Algebraic K-Theory and Sums-of-Squares Formulas

Daniel Dugger, Daniel C. Isaksen

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Abstract. We prove a result about the existence of certain 'sums-of-squares' formulas over a field F. A classical theorem uses topological K-theory to show that if such a formula exists over $\mathbb{R}$, then certain powers of 2 must divide certain binomial coefficients. In this paper we use algebraic K-theory to extend the result to all fields not of characteristic 2.

1. Introduction

Let F be a field. A classical problem asks for which values of r, s, and n does there exist an identity of the form

$$\left(x_1^2 + \cdots + x_r^2\right)\left(y_1^2 + \cdots + y_s^2\right) = z_1^2 + \cdots + z_n^2$$

in the polynomial ring $F[x_1, \ldots, x_r, y_1, \ldots, y_s]$, where the z_i's are bilinear expressions in the x's and y's. Such an identity is called a sums-of-squares formula of type $[\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{n}]$. For the history of this problem, see the expository papers [L, Sh].

The main theorem of this paper is the following:

Theorem 1.1. *Assume F is not of characteristic 2. If a sums-of-squares formula of type $[r, s, n]$ exists over F, then $2^{\lfloor \frac{s-1}{2} \rfloor - i + 1}$ divides $\binom{n}{i}$ for $n - r < i \leq \lfloor \frac{s-1}{2} \rfloor$.*

As one specific application, the theorem shows that a formula of type $[13, 13, 16]$ cannot exist over any field of characteristic not equal to 2. Previously this had only been known in characteristic zero. (Note that the case $\mathrm{char}(F) = 2$, which is not covered by the theorem, is trivial: formulas of type $[r, s, 1]$ always exist). In the case $F = \mathbb{R}$, the above theorem was essentially proven by Atiyah [At] as an early application of complex K-theory; the relevance of Atiyah's paper to the sums-of-squares problem was only later pointed out by Yuzvinsky [Y]. The result for characteristic zero fields can be deduced from the case $F = \mathbb{R}$ by an algebraic argument due to K. Y. Lam and T. Y. Lam (see [Sh]). Thus, our contribution is the extension to fields of non-zero characteristic. In this sense the present paper is a natural sequel to [DI], which extended another classical condition about sums-of-squares. We note that sums-of-squares formulas in characteristic p were first seriously investigated in [Ad1, Ad2].

Our proof of Theorem 1.1, given in Section 2, is a modification of Atiyah's original argument. The existence of a sums-of-squares formula allows one to make conclusions about the geometric dimension of certain algebraic vector bundles. A computation of algebraic K-theory (in fact just algebraic K^0), given in Section 3, determines restrictions on what that geometric dimension can be—and this yields the theorem.

Atiyah's result for $F = \mathbb{R}$ is actually slightly better than our Theorem 1.1. The use of topological KO-theory rather than complex K-theory yields an extra power of 2 dividing some of the binomial coefficients. It seems likely that this stronger result holds in non-zero characteristic as well and that it could be proved with Hermitian algebraic K-theory.

1.2. RESTATEMENT OF THE MAIN THEOREM. The condition on binomial coefficients from Theorem 1.1 can be reformulated in a slightly different way. This second formulation surfaces often, and it's what arises naturally in our proof. We record it here for the reader's convenience. Each of the following observations is a consequence of the previous one:

