The Microstates Free Entropy Dimension of any DT–operator is 2

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Abstract. Suppose that μ is an arbitrary Borel measure on $\mathbb{C}$ with compact support and $c > 0$. If Z is a DT(μ, c)–operator as defined by Dykema and Haagerup in [6], then the microstates free entropy dimension of Z is 2.

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1 Introduction.

DT–operators were introduced by Dykema and Haagerup in their work on invariant subspaces of certain operators in a II_1 factor [5, 6]. A DT–operator Z is specified by two parameters, μ and c, where $c > 0$ and μ is a Borel probability measure on $\mathbb{C}$ with compact support. Roughly, the operator Z is determined by stating that its $*$–distribution is the same as the limit $*$–distribution as $N \to \infty$ of random matrices

$$Z_N = D_N + cT_N,$$

where D_N are diagonal $N \times N$ matrices whose spectral measures converge to μ in distribution, while T_N is a strictly upper triangular random $N \times N$ matrix with i.i.d. Gaussian entries. Equivalently, (see [15], [12], [6] and the appendix of [7]), Z can be viewed as a sum $Z = d + cT$, where d is a normal operator with spectral measure μ contained in a diffuse von Neumann algebra A, and T is an A-valued circular operator with a certain covariance. Finally, a result of Śniady [14] shows that a DT(μ, c)–operator is one whose free entropy is maximized among all those operators having Brown measure equal to μ and with a fixed off–diagonality.
If we write $Z = d+cT$ as above, it is clear that $W^*(Z) \subset W^*(d, T) \subseteq W^*(A \cup \{T\})$, while a simple computation shows $W^*(A \cup \{T\}) = L(\mathbb{F}_2)$. By Lemma 6.2 of [6], for any μ we may choose d having trace of spectral measure equal to μ and so that

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$d, T \in W^*(Z)$; by [7], $A \subseteq W^*(T)$, so we always have $W^*(Z) \cong L(\mathbb{F}_2)$. Thus Z can be viewed as an interesting generator for this free group factor.

In order to test the hypothesis that Voiculescu's free entropy dimension δ_0 [16, 17, 20] is the same for any sets of generators of a von Neumann algebra, it is important to decide whether the free entropy dimension of Z is 2 ($L(\mathbb{F}_2)$ clearly has another set of generators of free entropy dimension 2).

For another version of free entropy dimension, also defined by Voiculescu, called the non-microstates free entropy dimension [18], L. Aagaard has recently shown [1] that the dimension of Z is indeed 2. It is known by [4] that the non-microstates free entropy dimension dominates δ_0 but at present it is open whether the reverse inequality holds. Thus, Aagaard's result does not solve the question for the original microstates definition.

In this paper, we show that, indeed, $\delta_0(Z) = 2$. Our proof uses an equivalent packing number formulation of the microstates free entropy dimension, due to Jung [8]. In this approach, to get the nontrivial lower bound on $\delta_0(Z)$, one must have lower bounds on the ϵ–packing numbers of spaces of matricial microstates for Z, which are in turn obtained by lower bounds on the volume of ϵ–neighborhoods of these microstate spaces. The kth microstate space is the set $\Gamma(Z; m, k, \gamma)$, for $m, k \in \mathbb{N}$ and $\gamma > 0$, of all $k \times k$ complex matrices whose $*$–moments up to order m are γ–close to the values of the corresponding $*$–moments of Z, and the volumes are for Lebesgue measure λ_k on $M_k(\mathbb{C})$ viewed as a Euclidean space of real dimension $2k^2$ with coordinates corresponding to the real and imaginary parts of the entries of a matrix.

In order to outline how we get these lower bounds on volumes, let us for convenience take Z equal to the $\mathrm{DT}(\delta_0, 1)$–operator T. A key result that we use is a recent one of Aagaard and Haagerup [2], showing that a certain ϵ–perturbation of T has Brown measure uniformly distributed on the disk of radius $r_\epsilon := 1/\sqrt{\log(1+\epsilon^{-2})}$ centered at the origin; note how slowly this disk shrinks as ϵ approaches zero. Applying a result of Śniady [13] to this situation, we find matrices $A_k \in M_k(\mathbb{C})$ that lie in ϵ–neighborhoods of microstate spaces for T, whose eigenvalues are close to uniformly distributed (as k gets large) in the disk of radius r_ϵ. Thus, in order to get a lower bound on the volume of a 2ϵ–neighborhood of a microstate space for T, it will suffice to get a lower bound on the volume of a unitary orbit of an ϵ–neighborhood of A_k.

Every element of $M_k(\mathbb{C})$ has an upper triangular matrix in its unitary orbit. Thus, letting $T_k(\mathbb{C})$ denote the set of upper triangular matrices in $M_k(\mathbb{C})$, there is a measure ν_k on $T_k(\mathbb{C})$ such that $\lambda_k(\mathcal{O}) = \nu_k(\mathcal{O} \cap T_k)$ for every $\mathcal{O} \subseteq M_k(\mathbb{C})$ invariant under unitary conjugation. Freeman Dyson identified such a measure ν_k (see Appendix 35 of [11]), and showed that if we view $T_k(\mathbb{C})$ as a Euclidean space of real dimension $k(k-1)$ with coordinates corresponding to the real and imaginary parts of the matrix entries lying on and above the diagonal, then ν_k is absolutely continuous with respect to Lebesgue measure on $T_k(\mathbb{C})$ and has density given at $B = (b_{ij})_{1 \le i,j \le k} \in T_k(\mathbb{C})$ by

$$C_k \prod_{1 \le p < q \le k} |b_{pp} - b_{qq}|^2, \tag{1}$$

where the constant is

$$C_k = \frac{\pi^{k(k-1)/2}}{\prod_{j=1}^k j!}. \tag{2}$$

We will use this measure of Dyson to find lower bound on the volume of unitary orbits of an ϵ–neighborhood of A_k, and we may take A_k to be upper triangular. However, so far we only have information about the eigenvalues of A_k, namely the diagonal part of it. Loosely speaking, in order to get a handle on the part strictly above the diagonal, we use a result of Dykema and Haagerup [6] to realize T as an upper triangular matrix

