Convexity, Valuations and Prüfer Extensions in Real Algebra

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Abstract. We analyse the interplay between real valuations, Prüfer extensions and convexity with respect to various preorderings on a given commutative ring. We study all this first in preordered rings in general, then in f-rings. Most often Prüfer extensions and real valuations abound whenever a preordering is present. The next logical step, to focus on the more narrow class of real closed rings, is not yet taken, except in some examples.

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Contents

Introduction

The present paper is based on the book "Manis valuations and Prüfer extensions I" [KZ_1] by the same authors. The book provides details about all terms used here without explanation. But let us emphasize that a "ring" always means a commutative ring with 1, and a *ring extension* $A \subset R$ consists of a ring R and a subring A of R, where, of course, we always demand that the unit element of R coincides with the unit element of A.

The strength and versality of the concept of a *Prüfer extension* seems to depend a great deal on the many different ways we may look at these ring extensions and handle them. So we can say that a ring extension $A \subset R$ is Prüfer iff for every *overring* B of A in R, i.e. subring B of R containing A, the inclusion map $A \hookrightarrow B$ is an epimorphism in the category of rings, and then it follows that B is flat over A, cf. [KZ_1, Th.I.5.2, conditions (11) and (2)]. We can also say that $A \subset R$ is Prüfer iff every overring B of A in R is integrally closed in R [loc.cit., condition (4)].

On the other hand a Prüfer extension $A \subset R$ is determined by the family $S(R/A)$ of equivalence classes of all non trivial Manis valuations $v: R \to \Gamma \cup \infty$ on R (cf. [KZ_1, I §1]), such that $v(x) \geq 0$ for every $x \in A$, namely A is the intersection of the rings $A_v := \{x \in R \mid v(x) \geq 0\}$ with v running through $S(R/A)$. Further we can associate to each $v \in S(R/A)$ a prime ideal $\mathfrak{p} := \{x \in A \mid v(x) > 0\}$ of A , and then have

$$A_v \;=\; A_{[\mathfrak{p}]} :\;=\; A^R_{[\mathfrak{p}]} :\;=\; \{x \in R \mid \exists s \in A \setminus \mathfrak{p} \text{ with } sx \in A\}.$$

v is – up to equivalence – uniquely determined by $\mathfrak{p}$. We have a bijection $v \leftrightarrow \mathfrak{p}$ of $S(R/A)$ with the set $Y(R/A)$ of all R-*regular* prime ideals $\mathfrak{p}$ of A, i.e. prime ideals $\mathfrak{p}$ of A with $\mathfrak{p}R = R$. {Usually we do not distinguish between equivalent valuations. So we talk abusively of $S(R/A)$ as the set of non trivial Manis valuations of R over A.} Actually the $v \in S(R/A)$ are not just Manis valuations but PM (= "Prüfer-Manis") valuations. These have significantly better properties than Manis valuations in general, cf. [KZ_1, Chap.III].

We call $S(R/A)$ the *restricted PM-spectrum* of the Prüfer extension $A \subset R$ (cf.§1 below). We regard the restricted PM-spectra of Prüfer extensions as the good "complete" families of PM-valuations. In essence they are the same objects as Prüfer extensions.

The word "real algebra" in the title of the present paper is meant in a broad sense. It refers to a part of commutative algebra which is especially relevant for real algebraic geometry, real analytic geometry, and recent expansions of these topics, in particular for semialgebraic and subanalytic geometry and the now emerging *o*-minimal geometry (cf. e.g. [vd D], [vd D_1]).

Real algebra often is of non noetherian nature, but in compensation to this valuations abound. Usually these valuations are *real*, i.e. have a formally real residue class field (cf.§2 below).

A ring R has real valuations whenever R is *semireal*, i.e. -1 is not a sum of squares in R (cf.§2). We then define the *real holomorphy ring* $\mathrm{Hol}(R)$ of R as the intersection of the subrings A_v with v running through all real valuations on R. If R is a field, formally real, this is the customary definition of real holomorphy rings (e.g. [B,p.21]). In the ring case real holomorphy rings have been introduced in another way by M. Marshall, V. Powers and E. Becker ([Mar], [P], [BP]). But we will see in §3 (Cor.3.5) that their definition is equivalent to ours.

Now it can be proved under mild conditions on R, e.g. if $1+x^2$ is a unit in R for every $x \in R$, that $\mathrm{Hol}(R)$ is Prüfer in R, i.e. the extension $\mathrm{Hol}(R) \subset R$ is Prüfer (cf.§2 below). It follows that the restricted PM-spectrum $S(R/\mathrm{Hol}(R))$ is the set of all non trivial *special* (cf.[KZ_1, p.11]) real valuations on R. {Notice that every valuation v on R can be specialized to a special valuation without changing the ring A_v (loc.cit.). A Manis valuation is always special.}

Thus, under mild conditions on R, the non trivial special real valuations on R comprise one good complete family of PM-valuations on R. This fact already indicates that Prüfer extensions are bound to play a major role in real algebra.

An important albeit often difficult task in Prüfer theory is to get a hold on the complete subfamilies of $S(R/A)$ for a given Prüfer extension $A \subset R$. These are the restricted PM-spectra $S(R/B)$ with B running through the overrings of A in R. Thus there is much interest in describing and classifying these overrings of A in various ways.

Some work in this direction has been done in [KZ_1, Chapter II] by use of multiplicative ideal theory, but real algebra provides us with means which go beyond this general theory. In real algebra one very often deals with a preordering T (cf.§5 below) on a given ring R. {A case in point is that R comes as a ring of $\mathbb{R}$-valued functions on some set X, and T is the set of $f \in R$ with $f \geq 0$ everywhere on X. Here T is even a partial ordering of R, $T \cap (-T) = \{0\}$.} Then it is natural to look for T-convex subrings of R, (i.e. subrings which are convex with respect to T) and to study the T-convex hull $\mathrm{conv}_T(\Lambda)$ of a given subring Λ of R. The interplay between real valuations, Prüfer extensions and convexity for varying preorderings on R is the main theme of the present paper.

The smallest preordering in a given semireal ring R is the set $T_0 = \Sigma R^2$ of sums of squares in R. It turns out that $\mathrm{Hol}(R)$ is the smallest $\hat{T}_0$-convex subring $\mathrm{conv}_{\hat{T}_0}(\mathbb{Z})$ of R with respect to the saturation $\hat{T}_0$ (cf.§5, Def.2) of T_0 {This is essentially the *definition* of $\mathrm{Hol}(R)$ by Marshall et al. mentioned above.} Moreover, if every element of $1+T_0$ is a unit in R – an often made assumption in real algebra – then $\mathrm{Hol}(R)$ is Prüfer in R, as stated above, and every overring of $\mathrm{Hol}(R)$ in R is $\hat{T}_0$-convex in R (cf.Th.7.2 below).

Similar results can be obtained for other preorderings instead of T_0. Let (R,T) be any preordered ring. We equip every subring A of R with the preordering $T \cap A$. Convexity in A is always meant with respect to $T \cap A$. We say that A

has *bounded inversion*, if every element of $1 + (T \cap A)$ is a unit in A. If R has bounded inversion, it turns out that a subring A of R is convex in R iff A itself has bounded inversion and A is Prüfer in R (cf.Th.7.2 below). Further in this case every overring on A in R again has bounded inversion and is convex in R.

Thus the relations between convexity and the Prüfer property are excellent in the presence of bounded inversion. If bounded inversion does not hold, they are still friendly, as long as $\mathrm{Hol}(R)$ is Prüfer in R. This is testified by many results in the paper.

Given a preordered ring (R, T) and a subring A of R, it is also natural to look for overrings B of A in R such that A is convex and Prüfer in B. Here we quote the following two theorems, contained in our results in §7.

Theorem 0.1 (cf.Cor.7.7 below). Assume that A has bounded inversion. There exists a unique maximal overring D of A in R such that A is convex in D and D has bounded inversion. The other overrings B of A in R with this property are just *all* overrings of A in D.

Notice that Prüfer extensions are not mentioned in this theorem. But in fact D is the Prüfer hull (cf.[KZ$_1$, I §5]) $P(A, R)$ of A in R. It seems to be hard to prove the theorem without employing Prüfer theory and valuations at last. We also do not know whether an analogue of the theorem holds if we omit bounded inversion.

Theorem 0.2 (cf.Cor.7.10 below). There exists a unique maximal overring E of A in R such that A is Prüfer and convex in E. The other overrings of A in R with this property are just *all* overrings of A in E.

Notice that here no bounded inversion is needed. We call E the *Prüfer convexity cover* of A in the preordered ring $R = (R, T)$ and denote it by $P_c(A, R)$.

If we start with a preordered ring $A = (A, U)$ we may ask whether for every Prüfer extension $A \subset R$ there exists a unique preordering T of R with $T \cap A = U$. In this case, taking for R the (absolute) Prüfer hull $P(A)$ (cf.[KZ$_1$, I §5]), we have an absolute Prüfer convexity cover $P_c(A) := P_c(A, P(A))$ at our disposal. This happens, as we will explicate in §10, if A is an *f-ring*, i.e. a lattice ordered ring which is an ℓ-subring (= subring and sublattice) of a direct product of totally ordered rings.

Another natural idea is to classify Prüfer subrings of a given preordering $R = (R, T)$ by the amount of convexity in R they admit. Assume that A is already a convex Prüfer subring of R. Does there exist a unique maximal preordering $U \supset T$ on R such that A is U-convex in R? {Without the Prüfer assumption on A this question still makes sense but seems to be very hard.}

We will see in §13 that this question has a positive answer if R is an f-ring. Let us denote this maximal preordering $U \supset T$ by T_A. Also the following holds, provided $\mathrm{Hol}(R)$ is Prüfer in R. Every overring B of A in R is convex in R

(cf.Th.9.10), and $T_B \supset T_A$. There exists a unique smallest subring H of A such that H is Prüfer and convex in A (hence in R), and $T_H = T_A$. A subring B of R is T_A-convex in R iff $B \supset H$. No bounded inversion condition is needed here.

On the contents of the paper. In §1 we develop the notion of PM-spectrum $pm(R/A)$ and restricted PM-spectrum $S(R/A)$ for any ring extension $A \subset R$. The full PM-spectrum $pm(R/A)$ is needed for functorial reasons, but nearly everything of interest happens in the subset $S(R/A)$. Actually $pm(R/A)$ carries a natural topology (not Hausdorff), but for the purposes in this paper it suffices to handle $pm(R/A)$ as a poset (= partially ordered set) under the specialization relation $\rightsquigarrow$ of that topology. For non trivial PM-valuations v and w the relation $v \rightsquigarrow w$ just means that v is a coarsening of w. {We do not discuss the topology of $pm(R/A)$.} In §1 real algebra does not play any role.

In §2 – §8 we study convexity in a preordered ring $R = (R, T)$ and its relations to real valuations, real spectra, and Prüfer extensions. We start in §2 with the smallest preordering $T_0 = \Sigma R^2$ (using the convexity concept explicitly only later), then considered prime cones in §3 and advance to arbitrary preorderings in §4.

The prime cones of R are the points of the real spectrum Sper$R$. We are eager not to assume too much knowledge about real spectra and related real algebra on the reader's side. We quote results from that area often in a detailed way but, mostly, without proofs.

We study convexity not only for subrings of R but also for ideals of a given subring A of R and more generally for A-submodules of R. Generalizing the concept of a real valuation we also study T-convex valuations on R (cf.§5). The real valuations are just the T_0-convex valuations. {Of course, these concepts exist in real algebra for long, sometimes under other names.} All this seems to be necessary to understand convex Prüfer extensions.

In the last sections, §9 – §13, we turn from preordered rings in general to f-rings. As common for f-rings (cf. e.g.[BKW]), we exploit the interplay between the lattice structure and the ring structure of an f-ring. In particular we here most often meet absolute convexity (cf.§9,Def.1) instead of just convexity. So we obtain stronger results than in the general theory, some of them described above.

Prominent examples of f-rings are the ring $C(X)$ of continuous $\mathbb{R}$-valued functions on a topological space X and the ring $CS(M, k)$ of k-valued continuous semialgebraic functions on a semialgebraic subset M of k^n ($n \in \mathbb{N}$) for k a real closed field.

These rings are fertile ground for examples illustrating our results. They are real closed (in the sense of N. Schwartz, cf.[Sch_1]). As Schwartz has amply demonstrated [Sch_3], the category of real closed rings, much smaller than the category of f-rings, is flexible enough to be a good environment for studying $C(X)$, and for studying $CS(M,k)$ anyway. Thus a logical next step beyond the study in the present paper will be to focus on real closed rings. For lack of space and time we have to leave this to another occasion.

We also give only few examples involving $C(X)$ and none involving $CS(M,k)$. It would be well possible to be more prolific here. But especially the literature on the rings $C(X)$ is so vast, that it is difficult to do justice to them without writing a much longer paper. We will be content to describe the real holomorphy ring of $C(X)$ (4.13), the minimal elements of the restricted PM-spectrum of $C(X)$ over this ring (1.3, 2.1, 4.13), and the Prüfer hull of $C(X)$ (§11) in general.

Other rings well amenable to our methods are the rings of real C^r-functions on C^r-manifolds, $r \in \mathbb{N} \cup \{\infty\}$, although they are not f-rings.

References. The present paper is an immediate continuation of the book [KZ_1], which is constantly refered to. In these references we omit the label [KZ_1]. Thus, for example, "in Chapter II" means "in [KZ_1, Chapter II]", and "by Theorem I.5.2" means "by Theorem 5.2 in [KZ_1, Chapter I §5]". All other references, which occur also in [KZ_1], are cited here by the same labels as there.

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§1 The PM-spectrum of a ring as a partially ordered set

Let R be any ring (as always, commutative with 1).

Definition 1. The *PM-spectrum* of R is the set of equivalence classes of PM-valuations on R. We denote this set by $pm(R)$, and we denote the subset of equivalence classes of *non-trivial* PM-valuations on R by $S(R)$. We call $S(R)$ the *restricted PM-spectrum* of the ring A.

Usually we are sloppy and think of the elements of $pm(R)$ as valuations instead of classes of valuations, replacing an equivalence class by one of its members. We introduce on $pm(R)$ a partial ordering relation "$\rightsquigarrow$" as follows.

Definition 2. Let v and w be PM-valuations of R. We decree that $v \rightsquigarrow w$ if either both v and w are nontrivial and $A_w \subset A_v$, which means that v is a coarsening of w (cf. I §1, Def. 9), or v is trivial and $\operatorname{supp} v \subset \operatorname{supp} w$. $\square$

Remarks 1.1. a) We have a map $\operatorname{supp}: pm(R) \to \operatorname{Spec} R$ from $pm(R)$ to the Zariski spectrum $\operatorname{Spec} R$, sending a PM-valuation on R to its support. This map is compatible with the partial orderings on $pm(R)$ and $\operatorname{Spec} R$: If $v \rightsquigarrow w$ then $\operatorname{supp} v \subset \operatorname{supp} w$.

b) The restriction of the support map $\operatorname{supp}: pm(R) \to \operatorname{Spec} R$ to the subset $pm(R) \setminus S(R)$ of trivial valuations on R is an isomorphism of this poset with $\operatorname{Spec} R$. {"poset" is an abbreviation of "partially ordered set."}

c) Notice that $S(R)$ is something like a "forest". For every $v \in S(R)$ the set of all $w \in S(R)$ with $w \rightsquigarrow v$ is a chain (i.e. totally ordered). Indeed, these valuations w correspond uniquely with the R-overrings B of A_v such that $B \neq R$. Perhaps this chain does not have a minimal element. We should add on the bottom of the chain the trivial valuation v^* on R with $\operatorname{supp} v^* = \operatorname{supp} v$. The valuations v^* should be regarded as the roots of the trees of our forest. $\square$

This last remark indicates that it is not completely silly to include the trivial valuations in the PM-spectrum, although we are interested in nontrivial valuations. Other reasons will be indicated later.

Usually we will not use the full PM-spectrum $pm(R)$ but only the part consisting of those valuations $v \in pm(R)$ such that $A_v \supset A$ for a given subring A.

Definition 3. Let $A \subset R$ be a ring extension.

a) A *valuation on R over A* is a valuation v on R with $A_v \supset A$. In this case the *center of v on A* is the prime ideal $\mathfrak{p}_v \cap A$. We denote it by $cent_A(v)$.

b) The PM-*spectrum of R over A* (or: of the extension $A \subset R$) is the partially orderd subset consisting of the PM-valuations v on R over A. We denote this poset by $pm(R/A)$. The *restricted* PM-*spectrum of R over A* is the subposet $S(R) \cap pm(R/A)$ of $pm(R/A)$. We denote it by $S(R/A)$.

c) The *maximal restricted* PM-spectrum of R over A is the set of maximal elements in the poset $S(R/A)$. We denote it by $\omega(R/A)$. It consists of all non-trivial PM-valuations of R over A which are not proper coarsenings of other such valuations.

REMARK 1.2. Notice that, if v and w are elements of $pm(R/A)$ and $v \rightsquigarrow w$, then $cent_A(v) \subset cent_A(w)$. Also, if $v \in pm(R/A)$ and $\mathfrak{p} := cent_A(v)$, then $A_{[\mathfrak{p}]} \subset A_v$ and $\mathfrak{p}_v \cap A_{[\mathfrak{p}]} = \mathfrak{p}_{[\mathfrak{p}]}$. In the special case that $A \subset R$ is Prüfer the pair $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is Manis in R. Since this pair is dominated by $(A_v, \mathfrak{p}_v)$ we have $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]}) = (A_v, \mathfrak{p}_v)$ (cf. Th.I.2.4). It follows that, for $A \subset R$ Prüfer, the center map $cent_A \colon pm(R/A) \to \operatorname{Spec} A$ is an isomorphism from the poset $pm(R/A)$ to the poset $\operatorname{Spec} A$. {Of course, we know this for long.} It maps $S(R/A)$ onto the set $Y(R/A)$ of R-regular prime ideals of A, and $\omega(R/A)$ onto the set $\Omega(R/A)$ of maximal R-regular prime ideals of A. □

DEFINITION 4. If $A \subset R$ is Prüfer and $\mathfrak{p} \in \operatorname{Spec} A$, we denote the PM-valuation v of R over A with $cent_A(v) = \mathfrak{p}$ by $v_{\mathfrak{p}}$. If necessary, we more precisely write $v_{\mathfrak{p}}^R$ instead of $v_{\mathfrak{p}}$.

For a Prüfer extension $A \subset R$ the posets $pm(R/A)$ and $S(R/A)$ are nothing new for us. Here it is only a question of taste and comfort, whether we use the posets $\operatorname{Spec}(A)$ and $Y(R/A)$ or work directly with $pm(R/A)$ and $S(R/A)$. Recall that, if A is Prüfer in R, we have

$$A = \bigcap_{\mathfrak{p} \in Y(R/A)} A_{[\mathfrak{p}]} = \bigcap_{\mathfrak{p} \in \Omega(R/A)} A_{\mathfrak{p}},$$

hence

$$A = \bigcap_{v \in S(R/A)} A_v = \bigcap_{v \in \omega(R/A)} A_v.$$

In the same way any R-overring B of A is determined by the sets of valuations $S(R/B)$ and $\omega(R/B)$.

EXAMPLE 1.3. Let X be a completely regular Hausdorff space (cf. [GJ, 3.2]). Let $R := C(X)$, the ring of continuous $\mathbb{R}$-valued functions on X, and $A := C_b(X)$, the subring of bounded functions in R.*) As proved in the book [KZ_1], and before in [G_2], the extension $A \subset R$ is Prüfer (even Bezout, cf.II.10.8). In the following we describe the set $\Omega(R/A)$ of R-regular maximal ideals of A.

Every function $f \in A$ extends uniquely to a continuous function f^β on the Stone-Čech compactification βX of X (e.g. [GJ, §6]). Thus we may identify

*) In most of the literature on $C(X)$ this ring is denoted by $C^*(X)$. We have to refrain from this notation since, for any ring R, we denote – as in [KZ_1] – the group of units of R by R^*.

$A = C(\beta X)$. As is very well known, the points $p \in \beta X$ correspond uniquely with the maximal ideals $\mathfrak{p}$ of A via

$$\mathfrak{p} = \mathfrak{m}_p := \{f \in A \mid f^\beta(p) = 0\},$$

cf. [GJ, 7.2]. In particular, $A/\mathfrak{p} = \mathbb{R}$ for every $\mathfrak{p} \in \mathrm{Max}A$. The maximal ideals of $\mathfrak{P}$ of R also correspond uniquely with the points p of βX in the following way [GJ, 7.3]: For any $f \in R$ let $Z(f)$ denote the zero set $\{x \in X \mid f(x) = 0\}$. Then the maximal ideal $\mathfrak{P}$ of R corresponding with $p \in \beta X$ is

$$\mathfrak{P} = M^p := \{f \in R \mid p \in cl_{\beta X}(Z(f))\},$$

where $cl_{\beta X}(Z(f))$ denotes the topological closure of $Z(f)$ in βX. It follows that $M^p \cap A \subset \mathfrak{m}_p$.

By definition $\Omega(R/A)$ is the set of all ideals $\mathfrak{m}_p$ with $\mathfrak{m}_p R = R$. If $\mathfrak{m}_p R = R$ then even $\mathfrak{m}_p \cap R^* \neq \emptyset$. Indeed, we have an equation $1 = \sum\limits_{i=1}^{r} f_i g_i$ with $f_i \in \mathfrak{m}_p$, $g_i \in R$. Then $h := 1 + \sum\limits_{i-1}^{r} g_i^2$ is a unit in R and the functions $\frac{g_i}{h}$ are elements of A. Thus $\frac{1}{h} = \sum\limits_{i=1}^{r} f_i \frac{g_i}{h} \in \mathfrak{m}_p$. It is known that $\mathfrak{m}_p \cap R^* = \emptyset$ iff $R/M^p = \mathbb{R}$ [GJ, 7.9.(b)]. Further the set of points $p \in \beta X$ with $R/M^p = \mathbb{R}$ is known as the real compactification υX of X [GJ, 8.4]. Thus we have

$$\Omega(R/A) = \{\mathfrak{m}_p \mid p \in \beta X \setminus \upsilon X\}.$$

By the way, every $f \in C(X)$ extends uniquely to a continuous function on υX (loc.cit.). Thus we may replace X by υX without loss of generality, i.e. assume that X is realcompact. Then

$$\Omega(R/A) = \{\mathfrak{m}_p \mid p \in \beta X \setminus X\}.$$

In Example 2.1 below we will give a description (from scratch) of the Manis pair $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ associated with $\mathfrak{p} = \mathfrak{m}_p$ for any $p \in \beta X$. □

We return to an arbitrary ring extension $A \subset R$.

Theorem 1.4. Let $A \subset R$ be a Prüfer extension and B an R-overring.

i) For every PM-valuation w of R over A the special restriction $w|_B$ of w to B is a PM-valuation of B over A.

ii) The map $w \mapsto w|_B$ from $pm(R/A)$ to $pm(B/A)$ is an isomorphism of posets.

Proof. a) Let w be a PM-valuation on R over A. Then $v := w|_B$ is a special valuation on B with $A_v = A_w \cap B$ and $\mathfrak{p}_v = \mathfrak{p}_w \cap B$. In particular, v is a valuation over A. The set $B \setminus A_v$ is closed under multiplication. Thus A_v is

PM in B (cf. Prop. I.5.1.iii). Proposition III.6.6 tells us that v is Manis, hence PM. We have $cent_A(w) = cent_A(v)$.
b) Since the center maps from $pm(R/A)$ to $\operatorname{Spec} A$ and $pm(B/A)$ to $\operatorname{Spec} A$ both are isomorphisms of posets, we have a unique isomorphism of posets $\alpha\colon pm(R/A) \xrightarrow{\sim} pm(B/A)$ such that $cent_A(w) = cent_A(\alpha(w))$ for every $w \in pm(R/A)$. From $cent_A(w) = cent_A(w|_B)$ we conclude that $\alpha(w) = w|_B$. □

The theorem shows well that we sometimes should work with the full PM-spectrum $pm(R/A)$ instead of $S(R/A)$: In the situation of the proposition, whenever $R \neq B$, there exist nontrivial PM-valuations w on R over A such that $w|_B$ is trivial. (All PM-valuations w of R over B have this property.) Thus we do not have a decent map from $S(R/A)$ to $S(B/A)$.

Proposition 1.5.a. Let $B \subset R$ be a Prüfer extension. For every PM-valuation v on B there exists (up to equivalence) a unique PM-valuation w on R with $w|_B = v$.

Proof. The claim follows by applying Theorem 4 *) to the Prüfer extensions $A_v \subset B \subset R$.

Definition 5. In the situation of Proposition 5.a we denote the PM-valuation w on R with $w|_B = v$ by v^R, and we call v^R the *valuation induced by v on R*.

Proposition 1.5.b. If v_1 is a second PM-valuation on B and $v \rightsquigarrow v_1$ then $v^R \rightsquigarrow v_1^R$. Thus, if A is any subring of B, the map $v \mapsto v^R$ is an isomorphism from $pm(B/A)$ onto a sub-poset of $pm(R/A)$. It consists of all $w \in pm(R/A)$ such that $A_w \cap B$ is PM in B.

Proof. We obtain the first claim by applying again Theorem 4 to the extensions $A_{v_1} \subset B \subset R$. The second claim is obvious. □

If M is a subset of $pm(B/A)$ we denote the set $\{v^R \mid v \in M\}$ by M^R.

Theorem 1.6. Assume that $A \subset B$ is a convenient extension (cf. I §6, Def.2) and $B \subset R$ a Prüfer extension. Then the map $S(B/A) \to S(B/A)^R$, $v \mapsto v^R$, is an isomorphism of posets, the inverse map being $w \mapsto w|_B$. The set $S(R/A)$ is the disjoint union of $S(B/A)^R$ and $S(R/B)$. The extension $A \subset R$ is again convenient.

Proof. a) Let $w \in S(R/A)$ be given. If $A_w \supset B$, then $w \in S(R/B)$ and $w|_B$ is trivial. Otherwise $A_w \cap B \neq B$, and the extension $A_w \cap B \subset B$ is PM, since $A \subset B$ is convenient. Now Proposition 5.b tells us that $w = v^R$ for some

*) Reference to Theorem 1.4 in this section. In later sections we will refer to this theorem as "Theorem 1.4." instead of "Theorem 4".

$v \in S(B/A)$. Of course, $v = w|_B$. The isomorphism $pm(R/A) \xrightarrow{\sim} pm(B/A)$, $w \mapsto w|_B$, stated in Theorem 4, maps $S(R/A) \setminus S(R/B)$ onto $S(B/A)$.

b) It remains to prove that R is convenient over A. Let C be an R-overring of A such that $R \setminus C$ is closed under multiplication. We have to verify that C is PM in R.

The set $B \setminus (C \cap B)$ is closed under multiplication. Thus $C \cap B$ is PM in B. It follows that $C \cap B$ is Prüfer in R, hence convenient in R. Since $C \cap B \subset C \subset R$, and $R \setminus C$ is closed under multiplication, we conclude that C is PM in R. $\square$

Various examples of convenient extensions have been given in I, §6. In the case that $A \subset B$ is Prüfer, Theorem 6 boils down to Theorem 4.

We write down a consequence of Theorem 6 for maximal restricted PM-spectra.

Corollary 1.7. Let $A \subset B$ be a convenient extension and $B \subset R$ a Prüfer extension. Then

$$\omega(B/A)^R \subset \omega(R/A) \subset \omega(B/A)^R \cup \omega(R/B).$$

Proof. a) Let $v \in \omega(B/A)^R$ be given. If $w \in S(R/A)$ and $v^R \rightsquigarrow w$ then

$$B \cap A_w \subset B \cap A_{v^R} = A_v \subsetneqq B.$$

We conclude, say by Theorem 6, that $w = u^R$ for some $u \in S(B/A)$. Then $v = v^R|_B \rightsquigarrow w|_B = u$. Since v is maximal, we have $u = v$, and $w = v^R$. Thus v^R is maximal in $S(R/A)$.

b) Let $w \in \omega(R/A)$ be given. Then either $w \in S(R/B)$ or $w = v^R$ for some $v \in S(B/A)$. In the first case certainly $w \in \omega(R/B)$ and in the second case $v \in \omega(B/A)$. {N.B. It may well happen that a given $w \in \omega(R/B)$ is not maximal in $S(R/A)$.} $\square$

§2 Real valuations and real holomorphy rings

If R is a ring and m a natural number we denote the set of sums of m-th powers $x_1^m + \cdots + x_r^m$ in R ($r \in \mathbb{N}$, all $x_i \in R$) by ΣR^m. Notice that $1 + \Sigma R^m$ is a multiplicative subset of R. If m is odd, this set contains 0, hence is of no use. But for m even the set $1 + \Sigma R^m$ will deserve interest.

Let now K be a field. Recall that K is called *formally real* if $-1 \notin \Sigma K^2$. As is very well known ([AS]) this holds iff there exists a *total ordering* on K, by which we always mean a total ordering compatible with addition and multiplication.

We will also use the less known fact, first proved by Joly, that, given a natural number d, the field K is formally real iff $-1 \notin \Sigma K^{2d}$ ([J, (6.16)], cf. also [B_4]).

In the following R is any ring (commutative, with 1, as always).

Definition 1. A prime ideal $\mathfrak{p}$ of R is called *real* if the residue class field $k(\mathfrak{p}) = \mathrm{Quot}(R/\mathfrak{p})$ is formally real.

Remark. Clearly this is equivalent to the following condition: If $a_1, \dots, a_n$ are elements of R with $\sum\limits_{i=1}^{n} a_i^2 \in \mathfrak{p}$ then $a_i \in \mathfrak{p}$ for each $i \in \{1, \dots, n\}$.

Definition 2. A valuation v on R is called *real* if the residue class field $\kappa(v)$ (cf. I, §1) is formally real.

Remark. If v is a trivial valuation on R, then clearly v is real iff the prime ideal $\operatorname{supp} v$ is real. The notion of a real valuation may be viewed as refinement of the notion of real prime ideal.

Example 2.1 (cf. [G_2, Examples 1A and 1B]). Let $R := C(X)$ be the ring of all real-valued continuous functions on a completely regular Hausdorff space X. Let further α be an ultrafilter on the lattice $\mathcal{Z}(X)$ of zero sets $Z(f) = \{x \in X \mid f(x) = 0\}$ of all $f \in R$. Given $f, g \in C(X)$ we say that $f \le g$ at α if there exists $S \in \alpha$ such that $f(x) \le g(x)$ for every $x \in S$, i.e. $\{x \in X \mid f(x) \le g(x)\} \in \alpha$. Since α is an ultrafilter we have $f \le g$ on α or $g \le f$ on α or both. We introduce the following subsets of R.

$$A_\alpha := \{f \in R \mid \exists\, n \in \mathbb{N} \text{ with } |f| \le n \text{ at } \alpha\}.$$
$$I_\alpha := \{f \in R \mid \forall\, n \in \mathbb{N}\colon |f| \le \frac{1}{n} \text{ at } \alpha\}.$$
$$\mathfrak{q}_\alpha := \{f \in R \mid \exists\, S \in \alpha \quad \text{with } f|S = 0\}.$$

We speak of the $f \in A_\alpha$ as the functions *bounded* at α, of the $f \in I_\alpha$ as the functions *infinitesimal* at α, and of the $f \in \mathfrak{q}_\alpha$ as the functions *vanishing* at α.

It is immediate that A_α is a subring of R and $\mathfrak{q}_\alpha$ is a maximal ideal of R (cf.[GJ, 2.5]). It is also clear that I_α is an ideal of A_α. We claim that this ideal is maximal.

