Cohomology of Arithmetic Groups with Infinite Dimensional Coefficient Spaces

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Abstract. The cuspidal cohomology groups of arithmetic groups in certain infinite dimensional modules are computed. As a result we get a simultaneous generalization of the Patterson-Conjecture and the Lewis-Correspondence.

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Introduction

Let G be a semisimple Lie group and $\Gamma \subset G$ an arithmetic subgroup. For a finite dimensional representation (ρ, E) of G the cohomology groups $H^\bullet(\Gamma, E)$ are related to automorphic forms and have for this reason been studied by many authors. The case of infinite dimensional representations has only very recently come into focus, mostly in connection with the Patterson Conjecture on the divisor of the Selberg zeta function [7, 8, 9, 11, 17]. In this paper we want to show that the Patterson conjecture [7] is related to the Lewis correspondence [21], i.e., that the multiplicities of automorphic representations can be expressed in terms of cohomology groups with certain infinite dimensional coefficient spaces.

One way to put (a special case of) the Patterson conjecture for *cocompact torsion-free* Γ in a split group G is to say that the multiplicity $N_\Gamma(\pi)$ of an irreducible unitary principal series representation π in the space $L^2(\Gamma\backslash G)$ is given by

$$N_\Gamma(\pi) \ = \ \dim H^{d-r}(\Gamma, \pi^\omega),$$

where r is the rank of G and π^ω is the subspace of analytic vectors in π, finally, $d = \dim(G/K)$ is the dimension of the symmetric space attached to G, where K is a maximal compact subgroup.

Our main result states that this assertion can be generalized to all arithmetic groups provided the ordinary group cohomology is replaced by the cuspidal cohomology. It will probably also work for more general lattices, but we stick to arihmetic groups, because some of the constructions used in this paper, like the Borel-Serre compactification, or the decomposition of the regular G-representation on the space $L^2(\Gamma\backslash G)$, have in the literature only been formulated for arithmetic groups. The relation to the Lewis correspondence is as follows. In [25] Don Zagier states that the correspondence for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ can be interpreted as the identity

$$N_\Gamma(\pi) \ = \ \dim H^1_{par}(\Gamma, \pi^{\omega/2}),$$

where π is as before, $\pi^{\omega/2}$ is a slightly bigger space than π^ω and H^1_{par} is the parabolic cohomology. Since π is a unitary principal series representation it follows that $N_\Gamma(\pi)$ coincides with the multiplicity of π in the cuspidal part $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ of $L^2(\Gamma\backslash G)$. More precisely, the correspondence gives an isomorphism

$$\mathrm{Hom}_G\left(\pi, L^2_{\mathrm{cusp}}(\Gamma\backslash G)\right) \to H^1_{par}(\Gamma, \pi^{\omega/2}).$$

As a consequence of our main result we will get the following theorem.

Theorem 0.1 *For every Fuchsian group* Γ *we have*

$$N_\Gamma(\pi) \ = \ \dim H^1_{\mathrm{cusp}}(\Gamma, \pi^\omega_{it}).$$

Here $H^\bullet_{\mathrm{cusp}}$ is the cuspidal cohomology. For finite dimensional modules the cuspidal cohomology is a subspace of the parabolic cohomology.
The following is our main theorem.

Theorem 0.2 *Let* Γ *be a torsion-free arithmetic subgroup of a split semisimple Lie group* G*. Let* $\pi \in \hat{G}$ *be an irreducible unitary principal series representation. Then*

$$N_\Gamma(\pi) \ = \ \dim H^{d-r}_{\mathrm{cusp}}(\Gamma, \pi^\omega),$$

where $d = \dim G/K$ *and* r *is the real rank of* G*.*
For G *non-split the assertion remains true for a generic set of representations* π*.*

This raises many questions. For a finite dimensional representation E it is known that the cuspidal cohomology is a subspace of the parabolic cohomology. The same assertion for infinite dimensional E is wrong in general, see Corollary 5.2. Can one characterize those infinite dimensional E for which the cuspidal cohomology indeed injects into ordinary cohomology?
Another question suggests itself: in which sense does our construction in the case $\mathrm{PSL}_2(\mathbb{Z})$ coincide with the Lewis correspondence? To even formulate a conjecture we must assume two further conjectures. First assume that in the relevant cases cuspidal and parabolic cohomology coincide; next assume

that the parabolic cohomology with coefficients in π^ω agrees with parabolic cohomology in $\pi^{\omega/2}$. Let M_λ be the space of cusp forms of eigenvalue λ. Then our construction gives a map into the dual space of the cohomology, $\alpha : M_\lambda \to H^1_{par}(\Gamma, \pi^\omega)^*$. The Lewis construction on the other hand gives a map $\beta : M_\lambda \to H^1_{par}(\Gamma, \pi^\omega)$. Together they define a duality on M_λ. One is tempted to speculate that this duality coincides with the natural duality given by the integral on the upper half plane. If that were so, then the two maps α and β would determine each other.

1 Fuchsian groups

Let G be the group $\mathrm{SL}_2(\mathbb{R})/\pm 1$. For $s \in \mathbb{C}$ let π_s denote the principal series representation with parameter s. Recall that this representation can be viewed as the regular representation on the space of square integrable sections of a line bundle over $\mathbb{P}^1(\mathbb{R}) \cong G/P$, where P is the subgroup of upper triangular matrices. For $s \in i\mathbb{R}$ this representation will be irreducible unitary. For any admissible representation π of G let π^ω denote the space of analytic vectors in π. Then π^ω is a locally convex vector space with continuous G-representation ([18], p. 463). Let $\pi^{-\omega}$ be its continuous dual. For $\pi = \pi_s$ the space π_s^ω is the space of analytic sections of a line bundle over $\mathbb{P}^1(\mathbb{R})$. Let $\pi_s^{\omega/2}$ denote the space of sections which are smooth everywhere and analytic up to the possible exception of finitely many points. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})/\pm 1$ be the modular group. For an irreducible representation π of G let $N_\Gamma(\pi)$ be its multiplicity in $L^2(\Gamma\backslash G)$. Let $H^1_{par}(\Gamma, \pi_s^\omega)$ denote the *parabolic cohomology*, i.e., the subspace of $H^1(\Gamma, \pi_s^\omega)$ generated by all cocycles μ which vanish on parabolic elements. For the group $\Gamma = \mathrm{SL}_2(\mathbb{Z})/\pm 1$ this means that H^1_{par} consists of all cohomology classes which have a representing cocycle μ with $\mu\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 0$. In [25] D. Zagier stated that for $s \in i\mathbb{R}$,

$$N_\Gamma(\pi) \;=\; \dim H^1_{par}(\Gamma, \pi_s^{\omega/2}).$$

We will first relate this to the Patterson Conjecture for the cocompact case.