- By repeated use of Pascal's identity $\binom{c}{d} = \binom{c-1}{d-1} + \binom{c-1}{d}$, the number $\binom{n+i-1}{k+i}$ is a $\mathbb{Z}$-linear combination of the numbers $\binom{n}{k+1}, \binom{n}{k+2}, \ldots, \binom{n}{k+i}$. Similarly, $\binom{n}{k+i}$ is a $\mathbb{Z}$-linear combination of $\binom{n}{k+1}, \binom{n+1}{k+2}, \ldots, \binom{n+i-1}{k+i}$.
- An integer b is a common divisor of $\binom{n}{k+1}, \binom{n}{k+2}, \ldots, \binom{n}{k+i}$ if and only if it is a common divisor of $\binom{n}{k+1}, \binom{n+1}{k+2}, \ldots, \binom{n+i-1}{k+i}$.
- The series of statements
$$2^N \,\Big|\, \binom{n}{k+1},\, 2^{N-1} \,\Big|\, \binom{n}{k+2},\, \ldots,\, 2^{N-i+1} \,\Big|\, \binom{n}{k+i}$$
is equivalent to the series of statements
$$2^N \,\Big|\, \binom{n}{k+1},\, 2^{N-1} \,\Big|\, \binom{n+1}{k+2},\, \ldots,\, 2^{N-i+1} \,\Big|\, \binom{n+i-1}{k+i}.$$
- If N is a fixed integer, then 2^{N-i+1} divides $\binom{n}{i}$ for $n-r < i \leq N$ if and only if 2^{N-i+1} divides $\binom{r+i-1}{i}$ for $n-r < i \leq N$.

The last observation shows that Theorem 1.1 is equivalent to the theorem below. This is the form in which we'll actually prove the result.

THEOREM 1.3. *Suppose that F is not of characteristic 2. If a sums-of-squares formula of type $[r,s,n]$ exists over F, then $2^{\lfloor \frac{s-1}{2} \rfloor - i+1}$ divides the binomial coefficient $\binom{r+i-1}{i}$ for $n-r < i \leq \lfloor \frac{s-1}{2} \rfloor$.*

1.4. NOTATION. Throughout this paper $K^0(X)$ denotes the Grothendieck group of locally free coherent sheaves on the scheme X. This group is usually denoted $K_0(X)$ in the literature.

2. The main proof

In this section we fix a field F not of characteristic 2. Let q_k be the quadratic form on $\mathbb{A}^k$ defined by $q_k(x) = \sum_{i=1}^k x_i^2$. A sums-of-squares formula of type $[r,s,n]$ gives a bilinear map $\phi\colon \mathbb{A}^r \times \mathbb{A}^s \to \mathbb{A}^n$ such that $q_r(x)q_s(y) = q_n(\phi(x,y))$. We begin with a simple lemma:

Lemma 2.1. *Let $F \hookrightarrow E$ be a field extension, and let $y \in E^s$ be such that $q_s(y) \neq 0$. Then for $x \in E^r$ one has $\phi(x,y) = 0$ if and only if $x = 0$.*

Proof. Let $\langle -,- \rangle$ denote the inner product on E^k corresponding to the quadratic form q_k. Note that the sums-of-squares identity implies that

$$\langle \phi(x,y), \phi(x',y) \rangle = q_s(y)\langle x, x' \rangle$$

for any x and x' in E^r. If one had $\phi(x,y) = 0$ then the above formula would imply that $q_s(y)\langle x,x'\rangle = 0$ for every x'; but since $q_s(y) \neq 0$, this can only happen when $x = 0$. $\square$

Let V_q be the subvariety of $\mathbb{P}^{s-1}$ defined by $q_s(y) = 0$. Let ξ denote the restriction to V_q of the tautological line bundle $\mathcal{O}(-1)$ of $\mathbb{P}^{s-1}$.

Proposition 2.2. *If a sums-of-squares formula of type $[r,s,n]$ exists over F, then there is an algebraic vector bundle ζ on $\mathbb{P}^{s-1} - V_q$ of rank $n-r$ such that*

$$r[\xi] + [\zeta] = n$$

as elements of the Grothendieck group $K^0(\mathbb{P}^{s-1} - V_q)$ of locally free coherent sheaves on $\mathbb{P}^{s-1} - V_q$.