$$T = \frac{1}{\sqrt{N}} \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1N} \\ 0 & T_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & T_{N-1,N} \\ 0 & \cdots & 0 & T_{NN} \end{bmatrix}$$

of operators where each T_{ii} is a copy of T, each T_{ij} for $i < j$ is circular and the family $(T_{ij})_{1\le i\le j\le N}$ is $*$–free. Thus, A_k can be taken to be of the form

$$\begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1N} \\ 0 & B_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & B_{N-1,N} \\ 0 & \cdots & 0 & B_{NN} \end{bmatrix}$$

where each B_{ii} is upper triangular, where we have good knowledge of the eigenvalue distributions of each B_{ii} and where the B_{ij} for $i < j$ approximate $*$–free circular elements. Using the strengthened asymptotic freeness results of Voiculescu [19], we find enough approximants for these B_{ij}. Although we still have no real knowledge about the entries of the B_{ii} lying above the diagonal, these parts are of negligibly small dimension as N gets large, and we are able to get good enough lower bounds. The techniques we use for estimating integrals of the quantity (1) over certain regions are taken from [9].

2 MICROSTATES FOR Z WITH WELL–SPACED SPECTRAL DENSITIES

The following lemma is an application of the result of Aagaard and Haagerup [2] mentioned in the introduction in order to make perturbations of general DT–operators having Brown measure that is relatively well spread out. For an element a of a non-commutative probability space $(\mathcal{M}, \tau)$, we write $\|a\|_2$ for $\tau(a^*a)^{1/2}$.

LEMMA 2.1. *Let μ be a compactly supported Borel probability measure on $\mathbb{C}$ and let $c > 0$. Let Z be a $\mathrm{DT}(\mu, c)$–operator in a W*–noncommutative probability space $(\mathcal{M}, \tau)$. Let us write*

$$\mu = \nu + \sum_{i=1}^s a_i \delta_{z_i}$$

for some $s \in \{0\} \cup \mathbb{N} \cup \{\infty\}$, $z_i \in \mathbb{C}$ and $a_i > 0$, where ν is a diffuse measure and where $z_i \neq z_j$ if $i \neq j$. Consider the W–noncommutative probability space*

$$(\widetilde{\mathcal{M}}, \tilde{\tau}) = (\mathcal{M}, \tau) * (L(\mathbb{F}_2), \tau_{\mathbb{F}_2}).$$

Then for every $\epsilon > 0$, there is $\widetilde{Z}_\epsilon \in \widetilde{\mathcal{M}}$ such that $\|\widetilde{Z}_\epsilon - Z\|_2 \leq \epsilon c$ and where the Brown measure of $\widetilde{Z}_\epsilon$ is equal to

$$\sigma_\epsilon := \nu + \sum_{i=1}^{s} a_i \rho_{i,\epsilon},$$

where $\rho_{i,\epsilon}$ is the probability measure that is uniform distribution on the disk centered at z_i and having radius

$$r_i := c\sqrt{\frac{a_i}{\log(1 + a_i \epsilon^{-2})}}.$$

Finally, if $\delta > 0$ and if

$$X_\delta = \{(w_1, w_2) \in \mathbb{C}^2 \mid |w_1 - w_2| < \delta\},$$

then

$$(\sigma_\epsilon \times \sigma_\epsilon)(X_\delta) \leq (\nu \times \nu)(X_\delta) + 2\sum_{i=1}^{s} \min(a_i, \delta^2 c^{-2} \log\left(1 + a_i \epsilon^{-2}\right)). \qquad (3)$$

Proof. By results from [6], taking projections onto local spectral subspaces of Z, we find projections $p_j \in \mathcal{M}$ (for $0 \leq j < s+1$) such that

- $\sum_{j=0}^{s} p_j = 1$,
- $p_0 + p_1 + \cdots + p_k$ is Z–invariant for all integers k such that $0 \leq k < s+1$,
- $\tau(p_k) = \begin{cases} |\nu| & \text{if } k = 0 \\ a_k & \text{if } 1 \leq k < s+1, \end{cases}$
- In $(p_k \mathcal{M} p_k, \tau(p_k)^{-1} \tau\restriction_{p_k \mathcal{M} p_k})$, $p_k Z p_k$ is $\mathrm{DT}(|\nu|^{-1}\nu, c\sqrt{|\nu|})$ if $k = 0$ and is $\mathrm{DT}(\delta_{z_k}, c\sqrt{a_k})$ if $1 \leq k < s+1$.

Let $Y \in \widetilde{\mathcal{M}}$ be centered circular such that Y and Z are $*$–free and $\tilde{\tau}(Y^*Y) = 1$. Let

$$\widetilde{Z}_\epsilon = Z + \epsilon \sum_{i=1}^{s} a_i^{-1/2} c p_i Y p_i. \qquad (4)$$

Then $\|\widetilde{Z}_\epsilon - Z\|_2^2 = \epsilon^2 c^2 \sum_{i=1}^{s} a_i \leq \epsilon^2 c^2$. On the other hand, $\widetilde{Z}_\epsilon$ is upper triangular with respect to the projections $p_0, p_1, \ldots$; the Brown measure of $\widetilde{Z}_\epsilon$ is, therefore, equal to the Brown measure of its diagonal part

$$p_0 Z p_0 + \sum_{i=1}^{s} \left(p_i Z p_i + \epsilon\, a_i^{-1/2} c p_i Y p_i\right). \qquad (5)$$