Theorem 1.1 *Let $\Gamma \subset G$ be a discrete, cocompact and torsion-free subgroup, then for $s \in i\mathbb{R}$,*

$$N_\Gamma(\pi_s) \;=\; \dim H^1_{par}(\Gamma, \pi_s^\omega) \;=\; \dim H^1(\Gamma, \pi_s^\omega).$$

Proof: Since Γ does not contain parabolic elements the parabolic cohomology coincides with the ordinary group cohomology. The Patterson Conjecture [7, 12] shows that

$$N_\Gamma(\pi_s) \;=\; \dim H^1(\Gamma, \pi_s^{-\omega}) - 2\dim H^2(\Gamma, \pi_s^{-\omega}).$$

Poincaré duality [8] implies that the dimension of the space $H^j(\Gamma, \pi_s^\omega)$ equals the dimension of $H^{2-j}(\Gamma, \pi_s^{-\omega})$. The Theorem follows from the next lemma.

Lemma 1.2 *For every Fuchsian group we have $H^0(\Gamma, \pi_s^\omega) = 0$.*

Proof: For this recall that every $f \in \pi_s^\omega$ is a continuous function on G satisfying among other things, $f(nx) = f(x)$ for every $n \in N$, where N is the unipotent group of all matrices modulo ± 1 which are upper triangular with ones on the diagonal. If f is Γ-invariant, then $f \in C(G/\Gamma)$. By Moore's Theorem ([26], Thm. 2.2.6) it follows that the action of N on G/Γ is ergodic. In particular, this implies that f must be constant. Since $s \in i\mathbb{R}$ this implies that $f = 0$. $\square$

2 Arbitrary arithmetic groups

Throughout, let G be a semisimple Lie group with finite center and finitely many connected components.
Let Γ be an arithmetic subgroup of G and assume that Γ is torsion-free. Then Γ is the fundamental group of $\Gamma\backslash X$, where $X = G/K$ the symmetric space and every Γ-module M induces a local system or locally constant sheaf $\mathcal{M}$ on $\Gamma\backslash X$. In the étale picture the sheaf $\mathcal{M}$ equals $\mathcal{M} = \Gamma\backslash(X \times M)$, (diagonal action). Let $\overline{\Gamma\backslash X}$ denote the Borel-Serre compactification [3] of $\Gamma\backslash X$, then Γ also is the fundamental group of $\overline{\Gamma\backslash X}$ and M induces a sheaf also denoted by $\mathcal{M}$ on $\overline{\Gamma\backslash X}$. This notation is consistent as the sheaf on $\Gamma\backslash X$ is indeed the restriction of the one on $\overline{\Gamma\backslash X}$. Let $\partial(\Gamma\backslash X)$ denote the boundary of the Borel-Serre compactification. We have natural identifications

$$H^j(\Gamma, M) \;\cong\; H^j(\Gamma\backslash X, \mathcal{M}) \;\cong H^j(\overline{\Gamma\backslash X}, \mathcal{M}).$$

We define the *parabolic cohomology* of a Γ-module M to be the kernel of the restriction to the boundary, ie,

$$H^j_{par}(\Gamma, M) \;\stackrel{\text{def}}{=}\; \ker\left(H^j(\overline{\Gamma\backslash X}, \mathcal{M}) \to H^j(\partial(\Gamma\backslash X), \mathcal{M})\right).$$

The long exact sequence of the pair $(\overline{\Gamma\backslash X}, \partial(\Gamma\backslash X))$ gives rise to

$$\dots \to H^j_c(\Gamma\backslash X, \mathcal{M}) \to H^j(\Gamma\backslash X, \mathcal{M}) =$$

$$= H^j(\overline{\Gamma\backslash X}, \mathcal{M}) \to H^j(\partial(\Gamma\backslash X), \mathcal{M}) \to \dots$$

The image of the cohomology with compact supports under the natural map is called the *interior cohomology* of $\Gamma\backslash X$ and is denoted by $H^j_!(\Gamma\backslash X, \mathcal{M})$. The exactness of the above sequence shows that

$$H^j_{par}(\Gamma, M) \;\cong\; H^j_!(\Gamma\backslash X, \mathcal{M}).$$

Let E be a locally convex space. We shall write E' for its topological dual. We assume that Γ acts linearly and continuously on E. We will present a natural complex that computes the cohomology $H^\bullet(\Gamma, E)$.

Let $\mathcal{E}_\Gamma$ be the locally constant sheaf on $\Gamma\backslash X$ given by E. Then $\mathcal{E}_\Gamma$ has stalk E and $H^\bullet(\Gamma, E) = H^\bullet(X_\Gamma, \mathcal{E}_\Gamma)$.
Let $\Omega_\Gamma^0, \dots, \Omega_\Gamma^d$ be the sheaves of differential forms on X_Γ and let $\mathcal{E}_\Gamma^p$ be the sheaf locally given by

$$\mathcal{E}_\Gamma^p(U) \;=\; \Omega_\Gamma^p(U)\hat{\otimes}\mathcal{E}_\Gamma(U),$$

where $\hat{\otimes}$ denotes the completion of the algebraic tensor product $\otimes$ in the projective topology. Write $X_\Gamma = \Gamma\backslash G/K = \Gamma\backslash X$. Let d denote the exterior differential. Then $D = d \otimes 1$ is a differential on $\mathcal{E}_\Gamma^\bullet$ and

$$0 \to \mathcal{E}_\Gamma \xrightarrow{D} \mathcal{E}_\Gamma^0 \xrightarrow{D} \cdots \xrightarrow{D} \mathcal{E}_\Gamma^d \to 0$$

is a fine resolution of $\mathcal{E}_\Gamma$. Hence $H^\bullet(X_\Gamma, \mathcal{E}_\Gamma) = H^\bullet(X_\Gamma, \mathcal{E}_\Gamma^\bullet)$.
Let $\Omega^\bullet(X)$ be the space of differential forms on X. The complex $\mathcal{E}_\Gamma^\bullet(X_\Gamma)$ is isomorphic to the space of Γ-invariants $(\Omega^\bullet(X)\hat{\otimes}E)^\Gamma$. So we get

$$H^\bullet(\Gamma, E) \cong H^\bullet\left((\Omega^\bullet(X)\hat{\otimes}E)^\Gamma\right).$$

We can write

$$\Omega^p(X) \;=\; (C^\infty(G) \otimes \wedge^p\mathfrak{p}^*)^K,$$

where $\mathfrak{p}$ is the positive part in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{g}$ is the complexified Lie algebra of G and $\mathfrak{k}$ is the complexified Lie algebra of K. The group K acts on $\mathfrak{p}^*$ via the coadjoint representation and on $C^\infty(G)$ via right translations and Γ, or more precisely G, acts by left translations on $C^\infty(G)$.
From now on we assume that E is not only a Γ-module but is a topological vector space that carries a continuous G-representation. We say that E is *admissible* if every K-isotype $E(\tau)$, $\tau \in \hat{K}$ is finite dimensional. Let E^∞ denote the subspace of smooth vectors. We say that E is smooth if $E = E^\infty$. We then have

$$(C^\infty(G) \otimes \wedge^p\mathfrak{p}^*)\hat{\otimes}E \cong C^\infty(G)\hat{\otimes}(E \otimes \wedge^p\mathfrak{p}^*)$$

as a $G \times K$-module, where G acts diagonally on $C^\infty(G)$ by left translations and on E by the given representation. The group K acts diagonally on $C^\infty(G)$ by right translations and on $\wedge^p\mathfrak{p}^*$ via the coadjoint action.

