, having the property that the categories involved are Morita equivalent to the starting ones. In particular from a full and dense functor we obtain a faithful one. Conversely, a smash extension of categories gives rise to a Galois covering, with categories actually equivalent to the original ones. Consequently both procedures are mutual inverses up to Morita equivalence.

This Theorem explains the analogous spectral sequence arising in cohomology for both theories, see [5] and the generalisation [16], and [12].

We emphasize that similar results for an arbitrary group G can be obtained considering coalgebras and comodule categories. This approach will be detailed in a subsequent paper.

In the other direction, motivated by the above problem, we study the Morita equivalence of k-categories, obtaining a complete description of these equivalences. In other words, a Morita theorem for linear categories.

We consider modules over a k-category $\mathcal{C}$, that is k-functors from $\mathcal{C}$ to the category of k-vector spaces *i.e.* collections of vector spaces attached to the objects with "actions" of morphisms transforming vectors at the source of the morphism to vectors at the target. Notice that if $\mathcal{C}$ is a finite object set k-category it is well known and easy to prove that modules over $\mathcal{C}$ coincide with usual modules over the "matrix algebra" $a(\mathcal{C}) = \oplus_{x,y\in\mathcal{C}}\, {}_y\mathcal{C}_x$.

We introduce in this paper a general framework for Morita theory for k-categories. More precisely we establish processes which provide categories Morita equivalent to a starting one. We prove in the Appendix that up to equivalence of categories any Morita equivalence of k-categories is a composition of contractions and expansions of a given k-category, where contraction and expansion are processes generalizing a construction considered in [5]. More precisely, given a partition E of the set of objects of a k-category $\mathcal{C}$ by means of finite sets, the *contracted category* $\mathcal{C}_E$ along E has set of objects the sets of the partition while morphisms are provided by the direct sum of all the morphism spaces involved between two sets of the partition. The reverse construction is called *expansion*. Another process is related to the classical Morita theory for algebras, that is for each vertex we provide an endomorphism algebra Morita equivalent to the given one together with a corresponding Morita context, which enables us to modify the morphisms of the original category. In particular the matrix category of a given category is obtained in this way. A discussion of this processes in relation with karoubianisation and additivisation (see for instance [1, 18]) is also presented in the Appendix. We thank Alain Bruguières and Mariano Suarez Alvarez for useful conversations concerning this point.

Usually smash extensions are considered for algebras, see for instance [15]. We begin by extending this construction to k-categories, namely given a Hopf algebra H we consider a Hopf module structure on a k-category $\mathcal{C}$ which is provided by an H-module structure on each morphism space such that the composition maps of $\mathcal{C}$ are H-module maps - in particular the endomorphism algebra of each object is required to be an usual H-module algebra. Given a Hopf module k-category $\mathcal{C}$ we define the *smash category* $\mathcal{C}\#H$ in a coherent way with the algebra case.

We need this extension of the usual algebra setting to the categorical one in order to relate smash extensions to Galois coverings of k-categories as considered for instance in [3, 5, 7].

Notice that we can consider, as in the algebra case, a smash extension of a category as a Hopf Galois extension with the normal basis property and with trivial map σ, see [15, p. 101] and also [2, 11, 17]. It would be interesting to relate non trivial maps σ to an extended class of coverings of categories accordingly, we will not initiate this study in the present paper.

We define a *smash extension* of an H-module category $\mathcal{C}$ to be the natural functor from $\mathcal{C}$ to $\mathcal{C}\#H$. An expected compatibility result holds, namely if the number of objects of $\mathcal{C}$ is finite, the corresponding matrix algebra $a(\mathcal{C})$ has an usual smash extension provided by $a(\mathcal{C}\#H)$. The later algebra can indeed be considered since the category $\mathcal{C}\#H$ has also a finite number of objects, namely the set of objects of $\mathcal{C}$. Moreover, we have that $a(\mathcal{C})\#H = a\,(\mathcal{C}\#H)$.

We consider also Galois coverings of k-categories given by a group G, that is a k-category with a free G-action and the projection functor to the corresponding quotient category. More precisely, by definition a G-k-category $\mathcal{C}$ has a set action of G on the set of objects, and has linear maps ${}_y\mathcal{C}_x \to {}_{sy}\mathcal{C}_{sx}$ for each element s of G and each couple of objects x and y, verifying the usual axioms that we recall in the text. In other words we have a group morphism from G to the autofunctors of $\mathcal{C}$. In case $\mathcal{C}$ is object-finite, we infer a usual action of G by automorphisms of the algebra $a(\mathcal{C})$. A G-k-category is called *free* in case the set action on the objects is free, namely $sx = x$ implies $s = 1$. The quotient category is well defined only in this case and we recall its construction, see [3, 9, 7, 5, 4].

The group algebra kG is a Hopf algebra, hence we can consider kG-module categories. Notice that G-k-categories form a wider class than kG-module categories. In fact kG-module categories are G-k-categories which have trivial action of G on the set of objects.