In order to prove this, let $f \in R \setminus A_\alpha$ be given. There exists some $n \in \mathbb{N}$ such that

$$Z_1 := \{x \in X \mid \frac{1}{n} \leq |f(x)| \leq n\} \in \alpha.$$

Let $V := \{x \in X \mid \frac{1}{n+1} < |f(x)| < n+1\}$. Then $Z_0 := X \setminus V \in \mathcal{Z}(X)$ and $Z_0 \cap Z_1 = \emptyset$. Thus there exists some $h \in R$ with $h|Z_0 = 0$ and $h|Z_1 = 1$ {We do not need that X is completely regular for this, cf.[GJ, 1.15].} The function $g: X \to \mathbb{R}$ with $g = \frac{h}{f}$ on V and $g = 0$ on Z_0 is continuous, since the function $\frac{1}{f}$ on V is bounded and continuous. Thus $g \in R$. Since $fg \mid Z_1 = 1$ we conclude that $1 - fg \in \mathfrak{q}_\alpha \subset I_\alpha$.

Thus I_α is indeed a maximal ideal on R. Our binary relation "$\leq$ at α" induces a total ordering on the field A_α/I_α which clearly is archimedian. Thus $A_\alpha/I_\alpha = \mathbb{R}$.

Moreover, (A_α, I_α) is a Manis pair in R. For, if $f \in R \setminus A_\alpha$, we have $Y_n := \{x \in X \mid |f(x)| \geq n\} \in \alpha$ for every $n \in \mathbb{N}$. This implies that $\frac{1}{1+f^2} \leq \frac{1}{n}$, $\frac{f}{1+f^2} \leq \frac{1}{n}$ on Y_n, hence $\frac{1}{1+f^2} \in I_\alpha$ and $\frac{f}{1+f^2} \in I_\alpha$. We conclude that

$$f \cdot \frac{f}{1+f^2} = 1 - \frac{1}{1+f^2} \in A_\alpha \setminus I_\alpha.$$

Let $v_\alpha: R \twoheadrightarrow \Gamma_\alpha \cup \infty$ denote the associated Manis valuation on R. Then $\operatorname{supp} v_\alpha = \mathfrak{q}_\alpha$, $A_{v_\alpha} = A_\alpha$, $\mathfrak{p}_{v_\alpha} = I_\alpha$, and v_α has the residue class field $A_\alpha/I_\alpha = \mathbb{R}$ (cf. Prop.I.1.6 and Lemma 2.10 below), hence is real. v_α is trivial iff $\mathfrak{q}_\alpha = I_\alpha$ iff $R/\mathfrak{q}_\alpha = \mathbb{R}$.

The ultrafilters α on $\mathcal{Z}(X)$ can be identified with the points p of βX, cf. [GJ, 6.5]. Clearly $I_\alpha \cap A$ is the maximal ideal $\mathfrak{m}_p$ of A corresponding to the point $p = \alpha$ (cf.1.4 above). Since $A := C_b(X)$ is Prüfer in R, we conclude that (A_α, I_α) is the Manis pair $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ with $\mathfrak{p} = \mathfrak{m}_p$ in the notation of 1.4. The pair is trivial, i.e. $A_\alpha = R$, iff $p \in \upsilon X$. □

We look for a characterization of a valuation to be real in other terms. As before, R is any ring.

Proposition 2.2. Let v be a valuation on R. The following are equivalent
(1) v is real
(2) If $x_1, \ldots, x_n$ are finitely many elements of R then

$$v\left(\sum_{i=1}^{n} x_i^2\right) = \min_{1 \leq i \leq n} v(x_i^2).$$

(3) There exists a natural number d such that for any finite sequence $x_1, \ldots, x_n$ in R

$$v\left(\sum_{i=1}^{n} x_i^{2d}\right) \quad = \quad \min_{1\le i\le n} v(x_i^{2d}).$$

{N.B. $v(x_i^{2d}) = 2d\, v(x_i)$, of course.}

PROOF. (1) $\Rightarrow$ (2): We first study the case that R is a field. Let $x_1, \ldots, x_n \in R$ be given. We assume without loss of generality that $v(x_1) \le \cdots \le v(x_n)$ and $x_1 \ne 0$. We have $x_i = a_i x_1$ with $a_i \in A_v$, $a_1 = 1$. Since $A_v/\mathfrak{p}_v$ is a formally real field,

$$1 + a_1^2 + \cdots + a_n^2 \notin \mathfrak{p}_v.$$

Thus $v(1 + a_1^2 + \cdots + a_n^2) = 0$. This implies

$$v\left(\sum_{i=1}^{n} x_i^2\right) = v(x_1^2) = \min_{1\le i\le n} v(x_i^2).$$

Let now R be a ring and again $x_1, \ldots, x_n$ a finite sequence in R. Let $\mathfrak{q} := \operatorname{supp} v$, and – as always – let $\hat{v}$ denote the valuation induced by v on $k(\mathfrak{q})$. Then with $\overline{x}_i := x_i + \mathfrak{q} \in k(\mathfrak{q})$ we have

$$v\left(\sum_{1}^{n} x_i^2\right) = \hat{v}\left(\sum_{1}^{n} \overline{x}_i^2\right) = \quad \min_{1\le i\le n} \hat{v}(\overline{x}_i^2) \quad = \quad \min_{1\le i\le n} v(x_i^2).$$

(2) $\Rightarrow$ (3): trivial.
(3) $\Rightarrow$ (1): Let $A := A_v$, $\mathfrak{p} := \mathfrak{p}_v$, $\mathfrak{q} := \operatorname{supp} v$. Property (3) for the valuation $v: R \to \Gamma \cup \infty$ implies the same property for $\overline{v}: R/\mathfrak{q} \to \Gamma \cup \infty$. Thus we may assume in advance that $\mathfrak{q} = 0$, hence R is an integral domain.

Let $K := \operatorname{Quot} R$. The valuation v extends to a valuation $\hat{v}: K \to \Gamma \cup \infty$. We have $\kappa(v) = \kappa(\hat{v}) = A_{\hat{v}}/\mathfrak{p}_{\hat{v}}$. Exploiting property (3) for $x_1, \ldots, x_n \in A_{\hat{v}}$ we obtain

$$-1 \notin \Sigma\, \kappa(\hat{v})^{2d}.$$

Thus $\kappa(v) = \kappa(\hat{v})$ is formally real. □

COROLLARY 2.3. Let $v: R \to \Gamma \cup \infty$ be a real valuation on R and H a convex subgroup of R. Then v/H is again a real valuation. If H contains the characteristic subgroup $c_v(\Gamma)$ (cf. I, §1, Def 3), then also $v|H$ is real.

PROOF. It is immediate that property (2) in Proposition 1 is inherited by v/H and $v|H$ from v. □

COROLLARY 2.4. If v is a real valuation on R and B is a subring of R, then the valuations $v|B$ and $v|_B$ are again real.

Proof. $v|B$ inherits property (2) from v, hence is real. It follows by Corollary 3 that also $v|_B$ is real. □

Corollary 2.5. If v is a real valuation on R, then $\operatorname{supp} v$ is a real prime ideal on R.

Proof. This follows immediately from condition (2) in Proposition 2. □

We now start out to prove the remarkable fact that – under a mild condition on R – the set of all non trivial special real valuations on R coincides with the restricted PM-spectrum $S(R/A)$ over a suitable subring A of R which is Prüfer in R.

Definition 3. Let R be any ring. The *real holomorphy ring* $\mathrm{Hol}(R)$ *of* R is the intersection $\bigcap\limits_v A_v$ with v running through all real valuations on R. {If R has no real valuations, we read $\mathrm{Hol}(R) = R$.}

In this definition there is a lot of redundance. $\mathrm{Hol}(R)$ is already the intersection of the rings A_v with v running through all non trivial special real valuations on R.

We need a handy criterion for R which guarantees in sufficient generality that $\mathrm{Hol}(R)$ is Prüfer in R.

Definition 4. We say that R *has positive definite inversion*, if $\mathbb{Q} \subset R$ and if for every $x \in R$ there exists a non constant polynomial $F(t)$ in one variable t over $\mathbb{Q}$ (depending on x) which is positive definite on R_0 *) , hence on $\mathbb{R}$), such that $F(x)$ is a unit of R. {N.B. In this situation the highest coefficient of F is necessarily positive. Thus we may assume in addition that $F(t)$ is monic.}

Notice that, if R has positive definite inversion, then R is convenient over $\mathbb{Q}$ (cf. Scholium I.6.8).

Example. Assume that $\mathbb{Q} \subset R$ and for every $x \in R$ there exists some $d \in \mathbb{N}$ such that $1 + x^{2d} \in R^*$. Then R has positive definite inversion.

Theorem 2.6. If R has positive definite inversion then also $\mathrm{Hol}(R)$ has this property and $\mathrm{Hol}(R)$ is Prüfer in R.

Proof. Let $A := \mathrm{Hol}(R)$. Clearly $\mathbb{Q} \subset A$. If v is any real valuation then also $\mathbb{Q} \subset \kappa(v)$. Moreover, if $F(t) \in \mathbb{Q}[t]$ is a positive definite monic polynomial, then $F(t)$ has no zero in $\kappa(v)$, since $\kappa(v)$ can be embedded into a real closed field which then contains R_0. Thus every real valuation v is an F-valuation as

*) R_0 denotes the real closure of $\mathbb{Q}$, i.e. the field of real algebraic numbers.

defined in I, § 6 (cf. Def.5 there), and we know by Theorem I.6.13 that A is Prüfer in R.

If $x \in A$, and if $F(t) \in \mathbb{Q}[t]$ is positive definite and $F(x) \in R^*$, then $A \subset A_v$ and clearly $v(F(x)) = 0$ for every real valuation v. Thus $\frac{1}{F(x)} \in A$ and $F(x) \in A^*$. □

In Definition 4 we demanded that $\mathbb{Q} \subset R$. This condition, of course, is not an absolute necessity in order to guarantee that $\mathrm{Hol}(R)$ is Prüfer in R. For example, one can prove the following variant of Theorem 2.6 by the same arguments as above.

THEOREM 2.6′. Assume that for every $x \in R$ there exists some $d \in \mathbb{N}$ with $1 + x^{2d} \in R^*$. Then also $\mathrm{Hol}(R)$ has this property, and $\mathrm{Hol}(R)$ is Prüfer in R. □

COROLLARY 2.7. Under the hypothesis in Theorem 6 or 6′ every special real valuation on R is PM. Moreover, if X is any set of real valuations on R, the ring $\bigcap\limits_{v \in X} A_v$ is Prüfer in R. □

Positive definite inversion holds for many rings coming up in real algebra, namely the "strictly semireal rings", to be defined now.

DEFINITION 5. We call a ring R *strictly semireal*, if for every maximal ideal $\mathfrak{m}$ of R the field $R/\mathfrak{m}$ is formally real.[*)]

Here are other characterizations of strictly semireal rings in the style of Proposition 2 above.

PROPOSITION 2.8. For any ring R the following are equivalent.
(1) R is strictly semireal.
(2) $1 + \Sigma R^2 \subset R^*$.
(3) There exists a natural number d such that $1 + \Sigma R^{2d} \subset R^*$.

PROOF. $1 + \Sigma R^2 \subset R^*$ means that $(1 + \Sigma R^2) \cap \mathfrak{m} = \emptyset$ for every maximal ideal $\mathfrak{m}$ of R, and this means that -1 is not a sum of squares in any of the fields $R/\mathfrak{m}$. In the same way we see that $1 + \Sigma R^{2d} \subset R^*$ means that -1 is not a sum of $2d$-th powers in each of these fields. □

Comment. Our term "strictly semireal" alludes to property (2) in Proposition 8. Commonly a ring R is called *semireal* if $-1 \notin \Sigma R^2$ and called *real* if $a_1^2 + \cdots + a_r^2 \neq 0$ for any nonzero elements $a_1, \ldots, a_r$ of R [La$_1$, §2], [KS Chap III,

[*)] In I §6, Def.6 we coined the term "totally real" for this property. We now think it is better to reserve the label "totally real" for a ring R where the residue class fields $k(\mathfrak{p})$ of all prime ideals $\mathfrak{p}$ of R are formally real.

§2]. It may be tempting to call a ring R just "totally real" if $R/\mathfrak{m}$ is formally real for every $\mathfrak{m} \in \mathrm{Max}R$, but notice that such a ring is not necessarily real in the established terminology. Schwartz and Madden call our strictly semireal rings "rings having the weak bounded inversion property" [SchM, p.40]. This is a very suitable but lenghty term.

Corollary 2.9. If R is any ring and $d \in \mathbb{N}$, then the localisation $S_d^{-1}R$ with respect to $S_d := 1 + \Sigma R^{2d}$ is strictly semireal, and $S_d^{-1}R = S_1^{-1}R$. $\square$

In the following we need a lemma which could well have been proved in III, §1.

Lemma 2.10. If v is a PM-valuation on R then $\kappa(v) = A_v/\mathfrak{p}_v$.

Proof. We know by III, §1 that $\mathfrak{p}_v$ is a maximal ideal of A_v, hence $\overline{\mathfrak{p}}_v := \mathfrak{p}_v/\mathrm{supp}\, v$ is a maximal ideal of $\overline{A}_v := A_v/\mathrm{supp}\, v$. Proposition I.1.6 tells us that $\mathfrak{o}_v = (\overline{A}_v)_{\overline{\mathfrak{p}}_v}$. (This holds for any Manis valuation v.) Thus $\kappa(v) = \mathfrak{o}_v/\mathfrak{m}_v = A_v/\mathfrak{p}_v$ in our case. $\square$

Theorem 2.11. Assume that R is strictly semireal. Let $d \in \mathbb{N}$ be fixed and $T := \Sigma R^{2d}$. Then

$$\mathrm{Hol}(R) = \sum_{t \in T} \mathbb{Z}\frac{1}{1+t} \quad .$$

(Recall that $1 + T \subset R^*$.) $\mathrm{Hol}(R)$ is again strictly semireal.

Proof. Let $A := \sum\limits_{t \in T} \mathbb{Z}\frac{1}{1+t}$. This is a subring of A since for $t_1, t_2 \in T$

$$\frac{1}{1+t_1} \cdot \frac{1}{1+t_2} = \frac{1}{1+u}$$

with $u := t_1 + t_2 + t_1 t_2 \in T$. As in the proof of Proposition 2, (1) $\Rightarrow$ (2), we see that $v\left(\frac{1}{1+t}\right) \geq 0$ for every $t \in T$ and every real valuation v on R. Thus $A \subset \mathrm{Hol}(R)$.

From I, §6 we infer that A is Prüfer in R (I §6, Example 13). Let v be a PM-valuation on R with $A_v \supset A$. If $a_1, \dots, a_n$ are elements of A then $t := a_1^{2d} + \cdots + a_n^{2d} \in A_v$ and $\frac{1}{1+t} \in A \subset A_v$, hence $1 + t \in A_v^*$. Thus A_v is strictly semireal. Since $\mathfrak{p}_v$ is a maximal ideal of A_v, we conclude by Lemma 10 above that the field $\kappa(v)$ is formally real, i.e. v is a real valuation. It follows that $A_v \supset \mathrm{Hol}(R)$. Since A is the intersection of the rings A_v with v running through $S(R/A)$, we infer that $A \supset \mathrm{Hol}(R)$, and then that $A = \mathrm{Hol}(R)$.

If $t := a_1^{2d} + \cdots + a_r^{2d}$ with elements a_i of A then $1 + t \in A$ and $\frac{1}{1+t} \in A$, hence $1 + t \in A^*$. Thus A is strictly semireal. $\square$

Proposition 2.12. Assume that $A \subset R$ is a Prüfer extension and A is strictly semireal. Then every non trivial PM-valuation on R over A is real.

Proof. Let $\mathfrak{m}$ be an R-regular maximal ideal of A, and let v denote the associated PM-valuation on R with $A_v = A_{[\mathfrak{m}]}$, $\mathfrak{p}_v = \mathfrak{p}_{[\mathfrak{m}]}$. The natural map $A/\mathfrak{m} \to A_{[\mathfrak{m}]}/\mathfrak{p}_{[\mathfrak{m}]}$ is an isomorphism, since $A/\mathfrak{m}$ is already a field. It follows by Lemma 10 that $\kappa(v) = A/\mathfrak{m}$. By assumption this field is formally real. Thus v is real.

We now have proved that every $v \in \omega(R/A)$ is real. The other non trivial PM-valuations on R over A are coarsenings of these valuations, hence are again real, as observed in Corollary 3 above. $\square$

We now state the first main result of this section.

Theorem 2.13. Let R be a strictly semireal ring, and let $A := \mathrm{Hol}(R)$.
i) A is Prüfer in R and $S(R/A)$ is the set of all non trivial special real valuations on R.
ii) A is strictly semireal and $\mathrm{Hol}(A) = A$.
iii) The overrings of A in R are precisely all subrings of R which are strictly semireal and Prüfer in R.
iv) If B is an overring of A in R then $\mathrm{Hol}(B) = A$.

Proof. i): We know by Theorem 6 that A is Prüfer in R and by Theorem 11 that A is strictly semireal, finally by Proposition 12 that every $v \in S(R/A)$ is real. Conversely, if v is any real valuation on R, then $A_v \supset A$ by definition of $A = \mathrm{Hol}(R)$. If in addition v is special, then v is PM since A is Prüfer in R. Thus, if v is non trivial, $v \in S(R/A)$.
ii): We said already that A is strictly semireal, and now know, again by Theorems 6 and 11 (or by i)), that $\mathrm{Hol}(A)$ is strictly semireal and Prüfer in A. Since A is Prüfer in R we conclude that $\mathrm{Hol}(A)$ is Prüfer in R (cf. Th.I.5.6). Now Proposition 12 tells us that every $v \in S(R/\mathrm{Hol}(A))$ is real, hence A_v contains $A = \mathrm{Hol}(R)$. Since $\mathrm{Hol}(A)$ is the intersection of these rings A_v, we have $A \subset \mathrm{Hol}(A)$, i.e. $A = \mathrm{Hol}(A)$.
iii): Assume that B is a strictly semireal subring of R which is Prüfer in R. We see by the same arguments as in the proof of part i) that every $v \in S(R/B)$ is real. B is the intersection of the rings A_v of these valuations v. Thus $A := \mathrm{Hol}(R) \subset B$.

Conversely, if B is an overring of A in R, we have $1+t \in B$ and $\frac{1}{1+t} \in A \subset B$ for every $t \in \Sigma B^2$. Thus $1+\Sigma B^2 \subset B^*$, and we conclude by Proposition 2 that B is strictly semireal. Of course, B is also Prüfer in R, since A is Prüfer in R.
iv): Assume that $A \subset B \subset R$. Then both A and B are strictly semireal. Applying claim iii) to the Prüfer extension $A \subset B$ we learn that $\mathrm{Hol}(B) \subset A$, and then, that $\mathrm{Hol}(B)$ is Prüfer in A. Applying the same argument to the Prüfer extension $\mathrm{Hol}(B) \subset A$ we obtain that $\mathrm{Hol}(A) \subset \mathrm{Hol}(B)$. Since $\mathrm{Hol}(A) = A$ we conclude that $\mathrm{Hol}(B) = A$. $\square$

Scholium 2.14. Let R be a strictly semireal ring and B a subring of R which is Prüfer in R. The following are equivalent:

(1) B is strictly semireal.
(2) $S(R/B)$ consists of real valuations.
(2′) $\omega(R/B)$ consists of real valuations.
(3) $\mathrm{Hol}(R) \subset B$.

Proof. The equivalence (1) $\Leftrightarrow$ (3) has been stated in Theorem 13.iii, and the implication (3) $\Rightarrow$ (2) is clear by Theorem 13.i. (2) $\Rightarrow$ (2′) is trivial, and (2′) $\Rightarrow$ (3) is clear by definition of $\mathrm{Hol}(R)$. □

Theorem 2.15. Assume that $A \subset R$ is a Prüfer extension and A is strictly semireal. Then the ring R is strictly semireal.

Proof. Let $\mathfrak{Q}$ be a maximal ideal of R. We want to verify that the field $R/\mathfrak{Q}$ is formally real. We have $\mathfrak{Q} = \mathfrak{q}R$ with $\mathfrak{q} := \mathfrak{Q} \cap A$ (cf. Prop.I.4.6); and $\mathfrak{q}$ is a prime ideal of A. We choose a maximal ideal $\mathfrak{m}$ of A containing $\mathfrak{q}$. Then $\mathfrak{m}R \supset \mathfrak{Q}$.

1.Case: $\mathfrak{m}R \neq R$. This forces $\mathfrak{m}R = \mathfrak{Q}$, since $\mathfrak{Q}$ is maximal. Intersecting with A we obtain $\mathfrak{m} = \mathfrak{q}$. Since $A \subset R$ is ws we have $A_\mathfrak{m} = R_\mathfrak{Q}$ (I, §3 Def.1). This gives us $R/\mathfrak{Q} = A/\mathfrak{m}$, and $A/\mathfrak{m}$ is formally real.

2.Case: $\mathfrak{m}R = R$. Now there is a PM-valuation v on R with $A_v = A_{[\mathfrak{m}]}$, $\mathfrak{p}_v = \mathfrak{m}_{[\mathfrak{m}]}$. Proposition 12 tells us that v is real. v induces a valuation $\tilde{v}$ on $R_\mathfrak{m}$ with $A_{\tilde{v}} = A_\mathfrak{m}$, $\mathfrak{p}_{\tilde{v}} = \mathfrak{m}A_\mathfrak{m}$, and $\tilde{v}$ is again PM (and real, since $\kappa(\tilde{v}) = \kappa(v)$). Now we invoke Proposition I.1.3, which tells us that $R_\mathfrak{m}$ is a local ring with maximal ideal $\mathrm{supp}\,\tilde{v} = (\mathrm{supp}\,v)_\mathfrak{m}$. This implies that $\mathfrak{Q}_\mathfrak{m} \subset (\mathrm{supp}\,v)_\mathfrak{m}$. Taking preimages of these ideals under the localisation map $R \to R_\mathfrak{m}$ we obtain $\mathfrak{Q} \subset \mathrm{supp}\,v$, hence $\mathfrak{Q} = \mathrm{supp}\,v$, since $\mathfrak{Q}$ is maximal. We conclude by Corollary 5 that $\mathfrak{Q}$ is real, i.e. $R/\mathfrak{Q}$ is formally real. □

Comment. Theorems 13 and 15 together tell us that for a given strictly semireal ring R we have a smallest strictly semireal subring A of R such that A is Prüfer in R, namely $A = \mathrm{Hol}(R)$, and a biggest strictly semireal ring $U \supset R$ such that R is Prüfer in U, namely $U = P(R)$, the Prüfer hull of R. Every ring B between $\mathrm{Hol}(R)$ and $P(R)$ is again strictly semireal, and $\mathrm{Hol}(B) = \mathrm{Hol}(R)$, $P(B) = P(R)$. □

The following theorem may be regarded as the second main result of this section.

Theorem 2.16. Let $B \subset R$ be any Prüfer extension and let v be a real PM-valuation on B. Then the induced PM-valuation v^R on R (cf. §1, Def.5) is again real.

Proof. a) We first prove this in the special case that B is strictly semireal.

Let v be a real PM-valuation on B. Then $A_v \subset B$ is a Prüfer extension with $\omega(R/A_v) = \{v\}$. Since v is real we learn by Scholium 14 that A_v is a strictly semireal ring. The extension $A_v \subset R$ is again Prüfer and v^R is a PM-valuation on R over A_v. Proposition 12 tells us that v^R is real, provided this valuation is non trivial.

There remains the case that v^R is trivial. Then also v is trivial. The prime ideal $\mathfrak{q} := \operatorname{supp} v = \mathfrak{p}_v$ of B is real, and $\mathfrak{Q} := \operatorname{supp}(v^R)$ is a prime ideal of R with $\mathfrak{Q} \cap B = \mathfrak{q}$, hence $\mathfrak{Q} = R\mathfrak{q}$. Since $B \subset R$ is ws, we have $B_\mathfrak{q} = R_\mathfrak{Q}$. This implies $k(\mathfrak{Q}) = k(\mathfrak{q})$, which is a formally real field. Thus $\mathfrak{Q}$ is real, which means that the trivial valuation v^R is real.

b) We now prove the theorem in general. Let again v be a real valuation on B and $A := A_v$. Let $S := 1 + \Sigma A^2$. The extension $S^{-1}A \subset S^{-1}B$ is Prüfer and $S^{-1}A$ is strictly semireal. By Theorem 15 also $S^{-1}B$ is semireal (and $S^{-1}R$ as well). We have $v(s) = 0$ for every $s \in S$. Thus v extends uniquely to a valuation v' on $S^{-1}B$, and v' is PM and real, the latter since $\kappa(v') = \kappa(v)$. As proved in step a) the PM-valuation $w' := (v')^R$ on $S^{-1}R$ is again real. We have $w'(s) = 0$ for every $s \in S$, of course. Let $j_B : B \to S^{-1}B$ and $j_R : R \to S^{-1}R$ denote the localisation maps of B and R with respect to S, and let $w := w' \circ j_R$. This is a Manis valuation on R since $w(s) = w'(\frac{s}{1}) = 0$ for every $s \in S$. We have $j_R^{-1}(A_{w'}) = A_w$, $j_B^{-1}(A_{v'}) = A_v$, and $A_{w'} \cap S^{-1}B = A_{v'}$. It follows that $A_w \cap B = A_v$. In particular $A_w \supset A_v$ and thus $A_w \subset R$ is Prüfer, hence w is PM. It is now clear that $w|_B = v$, which means that $w = v^R$ (cf. §1, Def.5). We have $\kappa(w) = \kappa(w')$, and we conclude that w is real, since w' is real. □

Corollary 2.17. Let $B \subset R$ be a Prüfer extension. Assume also that $\operatorname{Hol}(B)$ is Prüfer in B (e.g. B has positive definite inversion, cf. Theorem 6). Then $B \cap \operatorname{Hol}(R) = \operatorname{Hol}(B)$.

Proof. If w is a real valuation on R then the restriction $u := w|B$ is a real valuation on B and $A_u = B \cap A_w$. Thus $\operatorname{Hol}(B) \subset B \cap A_w$. Taking intersections we conclude that $\operatorname{Hol}(B) \subset B \cap \operatorname{Hol}(R)$.

On the other hand, if v is a special valuation on B we have $\operatorname{Hol}(B) \subset A_v \subset B$, and we conclude that v is PM, since $\operatorname{Hol}(B)$ is assumed to be Prüfer in B. Now Theorem 16 tells us that the valuation $w := v^R$ is again real. We have $w|_B = v$, hence $A_v = B \cap A_w \supset B \cap \operatorname{Hol}(R)$. Taking intersections we obtain $\operatorname{Hol}(B) \supset B \cap \operatorname{Hol}(R)$. □

Remark 2.18. If R is any ring and B is a subring of R then $\operatorname{Hol}(B) \subset B \cap \operatorname{Hol}(R)$. This is clear by the argument at the beginning of the proof of Corollary 17. □

By use of Theorem 16 we can expand a part of Theorem 13 to more general rings.

Theorem 2.19. Let R be a ring with positive definite inversion. Assume that B is an overring of $\mathrm{Hol}(R)$ in R. Then B has positive definite inversion and $\mathrm{Hol}(B) = \mathrm{Hol}(R)$.

Proof. a) Let $A := \mathrm{Hol}(R)$. We have $\mathbb{Q} \subset A \subset B$. If $x \in B$ and $F(t) \in \mathbb{Q}[t]$ then $F(x) \in B$. If in addition $F(t)$ is positive definite and $F(x) \in R^*$ then $\frac{1}{F(x)} \in A$, as has been verified in the proof of Theorem 6. Thus $\frac{1}{F(x)} \in B$ and $F(x) \in B^*$. This proves that B has positive definite inversion.
b) As observed above (Remark 18), we have $\mathrm{Hol}(B) \subset \mathrm{Hol}(R) \cap B = A$. Since A is a subring of B, we also have $\mathrm{Hol}(A) \subset \mathrm{Hol}(B) \cap A = \mathrm{Hol}(B)$. Thus $\mathrm{Hol}(A) \subset \mathrm{Hol}(B) \subset A$.
c) We finally prove that $\mathrm{Hol}(A) = A$, and then will be done. Given a real valuation v on A we have to verify that $A_v = A$. Now $u := v|_A$ is again real and $A_v = A_u$. Thus we may replace v by u and assume henceforth that v is special.

The ring A has positive definite inversion by Theorem 6 or step a) above. Thus A is convenient, hence v is PM. By Theorem 16 the induced valuation $w := v^R$ is real. This implies $A_w \supset \mathrm{Hol}(R) = A$. On the other hand $w|_A = v$ by definition of w. This implies $A_v = A_w \cap A$. It follows that $A_v = A$. □

As in Theorem 6 we can replace here positive definite inverison by a slightly different condition and prove by the same arguments

Theorem 2.19′. Let R be a ring and B an overring of $\mathrm{Hol}(R)$ in R. Assume that for every $x \in R$ there exists some $d \in \mathbb{N}$ with $1 + x^{2d} \in R^*$. Then this holds for B too, and $\mathrm{Hol}(B) = \mathrm{Hol}(R)$. □

We now introduce "relative" real holomorphy rings. In real algebra some of these are often more relevant objects than the "absolute" holomorphy rings $\mathrm{Hol}(R)$.

Definition 6. Let R be a ring and Λ a subring of R. The *real holomorphy ring of R over Λ* is the intersection of the rings A_v with v running through all real valuations on R over Λ (i.e. with $A_v \supset \Lambda$). We denote this ring by $\mathrm{Hol}(R/\Lambda)$.

In this terminology we have $\mathrm{Hol}(R/\mathbb{Z}) = \mathrm{Hol}(R)$ provided $\mathbb{Z} \subset R$. {If $n \cdot 1_R = 0$ for some $n \in \mathbb{N}$ we have $\mathrm{Hol}(R) = R$, since there do not exist real valuations on R.} It is also clear that $\Lambda \cdot \mathrm{Hol}(R) \subset \mathrm{Hol}(R/\Lambda)$ for any subring Λ of R.

Proposition 2.20. Assume that $\mathrm{Hol}(R)$ is Prüfer in R. {This holds for example if R has positive definite inversion, cf. Theorem 6.} Then for any subring Λ of R we have

$$\mathrm{Hol}(R/\Lambda) \;=\; \Lambda \cdot \mathrm{Hol}(R).$$

Proof. $\mathrm{Hol}(R)$ is Prüfer in R by Theorem 6 (or Theorem 6′), hence $\mathrm{Hol}(R)\cdot\Lambda$ is Prüfer in R. It follows that $\mathrm{Hol}(R)\cdot\Lambda$ is the intersection of the rings A_v with v running through all non trivial PM-valuations on R with $\mathrm{Hol}(R)\cdot\Lambda \subset A_v$, i.e. with $\Lambda \subset A_v$ and $\mathrm{Hol}(R) \subset A_v$. These valuations are known to be real (cf. Theorem 13.i). We conclude that $\mathrm{Hol}(R)\Lambda \supset \mathrm{Hol}(R/\Lambda)$. We also have $\mathrm{Hol}(R)\Lambda \subset \mathrm{Hol}(R/\Lambda)$ as stated above. Thus both rings are equal. □

Corollary 2.21. Assume that $B \subset R$ is a Prüfer extension and B is strictly semireal. Then we have a factorisation (cf.II §7, Def.3)

$$\mathrm{Hol}(R/B) \;=\; \mathrm{Hol}(R) \times_{\mathrm{Hol}(B)} B.$$

Proof. Theorem 15 tells us that R is strictly real. Then Proposition 20 says that $\mathrm{Hol}(R/B) = \mathrm{Hol}(R)\cdot B$. Finally $\mathrm{Hol}(R)\cap B = \mathrm{Hol}(B)$ by Corollary 17. □

§3 Real valuations and prime cones

As before let R be any ring (commutative, with 1, as always).