Lemma 2.1 *For any locally convex complete topological vector space F we have*

$$C^\infty(G)\hat{\otimes}F \;\cong\; C^\infty(G, F),$$

where the right hand side denotes the space of all smooth maps from G to F.

Proof: See [14], Example 1 after Theorem 13. □

Thus we have a $G \times K$-action on the space $C^\infty(G, \wedge^p\mathfrak{p}^* \otimes E)$ given by

$$(g, k).f \;=\; (\mathrm{Ad}^*(k) \otimes g)\, L_g\, R_k\, f,$$

where $L_g f(x) = f(g^{-1}x)$ and $R_k f(x) = f(xk)$.
The map

$$\psi\colon C^\infty(G, \wedge^p \mathfrak{p}^* \otimes E) \ \to \ C^\infty(G, \wedge^p \mathfrak{p}^* \otimes E)$$

given by

$$\psi(f)(x) = (1 \otimes x^{-1}).f(x)$$

is an isomorphism to the same space with a different the $G \times K$ structure. Indeed, one computes,

$$\begin{aligned}
\psi((g,k).f)(x) &= (1 \otimes x^{-1})(g,k).f(x) \\
&= (1 \otimes x^{-1})(\mathrm{Ad}^*(k) \otimes g) f(g^{-1}xk) \\
&= (\mathrm{Ad}^*(k) \otimes k(g^{-1}xk)^{-1}) f(g^{-1}xk) \\
&= (\mathrm{Ad}^*(k) \otimes k) R_k L_g \psi(f)(x).
\end{aligned}$$

For a smooth G-representation F we write $H^\bullet(\mathfrak{g}, K, F)$ for the cohomology of the standard complex of $(\mathfrak{g}, K)$-cohomology [5]. Then $H^\bullet(\mathfrak{g}, K, F) = H^\bullet(\mathfrak{g}, K, F_K)$, where F_K is the $(\mathfrak{g}, K)$-module of K-finite vectors in F.
If we assume that E is smooth, we get from this

$$H^\bullet(\Gamma, E) \ = \ H^\bullet(\mathfrak{g}, K, C^\infty(\Gamma\backslash G)\hat{\otimes} E).$$

In the case of finite dimensional E one can replace $C^\infty(\Gamma\backslash G)$ with the space of functions of moderate growth [6]. This is of importance, since it leads to a decomposition of the cohomology space into the cuspidal part and contributions from the parabolic subgroups. To prove this, one starts with differential forms of moderate growth and applies ψ. For infinite dimensional E this proof does not work, since it is not clear that ψ should preserve moderate growth, even if one knows that the matrix coefficients of E have moderate growth.
By the Sobolev Lemma the space of smooth vectors $L^2(\Gamma\backslash G)^\infty$ of the natural unitary representation of G on $L^2(\Gamma\backslash G)$ is a subspace of $C^\infty(\Gamma\backslash G)$. The representation $L^2(\Gamma\backslash G)$ splits as $L^2(\Gamma\backslash G) = L^2_{disc} \oplus L^2_{cont}$, where $L^2_{disc} = \bigoplus_{\pi\in\hat{G}} N_\Gamma(\pi)\pi$ is a direct Hilbert sum of irreducible representations and L^2_{cont} is a finite sum of continuous Hilbert integrals. The space of cusp forms $L^2_{\mathrm{cusp}}(\Gamma\backslash G) = \bigoplus_{\pi\in\hat{G}} N_{\Gamma,\mathrm{cusp}}(\pi)\pi$ is a subspace of L^2_{disc}. Note that $L^2_{\mathrm{cusp}}(\Gamma\backslash G)^\infty$ is a closed subspace of $C^\infty(G)$. The *cuspidal cohomology* is defined by

$$H^\bullet_{\mathrm{cusp}}(\Gamma, E) \ = \ H^\bullet(\mathfrak{g}, K, L^2_{\mathrm{cusp}}(\Gamma\backslash G)^\infty \hat{\otimes} E).$$

For finite dimensional E it turns out that $H^\bullet_{\mathrm{cusp}}(\Gamma, E)$ coincides with the image in $H^\bullet(\mathfrak{g}, K, C^\infty(\Gamma\backslash G)\hat{\otimes} E)$ under the inclusion map. This comes about as a consequence of the fact that the cohomology can also be computed using functions of uniform moderate growth and that in the space of such functions, $L^2_{\mathrm{cusp}}(\Gamma\backslash G)^\infty$ has a G-complement. The Borel-conjecture [13] is a refinement of this assertion. For infinite dimensional E this injectivity does not hold in general, see Corollary 5.2.

We define the *reduced cuspidal cohomology* to be the image $\tilde{H}^\bullet_{\rm cusp}(\Gamma, E)$ of $H^\bullet_{\rm cusp}(\Gamma, E)$ in $H^\bullet(\Gamma, E)$. Finally, let $H_{(2)}(\Gamma, E)$ be the image of the space $H^\bullet(\mathfrak{g}, K, L^2(\Gamma\backslash G)^\infty \hat{\otimes} E)$ in $H^\bullet(\mathfrak{g}, K, C^\infty(\Gamma\backslash G)\hat{\otimes} E)$.