First we establish a comparison between two constructions obtained when starting with a graded category $\mathcal{C}$ over a finite group G. From one side the smash product category $\mathcal{C}\#k^G$ is defined in the present paper, and from the other side a smash product category $\mathcal{C}\#G$ has been considered in [4], actually the later

is the Galois covering of $\mathcal{C}$ corresponding to the grading. We show that $\mathcal{C}\#k^G$ and $\mathcal{C}\#G$ are not equivalent but Morita equivalent categories.
We note that starting with a Galois covering $\mathcal{C}$ of a category $\mathcal{B}$, the covering category $\mathcal{C}$ is $\mathcal{B}\#G$ (see [4] and the grading of $\mathcal{B}$ introduced there, first considered by E. Green in [10] for presented k-categories by a quiver with relations). Unfortunately $\mathcal{B}\#G$ has no natural kG-module category structure. However $\mathcal{B}\#G$ and $\mathcal{B}\#k^G$ are Morita equivalent and we perform the substitution. The later is a kG-module category using the left kG-module structure of k^G provided by $t\delta_s = \delta_{st^{-1}}$. In this way we associate to the starting Galois covering the smash extension $(\mathcal{B}\#k^G)\#kG$ of $\mathcal{B}\#k^G$.
The important point is that the later is Morita equivalent to $\mathcal{C}$ while $(\mathcal{B}\#k^G)\#kG$ is isomorphic to a matrix category that we introduce, which in turn is Morita equivalent to $\mathcal{B}$. Notice that this result is a categorical version of the Cohen Montgomery duality Theorem, see [6]. Hence we associate to the starting Galois covering $\mathcal{C} \to \mathcal{B}$ a smash extension with the same group and where the categories are replaced by Morita equivalent ones.
Second we focus to the reverse procedure, namely given a smash extension of categories with finite group G – that is a kG-module category $\mathcal{B}$ and the inclusion $\mathcal{B} \to \mathcal{C} = \mathcal{B}\#kG$ – we intend to associate a Galois covering to this data. For this purpose we consider the *inflated category* $I_F\mathcal{B}$ of a category $\mathcal{B}$ along a sequence $F = \{F_x\}$ of sets associated to the vertices of the original category : each object x of $\mathcal{B}_0$ provides $\mid F_x \mid$ new objects while the set of morphisms from (x,i) to (y,j) is precisely the vector space ${}_y\mathcal{B}_x$ with the obvious composition. For a finite group G the inflated category of a kG-module category – using the constant sequence of sets G – has a natural structure of a free G-k-category. The inflated category $I_G\mathcal{B}$ is Morita equivalent to the matrix category $M_{|G|}(\mathcal{B})$ by contraction and in turn the later is Morita equivalent to $\mathcal{B}$.
Moreover the categorical quotient of $I_G\mathcal{C}$ exists and in this way we obtain a Galois covering having the required properties with respect to the starting smash extension.

2 Hopf module categories

In this section we introduce the smash product of a category with a Hopf algebra and we specify this construction in case the Hopf algebra is the function algebra of a finite group G. We will obtain that the later is Morita equivalent to the smash product category defined in [4].
We recall (see for instance [15]) that for a Hopf algebra H over k, an H-module algebra A is a k-algebra which is simultaneously an H-module in such a way that the product map of A is a morphism of H-modules, where $A \otimes A$ is considered as an H-module through the comultiplication of H. Moreover we require that $h1_A = \epsilon(h)1_A$ for every $h \in H$.
We provide an analogous definition for a k-category $\mathcal{C}$ instead of an algebra.

Definition 2.1 *A k-category $\mathcal{C}$ is an H*-module category *if each morphism* *space is an H-module, each endomorphism algebra is an H-module algebra and composition maps are morphisms of H-modules, where as before the tensor product of H-modules is considered as an H-module via the comultiplication of H.*

Notice that analogously we may consider the structure of an H-comodule category. In case H is a finite dimensional Hopf algebra, we recall from [15] that there is a bijective vector space preserving correspondence between right H-modules and left H^*-comodules.

Remark 2.2 *Given a finite k-category $\mathcal{C}$, let $a(\mathcal{C})$ be the k-algebra obtained as the direct sum of all k-module morphisms of $\mathcal{C}$ equipped with the usual matrix product combined with the composition of $\mathcal{C}$. In case $\mathcal{C}$ is an H-module category $a(\mathcal{C})$ is an H-module algebra.*

Let $\mathcal{C}$ be an H-module category. We define the k-category $\mathcal{C}\#H$ as follows. The objets remain the same, while given two objects x and y we put ${}_y(\mathcal{C}\#H)_x = {}_y\mathcal{C}_x \otimes_k H$. The composition map for morphisms

$$_z(\mathcal{C}\#H)_y \ \otimes \ {}_y(\mathcal{C}\#H)_x \longrightarrow {}_z(\mathcal{C}\#H)_x$$

is given by

$$({}_z\varphi_y \otimes h) \circ ({}_y\psi_x \otimes h') = \sum {}_z\varphi_y \circ (h_1 \ {}_y\psi_x) \otimes h_2 h',$$

where the comultiplication Δ of H is given by $\Delta(h) = \sum h_1 \otimes h_2$ and $\circ$ denotes composition in $\mathcal{C}$. As before we have an immediate coherence result:

Proposition 2.3 *Let $\mathcal{C}$ be a finite object H-module category $\mathcal{C}$. Then the k-algebras $a(\mathcal{C})\#H$ and $a(\mathcal{C}\#H)$ are canonically isomorphic.*

Let now G be a group. A G-graded k-category $\mathcal{C}$ (see for instance [4]) is a k-category $\mathcal{C}$ such that each morphism space ${}_y\mathcal{C}_x$ is the direct sum of sub-vector spaces ${}_y\mathcal{C}_x^s$, indexed by elements $s \in G$ such that ${}_z\mathcal{C}_y{}^t \ {}_y\mathcal{C}_x{}^s \subseteq {}_y\mathcal{C}_x{}^{ts}$ for all $x, y \in \mathcal{C}$ and for all $s, t \in G$.
Notice that as in the algebra case, gradings of a k-category $\mathcal{C}$ by means of a group G are in one-to-one correspondence with kG-comodule category structures on $\mathcal{C}$. Let now G be a finite group, $\mathcal{C}$ be a G-graded k-category and consider the function algebra $k^G = (kG)^*$ which is a Hopf algebra. The category $\mathcal{C}$ is a k^G-module category, hence according to our previous definition we can consider $\mathcal{C}\#k^G$.
We want to compare this category with another construction of a k-category denoted $\mathcal{C}\#G$ which can be performed for an arbitrary group G, see [4] : the set of objects is $\mathcal{C}_0 \times G$ while the morphisms from (x, s) to (y, t) is the vector space ${}_y\mathcal{C}_x{}^{(t^{-1}s)}$. The composition of morphisms is well-defined as an immediate consequence of the definition of a graded category.