But in $(p_i\widetilde{\mathcal{M}}p_i, a_i^{-1}\tilde{\tau}\restriction_{p_i\widetilde{\mathcal{M}}p_i})$, the operator $\epsilon\, a_i^{-1/2}cp_iYp_i$ is a centered circular operator of second moment ϵ^2c^2 that is $*$–free from the $\mathrm{DT}(\delta_{z_i}, c\sqrt{a_i})$ operator p_iZp_i. Therefore, the random variable

$$p_iZp_i + \epsilon\, a_i^{-1/2}cp_iYp_i \tag{6}$$

has the same $*$–distribution as $z_iI + c\sqrt{a_i}(T + \epsilon\, a_i^{-1/2}Y)$, where T is a $\mathrm{DT}(\delta_0, 1)$–operator that is $*$–free from Y. By [2], the Brown measure of the random variable (6) is equal to $\rho_{i,\epsilon}$. This yields σ_ϵ for the Brown measure of the operator (5), hence of $\widetilde{Z}_\epsilon$ itself.
Finally, we have

$$(\sigma_\epsilon \times \sigma_\epsilon)(X_\delta) \leq (\nu \times \nu)(X_\delta) + 2\sum_{i=1}^{s} a_i(\sigma_\epsilon \times \rho_{i,\epsilon})(X_\delta) \tag{7}$$

and

$$(\sigma_\epsilon \times \rho_{i,\epsilon})(X_\delta) = \int_{\mathbb{C}} \rho_{i,\epsilon}(w + \delta\,\mathbb{D})d\sigma_\epsilon(w) \leq \min(1, \delta^2 r_i^{-2}), \tag{8}$$

where $\mathbb{D}$ is the unit disk in $\mathbb{C}$. Taken together, (7) and (8) yield the inequality (3). □

The next lemma uses a result of Śniady [13] to find matrix approximants of the operators appearing in Lemma 2.1.
In the following lemma and throughout this paper, for a matrix $A \in M_k(\mathbb{C})$ we let $|A|_2 = \mathrm{tr}_k(A^*A)^{1/2}$, where tr_k is the normalized trace on $M_k(\mathbb{C})$. Moreover, by the eigenvalue distribution of $A \in M_k(\mathbb{C})$ we mean its Brown measure, which is just the probability measure that is uniformly distributed on its list of eigenvalues $\lambda_1, \ldots, \lambda_k$, where these are listed according to (general) multiplicity, i.e. a value z is listed $\dim \bigcup_{n=1}^{\infty} \ker((A - zI)^n)$ times.

LEMMA 2.2. *Let μ be a compactly supported Borel probability measure on $\mathbb{C}$ and let $c > 0$. Then there exists a sequence $\langle y_k\rangle_{k=1}^{\infty}$ such that for any $\epsilon > 0$, there exists a sequence $\langle z_{k,\epsilon}\rangle_{k=1}^{\infty}$ such that*

- *$y_k, z_{k,\epsilon} \in M_k(\mathbb{C})$,*
- *$\|y_k\|$ and $\|z_{k,\epsilon}\|$ remain bounded as $k \to \infty$,*
- *$\limsup_{k\to\infty} |y_k - z_{k,\epsilon}|_2 \leq \epsilon c$,*
- *y_k converges in $*$–moments as $k \to \infty$ to a $\mathrm{DT}(\mu, c)$–operator,*
- *the eigenvalue distribution of $z_{k,\epsilon}$ converges weakly as $k \to \infty$ to the measure σ_ϵ described in Lemma 2.1.*

Proof. Let Z be a $\mathrm{DT}(\mu, c)$–operator, let $\widetilde{Y}$ be the operator $\sum_{i=1}^{s} a_i^{-1/2}cp_iYp_i$ appearing in (4) in the proof of the preceding lemma, so that $\widetilde{Z}_\epsilon = Z + \epsilon\widetilde{Y}$. Since Z can be constructed in $L(\mathbb{F}_2)$ and since free group factors can be embedded in the ultrapower R^ω of the hyperfinite II_1 factor, there are bounded sequences $\langle y_k\rangle_{k=1}^{\infty}$ and

$\langle d_k\rangle_{k=1}^{\infty}$ such that $y_k, d_k \in M_k(\mathbb{C})$ and such that the pair y_k, d_k converges in $*$–moments to the pair $Z, \widetilde{Y}$. Letting $\widetilde{z}_k = y_k + \epsilon d_k$, we have that $\widetilde{z}_k$ converges in $*$–moments to $\widetilde{Z}_\epsilon$ as $k \to \infty$. By Theorem 7 of [13], there is a sequence $\langle z_{k,\epsilon}\rangle_{k=1}^{\infty}$ with $z_{k,\epsilon} \in M_k(\mathbb{C})$ such that $\|z_{k,\epsilon} - \widetilde{z}_{k,\epsilon}\|$ tends to zero and the eigenvalue distribution of $z_{k,\epsilon}$ converges weakly as $k \to \infty$ to the Brown measure of $\widetilde{Z}_\epsilon$, namely, to σ_ϵ. $\square$

Suppose that $\lambda = \langle\lambda_j\rangle_{j=1}^{k}$ is a finite sequence of complex numbers. For each j, write $\lambda_j = a_j + ib_j$, $a_j, b_j \in \mathbb{R}$. Define $Q_\epsilon = \prod_{j=1}^{k}[a_j - \epsilon, a_j + \epsilon]$ and $R_\epsilon = \prod_{j=1}^{k}[b_j - \epsilon, b_j + \epsilon]$. Set

$$E_\epsilon(\lambda) = \int_{R_\epsilon}\left(\int_{Q_\epsilon}\prod_{\substack{1\le i,j\le k\\ i\ne j}}\left(|s_i - s_j| + |t_i - t_j|^2\right)^{1/2} ds\right)dt,$$

where $ds = ds_1 \cdots ds_k$ and $dt = dt_1 \cdots dt_k$.
The following lemma proves lower bounds for certain asymptotics of the quantities $E_\epsilon(\lambda)$. We will apply this lemma to the case when λ is the eigenvalue sequence of matrices like the $z_{k,\epsilon}$ found in Lemma 2.2.