Proposition 2.2 *We have the following inclusions of cohomology groups,*

$$\tilde{H}^\bullet_{\rm cusp}(\Gamma, E) \ \subset \ H^\bullet_{par}(\Gamma, E) \ \subset \ H^\bullet_{(2)}(\Gamma, E).$$

Proof: The cuspidal condition ensures that every cuspidal class vanishes on each homology class of the boundary. This implies the first conclusion. Since every parabolic class has a compactly supported representative, the second also follows. □

3 Gelfand Duality

Recall that a *Harish-Chandra module* is a $(\mathfrak{g}, K)$-module which is admissible and finitely generated. Every Harish-Chandra module is of finite length. For a Harish-Chandra module V write $\tilde{V}$ for its dual, ie, $\tilde{V} = (V^*)_K$, the K-finite vectors in the algebraic dual.
A *globalization* of a Harish-Chandra module V is a continuous representation of G on a complete locally convex vector space W such that V is isomorphic to the $(\mathfrak{g}, K)$-module of K-finite vectors W_K. It was shown in [18] that there is a minimal globalization $V^{\min}$ and a maximal globalization $V^{\max}$ such that for every globalization W there are unique functorial continuous linear G-maps

$$V^{\min} \ \hookrightarrow \ W \ \hookrightarrow \ V^{\max}.$$

The spaces $V^{\min}$ and $V^{\max}$ are given explicitly by

$$V^{\min} \ = \ C_c^\infty(G) \otimes_{\mathfrak{g},K} V$$

and

$$V^{\max} \ = \ \mathrm{Hom}_{\mathfrak{g},K}(\tilde{V}, C^\infty(G)).$$

The action of G on V^{max} is given by

$$g.\alpha(\tilde{v})(x) \ = \ \alpha(\tilde{v})(g^{-1}x).$$

Let $\hat{G}$ be the unitary dual of G, i.e., the set of all isomorphism classes of irreducible unitary representations of G. Note [18] that for $\pi \in \hat{G}$ we have $(\pi_K)^{\min} = \pi^\omega$ and $(\pi_K)^{\max} = \pi^{-\omega}$.
The following is a key result of this paper.

Theorem 3.1 *Let F be a smooth G-representation on a complete locally convex topological vector space. Then there is a functorial isomorphism*

$$H^\bullet(\mathfrak{g}, K, F\hat{\otimes} V^{\max}) \ \to \ \mathrm{Ext}^\bullet_{\mathfrak{g},K}(\tilde{V}, F),$$

where as usual one writes $\mathrm{Ext}^\bullet_{\mathfrak{g},K}(\tilde{V}, F)$ *for* $\mathrm{Ext}^\bullet_{\mathfrak{g},K}(\tilde{V}, F_K)$.

Proof: We have

$$\begin{aligned} F\hat{\otimes}V^{\max} &= F\hat{\otimes}\mathrm{Hom}_{\mathfrak{g},K}(\tilde{V},C^\infty(G)) \\ &= \mathrm{Hom}_{\mathfrak{g},K}(\tilde{V},F\hat{\otimes}C^\infty(G)) \\ &= \mathrm{Hom}_{\mathfrak{g},K}(\tilde{V},C^\infty(G,F)). \end{aligned}$$

Lemma 3.2 *The map*

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{g},K}\left(\tilde{V},C^\infty(G,F)\right)^{\mathfrak{g},K} &\to \mathrm{Hom}_{\mathfrak{g},K}\left(\tilde{V},F\right) \\ \phi &\mapsto \alpha, \end{aligned}$$

with $\alpha(\tilde{v})=\phi(\tilde{v})(1)$ is an isomorphism.

Proof: Note that ϕ satisfies

$$\phi(X.\tilde{v})(x) \;=\; \frac{d}{dt}\phi(\tilde{v})(x\,\exp(tX))\Big|_{t=0}, \qquad X\in\mathfrak{g},$$

since it is a $(\mathfrak{g},K)$-homomorphism. Further,

$$\frac{d}{dt}\phi(\tilde{v})(\exp(tX)\,x)\Big|_{t=0} \;=\; X.\phi(\tilde{v})(x), \qquad X\in\mathfrak{g},$$

since ϕ is $(\mathfrak{g},K)$-invariant. Similar identities hold for the K-action. This implies that α is a $(\mathfrak{g},K)$-homomorphism. Note that the $(\mathfrak{g},K)$-invariance of ϕ also leads to

$$\phi(\tilde{v})(gx) \;=\; g.\phi(\tilde{v})(x), \qquad g,x\in G.$$

Hence if $\alpha=0$ then $\phi=0$ so the map is injective. For surjectivity let α be given and define ϕ by $\phi(\tilde{v})(x)=x.\alpha(\tilde{v})$. Then ϕ maps to α. □

By the Lemma we get an isomorphism

$$H^0(\mathfrak{g},K,F\hat{\otimes}V^{\max}) \;\cong\; \mathrm{Hom}_{\mathfrak{g},K}(\tilde{V},F) \;\cong\; \mathrm{Ext}^0_{\mathfrak{g},K}(\tilde{V},F)$$

and thus a functorial isomorphism on the zeroth level. We will show that both sides in the theorem define universal δ-functors [19]. From this the theorem will follow. Fix V and let $S^j(F)=H^j(\mathfrak{g},K,F\hat{\otimes}V^{\max})$ as well as $T^j(F)=\mathrm{Ext}^j_{\mathfrak{g},K}(\tilde{V},F)$. We will show that $S^\bullet$ and $T^\bullet$ are universal δ-functors from the category $Rep_s^\infty(G)$ defined below to the category of complex vector spaces. The objects of $Rep_s^\infty(G)$ are smooth continuous representations of G on Hausdorff locally convex topological vector spaces and the morphisms are *strong morphisms*. A continuous G-morphism $f:A\to B$ is called strong morphism or *s-morphism* if (a) $\ker f$ and $\mathrm{im} f$ are closed topological direct summands and (b) f induces an isomorphism of $A/\ker f$ onto $f(A)$. Then by [5], Chapter IX, the category $Rep_s^\infty(G)$ is an abelian category with enough injectives. In fact, for $F\in Rep_s^\infty(G)$ the map $F\to C^\infty(G,F)$ mapping f to the

function $\alpha(x) = x.f$ is a monomorphism into the s-injective object $C^\infty(G, F)$ (cf. [5], Lemma IX.5.2), which is considered a G-module via $x\alpha(y) = \alpha(yx)$.
Let us consider $S^\bullet$ first. By Corollary IX.5.6 of [5] we have

$$S^\bullet(F) \;=\; H^\bullet(\mathfrak{g}, K, F\hat{\otimes}V^{\max}) \;\cong\; H_d^\bullet(G, F\hat{\otimes}V^{\max}),$$

where the right hand side is the differentiable cohomology. The functor $.\hat{\otimes}V^{\max}$ is s-exact and therefore $S^\bullet$ is a δ-functor. We show that it is erasable. For this it suffices to show that $S^j(C^\infty(G, F)) = 0$ for $j > 0$. Now

$$C^\infty(G, F)\hat{\otimes}V^{\max} \;\cong\; C^\infty(G)\hat{\otimes}F\hat{\otimes}V^{\max} \;\cong\; C^\infty(G, F\hat{\otimes}V^{\max})$$

