Arakelov Invariants of Riemann Surfaces

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Abstract. We derive closed formulas for the Arakelov-Green function and the Faltings delta-invariant of a compact Riemann surface.

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1. Introduction

The main goal of this paper is to give closed formulas for the Arakelov-Green function G and the Faltings delta-invariant δ of a compact Riemann surface. Both G and δ are of fundamental importance in the Arakelov theory of arithmetic surfaces [2] [8] and it is a central problem in this theory to relate these difficult invariants to more accessible ones. For example, in [8] Faltings gives formulas which relate G and δ for elliptic curves directly to theta functions and to the discriminant modular form. Formulas of a similar explicit nature were derived by Bost in [3] for Riemann surfaces of genus 2. As to the case of general genus, less specific but still quite explicit formulas are known due to Bost [3] (for the Arakelov-Green function) and to Bost and Gillet-Soulé [4] [10] (for the delta-invariant). We recall these results in Sections 2 and 4 below.

In the present paper we express G and δ in terms of two new invariants S and T. Both S and T are initially defined as the norms of certain isomorphisms between line bundles, but eventually we find that they admit a very explicit description in terms of theta functions. They are intimately related to the divisor $\mathcal{W}$ of Weierstrass points. Of these new invariants, the T is certainly the easiest one. We are able to calculate it for hyperelliptic Riemann surfaces [13], where it is essentially a power of the Petersson norm of the discriminant modular form. The invariant S is less easy and involves a certain integral over the Riemann surface. We believe that the approach using S and T is very suitable for obtaining numerical results. An example at the end of this paper, where we compute δ and a special value of G for a certain hyperelliptic Riemann surface of genus 3, is meant to illustrate this belief.

We start our discussion by recalling the definitions of G and δ. From now on until the end of section 4, we fix a compact Riemann surface X. Let g be its genus, which we assume to be positive. The space of holomorphic differentials $H^0(X,\Omega_X^1)$ carries a natural hermitian inner product $(\omega,\eta) \mapsto \frac{i}{2}\int_X \omega \wedge \overline{\eta}$. We fix this inner product once and for all. Let $\{\omega_1,\ldots,\omega_g\}$ be an orthonormal basis with respect to this inner product. We have then a fundamental (1,1)-form μ on X given by $\mu = \frac{i}{2g}\sum_{k=1}^g \omega_k \wedge \overline{\omega}_k$. It is verified immediately that the form μ does not depend on the choice of orthonormal basis, and hence is canonical. Using this form, one defines the Arakelov-Green function G on $X \times X$. This function gives the local intersections "at infinity" of two divisors in Arakelov theory [2].

Theorem 1.1. *(Arakelov) There exists a unique function $G : X \times X \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $P \in X$:*

(i) *the function $\log G(P,Q)$ is C^∞ for $Q \neq P$;*
(ii) *we can write $\log G(P,Q) = \log|z_P(Q)| + f(Q)$ locally about P, where z_P is a local coordinate about P and where f is C^∞ about P;*
(iii) *we have $\partial_Q\overline{\partial}_Q \log G(P,Q)^2 = 2\pi i\mu(Q)$ for $Q \neq P$;*
(iv) *we have $\int_X \log G(P,Q)\mu(Q) = 0$.*

Theorem 1.1 is proved in [2]. We call the function G the Arakelov-Green function of X. We note that by an application of Stokes' theorem one can prove the symmetry relation $G(P,Q) = G(Q,P)$ for any $P,Q \in X$.

Importantly, the Arakelov-Green function gives rise to certain canonical metrics on line bundles on X. First, consider line bundles of the form $O_X(P)$ with P a point on X. Let s be the canonical generating section of $O_X(P)$. We then define a smooth hermitian metric $\|\cdot\|_{O_X(P)}$ on $O_X(P)$ by putting $\|s\|_{O_X(P)}(Q) = G(P,Q)$ for any $Q \in X$. By property (iii) of the Arakelov-Green function, the curvature form of $O_X(P)$ is equal to μ. Second, it is clear that the function G can be used to put a hermitian metric on the line bundle $O_{X\times X}(\Delta_X)$, where Δ_X is the diagonal on $X \times X$, by putting $\|s\|(P,Q) = G(P,Q)$ for the canonical generating section s of $O_{X\times X}(\Delta_X)$. Restricting to the diagonal, we have a canonical adjunction isomorphism $O_{X\times X}(-\Delta_X)|_{\Delta_X} \xrightarrow{\sim} \Omega_X^1$. We define a hermitian metric $\|\cdot\|_{\mathrm{Ar}}$ on Ω_X^1 by insisting that this adjunction isomorphism be an isometry. It is proved in [2] that this gives a smooth hermitian metric on Ω_X^1, and that its curvature form is a multiple of μ. For the rest of the paper we shall take these metrics on $O_X(P)$ and Ω_X^1 (as well as on tensor product combinations of them) for granted and refer to them as Arakelov metrics.

Next we introduce the Faltings delta-invariant. Let $\mathcal{H}_g$ be the generalised Siegel upper half plane of complex symmetric $g \times g$-matrices with positive definite imaginary part. Let $\tau \in \mathcal{H}_g$ be a period matrix associated to a symplectic basis of $H_1(X,\mathbb{Z})$ and consider the analytic jacobian $\mathrm{Jac}(X) = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ associated to τ. We fix τ for the rest of our discussion. On $\mathbb{C}^g$ one has a theta function $\vartheta(z;\tau) = \sum_{n\in\mathbb{Z}^g} \exp(\pi i {}^t n\tau n + 2\pi i {}^t n z)$, giving rise to an effective divisor Θ_0 and a line bundle $O(\Theta_0)$ on $\mathrm{Jac}(X)$. Now consider on the other hand the set $\mathrm{Pic}_{g-1}(X)$ of divisor classes of degree $g-1$ on X. It comes with a special subset Θ given by the classes of effective divisors. A fundamental theorem of Abel-Jacobi-Riemann says that there is a canonical bijection $\mathrm{Pic}_{g-1}(X) \xrightarrow{\sim} \mathrm{Jac}(X)$ mapping Θ onto Θ_0. As a result, we can equip $\mathrm{Pic}_{g-1}(X)$ with the structure of a compact complex manifold, together with a divisor Θ and a line bundle $O(\Theta)$. We fix this structure for the rest of the discussion.

The function ϑ is not well-defined on $\mathrm{Pic}_{g-1}(X)$ or $\mathrm{Jac}(X)$. We can remedy this by putting $\|\vartheta\|(z;\tau) = (\det \mathrm{Im}\,\tau)^{1/4} \exp(-\pi {}^t y (\mathrm{Im}\,\tau)^{-1} y) |\vartheta(z;\tau)|$, with $y = \mathrm{Im}\, z$. One can check that $\|\vartheta\|$ descends to a function on $\mathrm{Jac}(X)$. By our identification $\mathrm{Pic}_{g-1}(X) \xrightarrow{\sim} \mathrm{Jac}(X)$ we obtain $\|\vartheta\|$ as a function on $\mathrm{Pic}_{g-1}(X)$. It can be checked that this function is independent of the choice of τ.

The delta-invariant is the constant appearing in the following theorem, due to Faltings (cf. [8], p. 402).