Proof. We'll write $q = q_s$ in this proof, for simplicity. Let $S = F[y_1,\dots,y_s]$ be the homogeneous coordinate ring of $\mathbb{P}^{s-1}$. By [H, Prop. II.2.5(b)] one has $\mathbb{P}^{s-1} - V_q = \operatorname{Spec} R$, where R is the subring of the localization S_q that consists of degree 0 homogeneous elements. The group $K^0(\mathbb{P}^{s-1} - V_q)$ is naturally isomorphic to the Grothendieck group of finitely-generated projective R-modules. Let P denote the subset of S_q consisting of homogeneous elements of degree -1, regarded as a module over R. Then P is projective and is the module of sections of the vector bundle ξ. To see explicitly that P is projective of rank 1, observe that there is a split-exact sequence $0 \to R^{s-1} \to R^s \xrightarrow{\pi} P \to 0$ where $\pi(p_1,\dots,p_s) = \sum p_i \cdot \frac{y_i}{q}$ and the splitting $\chi\colon P \to R^s$ is $\chi(f) = (y_1 f, y_2 f, \dots, y_s f)$.

From our bilinear map $\phi\colon \mathbb{A}^r \times \mathbb{A}^s \to \mathbb{A}^n$ we get linear forms $\phi(e_i,y) \in S^n$ for $1 \leq i \leq r$. Here e_i denotes the standard basis for F^r, and $y = (y_1,\dots,y_s)$ is the vector of indeterminates from S. If f belongs to P, then each component of $f \cdot \phi(e_i,y)$ is homogeneous of degree 0—hence lies in R.

Define a map $\alpha\colon P^r \to R^n$ by

$$(f_1,\dots,f_r) \mapsto f_1\phi(e_1,y) + f_2\phi(e_2,y) + \cdots + f_r\phi(e_r,y).$$

We can write $\alpha(f_1,\dots,f_r) = \phi((f_1,\dots,f_r),y)$, where the expression on the right means to formally substitute each f_i for x_i in the defining formula for ϕ. If $R \to E$ is any map of rings where E is a field, we claim that $\alpha \otimes_R E$ is an injective map $E^r \to E^n$. To see this, note that $R \to E$ may be extended to a map $u\colon S_q \to E$ (any map $\operatorname{Spec} E \to \mathbb{P}^{s-1} - V_q$ lifts to the affine variety $q \neq 0$, as the projection map from the latter to the former is a Zariski locally trivial bundle). One obtains an isomorphism $P \otimes_R E \to E$ by sending $f \otimes 1$ to $u(f)$. Using this, $\alpha \otimes_R E$ may be readily identified with the map $x \mapsto \phi(x, u(y))$. Now apply Lemma 2.1.

Since R is a domain, we may take E to be the quotient field of R. It follows that α is an inclusion. Let M denote its cokernel. The module M will play the role of ζ in the statement of the proposition, so to conclude the proof we only need show that M is projective. An inclusion of finitely-generated projectives $P_1 \hookrightarrow P_2$ has projective cokernel if and only if $P_1 \otimes_R E \to P_2 \otimes_R E$ is injective for every map $R \to E$ where E is a field (that is to say, the map has constant rank on the fibers)—this follows at once using [E, Ex. 6.2(iii),(v)]. As we have already verified this property for α, we are done. $\square$

Remark 2.3. The above algebraic proof hides some of the geometric intuition behind Proposition 2.2. We outline a different approach more in the spirit of [At].

Let $Gr_r(\mathbb{A}^n)$ denote the Grassmannian variety of r-planes in affine space $\mathbb{A}^n$. We claim that ϕ induces a map $f\colon \mathbb{P}^{s-1} - V_q \to Gr_r(\mathbb{A}^n)$ with the following behavior on points. Let $[y]$ be a point of $\mathbb{P}^{s-1}$ represented by a point y of $\mathbb{A}^s$ such that $q_s(y) \neq 0$. Then the map $\phi_y : x \mapsto \phi(x, y)$ is a linear inclusion by Lemma 2.1. Let $f([y])$ be the r-plane that is the image of ϕ_y. Since ϕ is bilinear, we get that $\phi_{\lambda y} = \lambda \cdot \phi_y$ for any scalar y. This shows that $f([y])$ is well-defined. We leave it as an exercise for the reader to carefully construct f as a map of schemes.