Definition 1 ([BCR, 7.1], [KS, III, §3], [La$_1$, §4]). A *prime cone* (= "*Ordnung*" in German) of R is a subset P of R with the following properties: $P + P \subset P$, $P \cdot P \subset P$, $P \cup (-P) = A$, $\mathfrak{q} := P \cap (-P)$ is a prime ideal of A. We call $\mathfrak{q}$ the *support of* P and write $\mathfrak{q} = \operatorname{supp} v$.

If R is a field and P a prime cone of R we have $P \cap (-P) = \{0\}$. Thus P is just the set of nonnegative elements of a total ordering of the field R, by which we always mean a total ordering compatible with addition and multiplication. We then call P itself an *ordering of* R.

In general, a prime cone P on R induces a total ordering $\overline{P}$ on the ring $\overline{R} := R/\mathfrak{q}$, $\mathfrak{q} = \operatorname{supp} v$, and then an ordering on $\operatorname{Quot}(\overline{R}) = k(\mathfrak{q})$ in the obvious way (loc.cit.). We denote this ordering of $k(\mathfrak{q})$ by $\hat{P}$.

Notice that P can be recovered from the pair $(\mathfrak{q}, \hat{P})$, since P is just the preimage of $\hat{P}$ under the natural homomorphism $R \to k(\mathfrak{q})$. Thus a prime cone P on the ring R is essentially the same object as pair $(\mathfrak{q}, Q)$ consisting of a prime ideal $\mathfrak{q}$ of R and an ordering Q of $k(\mathfrak{q})$.

Definition 2. The *real spectrum of* R is the set of all prime cones of R. We denote it by $\operatorname{Sper} R$.

We have a natural map

$$\operatorname{supp}\colon \operatorname{Sper} R \longrightarrow \operatorname{Spec} R$$

which sends a prime cone P on R to its support. The image of this map is the set $(\operatorname{Spec} R)_{re}$ of real prime ideals of R. Indeed, if $\mathfrak{q} \in \operatorname{Spec} R$, then $k(\mathfrak{q})$ carries at least one ordering iff $k(\mathfrak{q})$ is formally real. For any $\mathfrak{q} \in (\operatorname{Spec} R)_{re}$ the fibre $\operatorname{supp}^{-1}(\mathfrak{q})$ can be identified with $\operatorname{Sper} k(\mathfrak{q})$.

There lives a very useful topology on $\operatorname{Sper} R$, under which the support map becomes continuous. We will need this only later, cf. §4 below.

Prime cones give birth to real valuations, as we are going to explain now. We first consider the case that R is a field.

We recall some facts about convexity in an ordered field $K = (K, P)$, (cf. [La$_1$], [KS, Chap II], [BCR, 10.1]). We keep the ordering P fixed and stick to the usual notations involving the signs $<, \leq$. Thus $P = \{x \in K \mid x \geq 0\}$. Also $|x| := x$ if $x \geq 0$ and $|x| := -x$ if $x \leq 0$. A subset M of K is called *convex with respect to* P or P*-convex*, if for $a, b \in M$ with $a < b$ the whole interval $[a, b] := \{x \in K \mid a \leq x \leq b\}$ is contained in M.

Notice that an abelian subgroup M of $(K, +)$ is P-convex iff for $x \in M \cap P$ and $y, z \in P$ with $x = y + z$ we have $y \in M$ and $z \in M$.

If N is a second P-convex subgroup of $(K,+)$ then $M \subset N$ or $N \subset M$. Also K contains a smallest convex additive subgroup, which we denote by A_P. We have

$$\begin{aligned} A_P &= \{x \in K \mid \exists\, n \in \mathbb{N} \quad \text{with} \quad |x| \le n\} \\ &= \{x \in K \mid \exists\, n \in \mathbb{N} \quad \text{with} \quad n \pm x \in P\}. \end{aligned}$$

Clearly A_P is a subring of K. If x is an element of $K \setminus A_P$, then $|x| > n$ for every $n \in \mathbb{N}$, hence $|x^{-1}| < \frac{1}{n}$ for every $n \in \mathbb{N}$, and a fortiori $x^{-1} \in A_P$. This proves that A_P is a valuation domain of K (i.e. with $\mathrm{Quot}(A_P) = K$), and that

$$I_P\colon = \{x \in K \mid \forall\, n \in \mathbb{N}\colon |x| < \frac{1}{n}\} = \{x \in K \mid \forall\, n \in \mathbb{N}\colon 1 \pm nx \in P\}$$

is the maximal ideal of A_P.

If B is any P-convex subring of K then B is an overring of A_P in K and thus again a valuation domain of K. Moreover,

$$\{0\} \subset \mathfrak{m}_B \subset I_P \subset A_P \subset B \subset K,$$

and $\mathfrak{m}_B$ is a prime ideal of A_P.

Conversely we conclude easily from the fact $[0,1] \subset A_P$ that every A_P-submodule of K is P-convex in K. In particular, every overring B of A_P and every prime ideal of A_P is P-convex in K. *The overrings B of A_P in K are precisely all P-convex subrings* of K. Their maximal ideals $\mathfrak{m}_B$ are the prime ideals of A_P, and they are P-convex in A_P and in K.

More notations. Given a valuation ring B of K, let $\mathfrak{m}_B$ denote the maximal ideal of B. Let $\kappa(B)$ denote the residue class field $B/\mathfrak{m}_B$ of B and $\pi_B\colon B \twoheadrightarrow \kappa(B)$ denote the natural map from B to $\kappa(B)$. Further let v_B denote the canonical valuation associated to B with value group R^*/B^*. {In notations of I, §1 we have $\kappa(v_B) = \kappa(B)$.} For $B = A_P$ we briefly write $\kappa(P)$ instead of $\kappa(A_P)$. Thus $\kappa(P) = A_P/I_P$. In the same vein we write π_P and v_P instead of π_{A_P} and v_{A_P}.

The following facts are easily verified.

Lemma 3.1. Let B be a P-convex subring of K.
i) $Q\colon = \pi_B(P \cap B)$ is an ordering of $\kappa(B)$. In particular $\kappa(B)$ is formally real.
ii) The P-convex subrings C of K with $C \subset B$ correspond uniquely with the Q-convex subrings D of $\kappa(B)$ via $\pi_B(C) = D$ and $\pi_B^{-1}(D) = C$. We have $\pi_B(\mathfrak{m}_C) = \mathfrak{m}_D$ and $\pi_B^{-1}(\mathfrak{m}_D) = \mathfrak{m}_C$.
iii) In particular $\pi_B(A_P) = A_Q$, $\pi_B(I_P) = I_Q$, $\pi_B^{-1}(A_Q) = A_P$, $\pi_B^{-1}(I_Q) = I_P$. $\square$

We state a consequence of a famous theorem by Baer and Krull (cf. [La$_1$, Cor.3.11], [KS, II §7], [BCR, Th.10.1.10]).

Lemma 3.2. Let B be a valuation ring of K and let Q be an ordering (= prime cone) of $\kappa(B)$. Then there exists at least one ordering P of K such that B is P-convex and $\pi_B(B \cap P) = Q$. $\square$

The theorem of Baer-Krull (loc.cit.) gives moreover a precise description of all orderings P on K with this property. We do not need this now. We refer to the literature for a proof of Lemma 2.

We return to an arbitrary ring R and a prime cone P of R. Let $\mathfrak{q} := \operatorname{supp} P$.

Definition 3. As above, $\hat{P}$ denotes the ordering on $k(\mathfrak{q}) = \operatorname{Quot}(R/\mathfrak{q})$. Let $j_\mathfrak{q}: R \to k(\mathfrak{q})$ denote the natural homomorphisms from R to $k(\mathfrak{q})$. We introduce the valuation

$$v_P := v_{\hat{P}} \circ j_\mathfrak{q}$$

on R, the ring $A_P := j_\mathfrak{q}^{-1}(A_{\hat{P}})$, and the prime ideal $I_P := j_\mathfrak{q}^{-1}(I_{\hat{P}})$ of A_P. $\square$

For $v := v_P$ we have $\kappa(v) = \kappa(\hat{P})$, $A_v = A_P$, $\mathfrak{p}_v = I_P$, and $\operatorname{supp} v = \mathfrak{q} = \operatorname{supp} P$.

From the description of A_P and I_P above in the field case, i.e. of $A_{\hat{P}}$ and $I_{\hat{P}}$, we deduce immediately

Lemma 3.3.
$A_P = \{x \in R \mid \exists n \in \mathbb{N}: n \pm x \in P\}$,
$I_P = \{x \in R \mid \forall n \in \mathbb{N}: 1 \pm nx \in P\}$. $\square$

Theorem 3.4. a) The real valuations on R are, up to equivalence, the coarsenings of the valuations v_P with P running through $\operatorname{Sper} R$.
b) Given a prime cone P of R, the coarsenings w of v_P correspond one-to-one with the $\hat{P}$-convex subrings B of $k(\mathfrak{q})$, $\mathfrak{q} := \operatorname{supp} P$, via $w = v_B \circ j_\mathfrak{q}$.

Proof. If P is a prime cone of R then we know by Lemma 1.i that $v_{\hat{P}}$ is real, and conclude that v_P is real. Thus every coarsening of v_P is real (cf. Cor.2.3.).

Conversely, given a real valuation w on R we have a real valuation $\hat{w}$ on $k(\mathfrak{q})$, $\mathfrak{q} := \operatorname{supp} w$, with $w = \hat{w} \circ j_\mathfrak{q}$. Applying Lemma 2 to an ordering Q on $\kappa(\hat{w}) = \kappa(w)$ we learn that there exists an ordering P' on $k(\mathfrak{q})$ such that $A_{\hat{w}} = \mathfrak{o}_w$ is P'-convex in $k(\mathfrak{q})$. This implies that $\hat{w}$ is a coarsening of $v_{P'}$.

Let $P := j_\mathfrak{q}^{-1}(P')$. This is a prime cone on R with $\operatorname{supp} P = \mathfrak{q}$, $\hat{P} = P'$. It follows that $v_P = v_{P'} \circ j_\mathfrak{q}$, and we conclude that $w = \hat{w} \circ j_\mathfrak{q}$ is a coarsening of v_P. Moreover the coarsenings w of v_P correspond uniquely with the coarsenings u of $v_{\hat{P}}$ via $u = \hat{w}$, $w = u \circ j_\mathfrak{q}$, hence with the overrings of $\mathfrak{o}_P = A_{\hat{P}}$ in $k(\mathfrak{q})$. These are the $\hat{P}$-convex subrings of $k(\mathfrak{q})$. $\square$

Corollary 3.5. The real holomorphy ring $\mathrm{Hol}(R)$ of R is the intersection of the rings A_P with P running through $\mathrm{Sper}R$. Thus $\mathrm{Hol}(R)$ is the set of all $x \in R$, such that for every $P \in \mathrm{Sper}R$ there exists some $n \in \mathbb{N}$ with $n \pm x \in P$.

Proof. This follows from the definition of $\mathrm{Hol}(R)$ in §2 by taking into account Lemma 3 and Theorem 4.a. □

We continue to work with a single prime cone P on R, and we stick to the notations from above. In particular, $\mathfrak{q} := \mathrm{supp}\, P$.

We introduce a binary relation $\leq_P$ on R by defining $x \leq_P y$ iff $y - x \in P$. This relation is reflexive and transitive, but not antisymmetric: If $x \leq_P y$ and $y \leq_P x$ then $x \equiv y \mod \mathfrak{q}$ and vice versa. For any two elements x, y of R we have $x \leq_P y$ or $y \leq_P x$. We write $x <_P y$ if $x \leq_P y$ but not $x \equiv y \mod \mathfrak{q}$.

Given elements a, b of R with $a \leq_P b$ we introduce the "intervals"

$$[a, b]_P := \{x \in R \mid a \leq_P x \leq_P b\} \quad , \quad]a, b[_P := \{x \in R \mid a <_P x <_P b\}.$$

We say that a subset M of R is P-*convex* in R if for any two elements $a, b \in R$ with $a \leq_P b$ the interval $[a, b]_P$ is contained in R.

Notice that the prime cone $\overline{P} := P/\mathfrak{q} := \{x + \mathfrak{q} \mid x \in P\}$ on $R/\mathfrak{q}$ defines a total ordering $\leq_{\overline{P}}$ on the ring $R/\mathfrak{q}$, compatible with addition and multiplication. The P-convex subsets of R are the preimages of the $\overline{P}$-convex subsets of $R/\mathfrak{q}$ under the natural map $R \twoheadrightarrow R/\mathfrak{q}$. Thus the following is evident.

Remarks 3.6. i) Let M be a subgroup of $(R, +)$. Then M is P-convex iff for any two elements x, y of P with $x + y \in M$, we have $x \in M$ and (hence) $y \in M$.
ii) The P-convex additive subgroups of R form a chain under the inclusion relation. □

Lemma 3.7.
i) $\mathrm{supp}\, P$ is the smallest P-convex additive subgroup of R.
ii) A_P is the smallest P-convex additive subgroup M of R with $1 \in M$.
iii) I_P is the biggest P-convex additive subgroup M of R with $1 \notin M$.
iv) If M is any P-convex additive subgroup of R, the set $\{x \in R \mid xM \subset M\}$ is a P-convex subring of R.

Proof. i): Clear, since $\{0\}$ is the smallest $\overline{P}$-convex additive subgroup of $R/\mathfrak{q}$.
ii): An easy verification starting from the description of A_P in Lemma 3.
iii): We know by Lemma 1 that I_P is P-convex in R, and, of course, $1 \notin I_P$. Let M be any P-convex additive subgroup of R with $1 \notin M$. Suppose that $M \not\subset I_P$. We pick some $x \in M \cap P$ with $x \notin I_P$. We learn by Lemma 3 that there exists some $n \in \mathbb{N}$ with $1 - nx \notin P$, hence $nx - 1 = p \in P$. This

implies $1+p=nx\in M$. We conclude by the P-convexity of M that $1\in M$, a contradiction. Thus $M\subset I_P$.
iv): Again an easy verification. □

As a consequence of this lemma we state

Proposition 3.8.
i) $\operatorname{supp} P$ is the smallest and I_P is the biggest P-convex prime ideal of A_P.
ii) A_P is the smallest P-convex subring of R.
iii) Every P-convex additive subgroup of R is an A_P-submodule of R. □

Definition 4. Given an additive subgroup M of R we introduce the set

$$\operatorname{conv}_P(M) := \bigcup_{z\in P\cap M} [-z,z]_P.$$

This is the smallest P-convex subset of R containing M. We call $\operatorname{conv}_P(M)$ the P-*convex hull* of M (in R). □

Lemma 3.9. $\operatorname{conv}_P(M)$ is again an additive subgroup of R, and

$$\operatorname{conv}_P(M)=\{x\in M \mid \exists\, z\in P\cap M \quad \text{with} \quad z\pm x\in P\}.$$

If M is a subring of R, then $\operatorname{conv}_P(M)$ is a subring of R.

Proof. All this is easily verified. □

Theorem 3.10. a) If w is a coarsening (cf.I §1, Def.9) of the valuation v_P on R, then A_w is a P-convex subring of R.
b) For any subring Λ of R there exists a minimal coarsening w of v_P with $A_w\supset\Lambda$, and $A_w=\operatorname{conv}_P(\Lambda)$.

Proof. a): If w is a coarsening of v_P then $\operatorname{supp}(w)=\mathfrak{q}$. The induced valuation $\hat{w}$ on $k(\mathfrak{q})$ is a coarsening of $\hat{v}_P=v_{\hat{P}}$, and $w=\hat{w}\circ j_{\mathfrak{q}}$. The ring $A_{\hat{w}}$ is $\hat{P}$-convex in $k(\mathfrak{q})$. Thus $A_w=j_{\mathfrak{q}}^{-1}(A_{\hat{w}})$ is P-convex in R.
b): Let $\overline{\Lambda}:=j_{\mathfrak{q}}(\Lambda)=\Lambda+\mathfrak{q}/\mathfrak{q}$. This is a subring of $R/\mathfrak{q}$, hence of the field $k(\mathfrak{q})$. We introduce the convex hulls $B:=\operatorname{conv}_P(\Lambda)$ and $\hat{B}:=\operatorname{conv}_{\hat{P}}(\overline{\Lambda})$. Clearly $\hat{B}$ is the smallest $\hat{P}$-convex subring C of $k(\mathfrak{q})$ with $j_{\mathfrak{q}}^{-1}(C)=B$. There exists a unique coarsening u of $v_{\hat{P}}$ with $A_u=\hat{B}$. Then $w:=u\circ j_P$ is a coarsening of v_P with $A_w=B$, and this is the minimal coarsening of v_P with valuation ring B. Since for every coarsening w' of v_P the ring $A_{w'}$ is P-convex in R, it follows that w is also the minimal coarsening of v_P with $A_w\supset\Lambda$. □

Definition 5. We call the valuation w described in Theorem 10.b the *valuation associated with P over* Λ, and denote it by $v_{P,\Lambda}$.

Corollary 3.11. Let again Λ be a subring of R. The relative holomorphy ring $\mathrm{Hol}(R/\Lambda)$ (cf.§2) is the intersection of the rings $A_{v_{P,\Lambda}} = \mathrm{conv}_P(\Lambda)$ with P running through $\mathrm{Sper}R$. It is also the set of all $x \in R$ such that for every $P \in \mathrm{Sper}R$ there exists some $\lambda \in P \cap \Lambda$ with $\lambda \pm x \in P$.

Proof. The first claim follows from Theorems 10 and 4. The second claim then follows from the description of $\mathrm{conv}_P(\Lambda)$ in Lemma 9. □

We now look for P-convex prime ideals of P-convex subrings of R.

Definition 5. For any subring Λ of R we define

$I_P(\Lambda){:=}\ \{x \in R \mid 1{+}\Lambda x \subset P\} \ = \ \{x \in R \mid 1{\pm}\lambda x \in P \quad \text{for every} \quad \lambda \in \Lambda{\cap}P\}.$

Theorem 3.12.
a) If w is a coarsening of the valuation v_P on R, then $\mathfrak{p}_w$ is a P-convex[*)] prime ideal of A_w.
b) Let Λ be a subring of R and $w{:=}\ v_{P,\Lambda}$. Then $\mathfrak{p}_w = I_P(\Lambda)$. Moreover $I_P(\Lambda)$ is the maximal P-convex proper ideal of $A_w = \mathrm{conv}_P(\Lambda)$.

Proof. a): $\mathfrak{p}_{\hat{w}}$ is a $\hat{P}$-convex prime ideal of $A_{\hat{w}}$. Taking preimages under $j_{\mathfrak{q}}$ we see that the same holds for $\mathfrak{p}_w$ with respect to P and A_w.
b): Let $B{:=}\ \mathrm{conv}_P(\Lambda)$ and $\hat{B}{:=}\ \mathrm{conv}_{\hat{P}}(\overline{\Lambda})$ with $\overline{\Lambda}{:=}\ j_{\mathfrak{q}}(\Lambda)$. For any $x \in R$ we denote the image $j_{\mathfrak{q}}(x)$ by $\overline{x}$. As observed in the proof of Theorem 10, we have $B = A_w$ and $\hat{B} = A_{\hat{w}}$. From valuation theory over fields we know for $x \in (R \setminus \mathfrak{q}) \cap P$ that $\overline{x} \in \mathfrak{p}_{\hat{w}}$ iff $\overline{x}^{-1} \notin \hat{B}$. This means $\overline{x}^{-1} >_{\hat{P}} \overline{\lambda}$ for every $\lambda \in P \cap \Lambda$, i.e. $1 - \overline{\lambda}\overline{x} >_{\hat{P}} 0$. Since $\overline{x} >_{\hat{P}} 0$, this is equivalent to $1 - \overline{\lambda}\overline{x} \in \hat{P}$ for every $\lambda \in P \cap \Lambda$, hence to $1 - \lambda x \in P$ for every $\lambda \in P \cap \Lambda$. It follows easily that indeed

$$\mathfrak{p}_w = j_{\mathfrak{q}}^{-1}(\hat{\mathfrak{p}}_w) = I_P(\Lambda).$$

In particular we now know that $I_P(\Lambda)$ is a P-convex proper ideal of B. If $\mathfrak{a}$ is any such ideal, then for every $x \in \mathfrak{a}$ and $b \in B$ we have $bx \in]-1,1[_P$, hence $1 \pm bx \in P$. In particular $1 \pm \lambda x \in P$ for every $\lambda \in \Lambda$. Thus $x \in I_P(\Lambda)$. This proves that $\mathfrak{a} \subset I_P(\Lambda)$. □

In the case $\Lambda = R$ the theorem tells us the following.

Scholium 3.13. $I_P(R)$ is the maximal P-convex proper ideal of R. It is a prime ideal of R. More precisely, $I_P(R) = \mathfrak{p}_w$ for w the minimal coarsening of v_P with $A_w = R$, i.e. $w = v_{P,R}$. Thus

$$I_P(R) = \{x \in R \mid Rx \subset I_P\}.$$

[*)] Perhaps it would be more correct to call $\mathfrak{p}_w$ a $(P{\cap}A_w)$-convex ideal of A_w. But this is not really necessary, since A_w is P-convex in R.

The latter fact is also obvious from the definition $I_P(R) := \{x \in R \mid 1 + Rx \subset P\}$ and the description of I_P in Lemma 3. □

If Λ is a subring of R with $B := \mathrm{conv}_P(\Lambda) \neq R$ the following lemma exhibits two more P-convex ideals of B which both may be different from $I_P(\Lambda) = I_P(B)$.

Lemma 3.14. Let B be a P-convex subring of R with $B \neq R$. Then $R \setminus B$ is closed under multiplication, and the prime ideals $\mathfrak{p}_B$ and $\mathfrak{q}_B$ (cf.I §2, Def.2) of B are again P-convex.

Proof. a) We know by Theorem 10.b that $B = A_w$ for some valuation w on R. This implies that $R \setminus B$ is closed under multiplication.
b) Let $x \in R$, $z \in \mathfrak{p}_B$, and $0 \leq_P x \leq_P z$. There exists some $s \in R \setminus B$ with $sz \in B$. Eventually replacing s by $-s$ we may assume in addition that $s \in P$. Now $0 \leq_P sx \leq_P sz$. We conclude by the P-convexity of B that $sx \in B$, hence $x \in \mathfrak{p}_B$. This proves that $\mathfrak{p}_B$ is P-convex in R.
c) Let $x \in R$, $z \in \mathfrak{q}_B$, and $0 \leq_P x \leq_P z$. For any $s \in P$ we have $0 \leq_P sx \leq_P sz$ and $sz \in B$. This implies that $sx \in B$. It is now clear that $Rx \subset B$, hence $x \in \mathfrak{q}_B$. □

We look for cases where *every* R-overring of A_P is P-convex. We will verify this if R is convenient over $\mathrm{Hol}(R)$. Notice that, according to §2, this happens to be true if R has positive definite inversion, and also, if for every $x \in R$ there exists some $d \in \mathbb{N}$ with $1 + x^{2d} \in R^*$. Indeed, in these cases $\mathrm{Hol}(R)$ is even Prüfer in R (cf. Theorems 2.6 and 2.6′).

We need one more lemma of general nature.

Lemma 3.15. Assume that B is a P-convex subring of R and S a multiplicative subset of R. Then $B_{[S]}$ is again P-convex in R.

Proof. Let $0 \leq_P x \leq_P z$ and $z \in B_{[S]}$. We choose some $s \in S$ with $sz \in B$. Then $s^2 z \in B$ and $0 \leq_P s^2 x \leq_P s^2 z$. Since B is P-convex in R this implies that $s^2 x \in B$. Thus $x \in B_{[S]}$. □

Theorem 3.16. Assume that R is convenient over $\mathrm{Hol}(R)$. Then every R-overring B of A_P is P-convex and PM in R, and $\mathfrak{p}_B = I_P(B)$, provided $B \neq R$.

Proof. We may assume that $B \neq R$. Let $A := A_P$. The set $R \setminus A$ is closed under multiplication. A contains $\mathrm{Hol}(R)$, and R is convenient over $\mathrm{Hol}(R)$. Thus A is PM in R, hence B is PM in R. Let $\mathfrak{P}$ denote the unique R-regular maximal ideal of B (cf. III, §1), and $\mathfrak{p} := \mathfrak{P} \cap A$. Then $B = B_{[\mathfrak{P}]} = A_{[\mathfrak{p}]}$, since A is ws in B. We conclude by Lemma 15 that B is P-convex in R.

We now know by Lemma 14, that $\mathfrak{p}_B$ is P-convex in R, and then by Theorem 12, that $\mathfrak{p}_B \subset I_P(B)$. But $\mathfrak{p}_B$ is a maximal ideal of B, since B is PM in R (cf. Cor.III.1.4). This forces $\mathfrak{p}_B = I_P(B)$. $\square$

A remarkable fact here is that, given a subring B of R, there may exist various prime cones P of R such that B is P-convex. But the prime ideals $I_P(B)$ are all the same, at least if R is convenient over $\mathrm{Hol}(R)$.

Assuming again that R is convenient over $\mathrm{Hol}(R)$ we know that the special restriction $v_P^* := v_P|_R$ of v_P is a PM valuation. There remains the problem to find criteria on P which guarantee that the valuation v_P itself is PM. More generally we may ask for any given ring R and prime cone P of R whether the valuation v_P is special. We defer these questions to the next section, §4.

§4 A brief look at real spectra

Let R be any ring (commutative with 1, as always). In §3 we defined the real spectrum $\mathrm{Sper}R$ as the set of prime cones of R. We now will introduce a topology on $\mathrm{Sper}R$. For this we need some more notations in addition to the ones established in §3.

The proofs of all facts on real spectra stated below can be found in most texts on "abstract" semialgebraic geometry and related real algebra, in particular in [BCR], [KS], [La$_1$]. We will give some of these proofs for the convenience of the reader.

Notations. Given a prime cone P on R let $k(P)$ denote a fixed real closure of the residue class field $k(\mathfrak{q})$ of $\mathfrak{q} := \operatorname{supp} P$ with respect to the ordering $\hat{P}$ induced by P on $k(\mathfrak{q})$. Further let r_P denote the natural homomorphism $R \to R/\mathfrak{q} \hookrightarrow k(\mathfrak{q}) \hookrightarrow k(P)$ from R to $k(P)$. Finally, for any $f \in R$, we define the "value" $f(P)$ of f at P by $f(P) := r_P(f)$. Thus $f(P) = f + \mathfrak{q}$, regarded as an element of $k(P)$.

Given $f \in R$ and $P \in \mathrm{Sper}R$ we either have $f(P) > 0$ or $f(P) = 0$ or $f(P) < 0$. Here we refer to the unique ordering of $k(P)$ (which we do not give a name). Notice that $f(P) = 0$ means $f \in \operatorname{supp} P$, and that $f(P) \geq 0$ iff there is some $\xi \in k(P)$ with $f(P) = \xi^2$.

Remark 4.1. In these notations we can rewrite the definition of $\mathrm{conv}_P(\Lambda)$ and of $I_P(\Lambda)$ for any subring Λ of R (cf. §3) as follows.

$$\begin{aligned}\mathrm{conv}_P(\Lambda) &= \{f \in R \mid \exists\, \lambda \in \Lambda : |f(P)| \leq |\lambda(P)|\} \\ &= \{f \in R \mid \exists\, \mu \in \Lambda : |f(P)| < |\mu(P)|\}, \\ I_P(\Lambda) &= \{f \in R \mid \forall\, \lambda \in \Lambda : |f(P)\lambda(P)| \leq 1\} \\ &= \{f \in R \mid \forall\, \mu \in \Lambda : |f(P)\mu(P)| < 1\}.\end{aligned}$$

Here, of course, absolute values are meant with respect to the unique ordering of $k(P)$. □

If T is any subset of R, we define

$$\begin{aligned}\mathring{H}_R(T) &:= \{P \in \mathrm{Sper}R \mid f(P) > 0 \quad \text{for every} \quad f \in T\}, \\ \overline{H}_R(T) &:= \{P \in \mathrm{Sper}R \mid f(P) \geq 0 \quad \text{for every} \quad f \in T\} \\ &= \{P \in \mathrm{Sper}R \mid P \supset T\}, \\ Z_R(T) &:= \{P \in \mathrm{Sper}R \mid f(P) = 0 \quad \text{for every} \quad f \in T\}.\end{aligned}$$

If $T = \{f_1, \dots, f_r\}$ is finite, we more briefly write $\mathring{H}_R(f_1, \dots, f_r)$ etc. instead of $\mathring{H}_R(\{f_1, \dots, f_r\})$ etc. We usually suppress the subscript "R" if this does not lead to confusion. Notice that $Z(f) = \overline{H}(-f^2)$ and $Z(f_1,\dots,f_r) = Z(f_1^2+\cdots+f_r^2) = \overline{H}(-f_1^2-\cdots-f_r^2)$.

In fact we introduce two topologies on Sper$R$.

Definition 1. a) The *Harrison topology* $\mathcal{T}_{\mathrm{Har}}$ on Sper$R$ is the topology generated by $\mathfrak{H}_R := \{\overset{\circ}{H}_R(f) \mid f \in R\}$ as a subbasis of open sets.
b) A subset X of Sper$R$ is called *constructible* if X is a boolean combination in Sper$R$ of finitely many sets $\overset{\circ}{H}_R(f), f \in R$. We denote the set of all constructible subsets of Sper$R$ by $\mathcal{K}_R$. This is the boolean lattice of subsets of Sper$R$ generated by $\mathfrak{H}_R$.
c) The *constructible topology* $\mathcal{T}_{\mathrm{con}}$ on Sper$R$ is the topology generated by $\mathcal{K}_R$ as a basis of open sets. In this topology every $X \in \mathcal{K}_R$ is clopen, i.e. closed and open.

If nothing else is said we regard Sper$R$ as a topological space with respect to the Harrison topology $\mathcal{T}_{\mathrm{Har}}$, while $\mathcal{T}_{\mathrm{con}}$ will play only an auxiliary role. Of course, $\mathcal{T}_{\mathrm{con}}$ is a much finer topology than $\mathcal{T}_{\mathrm{Har}}$. We denote the topological space (Sper$R, \mathcal{T}_{\mathrm{Har}}$) simply by Sper$R$ and the space (Sper$R, \mathcal{T}_{\mathrm{con}}$) by $(\mathrm{Sper}R)_{\mathrm{con}}$.

$(\mathrm{Sper}R)_{\mathrm{con}}$ turns out to be a compact Hausdorff space. Thus Sper$R$ itself is quasicompact. Also, a constructible subset U of Sper$R$ is open iff U is the union of finitely many sets $\overset{\circ}{H}(f_1,\dots,f_r)$. We denote the family of open constructible subsets of Sper$R$ by $\overset{\circ}{\mathcal{K}}_R$ and the family of closed constructible subsets of Sper$R$ by $\overline{\mathcal{K}}_R$.

If R is a field then $\mathcal{T}_{\mathrm{con}}$ and $\mathcal{T}_{\mathrm{Har}}$ coincide, hence Sper$R$ is compact (= quasicompact and Hausdorff) in this case, but for R a ring Sper$R$ most often is not Hausdorff.

The support map supp : Sper$R \to \mathrm{Spec}\,R$ is easily seen to be continuous. Indeed, given $f \in R$, the basic open set $D(f) := \{\mathfrak{p} \in \mathrm{Spec}\,R \mid f \notin \mathfrak{p}\}$ of Spec R has the preimage $\{P \in \mathrm{Sper}R \mid f(P) \neq 0\} = \overset{\circ}{H}(f^2)$ under this map.