Theorem 1.2. *(Faltings) There is a constant $\delta = \delta(X)$ depending only on X such that the following holds. Let $\{\omega_1, \ldots, \omega_g\}$ be an orthonormal basis of $H^0(X, \Omega^1_X)$. Let $P_1, \ldots, P_g, Q$ be points on X with $P_1, \ldots, P_g$ pairwise distinct. Then the formula*

$$\|\vartheta\|(P_1 + \cdots + P_g - Q) = \exp(-\delta(X)/8) \cdot \frac{\|\det \omega_k(P_l)\|_{\mathrm{Ar}}}{\prod_{k<l} G(P_k, P_l)} \cdot \prod_{k=1}^{g} G(P_k, Q)$$

holds.

The definition of the delta-invariant may seem quite complicated, yet it plays an important role in Arakelov intersection theory and in the function theory of the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g. In fact, as has become clear from certain asymptotic results [14] [20], the function $\exp(-\delta(X))$ can be interpreted as a natural "distance" function on $\mathcal{M}_g$ measuring the distance to the Deligne-Mumford boundary. As to Arakelov theory, the delta-invariant plays the role of an archimedean contribution in the Noether formula for arithmetic surfaces [8] [18]. The idea that $\delta(X)$ gives a distance to the boundary is supported by this formula.

The plan of this paper is as follows. In Section 2 we state a proposition and observe that it leads quickly to a formula for G. In Section 3 we prove this proposition. In Section 4 we derive our closed formula for δ. Some applications of our results to Arakelov intersection theory are given in Section 5. We conclude with a numerical example in Section 6.

2. The Arakelov-Green function

As was mentioned in the Introduction, the Weierstrass points of X play an important role in our approach to G and δ. The idea of considering Weierstrass points in the context of Arakelov theory is not new, cf. [6] and [14] for example. We start by recalling how we obtain the divisor of Weierstrass points using a Wronskian differential on X. Let $\{\psi_1, \ldots, \psi_g\}$ be an (arbitrary) basis of

$H^0(X, \Omega^1_X)$. Let P be a point on X and let z be a local coordinate about P. Write $\psi_k = f_k(z)dz$ for $k = 1, \ldots, g$. We have a holomorphic function

$$W_z(\psi) = \det \left(\frac{1}{(l-1)!} \frac{d^{l-1} f_k}{dz^{l-1}} \right)_{1 \leq k,l \leq g}$$

locally about P from which we build the $g(g+1)/2$-fold holomorphic differential

$$\tilde{\psi} = W_z(\psi)(dz)^{\otimes g(g+1)/2}.$$

We call $\tilde{\psi}$ the Wronskian differential about P and it is readily checked that $\tilde{\psi}$ is independent of the choice of the local coordinate. In fact, and this is less trivial, the differential $\tilde{\psi}$ extends over X to give a non-zero global section of the line bundle $\Omega_X^{\otimes g(g+1)/2}$. A change of basis changes the Wronskian differential by a non-zero scalar factor and hence the divisor of a Wronskian differential $\tilde{\psi}$ on X is unique. We denote this divisor by $\mathcal{W}$, the divisor of Weierstrass points. This divisor is effective and we have $\deg \mathcal{W} = g^3 - g$. Writing $\mathcal{W} = \sum_{P \in X} w(P) \cdot P$ we call the integer $w(P)$ the *weight* at P. This weight can be calculated using gap sequences, but we shall not need this.

Now fix for the moment a $Q \in X$. We consider the map $\phi_Q : X \to \mathrm{Pic}_{g-1}(X)$ given by sending $P \mapsto [gP - Q]$. We put a smooth hermitian metric on $O(\Theta)$ by setting $\|s\| = \|\vartheta\|$ where s is the canonical generating section of $O(\Theta)$. We shall refer to this metric as the Arakelov metric on $O(\Theta)$. It can be verified by a short calculation using Riemann's bilinear relations that $\phi_Q^* O(\Theta)$ is a line bundle on X of degree g^3 and with curvature form a multiple of μ. In fact we can say more. It is a classical result (cf. for example [9], p. 31) that $\phi_Q^*(\Theta) = \mathcal{W} + g \cdot Q$. Hence we obtain the first statement of the next proposition.

Proposition 2.1. *We have a canonical isomorphism*

$$\sigma_Q : \phi_Q^*(O(\Theta)) \xrightarrow{\sim} O_X(\mathcal{W} + g \cdot Q)$$

of line bundles on X. When both sides are equipped with their Arakelov metrics, the isomorphism σ_Q has constant norm on X. This norm is independent of the choice of Q.

The proposition will be proven in the next section. Meanwhile, we observe that it leads quite quickly to a closed formula for G.

Definition 2.2. We define $S(X)$ to be the norm of σ_Q for any $Q \in X$.

In more concrete terms we have the following formula.

Corollary 2.3. *For any P, Q on X we have*

$$G(P,Q)^g \cdot \prod_{W \in \mathcal{W}} G(P,W) = S(X) \cdot \|\vartheta\|(gP - Q)\,,$$

where the Weierstrass points are counted with their weights.

It follows from this corollary that the function $\prod_{W \in \mathcal{W}} \|\vartheta\|(gP - W)$ does not vanish if P is not a Weierstrass point. Hence the following formula makes sense.

Theorem 2.4. *For any P, Q on X with P not a Weierstrass point we have*

$$G(P,Q)^g = S(X)^{1/g^2} \cdot \frac{\|\vartheta\|(gP-Q)}{\prod_{W \in \mathcal{W}} \|\vartheta\|(gP-W)^{1/g^3}} \,.$$

Here the Weierstrass points are counted with their weights.

Proof. This follows by applying the formula from Corollary 2.3 two times. First, take the (weighted) product over Q running through $\mathcal{W}$. This gives

$$\prod_{W \in \mathcal{W}} G(P,W)^{g^3} = S(X)^{g^3-g} \cdot \prod_{W \in \mathcal{W}} \|\vartheta\|(gP-W) \,.$$

Plug this in again in the formula from Corollary 2.3. This gives

$$G(P,Q)^g \cdot S(X)^{\frac{g^3-g}{g^3}} \cdot \prod_{W \in \mathcal{W}} \|\vartheta\|(gP-W)^{1/g^3} = S(X) \cdot \|\vartheta\|(gP-Q) \,,$$

and a little rewriting gives the result. □

Taking logarithms in Corollary 2.3 and then integrating against μ with respect to the variable P immediately gives the following explicit formula for $S(X)$.

Theorem 2.5. *For any fixed Q, the function $\log \|\vartheta\|(gP-Q)$ is integrable against μ, and the formula*

$$\log S(X) = -\int_X \log \|\vartheta\|(gP-Q) \cdot \mu(P)$$

holds.

The invariant $S(X)$ is readily calculated in the case $g = 1$.

Example 2.6. Suppose that $X = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ with $\operatorname{Im}\tau > 0$ is an elliptic curve. The form μ is given by $\mu = \frac{i}{2}(dz \wedge d\overline{z})/\operatorname{Im}\tau$ and we have

$$\log S(X) = -\int_X \log \|\vartheta\| \cdot \mu \,.$$

A calculation (see [15], p. 45 or for a different approach [14], p. 250) yields

$$\log S(X) = -\log((\operatorname{Im}\tau)^{1/4}|\eta(\tau)|) \,,$$

where $\eta(\tau)$ is the usual Dedekind eta-function $\eta(q) = q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$ in $q = \exp(2\pi i \tau)$.

In the case that $P = W$ is a Weierstrass point of X, the formula in Theorem 2.4 is still correct, provided that on the right hand side we take a limit for P approaching W. That this limit exists and that it indeed gives $G(W,Q)^g$ follows easily from the proof of Theorem 2.4.