The map f has a special property related to bundles. If η_r denotes the tautological r-plane bundle over the Grassmannian, we claim that ϕ induces a map of bundles $\tilde{f}\colon r\xi \to \eta_r$ covering the map f. To see this, note that the points of $r\xi$ (defined over some field E) correspond to equivalence classes of pairs $(y, a) \in \mathbb{A}^s \times \mathbb{A}^r$ with $q(y) \neq 0$, where $(\lambda y, a) \sim (y, \lambda a)$ for any λ in the field. The pair (y, a) gives us a line $\langle y \rangle \subseteq \mathbb{A}^s$ together with r points $a_1 y, a_2 y, \ldots, a_r y$ on the line.

One defines $\tilde{f}$ so that it sends (y, a) to the element of η_r represented by the vector $\phi(a, y)$ lying on the r-plane spanned by $\phi(e_1, y), \ldots, \phi(e_r, y)$. This respects the equivalence relation, as $\phi(\lambda a, y) = \phi(a, \lambda y)$. So we have described our map $\tilde{f}\colon r\xi \to \eta_r$. We again leave it to the reader to construct f as a map of schemes.

One readily checks that $\tilde{f}$ is a linear isomorphism on geometric fibers, using Lemma 2.1. So $\tilde{f}$ gives an isomorphism $r\xi \cong f^*\eta_r$ of bundles on $\mathbb{P}^{s-1} - V_q$.

The bundle η_r is a subbundle of the rank n trivial bundle, which we denote by n. Consider the quotient n/η_r, and set $\zeta = f^*(n/\eta_r)$. Since $n = [\eta_r] + [n/\eta_r]$ in $K^0(Gr_r(\mathbb{A}^n))$, application of f^* gives $n = [f^*\eta_r] + [\zeta]$ in $K^0(\mathbb{P}^{s-1} - V_q)$. Now recall that $f^*\eta_r \cong r\xi$. This gives the desired formula in Proposition 2.2.

The next task is to compute the Grothendieck group $K^0(\mathbb{P}^{s-1} - V_q)$. This becomes significantly easier if we assume that F contains a square root of -1. The reason for this is made clear in the next section.

Proposition 2.4. *Suppose that F contains a square root of -1 and is not of characteristic 2. Let $c = \lfloor \frac{s-1}{2} \rfloor$. Then $K^0(\mathbb{P}^{s-1} - V_q)$ is isomorphic to $\mathbb{Z}[\nu]/(2^c\nu, \nu^2 = -2\nu)$, where $\nu = [\xi] - 1$ generates the reduced Grothendieck group $\tilde{K}^0(\mathbb{P}^{s-1} - V_q) \cong \mathbb{Z}/2^c$.*

The proof of the above result will be deferred until the next section. Note that $K^0(\mathbb{P}^{s-1} - V_q)$ has the same form as the complex K-theory of real projective space $\mathbb{R}P^{s-1}$ [A, Thm. 7.3]. To complete the analogy, we point out that when $F = \mathbb{C}$ the space $\mathbb{C}P^{s-1} - V_q(\mathbb{C})$ is actually homotopy equivalent to $\mathbb{R}P^{s-1}$ [Lw, 6.3]. We also mention that for the special case where F is contained in $\mathbb{C}$, the above proposition was proved in [GR, Theorem, p. 303].
By accepting the above proposition for the moment, we can finish the

Proof of Theorem 1.3. Recall that one has operations γ^i on $\tilde{K}^0(X)$ for any scheme X [SGA6, Exp. V] (see also [AT] for a very clear explanation). If $\gamma_t = 1 + \gamma^1 t + \gamma^2 t^2 + \cdots$ denotes the generating function, then their basic properties are:

(i) $\gamma_t(a + b) = \gamma_t(a)\gamma_t(b)$.
(ii) For a line bundle L on X one has $\gamma_t([L] - 1) = 1 + t([L] - 1)$.
(iii) If E is an algebraic vector bundle on X of rank k then $\gamma^i([E] - k) = 0$ for $i > k$.