Notice that given a graded algebra A considered as a single object G-graded k-category, the preceding construction provides a category with as many objects as elements of G, even if G is infinite. If G is finite, the associated algebra is known to be the usual smash product algebra $A\#k^G$, see [4].
We will recall below the definition of the module category of a k-category in order to prove that in case of a finite group G the module categories over $\mathcal{C}\#k^G$ and $\mathcal{C}\#G$ are equivalent.
First we introduce a general setting which is interesting by itself.

DEFINITION 2.4 *Let $\mathcal{D}$ be a k-category equipped with a partition E of the set of objects $\mathcal{D}_0$ by means of finite sets $\{E_i\}_{i\in I}$. Then $\mathcal{D}_E$ is a new k-category obtained by* contraction *along the partition, more precisely I is the set of objects of $\mathcal{D}_E$ and morphisms are given by*

$$_j(\mathcal{D}_E)_i = \bigoplus_{y\in E_j \quad x\in E_i} {}_y\mathcal{D}_x.$$

Composition is given by matrix product combined with composition of the original category. Notice that the identity map of an object i is given by $\sum_{z\in E_i} {}_z1_z$, which makes sense since E_i is finite.

EXAMPLE 2.5 *Let A be an algebra and let F be a complete finite family of orthogonal idempotents in A (we don't require that the idempotents are primitive). Consider the category $\mathcal{D}$ with set of objects F and morphisms ${}_y\mathcal{D}_x = yAx$. Then the contracted category along the trivial partition with only one subset is a single object category having endomorphism algebra $\bigoplus_{x,y\in F} {}_y\mathcal{D}_x = \bigoplus_{x,y\in F} yAx = A$.*

We also observe that for a finite object k-category $\mathcal{C}$, the contracted category along the trivial partition is a single object category with endomorphism algebra precisely $a(\mathcal{C})$. More generally let E be a partition of $\mathcal{C}_0$, then the k-algebras $a(\mathcal{C})$ and $a(\mathcal{C}_E)$ are equal.
We will establish now a relation between $\mathcal{D}$ and $\mathcal{D}_E$ at the representation theory level of these categories. In order to do so we recall the definition of modules over a k-category.

DEFINITION 2.6 *Let $\mathcal{C}$ be a k-category. A left $\mathcal{C}$-module $\mathcal{M}$ is a collection of k-modules $\{{}_x\mathcal{M}\}_{x\in\mathcal{C}_0}$ provided with a left* action *of the k-modules of morphisms of $\mathcal{C}$, given by k-module maps ${}_y\mathcal{C}_x \otimes_k {}_x\mathcal{M} \to {}_y\mathcal{M}$, where the image of ${}_yf_x \otimes {}_xm$ is denoted ${}_yf_x\ {}_xm$, verifying the usual axioms:*

- ${}_zf_y\ ({}_yg_x\ {}_xm) = ({}_zf_y\ {}_yg_x)\ {}_xm$,
- ${}_x1_x\ {}_xm = {}_xm$.

In other words $\mathcal{M}$ is a covariant k-functor from $\mathcal{C}$ to the category of k-modules, the preceding explicit definition is useful for some detailed constructions. We denote by $\mathcal{C}-\mathrm{Mod}$ the category of left $\mathcal{C}$-modules. In case of a k-algebra A it is clear that A-modules considered as k-vector spaces equipped with an action of A coincide with $\mathbb{Z}$-modules provided with an A-action. Analogously, $\mathcal{C}$-modules as defined above are the same structures than $\mathbb{Z}$-functors from $\mathcal{C}$ to the category of $\mathbb{Z}$-modules.

Definition 2.7 *Two k-categories are said to be* Morita equivalent *if their left module categories are equivalent.*

Proposition 2.8 *Let $\mathcal{D}$ be a k-category and let E be a partition of the objects of $\mathcal{D}$ by means of finite sets. Then $\mathcal{D}$ and the contracted category $\mathcal{D}_E$ are Morita equivalent.*

We notice that this result is an extension of the well known fact that the category of modules over an algebra is isomorphic to the category of functors over the category of projective left modules provided by a direct sum decomposition of the free rank one left module, obtained for instance through a complete system of orthogonal idempotents of the algebra.
Proof. Let $\mathcal{M}$ be a $\mathcal{D}$-module and let $F\mathcal{M}$ be the following $\mathcal{D}_E$-module:

$$_iF\mathcal{M} = \bigoplus_{x\in E_i} {}_x\mathcal{M} \text{ for each } i \in I,$$

the action of a morphism $_jf_i = ({}_yf_x)_{x\in E_i, y\in E_j} \in {}_j(\mathcal{D}_E)_i$ on $_im = ({}_xm)_{x\in E_i} \in {}_iF(\mathcal{M})$ is obtained as a matrix by a column product, namely:

$$_jf_i \; {}_im = (\sum_{x\in E_i} {}_yf_x \; {}_xm)_{y\in E_j}.$$

A $\mathcal{D} - \mathrm{Mod}$ morphism $\phi : \mathcal{M} \to \mathcal{M}'$ is a natural transformation between both functors, *i.e.* a collection of k-maps $_x\phi : {}_x\mathcal{M} \to {}_x\mathcal{M}'$, satisfying compatibility conditions. We define $F\phi : F\mathcal{M} \to F\mathcal{M}'$ by:

$$_i(F\phi) = \bigoplus_{x\in E_i} {}_x\phi.$$

Conversely given a $\mathcal{D}_E$-module $\mathcal{N}$, let $G\mathcal{N} \in (\mathcal{D} - \mathrm{Mod})$ be the functor given by $_x(G\mathcal{N}) = e_x\,({}_i\mathcal{N})$, where i is unique element in I such that $x \in E_i$, and where e_x is the idempotent $|E_i| \times |E_i|$ - matrix with one in the (x, x) entry and zero elsewhere.
The action of $_yf_x \in {}_y\mathcal{D}_x$ on $_x(G\mathcal{N})$ is obtained as follows: let $i, j \in I$ be such that $x \in E_i$ and $y \in E_j$. Let $\overline{{}_yf_x} \in {}_j(\mathcal{D}_E)_i$ be the matrix with $_yf_x$ in the (y, x) entry and zero elsewhere. Then, for $e_xn \in {}_x(G\mathcal{N})$ we put $({}_yf_x)(e_xn) = {}_j(\overline{{}_yf_x})_i \; {}_i(e_xn) \in e_y\,({}_j\mathcal{N}) = {}_y(G\mathcal{N})$.
It is easy to verify that both compositions of F and G are the corresponding identity functors.

We will now apply the preceding result to the situation $\mathcal{D} = \mathcal{C}\#G$ using the partition provided by the orbits of the free G-action on the objects.

Theorem 2.9 *The k-categories $\mathcal{C}\#G$ and $\mathcal{C}\#k^G$ are Morita equivalent.*

Proof. We consider the contraction of $\mathcal{C}\#G$ along the partition provided by the orbits, namely for $x \in \mathcal{C}_0$ we put $E_x = \{(x,g) \mid g \in G\}$. Observe that for all $x \in \mathcal{C}_0$ the set E_x is finite since its cardinal is the order of the group G. Moreover the set of objects $((\mathcal{C}\#G)_E)_0$ of the contracted category is identified to $\mathcal{C}_0$.
The morphisms from x to y in the contracted category are $\bigoplus_{s,t\in G} {}_y\mathcal{C}_x{}^{t^{-1}s}$. On the other hand

$$ {}_y(\mathcal{C}\#k^G)_x = {}_y\mathcal{C}_x \otimes k^G = \bigoplus_{v\in G} {}_y\mathcal{C}_x{}^v \otimes k^G. $$

We assert that the contracted category $(\mathcal{C}\#G)_E$ and $\mathcal{C}\#k^G$ are isomorphic. The sets of objects already coincide. We define the functor L on the morphisms as follows. Let ${}_{(y,t)}f_{(x,s)}$ be an elementary matrix morphism of the contracted category. We put

$$ L\left({}_{(y,t)}f_{(x,s)}\right) = f \otimes \delta_s \in {}_y\mathcal{C}_x{}^{t^{-1}s} \otimes k^G. $$

It is not difficult to check that L is an isomorphism preserving composition.

Remark 2.10 *The categories $\mathcal{C}\#G$ and $\mathcal{C}\#k^G$ are not equivalent in general as the following simple example already shows : let A be the group algebra kC_2 of the cyclic group of order two C_2 and let $\mathcal{C}_A$ be the single object C_2-graded k-category with A as endomorphism algebra. The category $\mathcal{C}\#C_2$ has two objects that we denote $(*,1)$ and $(*,t)$, while $\mathcal{C}\#k^{C_2}$ has only one object $*$. If $\mathcal{C}\#G$ and $\mathcal{C}\#k^G$ were equivalent categories the algebras $End_{\mathcal{C}\#C_2}((*,1))$ and $End_{\mathcal{C}\#k^{C_2}}(*))$ would be isomorphic. However the former is isomorphic to k while the latter is the four dimensional algebra $End_{\mathcal{C}\#k^{C_2}}(*) = (k \bigoplus kt) \otimes k^{C_2}$.*

3 kG-module categories

Let G be a group and let $\mathcal{C}$ be a kG-module category. Using the Hopf algebra structure of kG and the preceding definitions we are able to construct the smash category $\mathcal{C}\#kG$. We have already noticed that if $\mathcal{C}$ is an object finite k-category then the algebra $a(\mathcal{C}\#kG)$ is the classical smash product algebra $a(\mathcal{C})\#kG$.
According to [4] a G-k-category $\mathcal{D}$ is a k-category with an action of G on the set of objects and, for each $s \in G$, a k-linear map $s : {}_y\mathcal{D}_x \to {}_{sy}\mathcal{D}_{sx}$ such that $s(gf) = s(g)s(f)$ and $t(sf) = (ts)f$ for any composable couple of morphisms g, f and any elements s, t of G. Such a category is called a free G-k-category in case the action of G on the objects is a free action, namely the only group element acting trivially on the category is the trivial element of G.

Remark 3.1 *Notice that kG-module categories are G-k-categories verifying that the action of G on the set of objects is trivial.*

We need to associate a free G-k-category to a kG-module category $\mathcal{C}$, in order to perform the quotient category as considered in [4]. For this purpose we consider *inflated categories* as follows.