We finish this section by discussing very shortly several other approaches to G that we know from the literature. First of all, it is quite natural to develop G in terms of the eigenvalues and eigenfunctions of a Laplacian associated to μ on X. This is the approach taken in [8], see especially Section 3 of that paper. Second, it is possible to express G in terms of abelian differentials of the second and third kind, see for example [15], Chapter II. Third, and this is perhaps most close to our approach since it also involves theta functions quite explicitly, there is an integral formula for G derived by Bost, cf. [3], Proposition 1. This interesting result reads as follows: let ν be the curvature form of $O(\Theta)$ on $\mathrm{Pic}_{g-1}(X)$. Then there is an invariant $A(X)$ of X such that for every P, Q on X the formula

$$\log G(P,Q) = \frac{1}{g!}\int_{\Theta+P-Q} \log \|\vartheta\| \cdot \nu^{g-1} + A(X)$$

holds. It would be interesting to have results that relate $A(X)$ and $S(X)$ to each other in a natural, conceptual way.

3. Proof of Proposition 2.1

Proposition 2.1 follows directly from Lemmas 3.1 and 3.2 below. We will be dealing, among other things, with the line bundle $\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X$ on X. We equip this line bundle with the constant metric deriving from the hermitian inner product $(\omega, \eta) \mapsto \frac{i}{2}\int_X \omega \wedge \overline{\eta}$ on $H^0(X, \Omega_X^1)$ that we introduced in Section 1. From now on, this metric will be taken for granted and we shall also refer to it as an Arakelov metric.

Lemma 3.1. *There is a canonical isomorphism of line bundles*

$$\rho : \Omega_X^{\otimes g(g+1)/2} \otimes \left(\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X\right)^\vee \xrightarrow{\sim} O_X(\mathcal{W})$$

on X. When both sides are equipped with their Arakelov metrics, the norm of this isomorphism is constant on X.

Proof. The Wronskian differential $\tilde{\psi}$ formed on an arbitrary basis $\{\psi_1, \ldots, \psi_g\}$ of $H^0(X, \Omega_X^1)$ leads to a morphism of line bundles

$$\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X \longrightarrow \Omega_X^{g(g+1)/2}$$

by setting

$$\xi_1 \wedge \ldots \wedge \xi_g \mapsto \frac{\xi_1 \wedge \ldots \wedge \xi_g}{\psi_1 \wedge \ldots \wedge \psi_g} \cdot \tilde{\psi}\,.$$

This gives a canonical section in $\Omega_X^{\otimes g(g+1)/2} \otimes \left(\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X\right)^\vee$ whose divisor is $\mathcal{W}$ and thus we obtain the required isomorphism. The norm is constant on X because both sides have the same curvature form, and the divisors of their canonical sections are equal. □

Lemma 3.2. *Let Q be an arbitrary point of X. There is a canonical isomorphism of line bundles*

$$\phi_Q^*(O(\Theta)) \xrightarrow{\sim} \left(\Omega_X^{\otimes g(g+1)/2} \otimes \left(\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X\right)^\vee\right) \otimes O_X(g \cdot Q)$$

on X. When both sides are equipped with their Arakelov metrics, the norm of this isomorphism is constant on X and equal to $\exp(\delta(X)/8)$.

Proof. We are done once we prove that

$$\exp(\delta(X)/8) \cdot \|\vartheta\|(gP - Q) = \|\tilde{\omega}\|_{\mathrm{Ar}}(P) \cdot G(P,Q)^g$$

for arbitrary P, Q on X, where $\tilde{\omega}$ is the Wronskian differential formed out of an orthonormal basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(X, \Omega_X^1)$ and where the norm $\|\tilde{\omega}\|_{\mathrm{Ar}}$ of $\tilde{\omega}$ is taken in the line bundle $\Omega_X^{\otimes g(g+1)/2}$ equipped with its Arakelov metric. The required formula follows from the formula in Theorem 1.2, by a computation which is also performed in [14], p. 233 and which runs as follows. Let P be a point on X, and choose a local coordinate z about P. By definition of the Arakelov metric on Ω_X^1 we have that $\lim_{Q \to P} |z(Q) - z(P)|/G(Q,P) = \|dz\|_{\mathrm{Ar}}(P)$. Letting $P_1, \ldots, P_g$ approach P in Theorem 1.2 we obtain

$$\begin{aligned}
\lim_{P_l \to P} \frac{\| \det \omega_k(P_l) \|_{\mathrm{Ar}}}{\prod_{k<l} G(P_k, P_l)} &= \lim_{P_l \to P} \left\{ \frac{\| \det \omega_k(P_l)\|_{\mathrm{Ar}}}{\prod_{k<l} |z(P_k) - z(P_l)|} \cdot \frac{\prod_{k<l} |z(P_k) - z(P_l)|}{\prod_{k<l} G(P_k, P_l)} \right\} \\
&= \left\{ \lim_{P_l \to P} \frac{| \det \omega_k(P_l)|}{\prod_{k<l} |z(P_k) - z(P_l)|} \right\} \cdot \|dz\|_{\mathrm{Ar}}^{g+g(g-1)/2}(P) \\
&= |W_z(\omega)(P)| \cdot \|dz\|_{\mathrm{Ar}}^{g(g+1)/2}(P) \\
&= \|\tilde{\omega}\|_{\mathrm{Ar}}(P) \, .
\end{aligned}$$

The required formula is therefore just a limiting case of Theorem 1.2 where all P_k approach P. $\square$

4. The Faltings delta-invariant

In the present section we will express the Faltings delta-invariant $\delta(X)$ in terms of $S(X)$ and a second invariant $T(X)$. The significance of our formula is that the constant $T(X)$ is in a sense "classical" and easy to calculate numerically.
To start our discussion, we observe that it follows from the previous sections that (multiples of) the divisor $\mathcal{W}$ of Weierstrass points appear as a divisor of a section of a line bundle in various different situations. We will take advantage of this fact and take combinations until we obtain an isomorphism of line bundles whose norm is easy to measure.
First of all, recall (this is Proposition 2.1) that we have for any Q on X a canonical isomorphism

$$\sigma_Q : \phi_Q^*(O(\Theta)) \xrightarrow{\sim} O_X(\mathcal{W} + g \cdot Q) \, .$$

Taking the (weighted) tensor product over the Weierstrass points of X, we obtain a canonical isomorphism

$$\sigma_{\mathcal{W}} : \bigotimes_{W \in \mathcal{W}} \phi_W^*(O(\Theta)) \xrightarrow{\sim} O_X(g^3 \cdot \mathcal{W}) \, .$$

Second, recall that by Lemma 3.1 we have a canonical isomorphism

$$\rho : \Omega_X^{\otimes g(g+1)/2} \otimes \left(\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X \right)^\vee \xrightarrow{\sim} O_X(\mathcal{W}) \, .$$

Thirdly, taking a closer look at Lemma 3.2 we see that the proof in fact implies that we have on $X \times X$ a canonical isomorphism

$$\sigma : \Phi^*(O(\Theta)) \xrightarrow{\sim} O_{X\times X}(\mathcal{W} \cdot X + g \cdot \Delta_X)$$

where $\Phi : X \times X \to \mathrm{Pic}_{g-1}(X)$ is the map sending $(P,Q) \mapsto [gP - Q]$ and where again Δ_X is the diagonal on $X \times X$. Restricting σ to the diagonal, and using the adjunction isomorphism, we obtain a canonical isomorphism

$$\sigma|_\Delta : \Phi^*(O(\Theta))|_{\Delta_X} \otimes \Omega_X^{\otimes g} \xrightarrow{\sim} O_X(\mathcal{W}) \, .$$

Taking suitable combinations of $\sigma_\mathcal{W}, \rho$ and $\sigma|_\Delta$ we obtain

Proposition 4.1. *There is a canonical isomorphism of (fractional) line bundles*

$$\tau : \left(\Phi^*(O(\Theta))|_{\Delta_X} \otimes \Omega_X^{\otimes g}\right)^{\otimes(g+1)} \xrightarrow{\sim}$$
$$\left(\bigotimes_W \phi_W^*(O(\Theta))\right)^{\otimes(g-1)/g^3} \otimes \left(\Omega_X^{\otimes g(g+1)/2} \otimes \left(\wedge^g H^0(X,\Omega_X^1) \otimes_\mathbb{C} O_X\right)^\vee\right)^{\otimes 2}$$

on X.