The third property follows from the preceding two via the splitting principle.
If a sums-of-squares identity of type $[r, s, n]$ exists over a field F, then it also exists over any field containing F. So we may assume F contains a square root of -1. If we write $X = \mathbb{P}^{s-1} - V_q$, then by Proposition 2.2 there is a rank $n - r$ bundle ζ on X such that $r[\xi] + [\zeta] = n$ in $K^0(X)$. This may also be written as $r([\xi] - 1) + ([\zeta] - (n - r)) = 0$ in $\tilde{K}^0(X)$. Setting $\nu = [\xi] - 1$ and applying the operation γ_t we have

$$\gamma_t(\nu)^r \cdot \gamma_t([\zeta] - (n - r)) = 1$$

or

$$\gamma_t([\zeta] - (n - r)) = \gamma_t(\nu)^{-r} = (1 + t\nu)^{-r}.$$

The coefficient of t^i on the right-hand-side is $(-1)^i\binom{r+i-1}{i}\nu^i$, which is the same as $-2^{i-1}\binom{r+i-1}{i}\nu$ using the relation $\nu^2 = -2\nu$. Finally, since ζ has rank $n - r$ we know that $\gamma^i([\zeta] - (n - r)) = 0$ for $i > n - r$. In light of Proposition 2.4, this means that 2^c divides $2^{i-1}\binom{r+i-1}{i}$ for $i > n - r$, where $c = \lfloor \frac{s-1}{2} \rfloor$. When $i - 1 < c$, we can rearrange the powers of 2 to conclude that 2^{c-i+1} divides $\binom{r+i-1}{i}$ for $n - r < i \leq c$. □

3. K-theory of deleted quadrics

The rest of the paper deals with the K-theoretic computation stated in Proposition 2.4. This computation is entirely straightforward, and could have been done in the 1970's. We do not know of a reference, however.

Let $Q_{n-1} \hookrightarrow \mathbb{P}^n$ be the split quadric defined by one of the equations

$$a_1b_1+\cdots+a_kb_k = 0 \ (n = 2k-1) \qquad \text{or} \qquad a_1b_1+\cdots+a_kb_k+c^2 = 0 \ (n = 2k).$$

Beware that in general Q_{n-1} is not the same as the variety V_q of the previous section. However, if F contains a square root i of -1 then one can write $x^2 + y^2 = (x + iy)(x - iy)$. After a change of variables the quadric V_q becomes isomorphic to Q_{n-1}. These 'split' quadrics Q_{n-1} are simpler to compute with, and we can analyze the K-theory of these varieties even if F does not contain a square root of -1.

Write $DQ_n = \mathbb{P}^n - Q_{n-1}$, and let ξ be the restriction to DQ_n of the tautological line bundle $\mathcal{O}(-1)$ of $\mathbb{P}^n$. In this section we calculate $K^0(DQ_n)$ over any ground field F not of characteristic 2. Proposition 2.4 is an immediate corollary of this more general result:

Theorem 3.1. *Let F be a field of characteristic not 2. The ring $K^0(DQ_n)$ is isomorphic to $\mathbb{Z}[\nu]/(2^c\nu, \nu^2 = -2\nu)$, where $\nu = [\xi] - 1$ generates the reduced group $\tilde{K}^0(DQ_n) \cong \mathbb{Z}/2^c$ and $c = \lfloor \frac{n}{2} \rfloor$.*

Remark 3.2. We remark again that we are writing $K^0(X)$ for what is usually denoted $K_0(X)$ in the algebraic K-theory literature. We prefer this notation partly because it helps accentuate the relationship with topological K-theory.