Every ring homomorphism $\varphi: R \to R'$ gives us a map

$$\mathrm{Sper}(\varphi) = \varphi^*: \quad \mathrm{Sper}R' \longrightarrow \mathrm{Sper}R,$$

defined by $\varphi^*(P') = \varphi^{-1}(P')$ for P' a prime cone of R'. It is easily seen (loc.cit.) that Sper(φ) is continuous with respect to the Harrison topology and also with respect to the constructible topology on both sets. In other terms, if $X \in \mathcal{K}_R$ (resp. $\overset{\circ}{\mathcal{K}}_R$, resp. $\overline{\mathcal{K}}_R$) then $(\varphi^*)^{-1}(X) \in \mathcal{K}_{R'}$ (resp. $\overset{\circ}{\mathcal{K}}_{R'}$, resp. $\overline{\mathcal{K}}_{R'}$).

Notice also that supp $(\varphi^{-1}(P')) = \varphi^{-1}(\mathrm{supp}\,\varphi)$. Thus we have a commutative square of continuous maps

$$\begin{array}{ccc} \mathrm{Sper} R' & \overset{\mathrm{Sper}(\varphi)}{\longrightarrow} & \mathrm{Sper} R \\ \downarrow \text{supp} & & \downarrow \text{supp} \\ \mathrm{Spec}\, R' & \underset{\mathrm{Spec}\,(\varphi)}{\longrightarrow} & \mathrm{Spec}\, R. \end{array}$$

Before continuing our discussion of properties of real spectra, we give an application of the compactness of $(\mathrm{Sper} R)_{\mathrm{con}}$ to the theory of relative real holomorphy rings, displayed in §2 and §3, by improving Corollary 3.11.

Theorem 4.2. Let Λ be any subring of the ring R. Given an element f of R, the following are equivalent.
(i) $f \in \mathrm{Hol}(R/\Lambda)$.
(ii) There exists some $\lambda \in \Lambda$ with $|f(P)| \leq |\lambda(P)|$ for every $P \in \mathrm{Sper} R$.
(iii) There exists some $\mu \in \Lambda$ with $1 + \mu^2 \pm f \in P$ for every $P \in \mathrm{Sper} R$.

Proof. The implication (iii) $\Rightarrow$ (ii) is trivial, and (ii) $\Rightarrow$ (i) is obvious by Corollary 3.11.
(i) $\Rightarrow$ (iii): For every $P \in \mathrm{Sper} R$ we choose an element $\lambda_P \in P$ with $\lambda_P \pm f \in P$. This is possible by Corollary 3.11. Then also $1 + \lambda_P^2 \pm f \in P$. In other terms, $P \in \overline{H}(1 + \lambda_P^2 + f,\, 1 + \lambda_P^2 - f)$. Thus $\mathrm{Sper} R$ is covered by the sets $X_P := \overline{H}(1 + \lambda_P^2 + f,\, 1 + \lambda_P^2 - f)$ with P running through $\mathrm{Sper} R$. Since $(\mathrm{Sper} R)_{\mathrm{con}}$ is compact, there exist finitely many points $P_1, \dots, P_r$ in $\mathrm{Sper} R$ such that

$$\mathrm{Sper} R = X_{P_1} \cup \cdots \cup X_{P_r}.$$

Let $\gamma := \lambda_{P_1}^2 + \cdots + \lambda_{P_r}^2 \in \Lambda$. Clearly $1 + (1+\gamma)^2 \pm f \in P$ for every $P \in \mathrm{Sper} R$. $\square$

Applying the theorem to $\Lambda = \mathbb{Z}$ we obtain

Corollary 4.3. $\mathrm{Hol}(R)$ is the set of all $f \in R$ such that there exists some $n \in \mathbb{N}$ with $n \pm f \in P$, i.e. $|f(P)| \leq n$, for every $P \in \mathrm{Sper} R$. $\square$

We return to the study of the space $\mathrm{Sper} R$ for R any ring. As in any topological space we say that a point $Q \in \mathrm{Sper} R$ is a *specialization* of a point $P \in \mathrm{Sper} R$ if Q lies in the closure $\overline{\{P\}}$ of the one-point set $\{P\}$.

Proposition 4.4. If P and Q are prime cones of R, then Q is a specialization of P (in $\mathrm{Sper} R$) iff $P \subset Q$.

Proof. $Q \in \overline{\{P\}}$ iff for every open subset U of $\mathrm{Sper} R$ with $Q \in U$ also $P \in U$. It suffices to know this for the $U \in \mathfrak{H}_R$. Thus $Q \in \overline{\{P\}}$ iff for every $f \in R$ with

$f(Q) > 0$ also $f(P) > 0$; in other terms, iff for every $g \in R$ with $g(P) \geq 0$ we have $g(Q) \geq 0$. {Take $g = -f$.} This means that $P \subset Q$. $\square$

In the following P is a fixed prime cone of R. How do we obtain the prime cones $Q \supset P$? As in §3, let $\mathfrak{q}$ denote the support of P, $\mathfrak{q} = P \cap (-P)$. Recall from §3 that $\mathfrak{q}$ is the smallest P-convex additive subgroup of R.

LEMMA 4.5. Let $\mathfrak{a}$ be a P-convex additive subgroup of R and $T := P + \mathfrak{a}$. Then $T = P \cup \mathfrak{a}$ and $T \cap (-T) = \mathfrak{a}$.

PROOF. i) Let $p \in P$ and $a \in \mathfrak{a}$. If $p + a \notin P$ then $-(p + a) \in P$ and $-a = p - (p + a) \in \mathfrak{a}$. Since $\mathfrak{a}$ is P-convex, it follows that $-(p + a) \in \mathfrak{a}$, hence $p + a \in \mathfrak{a}$. This proves that $T = P \cup \mathfrak{a}$.
ii) Of course, $\mathfrak{a} \subset T \cap (-T)$. Let $x \in T$ be given, and assume that $x \notin \mathfrak{a}$. Then, as just proved, $x \in P$. But $x \notin -P$ since $P \cap (-P) \subset \mathfrak{a}$. Thus $x \notin -T$. This proves that $T \cap (-T) = \mathfrak{a}$. $\square$

THEOREM 4.6. The prime cones $Q \supset P$ correspond uniquely with the P-convex prime ideals $\mathfrak{r}$ of R via

$$Q = P + \mathfrak{r} = P \cup \mathfrak{r}\,, \quad \mathfrak{r} = \operatorname{supp} Q.$$

PROOF. a) If $\mathfrak{r}$ is a P-convex prime ideal of R then $Q := P + \mathfrak{r}$ is closed under addition and multiplication and $Q \cup (-Q) = R$. By Lemma 5 we know that $Q \cap (-Q) = \mathfrak{r}$. Thus Q is a prime cone with support $\mathfrak{r}$. Also $Q = P \cup \mathfrak{r}$ by Lemma 5.
b) Let Q be a prime cone of R containing P. Then $\mathfrak{r} := \operatorname{supp} Q$ is a Q-convex prime ideal of R. Since $P \subset Q$, it follows that $\mathfrak{r}$ is P-convex. We have $P + \mathfrak{r} \subset Q$. Let $f \in Q$ be given, and assume that $f \notin P$. Then $-f \in P \subset Q$, hence $f \in \mathfrak{r}$. We conclude that $Q \subset P \cup \mathfrak{r}$. Thus $Q = P + \mathfrak{r} = P \cup \mathfrak{r}$. $\square$

As observed in §3, the P-convex prime ideals of R form a chain under the inclusion relation. We know by §3 that $I_P(R)$ is the maximal element of this chain (cf. Scholium 3.13). Thus we infer from Proposition 4 and Theorem 6 the following

COROLLARY 4.7. The specialisations of $P \in \operatorname{Sper} R$ form a chain under the specialisation relation. In other terms, if Q_1 and Q_2 are prime cones with $P \subset Q_1$ and $P \subset Q_2$, then $Q_1 \subset Q_2$ or $Q_2 \subset Q_1$. The maximal specialisation of P is

$$P^* := \; P \cup I_P(R) \; = \; P + I_P(R). \qquad \square$$

Thus P^* is the unique closed point of $\operatorname{Sper} R$ in the set $\overline{\{P\}}$ of specialisations of P. We now analyze the situation that P itself is a closed point of $\operatorname{Sper} R$. This will give an answer to the question posed at the end of §3.

DEFINITION 2. a) Let Λ be a subring of R. We say that R is *archimedean over* Λ *with respect to* P if $\text{conv}_P(\Lambda) = R$, i.e. for every $f \in R$ there exists some $\lambda \in \Lambda$ with $|f(P)| \leq |\lambda(P)|$.
b) If K is a real closed field and Λ a subring of K, we say that K *is archimedean over* Λ if this holds with respect to the unique ordering of K.

THEOREM 4.8. Let P be a prime cone of R, and $\mathfrak{q} := \text{supp}\, P$. The following are equivalent.
(i) P is a closed point of $\text{Sper}R$.
(ii) $\mathfrak{q} = I_P(R)$.
(ii′) $\mathfrak{q}$ is the only proper P-convex ideal of R.
(iii) The field $k(\mathfrak{q})$ is archimedean over $R/\mathfrak{q}$ with respect to $\hat{P}$.
(iv) $k(P)$ is archimedean over $R/\mathfrak{q}$.
(v) The valuation v_P is special.

PROOF. The equivalence (i) $\Leftrightarrow$ (ii) is evident from Corollary 7, and (ii) $\Leftrightarrow$ (ii′) follows from the general observation (cf. §3) that $I_P(R)$ is the biggest proper P-convex ideal of R while $\mathfrak{q}$ is the smallest one. The equivalence (iii) $\Leftrightarrow$ (iv) follows from the well known fact that $k(P)$ is archimedean over $k(\mathfrak{q})$ since $k(P)$ is algebraic over $k(\mathfrak{q})$.
(ii′) $\Leftrightarrow$ (iii): Recall that for every $f \in R$ the image of $f + \mathfrak{q}$ of f in $\overline{R} := R/\mathfrak{q}$ has been denoted by $f(P)$. Recall also that the ordering $\hat{P}$ induced by P on $k(\mathfrak{q})$ is just the restriction of the unique ordering of $k(P)$ to $k(\mathfrak{q})$. A general element of $k(\mathfrak{q})$ has the form $\frac{f(P)}{g(P)}$ with $f, g \in R$ and $g \notin \mathfrak{q}$. The field $k(\mathfrak{q})$ is archimedean over $\overline{R}$ with respect to $\hat{P}$ iff for every such elements f, g there exists some $h \in R$ with $\left|\frac{f(P)}{g(P)}\right| \leq |h(P)|$. This property can also be stated as follows: $\text{conv}_{\overline{P}}(\overline{g}\overline{R}) = \overline{R}$ for every $g \in R \setminus \mathfrak{q}$ where $\overline{g} := g + \mathfrak{q}$. Translating back to R we see that (iii) means that $\text{conv}_P(gR) = R$ for every $g \in R \setminus \mathfrak{q}$. Clearly this holds iff $\mathfrak{q}$ is the only proper P-convex ideal of R.
(ii) $\Leftrightarrow$ (v): Let $v := v_P$ and $A := A_P = A_v$. We have $\mathfrak{p}_v = I_P$ and $\text{supp}\, v = \text{supp}\, P = \mathfrak{q}$. We first study the case that $A = R$. Now $I_P = I_P(R)$, and v is special iff v is trivial. This means that $\text{supp}\, v = \mathfrak{p}_v$, i.e. $\mathfrak{q} = I_P(R)$ in our case.

From now on we may assume that $A \neq R$. By Scholium 3.13 we have

$$I_P(R) = \{x \in R \mid Rx \subset I_P\} \ = \ \{x \in R \mid \forall\, y \in R{:}\, v(xy) > 0\}.$$

Since there exists some $z \in R$ with $v(z) < 0$, it follows that

$$I_P(R) = \{x \in R \mid \forall\, y \in R{:}\, v(xy) \geq 0\} \ = \ \{x \in R \mid Rx \subset A\}.$$

Thus $I_P(R)$ is the conductor $\mathfrak{q}_A$ of R in A. Proposition I.2.2 tells us that v is special iff $\text{supp}\, v = \mathfrak{q}_A$. This means $\mathfrak{q} = I_P(R)$ in our case. $\square$

Taking into account the study of real valuations in §3 we obtain

Corollary 4.9. Assume that R is convenient over $\mathrm{Hol}(R)$. Then the nontrivial real PM-valuations on R are precisely the coarsenings of the valuations v_P with P running through the closed points of $\mathrm{Sper}R$. $\square$

Lemma 4.10. Assume that P and Q are prime cones of R with $P \subset Q$.
a) For every subring Λ of R, we have $\mathrm{conv}_P(\Lambda) = \mathrm{conv}_Q(\Lambda)$ and $I_P(\Lambda) = I_Q(\Lambda)$. In particular, choosing $\Lambda = \mathbb{Z}$, we have $A_P = A_Q$ and $I_P = I_Q$.
b) If M is any additive subgroup of R then $\mathrm{conv}_P(M) = \mathrm{conv}_Q(M)$.

Proof. a): First notice that for any elements $f \in R$, $g \in R$ we have $|f(P)| < |g(P)|$ iff $(g^2 - f^2)(P) > 0$ and $|f(P)| \le |g(P)|$ iff $(g^2 - f^2)(P) \ge 0$. Thus $|f(Q)| < |g(Q)|$ implies $|f(P)| < |g(P)|$, and $|f(P)| \le |g(P)|$ implies $|f(Q)| \le |g(Q)|$. The assertions now follow from the various ways to characterize the elements of $\mathrm{conv}_P(\Lambda)$, $I_P(\Lambda)$, $\dots$ either by weak inequalities ($\le$) or by strong inequalities ($<$), cf. Remark 4.1 above.
b): This can be proved in the same way. $\square$

Definition 3. a) If $v\colon R \to \Gamma \cup \infty$ is any valuation on R we denote the valuation $v|c_v(\Gamma)\colon R \to c_v(\Gamma)$ (cf. notations in I, §2) by v^*, and we call v^* the *special valuation associated to* v. {N.B. We have $v^* = v|_{R}\cdot$}
b) If P is any prime cone on R we denote the maximal specialisation of P in $\mathrm{Sper}R$ (i.e. the unique closed point of $\overline{\{P\}}$) by P^*, as we did already above (Corollary 7).

Proposition 4.11. Assume that R is convenient over $\mathrm{Hol}(R)$. Given a prime cone P of R, the valuations $(v_P)^*$ and v_{P^*} are equivalent.

Proof. Let $v := v_P$, $u := v_{P^*}$. By Theorem 8 we know that u is special. By Lemma 10.a we have

$$A_v = A_P = A_{P^*} = A_u \quad , \quad \mathfrak{p}_v = I_P = I_{P^*} = \mathfrak{p}_u.$$

Both u and v^* are special valuations on R over $\mathrm{Hol}(R)$, hence are PM-valuations. We have $A_{v^*} = A_v = A_u$, $\mathfrak{p}_{v^*} = \mathfrak{p}_v = \mathfrak{p}_u$. We conclude (by I, §2) that u and v^* are equivalent. $\square$

Open problem. Does $(v_P)^* \sim v_{P^*}$ hold for any ring R and prime cone P of R?

Example 4.12 (The real spectra of $C(X)$ and $C_b(X)$). Let X be a completely regular Hausdorff space. Then the ring $R := C(X)$ is real closed in the sense of Schwartz (cf. [Sch], [Sch$_1$]). This implies that the support map $\mathrm{supp}\colon \mathrm{Sper}R \to \mathrm{Spec}\, R$ is a homeomorphism (loc.cit.). By restriction we obtain a bijection from the set $(\mathrm{Sper}R)^{\max}$ of closed points of $\mathrm{Sper}R$ to the set of closed points $(\mathrm{Spec}\, R)^{\max} = \mathrm{Max}R$ of $\mathrm{Spec}\, R$. On the other hand we have a bijection $\beta X \xrightarrow{\sim} \mathrm{Max}R$, $p \mapsto M^p$ (cf.1.4 above).

Let us regard βX as the set of ultrafilters α on the lattice $\mathcal{Z}(X)$. By what has been said there corresponds to each ultrafilter $\alpha \in \beta X$ a unique prime cone P_α of R with $\operatorname{supp} P_\alpha = M^\alpha$. We now describe this prime cone P_α. If $f \in R$ is given then both the sets $\{f \geq 0\} := \{x \in X \mid f(x) \geq 0\}$ and $\{-f \geq 0\}$ are elements of $\mathcal{Z}(X)$, and their union is X. Thus at least one of these sets is an element of α. Let

$$P := \{f \in R \mid \{f \geq 0\} \in \alpha\}.$$

Then we know already that $P \cup (-P) = R$. Clearly $P + P \subset P$ and $P \cdot P \subset P$. Also

$$P \cap (-P) = \{f \in R \mid Z(f) \in \alpha\} = M^\alpha$$

(cf.[GJ,§6]). Thus P is a prime cone of R with support M^α. We conclude that $P = P_\alpha$.

If α is not an ultrafilter but just a prime filter on the lattice $\mathcal{Z}(X)$ then we still see as above that

$$P_\alpha := \{f \in R \mid \{f \geq 0\} \in \alpha\}$$

is a prime cone on R. But not every prime cone of R is one of these P_α. The map $\alpha \mapsto P_\alpha$ is a bijection from the set of prime filters on $\mathcal{Z}(X)$ to a proconstructible subset of Sper$R$, the so called *real z-Spectrum* z-Sper$R$, cf.[Sch$_3$]. Under the support map we have a homeomorphism from z-Sper$R$ to the space z-Spec R constisting of the z-prime ideals of R, which have already much been studied in [GJ].

The ring $A := C_b(X)$ of bounded continuous real functions on X is again real closed. But now the situation is simpler. We have a bijection $\beta X \xrightarrow{\sim} \mathrm{Max}A$, $\alpha \mapsto \mathfrak{m}_\alpha$ (cf.1.4) and a bijection $(\mathrm{Sper}A)^{\max} \xrightarrow{\sim} \mathrm{Max}A$ by the support map. Thus to every $\alpha \in \beta X$ there corresponds a unique prime cone $P'_\alpha \in (\mathrm{Sper}A)^{\max}$ with $\operatorname{supp} P'_\alpha = \mathfrak{m}_\alpha$. We have

$$\mathfrak{m}_\alpha = \{f \in A \mid f^\beta(\alpha) = 0\}$$

and guess easily that

$$P'_\alpha = \{f \in A \mid f^\beta(\alpha) \geq 0\}.$$

Also $A/\mathfrak{m}_\alpha = \mathbb{R}$, hence $k(P_\alpha) = \mathbb{R}$. Clearly $A \cap P_\alpha \subset P'_\alpha$. Thus P'_α is the maximal specialization of $A \cap P_\alpha$ in the real spectrum Sper$A$, i.e.
$P'_\alpha = (A \cap P_\alpha)^*$.

Example 4.13 (The special real valuations and the real holomorphy ring of $C(X)$). Let again X be a complete regular Hausdorff space, $R := C(X)$, $A := C_b(X)$. We retain the notations from 4.12. For every $\alpha \in \beta X$ we denote the valuation v_{P_α} more briefly by v_α. Since P_α is a closed point of Sper$R$, this valuation is special. Now $1 + R^2 \subset R^*$. Thus we know, say by §2, that Hol(R) is Prüfer in R. This implies that every v_α is a PM-valuation, hence

$v_\alpha \in pm(R/\mathrm{Hol}(R))$. Corollary 9 tells us that $pm(R/\mathrm{Hol}(R))$ is the set of coarsenings of the valuations v_α with α running through βX.

Let $A_\alpha := A_{v_\alpha}$ and $I_\alpha := \mathfrak{p}_{v_\alpha}$. We know by Lemma 3.3 that

$$A_\alpha = \{f \in R \mid \exists\, n \in \mathbb{N}: \quad n \pm f \in P_\alpha\},$$

$$I_\alpha = \{f \in R \mid \forall\, n \in \mathbb{N}: \quad \frac{1}{n} \pm f \in P_\alpha\}.$$

For every $f \in R$ and $n \in \mathbb{N}$ we introduce the set

$$\begin{aligned} Z_n(f) &:= \{x \in X \mid n + f(x) \geq 0\} \cap \{x \in X \mid n - f(x) \geq 0\} \\ &= \{x \in X \mid |f(x)| \leq n\}. \end{aligned}$$

From the description of P_α above we read off that $f \in A_\alpha$ iff $Z_n(f)$ is an element of the ultrafilter α for some $n \in \mathbb{N}$. Thus A_α coincides with the subring A_α of R as defined in 2.1. In the same way we see that I_α is the ideal of A_α considered there and that $\mathrm{supp}\,(v_\alpha)$ is the ideal $\mathfrak{q}_\alpha$ of R considered there.

Using 2.1 we conclude that v_α is the PM-valuation of R over A corresponding to the prime ideal $\mathfrak{m}_\alpha$ of A. Thus $pm(R/\mathrm{Hol}R) = pm(R/A)$. This forces $\mathrm{Hol}(R) = A$. Using also 1.4 we conclude that

$$\omega(R/A) = \{v_\alpha \mid \alpha \in \beta X \setminus \upsilon X\}.$$

The result $\mathrm{Hol}(R) = A$ can also be verified as follows, using less information about the real valuations on R: We know by Corollary 3 above that a given element f of R is in $\mathrm{Hol}(R)$ iff there exists some $n \in \mathbb{N}$ such that $n \pm f \in P$ for every $P \in \mathrm{Sper} R$. Here we may replace $\mathrm{Sper} R$ by $(\mathrm{Sper} R)^{\max}$. Thus we see that $f \in \mathrm{Hol}(R)$ iff there exists some $n \in \mathbb{N}$ with $Z_n(f) \in \alpha$ for every ultrafilter α of the lattice $\mathcal{Z}(X)$. This means that $Z_n(f) = X$ for some $n \in \mathbb{N}$, i.e. f is bounded. □

§5 Convexity of subrings and of valuations

Let R be any ring. A subset T of R is called a *preordering* of R (or: a *cone* of R [BCR, p.86]), if T is closed under addition and multiplication and contains the set $R^2 = \{x^2 \mid x \in R\}$. We call a preordering T *proper* if $-1 \notin T$.

We associate with a preordering T of R a binary relation $\leq_T$ on R, defined by

$$f \leq_T g \iff g - f \in T.$$

This relation is transitive and reflexive but in general not antisymmetric. We define the *support* of T as the set

$$\operatorname{supp} T \;=\; T \cap (-T).$$

This is an additive subgroup of R. Clearly $f \leq_T g$ and $g \leq_T f$ iff $f-g \in \operatorname{supp} T$.

Of course, the prime cones $P \in \operatorname{Sper} R$ are preorderings, but there are many more. The intersection of any family of preorderings is again a preordering. In particular R has a smallest preordering, which we denote by T_0. Clearly $T_0 = \Sigma R^2$.

In the following T is a fixed preordering of R.

Definition 1. a) A subset M of R is called T-*convex* (in R) if for any three elements x, y, z of R with $x \leq_T y \leq_T z$ and $x \in M$, $z \in M$, also $y \in M$.
b) If U is any subset of R there clearly exists a smallest T-convex subset M of R containing U. We call M the T-*convex hull* of U, and we write $M = \operatorname{conv}_T(U)$. □

Remark. An additive subgroup M of R is T-convex iff for all $s \in T$, $t \in T$ with $s + t \in M$ we have $s \in M$ and (hence) $t \in M$. □

It is obvious that $\operatorname{supp} T = T \cap (-T)$ is the smallest T-convex additive subgroup of R. Notice also that the set $T - T$, consisting of the differences $t_1 - t_2$ of elements t_1, t_2 of T, is a T-convex subring of R, and that $\operatorname{supp} T$ is an ideal of the ring $T - T$.

If 2 is a unit in R we have $T - T = R$, as follows from the identity

$$x = 2\left[\left(\frac{1+x}{2}\right)^2 - \left(\frac{x}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right].$$

Later only rings with 2 a unit will really matter, but we can avoid this assumption here by enlarging T slightly.

Lemma 5.1. $T' := \{x \in R \mid \exists\, n \in \mathbb{N}: 2^n x \in T'\}$ is again a preordering of R. It is proper iff T is proper. $\square$

We omit the easy proof. We call T' the *2-saturation* of T, and we call T *2-saturated* if $T' = T$.

If T is 2-saturated, then $\operatorname{supp} T$ is an ideal of R due to the identity

$$2xy = (1+x)^2 y - x^2 y - y.$$

Of course, if $2 \in R^*$, then every preordering of R is 2-saturated. Notice also that every prime cone is a 2-saturated preordering.

Given a subring Λ and a preordering T of R we strive for an understanding and a handy description of the convex hull $\operatorname{conv}_T(\Lambda)$ of Λ in R. We introduce a new notation for this,

$$C(T, R/\Lambda) :=\ \operatorname{conv}_T(\Lambda),$$

which reflects that $\operatorname{conv}_T(\Lambda)$ also depends on the ambient ring R. It is easily seen that $C(T, R/\Lambda)$ is the set of all $x \in R$ with $\lambda_1 \underset{T}{\leq} x \underset{T}{\leq} \lambda_2$ for some elements λ_1, λ_2 of Λ. From this it is immediate that $C(T, R/\Lambda)$ is an additive subgroup of R. We also introduce the set

$$\begin{aligned} A(T, R/\Lambda) :&= \{x \in R \mid \exists\, \lambda \in T \cap \Lambda: \lambda \pm x \in T\} \\ &= \{x \in R \mid \exists\, \lambda \in T \cap \Lambda: -\lambda \leq_T x \leq_T \lambda\}. \end{aligned}$$

We use the abbreviations $C(T, R) := C(T, R/\mathbb{Z}1_R)$ and $A(T, R) := A(T, R/\mathbb{Z}1_R)$.

Given an additive subgroup M of R let M' denote the 2-saturation of M in R, i.e. the additive group consisting of all $x \in R$ such that $2^n x \in M$ for some $n \in \mathbb{N}_0$. If M is a subring of R then also M' is a subring of R.

Proposition 5.2. a) $A(T, R/\Lambda)$ is a T-convex subring of R contained in $C(T, R/\Lambda)$.
b) $C(T, R/\Lambda) = \Lambda + A(T, R/\Lambda)$.
c) $C(T, R/\Lambda) = A(T, R/\Lambda)$ iff Λ is generated by $\Lambda \cap T$ as an additive group, i.e., $\Lambda = (\Lambda \cap T) - (\Lambda \cap T)$.
d) $C(T, R) = A(T, R)$, and this is the smallest T-convex subring of R.
e) If T contains the 2-saturated hull T_0' of $T_0 = \Sigma R^2$ (e.g. T itself is 2-saturated), then $C(T, R/\Lambda) = A(T, R/\Lambda)$.
f) Without any extra assumption on T and Λ we have $A(T, R/\Lambda)' = C(T, R/\Lambda)'$ $= A(T', R/\Lambda) = C(T', R/\Lambda)$.

Proof. a) We first prove that $A(T, R/\Lambda)$ is a subring of R. Given elements x and y of $A(T, R/\Lambda)$ we choose elements λ and μ in $\Lambda \cap T$ such that $\lambda \pm x \in T$ and $\mu \pm y \in T$. Then we have

$$(\lambda + \mu) \pm (x - y) \in T,$$

which proves that $x - y \in A(T, R/\Lambda)$.

Moreover we have

$$(\lambda + x)(\mu + y) = \lambda\mu + \lambda y + \mu x + xy \in T$$

and

$$\lambda(\mu - y) \in T \quad , \quad \mu(\lambda - x) \in T.$$

By adding we obtain

$$3\lambda\mu + xy \in T.$$

Replacing x by $-x$ we obtain $3\lambda\mu - xy \in T$. This proves that $xy \in A(T, R/\Lambda)$.

Thus $A(T, R/\Lambda)$ is a subring of R. It is clear from the definition of $A(T, R/\Lambda)$ that this ring is contained in the T-convex hull $C(T, R/\Lambda)$ of Λ in R. Given elements x_1, x_2 of $A(T, R/\Lambda)$ and $y \in R$ with $x_1 \underset{T}{\leq} y \underset{T}{\leq} x_2$, we have elements λ_1, λ_2 of $\Lambda \cap T$ such that $-\lambda_1 \underset{T}{\leq} x_1 \underset{T}{\leq} \lambda_1$ and $-\lambda_2 \underset{T}{\leq} x_1 \underset{T}{\leq} \lambda_2$. These inequalities imply

$$-(\lambda_1 + \lambda_2) \underset{T}{\leq} x_1 \underset{T}{\leq} y \underset{T}{\leq} x_2 \underset{T}{\leq} (\lambda_1 + \lambda_2).$$

Thus $y \in A(T, R/\Lambda)$. This proves that $A(T, R/\Lambda)$ is T-convex in R.

b): It is evident that the additive group $M := \Lambda + A(T, R/\Lambda)$ is contained in $C(T, R/\Lambda)$. We are done if we verify that M is T-convex in R.

Let $s, t \in T$ be given with $s + t \in M$, hence $s + t = \lambda + x$ with $\lambda \in \Lambda$, $x \in A(T, R/\Lambda)$. We have $0 \underset{T}{\leq} s \underset{T}{\leq} \lambda + x$. There exists some $\mu \in \Lambda$ with $x \underset{T}{\leq} \mu$. Then $0 \underset{T}{\leq} s \underset{T}{\leq} \lambda + \mu$, and thus $\lambda + \mu \in \Lambda \cap T$. This proves that $s \in A(T, R/\Lambda) \subset M$.

c): $A(T, R/\Lambda) = C(T, R/\Lambda)$ means that $\Lambda \subset A(T, R/\Lambda)$. This is certainly true if $\Lambda = (\Lambda \cap T) - (\Lambda \cap T)$, since $\Lambda \cap T \subset A(T, R/\Lambda)$ by definition of $A(T, R/\Lambda)$.

It remains to verify that the inclusion $\Lambda \subset A(T, R/\Lambda)$ implies $\Lambda = (\Lambda \cap T) - (\Lambda \cap T)$. Let $\lambda \in \Lambda$ be given. There exists some $\mu \in \Lambda \cap T$ such that $\mu \pm \lambda \in T$. Then $\lambda = \mu - (\mu - \lambda)$, and both $\mu, \mu - \lambda \in \Lambda \cap T$.

d): Applying c) to $\Lambda = \mathbb{Z} \cdot 1_R$ we see that $C(T, R) = A(T, R)$. By definition $C(T, R)$ is the smallest T-convex additive subgroup of R containing 1_R, hence also the smallest T-convex subring of R.

e): For every $\lambda \in \Lambda$ we have

$$2(\lambda^2 + 1 \pm \lambda) = \lambda^2 + 1 + (\lambda \pm 1)^2 \in T_0,$$

hence $\lambda^2 + 1 \pm \lambda \in T_0' \subset T$. This implies $\Lambda \subset A(T, R/\Lambda)$, hence $C(T, R/\Lambda) = A(T, R/\Lambda)$.

f): We first verify that $A(T, R/\Lambda)' = A(T', R/\Lambda)$. Given $x \in A(T, R/\Lambda)'$, we have some $n \in \mathbb{N}$ with $2^n x \in A(T, R/\Lambda)$, hence $\lambda \pm 2^n x \in T$ for some $\lambda \in T \cap \Lambda$. It follows that $2^n(\lambda \pm x) \in T$, hence $\lambda \pm x \in T'$, hence $x \in A(T', R/\Lambda)$.