Lemma 2.3. *Let μ and c be as in Lemma 2.1. For each $\epsilon > 0$ and $k \in \mathbb{N}$, let $\lambda^{(k,\epsilon)} = \langle\lambda_1^{(k,\epsilon)}, \ldots, \lambda_{n(k)}^{(k,\epsilon)}\rangle$ be a finite sequence of complex numbers and assume that for every $\epsilon > 0$,*

$$\sup_{k\in\mathbb{N},\,1\le j\le n(k)} |\lambda_j^{(k,\epsilon)}| < \infty$$

and the probability measures

$$\frac{1}{n(k)}\sum_{j=1}^{n(k)}\delta_{\lambda_j^{(k,\epsilon)}} \tag{9}$$

converge weakly to the measure σ_ϵ of Lemma 2.1 as $k \to \infty$. Let

$$f(\epsilon) = \liminf_{k\to\infty} n(k)^{-2}\log(E_\epsilon(\lambda^{(k,\epsilon)})).$$

Then

$$\liminf_{\epsilon\to 0}\left(\frac{f(\epsilon)}{|\log\epsilon|}\right) \ge 0. \tag{10}$$

Proof. Note that we must have $n(k) \to \infty$ as $k \to \infty$. Given $\epsilon > 0$ small, take $1 \ge \delta > 3\epsilon$. Define

$$W_{k,\epsilon} = \{(i,j) \in \{1,\ldots,n(k)\}^2 \mid i \ne j,\, |\lambda_i^{(k,\epsilon)} - \lambda_j^{(k,\epsilon)}| < \delta\}.$$

Writing for each $1 \le j \le k$, $\lambda_j^{(k,\epsilon)} = a_j + ib_j$ where $a_j, b_j \in \mathbb{R}$ define $Q_{\epsilon,k} =$

$\prod_{j=1}^{n(k)}[a_j-\epsilon,a_j+\epsilon]$, $R_{\epsilon,k}=\prod_{j=1}^{n(k)}[b_j-\epsilon,b_j+\epsilon]$, and $K_{\epsilon,k}=Q_{\epsilon,k}\times R_{\epsilon,k}$. Now

$$\begin{aligned}
E_\epsilon(\lambda^{(k,\epsilon)}) &= \int_{K_{\epsilon,k}}\prod_{i\neq j}(|s_i-s_j|^2+|t_i-t_j|^2)^{1/2}dsdt\\
&\geq (\delta-\sqrt{8}\epsilon)^{n(k)^2-\#W_{k,\epsilon}}\int_{K_{\epsilon,k}}\prod_{(i,j)\in W_{k,\epsilon}}(|s_i-s_j|^2+|t_i-t_j|^2)^{1/2}dsdt\\
&\geq (\delta-3\epsilon)^{n(k)^2-\#W_{k,\epsilon}}\left(\int_{Q_{\epsilon,k}}\prod_{(i,j)\in W_{k,\epsilon}}|s_i-s_j|ds\right)\cdot\\
&\qquad\qquad\left(\int_{R_{\epsilon,k}}\prod_{(i,j)\in W_{k,\epsilon}}|t_i-t_j|dt\right),
\end{aligned}$$

where $ds=ds_1\cdots ds_{n(k)}$ and $dt=dt_1\cdots dt_{n(k)}$.
We now wish to find a lower bounds for the two integrals in the above expression. By Fubini's Theorem we can assume $a_1\leq a_2\leq\cdots\leq a_{n(k)}$. Let

$$[-\epsilon,\epsilon]_<^{n(k)}=\{(x_1,\dots,x_{n(k)})\in[-\epsilon,\epsilon]^{n(k)}\mid x_1<x_2<\cdots<x_{n(k)}\}.$$

Then by the change of variables $[-\epsilon,\epsilon]_<^{n(k)}\ni(x_1,\dots,x_{n(k)})\mapsto(a_1+x_1,\dots,a_{n(k)}+x_{n(k)})\in Q_{\epsilon,k}$ and Selberg's Integral Formula it follows that

$$\begin{aligned}
\int_{Q_{\epsilon,k}}\prod_{(i,j)\in W_{k,\epsilon}}|s_i-s_j|ds &\geq \int_{[-\epsilon,\epsilon]_<^{n(k)}}\prod_{(i,j)\in W_{k,\epsilon}}|x_i-x_j|dx_1\cdots dx_{n(k)}\\
&\geq (2\epsilon)^{-(n(k)^2-n(k)-\#W_{k,\epsilon})}\cdot\int_{[-\epsilon,\epsilon]_<^{n(k)}}\prod_{i\neq j}|x_i-x_j|dx_1\cdots dx_{n(k)}\\
&= \frac{(2\epsilon)^{-(n(k)^2-n(k)-\#W_{k,\epsilon})}}{n(k)!}\cdot\int_{[-\epsilon,\epsilon]^{n(k)}}\prod_{i\neq j}|x_i-x_j|dx_1\cdots dx_{n(k)}\\
&= \frac{(2\epsilon)^{n(k)+\#W_{k,\epsilon}}}{n(k)!}\cdot\prod_{j=0}^{n(k)-1}\frac{\Gamma(j+2)\Gamma(j+1)^2}{\Gamma(n(k)+j+1)},
\end{aligned}$$

The same lower bound applies to $\int_{R_{\epsilon,k}}\prod_{(i,j)\in W_{k,\epsilon}}|t_i-t_j|dt$ so that combining these two we get

$$\begin{aligned}
E_\epsilon(\lambda^{(k,\epsilon)}) &\geq (\delta-3\epsilon)^{n(k)^2-\#W_{k,\epsilon}}\left(\frac{(2\epsilon)^{n(k)+\#W_{k,\epsilon}}}{n(k)!}\cdot\prod_{j=0}^{n(k)-1}\frac{\Gamma(j+2)\Gamma(j+1)^2}{\Gamma(n(k)+j+1)}\right)^2\\
&\geq (\delta-3\epsilon)^{n(k)^2}\left(\frac{(2\epsilon)^{n(k)+\#W_{k,\epsilon}}}{n(k)!}\cdot\prod_{j=0}^{n(k)-1}\frac{\Gamma(j+2)\Gamma(j+1)^2}{\Gamma(n(k)+j+1)}\right)^2.
\end{aligned}$$