Definition 3.2 *Let $\mathcal{C}$ be a k-category and let $F = (F_x)_{x\in\mathcal{C}_0}$ be a sequence of sets associated to the objects of $\mathcal{C}$. The set of objects of the inflated category $I_F\mathcal{C}$ is*

$$\{(x,i) \mid x \in \mathcal{C}_0 \text{ and } i \in F_x\}$$

while ${}_{(y,j)}(I_F\mathcal{C})_{(x,i)} = {}_y\mathcal{C}_x$ with the obvious composition provided by the composition of $\mathcal{C}$. Alternatively, consider F as a map φ from a set to $\mathcal{C}_0$ such that the fiber over each object x is F_x. The set of objects of the inflated category is the fiber product of $\mathcal{C}_0$ with this set over φ.

Remark 3.3 *Clearly an inflated category is equivalent to the original category since all the objects with the same first coordinate are isomorphic. Hence a choice of one object in each set $\{(x,i) \mid i \in F_x\}$ provides a full sub-category of $I_F\mathcal{C}$ which is isomorphic to $\mathcal{C}$.*

In case $\mathcal{C}$ is a kG-module category we use the constant sequence of sets provided by the underlying set of G. We obtain a free action of G on the objects of the inflated category $I_G\mathcal{C}$ by translation on the second coordinate. Moreover the original action of G on each morphism set of $\mathcal{C}$ provides a free G-k-category structure on the inflated category. More precisely the G-action on the category $I_G\mathcal{C}$ is obtained through maps for each $u \in G$ as follows:

$$u : {}_{(y,t)}I_G\mathcal{C}_{(x,s)} \to {}_{(y,ut)}I_G\mathcal{C}_{(x,us)}$$

$$u\left({}_{(y,t)}f_{(x,s)}\right) = {}_{(y,ut)}(u\left({}_yf_x\right))_{(x,us)}.$$

As a next step we notice that the free G-k-category $I_G\mathcal{C}$ has a skew category $(I_G\mathcal{C})[G]$ associated to it. In fact any G-k-category has a related skew category defined in [4]. We recall that $(I_G\mathcal{C})[G]_0 = (I_G\mathcal{C})_0 = \mathcal{C}_0 \times G$. For $x, y \in \mathcal{C}_0$ $t, s \in G$ we have

$${}_{(y,t)}(I_G\mathcal{C})[G]_{(x,s)} = \bigoplus_{u\in G} {}_{(y,ut)}(I_G\mathcal{C})_{(x,s)} = \bigoplus_{u\in G} {}_y\mathcal{C}_x = {}_y\mathcal{C}_x \times G.$$

We are going to compare the categories $\mathcal{C}\#kG$ and $(I_G\mathcal{C})[G]$. In order to do so we consider the intermediate quotient category $(I_G\mathcal{C})/G$ (see [4, Definition 2.1]). We recall the definition of $\mathcal{D}/G$, where $\mathcal{D}$ is a free G-k-category: the set of objects is the set of G-orbits of $\mathcal{D}_0$, while the k-module of morphisms in $\mathcal{D}/G$ from the orbit α to the orbit β is

$${}_\beta(\mathcal{D}/G)_\alpha = \left(\bigoplus_{b\in\beta, a\in\alpha} {}_b\mathcal{D}_a\right)/G.$$

Recall that X/G denotes the module of coinvariants of a kG-module X, namely the quotient of X by $(\mathrm{Ker}\,\epsilon)X$ where $\epsilon : kG \to k$ the augmentation map. Composition is well defined precisely because the action of G is free on the objects, more explicitly, for $g \in {}_d\mathcal{D}_c$ and $f \in {}_b\mathcal{D}_a$ where b and c are objects in the same G-orbit, let s be the unique element of G such that $sb = c$. Then $[g][f] = [g\ (sf)] = [(s^{-1}g)\ f]$.

Lemma 3.4 *The k-categories $\mathcal{C}\#kG$ and $(I_G\mathcal{C})/G$ are isomorphic.*

Proof. Clearly the set of objects can be identified. Given a morphism $({}_yf_x \otimes u) \in {}_y(\mathcal{C}\#kG)_x$ we associate to it the class $[f]$ of the morphism $f \in {}_{(y,1)}(I_G\mathcal{C})_{(x,u)}$. Notice that in the smash category we have

$$({}_zg_y \otimes v)({}_yf_x \otimes u) = {}_zg_y\ v({}_yf_x) \otimes vu$$

which has image $[{}_zg_y\ v({}_yf_x)]$. The composition in the quotient provides precisely $[g][f] = [g\ vf]$. The inverse functor is also clear.

Since $(I_G\mathcal{C})/G$ and $(I_G\mathcal{C})[G]$ are equivalent (see [4]), we obtain the following:

Proposition 3.5 *The categories $\mathcal{C}\#kG$ and $(I_G\mathcal{C})[G]$ are equivalent.*

4 From Galois coverings to smash extensions and vice versa

Our aim is to relate kG-smash extensions and Galois coverings for a finite group G. Recall that it has been proved in [4] that any Galois covering with group G of a k-category $\mathcal{B}$ is obtained *via* a G-grading of $\mathcal{B}$, we have that $\mathcal{C} = \mathcal{B}\#G$ is the corresponding Galois covering of $\mathcal{B}$. We have already noticed that for a finite group G a G-grading of a k-category $\mathcal{B}$ is the same thing than a k^G-module category structure on $\mathcal{B}$.
However neither $\mathcal{B}$ nor $\mathcal{B}\#G$ have a natural kG-module category structure which could provide a smash extension. We have proven before that $\mathcal{B}\#k^G$ is Morita equivalent to the category $\mathcal{B}\#G$. The advantage of $\mathcal{B}\#k^G$ is that it has a natural kG-module category structure provided by the left kG-module structure of k^G given by $t\delta_s = \delta_{st^{-1}}$.
In this way we associate to the starting Galois covering $\mathcal{B}\#G$ of $\mathcal{B}$ the smash extension $(\mathcal{B}\#k^G) \to (\mathcal{B}\#k^G)\#kG$. In [17] the authors describe when a given Hopf-Galois extension is of this type (in the case of algebras). We will prove that the later is isomorphic to an *ad-hoc* category $M_{|G|}(\mathcal{B})$ which happens to be Morita equivalent to $\mathcal{B}$.