Conversely, if $x \in A(T', R/\Lambda)$ we have some $\lambda \in T' \cap \Lambda$ with $\lambda \pm x \in T'$ and then some $n \in \mathbb{N}$ with $2^n \lambda \in T \cap \Lambda$ and $2^n \lambda \pm 2^n x \in T$. Thus $2^n x \in A(T, R/\Lambda)$, and $x \in A(T, R/\Lambda)'$.

This completes the proof that $A(T, R/\Lambda)' = A(T', R/\Lambda)$. Now observe that $A(T, R/\Lambda) \subset C(T, R/\Lambda) \subset C(T', R/\Lambda)$. As proved above, $C(T', R/\Lambda) = A(T', R/\Lambda) = A(T, R/\Lambda)'$. In particular we know that $C(T', R/\Lambda)$ is 2-saturated. It follows that

$$A(T, R/\Lambda)' \subset C(T, R/\Lambda)' \subset C(T', R/\Lambda) = A(T, R/\Lambda)'.$$

Thus the groups $A(T, R/\Lambda)'$, $A(T', R/\Lambda)$, $C(T, R/\Lambda)'$, $C(T', R/\Lambda)$ are all the same. $\square$

We aim at a description of the rings between $\mathrm{Hol}(R)$ and R by T-convexity for varying preorderings T in the case that $\mathrm{Hol}(R)$ is Prüfer in R. Here preorderings will play a dominant role which are "saturated" in the sense of the following definition.

Definition 2. The *saturation* $\hat{T}$ of a preordering T of R is the intersection of all prime cones $P \supset T$ of R. In other terms,

$$\hat{T} = \{f \in R \mid \forall\, P \in \overline{H}_R(T) \colon f(P) \geq 0\}.$$

T is called *saturated* if $\hat{T} = T$. $\square$

Of course, $\hat{T}$ is always 2-saturated. More generally $\hat{T}$ is saturated with respect to the multiplicative subset $1 + T$ of R, i.e. for any $x \in R$, $t \in T$:

$$(1 + t)x \in \hat{T} \Longrightarrow x \in \hat{T}.$$

Notice that the saturation $\hat{T}_0$ of $T_0 = \Sigma R^2$ is the set of all $f \in R$ which are nonnegative on $\mathrm{Sper} R$. Thus, taking into account Proposition 2, the description of $\mathrm{Hol}(R/\Lambda)$ in Theorem 4.2 can be read as follows.

Scholium 5.3. For any ring extension $\Lambda \subset R$

$$\mathrm{Hol}(R/\Lambda) = A(\hat{T}_0, R/\Lambda) = C(\hat{T}_0, R/\Lambda). \qquad \square$$

Every proper preordering of a field is saturated, as is very well known ([BCR, p.9], [KS, p.2]). In the field case we have $T \cap (-T) = \{0\}$. Then a proper preordering is a partial ordering of the field in the usual sense.

We recall without proof the famous *abstract Positivstellensatz* about an algebraic description of $\hat{T}$ in terms of T for R an arbitrary ring.

THEOREM 5.4. (cf.[BCR, p.92], [KS, p.143]). If T is any preordering of R and $a \in R$, the following are equivalent.
(1) $a \in \hat{T}$.
(2) $-a^{2n} \in T - aT$ for some $n \in \mathbb{N}_0$.
(3) There exist $t, t' \in T$ and $n \in \mathbb{N}_0$ with $a(a^{2n} + t) = t'$. □

The theorem tells us in particular (take $a = -1$) that for T proper, i.e. $-1 \notin T$, also $\hat{T}$ is proper. It follows that for a proper preordering T there always exists some prime cone $P \supset T$.

In order to get a somewhat "geometric" understanding of saturated preorderings we introduce more terminology.

DEFINITIONS 3. a) Given any subset X of Sper$R$, let $P(X)$ denote the intersection of the prime cones $P \in X$. In other terms,

$$P(X) := \{f \in R \mid \forall x \in X\colon\ f(x) \geq 0\}.$$

In particular, for every $x \in X$, $P(\{x\})$ is the point x itself, viewed as a prime cone, $P(\{x\}) = P_x$.
b) We call a subset X of Sper$R$ *basic closed*, if

$$X \ = \ \overline{H}_R(\Phi) \ = \ \{x \in \mathrm{Sper}R \mid f(x) \geq 0 \quad \text{for every} \quad f \in \Phi\}$$

for some subset Φ of R, i.e. X is the intersection of a family of "principal closed" sets $\overline{H}_R(f) = \{x \in \mathrm{Sper}R \mid f(x) \geq 0\}$. □
c) If X is any subset of Sper$R$, let $\hat{X}$ denote the smallest basic closed subset of Sper$R$ containing X, i.e. the intersection of all principal closed sets $\overline{H}_R(f)$ containing X. We call $\hat{X}$ the *basic closed hull* of X.
d) If Φ is any subset of R, there exists a smallest preordering T containing Φ. This is the semiring generated by $\Phi \cup R^2$ in R. We call T the *preordering generated by* Φ, and write $T = T(\Phi)$.

REMARKS 5.5. i) For every $X \subset \mathrm{Sper}R$ the set $P(X)$ is a saturated preordering of R and $\overline{H}_R(P(X)) = \hat{X}$. It follows that $P(\hat{X}) = P(X)$. Moreover $\hat{X}$ is the unique maximal subset Y of Sper$R$ with $P(Y) = P(X)$.
ii) If Φ is any subset of R then $\overline{H}_R(\widehat{T(\Phi)}) = \overline{H}_R(\Phi)$. Moreover $\widehat{T(\Phi)}$ is the unique maximal subset U of R with $\overline{H}_R(U) = \overline{H}_R(\Phi)$.
iii) The basic closed subsets Z of Sper$R$ correspond uniquely with the saturated preorderings T of R via $T = P(Z)$ and $Z = \overline{H}_R(T)$.

All this can be verified easily in a straightforward way. $\square$

If X is any subset of Sper$R$ we call a $P(X)$-*convex subset* M of R also X-*convex*. In the case that term X is a one-point set $\{x\}$, we use the term "x-convex". {Thus x-convexity is the same as P-convexity for $P = x$, regarded as prime cone.}

Instead of $A(P(X), R/\Lambda)$ we write $A_X(R/\Lambda)$. Thus

$$A_X(R/\Lambda) = \{f \in R \mid \exists\, \lambda \in \Lambda \quad \text{such that} \quad |f(x)| \leq \lambda(x) \quad \text{for every } x \in X\}.$$

{Read $A_X(R/\Lambda) = R$ if X is empty.} By Proposition 2 we have
$C(P(X), R/\Lambda) = A_X(R/\Lambda)$.

Let again T be any preordering of a ring R. There exists a by now well known and well developed theory of T-convex prime ideals of R which we will need below (cf. [Br], [Br$_1$], [KS, Chap.III, §10]). The main result can be subsumed in the following theorem.

Theorem 5.6. a) Let T be a proper preordering of R and $\mathfrak{p}$ a prime ideal of R. Then $\mathfrak{p}$ is T-convex iff $\mathfrak{p}$ is $\hat{T}$-convex. In this case there exists a prime cone $P \supset T$ such that $\mathfrak{p}$ is P-convex.
b) Let X be a closed subset of Sper$R$. The X-convex prime ideals of R are precisely the supports supp(P) of the prime cones $P \in X$. $\square$

We do not give the proof here,*) refering the reader to [KS, Chap.III, §10] for this, but we state two key observations leading to the theorem.

Proposition 5.7 ([KS, p.148]). Let T be any preordering of R. The maximal proper T-convex ideals of R are the ideals $\mathfrak{a}$ of R which are maximal with the property $\mathfrak{a} \cap (1 + T) = \emptyset$. They are prime. $\square$

{N.B. This holds also in the case that $-1 \in T$. Then R itself is the only T-convex ideal of R.}

Proposition 5.8 (A. Klapper, cf. [Br, p.63], [KS, p.149]). Let T_1 and T_2 be preorderings of R and $\mathfrak{p}$ a prime ideal of R. Assume that $\mathfrak{p}$ is $(T_1 \cap T_2)$-convex. Then $\mathfrak{p}$ is T_1-convex or T_2-convex. $\square$

For later use we also mention

Lemma 5.9. Let T be a proper preordering of R and $\mathfrak{a}$ a T-convex proper ideal of R. Then $T_1 := T + \mathfrak{a}$ is again a proper preordering of R and $T_1 \cap (-T_1) = \mathfrak{a}$. The image $\overline{T} = T_1/\mathfrak{a}$ of T in $R/\mathfrak{a}$ is a proper preordering of $R/\mathfrak{a}$, and $\overline{T} \cap (-\overline{T}) = \{0\}$.

*) In fact part a) will be proved below as a special case of Theorem 16.

We leave the easy proof to the reader. □

As before let T be a fixed preordering of R.

Definition 4. We say that a *valuation* $v: R \to \Gamma \cup \infty$ is T-*convex* if the prime ideal $\operatorname{supp} v$ is T-convex in R and, for every $\gamma \in \Gamma$, the additive group $I_{\gamma,v} = \{x \in R \mid v(x) \geq \gamma\}$ is T-convex in R. In other terms, v is T-convex iff for any elements x, y of R with $0 \leq_T y \leq_T x$ we have $v(y) \geq v(x)$. If $T = P(X)$ for some set $X \subset \mathrm{Sper} R$, we also use the term "$X$-convex" instead of T-convex.

Comment. In the – not very extended – literature these valuations are usually called "compatible with T". The term "T-convex" looks more imaginative, in particular if one follows the philosophy (as we do) that valuations are refinements of prime ideals.

Several observations on real valuations stated in §2 extend readily to T-convex valuations.

Remarks 5.10. Let $v: R \to \Gamma \cup \infty$ be a valuation.
i) The following are clearly equivalent.
  (1) v is T-convex.
  (2) If $x \in T$ and $y \in T$ then $v(x) \geq v(x+y)$.
  (3) If $x \in T$ and $y \in T$ then $v(x+y) = \min(v(x), v(y))$.
In particular, v is T_0-convex iff v is real (cf. Prop.2.2.). Every T-convex valuation is real.
ii) If T is improper, i.e. $-1 \in T$, there do not exist T-convex valuations.
iii) If v is trivial then v is T-convex iff $\operatorname{supp} v$ is T-convex in R. The T_0-convex prime ideals are just the real prime ideals.
iv) If v is T-convex, both A_v and $\mathfrak{p}_v$ are T-convex in R.
v) Assume that v is T-convex. For every convex subgroup H of Γ the coarsening v/H is again T-convex. If H contains the characteristic subgroups $c_v(\Gamma)$ then also $v|H$ is T-convex.
vi) If B is a subring of R and v is T-convex, then both the valuations $v|B$ and $v|_B$ are $(T \cap B)$-convex. □

In the case of Manis valuations we have very handy criteria for T-convexity.

Theorem 5.11. Let v be a Manis valuation on R.
i) The following are equivalent.
  (1) v is T-convex.
  (2) $\mathfrak{p}_v$ is T-convex in R.
  (3) $\mathfrak{p}_v$ is $(T \cap A_v)$-convex in A_v.
ii) If v is non trivial, then (1) – (3) are also equivalent to
  (4) A_v is T-convex in R.

PROOF. If v is trivial the equivalence of (1), (2), (3) is evident. Henceforth we assume that v is not trivial. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) are evident from the definition of T-convexity of valuations (cf. Def.4 above). The implication (2) $\Rightarrow$ (3) is trivial.

(4) $\Rightarrow$ (1): Assume that A_v is T-convex in R. Let $x, y \in R$ be given with $0 \leq_T y \leq_T x$ and $v(x) \neq \infty$. We choose some $z \in R$ with $v(xz) = 0$. This is possible since v is Manis. We have $0 \leq_T (yz)^2 \leq_T (xz)^2$. {Notice that $x^2 - y^2 = (x-y)(x+y) \in T$.} Since $(xz)^2 \in A_v$ and A_v is T-convex, it follows that $(yz)^2 \in A_v$, hence $2v(yz) \geq 0$, hence $v(y) \geq -v(z) = v(x)$. This proves that $I_{v,\gamma}$ is T-convex in R for every $\gamma \in \Gamma_v$. The support of v is the intersection of all these $I_{v,\gamma}$, since v is not trivial. Thus $\operatorname{supp} v$ is T-convex in R. This finishes the proof that v is T-convex.

(2) $\Rightarrow$ (4): Assume that $\mathfrak{p}_v$ is T-convex in R. Since v is Manis we have $A_v = \{x \in R \mid x\mathfrak{p}_v \subset \mathfrak{p}_v\}$. Let $0 \leq_T y \leq_T x$ and $x \in A_v$. For every $z \in \mathfrak{p}_v$ this implies $0 \leq_T (yz)^2 \leq_T (xz)^2 \in \mathfrak{p}_v$. Since $\mathfrak{p}_v$ is T-convex in R, we conclude that $(yz)^2 \in \mathfrak{p}_v$, and then that $yz \in \mathfrak{p}_v$. This proves that $y\mathfrak{p}_v \subset \mathfrak{p}_v$, hence $y \in A_v$.

(3) $\Rightarrow$ (2): Assume that $\mathfrak{p}_v$ is $(T \cap A_v)$-convex in A_v. We verify that $\mathfrak{p}_v$ is T-convex in R. Let $x \in \mathfrak{p}_v$ and $y \in R$ be given with $0 \leq_T y \leq_T x$. Suppose that $y \notin \mathfrak{p}_v$, i.e. $v(y) \leq 0$. We choose some $z \in R$ with $v(yz) = 0$. Then $0 \leq_T (yz)^2 \leq_T (xz)^2$. Now $z \in A_v$, hence $(xz)^2 \in \mathfrak{p}_v$, and $(yz)^2 \in A_v$. It follows that $(yz)^2 \in \mathfrak{p}_v$, hence $yz \in \mathfrak{p}_v$. This contradicts $v(yz) = 0$. Thus $\mathfrak{p}_v$ is indeed T-convex in R. $\square$

Another proof of Theorem 11 can be found in [Z_1, §2].

COROLLARY 5.12. Let U be a preordering (= partial ordering) of a field K. A valuation v on K is U-convex iff the valuation domain A_v is U-convex in K.

PROOF. v is Manis. If v is nontrivial the claim is covered by Theorem 11.ii. If v is trivial, $\mathfrak{p}_v = \operatorname{supp} v = \{0\}$, which is U-convex. Now the claim is covered by Theorem 11.i. $\square$

COROLLARY 5.13. Assume that T and U are preorderings on R and that v is a Manis valuation on R which is $(T \cap U)$-convex. Then v is T-convex or U-convex.

PROOF. We work with condition (3) in Theorem 11. We know that $\mathfrak{p}_v$ is $(T \cap U \cap A_v)$-convex in A_v, and we conclude that $\mathfrak{p}_v$ is $T \cap A_v$-convex or $U \cap A_v$-convex in A_v by Proposition 8 above. $\square$

Returning to valuations which are not necessarily Manis we now prove a lemma by which the study of T-convex valuations on R can be reduced to the study of U-convex valuations for preorderings U on suitable residue class fields of R.

Lemma 5.14. Let T be a proper preordering of R and v a valuation on R. We assume that $\mathfrak{q} := \operatorname{supp} v$ is T-convex.
i) $T_1 := T + \mathfrak{q}$ is a proper preordering of R and $T_1 \cap (-T_1) = \mathfrak{q}$.
ii) Let $\overline{T} := T_1/\mathfrak{q}$ denote the image of T and of T_1 in $\overline{R} := R/\mathfrak{q}$. Then the subset

$$U := \left\{ \frac{\overline{x}}{\overline{s}^2} \mid x \in T,\ s \in R \setminus \mathfrak{q} \right\}$$

of the field $k(\mathfrak{q})$ is a proper preordering (= partial ordering) of $k(\mathfrak{q})$, and $T_2 := j_{\mathfrak{q}}^{-1}(U)$ is a proper preordering of R. {Here, of course, $\overline{x} := j_{\mathfrak{q}}(x)$, $\overline{s} := j_{\mathfrak{q}}(s)$, the images of x and s in $k(\mathfrak{q})$.} We have $T \subset T_1 \subset T_2$ and $T_2 \cap (-T_2) = \mathfrak{q}$.
iii): $T_2 = \{x \in R \mid \exists\, s \in R \setminus \mathfrak{q}\colon\ s^2 x \in T_1\}$.
iv) As always (cf.I, §1) we denote the valuations induced by v on $\overline{R}$ and $k(\mathfrak{q})$ by $\overline{v}$ and $\hat{v}$ respectively. The following are equivalent:
(1) v is T-convex.
(2) $\overline{v}$ is $\overline{T}$-convex.
(3) $\hat{v}$ is U-convex.
(4) v is T_2-convex.
(5) v is T_1-convex.

Proof. i): This is covered by Lemma 9 above.
ii): We know by Lemma 9 that $\overline{T}$ is a preordering of $\overline{R}$ with $\overline{T} \cap (-\overline{T}) = \{0\}$. It then is a straightforward verification that U is a proper preordering of $k(\mathfrak{q})$. We have $\overline{T} \subset U \cap \overline{R}$, hence $T_1 = j_{\mathfrak{q}}^{-1}(\overline{T}) \subset j_{\mathfrak{q}}^{-1}(U) =: T_2$. Also $T_2 \cap (-T_2) = j_{\mathfrak{q}}^{-1}(U \cap (-U)) = \mathfrak{q}$.
iii): An easy verification.
iv): (1) $\Leftrightarrow$ (2) is completely obvious by using, say, condition (3) in Remark 10.i characterizing convexity of valuations. The implications (4) $\Rightarrow$ (5) $\Rightarrow$ (1) are trivial since $T \subset T_1 \subset T_2$, and (3) $\Rightarrow$ (4) is immediate, due to the fact that $v = \hat{v} \circ j_{\mathfrak{q}}$ and $T_2 = j_{\mathfrak{q}}^{-1}(U)$.
(1) $\Rightarrow$ (3): Let $\xi_1, \xi_2 \in U$ be given. We verify condition (3) in Remark 10.i. We write

$$\xi_1 = \frac{\overline{t_1}}{\overline{s}^2} \quad , \quad \xi_2 = \frac{\overline{t_2}}{\overline{s}^2}$$

with $t_1, t_2 \in T$, $s \in R \setminus \mathfrak{q}$. Then

$$\xi_1 + \xi_2 = \frac{\overline{t_1 + t_2}}{\overline{s}^2},$$

and $v(s) \neq \infty$, $\hat{v}(\xi_1 + \xi_2) = v(t_1 + t_2) - 2v(s) = \min(v(t_1), v(t_2)) - 2v(s)$
$= \min(v(t_1) - 2v(s),\ v(t_2) - 2v(s)) = \min(\hat{v}(\xi_1),\ \hat{v}(\xi_2))$. $\square$

As a modest first application of Lemma 14 we analyse T-convexity for valuations in the case that T is a prime cone.

DEFINITION 5. Given valuations v and w on R, we write $v \leq w$ if w is a coarser than v (cf.I §1, Def.9).

Notice that $v \sim w$ iff $v \leq w$ and $w \leq v$.

THEOREM 5.15. Let P be a prime cone of R and v a valuation on R.
i) v is P-convex and $\operatorname{supp} v = \operatorname{supp} P$ iff $v_P \leq v$.
ii) v is P-convex iff there exists some prime cone $\tilde{P} \supset P$ such that $v_{\tilde{P}} \leq v$.

PROOF. i): Let $\mathfrak{q} := \operatorname{supp} P$. This is a P-convex prime ideal of R, in fact the smallest one. If $v_P \leq v$ then $\operatorname{supp} v = \operatorname{supp} v_P = \mathfrak{q}$. Thus we may assume from start that $\operatorname{supp} v = \mathfrak{q}$. Lemma 14 tells us that v is P-convex iff the valuation $\hat{v}$ on $k(\mathfrak{q})$ is $\hat{P}$-convex. {Here $\hat{P}$ denotes the ordering induced by P on $k(\mathfrak{q})$, as has been decreed in §3.} By Corollary 12 $\hat{v}$ is $\hat{P}$-convex iff the valuation ring $A_{\hat{v}} = \mathfrak{o}_v$ is $\hat{P}$-convex in $k(\mathfrak{q})$. This happens to be true iff $v_{\hat{P}} \leq \hat{v}$. Since $v_{\hat{P}} \circ j_{\mathfrak{q}} = v_P$ and $\hat{v} \circ j_{\mathfrak{q}} = v$, we have $v_{\hat{P}} \leq \hat{v}$ iff $v_P \leq v$.
ii): If there exists some prime cone $\tilde{P} \supset P$ with $v_{\tilde{P}} \leq v$ then v is $\tilde{P}$-convex, as we have proved, hence v is P-convex. Conversely, assume that v is P-convex. Then $\mathfrak{p} := \operatorname{supp} v$ is P-convex (hence $\mathfrak{q} \subset \mathfrak{p}$). $\tilde{P} := P \cup \mathfrak{p} = P + \mathfrak{p}$ is a prime cone of R containing P, and $\operatorname{supp} \tilde{P} = \mathfrak{p} = \operatorname{supp} v$ (cf.Th.4.6). We claim that v is $\tilde{P}$-convex, and then will know by i) that $v_{\tilde{P}} \leq v$.

This is pretty obvious. If $\tilde{x}, \tilde{y} \in \tilde{P}$, we have $\tilde{x} = x+a$, $\tilde{y} = y+b$ with $x, y \in P$ and $a, b \in \mathfrak{p}$. Then $v(\tilde{x}) = v(x)$, $v(\tilde{y}) = v(y)$, $v(\tilde{x}+\tilde{y}) = v(x+y)$, since also $a+b \in \mathfrak{p}$. We conclude that $v(\tilde{x} + \tilde{y}) = v(x + y) = \min(v(x), v(y)) = \min(v(\tilde{x}), v(\tilde{y}))$, which proves that v is $\tilde{P}$-convex. □

As before, let T be a preordering of R.

THEOREM 5.16. Assume that v is a T-convex valuation on R. Then there exists a prime cone $P \supset T$ of R such that v is P-convex and $\operatorname{supp} P = \operatorname{supp} v$ {hence $v_P \leq v$ by Th.15}.

PROOF. a) We first prove this in the case that $R = K$ is a field. Let $B := A_v$, $\mathfrak{m} := \mathfrak{p}_v$, and $U := T \cap B$. Then B is a T-convex Krull valuation ring of K with maximal ideal $\mathfrak{m}$, and $\mathfrak{m}$ is U-convex in B. By Lemma 9 we know that $U_1 := U + \mathfrak{m}$ is a proper preordering of B and that its image $U_1/\mathfrak{m} = \overline{U}$ in the residue class field $\kappa(B) = B/\mathfrak{m}$ is a proper preordering (= partial ordering) of $\kappa(B)$. We choose a prime cone (= total ordering) $\overline{Q}$ of $\kappa(B)$ containing $\overline{U}$. {Usually this can be done in several ways.} Let $\pi: B \twoheadrightarrow \kappa(B)$ denote the residue class homomorphism from B to $\kappa(B)$. $Q := \pi^{-1}(\overline{Q})$ is a prime cone of B with $T_1 \subset Q$, $\operatorname{supp} Q = \mathfrak{m}$ and $U \subset Q$.

We now invoke the Baer-Krull theorem connecting ordering of K and $\kappa(B)$ in full strength (cf. [La, Cor.3.11], [KS, II §7], [BCR, Th.10.1.10])*). The theorem

*) We stated a rough version of this theorem already above, cf. Lemma 3.2.

can be quoted as follows. Given a group homomorphism $\chi\colon K^* \to \{\pm 1\}$ with $\chi(Q \cap B^*) = \{1\}$ and $\chi(-1) = -1$, there exists a unique prime cone (= ordering) P of K such that B is P-convex and $\operatorname{sign}_P(a) = \chi(a)$ for every $a \in K^*$.

We choose $\chi\colon K^* \to \{\pm 1\}$ in such a way that also $\chi(T \cap K^*) = 1$. By elementary character theory on the group K^*/K^{*2} this is possible, since we have $T \cap B^* \subset Q \cap B^*$ and $-1 \notin (Q \cap B^*) \cdot (T \cap K^*)$. The resulting ordering (= prime cone) P of K contains T, and B is P-convex in K, hence v is P-convex. This completes the proof for $R = K$ a field.

b) We prove the theorem in general. We are given a preordering T and a T-convex valuation v on R. The prime ideal $\mathfrak{q} := \operatorname{supp} v$ is T-convex. Thus Lemma 14 applies. We have a proper preordering U on $k(\mathfrak{q})$ as described there in part ii), and we know by part iii) of the lemma that the valuation $\hat{v}$ on $k(\mathfrak{q})$ is U-convex. As proved above in part a), there exists a prime cone (= ordering) Q on $k(\mathfrak{q})$ containing U such that $\hat{v}$ is Q-convex. It follows that $P := j_{\mathfrak{q}}^{-1}(Q)$ is a prime cone on R with $P \supset T_2 := j_{\mathfrak{q}}^{-1}(U)$, and that $v = \hat{v} \circ j_{\mathfrak{q}}$ is P-convex. As stated in the lemma, $T \subset T_2$, hence $T \subset P$. □

Notice that for v a trivial valuation the theorem boils down to part a) of Theorem 6.

Corollary 5.17. Every T-convex valuation v on R is $\hat{T}$-convex. □

This follows immediately from Theorem 16. It may be of interest – or at least amusing – to see a second proof of Corollary 17, which is based on the Positivstellensatz Theorem 4.

Second proof of Corollary 5.17 (cf.[Z_1, §2]). Suppose that v is T-convex but not $\hat{T}$-convex. We have elements a, b in $\hat{T}$ with

$$v(a+b) > \min(v(a), v(b)). \tag{1}$$

In particular $v(a) \neq \infty$, $v(b) \neq \infty$. By Theorem 4 we have natural numbers m, n and elements u, u', w, w' in T such that

$$au = a^{2m} + u' \quad , \quad bw = b^{2n} + w'.$$

Then $au \in T$, $bw \in T$ and

$$v(au) = \min(v(a^{2m}), v(u')) < \infty \ , \quad v(bw) = \min(v(b^{2n}), v(w')) < \infty.$$

Let $c := a(aubw)$, $d := b(aubw)$. We have $c \in T$, $d \in T$ and

$$v(c+d) \ = \ \min(v(c), v(d)) \ = \ \min(v(a), v(b)) + v(aubw). \tag{2}$$

On the other hand, $c + d = (a+b)aubw$, hence

$$v(c+d) \ = v(a+b) + v(aubw). \tag{3}$$

Since $v(aubw) \neq \infty$, we conclude from (2) and (3) that

$$v(a+b) \ = \ \min(v(a), v(b)), \tag{4}$$

in contradiction to (1). Thus v is $\hat{T}$-convex. □

§6 Convexity of overrings of real holomorphy rings

In this section Λ is a subring of a ring R and T a preordering of R. In §2 we defined the real holomorphy ring $\mathrm{Hol}(R/\Lambda)$ of R over Λ (§2, Def.6). We now generalise this definition.

Definition 1. a) The *T-holomorphy ring* $\mathrm{Hol}_T(R/\Lambda)$ of R over Λ is the intersection of the rings A_v with v running through all T-convex valuations of R over Λ (i.e. with $\Lambda \subset A_v$).
b) If $T = P(X)$ for some set $X \subset \mathrm{Sper}R$ we denote this ring also by $\mathrm{Hol}_X(R/\Lambda)$ and call it the holomorphy ring of the extension $\Lambda \subset R$ over X.
c) In the case $\Lambda = \mathbb{Z}1_R$ we write $\mathrm{Hol}_T(R)$ and $\mathrm{Hol}_X(R)$ instead of $\mathrm{Hol}_T(R/\Lambda)$, $\mathrm{Hol}_X(R/\Lambda)$. We call $\mathrm{Hol}_T(R)$ the *T-holomorphy ring of* R and $\mathrm{Hol}_X(R)$ the *X-holomorphy ring of* R.

Remarks 6.1. i) We know by Corollary 5.17 that

$$\mathrm{Hol}_T(R/\Lambda) = \mathrm{Hol}_{\hat{T}}(R/\Lambda).$$

ii) For the smallest preordering $T_0 = \Sigma R^2$ we have $\mathrm{Hol}_{T_0}(R/\Lambda) = \mathrm{Hol}(R/\Lambda) = \mathrm{Hol}_{\mathrm{Sper}R}(R/\Lambda)$.
iii) If $\mathrm{Hol}_T(R)$ is Prüfer in R then

$$\mathrm{Hol}_T(R/\Lambda) = \Lambda \cdot \mathrm{Hol}_T(R).$$

This can be verified by a straightforward modification of the proof of Proposition 2.20 (which settles the case $T = \Sigma R^2$). $\square$

Given a prime cone P of R we introduced in §3 (cf.Def.5 there) the P-convex valuation $v_{P,\Lambda}$. It has the valuation ring

$$A_{v_{P,\Lambda}} = \mathrm{conv}_P(\Lambda) = C(P, R/\Lambda)$$

and the center $\mathfrak{p}_{v_{P,\Lambda}} = I_P(\Lambda)$. Using these valuations we now obtain a simple description of $\mathrm{Hol}_P(R/\Lambda)$, starting from Theorem 5.15.

Theorem 6.2. Let P be any prime cone of R.
a) A valuation v of R is P-convex and $\Lambda \subset A_v$ iff there exists some prime cone $\tilde{P} \supset P$ with $v_{\tilde{P},\Lambda} \le v$.
b) For every such valuation v we have $A_v \supset \mathrm{Hol}_P(R/\Lambda)$, and

$$\mathrm{Hol}_P(R/\Lambda) = C(P, R/\Lambda) = A(P, R/\Lambda).$$

Also $\mathrm{Hol}_Q(R/\Lambda) = \mathrm{Hol}_P(R/\Lambda)$ for every prime cone $Q \supset P$.

Proof. Claim a) follows immediately Theorem 5.15 which settles the case $\Lambda = \mathbb{Z} \cdot 1_R$.

b): If $v_{Q,\Lambda} \leq v$ then $A_v \supset A_{v_{Q,\Lambda}} = C(Q, R/\Lambda)$. As observed in §5, the Q-convex hull $C(Q, R/\Lambda)$ of Λ with respect to Q does not change if we replace Q by P, and also coincides with the ring $A(Q, R/\Lambda) = A(P, R/\Lambda)$. $\square$

Theorem 6.3. As before, let T be any preordering of R.
a) $\mathrm{Hol}_T(R/\Lambda)$ is the intersection of the rings $\mathrm{Hol}_P(R/\Lambda)$ with P running through the set $\overline{H}_R(T)$ of prime cones $P \supset T$.
b) Given $f \in R$, the following are equivalent.
(1) $f \in \mathrm{Hol}_T(R/\Lambda)$.
(2) $\exists\, \lambda \in \Lambda\colon |f(P)| \leq |\lambda(P)|$ for every $P \in \overline{H}_R(T)$.
(3) $\exists\, \mu \in \Lambda\colon \quad 1 + \mu^2 \pm f \in \hat{T}$.
c) $\mathrm{Hol}_T(R/\Lambda) = C(\hat{T}, R/\Lambda) = A(\hat{T}, R/\Lambda)$.