3.3. Basic facts about K-theory. Let X be a scheme. As usual $K^0(X)$ denotes the Grothendieck group of locally free coherent sheaves, and $G_0(X)$ (also called $K_0'(X)$) is the Grothendieck group of coherent sheaves [Q, Section 7]. Topologically speaking, $K^0(-)$ is the analog of the usual complex K-theory functor $KU^0(-)$ whereas G_0 is something like a Borel-Moore version of *KU-homology*.

Note that there is an obvious map $\alpha\colon K^0(X) \to G_0(X)$ coming from the inclusion of locally free coherent sheaves into all coherent sheaves. When X is nonsingular, α is an isomorphism whose inverse $\beta\colon G_0(X) \to K^0(X)$ is constructed in the following way [H, Exercise III.6.9]. If $\mathcal{F}$ is a coherent sheaf on X, there exists a resolution

$$0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

in which the $\mathcal{E}_i$'s are locally free and coherent. One defines $\beta(\mathcal{F}) = \sum_i (-1)^i[\mathcal{E}_i]$. This does not depend on the choice of resolution, and now $\alpha\beta$ and $\beta\alpha$ are obviously the identities. This is 'Poincare duality' for K-theory.

Since we will only be dealing with smooth schemes, we are now going to blur the distinction between G_0 and K^0. If $\mathcal{F}$ is a coherent sheaf on X, we will write $[\mathcal{F}]$ for the class that it represents in $K^0(X)$, although we more literally mean $\beta([\mathcal{F}])$. As an easy exercise, check that if $i\colon U \hookrightarrow X$ is an open immersion then the image of $[\mathcal{F}]$ under $i^*\colon K^0(X) \to K^0(U)$ is the same as $[\mathcal{F}\,|_U]$. We will use this fact often.

If $j\colon Z \hookrightarrow X$ is a smooth embedding and $i\colon X - Z \hookrightarrow X$ is the complement, there is a Gysin sequence [Q, Prop. 7.3.2]

$$\cdots \to K^{-1}(X-Z) \longrightarrow K^0(Z) \xrightarrow{j_!} K^0(X) \xrightarrow{i^*} K^0(X-Z) \longrightarrow 0.$$

(Here $K^{-1}(X-Z)$ denotes the group usually called $K_1(X-Z)$, and i^* is surjective because X is regular). The map $j_!$ is known as the Gysin map. If $\mathcal{F}$ is a coherent sheaf, then $j_!([\mathcal{F}])$ equals the class of its pushforward $j_*(\mathcal{F})$ (also known as extension by zero). Note that the pushforward of coherent sheaves is exact for closed immersions.

3.4. Basic facts about $\mathbb{P}^n$. If Z is a degree d hypersurface in $\mathbb{P}^n$, then the structure sheaf $\mathcal{O}_Z$ can be pushed forward to $\mathbb{P}^n$ along the inclusion $Z \to \mathbb{P}^n$; we will still write this pushforward as $\mathcal{O}_Z$. It has a very simple resolution of the form $0 \to \mathcal{O}(-d) \to \mathcal{O} \to \mathcal{O}_Z \to 0$, where $\mathcal{O}$ is the trivial rank 1 bundle on $\mathbb{P}^n$ and $\mathcal{O}(-d)$ is the d-fold tensor power of the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^n$. So $[\mathcal{O}_Z]$ equals $[\mathcal{O}] - [\mathcal{O}(-d)]$ in $K^0(\mathbb{P}^n)$. From now on we'll write $[\mathcal{O}] = 1$.