and therefore for $j > 0$,

$$S^j(C^\infty(G, F)) \;\cong\; H_d^j(G, C^\infty(G, F\hat{\otimes}V^{\max})) \;=\; 0,$$

since $C^\infty(G, F\hat{\otimes}V^{\max})$ is s-injective. Thus $S^\bullet$ is erasable and therefore universal.
Next consider $T^\bullet(F) = \mathrm{Ext}^\bullet_{\mathfrak{g},K}(\tilde{V}, F)$. Since an exact sequence of smooth representations gives an exact sequence of $(\mathfrak{g}, K)$-modules, it follows that $T^\bullet$ is a δ-functor.To show that it is erasable let $j > 0$. Then

$$\begin{aligned}
T^j(C^\infty(G, F)) &= \mathrm{Ext}^j_{\mathfrak{g},K}(\tilde{V}, C^\infty(G)\hat{\otimes}F) \\
&= H^j(\mathfrak{g}, K, \mathrm{Hom}_{\mathbb{C}}(\tilde{V}, C^\infty(G))\hat{\otimes}F) \\
&= H_d^j(G, \mathrm{Hom}_{\mathbb{C}}(\tilde{V}, C^\infty(G))\hat{\otimes}F) \\
&= H_d^j(G, \mathrm{Hom}_{\mathbb{C}}(\tilde{V}, C^\infty(G)))\hat{\otimes}F \\
&= \mathrm{Ext}^j_{\mathfrak{g},K}(\tilde{V}, C^\infty(G))\hat{\otimes}F.
\end{aligned}$$

By Theorem 6.13 of [18] we have $\mathrm{Ext}^j_{\mathfrak{g},K}(\tilde{V}, C^\infty(G)) = 0$. The Theorem is proven. $\square$

Choosing $C^\infty(\Gamma\backslash G)$ and $L^2_{\mathrm{cusp}}(\Gamma\backslash G)^\infty$ for F in Theorem 3.1 gives the following Corollary.

COROLLARY 3.3 (i)

$$H^p(\Gamma, V^{\max}) \;\cong\; \mathrm{Ext}^p_{\mathfrak{g},K}(\tilde{V}, C^\infty(\Gamma\backslash G)).$$

For Γ *cocompact and* $p = 0$ *this is known under the name* Gelfand Duality.

(ii)

$$H^\bullet_{\mathrm{cusp}}(\Gamma, V^{\max}) \;\cong\; \mathrm{Ext}^\bullet_{\mathfrak{g},K}(\tilde{V}, L^2_{\mathrm{cusp}}(\Gamma\backslash G)^\infty).$$

4 The case of the maximal globalization

The space of cusp forms decomposes discretely,

$$L^2_{\rm cusp}(\Gamma\backslash G) \;=\; \bigoplus_{\pi\in\hat G} N_{\Gamma,{\rm cusp}}(\pi)\pi.$$

Suppose that V has an infinitesimal character χ. Let $\hat G(\chi)$ be the set of all irreducible unitary representations of G with infinitesimal character χ. It is easy to see that

$$\begin{aligned}
{\rm Ext}^\bullet_{\mathfrak g,K}(\tilde V, L^2_{\rm cusp}(\Gamma\backslash G)^\infty) &= {\rm Ext}^\bullet_{\mathfrak g,K}\left(\tilde V, \bigoplus_{\pi\in\hat G(\chi)} N_{\Gamma,{\rm cusp}}(\pi)\pi_K\right)\\
&= \bigoplus_{\pi\in\hat G(\chi)} N_{\Gamma,{\rm cusp}}(\pi)\,{\rm Ext}^\bullet_{\mathfrak g,K}\left(\tilde V,\pi_K\right)\\
&= \bigoplus_{\pi\in\hat G(\chi)} N_{\Gamma,{\rm cusp}}(\pi)\,{\rm Ext}^\bullet_{\mathfrak g,K}\left(\tilde\pi_K, V\right)
\end{aligned}$$

The last line follows by dualizing.
Let P be a parabolic subgroup and $\mathfrak m\oplus\mathfrak a\oplus\mathfrak n$ a Langlands decomposition of its Lie algebra.

Lemma 4.1 *For a $(\mathfrak g,K)$-module π and a $(\mathfrak a\oplus\mathfrak m,K_M)$-module U we have*

$$\operatorname{Hom}_{\mathfrak g,K}(\pi,\operatorname{Ind}_P^G(U)) \;\cong\; \operatorname{Hom}_{\mathfrak a\oplus\mathfrak m,K_M}(H_0(\mathfrak n,\pi), U\otimes\mathbb C_{\rho_P}),$$

where $\mathbb C_{\rho_P}$ is the one dimensional A-module given by ρ_P.

Proof: See [15] page 101. □

Lemma 4.2 *Let $\mathcal C$ be an abelian category with enough injectives. Let $\mathfrak a$ be a finite dimensional abelian complex Lie algebra and let T be a covariant left exact functor from $\mathcal C$ to the category of $\mathfrak a$-modules. Assume that T maps injectives to $\mathfrak a$-acyclics and that T has finite cohomological dimension, i.e., that $R^pT=0$ for p large. Let M be an object of $\mathcal C$ such that $R^pT(M)$ is finite dimensional for every p. Let $H_{\mathfrak a}$ denote the functor $H^0(\mathfrak a,\cdot)$. Then*

$$\sum_{p\ge0}\binom{p}{r}(-1)^{p+r}\,\dim R^p(H_{\mathfrak a}\circ T)(M) \;=\; \sum_{p\ge0}(-1)^p\,\dim H^0(\mathfrak a, R^pT(M)),$$

where $r=\dim\mathfrak a$. If M is T-acyclic, then these alternating sums degenerate to

$$\dim R^r(H_{\mathfrak a}\circ T)(M) \;=\; \dim H^0(\mathfrak a, T(M)).$$

Proof: Split $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{b}_1$, where $\dim \mathfrak{a}_1 = 1$. Consider the Grothendieck spectral sequence with $E_2^{p,q} = H^p(\mathfrak{a}_1, R^q(H_{\mathfrak{b}_1} \circ T)(M))$ which abuts to $R^{p+q}(H_{\mathfrak{a}} \circ T)(M)$ (see [19], Theorem XX.9.6). Since $\mathfrak{a}_1$ is one dimensional, for any finite dimensional $\mathfrak{a}_1$-module V we have $H^0(\mathfrak{a}, V) \cong H^1(\mathfrak{a}, V)$ and $H^p(\mathfrak{a}_1, V) = 0$ if $p > 1$. This implies $E_2^{0,q} \cong E_2^{1,q}$ and $E_2^{p,q} = 0$ for $p \notin \{0,1\}$. The spectral sequence therefore degenerates and