Proof. a): This follows from the fact that every T-convex valuation v on R is P-convex for some prime cone $P \supset T$, cf. Theorem 5.16.
b): The proof runs in the same way as the proof of Theorem 4.2, which settled the case $T = T_0$.
c): We know by Proposition 5.2 that $C(\hat{T}, R/\Lambda) = A(\hat{T}, R/\Lambda)$. If $f \in \mathrm{Hol}_T(R/\Lambda)$ then condition (3) in b) is fulfilled, hence $f \in A(\hat{T}, R/\Lambda)$. Conversely, if $f \in A(\hat{T}, R/\Lambda)$ we have $-\lambda \leq_{\hat{T}} f \leq_{\hat{T}} \lambda$ for some $\lambda \in \Lambda$. This implies condition (2) in b), hence $f \in \mathrm{Hol}_T(R/\Lambda)$. $\square$

Corollary 6.4. Every $\hat{T}$-convex subring B of R is integrally closed in R.

Proof. We know by Theorem 3 that $B = \mathrm{Hol}_T(R/B)$. Thus B is an intersection of rings A_w with w running through a set of valuations on R. Each A_w is integrally closed in R (cf.Th.I.2.1). Thus B is integrally closed in R. $\square$

Remark. This corollary can be proved in a more direct way, cf.[KS, III §11, Satz 1] or §8 below.

We now turn to a study of T-convexity for subrings of R which are Prüfer in R. This will be a lot easier than studying T-convex subrings in general. We start with a general lemma on localizations.

Lemma 6.5. Let A be a subring of R, M an additive subgroup of A, and S a multiplicative subset of A with $sM \subset M$ for every $s \in S$. We define

$$M_{[S]} := \{x \in R \mid \exists\, s \in S\colon sx \in M\}$$

and, as always,

$$A_{[S]} := \{x \in R \mid \exists\, s \in S\colon sx \in A\}.$$

i) $M_{[S]}$ is an additive subgroup of $A_{[S]}$. If M is an ideal of A then $M_{[S]}$ is an ideal of $A_{[S]}$.
ii) If M is $(T \cap A)$-convex in A then $M_{[S]}$ is $(T \cap A_{[S]})$-convex in $A_{[S]}$.

iii) If $M_{[S]}$ is $(T \cap A_{[S]})$-convex in $A_{[S]}$ and $M_{[S]} \cap A = M$ then M is $(T \cap A)$-convex in A.

Proof. i): evident.
ii): Let $x \in M_{[S]}$ and $y \in A_{[S]}$ be given with $0 \leq_T y \leq_T x$. We choose some $s \in S$ with $sy \in A$ and $sx \in M$. Then $0 \leq_T s^2y \leq_T s^2x \in M$ and $s^2y \in A$. Since M is assumed to be $(T \cap A)$-convex in A, we conclude that $s^2y \in M$, hence $y \in M_{[S]}$. Thus $M_{[S]}$ is $(T \cap A_{[S]})$-convex in $A_{[S]}$.
iii): Let $x \in M$, $y \in A$ and $0 \leq_T y \leq_T x$. Since $M_{[S]}$ is assumed to be $(T \cap A_{[S]})$-convex in $A_{[S]}$, we conclude that $y \in M_{[S]} \cap A = M$. Thus M is $(T \cap A)$-convex in A. $\square$

We will use two special cases of this lemma, stated as follows.

Lemma 6.6. Let A be a subring of R and $\mathfrak{p}$ a prime ideal of A.
i) If A is T-convex in R then $A_{[\mathfrak{p}]}$ is T-convex in R.
ii) $\mathfrak{p}_{[\mathfrak{p}]}$ is $(T \cap A_{[\mathfrak{p}]})$-convex in $A_{[\mathfrak{p}]}$ iff $\mathfrak{p}$ is $(T \cap A)$-convex in A.

Proof. i): Apply Lemma 5 choosing $A, R, A \setminus \mathfrak{p}$ for M, A, S.
ii): Apply the lemma choosing $\mathfrak{p}, A, A \setminus \mathfrak{p}$ for M, A, S. $\square$

Theorem 6.7. Assume that A is a Prüfer subring of R. The following are equivalent.
(1) A is T-convex in R.
(2) For every R-regular maximal (or: prime) ideal $\mathfrak{p}$ of A the ring $A_{[\mathfrak{p}]}$ is T-convex in R.
(3) For every R-regular maximal (or: prime) ideal $\mathfrak{p}$ of A the ideal $\mathfrak{p}_{[\mathfrak{p}]}$ of $A_{[\mathfrak{p}]}$ is $(T \cap A_{[\mathfrak{p}]})$-convex in $A_{[\mathfrak{p}]}$.
(4) Every non trivial PM-valuation v of R over A is T-convex.
(5) Each R-regular maximal (or: prime) ideal of A is $(T \cap A)$-convex in A.
(6) Each R-regular maximal (or: prime) ideal of A is T-convex in R.
(7) A is $\hat{T}$-convex in R.

Proof. We may assume that $A \neq R$.
(1) $\Rightarrow$ (2): Evident by Lemma 6.6.i.
(2) $\Rightarrow$ (1): Clear, since A is the intersection of the rings $A_{[\mathfrak{p}]}$ with $\mathfrak{p}$ running through $\Omega(R/A)$.
(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4): This holds by Theorem 5.11.
(3) $\Leftrightarrow$ (5): Evident by Lemma 6.6.ii.
We now have verified the equivalence of (1), (2), (3), (4), (5).
(1) $\Rightarrow$ (6): If $\mathfrak{p}$ is an R-regular prime ideal of A then $\mathfrak{p}$ is $(T \cap A)$-convex in A by (5) and A is T-convex in R. Thus $\mathfrak{p}$ is T-convex in R.
(6) $\Rightarrow$ (5): trivial.
(7) $\Rightarrow$ (1): trivial.

(4) $\Rightarrow$ (7): We know by Corollary 5.17 that $v_{\mathfrak{p}}$ is $\hat{T}$-convex for every $\mathfrak{p} \in \Omega(R/A)$. Using the implication (4) $\Rightarrow$ (1) for $\hat{T}$ instead of T we see that A is $\hat{T}$-convex in R. □

Corollary 6.8. Let A be a Prüfer subring of R, and let C denote the T-convex hull of A in R, $C = C(T, R/\Lambda)$. Assume that C is a subring of R. {N.B. This is known to be true under very mild additional assumptions, cf. Prop.5.2.}
a) Then $S(R/C)$ *) is the set of all T-convex valuations $v \in S(R/A)$.
b) $C = \mathrm{Hol}_T(R/A)$, and $C = \bigcap\limits_{\mathfrak{p}} A^R_{[\mathfrak{p}]}$ with $\mathfrak{p}$ running through the set of R-regular prime ideals $\mathfrak{p}$ of A which are T-convex (i.e. $(T \cap A)$-convex)) in A. □

Proof. Claim a) follows immediately from the equivalence (1) $\Leftrightarrow$ (4) in Theorem 7. We then have $C = \mathrm{Hol}_T(R/A)$ by the very definition of the relative real holomorphy ring $\mathrm{Hol}_T(R/A)$. The last statement in the corollary is evident due to the 1-1-correspondence of PM-valuations v of R over A with the R-regular prime ideals $\mathfrak{p}$ of A. □

We arrive at a theorem which demonstrates well the friendly relation between T-convexity and the Prüfer condition.

Theorem 6.9. Let A be a T-convex subring of R. Then A is Prüfer in R iff every R-overring of A is $\hat{T}$-convex in R.

Proof. a) Assume that A is Prüfer in R. Let B be an R-overring of R. The ring B inherits property (4) in Theorem 7 from A, hence is $\hat{T}$-convex in R by that theorem.
b) If every R-overring of A is $\hat{T}$-convex in R then each such ring is integrally closed in R, as stated above (Corollary 4). Thus A is Prüfer in R (cf. Theorem I.5.2).

Corollary 6.10. Let Λ be a subring of R. Assume that $\mathrm{Hol}_T(R/\Lambda)$ is Prüfer in R. Then the $\hat{T}$-convex subrings of R containing Λ are precisely the overrings of $\mathrm{Hol}_T(R/\Lambda)$ in R.

Proof. We know by Theorem 3 that $\mathrm{Hol}_T(R/\Lambda)$ is the $\hat{T}$-convex hull $C(\hat{T}, R/\Lambda)$ of Λ in R. Now apply Theorem 9. □

Remark. If R has positive definite inversion, or, if for every $x \in R$ there exists some $d \in \mathbb{N}$ with $1 + x^{2d} \in R^*$, we know by §2 that $\mathrm{Hol}(R)$ is Prüfer in R, hence $\mathrm{Hol}_T(R)$ is Prüfer in R, and Corollary 10 applies. Thus we have a good

*) Recall that $S(R/C)$ denotes the restricted PM-spectrum of R over C (§1).

hold on $\hat{T}$-convexity under conditions which, regarded from the view-point of real algebra, are mild.

Our proof of Theorem 7 (and hence Theorem 9) is based a great deal on Lemma 6 above. The lemma also leads us to a supplement to the theory of convex valuations developed in §5.

Proposition 6.11. Let B be a Prüfer subring of R which is T-convex in R, and let v be a $(T \cap B)$-convex PM-valuation on B. Then the induced valuation v^R on R (cf. §1, Def.5) is T-convex.

Proof: Let $A := A_v$, $\mathfrak{p} := \mathfrak{p}_v$, $w := v^R$. Since v is the special restriction $w|_B$ of w to B, we have $A_w \cap B = A$, $\mathfrak{p}_w \cap B = \mathfrak{p}$. Now A is Prüfer in R, and $A \subset A_w \subset R$. Thus $A_w = A^R_{[\mathfrak{p}]}$, $\mathfrak{p}_w = \mathfrak{p}^R_{[\mathfrak{p}]}$. The ring A is T-convex in B, hence in R. Further $\mathfrak{p}$ is T-convex in A, hence in R. By Lemma 6 it follows that A_w is T-convex in R and $\mathfrak{p}_w$ is T-convex in A_w. We conclude by Theorem 5.11 that the Manis valuation w is T-convex. $\square$

§7 The case of bounded inversion; convexity covers

Definition 1. Let (R,T) be a preordered ring, i.e. a ring R equipped with a preordering T. We say that (R,T) has *bounded inversion*, if $1+t$ is a unit of R for every $t \in T$, in short, $1+T \subset R^*$. If A is a subring of R, we say that A has *bounded inversion with respect to* T, if $(A, T\cap A)$ has bounded inversion, i.e. $1+(T\cap A) \subset A^*$.

The theory of T-convex Prüfer subrings of R turns out to be particularly nice and good natured if (R,T) has bounded inversion, as we will explicate now.

We first observe that (R,T) has bounded inversion iff $(R,\hat{T})$ has bounded inversion, due to the following proposition.

Proposition 7.1. Given a preordering T on a ring R, the following are equivalent.
(1) $1+T \subset R^*$
(2) Every maximal ideal $\mathfrak{m}$ of R is T-convex in R.
(3) $1+\hat{T} \subset R^*$.

Proof. (1) $\Rightarrow$ (2): This follows from Proposition 5.7.*)
(2) $\Rightarrow$ (3): If $\mathfrak{m}$ is a maximal ideal of R then $\mathfrak{m}$ is T-convex in R, hence $\hat{T}$-convex in R (cf.Th.5.6). It follows that $\mathfrak{m}\cap(1+\hat{T}) = \emptyset$. Since this holds for every maximal ideal of $\mathfrak{m}$, the set $1+\hat{T}$ consists of units of R.
(3) $\Rightarrow$ (1): trivial. □

Thus, in the bounded inversion situation, we most often can switch from T to $\hat{T}$ and back.

Theorem 7.2. Let A be a subring of R.
i) The following are equivalent.
(1) A is Prüfer in R and $1+(T\cap A) \subset A^*$.
(2) A is Prüfer in R and $1+(\hat{T}\cap A) \subset A^*$.
(3) A is T-convex in R and $1+T \subset R^*$.
(4) A is $\hat{T}$-convex in R and $1+\hat{T} \subset R^*$.
ii) If (1) – (4) hold, every R-overring B of A is $\hat{T}$-convex in R and $B = S^{-1}A$ with $S := T\cap A\cap B^*$.

Proof. a) We assume (1), i.e. $A \subset R$ is Prüfer and $1+(T\cap A) \subset A^*$. By Proposition 1 every maximal ideal $\mathfrak{m}$ of A is $(T\cap A)$-convex in A. Thus condition (5) in Theorem 6.7 holds and A is $\hat{T}$-convex and (hence) T-convex in R by that theorem. Applying Theorem 6.7 to $\hat{T}$ instead of T we learn that (2) holds. Since the implication (2) $\Rightarrow$ (1) is trivial we now know that (1) $\Leftrightarrow$ (2).

*) A direct proof can be found in [Z_1, p.5804 f].

b) Assuming (1) we prove that $1+T \subset R^*$. To this end let $\mathfrak{Q}$ be a maximal ideal of R. We verify that $\mathfrak{Q}$ is T-convex in R and then will be done by Proposition 1.

Let $\mathfrak{q} := \mathfrak{Q} \cap A$. Since A is ws in R, we have $A_{[\mathfrak{q}]} = R$ and $\mathfrak{q}_{[\mathfrak{q}]} = \mathfrak{Q}$ (cf.Th.I.4.8). By Lemma 6.6 it suffices to verify that $\mathfrak{q}$ is $(T \cap A)$-convex in A. We choose a maximal ideal $\mathfrak{m}$ of A containing $\mathfrak{q}$.

Case 1. $\mathfrak{m}R \neq R$. We have $\mathfrak{Q} = R\mathfrak{q} \subset R\mathfrak{m}$ (cf.Th.I.4.8). Since $\mathfrak{Q}$ is a maximal ideal of R it follows that $R\mathfrak{q} = R\mathfrak{m}$ and then, again by Th.I.4.8., that $\mathfrak{q} = \mathfrak{m}$. The ideal $\mathfrak{m}$ is $(T \cap A)$-convex in A, due to (1) and Lemma 6.6.

Case 2. $\mathfrak{m}R = R$. We have a Manis valuation v on R with $A_v = A_{[\mathfrak{m}]}$ and $\mathfrak{p}_v = \mathfrak{p}_{[\mathfrak{m}]}$. It follows by Proposition I.1.3 that $(\operatorname{supp} v)_{\mathfrak{m}}$ is a maximal ideal of $R_{\mathfrak{m}}$. Now $\mathfrak{Q}_{\mathfrak{m}}$ is an ideal of $R_{\mathfrak{m}}$ contained in the center $\mathfrak{p}_{\mathfrak{m}}$ of the Manis valuation $\tilde{v}$ induced by v on $R_{\mathfrak{m}}$. Thus $\mathfrak{Q}_{\mathfrak{m}} \subset \operatorname{supp}(\tilde{v}) = (\operatorname{supp} v)_{\mathfrak{m}}$. This implies $\mathfrak{Q} \subset R \cap (\operatorname{supp} v)_{\mathfrak{m}} = \operatorname{supp} v$, and then $\mathfrak{Q} = \operatorname{supp} v$, since $\mathfrak{Q}$ is a maximal ideal of R. Thus $\operatorname{supp} v = \mathfrak{q}_{[\mathfrak{q}]}$.

Since $1+(T \cap A) \subset A^*$, the ideal $\mathfrak{m}$ is $(T \cap A)$-convex in A, due to Proposition 1. Now Lemma 6.6 tells us that $\mathfrak{m}_{[\mathfrak{m}]} = \mathfrak{p}_v$ is $(T \cap A_{[\mathfrak{m}]})$-convex in $A_{[\mathfrak{m}]} = A_v$. We conclude by Theorem 5.11 that the valuation v is T-convex. It follows that $\operatorname{supp} v = \mathfrak{Q}_{[\mathfrak{q}]}$ is T-convex in R.
We have proved the implication (1) $\Rightarrow$ (3) in part i) of the theorem. Changing from T to $\hat{T}$ we also know that (2) $\Rightarrow$ (4). The implication (4) $\Rightarrow$ (3) is trivial. Altogether we have proved the implications (1) $\Leftrightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (3).

c) We finally prove that condition (3) implies (1) and all the assertions listed in part ii) of the theorem, and then will be done. Thus assume that that A is T-convex in R and $1+T \subset R^*$. For every $t \in T$ we have $0 \leq_T \frac{1}{1+t} \leq_T 1$. It follows that $\frac{1}{1+t} \in A$. In particular $1+x^2 \in R^*$ and $\frac{1}{1+x^2} \in A$ for every $x \in R$. Thus A is Prüfer in R, as is clear already by I §6, Example 13. (Take $d=2$ there.) For $t \in A \cap T$ we have $1+t \in A$ and $(1+t)^{-1} \in A$, hence $1+t \in A^*$.

Let B be an R-overring of A. If $t \in T \cap B$ then $\frac{1}{1+t} \in A \subset B$, hence $1+t \in B^*$. By the proved implication (1) $\Rightarrow$ (3) from above it follows that B is T-convex in R.

Let $b \in B$ be given. Then $s := \frac{1}{1+b^2} \in A$. Also $0 \leq_T \frac{2b}{1+b^2} \leq_T 1$, hence $a := 2bs \in A$. We have $s \in S := T \cap A \cap B^*$ and, of course, $2 \in S$. Thus $b = \frac{a}{2s} \in S^{-1}A$. We have proved all claims of the theorem. $\square$

Corollary 7.3. Let A be a Prüfer subring of R and B an overring of A in R. Then the T-convex hull $C(T, R/B)$ coincides with the saturation

$$B_{[S]} := \{x \in R \mid \exists\, s \in S \colon sx \in B\},$$

where $S := 1 + (T \cap B)$.

Proof. a) We equip the localisation $S^{-1}R$ with the preordering $S^{-1}T = \{\frac{t}{s} \mid t \in T, s \in S\}$. One easily checks that $(S^{-1}T) \cap (S^{-1}A) = S^{-1}(T \cap A)$. Applying Theorem 2 to the Prüfer extension $S^{-1}A \subset S^{-1}R$ we learn that $S^{-1}B$ is $S^{-1}T$-convex in $S^{-1}R$. Taking preimages in R we see that $B_{[S]}$ is $T_{[S]}$-convex in R, where $T_{[S]}$ denotes the preimage of $S^{-1}T$ in R. Now $T \subset T_{[S]}$. Thus $B_{[S]}$ is T-convex in R. This proves that $C(T, R/B) \subset B_{[S]}$.
b) Let $x \in B_{[S]}$ be given. There exists some $s \in S$ with $sx \in B$, $s = 1 + t$ with $t \in T \cap A$. We conclude from $0 \leq_T x^2 \leq_T s^2x^2 \in B$ that $x^2 \in C(T, R/B)$. Now B is integrally closed in R, since A is Prüfer in R. Thus $x \in C(T, R/B)$. This proves that $B_{[S]} \subset C(T, R/B)$. □

In the following we fix a preordered ring (R, T). As common in the case of ordered structures we suppress the ordering in the notation (since it is fixed), simply writing R for the pair (R, T). The subset T of R will usually be denoted by R^+. Any subring B of R is again regarded as a preordered ring, with $B^+ = T \cap B$. If we say that B has bounded inversion, we of course mean bounded inversion with respect to B^+.

Definition 2. For any subring B of R let C_B denote the smallest subring of B which is convex (= T-convex) in B. Thus, in former notation, $C_B = C(T \cap B, B) = C(T \cap B, B/\mathbb{Z})$. {Recall Prop.5.2.d.}

Proposition 7.4. Let B be a subring of R.
i) $C_B = \{x \in B \mid \exists\, n \in \mathbb{N}\colon -n \leq_T x \leq_T n\}$.
ii) C_B is contained in the real holomorphy ring $\mathrm{Hol}_{B^+}(B)$.
iii) If C_B is Prüfer in B, then $C_B = \mathrm{Hol}_{B^+}(B)$.
iv) If B has bounded inversion, then C_B is Prüfer in B and $C_B = \sum\limits_{t \in B^+} \mathbb{Z}\, \frac{1}{1+t}$.

Proof. i): Clear by Proposition 5.2.d.
ii): $\mathrm{Hol}_{B^+}(B)$ is a subring of B which is $(B^+)^\wedge$-convex in B (cf.Th.6.3.c), hence B^+-convex in B. This forces $C_B \subset \mathrm{Hol}_{B^+}(B)$.
iii): C_B is the intersection of the rings A_v with v running through the non-trivial PM-valuations of B over C_B. These are B^+-convex (cf.Th.6.7). Thus $\mathrm{Hol}_{B^+}(B) \subset C_B$. Since the reverse inclusion holds anyway, as just proved, $\mathrm{Hol}_{B^+}(B) = C_B$.
iv): The proof of Theorem 2.11 extends readily to the present situation. It gives us $\mathrm{Hol}_{B^+}(B) = \sum\limits_{t \in B^+} \mathbb{Z}\, \frac{1}{1+t}$, verifying in between that the right hand side is a Prüfer subring of B. We have $0 \leq_T \frac{1}{1+t} \leq_T 1$ for every $t \in B^+$. Thus $\mathrm{Hol}_{B^+}(B) \subset C_B$. Since $C_B \subset \mathrm{Hol}_{B^+}(B)$ anyway, both rings coincide. □

Up to now we have been rather pedantic using the term "B^+-convex" instead of just "convex". The reason was that also the saturated preordering $(B^+)^\wedge$ came into play. In the following the term "convex" will always refer to the given preordering $T = R^+$ of R.

Remark 7.5. If A and B are subrings of R with $A \subset B$, then $C_A \subset C_B$. Indeed, $A \cap C_B$ is convex in A, hence $C_A \subset A \cap C_B$.

Theorem 7.6. Let A and B be subrings of R with $A \subset B$. The following are equivalent.
(1) A has bounded inversion, and A is Prüfer in B.
(2) B has bounded inversion, and A is convex in B.
(3) Both A and B have bounded inversion, and $C_A = C_B$.

Proof. The equivalence (1) $\iff$ (2) is a restatement of (1) $\iff$ (3) in Theorem 2.
(1) $\wedge$ (2) $\Rightarrow$ (3): By assumption (1) and (2) both A and B have bounded inversion, and A is convex in B. Since C_A is convex in A we conclude that C_A is convex in B, and then, that $C_B \subset C_A$. Thus $C_A = C_B$.
(3) $\Rightarrow$ (1): Applying the implication (2) $\Rightarrow$ (1) to C_B and B, we see that $C_A = C_B$ is Prüfer in B. {This had already been stated in Prop.4.} Since $C_A \subset A \subset B$, it follows that A is Prüfer in B. $\square$

Corollary 7.7. Let A be a subring of R, and let D denote the Prüfer hull of A in R, $D = P(A, R)$ (cf.I, §5, Def.2). Assume that A has bounded inversion.
a) Every overring B of A in D has bounded inversion and is convex in D, and $C_B = C_A$.
b) D is the unique maximal overring B of A in R such that B has bounded inversion and $C_B = C_A$.
c) D is the unique maximal overring B of A such that A is convex in B and B has bounded inversion.
d) C_A has bounded inversion, and D is the Prüfer hull of C_A in R. The overrings of C_A in D are precisely all subrings B of R such that $C_B = C_A$ and B has bounded inversion.

Proof. a): If B is an overring of A in D, then A is Prüfer in B. Thus, by Theorem 6, B has bounded inversion and $C_A = C_B$. In particular, D has bounded inversion and $C_A = C_D$. Applying Theorem 6 to B and D we see that B is convex in D.
b): If B is an overring of A in R with bounded inversion and $C_A = C_B$, then A is Prüfer in B by Theorem 6, hence $B \subset D$.
c): If B is an overring of A in R with bounded inversion such that A is convex in B, then again A is Prüfer in B by Theorem 6, hence $B \subset D$.
d): C_A is convex in A, hence is Prüfer in A (cf.Th.6 or Prop.4). Thus D is also the Prüfer hull of C_A in R. Now apply what has been proved about the extension $A \subset R$ to the extension $C_A \subset R$, taking into account the trivial fact that $C_A = C_B$ implies $C_A \subset B$. $\square$

The corollary tells us in particular (part c) that A has a unique maximal overring D such that A is convex in D and D has bounded inversion. Does there hold something similar without the inverse boundedness condition? The answer is "Yes" provided A is Prüfer in R, as we are going to explain. We now denote the basic subring of R to start with Λ instead of A, since the letter A will turn up with another meaning.

Let Λ be a subring of R. We denote the subring $A(R^+, R/\Lambda)$ and the additive subgroup $C(R^+, R/\Lambda)$ (cf.§5) briefly by $A(R/\Lambda)$ and $C(R/\Lambda)$ respectively. Recall from Proposition 5.2 that $C(R/\Lambda) = \Lambda + A(R/\Lambda)$. We need the following easy

Lemma 7.8. Let B be an overring of Λ in R. Then $A(B/\Lambda) = B \cap A(R/\Lambda)$ and $C(B/\Lambda) = B \cap C(R/\Lambda)$.

Proof. The first equality is evident from the definition of $A(B/\Lambda)$ and $A(R/\Lambda)$ in §5. The second one now follows since $B \cap [\Lambda + A(R/\Lambda] = \Lambda + [B \cap A(R/\Lambda)]$. $\square$

Definition 3. Assume that Λ is Prüfer in R. The *convexity cover of* Λ *in* R is the polar $C(R/\Lambda)^\circ$ of $C(R/\Lambda)$ over Λ in R, i.e. the unique maximal R-overring E of Λ with $C(R/\Lambda) \cap E = \Lambda$ (cf.II, §7). We denote the convexity cover by $CC(R/\Lambda)$.*)

Recall that the polar I° is defined for any Λ-overmodule I of Λ in R. Thus we do not need to assume here that $C(R/\Lambda)$ itself is a subring of R.

The name "convexity cover" is justified by the following theorem.

Theorem 7.9. Assume that Λ is Prüfer in R. Let B be any R-overring of Λ. Then Λ is convex in B iff $B \subset CC(R/\Lambda)$. Thus $CC(R/\Lambda)$ is the unique maximal overring E of Λ in R such that Λ is convex in E.

Proof. Let B be any R-overring of Λ. By the lemma we have $C(B/\Lambda) = B \cap C(R/\Lambda)$. Thus Λ is convex in B iff $B \cap C(R/\Lambda) = \Lambda$. This means that $B \subset C(R/\Lambda)^\circ$. $\square$

If Λ is any subring of R then Theorem 9 still gives us the following.

Corollary 7.10. There exists a unique maximal R-overring E of Λ such that Λ is Prüfer and convex in E, namely $E = CC(P(\Lambda, R)/\Lambda)$. $\square$

Definition 4. We call this R-overring E of Λ the *Prüfer convexity cover of* Λ *in* R, and denote it by $P_c(\Lambda, R)$. $\square$

Scholium 7.11. If B_1 and B_2 are overrings of Λ in R such that Λ is Prüfer and convex in B_1 and in B_2 then Λ is also Prüfer and convex in B_1B_2. Indeed, B_1 and B_2 are both subrings of $P_c(A, R)$. Thus $B_1B_2 \subset P_c(A, R)$. $\square$

*) More precisely we write $CC(T, R/\Lambda)$, with $T = R^+$, if necessary.

We do not have such a result for “convex” alone, omitting the Prüfer condition.

In §10 we will meet a situation where a preordered (in fact partially ordered) ring A is given, such that the preordering extends to the Prüfer hull $P(A)$ in a natural way. Then we will have an *“absolute” Prüfer convexity cover* $P_c(A) := P_c(A, P(A))$ at our disposal, which is the unique maximal Prüfer extension E of A such that A is convex in E.

§8 Convexity of submodules

As before (R,T) is a preordered ring. But now we fix a subring A of R and study T-convexity for A-submodules of R instead of subrings. We will use this to develop more criteria that A is Prüfer and T-convex in R, and to find more properties of such extensions $A \subset R$. Large parts of this section may be read as a supplement to our multiplicative ideal theory in Chapter II in the presence of a preordering.

As we already did in part of the preceding section we usually simplify notation by saying "convex" instead of "T-convex", and writing $C(R/A)$ instead of $C(T,R/A)$ etc. This will cause no harm as long as we keep the preordering T fixed.

We start with an important observation by Brumfiel in his book [Br]. Brumfiel there only considers the case that T is a partial ordering of R, i.e. $T \cap (-T) = \{0\}$, but his arguments go through more generally for a preordering T.

Proposition 8.1. Let $u_1, \dots, u_{2n}, t$ be indeterminates over $\mathbb{Q}$, $u := (u_1, \dots, u_{2n})$, and $f(t) := t^{2n} + u_1 t^{2n-1} + \cdots + u_n$. Then there exists some $k \in \mathbb{N}$, polynomials $b^+(u), b^-(u) \in \mathbb{Q}[u]$, and polynomials $h_i^+(u,t), h_i^-(u,t) \in \mathbb{Q}[u,t]$, $1 \le i \le k$, such that

$$t - b^+(u) + \sum_{i=1}^{k} h_i^+(u,t)^2 = f(t),$$
$$b^-(u) - t + \sum_{i=1}^{k} h_i^-(u,t)^2 = f(t).$$

The proof runs by induction on n, cf. [Br, p.123 ff]. □

Inserting for the u_i elements a_i of our subring A of R we obtain the following corollary.

Corollary 8.2. Assume that $\mathbb{Q} \subset R$. If $\alpha \in R$ and $f(t) = t^{2n} + a_1 t^{2n-1} + \cdots + a_{2n}$ is a monic polynomial of even degree over A with $f(\alpha) \le_T 0$, then

$$b^-(a_1, \dots, a_{2n}) \ \le_T \ \alpha \ \le_T \ b^+(a_1, \dots, a_{2n}).$$

Thus α is an element of the convex closure $C(R/A)$ of A in R.*) □

In particular we have

Corollary 8.3. If $\mathbb{Q} \subset R$, and A is convex in R, then A is integrally closed in R. □

*) Notice that $\mathbb{Q} \subset C(R/A)$.

It is possible to weaken the condition $\mathbb{Q} \subset R$ in Corollary 3 considerably.

Proposition 8.4. *Assume that A is convex in R and 2-saturated in R (i.e., for every $x \in R$, $2x \in A \Rightarrow x \in A$). Then A is integrally closed in R.*

Proof Let $\tilde{R} := \mathbb{Q} \otimes_{\mathbb{Z}} R$ and $\tilde{A} := \mathbb{Q} \otimes_{\mathbb{Z}} A$. As usual, we regard R as a subring of $\tilde{R}$. Then $A \subset \tilde{A}$. The preordering T extends to a preordering $\tilde{T}$ of $\tilde{R}$, and $\tilde{A}$ is $\tilde{T}$-convex in $\tilde{R}$, as is easily seen, since A is assumed to be T-convex in R.

Let $x \in R$ be integral over A. Then x is integral over $\tilde{A}$, and we know by Corollary 3 that $x \in \tilde{A}$. Thus $nx \in A$ for some $n \in \mathbb{N}$. We have $0 \underset{T}{\leq} x^2 \underset{T}{\leq} n^2x^2 \in A$. Since A is T-convex in R, it follows that $x^2 \in A$. Also $1+x$ is integral over A, and thus $(1+x)^2 \in A$. We conclude that $2x = (1+x)^2 - x^2 \in A$, and then, that $x \in A$, since A is 2-saturated in R. □

Here is another observation about convexity in R. If M is any subset of R, we define

$$[A\colon M] := [A\colon_R M] := \{y \in R \mid yx \in A \quad \text{for every } x \in M\}$$

(thus $[A\colon M] = [A\colon AM]$).

Proposition 8.5. *Assume again that A is convex and 2-saturated in R.*

a) For every subset M of R the A-module $[A\colon M]$ is convex and 2-saturated in R.

b) Every R-invertible A-submodule of R is convex and 2-saturated in R.

Proof. a): Since $[A\colon M]$ is the intersection of the A-modules $[A\colon x]$ with x running through M, it suffices to prove the claim for $M = \{x\}$ with x a given element of R.

If $y \in R$ and $2y \in [A\colon x]$, then $2xy \in A$, hence $xy \in A$, i.e. $y \in [A\colon x]$. Thus $[A\colon x]$ is 2-saturated in R.