Now suppose that $Z \hookrightarrow \mathbb{P}^n$ is a complete intersection, defined by the regular sequence of homogeneous equations $f_1, \dots, f_r \in k[x_0, \dots, x_n]$. Let f_i have degree d_i. The module $k[x_0, \dots, x_n]/(f_1, \dots, f_r)$ is resolved by the Koszul complex, which gives a locally free resolution of $\mathcal{O}_Z$. It follows that

$$[\mathcal{O}_Z] = \big(1 - [\mathcal{O}(-d_1)]\big)\big(1 - [\mathcal{O}(-d_2)]\big)\cdots\big(1 - [\mathcal{O}(-d_r)]\big) \tag{3.4}$$

in $K^0(\mathbb{P}^n)$. In particular, note that for a linear subspace $\mathbb{P}^i \hookrightarrow \mathbb{P}^n$ one has

$$[\mathcal{O}_{\mathbb{P}^i}] = \Big(1 - [\mathcal{O}(-1)]\Big)^{n-i}$$

because $\mathbb{P}^i$ is defined by $n - i$ linear equations.

One can compute that $K^0(\mathbb{P}^n) \cong \mathbb{Z}^{n+1}$, with generators $[\mathcal{O}_{\mathbb{P}^0}], [\mathcal{O}_{\mathbb{P}^1}], \dots, [\mathcal{O}_{\mathbb{P}^n}]$ (see [Q, Th. 8.2.1], as one source). If $t = 1 - [\mathcal{O}(-1)]$, then the previous paragraph tells us that $K^0(\mathbb{P}^n) \cong \mathbb{Z}[t]/(t^n)$ as rings. Here t^k corresponds to $[\mathcal{O}_{\mathbb{P}^{n-k}}]$.

3.5. Computations. Let $n = 2k$. Recall that Q_{2k-1} denotes the quadric in $\mathbb{P}^{2k}$ defined by $a_1b_1 + \cdots + a_kb_k + c^2 = 0$. The Chow ring $\mathrm{CH}^*(Q_{2k-1})$ consists of a copy of $\mathbb{Z}$ in every dimension (see [DI, Appendix A] or [HP, XIII.4–5], for example). The generators in dimensions k through $2k-1$ are represented by subvarieties of Q_{2k-1} which correspond to linear subvarieties $\mathbb{P}^{k-1}, \mathbb{P}^{k-2}, \dots, \mathbb{P}^0$ under the embedding $Q_{2k-1} \hookrightarrow \mathbb{P}^{2k}$. In terms of equations, $\mathbb{P}^{k-i}$ is defined by $c = b_1 = \cdots = b_k = 0$ together with $0 = a_k = a_{k-1} = \cdots = a_{k-i+2}$. The generators of the Chow ring in degrees 0 through $k-1$ are represented by subvarieties $Z_i \hookrightarrow \mathbb{P}^{2k}$ ($k \le i \le 2k-1$), where Z_i is defined by the equations

$$0 = b_1 = b_2 = \cdots = b_{2k-1-i}, \qquad a_1b_1 + \cdots + a_kb_k + c^2 = 0.$$

Note that $Z_{2k-1} = Q_{2k-1}$.

The following result is proven in [R, pp. 128-129] (see especially the first paragraph on page 129):

PROPOSITION 3.6. *The group* $K^0(Q_{2k-1})$ *is isomorphic to* $\mathbb{Z}^{2k}$*, with generators* $[\mathcal{O}_{\mathbb{P}^0}], \ldots, [\mathcal{O}_{\mathbb{P}^{k-1}}]$ *and* $[\mathcal{O}_{Z_k}]$*, …,* $[\mathcal{O}_{Z_{2k-1}}]$.

It is worth noting that to prove Theorem 3.1 we don't actually need to know that $K^0(Q_{2k-1})$ is free—all we need is the list of generators.