Our results thus far imply that τ has a constant norm for the Arakelov metrics on both sides.

Definition 4.2. We define $T(X)$ to be the norm of τ on X.

The constant $T(X)$ admits the following concrete description using a local coordinate.

Proposition 4.3. *Let $P \in X$ not a Weierstrass point and let z be a local coordinate about P. Define $\|F_z\|(P)$ as*

$$\|F_z\|(P) = \lim_{Q\to P} \frac{\|\vartheta\|(gP-Q)}{|z(P)-z(Q)|^g} \, .$$

This limit exists and is non-zero. Further, let $\{\omega_1, \ldots, \omega_g\}$ be an orthonormal basis of $H^0(X,\Omega_X^1)$. Then the formula

$$T(X) = \|F_z\|(P)^{-(g+1)} \cdot \prod_{W\in\mathcal{W}} \|\vartheta\|(gP-W)^{(g-1)/g^3} \cdot |W_z(\omega)(P)|^2$$

holds, where $W_z(\omega)$ is the determinant of the Wronskian of $\{\omega_1, \ldots, \omega_g\}$ with respect to z.

In particular, the evaluation of $T(X)$ for a given X only involves the evaluation of certain classical functions at an arbitrary (non-Weierstrass) point of X.

Proof. Let F be the canonical section of $\Phi^*(O(\Theta))|_{\Delta_X} \otimes \Omega_X^{\otimes g}$ coming from the canonical section in $\Phi^*(O(\Theta))$ and the canonical generating section of $O_{X\times X}(\Delta_X)$ using the adjunction isomorphism. For its norm we have $\|F\| = \|F_z\| \cdot \|dz\|_{\mathrm{Ar}}^g$ in the local coordinate z. We see from the isomorphism $\sigma|_\Delta$

that $\|F\|(P)$ does not vanish if P is not a Weierstrass point. Next, the canonical section of $\bigotimes_{W\in\mathcal{W}}\phi_W^*O(\Theta)$ has norm $\prod_{W\in\mathcal{W}}\|\vartheta\|(gP-W)$ at P. Finally, the canonical section of $\Omega_X^{\otimes g(g+1)/2}\otimes\left(\wedge^g H^0(X,\Omega_X^1)\otimes_{\mathbb{C}} O_X\right)^{\vee}$ has norm $\|\tilde{\omega}\|_{\mathrm{Ar}}=|W_z(\omega)|\cdot\|dz\|_{\mathrm{Ar}}^{g(g+1)/2}$. The proposition follows by taking the appropriate combinations of these norms. □

Considering the norms of the three isomorphisms $\sigma_{\mathcal{W}},\rho$ and $\sigma|_\Delta$ one sees that they are directly expressible in terms of $\exp(\delta)$ and $S(X)$. Hence the same holds for the norm $T(X)$ of τ. Viewing things a little differently, we obtain a formula for $\exp(\delta)$ in terms of $S(X)$ and $T(X)$.

Theorem 4.4. *The formula*

$$\exp(\delta(X)/4)=S(X)^{-(g-1)/g^3}\cdot T(X)$$

holds.

Proof. The norm of $\sigma_{\mathcal{W}}$ is equal to $S(X)^{g^3-g}$. The norm of ρ is equal to $S(X)\exp(-\delta(X)/8)$ as becomes clear by decomposing again the isomorphism from Proposition 2.1, which has norm $S(X)$, into the isomorphisms from Lemmas 3.1 and 3.2. Lastly, the norm of $\sigma|_\Delta$ is equal to $S(X)$ since σ has this norm and the restriction to the diagonal using the adjunction isomorphism is an isometry. We obtain the required formula by just combining. □

We want to finish this section with a second formula for $T(X)$, involving only first order derivatives of the theta function. It is based on a function $\|J\|$ on $\mathrm{Sym}^g X$ introduced by Guàrdia in [11].

Let $\tau\in\mathcal{H}_g$ be a period matrix associated to a symplectic basis of $H_1(X,\mathbb{Z})$ and consider again the analytic jacobian $\mathrm{Jac}(X)=\mathbb{C}^g/\mathbb{Z}^g+\tau\mathbb{Z}^g$. For $w_1,\ldots,w_g\in\mathbb{C}^g$ we put

$$J(w_1,\ldots,w_g)=\det\left(\frac{\partial\vartheta}{\partial z_k}(w_l)\right)$$

and

$$\|J\|(w_1,\ldots,w_g)=(\det\mathrm{Im}\,\tau)^{\frac{g+2}{4}}\exp(-\pi\sum_{k=1}^{g}{}^t y_k(\mathrm{Im}\,\tau)^{-1}y_k)\cdot|J(w_1,\ldots,w_g)|\,.$$

Here in the latter formula $y_k=\mathrm{Im}\,w_k$ for $k=1,\ldots,g$. It can be checked that the function $\|J\|(w_1,\ldots,w_g)$ depends only on the classes in $\mathrm{Jac}(X)$ of the vectors w_k. Now let $P_1,\ldots,P_g$ be a set of g points on X. We take vectors $w_1,\ldots,w_g\in\mathbb{C}^g$ such that for all $k=1,\ldots,g$ the class $[w_k]\in\mathrm{Jac}(X)$ corresponds to $[\sum_{l=1, l\neq k}^{g}P_l]\in\mathrm{Pic}_{g-1}(X)$ under the Abel-Jacobi-Riemann correspondence $\mathrm{Pic}_{g-1}(X)\leftrightarrow\mathrm{Jac}(X)$. We then put $\|J\|(P_1,\ldots,P_g)=\|J\|(w_1,\ldots,w_g)$. One may check that this does not depend on the choice of the period matrix τ. The function $\|J\|$ has the following geometrical property: we have $\|J\|(P_1,\ldots,P_g)=0$ if and only if $P_1,\ldots,P_g$ are linearly dependent on the image of X under the canonical map $X\to\mathbb{P}(H^0(X,\Omega_X^1)^\vee)$. We refer to [11] for a proof of the following theorem.

Theorem 4.5. *Let $P_1, \ldots, P_g, Q$ be points on X with $P_1, \ldots, P_g$ pairwise distinct. Then the formula*

$$\|\vartheta\|(P_1 + \cdots + P_g - Q)^{g-1} = \exp(\delta(X)/8) \cdot \|J\|(P_1, \ldots, P_g) \cdot \frac{\prod_{k=1}^g G(P_k, Q)^{g-1}}{\prod_{k<l} G(P_k, P_l)}$$

holds.

A combination of Theorems 2.4, 4.4 and 4.5 yields the following formula for $T(X)$.