$$\begin{aligned}
& \sum_{p\geq 0} \binom{p}{r} (-1)^{p+r} \dim R^p(H_{\mathfrak{a}} \circ T)(M) \\
= & \sum_{p\geq 0} \binom{p}{r} (-1)^{p+r} \dim E_2^{0,p} + \sum_{p\geq 1} \binom{p}{r} (-1)^{p+r} \dim E_2^{1,p-1} \\
= & \sum_{p\geq 0} \binom{p}{r} (-1)^{p+r} \dim E_2^{0,p} + \sum_{p\geq 0} \binom{p+1}{r} (-1)^{p+r-1} \dim E_2^{1,p} \\
= & \sum_{p\geq 0} \left(\binom{p+1}{r} - \binom{p}{r} \right) (-1)^{p+r-1} \dim E_2^{0,p} \\
= & \sum_{p\geq 0} \binom{p}{r-1} (-1)^{p+r-1} \dim E_2^{0,p} \\
= & \sum_{p\geq 0} \binom{p}{r-1} (-1)^{p+r-1} \dim H^0(\mathfrak{a}_1, R^p(H_{\mathfrak{b}_1} \circ T)(M)).
\end{aligned}$$

Next we split $\mathfrak{b}_1 = \mathfrak{a}_2 \oplus \mathfrak{b}_2$, where $\mathfrak{a}_2$ is one-dimensional. Since the $\mathfrak{a}_1$-action commutes with the a_2-action the isomorphism

$$H^0(\mathfrak{a}_2, R^p(H_{\mathfrak{b}_2} \circ T)(M)) \;\cong\; H^1(\mathfrak{a}_2, R^p(H_{\mathfrak{b}_2} \circ T)(M))$$

is an $\mathfrak{a}_1$-isomorphism. Therefore we apply the same argument to get down to

$$\sum_{p\geq 0} \binom{p}{r-2} (-1)^{p+r-2} \dim H^0(\mathfrak{a}_1 \oplus \mathfrak{a}_2, R^p(H_{\mathfrak{b}_2} \circ T)(M)).$$

Iteration gives the claim.
To get the last assertion of the lemma, note that if M is T-acyclic, then following the inductive argument above, one sees that $R^p(H_{\mathfrak{a}} \circ T(M) = 0$ for $p > r$. □

Let P be a minimal parabolic subgroup of G so that M is compact. For a unitary irreducible representation σ of M and linear functional $\nu \in i\mathfrak{a}^*$ we obtain the unitary principal series representation $\pi_{\sigma,\nu}$ of G induced from P. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{m} = \mathrm{Lie}_{\mathbb{C}}(M)$. Then $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\Lambda_\sigma \in \mathfrak{t}^*$ be a representative of the infinitesimal character of σ. Then $\Lambda_\sigma + \nu \in \mathfrak{h}^*$ is a representative of the infinitesimal character of $\pi_{\sigma,\nu}$.

We say that the parameters (σ,ν) are *generic* if $\pi_{\sigma,\nu}$ is irreducible and for any two w, w' in the Weyl group of $(\mathfrak{h},\mathfrak{g})$ the linear functional

$$w(\Lambda_\sigma+\nu)|_{\mathfrak{a}} - w'(\Lambda_\sigma+\nu)|_{\mathfrak{a}}$$

on $\mathfrak{a}$ is not a positive integer linear combination of positive roots.

Theorem 4.3 *If G is $\mathbb{R}$-split and $\pi_{\sigma,\nu}$ is irreducible, we have*

$$N_\Gamma(\pi_{\sigma,\nu}) \;=\; H^r_{\rm cusp}(\Gamma,\pi^{-\omega}_{\sigma,\nu}).$$

If G is not split, the assertions remains true if the parameters (σ,ν) are generic.

Proof: First note that since G is split, the decomposition of $L^2(\Gamma\backslash G)$ as in [20, 1] implies that for imaginary ν one has $N_\Gamma(\pi_{\sigma,\nu}) = N_{\Gamma,{\rm cusp}}(\pi_{\sigma,\nu})$, since the Eisenstein series are regular at imaginary ν. Applying Lemma 4.1 with $\pi = L^2_{\rm cusp}(\Gamma\backslash G)^\infty$ and ${\rm Ind}_P^G(U) = \pi_{\sigma,\nu}$ we find

$$\begin{aligned}
&{\rm Hom}_{\mathfrak{a}\oplus\mathfrak{m},M}\left(H_0(\mathfrak{n}, L^2_{\rm cusp}(\Gamma\backslash G)^\infty_K), \sigma\otimes(\nu+\rho_P)\right)\\
&\qquad\cong\; {\rm Hom}_{\mathfrak{g},K}\left(L^2_{\rm cusp}(\Gamma\backslash G)^\infty, \pi_{\sigma,\nu}\right)\\
&\qquad\cong\; {\rm Hom}_{\mathfrak{g},K}\left(\tilde\pi_{\sigma,\nu}, L^2_{\rm cusp}(\Gamma\backslash G)^\infty\right)
\end{aligned}$$

so that

$$N_{\Gamma,{\rm cusp}}(\pi_{\sigma,\nu}) = \dim {\rm Hom}_{\mathfrak{a}\oplus\mathfrak{m},M}\left(H_0(\mathfrak{n}, L^2_{\rm cusp}(\Gamma\backslash G)^\infty_K), \sigma\otimes(\nu+\rho_P)\right).$$

In order to calculate the latter we apply Lemma 4.2 to the category $\mathcal{C}$ of $(\mathfrak{g},K)$ modules and

$$\begin{aligned}
T(V) &= {\rm Hom}_M(H_0(\mathfrak{n},\tilde V), U\otimes\mathbb{C}_{\rho_P})\\
&= (H_0(\mathfrak{n},\tilde V)^*\otimes U\otimes\mathbb{C}_{\rho_P})^M.
\end{aligned}$$

The conditions of the Lemma 4.2 are easily seen to be satisfied since $H_0(\mathfrak{n},\cdot)$ maps injectives to injectives and $H^0(M,\cdot)$ is exact. Note that in the case of a representation π of G,

$$\begin{aligned}
H_{\mathfrak{a}}\circ T(\pi) &\cong {\rm Hom}_{\mathfrak{a}\oplus\mathfrak{m},M}(H_0(\mathfrak{n},\tilde\pi_K), U\otimes\mathbb{C}_{\rho_P})\\
&\cong {\rm Hom}_{\mathfrak{g},K}(\tilde\pi, {\rm Ind}_P^G(U))
\end{aligned}$$

by Lemma 4.1. From this we obtain

$${\rm Ext}^j_{\mathfrak{g},K}(\tilde\pi, {\rm Ind}_P^G(U)) = R^p(H_{\mathfrak{a}}\circ T)(\pi).$$