Proof of Theorem 3.1 when n is even. Set $n = 2k$. To calculate $K^0(DQ_{2k})$ we must analyze the localization sequence

$$\cdots \to K^0(Q_{2k-1}) \xrightarrow{j_!} K^0(\mathbb{P}^{2k}) \to K^0(DQ_{2k}) \to 0.$$

The image of $j_! : K^0(Q_{2k-1}) \to K^0(\mathbb{P}^{2k})$ is precisely the subgroup generated by $[\mathcal{O}_{\mathbb{P}^0}], \ldots, [\mathcal{O}_{\mathbb{P}^{k-1}}]$ and $[\mathcal{O}_{Z_k}]$, …, $[\mathcal{O}_{Z_{2k-1}}]$. Since $\mathbb{P}^i$ is a complete intersection defined by $2k - i$ linear equations, formula (3.4) tells us that $[\mathcal{O}_{\mathbb{P}^i}] = t^{2k-i}$ for $0 \le i \le k-1$.
Now, Z_{2k-1} is a degree 2 hypersurface in $\mathbb{P}^{2k}$, and so $[\mathcal{O}_{Z_{2k-1}}]$ equals $1-[\mathcal{O}(-2)]$. Note that

$$1 - [\mathcal{O}(-2)] = 2(1 - [\mathcal{O}(-1)]) - (1 - [\mathcal{O}(-1)])^2 = 2t - t^2.$$

In a similar way one notes that Z_i is a complete intersection defined by $2k-1-i$ linear equations and one degree 2 equation, so formula (3.4) tells us that

$$[\mathcal{O}_{Z_i}] = (1 - [\mathcal{O}(-1)])^{2k-1-i} \cdot (1 - [\mathcal{O}(-2)]) = t^{2k-1-i}(2t - t^2).$$

The calculations in the previous two paragraphs imply that the kernel of the map $K^0(\mathbb{P}^{2k}) \to K^0(DQ_{2k})$ is the ideal generated by $2t-t^2$ and t^{k+1}. This ideal is equal to the ideal generated by $2t - t^2$ and $2^k t$, so $K^0(DQ_{2k})$ is isomorphic to $\mathbb{Z}[t]/(2^k t, 2t - t^2)$. If we substitute $\nu = [\xi] - 1 = -t$, we find $\nu^2 = -2\nu$.
To find $\tilde{K}^0(DQ_{2k})$, we just have to take the additive quotient of $K^0(DQ_{2k})$ by the subgroup generated by 1. This quotient is isomorphic to $\mathbb{Z}/2^k$ and is generated by ν. □

This completes the proof of Theorem 3.1 in the case where n is even. The computation when n is odd is very similar:

Proof of Theorem 3.1 when n is odd. In this case Q_{n-1} is defined by the equation $a_1 b_1 + \cdots + a_k b_k = 0$ with $k = \frac{n+1}{2}$. The Chow ring $\mathrm{CH}^*(Q_{n-1})$ consists of $\mathbb{Z}$ in every dimension except for $k-1$, which is $\mathbb{Z} \oplus \mathbb{Z}$. The generators are the Z_i's ($k-1 \le i \le 2k-2$) defined analogously to before, together with the linear subvarieties $\mathbb{P}^0, \mathbb{P}^1, \ldots, \mathbb{P}^{k-1}$. By [R, pp. 128–129], the group $K^0(Q_{n-1})$ is again free of rank $2k$ on the generators $[\mathcal{O}_{Z_i}]$ and $[\mathcal{O}_{\mathbb{P}^i}]$. One finds that $K^0(DQ_n)$ is isomorphic to $\mathbb{Z}[t]/(2t - t^2, t^k) = \mathbb{Z}[t]/(2t - t^2, 2^{k-1}t)$. Everything else is as before. □

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Daniel Dugger
Department of Mathematics
University of Oregon
Eugene, OR 97403, USA
ddugger@math.uoregon.edu

Daniel C. Isaksen
Department of Mathematics
Wayne State University
Detroit, MI 48202, USA
isaksen@math.wayne.edu