Proposition 4.6. *Let $P_1, \ldots, P_g, Q$ be generic points on X. Then the formula*

$$T(X) = \left(\frac{\|\vartheta\|(P_1 + \cdots + P_g - Q)}{\prod_{k=1}^g \|\vartheta\|(gP_k - Q)^{1/g}} \right)^{2g-2} \cdot \\ \cdot \left(\frac{\prod_{k \neq l} \|\vartheta\|(gP_k - P_l)^{1/g}}{\|J\|(P_1, \ldots, P_g)^2} \right) \cdot \prod_{W \in \mathcal{W}} \prod_{k=1}^g \|\vartheta\|(gP_k - W)^{(g-1)/g^4}$$

holds. Again the Weierstrass points are counted with their weights.

Let us make the invariant $T(X)$ explicit in the case that X is an elliptic curve. Writing $X = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ with $\mathrm{Im}\,\tau > 0$ we obtain from either Proposition 4.3 or 4.6 that

$$T(X) = (\mathrm{Im}\,\tau)^{-3/2} \exp(\pi \mathrm{Im}\,\tau/2) \cdot |\frac{\partial\vartheta}{\partial z}(\frac{1+\tau}{2}; \tau)|^{-2}.$$

By Jacobi's derivative formula (cf. [19], Chapter I, §13) we can rewrite this as

$$T(X) = (2\pi)^{-2} \cdot ((\mathrm{Im}\,\tau)^6 |\Delta(\tau)|)^{-1/4}$$

where Δ is the discriminant modular form $\Delta(q) = \eta(q)^{24} = q \prod_{n=1}^\infty (1 - q^n)^{24}$ in $q = \exp(2\pi i \tau)$. Using Theorem 4.4 we obtain

$$\delta(X) = -\log((\mathrm{Im}\,\tau)^6 |\Delta(\tau)|) - 8 \log(2\pi)$$

which is well-known, see [8], p. 417.

In [13] we obtain a generalisation of the above formula for $T(X)$ to the case where X is a hyperelliptic Riemann surface of genus $g \geq 2$. The result is expressed in terms of the discriminant modular form φ_g on the generalised Siegel upper half plane $\mathcal{H}_g$ as defined in [17], Section 3. This is a modular form on $\Gamma_g(2) = \{\gamma \in \mathrm{Sp}(2g, \mathbb{Z}) : \gamma \equiv I_{2g} \mod 2\}$ of weight $4r$, where $r = \binom{2g+1}{g+1}$, generalising the usual discriminant modular form Δ in genus 1.

Theorem 4.7. *Let X be a hyperelliptic Riemann surface of genus $g \geq 2$. Choose an ordering of the Weierstrass points on X and construct a canonical symplectic basis of $H_1(X, \mathbb{Z})$ starting with this ordering (cf.* [19], *Chapter IIIa, §5). Let $\tau \in \mathcal{H}_g$ be a period matrix of X associated to this canonical basis and put $\Delta_g(\tau) = 2^{-(4g+4)n} \cdot \varphi_g(\tau)$ where $n = \binom{2g}{g+1}$. Then $\Delta_g(\tau)$ is non-zero and the formula*

$$T(X) = (2\pi)^{-2g} \cdot ((\mathrm{Im}\,\tau)^{2r} |\Delta_g(\tau)|)^{-\frac{3g-1}{8ng}}$$

holds.

It is an intriguing question whether the invariant $T(X)$ admits of a simple description in terms of modular forms for a *general* Riemann surface X of genus g.
We finish this section by remarking that a closed formula of a quite different type can be given for δ using work of Bost [4] and Gillet-Soulé [10]. The point of view leading to this formula is the Riemannian manifold structure on X deriving from μ. Let ds^2 be the metric on X given in conformal coordinates by $ds^2 = 2h_{z\bar{z}}dzd\bar{z}$ with $h_{z\bar{z}} = \|dz\|_{\mathrm{Ar}}^{-2}$. Let $\det' \Delta_h$ be the zeta regularised determinant of the Laplace operator with respect to this metric, and let $\mathrm{vol}(X,h)$ be the volume of X. Then the formula

$$\delta(X) = c(g) - 6 \log \frac{\det \Delta_h'}{\mathrm{vol}(X,h)}$$

holds, where $c(g)$ is a constant depending only on g. It would be interesting to know whether the terms occurring in this formula can be naturally related to the constants $S(X)$ and $T(X)$ which are the subject of this paper.

5. Applications to intersection theory

In this section we discuss several applications of our results to Arakelov intersection theory. Let $p : \mathcal{X} \to B$ be an arithmetic surface over the spectrum B of the ring of integers of a number field K. For us this means that $\mathcal{X}$ is a regular scheme and that p is a proper and flat relative curve whose generic fiber is smooth and geometrically connected. We denote this generic fiber by $\mathcal{X}_K$. We assume that the reader is familiar with the basic notions and statements in the Arakelov intersection theory on $\mathcal{X}$, as explained in [2] or [8].
We let g be the genus of $\mathcal{X}_K$, and assume that it is positive. We fix a K-basis $\{\psi_1, \ldots, \psi_g\}$ of regular differential 1-forms on $\mathcal{X}_K$. Looking back at the discussion at the beginning of Section 2, which was purely algebraic, we note that a non-zero Wronskian differential $\tilde{\psi}$ can be formed out of this basis. Its divisor $\mathrm{div}\,\tilde{\psi}$ is an effective K-divisor on $\mathcal{X}_K$ and we have, completely analogous to Lemma 3.1, a canonical isomorphism

$$\Omega_{\mathcal{X}_K}^{\otimes g(g+1)/2} \otimes_{O_{\mathcal{X}_K}} \left(\wedge^g H^0(\mathcal{X}_K, \Omega^1_{\mathcal{X}_K}) \otimes_K O_{\mathcal{X}_K}\right)^\vee \xrightarrow{\sim} O_{\mathcal{X}_K}(\mathrm{div}\,\tilde{\psi})$$

of invertible sheaves on $\mathcal{X}_K$. We denote by $\mathcal{W}$ the Zariski closure of $\mathrm{div}\,\tilde{\psi}$ in $\mathcal{X}$. Let $\omega_{\mathcal{X}/B}$ be the relative dualising sheaf of $p : \mathcal{X} \to B$.

Lemma 5.1. *The above isomorphism extends to a canonical isomorphism*

$$\rho : \omega_{\mathcal{X}/B}^{\otimes g(g+1)/2} \otimes_{O_\mathcal{X}} \left(p^*(\det p_* \omega_{\mathcal{X}/B})\right)^\vee \xrightarrow{\sim} O_\mathcal{X}(\mathcal{V} + \mathcal{W})$$

of invertible sheaves on $\mathcal{X}$, for some effective divisor $\mathcal{V}$ whose support is entirely contained in the fibers of p.