Now Lemma 4.2 shows that

$$\dim {\rm Ext}^r_{\mathfrak{g},K}(\tilde\pi, {\rm Ind}_P^G(U))$$

equals

$$\dim \mathrm{Hom}_{AM}(H_0(\mathfrak{n},\tilde{\pi}_K), U\otimes \mathbb{C}_{\rho_P}).$$

Suppose that π is an irreducible summand in $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$. If G is split, the set of π which share the same infinitesimal character as $\pi_{\sigma,\nu}$ equals the set of all $\pi_{\xi,w\nu}$, where w ranges over the Weyl group and $\xi\in\hat{M}$. Then the space $\mathrm{Hom}_{AM}(H_j(\mathfrak{n},\tilde{\pi}_K), U\otimes\mathbb{C}_{\rho_P})$ is only non-zero for $\pi=\pi_{\xi,w\nu}$ for some w. But then Proposition 2.32 of [15] implies that $H_j(\mathfrak{n},\tilde{\pi}_K)$ is zero unless $j=0$. The same conclusion is assured in the non-split case by the genericity condition. Now the proof is completed by the following calculation:

$$\begin{aligned}
N_{\Gamma,\mathrm{cusp}}(\pi_{\sigma,\nu}) &= \dim \mathrm{Hom}_{\mathfrak{a}\oplus\mathfrak{m},M}(H_0(\mathfrak{n},L^2_{\mathrm{cusp}}(\Gamma\backslash G)^\infty_K),\sigma\otimes(\nu+\rho_P))\\
&= \dim \mathrm{Ext}^r_{\mathfrak{g},K}(L^2_{\mathrm{cusp}}(\Gamma\backslash G),\pi_{\sigma,\nu})\\
&= \dim \mathrm{Ext}^r_{\mathfrak{g},K}(\tilde{\pi}_{\sigma,\nu},L^2_{\mathrm{cusp}}(\Gamma\backslash G))\\
&= \dim H^r_{\mathrm{cusp}}(\Gamma,\pi^{-\omega}_{\sigma,\nu}),
\end{aligned}$$

where the last equality is a consequence of Corollary 3.3(ii), applied to $V=\pi_{\sigma,\nu}$. □

5 Poincaré duality

In order to conclude the main Theorem it suffices to prove the following Poincaré duality.

Theorem 5.1 *(Poincaré duality)*
For every Harish-Chandra module V,

$$H^j_{\mathrm{cusp}}(\Gamma,V^{\max}) \;\cong\; H^{d-j}_{\mathrm{cusp}}(\Gamma,\tilde{V}^{\min})^*,$$

and both spaces are finite dimensional.

Before we prove the theorem, we add a Corollary.

Corollary 5.2 *Let Γ be a torsion-free non-uniform lattice in $G=\mathrm{PSL}_2(\mathbb{R})$ and $\pi\in\hat{G}$ a principal series representation with $N_{\Gamma,\mathrm{cusp}}(\pi)\neq 0$. Let $E=\tilde{\pi}_K^{min}$. Then the natural map $H^\bullet_{\mathrm{cusp}}(\Gamma,E)\to H^\bullet(\Gamma,E)$ is not injective.*

Proof of the Corollary: We have $H^0_{\mathrm{cusp}}(\Gamma,\pi^{max})\neq 0$ and by the Poincaré duality, $H^2_{\mathrm{cusp}}(\Gamma,E)\neq 0$. However, as the cohomological dimension of Γ is 1, it follows that $H^2(\Gamma,E)=0$. □

Proof of the Theorem: A *duality* between two complex vector spaces E,F is a bilinear pairing,

$$\langle .,.\rangle : E\times F \;\to\; \mathbb{C}$$

which is *non-degenerate*, i.e., for every $e \in E$ and every $f \in F$,

$$\langle e, F\rangle = 0 \quad \Rightarrow \quad e = 0,$$
$$\langle E, f\rangle = 0 \quad \Rightarrow \quad f = 0.$$

We say that E and F are *in duality* if there is a duality between them. Note that if E and F are in duality and one of them is finite dimensional, then the other also is and their dimensions agree. The pairing is called *perfect* if it induces isomorphisms $E \cong F^*$ and $F \cong E^*$. If E and F are topological vector spaces then the pairing is called *topologically perfect* if it induces topological isomorphisms $E \cong F'$ and $F \cong E'$, where the dual spaces are equipped with the strong dual topology.

Now suppose that V and W are $\mathfrak{g}, K$-modules in duality through a $\mathfrak{g}, K$-invariant pairing. Recall the canonical complex defining $\mathfrak{g}, K$-cohomology which is given by $C^q(V) = \mathrm{Hom}_K(\wedge^q(\mathfrak{g}/\mathfrak{k}), V) = (\wedge^q(\mathfrak{g}/\mathfrak{k})^* \otimes V)^K$. Let $d = \dim G/K$. The prescription $\langle y \otimes v, y' \otimes w\rangle = (-1)^q \langle v, w\rangle\, y \wedge y'$ gives a pairing from $C^q(V) \times C^{d-q}(W)$ to $\wedge^d(\mathfrak{g}/\mathfrak{k})^* \cong \mathbb{C}$. Let $d : C^q \to C^{q+1}$ be the differential, then one sees [5], $\langle da, b\rangle = \langle a, db\rangle$.

Let π be an irreducible unitary representation of G. Then the spaces π^∞ and $\tilde{\pi}^{-\infty}$ are each other's strong duals [10]. The same holds for $V^{\max}$ and $\tilde{V}^{\min}$ [18].

LEMMA 5.3 *The spaces $A = \pi^{-\infty} \hat{\otimes} V^{\max}$ and $B = \tilde{\pi}^\infty \hat{\otimes} \tilde{V}^{\min}$ are each other's strong duals. Both of them are LF-spaces.*

PROOF: Since $C^\infty(G)$ is nuclear and Fréchet and $\tilde{\pi}$ is a Hilbert space the results of [24], §III.50, allow us to conclude that $C^\infty(G, \tilde{\pi})' = C^\infty(G) \hat{\otimes} \tilde{\pi}$ is nuclear which is then true also for $C^\infty(G, \tilde{\pi})$. Now the embedding of $\tilde{\pi}^\infty$ into $C^\infty(G, \tilde{\pi})$ shows the nuclearity of $\tilde{\pi}^\infty$.