Let $s, t \in T$ be given with $s + t \in [A\colon x]$. Then $0 \underset{T}{\leq} s^2x^2 \underset{T}{\leq} (s+t)^2x^2 \in A$. Thus $(sx)^2 \in A$. By Proposition 4 we infer that $sx \in A$, i.e. $s \in [A\colon x]$. This proves that $[A\colon x]$ is convex in R.

b): If I is an R-invertible A-submodule of R then $I = [A\colon I^{-1}]$, and part a) applies. □

Remark 8.6. Assume that A is convex in R and $2 \in R^*$. Then $2 \in A^*$, hence A is 2-saturated in R.

Proof. $0 \underset{T}{\leq} \frac{1}{2} \underset{T}{\leq} 1 \in A$, hence $\frac{1}{2} \in A$. □

Thus the assumption in Propositions 4 and 5, that A is 2-saturated in R, is a very mild one.

Theorem 8.7. The following are equivalent.
(i) A is Prüfer, convex and 2-saturated in R.
(ii) Every R-regular A-submodule of R is convex and 2-saturated in R.
(iii) For every $x \in R$ the A-module $A + Ax^2$ is convex and 2-saturated in R.
(iv) Every R-overring of A is convex and 2-saturated in R.

Proof. (i) $\Rightarrow$ (ii): It suffices to study finitely generated R-regular A-modules. These are invertible in R, hence, according to Proposition 5, are convex and 2-saturated in R.
(ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv): trivial.
(iii) $\Rightarrow$ (i): By assumption $A = A + 0 \cdot A$ is convex and 2-saturated in R, and A is integrally closed in R due to Proposition 4. Let $x \in R$ be given. We have $-1 - x^2 \leq_T 2x \leq_T 1 + x^2$ and conclude by (iii) that $2x \in A + Ax^2$, then, that $x \in A + Ax^2$. Now Theorem I.5.2 tells us that A is Prüfer in R.
(iv) $\Rightarrow$ (i): Let B be an overring of A in R. By assumption B is convex and 2-saturated in R. Thus, by Proposition 4, B is integrally closed in R. We conclude by Theorem I.5.2 that A is Prüfer in R. $\square$

Remarks 8.8. i) If $2 \in R^*$ we may drop the 2-saturation assumption in all conditions (i) – (iv), since now convexity of A implies $2 \in A^*$ (cf. Remark 8.6 above). Then every A-submodule of R is 2-saturated.
ii) If $2 \in R^*$ and A is convex in R, the theorem tells us in particular that A is Prüfer in R iff every R-overring of A is convex in R. This improves Theorem 6.9 in the case $2 \in R^*$. $\square$

We now strive for criteria which start with a mild general assumption on T and the extension $A \subset R$, and then decide whether A is T-convex and Prüfer in R by looking for $(T \cap A)$-convexity in A of suitable R-regular ideals of A. One such criterion had already been given within Theorem 6.7, cf. there (1) $\Leftrightarrow$ (5).

Theorem 8.9. Assume that S is a multiplicative subset of A. Assume further that $2 \in S$, and every element of S is a nonzero divisor in A. Let $R := S^{-1}A$. The following are equivalent.
(i) A is Prüfer and convex in R.
(ii) For every $a \in A$ and $s \in S$ the ideal $As^2 + Aa^2$ is convex (i.e. $A \cap T$-convex) in A.

Proof. (i) $\Rightarrow$ (ii): Let $a \in A$ and $s \in S$ be given. Take $x := \frac{a}{s^2}$. The module $A + Ax^2$ is convex in R by Theorem 7. The map $z \mapsto s^2 z$ from R to R is an automorphism of the preordered abelian group $(R, +, T)$. Thus $As^2 + Aa^2 = s^2(A + Ax^2)$ is convex in R, hence in A.
(ii) $\Rightarrow$ (i): a) We first verify that 2 is a unit in A. Let $x := \frac{1}{2}$. Then $x \in R = S^{-1}A$ and $a := 4x \in A$. We have $0 \leq a \leq 4$, and $A \cdot 4 = A \cdot 2^2 + A \cdot 0$ is convex in A. Thus $a \in 4A$, hence $x \in A$.

b) We start out to prove that A is convex in R. {This is the main task!} Let $x \in R$ and $b \in A$ be given with $0 \underset{T}{\leq} x \underset{T}{\leq} b$. Write $x = \frac{a}{s}$ with $a \in A$, $s \in S$. We have

$$0 \underset{T}{\leq} a^2 \underset{T}{\leq} b^2 s^2 \underset{T}{\leq} s^4 + b^2 s^2.$$

Since $As^4 + Ab^2 s^2$ is convex in A, this implies $a^2 \in As^4 + Ab^2 s^2$, hence $x^2 \in As^2 + Ab^2 \subset A$.
Since $0 \underset{T}{\leq} x+1 \underset{T}{\leq} b+1 \in A$, also $(1+x)^2 \in A$, and thus $x = \frac{1}{2}[(1+x)^2 - x^2] \in A$. A is convex in R.

c) We finally prove for any $x \in R$ that $A + Ax^2$ is convex in R. Then we will know by Theorem 7 and Remark 8.i that A is Prüfer in R, and will be done.

Write $x = \frac{a}{s}$ with $a \in A$, $s \in S$. By assumption the A-module $Aa^2 + As^2$ is convex in A, hence convex in R. Thus also $A + Ax^2 = s^{-2}(Aa^2 + As^2)$ is convex in R. □

Lemma 8.10. Let I, J, K be A-submodules of R with $I \subset J$.
a) If I is 2-saturated in J, then $[I\colon K]$ is 2-saturated in $[J\colon K]$.
b) If the A-module K is generated by $K \cap T$ and I is convex in J, then $[I\colon K]$ is convex in $[J\colon K]$.

Proof. a): Let $x \in [J\colon K]$ and $2x \in [I\colon K]$. For any $s \in K$ we have $2sx \in I$, $sx \in J$, hence $sx \in I$. Thus $x \in [I\colon K]$.
b): Let $M\colon = K \cap T$. Let $x \in [J\colon K]$ and $y \in [I\colon K]$ be given with $0 \underset{T}{\leq} x \underset{T}{\leq} y$. For any $s \in M$ we have $0 \underset{T}{\leq} sx \underset{T}{\leq} sy$ and $sx \in J$, $sy \in I$. It follows that $sx \in I$. Since the A-module K is generated by M, we conclude that $x \in [I\colon K]$. □

Definition 1. We say that an A-submodule I of R is *T-invertible in R*, or *(R,T)-invertible*, if I is R-invertible and both I and I^{-1} are generated by $I \cap T$ and $I^{-1} \cap T$ respectively.

Notice that the product IJ of any two (R,T)-invertible A-submodules I, J of R is again (R,T)-invertible.

Examples 8.11. i) Assume that A is Prüfer in R. Then, for every R-invertible A-module I, the module I^2 is T-invertible in R. Indeed, write $I = Aa_1 + \cdots + Aa_n$. Then $I^2 = Aa_1^2 + \cdots + Aa_n^2$ (cf. Prop.II.1.8), and $a_1^2, \ldots, a_n^2 \in T$. Also I^{-2} is generated by $T \cap I^{-2}$.
ii) If $A \subset R$ is any ring extension and P is a prime cone of R then clearly every R-invertible A-submodule of R is P-invertible in R.

Lemma 8.12. Let I, J, K be A-submodules of R with $I \subset J$. Assume that K is T-invertible in R. Then I is convex in J iff IK is convex in JK, and I is 2-saturated in J iff IK is 2-saturated in JK.

Proof. This follows from Lemma 10, since, for any A-module $\mathfrak{a}$ in R, we have $\mathfrak{a}K = [\mathfrak{a}\colon K^{-1}]$ and $\mathfrak{a}K^{-1} = [\mathfrak{a}\colon K]$. □

Lemma 8.13. Let I be an A-submodule of R which is T-invertible in R. Then I is convex in A iff A is convex in R, and I is 2-saturated in A iff A is 2-saturated in R.

Proof. Apply Lemma 12 to the A-modules A, R, I. □

Definition 2. We call the ring extension $A \subset R$ T-*tight*, or say that A is T-*tight in* R, if for every $x \in R$ there exists some (R,T)-invertible ideal I of A with $Ix \subset A$.

Examples 8.14. i) If $A \subset R$ is a ring extension and $R = S^{-1}A$ with $S = A \cap R^*$, the ring A is T-tight in R for any preordering T of R. Indeed, if $x = \frac{a}{s} \in R$ is given $(a \in A, s \in S)$, then $(As^2)x \subset A$, and As^2 is T-invertible in R.
ii) If A is Prüfer in R then, for every preordering T of R, A is T-tight in R. Indeed, let $x \in R$ be given. Choose an R-invertible ideal I of A with $Ix \subset A$. Then, as observed above (Example 12.ii), I^2 is T-invertible in R and $I^2x \subset A$. □

Lemma 8.15. If for any $x \in R$ there exists an (R,T)-invertible *convex* ideal I of A with $Ix \subset A$, then A is convex in R.

Proof. Let $x \in R$, $a \in A$ be given with $0 \leq_T x \leq_T a$. By the assumption, there exists an (R,T)-invertible convex ideal I of A such that $Ix \in A$, i.e. $x \in I^{-1}$. By Lemma 12, we see that I is convex in A iff A is convex in I^{-1}. Hence $x \in A$. Therefore, A is convex in R. □

Theorem 8.16. Assume that A is T-tight in R. The following are equivalent.
(i) A is Prüfer and 2-saturated in R.
(ii) Every R-regular ideal of A is 2-saturated and convex in A.
(iii) If $a \in A$ and I is an (R,T)-invertible ideal of A, then the ideal $I + Aa$ is 2-saturated and convex in A.
(iii′) Every (R,T)-invertible ideal K of A contains an (R,T)-invertible ideal I of A such that for every $a \in A$ the ideal $I + aA$ is 2-saturated and convex in A.
(iv) If I and J are finitely generated ideals of A and I^2 is (R,T)-invertible, then $I^2 + J^2$ is 2-saturated and convex in A.

Proof. (i) ⇒ (ii): Clear by Theorem 7.
(ii) ⇒ (iii) ⇒ (iii′) and (ii) ⇒ (iv): trivial.
(iii′) ⇒ (iii): We prove that any ideal J of A containing an (R,T)-invertible ideal I of A with the property listed in (iii′) is 2-saturated and convex in A.

Let $x \in A$ be given with $2x = a \in J$. Since $I + Aa$ is 2-saturated in A, we conclude that $x \in I + Aa \subset J$. Thus J is 2-saturated in A.

Let $x \in A$, $a \in J$ be given with $0 \leq_T x \leq_T a$. Again, since $I + Aa$ is convex in A, we conclude that $x \in I + Aa \subset J$. Thus J is convex in A.

(iii) $\Rightarrow$ (i): (a) Since by assumption every (R,T)-invertible ideal of A is convex in A, we know by Lemma 15 that A is convex in R.

(b) Let $x \in R$ be given. Since A is T-tight in R there exists some (R,T)-invertible ideal I of A having the property listed in (iii) with $Ix \subset A$. Then $I \subset I(A+Ax) \subset A$. As just proved, $I(A+Ax)$ is 2-saturated and convex in A, hence in R by (a). We conclude by Lemma 13 that $A + Ax$ is 2-saturated and convex in R. It follows by Theorem 7 (cf. there (iii) $\Rightarrow$ (ii)), that A is Prüfer in R.
(iv) $\Rightarrow$ (i): (a) We prove first that A is convex in R. Let $x \in R$ be given. We choose an (R,T)-invertible ideal I of A with $J\colon= Ix \subset A$. By assumption, $I^2 = I^2 + A \cdot 0^2$ is 2-saturated and convex in A, and $I^2x \subset A$. Hence A is convex in R by Lemma 15.

(b) We show that A is Prüfer in R. Let $x \in R$ be given. We again choose an (R,T)-invertible ideal I of A with $J\colon= Ix \subset A$. By assumption, $I^2 + J^2 = I^2(A + Ax^2)$ is 2-saturated and convex in A, hence in R. Taking again into account that I^2 is (R,T)-invertible, we conclude by Lemma 13 that $A + Ax^2$ is 2-saturated and convex in R. Now Theorem 7 tells us that A is Prüfer in R. $\square$

It is the somewhat artificial looking condition (iii′) in this theorem which will turn out to be useful later (cf.Th.9.12 and Th.9.13), more than the less complicated condition (iii).

§9 Prüfer subrings and absolute convexity in f-rings

In f-rings, to be defined and discussed below, the theory of Prüfer subrings seems to be particularly well amenable to our methods. It is traditional to study f-rings within the category of lattice ordered rings. This category is slightly outside the framework we have used in §5 – §8. Thus some words of explanation are in order. Our main reference for lattice ordered rings and groups, and in particular for f-rings, is the book [BKW] by Bigard, Keimel and Wolfenstein.

We start with an abelian group G, using the additive notation. Assume that G is (partially) ordered in the usual sense, the ordering being compatible with addition. Thus $x \le y$ implies $x+z \le y+z$ and $-y \le -x$. We write $G^+ := \{x \in G \mid x \ge 0\}$, and we have $G^+ + G^+ \subset G^+$, $G^+ \cap (-G^+) = \{0\}$.

G is called *lattice-ordered* if G is a lattice with respect to its ordering. This means that the infimum and supremum

$$x \wedge y := \inf(x,y)\ , \quad x \vee y := \sup(x,y)$$

exist for any two elements x, y of G. As is well known, the lattice G is then automatically distributive [BKW, 1.2.14], and the group G has no torsion [BKW, 1.2.13].

We assume henceforth that G is a lattice ordered group. Clearly, for any $x, y, z \in G$ we have

$$(x+z) \wedge (y+z) = (x \wedge y) + z, \quad (x+z) \vee (y+z) = (x \vee y) + z,$$

and $(-x) \wedge (-y) = -(x \vee y)$.

For any $x \in G$ we define $x^+ := x \vee 0$, $x^- := (-x) \vee 0$. We have $x = x^+ - x^-$. Moreover, if $x = y - z$ with $y, z \in G$, then $y = x^+$, $z = x^-$ iff $y \wedge z = 0$, cf.[BKW, 1.3.4].

The *absolute value* $|x|$ of $x \in G$ is defined by $|x| := x \vee (-x)$. One proves easily that $|x| = x_+ + x_-$ [BKW, 1.3.10], more generally [BKW, 1.3.12],

$$|x-y| = (x \vee y) - (x \wedge y).$$

Of course, $|x| = 0$ iff $x = 0$, and $|x| = x$ iff $x \ge 0$.

We explicitly mention the following three facts about absolute values. Here x, y are any elements of G, and $n \in \mathbb{N}$ (The label "LO" alludes to "lattice ordered").

(LO1) $$|x| \le |y| \iff -y \le x \le y.$$

Indeed, $x \vee (-x) \le y$ means that $x \le y$ and $-x \le y$, hence $x \le y$ and $-y \le x$.

(LO2) $$-|x| - |y| \;\le\; x \wedge y \;\le\; x \vee y \;\le\; |x| + |y|$$

This follows from the trivial estimates $-|x|-|y| \le x \le |x|+|y|$ and $-|x|-|y| \le y \le |x| + |y|$.

(LO3) $$(nx)_+ \;=\; nx_+\,,\; (nx)_- \;=\; nx_-, \quad \text{hence} \quad |nx| = n|x|.$$

cf. [BKW, 1.3.7].

We now introduce a key notion for everything to follow.

Definition 1. We call a subgroup M of G *absolutely convex in* G, if $|x| \le |a|$ implies $x \in M$ for any two elements x of G and a of M. (In [BKW] the term "solid" is used for our "absolute convex".)

On the other hand, convexity in G is defined as in §5, Definition 1. Of course, absolute convexity is a stronger property than convexity.

We will need three lemmas about absolutely convex subgroups, the first and the second being very easy.

Lemma 9.1. Every absolutely convex subgroup M of G is 2-saturated in G.

Proof. Let $x \in G$ be given with $2x \in M$. Then $2|x| = |2x|$ (cf. LO3 above), and $0 \le |x| \le 2|x|$. It follows that $x \in M$. $\square$

Lemma 9.2. Assume that M is a convex subgroup of the lattice ordered abelian group G. The following are equivalent.
(i) M is a sublattice of G (i.e. $x \wedge y \in M$ and $x \vee y \in M$ for any two elements x, y of M).
(ii) M is absolutely convex in G.
(iii) If $x \in M$ then $|x| \in M$.

Proof. (i) $\Rightarrow$ (ii): Let $a \in M$ and $x \in G$ be given with $|x| \le |a|$. Then $|a| = a \vee (-a) \in M$, and we conclude from $0 \le |x| \le |a|$ and the convexity of M that $|x| \in M$, then from $-|x| \le x \le |x|$ (cf. LO1) that $x \in M$.
(ii) $\Rightarrow$ (iii): trivial.
(iii) $\Rightarrow$ (i): Let $a, b \in M$ be given. By assumption then $|a| \in M$, $|b| \in M$. As stated above (LO2), $-|a| - |b| \;\le\; a \wedge b \;\le\; a \vee b \;\le\; |a| + |b|$. Since M is convex in G, this implies $a \wedge b \in M$, $a \vee b \in M$. $\square$

Lemma 9.3. Let I, J, K be absolutely convex subgroups of G. Then the subgroup $J + K$ is again absolutely convex and

$$I \cap (J + K) = (I \cap J) + (I \cap K).$$

This can be extracted from [BKW, Chap.2]. We give a direct proof of the theorem for the convenience of the reader, following arguments in [Ban, p.130 f].

Proof. i) We first verify the following: Let $a \in J^+$, $b \in K^+$, $y \in I^+$ and $y \leq a+b$. Then $y \in (I^+ \cap J^+) + (I^+ \cap K^+)$.

Starting with the triviality $y = a \wedge y + (y - a \wedge y)$, we obtain $y = a \wedge y + y + (-a) \vee (-y)$ and then

$$(*) \qquad y = a \wedge y \ + \ (y-a) \vee 0.$$

Now $0 \leq a \wedge y \leq y$ and $0 \leq a \wedge y \leq a$. Thus $a \wedge y \in I^+ \cap J^+$. We read off from $(*)$ that $(y-a) \vee 0 \in I^+$. Further $y - a \leq b$, hence $0 \leq (y-a) \vee 0 \leq b \vee 0 \in K^+$, hence $(y-a) \vee 0 \in K^+$, and we conclude that $(y-a) \vee 0 \in I^+ \cap K^+$.
ii) We use part i) with $I = G$ to verify that $J + K$ is absolutely convex in G. Let $x \in G$, $a \in J$, $b \in K$ be given with $|x| \leq |a+b|$. Then $0 \leq x^+ \leq |x| \leq |a+b| \leq |a| + |b|$. This implies, as proved, that $x^+ \in J + K$ and $|x| \in J + K$. Thus $x = 2x^+ - |x| \ \in \ J + K$.
c) Let now $a \in I \cap (J+K)$ be given. We have $a = b + c$ with $b \in J$, $c \in K$. Then we conclude from $|a| \leq |b| + |c|$ by (i) that $|a| \in (I \cap J) + (I \cap K)$. The groups $I \cap J$ and $I \cap K$ are absolutely convex in G. Thus, as proved, $(I \cap J) + (I \cap K)$ is absolutely convex in G. It follows that $a \in (I \cap J) + (I \cap K)$. This proves $I \cap (J + K) = (I \cap J) + (I \cap K)$. □

We now switch to lattice ordered rings. A *ring* R (here always commutative, with 1) is called *lattice ordered*, if the set R is equipped with a partial ordering, which makes $(R, +)$ a lattice ordered abelian group, and such that $xy \geq 0$ for any two elements $x \geq 0$, $y \geq 0$ of R. Thus for $T := R^+$ the properties $T + T \subset T$, $T \cdot T \subset T$, $T \cap (-T) = \{0\}$ hold, but we do not demand that $x^2 \in T$ for $x \in R$.

We call T an *ordering* of R and sometimes speak of "the lattice ordered ring (R, T)".

A subring A of R is called an *ℓ-subring*, if A is a subring and a sublattice of R. We know by Lemma 2 that the absolutely convex subrings of R coincide with the convex ℓ-subrings of R.

A subset $\mathfrak{a}$ of R is called an *ℓ-ideal*, if $\mathfrak{a}$ is a *convex* ideal of R and a *sublattice* of R,*) equivalently (Lemma 2), if $\mathfrak{a}$ is an *absolutely convex* ideal of R.

Proposition 9.4. Let $A \subset R$ be a weakly surjective ring extension. Assume that A is lattice ordered and every R-regular ideal of A is absolutely convex in A (i.e. an ℓ-ideal). Then A is Prüfer in R.

*) The unitiated reader may object to this terminology, insisting that "ℓ" should just mean "sublattice". But observe that the ℓ-ideals, as defined here, are the kernels of the homomorphisms between lattice-ordered rings, cf.[BKW, §8.3].

Proof. It follows from Lemma 3, applied to the lattice-ordered group $(A,+)$, that the lattice of R-regular ideals of A is distributive. Theorem II.2.8 tells us that A is Prüfer in R. $\square$

This proposition should be regarded as a preliminary result, already indicating that there are friendly relations between absolute convexity and the Prüfer property. The assumption that A is lattice ordered seems to be too weak to allow a good theory of Prüfer extensions beyond our results in Chapters I and II. But if A is an f-ring, to be defined in a minute, we will see later that the situation described in Proposition 4 is met rather often, for example for *every* Prüfer extension $A \subset R$ in case A has bounded inversion (cf.Theorems 9.15 and 10.12).

If $(C_\alpha \mid \alpha \in X)$ is a family of lattice ordered rings, the direct product $\prod\limits_{\alpha\in X} C_\alpha$ is again a lattice ordered ring in the obvious way: We equip the ring $C := \prod\limits_{\alpha\in X} C_\alpha$ with the ordering $f \le g \iff f(\alpha) \le g(\alpha)$ for every $\alpha \in X$, and we have, for $f, g \in C$, $\alpha \in X$,

$$(f \wedge g)(\alpha) = f(\alpha) \wedge g(\alpha) \quad , \quad (f \vee g)(\alpha) = f(\alpha) \vee g(\alpha).$$

{Explanation: If $h \in C$, we denote the component of h at the index α by $h(\alpha)$. Thus h is the family $(h(\alpha) \mid \alpha \in X)$.} Notice also that $f^+(\alpha) = f(\alpha)^+$, $f^-(\alpha) = f(\alpha)^-$, and $|f|(\alpha) = |f(\alpha)|$.

Definition 2 [BKW, 9.11]. A lattice ordered ring R is called an f-*ring* if there exists a family $(C_\alpha \mid \alpha \in X)$ of *totally ordered* rings C_α, such that R is isomorphic (as an ordered ring) to an ℓ-subring of $\prod\limits_{\alpha\in X} C_\alpha$.

The following rules clearly hold in a totally ordered ring, hence in any f-ring R.
(F1) If $x \ge 0$ then $x(a \wedge b) = (xa) \wedge (xb)$.
(F2) If $x \ge 0$ then $x(a \vee b) = (xa) \vee (xb)$.
(F3) $|ab| = |a|\,|b|$.
(F4) $a^2 = |a|^2$.
(F5) If $a \ge 0$, $b \ge 0$, $x \ge 0$, $a \wedge b = 0$, then $a \wedge bx = 0$.
(F6) If $a \wedge b = 0$ then $ab = 0$.
(F7) $a + b = (a \wedge b) + (a \vee b)$.
(F8) $ab = (a \wedge b)(a \vee b)$.

Remarks. i) In any lattice ordered ring R the following weaker rules hold [BKW, 8.1.4]:
1) If $x \ge 0$ then $x(a \wedge b) \le xa \wedge xb$, $x(a \vee b) \ge xa \vee xb$.
2) $|ab| \le |a|\,|b|$

ii) It is known that *each* of the rules (F1), (F2), (F3), (F5) *characterizes* f-rings within the category of lattice ordered rings, thus allowing a more intrinsic definition of f-rings than Definition 2 above. {[BKW, p.173, 175 f]. Notice that, contrary to [BKW], our rings are always assumed to have a unit element. Thus [BKW, 9.1.14] applies.} □

In an f-ring R we have $x^2 \geq 0$ for every $x \in R$ (cf. F4). Thus
$R^+ = \{x \in R \mid x \geq 0\}$ is a partial ordering of R in the sense of §5, i.e. $T = R^+$ is a preordering of R with $T \cap (-T) = \{0\}$.

In the following WE ASSUME THAT R IS AN F-RING AND A IS A SUBRING OF R, if nothing else is said.

PROPOSITION 9.5. The following are equivalent.
(i) A is absolutely convex in R.
(ii) A is a convex ℓ-subring of R.
(iii) A is 2-saturated and convex in R.
(iv) A is convex and integrally closed in R.
(v) A is convex in R. If $x \in R$ and $x^2 \in A$ then $x \in A$.

PROOF. The implications (i) $\Rightarrow$ (iii) and (i) $\Leftrightarrow$ (ii) are covered by Lemmas 1 and 2, and (iii) $\Rightarrow$ (iv) is covered by Proposition 8.4. (iv) $\Rightarrow$ (v) is trivial.
(v) $\Rightarrow$ (i): If $x \in A$ then $|x|^2 = x^2 \in A$ by F4, hence $|x| \in A$. Lemma 2 tells us that A is absolutely convex in R. □

COROLLARY 9.6. If A is Prüfer and convex in R then A is absolutely convex in R. □

If M and I are subsets of R let $[I\colon M]$ or, if necessary, more precisely $[I\colon_R M]$ denote the set of all $x \in R$ with $xM \subset I$. Notice that, if I is an additive subgroup of R or an A-submodule of R, then also $[I\colon M]$ is an additive subgroup resp. an A-submodule of R.

DEFINITION 2. Let I, J be additive subgroups of R with $I \subset J$. We say that I is *absolutely convex in* J, if

$$x \in J,\ a \in I,\ |x| \leq |a| \Longrightarrow x \in I.$$

{The point here is that J is not assumed to be a sublattice of R. Thus the definition goes beyond Definition 1.}

LEMMA 9.7. Let I and J be additive subgroups of R with $I \subset J$. Assume that I is absolutely convex in J.
a) If M is any subset of R^+ then $[I\colon M]$ is absolutely convex in $[J\colon M]$.
b) If K is an additive subgroup and a sublattice of R, then $[I\colon K]$ is absolutely convex in $[J\colon K]$.

PROOF. a): Let $x \in [I\colon M]$ and $y \in [J\colon M]$ be given with $0 \leq |y| \leq |x|$. For every $s \in M$ we have (using F3)

$$0 \leq s|y| = |sy| \leq s|x| = |sx|,$$

and $sx \in I$, $sy \in J$. Since I is absolutely convex in J, this implies $sy \in I$. Thus $y \in [I\colon M]$.
b): If $x \in K$, then $x = x^+ - x^-$ and $x^+ \in K$, $x^- \in K$. Thus $[I\colon K] = [I\colon K^+]$ and $[J\colon K] = [J\colon K^+]$. The claim now follows from a). $\square$

LEMMA 9.8. Assume that I is an absolutely convex additive subgroup of R.
a) $[I\colon x] = [I\colon |x|]$ for every $x \in R$.
b) For any subset K of R the additive group $[I\colon K]$ is absolutely convex in R.

PROOF. a): Let $y \in [I\colon x]$ be given. We have $xy \in I$, hence (using F3)

$$|x|y^+ + |x|y^- = |x|\ |y| = |xy| \in I.$$

It follows that $|x|y_+$ and $|x|y_-$ both are elements of I. We conclude that $y = y^+ - y^- \in [I\colon |x|]$. This proves that $[I\colon x] \subset [I\colon |x|]$.

Let now $z \in [I\colon |x|]$ be given. Then $|zx| = |z \cdot |x|\ | \in I$, hence $zx \in I$, i.e. $z \in [I\colon x]$. This proves that $[I\colon |x|] \subset [I\colon x]$.
b): Let $M\colon= \{|x|\colon x \in K\}$. Using a) we obtain

$$[I\colon K] = \bigcap_{x \in K} [I\colon x] = \bigcap_{x \in K} [I\colon |x|] = [I\colon M].$$

Now apply Lemma 7.a with $J = R$. $\square$

LEMMA 9.9. Assume that A is absolutely convex in R. Then every R-invertible A-submodule of R is absolutely convex in R.

PROOF. Let K be such an A-submodule. Then $K = [A\colon K^{-1}]$, and Lemma 8 applies. $\square$

THEOREM 9.10. The following are equivalent.
(1) A is Prüfer and convex in R.
(2) Every R-regular A-submodule of R is absolutely convex in R.
(3) For every $x \in R$ the A-module $A + Ax^2$ is absolutely convex in R.
(4) Every overring of A in R is absolutely convex in R.

PROOF. (1) $\Rightarrow$ (2): It suffices to prove that a given finitely generated R-regular A-submodule I is absolutely convex in R. Since A is Prüfer in R the A-module I is R-invertible. We know by Corollary 6 that A is absolutely convex in R. Now Lemma 9 tells us that I is absolutely convex in R.

(2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4): trivial.
(3) $\Rightarrow$ (1): It suffices to prove that A is Prüfer in R. By assumption $A = A + A \cdot 0$ is absolutely convex in R. We conlcude by Proposition 5 that A is integrally closed in R. Let $x \in R$ be given. We have $-1 - x^2 \leq 2|x| \leq 1 + x^2$. By (3) it follows that $2|x| \in A + Ax^2$, then that $|x| \in A + Ax^2$, finally that $x \in A + Ax^2$. Now Theorem I.5.2 tells us that A is Prüfer in R.
(4) $\Rightarrow$ (1): Let B be an R-overring of A. By assumption B is absolutely convex in R. It follows by Proposition 5 that B is integrally closed in R, then by Theorem I.5.2 that A is Prüfer in R. □

Lemma 9.11. Assume that A is absolutely convex in R.
a) Every R-invertible A-submodule I of R is R^+-invertible (cf. §8, Def.1) in R.
b) If A is tight in R, then A is R^+-tight in R (cf. §8, Def.2).

Proof. a): We know by Lemma 9 that I is absolutely convex in R. The same holds for I^{-1}. Since both I and I^{-1} are sublattices of R, they certainly are generated (as A-modules) by I^+ and $(I^{-1})^+$ respectively. Thus I is R^+-invertible in R.
b): Now obvious. □

Theorem 9.12. Assuming that A is an ℓ-subring of R, the following are equivalent.
(1) A is Prüfer and convex in R {hence absolutely convex in R by Lemma 2 or Cor.6}.
(2) A is tight in R, and every R-regular ideal of A is an ℓ-ideal of A.
(3) A is tight in R. For every R-invertible ideal I of A and every $a \in A$ the set $I + Aa$ is an ℓ-ideal of A.
(3′) A is tight in R. Every R-invertible ideal K of A contains an R-invertible ideal I of A such that $I + Aa$ is an ℓ-ideal of A for every $a \in A$.
(4) A is tight in R. For any two finitely generated ideals I, J of A with I invertible in R the set $I^2 + J^2$ is an ℓ-ideal of A.