Using

$$\lim_{k\to\infty} n(k)^{-2}\log(\prod_{j=0}^{n(k)-1}\frac{\Gamma(j+2)\Gamma(j+1)^2}{\Gamma(n(k)+j+1)}) = -2\log 2,$$

we find

$$f(\epsilon) \geq \log(\delta - 3\epsilon) + 2\log(2\epsilon)\limsup_{k\to\infty}\frac{\#W_{k,\epsilon}}{n(k)^2} - 4\log 2.$$

Since the measures (9) converge weakly to σ_ϵ, by standard approximation techniques one sees

$$\lim_{k\to\infty}\frac{\#W_{k,\epsilon}}{n(k)^2} = (\sigma_\epsilon \times \sigma_\epsilon)(X_\delta),$$

where X_δ is as in Lemma 2.1. As $\epsilon \to 0$ choose $\delta = \frac{1}{|\log\epsilon|}$, so that $\delta^2\log(1+a\epsilon^{-2}) \to 0$ for all $a > 0$ and $\frac{\epsilon}{\delta} \to 0$ and $\frac{\log\delta}{\log\epsilon} \to 0$. Using the upper bound (3) and the fact that ν is diffuse, we get

$$\lim_{\epsilon\to 0}(\sigma_\epsilon \times \sigma_\epsilon)(X_\delta) = 0.$$

Now one easily verifies that (10) holds. $\square$

3 The Main Result

Before beginning the main result first a few comments on a packing formulation for microstates free entropy dimension are in order. If $X = \{x_1, \ldots, x_n\}$ is an n-tuple of selfadjoint elements in a tracial von Neumann algebra, then the free entropy dimension (as defined by Voiculescu [17]) is given by the formula

$$\delta_0(X) = n + \limsup_{\epsilon\to 0}\frac{\chi(x_1+\epsilon s_1, \ldots, x_n+\epsilon s_n : s_1, \ldots, s_n)}{|\log\epsilon|}$$

where $\{s_1, \ldots, s_n\}$ is a semicircular family free from X. The packing formulation found in [8] and modified slightly in [10] (to remove the norm restriction on microstates), is

$$\delta_0(X) = \limsup_{\epsilon\to 0}\frac{\mathbb{P}_\epsilon(X)}{|\log\epsilon|},$$

where

$$\mathbb{P}_\epsilon(X) = \inf_{m\in\mathbb{N},\,\gamma>0}\limsup_{k\to\infty} k^{-2}\log P_\epsilon(\Gamma(X; m, k, \gamma)). \tag{11}$$

Here, $\Gamma(X; m, k, \gamma) \subseteq (M_k(\mathbb{C})_{s.a.})^n$ is the microstate space of Voiculescu [16], but taken without norm restriction, as considered in [3], and P_ϵ is the packing number with respect to the metric arising from the normalized trace.

Let $Y = \{y_1, \ldots, y_n\}$ be an arbitrary n-tuple of (possibly nonselfadjoint) elements in a tracial von Neumann algebra. Now the definition of $\mathbb{P}_\epsilon$ makes perfect sense for the set Y if we replace the microstate space in (11) with the non-selfadjoint $*$-microstate space $\Gamma(Y; m, k, \gamma) \subseteq (M_k(\mathbb{C}))^n$, which is the set of all n–tuples of $k \times k$ matrices whose $*$–moments up to order m approximate those of Y within tolerance of γ. Let us (temporarily) denote the quantity so obtained by $\overline{\mathbb{P}_\epsilon}(Y)$ and define

$$\overline{\delta_0}(Y) = \limsup_{\epsilon \to 0} \frac{\overline{\mathbb{P}_\epsilon}(Y)}{|\log \epsilon|}. \tag{12}$$

It is easy to see that if X is a set of selfadjoints, then $\overline{\mathbb{P}_\epsilon}(X) \geq \mathbb{P}_\epsilon(X) \geq \overline{\mathbb{P}_{2\epsilon}}(X)$ and that in the nonselfadjoint setting the quantity (12) is a $*$-algebraic invariant, so that

$$\begin{aligned}
\delta_0(\mathrm{Re}(y_1),&\mathrm{Im}(y_1), \ldots, \mathrm{Re}(y_n), \mathrm{Im}(y_n)) = \\
&= \limsup_{\epsilon \to 0} \frac{\mathbb{P}_\epsilon(\mathrm{Re}(y_1), \mathrm{Im}(y_1), \ldots, \mathrm{Re}(y_n), \mathrm{Im}(y_n))}{|\log \epsilon|} \\
&= \limsup_{\epsilon \to 0} \frac{\overline{\mathbb{P}_\epsilon}(\mathrm{Re}(y_1), \mathrm{Im}(y_1), \ldots, \mathrm{Re}(y_n), \mathrm{Im}(y_n))}{|\log \epsilon|} \\
&= \limsup_{\epsilon \to 0} \frac{\overline{\mathbb{P}_\epsilon}(Y)}{|\log \epsilon|} = \overline{\delta_0}(Y),
\end{aligned}$$

where $\mathrm{Re}(y_i)$ and $\mathrm{Im}(y_i)$ are the real and imaginary parts of y_i. Moreover, if X is set of selfadjoints, then

$$\delta_0(X) = \limsup_{\epsilon \to 0} \frac{\mathbb{P}_\epsilon(X)}{|\log \epsilon|} = \limsup_{\epsilon \to 0} \frac{\overline{\mathbb{P}_\epsilon}(X)}{|\log \epsilon|} = \overline{\delta_0}(X).$$

The following notational conventions, which will be used in the remainder of this paper, are, therefore, justified: for any finite set of operators Y (selfadjoint or otherwise) in a tracial von Neumann algebra we will write $\mathbb{P}_\epsilon(Y)$ for the packing quantity derived from the nonselfadjoint microstates (that was denoted $\overline{\mathbb{P}_\epsilon}(Y)$ above) and we will write $\delta_0(Y)$ for the free entropy dimension of Y that was denoted $\overline{\delta_0}(Y)$ above.
In the proof of the main result, we will use $E_\epsilon(A)$ for $A \in M_k(\mathbb{C})$ to mean $E_\epsilon(\lambda)$, where $\lambda = \langle \lambda_j \rangle_{j=1}^k$ are the eigenvalues of A listed according to general multiplicity (see the description immediately before Lemma 2.2). Notice that this is independent of the choice of λ since $E_\epsilon(\lambda \circ \sigma) = E_\epsilon(\lambda)$ for any permutation σ of $\{1, \ldots, k\}$.