Proof. The idea for the proof is taken from [1], p. 1298, where an analogous result is proven for the function field case. We recall that $\tilde{\psi}$ is given in a local coordinate z by $W_z(\psi)(dz)^{\otimes g(g+1)/2}$ where

$$W_z(\psi) = \det \left(\frac{1}{(l-1)!} \frac{d^{l-1} f_k}{dz^{l-1}} \right)_{1 \leq k,l \leq g}$$

if the ψ_k are locally written as $\psi_k = f_k(z)dz$ for $k = 1, \ldots, g$. On $\mathcal{X}_K$ this gives rise to a morphism of invertible sheaves

$$\wedge^g H^0(\mathcal{X}_K, \Omega^1_{\mathcal{X}_K}) \otimes_K O_{\mathcal{X}_K} \longrightarrow \Omega^{\otimes g(g+1)/2}_{\mathcal{X}_K}$$

by setting

$$\xi_1 \wedge \ldots \wedge \xi_g \mapsto \frac{\xi_1 \wedge \ldots \wedge \xi_g}{\psi_1 \wedge \ldots \wedge \psi_g} \cdot \tilde{\psi}$$

(cf. the proof of Lemma 3.1). Now note that the construction of $\tilde{\psi}$ is valid for smooth proper curves over any base scheme. As a result, by modifying the basis $\{\psi_1, \ldots, \psi_g\}$ if necessary, the above morphism extends canonically at least over the open dense subscheme of $\mathcal{X}$ where p is smooth. Automatically it extends then further to give a canonical morphism $p^*(\det p_* \omega_{\mathcal{X}/B}) \to \omega^{\otimes g(g+1)/2}_{\mathcal{X}/B}$ on the whole of $\mathcal{X}$. Multiplying by $(p^*(\det p_* \omega_{\mathcal{X}/B}))^\vee$ we obtain from this a morphism

$$O_{\mathcal{X}} \longrightarrow \omega^{\otimes g(g+1)/2}_{\mathcal{X}/B} \otimes_{O_{\mathcal{X}}} \left(p^*(\det p_* \omega_{\mathcal{X}/B}) \right)^\vee .$$

The image of 1 is a section whose divisor is a divisor $\mathcal{V} + \mathcal{W}$ where $\mathcal{V}$ is effective and has support entirely contained in the fibers of p. This gives the lemma. $\square$

The divisor $\mathcal{V}$ is an invariant of the arithmetic surface $p : \mathcal{X} \to B$ and we shall use it without further mention in the sequel.

Example 5.2. In the case that $g = 1$, the morphism $p^* p_* \omega_{\mathcal{X}/B} \to \omega_{\mathcal{X}/B}$ in the above proof is just the natural morphism, as is readily checked. According to [16], Corollary 3.27, if $p : \mathcal{X} \to B$ is a minimal arithmetic surface, then the natural morphism $p^* p_* \omega_{\mathcal{X}/B} \to \omega_{\mathcal{X}/B}$ is in fact an isomorphism. Hence we find $\mathcal{V} = \varnothing$ in this case.

We want to translate the isomorphism ρ of Lemma 5.1 into an equality of Arakelov divisors on $\mathcal{X}$. For this we need a notation for the norm of ρ at the various complex embeddings of K.

Definition 5.3. Let X be a compact Riemann surface of positive genus. We denote by $R(X)$ the norm of the isomorphism ρ from Lemma 3.1.

It follows from our discussion so far that $R(X) = S(X) \cdot \exp(-\delta(X)/8)$. Now let's turn back to our arithmetic surface $p : \mathcal{X} \to B$. We recall from [2] [8] that both sides of the isomorphism ρ from Lemma 5.1 come equipped with a canonical structure of metrised invertible sheaf, and that to each non-zero rational section of such a sheaf we can associate its Arakelov divisor. For each complex embedding σ of K we denote by X_σ the compact Riemann surface $(\mathcal{X}_K \otimes_{K,\sigma} \mathbb{C})(\mathbb{C})$ obtained from base changing $\mathcal{X}_K$ to $\mathbb{C}$ along σ. We denote by F_σ the corresponding Arakelov fiber. The next proposition follows easily from Lemma 5.1 and from the fact that ρ has constant norm $R(X_\sigma)$ on X_σ.

Proposition 5.4. *We have an equality*

$$\frac{1}{2}g(g+1)\omega_{\mathcal{X}/B} = \mathcal{V} + \mathcal{W} + \sum_\sigma \log R(X_\sigma)\cdot F_\sigma + p^*(\det p_*\omega_{\mathcal{X}/B})$$

of Arakelov divisors on $\mathcal{X}$. Here the sum runs over the complex embeddings of K.

This proposition can be used to deduce some interesting formulas involving Arakelov intersection numbers.

Definition 5.5. We define a function R on the set of closed fibers of $p:\mathcal{X}\to B$ as follows. Let s be a closed point of B. If $g=1$, we put $R(\mathcal{X}_s)=0$. If $g\geq 2$, then we define $R(\mathcal{X}_s)$ by the equality $(2g-2)\cdot\log R(\mathcal{X}_s) = (\mathcal{V}_s,\omega_{\mathcal{X}/B})\cdot\log\#k(s)$, where $(\mathcal{V}_s,\omega_{\mathcal{X}/B})$ is the usual intersection number of the divisors $\mathcal{V}$ and $\omega_{\mathcal{X}/B}$ above s, and where $k(s)$ is the residue field at s.

As the next proposition implies, the function R can be seen as an analogue of the previously defined R for compact Riemann surfaces. The quantity $\widehat{\deg}\det p_*\omega_{\mathcal{X}/B}$ is the usual Arakelov degree of the metrised invertible sheaf $\det p_*\omega_{\mathcal{X}/B}$ on B (i.e., $[K:\mathbb{Q}]$ times the Faltings height of $p:\mathcal{X}\to B$).

Proposition 5.6. *Assume that $p:\mathcal{X}\to B$ is a semi-stable arithmetic surface. Then for the self-intersection of the relative dualising sheaf we have a lower bound*

$$(\omega,\omega) \geq \frac{8(g-1)}{(2g-1)(g+1)}\cdot\left(\sum_s \log R(\mathcal{X}_s) + \sum_\sigma \log R(X_\sigma) + \widehat{\deg}\det p_*\omega_{\mathcal{X}/B}\right).$$

Here the first sum runs over the closed points $s\in B$, and the second sum runs over the complex embeddings of K.

Proof. In the case $g=1$, the lower bound is trivially satisfied since we have $(\omega_{\mathcal{X}/B},\omega_{\mathcal{X}/B})=0$ in this case by [8], Theorem 7. So assume that $g\geq 2$. We take the equality from Proposition 5.1 and intersect the divisors on both sides with $\omega_{\mathcal{X}/B}$. This gives that $\frac{1}{2}g(g+1)(\omega_{\mathcal{X}/B},\omega_{\mathcal{X}/B})$ can be written as

$$(\mathcal{W},\omega_{\mathcal{X}/B}) + (2g-2)\cdot\left(\sum_s \log R(\mathcal{X}_s) + \sum_\sigma \log R(X_\sigma) + \widehat{\deg}\det p_*\omega_{\mathcal{X}/B}\right).$$

For the term $(\mathcal{W},\omega_{\mathcal{X}/B})$ we have by [8], Theorem 5 the lower bound

$$(\mathcal{W},\omega_{\mathcal{X}/B}) \geq \frac{g^3-g}{2g(2g-2)}(\omega_{\mathcal{X}/B},\omega_{\mathcal{X}/B}) = \frac{1}{4}(g+1)(\omega_{\mathcal{X}/B},\omega_{\mathcal{X}/B})$$

since the generic degree of $\mathcal{W}$ is g^3-g. Using this in the first equality gives the required lower bound. □

We remark that for a semi-stable arithmetic surface $p:\mathcal{X}\to B$ the numbers $\log R(\mathcal{X}_s)$ are always non-negative. Lower bounds of a similar type for $(\omega_{\mathcal{X}/B},\omega_{\mathcal{X}/B})$ can be found in [6]. The problem with the above proposition is that the right hand side may be negative, and then the lower bound becomes useless in view of the fundamental inequality $(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) \geq 0$ proved by Faltings [8]. However, for any fixed $g \geq 2$ the invariant $\log R(X)$ can become arbitrarily large, as the next proposition shows.