Since $\tilde{V}$ is finitely generated one can embed the space

$$V^{\max} = \mathrm{Hom}_{\mathfrak{g},K}(\tilde{V}, C^\infty(G))$$

into a strict inductive limit $\varinjlim_j \mathrm{Hom}(\tilde{V}^j, C^\infty(G))$ with finite dimensional V^j's. Then the nuclearity of $V^{\max}$ follows from the nuclearity of

$$\mathrm{Hom}(\tilde{V}^j, C^\infty(G)) = (\tilde{V}^j)^* \otimes C^\infty(G).$$

We conclude that the spaces $V^{\max}$ and $\tilde{\pi}^\infty$ are nuclear Fréchet spaces. Their duals $\pi^{-\infty}$ and $\tilde{V}^{\min}$ are LF-spaces (see [14], Introduction IV). Therefore they all are barreled ([22], p. 61). By [22], p. 119 we know that the inductive completions of the tensor products $\pi^{-\infty} \bar{\otimes} V^{\max}$ and $\pi^\infty \bar{\otimes} V^{\min}$ are barreled. Since $V^{\max}$ and π^∞ are nuclear, these inductive completions coincide with the projective completions. So A and B are barreled. By Theorem 14 of [14] it follows that the strong duals A' and B' are complete and by the Corollary to Lemma 9 of [14] it follows that $A' = B$ and $B' = A$. Finally, Lemma 9 of [14] implies that A and B are LF-spaces. $\square$

Proposition 5.4 *For every $\pi \in \hat{G}$ and every Harish-Chandra module V the vector spaces $H^q(\mathfrak{g}, K, \pi^{-\infty}\hat{\otimes} V^{\max})$ and $H^q(\mathfrak{g}, K, \tilde{\pi}^{\infty}\hat{\otimes}\tilde{V}^{\min})$ are finite dimensional. The above pairing between their canonical complexes induces a duality between them, so*

$$H^q(\mathfrak{g}, K, \pi^{-\infty}\hat{\otimes} V^{\max}) \;\cong\; H^{d-q}(\mathfrak{g}, K, \tilde{\pi}^{\infty}\hat{\otimes}\tilde{V}^{\min})^*.$$

Proof: Note that by Theorem 3.1,

$$H^q(\mathfrak{g}, K, \pi^{-\infty}\hat{\otimes} V^{\max}) \cong \mathrm{Ext}^q_{\mathfrak{g},K}(\tilde{V}, \pi^{-\infty}) \cong \mathrm{Ext}^q_{\mathfrak{g},K}(\tilde{V}, \pi_K)$$

and the latter space is finite dimensional ([5], Proposition I.2.8). The proposition will thus follow from the next lemma.

Lemma 5.5 *Let A, B be smooth representations of G. Suppose that A and B are LF-spaces and that they are in perfect topological duality through a G-invariant pairing. Assume that $H^\bullet(\mathfrak{g}, K, A)$ is finite dimensional. Then the natural pairing between $H^q(\mathfrak{g}, K, A)$ and $H^{d-q}(\mathfrak{g}, K, B)$ is perfect.*

Proof: We only have to show that the pairing is non-degenerate. We will start by showing that the induced map $H^{d-q}(\mathfrak{g}, K, B)$ to $H^q(\mathfrak{g}, K, A)^*$ is injective. So let $b \in Z^{d-q}(B) = C^{d-q}(B) \cap \ker d$ with $\langle a, b\rangle = 0$ for every $a \in Z^q(A)$. Define a map $\psi\colon d(C^q(A)) \to \mathbb{C}$ by

$$\psi(da) \;=\; \langle a, b\rangle\,.$$

We now show that the image $d(C^q(A))$ is a closed subspace of $C^{q+1}(A)$ and that the map $C^q(A)/\ker d \to d(C^q(A))$ is a topological isomorphism. For this let E be a finite dimensional subspace of $Z^{q+1}(A)$ that bijects to $H^{q+1}(\mathfrak{g}, K, A)$. Since E is finite dimensional, it is closed. The map $\eta = d + 1\colon C^q(A) \oplus E \to Z^{q+1}(A)$ is continuous and surjective. Since $C^q(A)$ and $Z^{q+1}(A)$ are LF-spaces, the map η is open (see [24], p. 78), hence it induces a topological isomorphism $(C^q(A)/\ker d) \oplus E \to Z^{q+1}(A)$. This implies that $d(C^q(A))$ is closed and $C^q(A)/\ker d \to d(C^q(A))$ is a topological isomorphism. Consequently, the map ψ is continuous. Hence it extends to a continuous linear map on $C^{q+1}(A)$. Therefore, it is given by an element f of $C^{d-q-1}(A)$, so

$$\langle a, b\rangle \;=\; \langle da, f\rangle \;=\; \langle a, df\rangle$$

for every $a \in C^q(A)$. We conclude $b = df$ and thus the non-degeneracy on one side. In particular it follows that $H^\bullet(\mathfrak{g}, K, B)$ is finite dimensional as well. The claim now follows by symmetry. □

We will now deduce Theorem 5.1. We have

$$\begin{aligned} H^q_{\rm cusp}(\Gamma, V^{\max}) &\cong \bigoplus_{\pi\in\hat{G}(\chi)} N_{\Gamma,{\rm cusp}}(\pi)\,{\rm Ext}^q_{\mathfrak{g},K}(\tilde{V}, \pi_K) \\ &\cong \bigoplus_{\pi\in\hat{G}(\chi)} N_{\Gamma,{\rm cusp}}(\pi)\, H^q(\mathfrak{g}, K, \pi^{-\infty}\hat{\otimes}V^{\max}) \\ &\cong \bigoplus_{\pi\in\hat{G}(\chi)} N_{\Gamma,{\rm cusp}}(\pi)\, H^{d-q}(\mathfrak{g}, K, \tilde{\pi}^{\infty}\hat{\otimes}\tilde{V}^{\min})^* \\ &\cong \bigoplus_{\pi\in\hat{G}(\chi)} N_{\Gamma,{\rm cusp}}(\pi)\, H^{d-q}(\mathfrak{g}, K, \pi^{\infty}\hat{\otimes}\tilde{V}^{\min})^* \\ &\cong H^{d-q}_{\rm cusp}(\Gamma, \tilde{V}^{\min})^*. \end{aligned}$$

In the second to last step we have used the fact that $L^2_{\rm cusp}$ is self-dual. Theorem 5.1 and thus Theorem 0.2 follow.

It remains to deduce Theorem 0.1. For Γ torsion-free arithmetic it follows directly from Theorem 0.2 and Lemma 1.2. Since the Borel-Serre compactification exists for arbitrary Fuchsian groups, the proof runs through and we also get Theorem 0.1 for torsion-free Fuchsian groups. An arbitrary Fuchsian group Γ has a finite index subgroup Γ' which is torsion-free. An inspection shows that all our constructions allow descent from Γ'-invariants to Γ-invariants and thus Theorem 0.1 follows. □

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