THEOREM 3.1. *Let Z be a $\mathrm{DT}(\mu, c)$–operator, for any compactly supported Borel probability measure μ on the complex plane and any $c > 0$. Then $\delta_0(Z) = 2$.*

Proof. Obviously $\delta_0(Z) \leq 2$ so it suffices to show the reverse inequality.
We may without loss of generality assume $c = 1$ (see Proposition 2.12 of [6]). Fix $N \in \mathbb{N}$ with $N \geq 2$. By Theorem 4.12 of [6],

$$\begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1N} \\ 0 & B_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & B_{N-1,N} \\ 0 & \cdots & 0 & B_{NN} \end{bmatrix} \in \mathcal{M} \otimes M_N(\mathbb{C}) \tag{13}$$

is a $\mathrm{DT}(\mu,1)$–operator where $\{B_{11},\ldots,B_{NN}\}\cup\langle B_{ij}\rangle_{1\le i<j\le N}$ is a $*$-free family in $\mathcal{M}$, the B_{ii} are $\mathrm{DT}(\mu,\frac{1}{\sqrt{N}})$–operators, and each B_{ij} is circular with $\varphi(|B_{ij}^2|)=\frac{1}{N}$. From this we see that finding microstates for Z is equivalent to finding microstates for the operator (13) in $\mathcal{M}\otimes M_N(\mathbb{C})$.

Consider the sequence $\langle y_k\rangle_{k=1}^\infty$ constructed in Lemma 3.2 and for each $\epsilon>0$ small enough, the corresponding sequence $\langle z_{k,\epsilon}\rangle_{k=1}^\infty$. Let $R>1$, $m\in\mathbb{N}$, $\gamma>0$ and take $\gamma'=\gamma/16^m(R+1)^m>0$. By Corollary 2.11 of [19] there exist $k\times k$ complex unitary matrices $u_{1k},u_{2k},\ldots,u_{kk}$ such that $\{u_{1k}y_ku_{1k}^*,\ldots,u_{Nk}y_ku_{Nk}^*\}$ is an (m,γ')–$*$–free family in $M_k(\mathbb{C})$. Also,by an application of Corollary 2.14 of [19], there exists a set $\Omega_k\subset\Gamma_R(\langle B_{ij}\rangle_{1\le i<j\le N};m,k,\gamma')$ such that for any $\langle\eta_{ij}\rangle_{1\le i<j\le N}\in\Omega_k$,

$$\{u_{1k}y_ku_{1k}^*,\ldots,u_{Nk}y_ku_{Nk}^*\}\cup\langle\eta_{ij}\rangle_{1\le i<j\le N}$$

is an (m,γ')-$*$ free family and such that

$$\liminf_{k\to\infty}\left(k^{-2}\cdot\log(\mathrm{vol}(\Omega_k))+\frac{N(N-1)}{2}\cdot\log k\right)\ge$$
$$\ge\chi(\langle\mathrm{Re}B_{ij}\rangle_{1\le i<j\le N},\langle\mathrm{Im}B_{ij}\rangle_{1\le i<j\le N})>-\infty,$$

where the volume is computed with respect to the product of the Euclidean norm $k^{1/2}|\cdot|_2$. Since the operator (13) is a copy of Z, for any $\langle\eta_{ij}\rangle_{1\le i<j\le N}\in\Omega_k$ we have

$$\begin{bmatrix} u_{1k}y_ku_{1k}^* & \eta_{12} & \cdots & \eta_{1N}\\ 0 & u_{2k}y_2u_{2k}^* & \ddots & \vdots\\ \vdots & \ddots & \ddots & \eta_{N-1,N}\\ 0 & \cdots & 0 & u_{Nk}y_ku_{Nk}^*\end{bmatrix}\in\Gamma(Z;m,Nk,\gamma).$$

Because every complex matrix can be put into an upper-triangular form with respect to an orthonormal basis, we can find for each $1\le j\le N$, a $k\times k$ unitary matrix v_{jk} such that $v_{jk}u_{jk}z_{k,\epsilon}u_{jk}^*v_{jk}^*$ is upper triangular. Observe now that for any $\langle\eta_{ij}\rangle_{1\le i<j\le n}\in\Omega_k$, the product of matrices

$$\begin{bmatrix} v_{1k} & 0 & \cdots & 0\\ 0 & v_{2k} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & v_{Nk}\end{bmatrix}\begin{bmatrix} u_{1k}y_ku_{1k}^* & \eta_{12} & \cdots & \eta_{1N}\\ 0 & u_{2k}y_ku_{2k}^* & \ddots & \vdots\\ \vdots & \ddots & \ddots & \eta_{N-1,N}\\ 0 & \cdots & 0 & u_{Nk}y_ku_{Nk}^*\end{bmatrix}\cdot$$
$$\cdot\begin{bmatrix} v_{1k}^* & 0 & \cdots & 0\\ 0 & v_{2k}^* & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & v_{Nk}^*.\end{bmatrix}$$

is also an element of $\Gamma(Z; m, Nk, \gamma)$ and is equal to

$$\begin{bmatrix} v_{1k}u_{1k}y_k u_{1k}^* v_{1k}^* & v_{1k}\eta_{12}v_{2k}^* & \cdots & v_{1j}\eta_{1N}v_{2k}^* \\ 0 & v_{2j}u_{2k}y_k u_{2k}^* v_{2k}^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & v_{(N-1),k}\eta_{N-1,N}v_{Nk}^* \\ 0 & \cdots & 0 & v_{Nk}u_{Nk}y_k u_{Nk}^* v_{Nk}^* \end{bmatrix}.$$