Definition 4.1 *Let $\mathcal{B}$ be a k-category and let n be a sequence of positive integers $(n_x)_{x\in\mathcal{B}_0}$. The objects of the matrix category $M_n(\mathcal{B})$ remain the same objects of $\mathcal{B}$. The set of morphisms from x to y is the vector space of n_x-columns and n_y-rows rectangular matrices with entries in ${}_y\mathcal{B}_x$. Composition of morphisms is given by the matrix product combined with the composition in $\mathcal{B}$.*

Remark 4.2 *In case the starting category $\mathcal{B}$ is a single object category provided by an algebra B, the matrix category has one object with endomorphism algebra precisely the usual algebra of matrices $M_n(B)$.*
Notice that the matrix category that we consider is not the category $\mathrm{Mat}(\mathcal{C})$ *defined by Mitchell in [13]. In fact* $\mathrm{Mat}(\mathcal{C})$ *corresponds to the additivisation of* $\mathcal{C}$ *(see the Appendix).*

We need the next result in order to have that the smash extension associated to a Galois covering has categories Morita equivalent to the original ones. In fact this result is also a categorical generalization of Cohen Montgomery duality Theorem [6].

Lemma 4.3 *Let $\mathcal{B}$ be a G-graded category and let n be the order of G. Then the categories $(\mathcal{B}\#k^G)\#kG$ and $M_n(\mathcal{B})$ are isomorphic.*

Proof. Both sets of objects coincide. Given two objects x and y we define two linear maps:

$$\phi : {}_y\mathcal{B}_x \otimes k^G \otimes kG \to {}_y(M_n(\mathcal{B}))_x\,,$$
$$\psi : {}_y(M_n(\mathcal{B}))_x \to {}_y\mathcal{B}_x \otimes k^G \otimes kG.$$

Given an homogeneous element

$$(f \otimes \delta_g \otimes h) \in {}_y\mathcal{B}_x \otimes k^G \otimes kG,$$

where f has degree r and $g, h \in G$ we put

$$\phi(f \otimes \delta_g \otimes h) = f\ {}_{rg}E_{gh},$$

where ${}_{rg}E_{gh}$ is the elementary matrix with 1 in the (rg, gh)-spot and 0 elsewhere. It is straightforward to verify that ϕ is well-behaved with respect to compositions.
We also define ψ on elementary morphisms as follows:

$$\psi(f\ {}_gE_h) = f \otimes \delta_{r^{-1}g} \otimes g^{-1}rh,$$

where r is the degree of f.

Next we have to prove that $M_n(\mathcal{B})$ is Morita equivalent to $\mathcal{B}$. In order to do so we develop some Morita theory for k-categories which is interesting by itself. When we restrict the following theory to a particular object, it will coincide with the classical theory, see for instance [19, p.326]. Moreover, in case of a finite object set k-categories both Morita theories coincide using the associated algebras that we have previously described.
Let $\mathcal{C}$ be a k-category. For simplicity for a given object x we denote by A_x the k-algebra ${}_x\mathcal{C}_x$. For each x, let B_x be a k-algebra such that there is a (B_x, A_x)-bimodule P_x and a (A_x, B_x)-bimodule Q_x verifying that $P_x \otimes_{A_x} Q_x \cong B_x$ as B_x-bimodules and $Q_x \otimes_{B_x} P_x \cong A_x$ as A_x-bimodules. In other words for each

object we assume that we have a Morita context providing that A_x and B_x are Morita equivalent. Note that it follows from the assumptions that P_x is projective and finitely generated on both sides, see for instance [19].
Using the preceding data we modify the morphisms in order to define a new k-category $\mathcal{D}$ which will be Morita equivalent to $\mathcal{C}$. In particular the endomorphism algebra of each object x will turn out to be B_x.
More precisely the set of objects of $\mathcal{D}$ remains the set of objects of $\mathcal{C}$ while for morphisms we put

$$ {}_y\mathcal{D}_x = P_y \otimes_{A_y} {}_y\mathcal{C}_x \otimes_{A_x} Q_x. $$

Notice that for $x = y$ we have ${}_x\mathcal{D}_x \cong B_x$. In order to define composition in $\mathcal{D}$ we need to provide a map

$$ (P_z \otimes_{A_z} {}_z\mathcal{C}_y \otimes_{A_y} Q_y) \otimes_k (P_y \otimes_{A_y} {}_y\mathcal{C}_y \otimes_{A_x} Q_x) \longrightarrow P_z \otimes_{A_z} {}_z\mathcal{C}_x \otimes_{A_x} Q_x, $$

For this purpose let φ_x be a fixed A_x-bimodule isomorphism from $Q_x \otimes_{B_x} P_x$ to A_x and consider ϕ_x the composition the projection $Q_x \otimes_k P_x \to Q_x \otimes_{B_x} P_x$ followed by φ_x. Then composition is defined as follows

$$ (p_z \otimes g \otimes q_y)(p_y \otimes f \otimes q_x) = p_z \otimes g\,[\phi_y(q_y \otimes p_y)]\,f \otimes q_x. $$

This composition is associative since the use of the morphisms ϕ do not interfere in case of composition of three maps.