Proposition 5.7. *Let X_t be a holomorphic family of compact Riemann surfaces of genus $g \geq 2$ over the punctured disk, degenerating as $t \to 0$ to the union of two Riemann surfaces of positive genera g_1, g_2 with two points identified. Suppose that neither of these two points was a Weierstrass point. Then the formula*

$$\log R(X_t) = -\frac{g_1 g_2}{2g} \log |t| + O(1)$$

holds as $t \to 0$.

For a proof we refer to the author's thesis [12].
The next application we have in mind is a formula for the self-intersection of a point. In order to derive this formula it is convenient to use the machinery of the determinant of cohomology $\det Rp_*(\cdot)$ and the Deligne bracket $\langle \cdot, \cdot \rangle$, for which we refer to [7]. We will use that for any section $P : B \to \mathcal{X}$ and any invertible sheaf L on $\mathcal{X}$ we have canonical isomorphisms $\langle O_\mathcal{X}(P), L\rangle \xrightarrow{\sim} P^*L$ and $\langle P, \omega_{\mathcal{X}/B}\rangle \xrightarrow{\sim} \langle P, P\rangle^{\otimes -1}$. The latter is sometimes called the adjunction formula. Moreover, we have a Riemann-Roch theorem in the following form: for each invertible sheaf L on $\mathcal{X}$ there is a canonical isomorphism $(\det Rp_*L)^{\otimes 2} \xrightarrow{\sim} \langle L, L \otimes \omega_{\mathcal{X}/B}^{-1}\rangle \otimes (\det p_*\omega_{\mathcal{X}/B})^{\otimes 2}$.

Lemma 5.8. *Let P be a section of p, not a Weierstrass point on the generic fiber. Then we have a canonical isomorphism*

$$\upsilon : P^*(O_\mathcal{X}(\mathcal{V} + \mathcal{W}))^{\otimes 2} \xrightarrow{\sim} (\det Rp_*O_\mathcal{X}(gP))^{\otimes -2}$$

of line bundles on B. When restricted to the generic fiber, the left hand side gets identified with $O_{\mathrm{Spec} K}^{\otimes 2}$ and the right hand side gets identified with $H^0(\mathcal{X}_K, O_{\mathcal{X}_K}(gP))^{\otimes -2}$. The latter has a canonical trivialising section 1 and the isomorphism υ, when restricted to the generic fiber, sends the 1 of $O_{\mathrm{Spec} K}^{\otimes 2}$ to the 1 of $H^0(\mathcal{X}_K, O_{\mathcal{X}_K}(gP))^{\otimes -2}$.

Proof. The Riemann-Roch theorem applied to the invertible sheaf $O_\mathcal{X}(gP)$ gives a canonical isomorphism

$$(\det Rp_*O_\mathcal{X}(gP))^{\otimes 2} \xrightarrow{\sim} \langle O_\mathcal{X}(gP), O_\mathcal{X}(gP) \otimes \omega_{\mathcal{X}/B}^{-1}\rangle \otimes (\det p_*\omega_{\mathcal{X}/B})^{\otimes 2}.$$

By the adjunction formula, the right hand side can be canonically identified with $\langle P, P\rangle^{\otimes g(g+1)} \otimes (\det p_*\omega_{\mathcal{X}/B})^{\otimes 2}$, giving a canonical isomorphism

$$(\det Rp_*O_\mathcal{X}(gP))^{\otimes 2} \xrightarrow{\sim} \langle P, P\rangle^{\otimes g(g+1)} \otimes (\det p_*\omega_{\mathcal{X}/B})^{\otimes 2}.$$

On the other hand, pulling back the isomorphism ρ from Lemma 5.1 along P and using once more the adjunction formula we find a canonical isomorphism

$$\langle P, P\rangle^{\otimes -g(g+1)/2} \xrightarrow{\sim} \langle \mathcal{V} + \mathcal{W}, P\rangle \otimes \det p_*\omega_{\mathcal{X}/B}$$

and hence

$$\langle P,P\rangle^{\otimes g(g+1)} \xrightarrow{\sim} (P^* O_{\mathcal{X}}(\mathcal{V}+\mathcal{W}))^{\otimes -2} \otimes (\det p_* \omega_{\mathcal{X}/B})^{\otimes -2} .$$

The isomorphism v follows by combining these isomorphisms. Since P is not a Weierstrass point on the generic fiber, we have that $H^0(\mathcal{X}_K, gP)$ is 1-dimensional and hence is generated by its canonical section 1. The last statement of the lemma follows then by carefully spelling out all the isomorphisms. □

Proposition 5.9. *Let P be a section of p, not a Weierstrass point on the generic fiber. Then $R^1 p_* O_{\mathcal{X}}(gP)$ is a torsion module on B and the self-intersection $-\frac{1}{2}g(g+1)(P,P)$ is given by*

$$-\sum_\sigma \log G(P_\sigma, \mathcal{W}_\sigma) + \log \# R^1 p_* O_{\mathcal{X}}(gP) + \sum_\sigma \log R(X_\sigma) + \widehat{\deg} \det p_* \omega_{\mathcal{X}/B} .$$

Here σ runs through the complex embeddings of K.

Proof. That $R^1 p_* O_{\mathcal{X}}(gP)$ is a torsion module on B follows since we have $H^1(\mathcal{X}_K, gP) = 0$ on the generic fiber. As to the formula, we take the equality from Proposition 5.1 and intersect the divisors on both sides with P. By the Arakelov adjunction formula $(\omega, P) = -(P,P)$ we obtain

$$-\frac{1}{2}g(g+1)(P,P) = (\mathcal{V}+\mathcal{W}, P) + \sum_\sigma \log R(X_\sigma) + \widehat{\deg} \det p_* \omega_{\mathcal{X}/B} .$$

It remains therefore to see that $(\mathcal{V}+\mathcal{W},P)_{\mathrm{fin}} = \log \# R^1 p_* O_{\mathcal{X}}(gP)$. For this we invoke Lemma 5.8. It follows from the description of v on the generic fiber that in fact v is the natural isomorphism over the open dense subscheme of B where P does not meet $\mathcal{V}+\mathcal{W}$. Now for any closed point s of B denote by e_s the length at s of $R^1 p_* O_{\mathcal{X}}(gP)$. Then if we let $D = \sum_s e_s \cdot s$, the invertible sheaf $\det R^1 p_* O_{\mathcal{X}}(gP)$ gets identified with $O_B(D)$ and, since $\det R^0 p_* O_{\mathcal{X}}(gP)$ is trivialised by the section 1, the determinant of cohomology $\det Rp_* O_{\mathcal{X}}(gP)$ gets identified with $O_B(-D)$. By Lemma 5.8, for any closed point s the length e_s coincides with the intersection multiplicity of P and $\mathcal{V}+\mathcal{W}$ at s and consequently $(\mathcal{V}+\mathcal{W},P)_{\mathrm{fin}} = \sum_s e_s \log \# k(s) = \log \# R^1 p_* O_{\mathcal{X}}(gP)$. □

6. A numerical example

In this section we use the results of Sections 2 and 4 to calculate the Faltings height and the self-intersection of the relative dualising sheaf of a certain hyperelliptic curve of genus 3 defined over the rationals. We start with two theoretical results, both of which can be proved by methods similar to those used in [5], Section 3.