Moreover,

$$|v_{jk}u_{jk}z_{k,\epsilon}u_{jk}^* v_{jk}^* - v_{jk}u_{jk}y_k u_{jk}^* v_{jk}^*|_2 = |z_{k,\epsilon} - y_k|_2$$

and $\limsup_{k\to\infty} |z_{k,\epsilon} - y_k|_2 \leq \epsilon/\sqrt{N}$. Therefore, for k sufficiently large and for each $1 \leq j \leq N$ we have $|v_{jk}u_{jk}z_{k,\epsilon}u_{jk}^* v_{jk}^* - v_{jk}u_{jk}y_k u_{jk}^* v_{jk}^*|_2 \leq \epsilon$. Set $d_{jk} = v_{jk}u_{jk}z_{k,\epsilon}u_{jk}^* v_{jk}^*$, and denote by G_k the set of all $Nk \times Nk$ matrices of the form

$$\begin{bmatrix} d_{1k} & v_{1k}\eta_{12}v_{2k}^* & \cdots & v_{1j}\eta_{1N}v_{Nk}^* \\ 0 & d_{2k} & \ddots & \vdots \\ \vdots & \ddots & \ddots & v_{(N-1),k}\eta_{N-1,N}v_{Nk}^* \\ 0 & \cdots & 0 & d_{Nk} \end{bmatrix}$$

where $\langle \eta_{ij} \rangle_{1\leq i<j\leq N} \in \Omega_k$. Notice that each d_{jk} is upper triangular and its eigenvalue distribution is exactly the same as that of $z_{k,\epsilon}$. For k sufficiently large, the set G_k lies in the ϵ-neighborhood of $\Gamma(Z; m, Nk, \gamma)$. Let $\theta(G_k)$ denote the unitary orbit of G_k in $M_{Nk}(\mathbb{C})$. We will now find lower bounds for the ϵ-packing numbers of $\theta(G_k)$ and thus, ones for $\Gamma(Z; m, Nk, \gamma)$.
Denote by $H_k \subset M_{Nk}(\mathbb{C})$ all matrices of the form

$$\begin{bmatrix} 0 & v_{1k}\eta_{12}v_{2k}^* & \cdots & v_{1j}\eta_{1N}v_{Nk}^* \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & v_{(N-1),k}\eta_{N-1,N}v_{Nk}^* \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

where $\langle \eta_{ij} \rangle_{1\leq i<j\leq N} \in \Omega_k$. Notice that H_k is isometric to the space of all matrices of the form

$$\begin{bmatrix} 0 & \eta_{12} & \cdots & \eta_{1N} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \eta_{N-1,N} \\ 0 & \cdot & \cdots & 0 \end{bmatrix}$$

where $\langle \eta_{ij} \rangle_{1\leq i<j\leq N} \in \Omega_k$. It follows that H_k must also have the same volume as the above subspace, computed in the obvious ambient Hilbert space of block upper triangular matrices obeying the above decomposition. Recall that for $n \in \mathbb{N}$, $T_n(\mathbb{C})$ denotes the set of uppertriangular matrices in $M_n(\mathbb{C})$; let $T_{n,<}(\mathbb{C})$ denote the matrices in $T_n(\mathbb{C})$ that have zero diagonal, i.e. the strictly upper triangular matrices in $M_n(\mathbb{C})$. Denote by W_k the subset of $T_{Nk,<}(\mathbb{C})$ consisting of all matrices x such that $|x|_2 < \epsilon$ and $x_{ij} = 0$ whenever $1 \leq p < q \leq N$ and $(p-1)k < i \leq pk$ and $(q-1)k < j \leq qk$. Thus, W_k consists of $N \times N$ diagonal matrices whose diagonal entries are strictly upper triangular $k \times k$ matrices. Denote by D_k the subset of diagonal matrices x of $M_{Nk}(\mathbb{C})$ such that $|x|_2 < \epsilon\sqrt{2}$. It follows that if f_k is the matrix

$$\begin{bmatrix} d_{1k} & 0 & \cdots & 0 \\ 0 & d_{2k} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & d_{Nk} \end{bmatrix}$$

then $f_k + D_k + W_k + H_k \subset \mathcal{N}_{3\epsilon}(G_k)$, where the 3ϵ neighborhood is taken in the ambient space $T_{Nk}(\mathbb{C})$ with respect to the metric induced by $|\cdot|_2$. Now observe that the space of diagonal $Nk \times Nk$ matrices and $T_{Nk,<}(\mathbb{C})$ are orthogonal subspaces of $T_{Nk}(\mathbb{C})$. Let $\theta_{3\epsilon}(G_k)$ denote the 3ϵ–neighborhood of the unitary orbit $\theta(G_k)$ of G_k. Thus, denoting by dX Lebesgue measure on $T_{Nk}(\mathbb{C})$ where $X = \langle x_{ij}\rangle_{1\leq i\leq j\leq k}$, using Dyson's formula we have

$$\begin{aligned} \mathrm{vol}(\theta_{3\epsilon}(G_k)) &\geq C_{Nk} \cdot \int_{f_k+D_k+W_k+H_k} \prod_{1\leq i<j\leq Nk} |x_{ii} - x_{jj}|^2 dX \\ &= C_{Nk} \cdot \mathrm{vol}(W_k + H_k) \cdot \int_{f_k+D_k} \prod_{1\leq i<j\leq Nk} |x_{ii} - x_{jj}|^2 dx_{11} \cdots dx_{(Nk)(Nk)} \\ &\geq C_{Nk} \cdot \mathrm{vol}(W_k + H_k) \cdot E_\epsilon(z_{k,\epsilon} \otimes I_N), \end{aligned} \tag{14}$$