Proposition 4.4 *Let $\mathcal{C}$ and $\mathcal{D}$ be k-categories as above. Then $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent.*

Proof. For a $\mathcal{C}$-module $\mathcal{M}$ we define the $\mathcal{D}$-module $F\mathcal{M}$ as follows:

$$ {}_x(F\mathcal{M}) = P_x \otimes_{A_x} {}_x\mathcal{M}, \text{ which is already a left } B_x\text{-module.} $$

The left action ${}_y\mathcal{D}_x \otimes {}_x(F\mathcal{M}) \to {}_y(F\mathcal{M})$ is obtained using the following morphism induced by ϕ_x

$$ \left(P_y \otimes_{A_y} {}_y\mathcal{C}_x \otimes_{A_x} Q_x\right) \otimes_k (P_x \otimes_{A_x} {}_x\mathcal{M}) \longrightarrow P_y \otimes_{A_y} {}_y\mathcal{C}_x \otimes_k A_x \otimes_k {}_x\mathcal{M} $$

and the actions of A_x and of ${}_y\mathcal{C}_x$ on ${}_x\mathcal{M}$. We then obtain a map with target ${}_y(F\mathcal{M})$. This defines clearly a $\mathcal{D}$-module structure.
Similarly we obtain a functor G in the reverse direction which is already an equivalent inverse for F.

We apply now this Proposition to a k-category $\mathcal{C}$ and the category obtained from $\mathcal{C}$ by replacing each endomorphism algebra by matrix algebras over it. For each object x in $\mathcal{C}_0$ consider the k-algebra $B_x = M_n(A_x)$. The bimodule ${}_{M_n(A_x)}(P_x)_{A_x}$ is the left ideal of $M_n(A_x)$ given by the first column and zero elsewhere, while ${}_{A_x}(Q_x)_{M_n(A_x)}$ is given by the analogous right ideal provided by the first row. Then the category $\mathcal{D}$ defined above is precisely $M_n(\mathcal{C})$.

Corollary 4.5 *$\mathcal{C}$ and $M_n(\mathcal{C})$ are Morita equivalent.*

Remark 4.6 *An analogous Morita equivalence still hold when the integer n is replaced by a sequence of positive integers $(n_x)_{x\in\mathcal{C}_0}$.*

The applications of Morita theory for categories developed above covers a larger spectra than the one considered in this paper. We have produced several sorts of Morita equivalences for categories, namely expansion, contraction and the Morita context for categories described above. We will prove the next result in the Appendix.

Theorem 4.7 *Let $\mathcal{C}$ and $\mathcal{D}$ be Morita equivalent k-categories. Up to equivalence of categories, $\mathcal{D}$ is obtained from $\mathcal{C}$ by contractions and expansions.*

Example 4.8 *Let A be a k-algebra and $\mathcal{C}_A$ the corresponding single object category. It is well known that the following k-category $M\mathcal{C}_A$ is Morita equivalent to $\mathcal{C}_A$: objects are all the positive integers $[n]$ and the morphisms from $[n]$ to $[m]$ are the matrices with n columns, m rows, and with A entries.*
At each object $[n]$ choose the system of n idempotents provided by the elementary matrices which are zero except in a diagonal spot where the value is the unit of the algebra. The expansion process through this choice provides a category with numerable set of objects, morphisms are A between any couple of objects, they are all isomorphic, consequently this category is equivalent to $\mathcal{C}_A$. This way a Morita equivalence (up to equivalence) between $\mathcal{C}_A$ and $M\mathcal{C}_A$ is obtained using the expansion process.
Conversely, in order to obtain $M\mathcal{C}_A$ from $\mathcal{C}_A$, first inflate $\mathcal{C}_A$ using the set of positive integers. Then consider the partition by means of the finite sets having all the positive integers cardinality, namely $\{1\}, \{2,3\}, \{4,5,6\}, \dots$. Finally the contraction along this partition provides precisely $M\mathcal{C}_A$.

We provide now an alternative proof of the fact that a matrix category is Morita equivalent to the original one. It provides also evidence for Theorem 4.7 concerning the structure of the Morita equivalence functors. First consider the inflated category using the sequence of positive integers defining the matrix category. We have shown before that this category is equivalent to the original one. Secondly perform the contraction of this inflated category along the finite sets partition provided by couples having the same first coordinate. This category is the matrix category. Since we know that a contracted category is Morita equivalent to the original one, this provides a proof that a matrix category is Morita equivalent to the the starting category, avoiding the use of Morita contexts. The alternative proof we have presented indicate how classical Morita equivalence between algebras can be obtained by means of contractions, expansions and equivalences of categories. More precisely Theorem 4.7 states that classical Morita theory can be replaced by those processes.
The results that we have obtained provide the following

Theorem 4.9 *Let $\mathcal{C} \longrightarrow \mathcal{B}$ be a Galois covering of categories with finite group G. The associated smash extension $\mathcal{B}\#k^G \longrightarrow (\mathcal{B}\#k^G)\#kG$ verifies that $\mathcal{B}\#k^G$ is Morita equivalent to $\mathcal{C}$ and $(\mathcal{B}\#k^G)\#kG$ is Morita equivalent to $\mathcal{B}$.*

Finally notice that the proof of a converse for this result is a direct consequence of the discussion we have made in the previous section:

Theorem 4.10 *Let $\mathcal{C} \longrightarrow \mathcal{B}$ be a smash extension with finite group G. The corresponding Galois covering $I_G\mathcal{C} \longrightarrow (I_G\mathcal{C})/G$ verifies that $I_G\mathcal{C}$ is equivalent to $\mathcal{C}$ and that $(I_G\mathcal{C})/G$ is equivalent to $\mathcal{B}$.*

Proof. Indeed an inflated category is isomorphic to the original one; moreover $\mathcal{B} = \mathcal{C}\#kG$ and by Lemma 3.4 this category is isomorphic to $(I_G\mathcal{C})/G$.