Let K be a number field, and let O_K be its ring of integers. For a non-zero element $a \in O_K$ and a prime ideal $\wp$ of O_K we denote by $v_\wp(a)$ the exponent of $\wp$ in the prime ideal decomposition of $a \cdot O_K$. Let $f \in O_K[x]$ be a monic polynomial of degree 5 with $f(0)$ and $f(1)$ units in O_K and put $g(x) = x(x-1) + 4f(x)$.

Proposition 6.1. *Suppose that the discriminant Δ of g is non-zero, that we have $v_\wp(\Delta) = 0$ or 1 for each prime ideal $\wp$ of residue characteristic $\neq 2$, and that g mod $\wp$ has a unique multiple root of multiplicity 2 for prime ideals $\wp$ with residue characteristic $\neq 2$ and with $v_\wp(\Delta) = 1$. Then the equation*

$$C_f \,:\, y^2 = x(x-1)f(x)$$

defines a hyperelliptic curve of genus 3 over K. It extends to a semi-stable arithmetic surface $p : \mathcal{X} \to B = \mathrm{Spec}(O_K)$. We have that $\mathcal{X}$ has bad reduction at $\wp$ if and only if $\wp$ has residue characteristic $\neq 2$ and $v_\wp(\Delta) = 1$. For such $\wp$, the fiber at $\wp$ is an irreducible curve with a single double point. The differentials $dx/y, xdx/y, x^2dx/y$ form a basis of the free O_B-module $p_*\omega_{\mathcal{X}/B}$. The points W_0, W_1 on C_f given by $x = 0$ and $x = 1$ extend to disjoint sections of p.

As to the Faltings height and the self-intersection of the relative dualising sheaf of C_f we have the following. For a complex embedding σ of K we denote by $X_{f,\sigma}$ the compact Riemann surface $(C_f \otimes_{K,\sigma} \mathbb{C})(\mathbb{C})$ obtained by base changing C_f to $\mathbb{C}$ along σ. For each σ, we choose a symplectic basis of $H_1(X_{f,\sigma}, \mathbb{Z})$ and form the period matrix $\Omega_\sigma = (\Omega_{1,\sigma}|\Omega_{2\sigma})$ for $dx/y, xdx/y$ and x^2dx/y on this basis. We further put $\tau_\sigma = \Omega_{1\sigma}^{-1}\Omega_{2\sigma}$.

Proposition 6.2. *The degree of $\det p_*\omega_{\mathcal{X}/B}$ satisfies*

$$\widehat{\deg} \det p_*\omega_{\mathcal{X}/B} = -\frac{1}{2}\sum_\sigma \log\left(|\det \Omega_{1\sigma}|^2(\det \mathrm{Im}\,\tau_\sigma)\right) .$$

For the self-intersection of the relative dualising sheaf we have the formula

$$(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) = 24 \sum_\sigma \log G_\sigma(W_0, W_1)\,,$$

where G_σ denotes the Arakelov-Green function on $X_{f,\sigma}$.

We apply these propositions to a concrete example. We choose $K = \mathbb{Q}$ and $f(x) = x^5 + 6x^4 + 4x^3 - 6x^2 - 5x - 1$. It can be checked that $g(x) = x(x-1) + 4f(x)$ satisfies the conditions of Proposition 6.1. The corresponding hyperelliptic curve C_f has bad reduction at the primes $p = 37, p = 701$ and $p = 14717$. Let X_f be the compact Riemann surface obtained from base changing C_f to the complex numbers. We choose an ordering of the Weierstrass points of X and as in [19], Chapter IIIa, §5 this gives us a canonical way to construct a symplectic basis for $H_1(X_f, \mathbb{Z})$. We have computed the periods with respect to this basis of the differentials $dx/y, xdx/y$ and x^2dx/y. Using Proposition 6.2 we easily obtain

$$\widehat{\deg} \det p_*\omega_{\mathcal{X}/B} = -1.280295247656532068\ldots$$

which is the Faltings height of C_f. Next we take a look at the self-intersection of the relative dualising sheaf. According to Proposition 6.2 we need to calculate $G(W_0, W_1)$. We apply Theorem 2.4 where we carefully take a limit for P approaching W_0. Using theory as developed for example in [19], Chapter IIIa it is possible to make the Abel-Jacobi-Riemann correspondence $\mathrm{Pic}_2(X_f) \leftrightarrow$

$\mathrm{Jac}(X_f)$ completely explicit. This makes it easy to carry out the theta function evaluations that are needed to compute $G(W_0, W_1)$. The calculation of $S(X_f)$ is, however, considerably harder. We recall that in our case $S(X_f)$ is given by

$$\log S(X_f) = -\int_{X_f} \log \|\vartheta\|(3P - Q) \cdot \mu(P)$$

where μ is the Arakelov metric and where Q is any point on X_f. We want to express μ in terms of the coordinates x, y but then it immediately becomes clear that the integrand will diverge at the Weierstrass point at infinity. However, by taking logarithms in Theorem 2.4 and integrating against $\mu(Q)$ we find the alternative formula

$$\log S(X_f) = -9 \int_X \log \|\vartheta\|(3P - Q) \cdot \mu(Q) + \frac{1}{3} \sum_{W \in \mathcal{W}} \log \|\vartheta\|(3P - W)\,,$$

valid for any non-Weierstrass point P on X, in which the integrand behaves better. In fact, the integrand now only has a singularity at $Q = P$. Let $\Omega = (\Omega_1|\Omega_2)$ be the period matrix of X_f referred to above and put $\tau = \Omega_1^{-1}\Omega_2$. Let (μ_{kl}) be the matrix given by $(\mu_{kl}) = \left(\overline{\Omega}_1 (\mathrm{Im}\,\tau)^t \Omega_1\right)^{-1}$. An application of Riemann's bilinear relations yields $\mu = \frac{i}{6} \sum \mu_{kl}\, \psi_k \wedge \overline{\psi_l}$ with $\psi_1 = dx/y, \psi_2 = xdx/y$ and $\psi_3 = x^2 dx/y$. Writing $x = u + iv$ with $u, v \in \mathbb{R}$ we can rewrite this as the real 2-form

$$\begin{aligned} \mu =& \frac{1}{3} \left(\mu_{11} + 2\mu_{12}u + 2\mu_{13}(u^2 - v^2) + \mu_{22}(u^2 + v^2)\right. \\ & \left. + 2\mu_{23}u(u^2 + v^2) + \mu_{33}(u^2 + v^2)^2\right) \cdot \frac{dudv}{|h(u + iv)|}\,, \end{aligned}$$

where $h(x) = x(x-1)g(x)$. Using a computer algebra package, we have evaluated the integral. This is a slow process, because one has to take care of the logarithmic singularity. On the other hand, it is possible to check the answers by trying various choices of P. We found that within reasonable time limits we can only reach an accuracy within $\pm$ 0.005. The end result is

$$\log S(X_f) = 17.57\ldots$$

Using this we find the approximation

$$G(W_0, W_1) = 2.33\ldots$$

and finally

$$(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) = 20.32\ldots$$

It is almost no extra effort to compute also the delta-invariant of X_f. Using Theorem 4.7 we obtain, first of all,

$$\log T(X_f) = -4.44361200473681284\ldots$$

With Theorem 4.4 and our value above for $\log S(X_f)$ we get as a result

$$\delta(X_f) = -33.40\ldots$$

The reader may check that the Noether formula [18] is verified by our numerical results.

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