where the constant C_{Nk} is as in 2 and where $\mathrm{vol}(\theta_{3\epsilon}(G_k))$ is computed in $M_{Nk}(\mathbb{C})$ and $\mathrm{vol}(W_k + H_k)$ is computed in $T_{Nk,<}(\mathbb{C})$, both being Euclidean volumes corresponding to the norms $(Nk)^{1/2}|\cdot|_2$. Clearly $\theta_{3\epsilon}(G_k) \subset \mathcal{N}_{4\epsilon}(\Gamma(Z; m, Nk, \gamma))$, so (14) gives a lower bound on $\mathrm{vol}(\mathcal{N}_{4\epsilon}(\Gamma(Z; m, Nk, \gamma))$ as well.
Using (14) and the standard volume comparison test, we have

$$\begin{aligned} P_\epsilon(\Gamma(Z;m, Nk, \gamma)) &\geq \frac{\mathrm{vol}(\mathcal{N}_{4\epsilon}(\Gamma(Z; m, Nk, \gamma)))}{\mathrm{vol}(\mathcal{B}_{6\epsilon})} \\ &\geq C_{Nk} \cdot E_\epsilon(z_{k,\epsilon} \otimes I_N) \cdot \mathrm{vol}(W_k + H_k) \cdot \frac{\Gamma((Nk)^2 + 1)}{\pi^{(Nk)^2}(6(Nk)^{1/2}\epsilon)^{2(Nk)^2}}, \end{aligned}$$

where $\mathcal{B}_{6\epsilon}$ is a ball in $M_{Nk}(\mathbb{C})$ of radius 6ϵ with respect to $|\cdot|_2$, and we are computing volumes corresponding to the Euclidean norm $(Nk)^{1/2}|\cdot|_2$. Since W_k and H_k are orthogonal, we have $\mathrm{vol}(W_k + H_k) = \mathrm{vol}(W_k)\mathrm{vol}(H_k)$, where each volume is taken in the subspace of appropriate dimension. But W_k is a ball of radius $(Nk)^{1/2}\epsilon$ in a space of real dimension $Nk(k-1)$, so

$$\mathrm{vol}(W_k + H_k) = \frac{\pi^{\frac{Nk(k-1)}{2}}((Nk)^{1/2}\epsilon)^{Nk(k-1)}}{\Gamma(\frac{Nk(k-1)}{2} + 1)} \cdot (N^{1/2})^{k^2N(N-1)}\mathrm{vol}(\Omega_k).$$

Applying Stirling's formula, we find

$$\begin{aligned}
\mathbb{P}_\epsilon(Z;m,\gamma) &\geq \liminf_{k\to\infty}(Nk)^{-2}\log P_\epsilon(\Gamma(Z;m,Nk,\gamma)) \\
&\geq \liminf_{k\to\infty}(Nk)^{-2}\log(E_\epsilon(z_{k,\epsilon}\otimes I_N)) \\
&\quad +\liminf_{k\to\infty}\Bigg((Nk)^{-2}\log(C_{Nk})+\frac{1}{2N}\log k+\frac{1}{N}\log\epsilon \\
&\qquad -\frac{1}{2N}\log(\frac{Nk(k-1)}{2})+\log((Nk)^2)-\log k \\
&\qquad -2\log\epsilon+(Nk)^{-2}\log(\mathrm{vol}(\Omega_k))\Bigg)+K_1 \\
&= \liminf_{k\to\infty}(Nk)^{-2}\log(E_\epsilon(z_{k,\epsilon}\otimes I_N)) \\
&\quad +\liminf_{k\to\infty}\left((Nk)^{-2}\log(C_{Nk})+\frac{1}{2}\log k\right) \\
&\quad +\liminf_{k\to\infty}\left((Nk)^{-2}\log(\mathrm{vol}(\Omega_k))+(\frac{1}{2}-\frac{1}{2N})\log k\right) \\
&\quad +(2-N^{-1})|\log\epsilon|+K_2 \\
&= \liminf_{k\to\infty}(Nk)^{-2}\log(E_\epsilon(z_{k,\epsilon}\otimes I_N)) \\
&\quad +N^{-2}\chi(\langle \mathrm{Re}B_{ij}\rangle_{1\leq i<j\leq N},\langle \mathrm{Im}B_{ij}\rangle_{1\leq i<j\leq N}) \\
&\quad +(2-N^{-1})|\log\epsilon|+K_3,
\end{aligned}$$

where K_1, K_2 and K_3 are constants independent of ϵ, m and γ. Taking $m\to\infty$ and $\gamma\to 0$, we get

$$\begin{aligned}
\mathbb{P}_\epsilon(Z) &\geq \liminf_{k\to\infty}(Nk)^{-2}\log(E_\epsilon(z_{k,\epsilon}\otimes I_N)) \\
&\quad +N^{-2}\chi(\langle \mathrm{Re}B_{ij}\rangle_{1\leq i<j\leq N},\langle \mathrm{Im}B_{ij}\rangle_{1\leq i<j\leq N}) \\
&\quad +(2-N^{-1})|\log\epsilon|+K_3.
\end{aligned}$$

Since the eigenvalue distribution of $z_{k,\epsilon}\otimes I_N$ converges as $k\to\infty$ to the measure σ_ϵ of Lemma 2.1, dividing by $|\log\epsilon|$ and applying Lemma 2.3 now yields

$$\delta_0(Z)=\limsup_{\epsilon\to 0}\frac{\mathbb{P}_\epsilon(Z)}{|\log\epsilon|}\geq\liminf_{\epsilon\to 0}\frac{f(\epsilon)}{|\log\epsilon|}+2-N^{-1}\geq 2-N^{-1}.$$

Since N was arbitrary, it follows that $\delta_0(Z)\geq 2$, thereby completing the proof. □

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