5 Appendix: Morita equivalence of categories over a field

We have considered in this paper several procedures that we can apply to a k-category. We briefly recall and relate them with the karoubianisation (also called idempotent completion) and the additivisation (or additive completion), see for instance the appendix of [18].
The *inflation* procedure clearly provides an equivalent category : given a set F_x over each object x of the k-category $\mathcal{C}$, the objects of the inflated category $I_F\mathcal{C}$ are the couples (x, i) with $i \in F_x$. Morphisms from (x, i) to (y, j) remain the morphisms from x to y. Consequently objects with the same first coordinate are isomorphic in the inflated category. Choosing one of them above each object of the original category $\mathcal{C}$ provides a full subcategory of the inflated one, which is isomorphic to $\mathcal{C}$.
The *skeletonisation* procedure consists in choosing precisely one object in each isomorphism set of objects and considering the corresponding full subcategory. Clearly any category is isomorphic to an inflation of its skeleton. Skeletons of the same category are isomorphic, as well as skeletons of equivalent categories. Those remarks show that up to isomorphism of categories, any equivalence of categories is the composition of a skeletonisation and an inflation procedure.
Concerning Morita equivalence, we have used *contraction* and *expansion.* In order to contract we need a partition of the objects of the k-category $\mathcal{C}$ by means of finite sets. The sets of the partition become the objects of the contracted category, and morphisms are provided by matrices of morphisms of $\mathcal{C}$. Conversely, in order to expand we choose a complete system of orthogonal idempotents for each endomorphism algebra at each object of the k-category (the trivial choice is given by just the identity morphism at each object). The set of objects of the expanded category is the disjoint union of all those finite sets of idempotents. Morphisms from e to f are $f_y\mathcal{C}_xe$, assuming e is an idempotent at x and f is an idempotent at y. Composition is given by the composition of $\mathcal{C}$.
We assert that the karoubianisation and the additivisation (see for instance [1, 18]) can be obtained through the previous procedures.

Recall that the karoubianisation of $\mathcal{C}$ replaces each object of $\mathcal{C}$ by *all* the idempotents of its endomorphism algebra, while the morphisms are defined as for the expansion process above.

Consider now the partition of the objects of the karoubianisation of $\mathcal{C}$ given by an idempotent and its complement, namely the sets $\{e, 1-e\}$ for each idempotent at each object of $\mathcal{C}$. The contraction along this partition provides a category equivalent to $\mathcal{C}$, since all the objects over a given object of $\mathcal{C}$ are isomorphic in the contraction of the karoubianisation. Concerning the additivisation, notice first that two constructions are in force which provide equivalent categories as follows.

The larger category is obtained from $\mathcal{C}$ by considering all the finite sequences of objects, and morphisms given through matrix morphisms of $\mathcal{C}$. Observe that two objects (i.e. two finite sequences) which differ by a transposition are isomorphic in this category, using the evident matrix morphism between them. Consequently the objects of the smaller construction are the objects of the previous one modulo permutation, namely the set of objects are finite sets of objects of $\mathcal{C}$ with positive integers coefficients attached. In other words objects are maps from $\mathcal{C}_0$ to $\mathbb{N}$ with finite support. Morphisms are once again matrix morphisms.

The observation above concerning finite sequences differing by a transposition shows that the larger additivisation completion is equivalent to the smaller one.

Finally the smaller additivisation of $\mathcal{C}$ can be expanded: choose the canonical complete orthogonal idempotent system at each object provided by the matrix endomorphism algebra. Of course the expanded category have several evident isomorphic objects which keeps trace of the original objects. A choice provides a full subcategory equivalent to $\mathcal{C}$.

It follows from this discussion that karoubianisation and additivisation provide Morita equivalent categories to a given category, using contraction and expansion processes, up to isomorphism of categories.

We denote $\widehat{\mathcal{C}}$ the completion of $\mathcal{C}$, namely the additivisation of the karoubianisation (or *vice-versa* since those procedures commute). We notice that two categories are Morita equivalent if and only if their completions are Morita equivalent.

Recall that a k-category is called *amenable* if it has finite coproducts and if idempotents split, see for instance [8]. It is well known and easy to prove that the completion $\widehat{\mathcal{C}}$ is amenable.

We provide now a proof of Theorem 4.7. We have shown that the completion of a k-category is obtained (up to equivalence) by expansions and contractions of the original one. Notice that $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ are Morita equivalent amenable categories. We recall now the proof that this implies that the categories $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ are already equivalent (a result known as "Freyd's version of Morita equivalence", see [13, p.18]): consider the full subcategory of representable $\widehat{\mathcal{C}}$-modules, namely modules of the form ${}_{-}\widehat{\mathcal{C}}_x$. This category is isomorphic to the opposite of the original one (this is well known and immediate to prove using

Yoneda's Lemma). Since $\widehat{\mathcal{C}}$ is amenable, representable $\widehat{\mathcal{C}}$-modules are precisely the small (or finitely generated) projective ones, see for instance [8, p. 119]. Finally the small projective modules are easily seen to be preserved by any equivalence of categories; consequently the opposite categories of $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ are equivalent, hence the categories themselves are also equivalent.

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