Proof. (1) $\Rightarrow$ (2): The extension $A \subset R$ is tight since it is Prüfer. It follows by Theorem 10 that every R-regular ideal of A is absolutely convex in R, hence is absolutely convex in A.
(2) $\Rightarrow$ (3) $\Rightarrow$ (3′): trivial.
(3′) $\Rightarrow$ (1): We first prove that A is absolutely convex in R. Let $x \in R$ and $a \in A$ be given with $0 \leq |x| \leq |a|$. Since A is tight in R there exists an R-invertible ideal K of A such that $Kx \subset A$. By (3′) K contains an R-invertible ideal I of R having the property listed in (3′), i.e. $I + aA$ is an l-ideal of A for every $a \in A$. In particular I is absolutely convex in A, hence a sublattice of A, hence a sublattice of R. By Lemma 7.b we conclude that $A = [I\colon I]$ is absolutely convex in $I^{-1} = [A\colon I]$. We now infer from $0 \leq |x| \leq |a|$ and $x \in K^{-1} \subset I^{-1}$ that $x \in A$. Thus A is absolutely convex in R.

Lemma 11 tells us that A is T-tight in R, with $T = R^+$, and moreover, that all R-invertible ideals of A are (R,T)-invertible. We conclude by Theorem 8.16, using there the implication (iii′) $\Rightarrow$ (i), that A is Prüfer in R.

(4) $\Rightarrow$ (1): The proof runs the same way as for the implication (3′) $\Rightarrow$ (1). We now work with I^2 instead of I for I an R-invertible ideal such that $Ix \subset A$, and we use the implication (iv) $\Rightarrow$ (i) in Theorem 8.16. $\square$

We also ask for criteria, in the vein of the preceding theorems 10 and 12, that A is Bezout and convex in R.

Theorem 9.13. a) The following are equivalent.
(1) A is Bezout and convex in R.
(2) For every $x \in R$ the A-module $A + Ax$ is principal and absolutely convex in R.
(3) A is an ℓ-subring of R, and $R = S^{-1}A$ with $S := A \cap R^*$. For every $a \in A$ and $s \in S$ the ideal $As + Aa$ of A is principal. For every $s \in S$ the ideal As is absolutely convex in A (i.e. an ℓ-ideal of A).
(3′) A is an ℓ-subring of R. There exists a multiplicative subset S of A with the following properties: $R = S^{-1}A$. For every $s \in S$ and $a \in A$ there exists some $t \in S$ such that $As + Aa = At$. For every $s \in A$ the ideal As is absolutely convex in A.

b) If $2 \in R^*$ then (1) – (3) are also equivalent to each of the following two conditions.
(4) $R = S^{-1}A$ with $S := A \cap R^*$. For every $s \in S$ and $a \in A$ the ideal $As^2 + Aa$ of A is principal and absolutely convex in A.
(4′) There exists a multiplicative subset S of A with $2 \in S$ and $R = S^{-1}A$, and such that, for every $a \in A$ and $s \in S$, the ideal $As^2 + Aa$ is principal and absolutely convex in A.

Comment. Given an f-ring A the somewhat artificial looking conditions (3′) and (4′) are useful for finding – theoretically – all Prüfer (hence Bezout) extensions $A \subset R$ such that R is an f-ring with $R^+ \cap A = A^+$ and A an ℓ-subring of R. Indeed, we will see in §10 (in a more general context) that, given a multiplicative subset S of A consisting of non-zero divisors of A, there exists a unique partial ordering on $R := S^{-1}A$ such that R is an f-ring, A is an ℓ-subring of R, and $R^+ \cap A = A^+$. (Actually it is not difficult, just an exercise, to give a direct proof of this fact.)

Proof of Theorem 9.13. (1) $\Rightarrow$ (2): Let $x \in R$ be given. Then $A + Ax$ is principal, since A is Bezout in R (cf.Th.II.10.2). It follows from Theorem 10 (cf. there (1) $\Rightarrow$ (2)) that $A + Ax$ is absolutely convex in R.
(2) $\Rightarrow$ (1): trivial.
(1) $\Rightarrow$ (3): Let $S := A \cap R^*$. Theorem II.10.16 tells us that $R = S^{-1}A$. We further know by Theorem 10 above (cf. there (1) $\Rightarrow$ (2)) that, for every $s \in S$

and $a \in A$, the ideal $As + Aa$ is absolutely convex in R, hence in A. Since A is Bezout in R, this ideal is also principal (cf. Th.II.10.2).
(3) $\Rightarrow$ (3′): The set $S := A \cap R^*$ has all the properties listed in (3′). This needs a verification only for the second one. Let $s \in S$ and $a \in A$ be given. By assumption (3), $As + Aa = At$ for some $t \in A$. We have $s = bt$ with some $b \in A$, and we conclude that $t \in A \cap R^* = S$.
(3′) $\Rightarrow$ (1): Let $a_1, \dots, a_r \in A$ and $s \in S$ be given. Then there exists some $t \in S$ such that $As + Aa_1 + \cdots + Aa_r = At$. Indeed, this holds for $r = 1$ by assumption (3′) and then follows for all r by an easy induction. Now Theorem 12 tells us (implication (3′) $\Rightarrow$ (1) there) that A is (Prüfer and) convex in R.
Let $x \in R$ be given. Write $x = \frac{a}{s}$ with $a \in A$, $s \in S$. Then $A + Ax = S^{-1}(As + Aa)$ and $As + Aa = At$ with $t \in S$. Thus the A-module $A + Ax$ is principal, and we conclude that A is Bezout in R (cf.Th.II.10.2).
(3) $\Rightarrow$ (4) $\Rightarrow$ (4′): trivial.
(4′) $\Rightarrow$ (1): We learn from Theorem 8.9 that A is convex in R. Let $x \in R$ be given. Write $x = \frac{a}{s^2}$ with $a \in A$, $s \in S$. The ideal $As^2 + Aa$ is principal by assumption (4′). Thus the module $A + Ax = s^{-2}(Aa + As^2)$ is principal. This proves that A is Bezout in R. □

Open Question. If A is a convex (hence absolutely convex) Prüfer subring of R, does it follow that A is Bezout in R?

We will now see that the answer is "Yes" if R or (equivalently) A has bounded inversion. Related to this, we will find more criteria, that A is Bezout in R, and results about such extensions more precise than those stated in Theorem 13.

We store our results in the following lengthy theorem 15. Here the dashed conditions (2′), (3′), (4′), (6′) are included in order to make the proof more transparent, while the undashed conditions (1) – (8) are the more interesting ones. For the proof we will need (a special case of) the following easy lemma.

Lemma 9.14. Let I be a 2-saturated additive subgroup of R. Assume that every $x \in R$ with $x^2 \in I$ is an element of I. Then I is a sublattice of R.

Proof. If $x \in I$ then $|x|^2 = x^2 \in I$, hence $|x| \in I$. It follows that $2x^+ = x + |x| \in I$ and then that $x^+ \in I$. Given elements $x, y \in I$ we conclude that

$$x \vee y = y + [(x - y) \vee 0] = y + (x - y)^+ \in I. \qquad \square$$

Theorem 9.15. The following are equivalent.
(1) A has bounded inversion and is Prüfer in R.
(2) R has bounded inversion. A is convex in R.
(2′) R has bounded inversion. A is absolutely convex in R.
(3) A is convex in R. For every $x \in R$, $A + Ax = A(1 + |x|)$.
(3′) A is absolutely convex in R. For every $x \in R$, $A + Ax = A(1 + |x|)$.

(4) For every $x \in R$, $A + Ax = A(1 + |x|)$, and this module is convex in R.
(4′) For every $x \in R$, $A+Ax = A(1+|x|)$, and this module is absolutely convex in R.
(5) R has bounded inversion. A is Bezout and convex in R.
(6) A is convex in R. For every $x \in R$, $A + Ax = A(1 \vee |x|)$.
(6′) For every $x \in R$ the module $A+Ax$ is absolutely convex in R, and $A+Ax = A(1 \vee |x|)$.
(7) $2 \in R^*$, and $R = S^{-1}A$ with $S := A \cap R^*$. For every $a \in A$, $s \in A$, the ideal $As^2 + Aa$ is an ℓ-ideal of A, and $As^2 + Aa = A(s^2 + |a|)$.
(8) There exists a multiplicative subset S of A such that $2 \in S$, $R = S^{-1}A$, and $As^2 + Aa$ is an ℓ-ideal of A for every $a \in A$ and $s \in S$.

Comment. Given an f-ring A, this time with bounded inversion, condition (8) is useful for finding – theoretically – all Prüfer (hence Bezout) extensions $A \subset R$ such that R is an f-ring with $R^+ \cap A = A^+$ and A is an ℓ-subring of R, cf. the comment following Theorem 13.

PROOF OF THEOREM 9.15.
(1) $\Leftrightarrow$ (2): This is covered by Theorem 7.2.
(2) $\Rightarrow$ (2′): $2 \in R^*$, since R has bounded inversion. $\frac{1}{2} \in A$, since A is convex in R. Thus A is 2-saturated in R. The ring A is also convex in R. By Proposition 5 we conclude that A is absolutely convex in R.
(2′) $\Rightarrow$ (2): trivial.

We now know that conditions (1), (2), (2′) are equivalent.

(1) $\wedge$ (2) $\Rightarrow$ (3′): A is Prüfer and convex in R. Let $x \in R$ be given. Theorem 10 tells us that the module $A + Ax$ is absolutely convex in R, since this module is R-regular. In particular, $|x| \in A + Ax$, hence $1 + |x| \in A + Ax$. This proves that $A(1 + |x|) \subset A + Ax$. On the other hand, $1 + |x| \in R^*$ by (2), and $(1+|x|)^{-1} \leq 1$, hence $(1+|x|)^{-1} \in A$. We also have $|x \cdot (1+|x|)^{-1}| \leq 1$, hence $x(1 + |x|)^{-1} \in A$. It follows that $1 \in A(1 + |x|)$ and $x \in A(1 + |x|)$, hence $A + Ax \subset A(1 + |x|)$. Thus $A + Ax = A(1 + |x|)$.
(3′) $\Rightarrow$ (3): trivial.
(3) $\Rightarrow$ (2): If $x \in R$ and $x \geq 1$ then, by (3),

$$A + Ax = A + A(x - 1) = A(1 + x - 1) = Ax.$$

Thus $1 \in Ax$, which implies $x \in R^*$. This proves that R has bounded inversion.

We now know that all conditions (1) – (3′) are equivalent.

(1) $\wedge$ (3′) $\Rightarrow$ (4′): A is Prüfer in R by (1) and absolutely convex in R by (3′). Theorem 10 tells us again that, for every $x \in R$, the module $A + Ax$ is absolutely convex in R. Also $A + Ax = A(1 + |x|)$ by (3′).
(4′) $\Rightarrow$ (4) $\Rightarrow$ (3): trivial.

We have proved the equivalence of all conditions (1) – (4′).

(2) $\wedge$ (4) $\Rightarrow$ (5): R has bounded inversion by (2). For every $x \in R$ the A-module $A + Ax$ is principal by (4). Thus A is Bezout in R (cf.Th.II.10.2). $A + Ax$ is also convex in R by (4). In particular ($x = 0$), A is convex in R.
(5) $\Rightarrow$ (2): trivial.
(4′) $\Rightarrow$ (6′): Let $x \in R$ be given. The module $A+Ax$ is absolutely convex in R, and $A+Ax = A(1+|x|)$. We have $1 \vee |x| \leq 1+|x|$ Thus $A(1+|x|) \supset A(1 \vee |x|)$. Now $1 \vee |x| = 1 + y$ with $y \in R^+$. Thus $A(1 \vee |x|) = A + Ay$, and this module is again absolutely convex in R. Since $1 + |x| \leq 2(1 \vee |x|)$ we infer that $A(1 + |x|) \subset A(1 \vee |x|)$, and conclude that $A(1 + |x|) = A(1 \vee |x|)$.
(6′) $\Rightarrow$ (6): trivial.
(6) $\Rightarrow$ (2): For every $x \in R$ with $x \geq 1$ we have $A + Ax = Ax$, since $1 \vee |x| = x$. It follows that $1 \in Ax$, hence $x \in R^*$. Thus R has bounded inversion.

We have proved the equivalence of all conditions (1) – (6′).

(1) – (6′) $\Rightarrow$ (7): R has bounded inversion, hence $2 \in R^*$. Since A is Bezout in R, we have $R = S^{-1}A$ with $S{:}= R^* \cap A$ (cf.Prop.II.10.16 or Th.13). Let $s \in S$ and $a \in A$ be given. By (3),

$$As^2 + Aa = s^2 \left(A + \frac{a}{s^2}\right) = s^2 A \left(A + \frac{|a|}{s^2}\right) = A(s^2 + |a|).$$

By (4) the module $A\left(1 + \frac{|a|}{s^2}\right)$ is absolutely convex in R. It follows that $A(s^2 + |a|)$ is absolutely convex in R, hence in A, i.e. $A(s^2 + |a|)$ is an ℓ-ideal of A.
(7) $\Rightarrow$ (8): trivial.
(8) $\Rightarrow$ (3): Theorem 8.9 tells us that A is Prüfer and convex in R. Let $x \in R$ be given. Write $x = \frac{a}{s^2}$ with $a \in A$, $s \in S$. Then

$$A + Ax = s^{-2}(As^2 + Aa) = s^{-2}A(s^2 + |a|) = A(1 + |x|). \qquad \square$$

§10 Rings of quotients of an f-ring

In the following A is an f-ring. We will study overrings of A in the complete ring of quotients $Q(A)$. For the general theory of $Q(A)$ we refer to Lambek's book [Lb]. (Some facts had been recapitulated in I §3.)

Recall that every element of $Q(A)$ can be represented by an A-module homomorphism $f\colon I \to A$ with I a dense ideal of A. More precisely

$$Q(A) = \varinjlim_{I \in \mathcal{D}(A)} \operatorname{Hom}_A(I, A)$$

with $\mathcal{D}(A)$ denoting the direct system of dense ideals of A, the ordering being given by reversed inclusion, $I \leq J$ iff $I \supset J$. Most often we will not distinguish between such a homomorphism $f\colon I \to A$ and the corresponding element $[f]$ of $Q(A)$.

Our first goal in the present section is to prove that *there exists a unique partial ordering U on $Q(A)$ which makes $Q(A)$ an f-ring in such a way that $U \cap A = A^+$ and A is an ℓ-subring of $Q(A)$.* This is an important result due to F.W. Anderson [And]. Anderson's paper is difficult to read since he establishes such a result also for certain non commutative f-rings. For the convenience of the reader we will write down a full proof in the much easier commutative case. We then will prove the same for suitable overrings R of A in $Q(A)$ instead of $Q(A)$ itself. Among these overrings will be all Prüfer extensions of A.

Whenever it seems appropriate we will work in an arbitrary overring R of A in $Q(A)$ instead of $Q(A)$ itself. Recall that, up to isomorphism over A, these rings are all the rings of quotients of A.

Lemma 10.1. Let $a \in A^+$, $b \in A$. Then $(ab)^+ = ab^+$ and $(ab)^- = ab^-$.

Proof. $ab = ab^+ - ab^-$. Applying the property (F1) from §9 we obtain $(ab^+) \wedge (ab^-) = a(b^+ \wedge b^-) = 0$. This proves the claim. □

Corollary 10.2. If a, b, s are elements of A with $a \geq 0$, $b \geq 0$, $a = bs$, then $a = bs^+$, $0 = bs^-$.

Proof. By the lemma we have $bs^+ = (bs)^+ = a$, $bs^- = (bs)^- = 0$. □

Definitions 1 a) We call a *subset* M of A *dense in* A, if the ideal AM generated by M is dense in A. This means that for every $x \in A$ with $x \neq 0$ there exists some $m \in M$ with $xm \neq 0$.
b) If I is any ideal of A let $I^{(2)}$ denote the set $\{a^2 \mid a \in I\}$. □

Lemma 10.3. If I is a dense ideal of A the set $I^{(2)}$ is also dense in A.

Proof. Let $x \in A$ be given with $xI^{(2)} = 0$. For any two elements a, b of I we have $xa^2 = 0$, $xb^2 = 0$, $x(a+b)^2 = 0$. It follows that $2xab = 0$, and then that $xab = 0$, since the additive group of A has no torsion. Thus $xI^2 = 0$. Since I is dense in A we conclude that $xI = 0$ and then that $x = 0$. $\square$

Corollary 10.4. If I is dense ideal of A then I^+ is dense in A. $\square$

Lemma 10.5. Let M be a subset of A^+ which is dense in A. Assume that x is an element of $Q(A)$ with $xM \subset A^+$. Then $x \cdot (A\colon x)^+ \subset A^+$.

Proof. Let $a \in (A\colon x)^+$ be given. If $d \in M$, then $(ax)d = (xd)a \in A^+$ and $ax \in A$. It follows that $(ax)^- d = 0$ by Corollary 2 above. Since M is dense in A we conclude that $(ax)^- = 0$, hence $ax \in A^+$. $\square$

In the following R *is an overring of* A *in* $Q(A)$. We introduce the set

$$U\colon= \{x \in R \mid x \cdot (A\colon x)^+ \subset A^+\}.$$

Due to Corollary 4 and Lemma 5 we can say, that U is the set of elements x of R such that there exists some dense subset M of A with $M \subset A^+$ and $Mx \subset A^+$.

Proposition 10.6.
i) U is a partial orderring of R with $x^2 \in U$ for every $x \in R$, and $U \cap A = A^+$.
ii) If T is any preordering of R with $T \cap A \subset A^+$ then $T \subset U$.

Proof. i): If $x \in U \cap (-U)$ then $x(A\colon x)^+$ is contained in $A^+ \cap (-A^+) = \{0\}$. Since $(A\colon x)^+$ is dense in A (cf.Cor.4), we conclude that $x = 0$. Thus $U \cap (-U) = \{0\}$.

Let $x, y \in U$ be given. We choose dense subsets M, N of A with $M \subset A^+$, $N \subset A^+$, $Mx \subset A^+$, $Ny \subset A^+$. The set $MN = \{uv \mid u \in M, v \in N\}$ is again dense in A and contained in A^+, and $MN(x+y) \subset A^+$, $MN(x \cdot y) \subset A^+$. Thus $U + U \subset U$ and $U \cdot U \subset U$.

Finally let $x \in U$ and $I\colon= (A\colon x)$. We know by Lemma 3 that the subset $I^{(2)}$ of A^+ is dense in A. Since $x^2 I^{(2)} \subset A^+$, we conclude that $x^2 \in U$.

If $x \in A$ then $(A\colon x) = A$. The condition $A^+ x \subset A^+$ means that $x \in A^+$. Thus $U \cap A = A^+$.

ii): Let T be a preordering of R with $T \cap A \subset A^+$. For any $x \in T$ we have $(A\colon x)^+ \cdot x \subset T \cap A \subset A^+$, hence $x \in U$. Thus $T \subset U$. $\square$

Remark. In part ii) of the theorem we do not fully need the assumption that T is a preordering of R. It suffices to know that T is a subset of R with $T \cdot T \subset T$ and $T \cap A \subset A^+$. $\square$

Definition 2. We call U the *canonical ordering on* R induced by the ordering A^+ of A. If necessary, we write U_R instead of U. Notice that $U_R = R \cap U_{Q(A)}$. $\square$

Lemma 10.7. Let M be a subset of A which is dense in A. Then M is dense in $Q(A)$.

Proof. Let $x \in Q(A)$ be given with $Mx = 0$. Then $M \cdot (A\colon x)x = 0$. This implies $(A\colon x)x = 0$ and then $x = 0$, since $(A\colon x)$ is dense in $Q(A)$. $\square$

Proposition 10.8. Assume that T is a partial ordering of R with $T \cap A = A^+$. Assume further that (R, T) is an f-ring. Then $T = U_R$.

Proof. We write $U\colon = U_R$. We know by Proposition 6 that $T \subset U$. We now prove that also $U \subset T$.

In the f-ring (R, T) we use standard notation from previous sections: $T = R^+$, $x \le y$ iff $y - x \in T$, etc. Let $x \in U$ be given. We have to verify that $x \ge 0$, i.e. $x^- = 0$. Suppose that $x^- \neq 0$. The set $M\colon = (A\colon x)^+$ is dense in A by Corollary 4, hence dense in R by Lemma 7. Thus there exists some $s \in M$ with $sx^- \neq 0$. Since R is an f-ring and $s \in A^+ \subset R^+$, we conclude by Lemma 1 that $(sx)^- = sx^- \neq 0$. But $sx \in U \cap A = A^+ \subset R^+$. This is a contradiction. Thus $x^- = 0$. $\square$

Definition 3. An *f-extension* of the f-ring A is an f-ring R which contains A as an ℓ-subring such that $R^+ \cap A = A^+$.

Theorem 10.9 (F.W. Anderson [And]). There exists a unique partial ordering T on $Q(A)$ such that $(Q(A), T)$ is an f-extension of A. This ordering T is the canonical ordering $U = U_{Q(A)}$ induced by A^+ on $Q(A)$.

Proof. We know by Proposition 8 that U is the only candidate for a partial ordering T on $Q(A)$ with these properties. We endow $Q(A)$ with the ordering U and write $U = Q(A)^+$.

Step 1. We first prove that $Q(A)$ is lattice ordered. Given $x \in Q(A)$ it suffices to verify that $x \vee 0 = \sup(x, 0)$ exists in $Q(A)$. We give an explicit construction of $x \vee 0$.

Claim. Let $a_1, \dots, a_n \in (A\colon x)^+$ and $b_1, \dots, b_n \in A$ be given with $\sum\limits_{i=1}^{n} a_i b_i = 0$. Then $\sum\limits_{i=1}^{n} (a_i x)^+ b_i = 0$.

Proof of the claim. Let $c \in (A\colon x)^+$. It follows by Lemma 1 from $(cx)a_i = c(a_i x)$ that $(cx)^+ a_i = (cxa_i)^+ = c(a_i x)^+$. Thus

$$c \sum_{i=1}^{n} (a_i x)^+ b_i = (cx)^+ \sum_{i=1}^{n} a_i b_i = 0.$$

Since $(A\colon x)^+$ is dense in A we obtain $\sum_{i=1}^{n}(a_i x)^+ b_i = 0$, as desired.

Thus there exists a well defined homomorphism $h\colon (A\colon x)^+ A \to A$ of A-modules with

$$h\left(\sum_{i=1}^{n} a_i b_i\right) := \sum_{i=1}^{n}(a_i x)^+ b_i$$

for all $n \in \mathbb{N}, a_i \in (A\colon x)^+$, $b_i \in A$. The map h may be viewed as an element of $Q(A)$. Notice that for every $a \in (A\colon x)^+$ we have $ah = ha = (ax)^+$.

We want to prove that $h = x \vee 0$. From $(A\colon x)^+ h \subset A^+$ we conclude that $h \geq 0$. For any $a \in (A\colon x)^+$ we have $(h - x)a = h(a) - xa = (xa)^+ - xa = (xa)^- \in A^+$. Thus $h \geq x$.

Let $y \in Q(A)$ be given with $y \geq 0$ and $y \geq x$. For any $a \in (A\colon x)^+ \cap (A\colon y)^+$ the products ax, ay are in A and $ay \geq 0$, $ay \geq ax$, hence $ay \geq (ax)^+$, where, of course, $(ax)^+$ means $\sup_A(ax, 0)$. It follows that $a(y - h) \geq (ax)^+ - ah = 0$. Since $(A\colon x)^+ \cap (A\colon y)^+$ is dense in A we conclude that $y - h \geq 0$, i.e. $y \geq h$. This finishes the proof that $h = x \vee 0$.

Step 2. We prove that A is a sublattice of $Q(A)$. It suffices to verify for a given $x \in A$ that the element h constructed in Step 1 coincides with $\sup_A(x, 0) = x^+$. We have $(A\colon x)^+ = A^+$, hence by Step 1, for any $a \in A^+$, $ah = (ax)^+ = ax^+$ (cf.Lemma 1). Since A^+ is dense in $Q(A)$ it follows that indeed $h = x^+$.

Step 3. We now may use the notation x^+, x^- for any $x \in A$ unambiguously, since x^+, x^- means the same by regarding x as an element of the lattice A or of the lattice $Q(A)$. Our proof in Step 1 tells us that, for any $x \in Q(A)$, $a \in (A\colon x)^+$ we have

$$(*) \qquad (ax)^+ = ax^+.$$

Indeed, this is just the statement that $h(a) = (ax)^+$ from Step 1. We now can prove that $Q(A)$ is an f-ring by verifying

$$(**) \qquad s(x \vee y) = (sx) \vee (sy)$$

for given elements $x, y \in Q(A)$ and $s \in Q(A)^+$. ([BKW, 9.1.10]; we mentioned this criterion for a lattice ordered ring to be an f-ring in §9.) Subtracting sy on both sides we see that it suffices to prove $(**)$ in the case $y = 0$, i.e.

$$(***) \qquad sx^+ = (sx)^+.$$

In order to verify this identity for given $x \in Q(A), s \in Q(A)^+$ we introduce the ideal $I := ((A\colon x)\colon s) \cap (A\colon sx)$, which is dense in A. $\{$Observe that $(A\colon x) \cdot (A\colon s) \subset ((A\colon x)\colon s)$.$\}$ For $a \in I^+$ we have, by use of $(*)$, $a(sx)^+ = (asx)^+$ since $a \in (A\colon sx)^+$, and $asx^+ = (asx)^+$ since $as \in (A\colon x)^+$. Thus $a[sx^+ - (sx)^+] = 0$

for every $a \in I^+$. Since I^+ is dense in $Q(A)$, we conclude that $sx^+ = (sx)^+$, as desired. This finishes the proof that $Q(A)$ is an f-ring. $\square$

We want to extend Theorem 9 to suitable subrings of $Q(A)$ containing A. These are the rings of type $A_{[\mathcal{F}]}$ occuring already in Theorem II.3.5 (with $R = Q(A)$ there), but now we use a more professional terminology.

Definition 4. Let A be any ring (commutative, with 1, as always). As previously let $J(A)$ denote the set of all ideals of A. We call a subset $\mathcal{F}$ of $J(A)$ a *filter on* A, if the following holds:
(1) $I \in \mathcal{F}, J \in J(A), I \subset J \ \Rightarrow\ J \in \mathcal{F}$.
(2) $I \in \mathcal{F}, J \in \mathcal{F} \ \Rightarrow\ I \cap J \in \mathcal{F}$.
(3) $A \in \mathcal{F}$.
We call a filter $\mathcal{F}$ *multiplicative* if instead of (2) the following stronger property holds:
(4) $I \in \mathcal{F}, J \in \mathcal{F} \ \Rightarrow\ IJ \in \mathcal{F}$. $\square$

We say that $\mathcal{F}$ is *of finite type* if the following holds.
(5) If $I \in \mathcal{F}$ there exists a finitely generated ideal I_0 of A with $I_0 \in \mathcal{F}$ and $I_0 \subset I$.

Notice that the subsets $\mathcal{F}$ of $J(A)$ considered in II, §3 with the properties R0-R2 (resp. R0-R3) there are just the multiplicative filters (resp. mutliplicative filters of finite type) on A.

Examples. 1) The set $\mathcal{D}(A)$ consisting of all dense ideals of A is a multiplicative filter on A.
2) If $A \subset R$ is any ring extension then the set $\mathcal{F}(R/A)$ of R-regular ideals of A is a multiplicative filter of finite type on A. $\square$

By definition we have

$$Q(A) = \varinjlim_{I \in \mathcal{D}(A)} \operatorname{Hom}_A(I, A).$$

If $\mathcal{F}$ is any filter on A contained in $\mathcal{D}(A)$ then we can form the ring

$$A_{\mathcal{F}} := \varinjlim_{I \in \mathcal{F}} \operatorname{Hom}_A(I, A).$$

in an analogous way. Since for any $I \in \mathcal{F}$ the natural map $\operatorname{Hom}_A(I, A) \to Q(A)$ is injective, we may – and will – regard $A_{\mathcal{F}}$ as a subring of $Q(A)$. For the smallest filter $\{A\}$ we obtain $A_{\{A\}} = A$. Thus $A \subset A_{\mathcal{F}} \subset Q(A)$. We have

$$A_{\mathcal{F}} \;=\; \{x \in Q(A) \mid (A{:}\,x) \in \mathcal{F}\} \;=\; \{x \in Q(A) \mid \exists\, I \in \mathcal{F} \text{ with } Ix \subset A\}.$$

Thus $A_{\mathcal{F}}$ is the ring $A_{[\mathcal{F}]}$ in the terminology of II, §3 (cf. Theorem II.3.5), with $R = Q(A)$ there.

Definition 4. We call a filter $\mathcal{F}$ on A *positively generated* if for any $I \in \mathcal{F}$ also $I^+A \in \mathcal{F}$.

Remark. If $\mathcal{F}$ is any filter on A then a *base* $\mathfrak{B}$ of $\mathcal{F}$ is a subset $\mathfrak{B}$ of $\mathcal{F}$ such that for every $I \in \mathcal{F}$ there exists some $K \in \mathfrak{B}$ with $K \subset I$. Of course, if $\mathcal{F}$ has a base $\mathfrak{B}$ such that $K^+A \in \mathcal{F}$ for every $K \in \mathfrak{B}$, then $\mathcal{F}$ is positively generated.

Examples 10.10. i) $\mathcal{D}(A)$ is positively generated. This is the content of Corollary 4 above.
ii) If $\mathcal{F}$ is a multiplicative filter of finite type then $\mathcal{F}$ is positively generated. Indeed, let $\mathfrak{B}$ be the set of finitely generated ideals $I \in \mathcal{F}$. It is a base of $\mathcal{F}$. If $I = Aa_1 + \cdots + Aa_n \in \mathcal{F}$, then $I^{n+1} \subset Aa_1^2 + \cdots + Aa_n^2 \subset I^+A$. Thus $I^+A \in \mathcal{F}$.
iii) Assume that $\mathcal{F}$ has a base $\mathfrak{B}$ consisting of ideals I which are sublattices of A. Then $\mathcal{F}$ is positively generated. Indeed, if $I \in \mathfrak{B}$ and $x \in I$, then $x = x^+ - x^-$ and $x^+, x^- \in I^+$. Thus $I = I^+A$.

Proposition 10.11. Assume that $\mathcal{F}$ is a positively generated multiplicative filter consisting of dense ideals.
i) $A_{\mathcal{F}}$ is an ℓ-subring of $Q(A)$. Thus, with the ordering $A^+_{\mathcal{F}} := A : f \cap Q(A)$ on $A_{\mathcal{F}}$, both $A \subset A_{\mathcal{F}}$ and $A_{\mathcal{F}} \subset Q(A)$ are f-extensions.
ii) Let $x \in Q(A)$. Then $x \in A^+_{\mathcal{F}}$ iff there exists some $I \in \mathcal{F}$ with $I^+x \subset A^+$.

Proof. i): We verify for a given $x \in A_{\mathcal{F}}$ that $x^+ = x \vee 0 \in A_{\mathcal{F}}$. We choose some $I \in \mathcal{F}$ with $Ix \subset A$. For $a \in I^+$ we have $ax^+ = (ax)^+ \in A^+$. Thus $(I^+A)x^+ \subset A$. Since $I^+A \in \mathcal{F}$ we conclude that $x^+ \in A_{\mathcal{F}}$.
ii): Let $R := A_{\mathcal{F}}$. If $x \in Q(A)$ and $I^+x \subset A^+$ for some $I \in \mathcal{F}$ then $x \in Q(A)^+$ by definition of the ordering of $Q(A)$, since $I \in \mathcal{D}(A)$. Also $x \in A_{\mathcal{F}} = R$, since $I^+A \in \mathcal{F}$ and $(I^+A)x \subset A$. Thus $x \in R \cap Q(A)^+ = R^+$. Conversely, if $x \in R^+$, we choose some $I \in \mathcal{F}$ with $Ix \in A$. Then $I^+x \subset R^+ \cap A = A^+$. □

We arrive at our main result in this section. It generalizes Theorem 9 to ws extensions of A.*) We write it down in an explicit way avoiding the technical notion of canonical ordering.

Theorem 10.12. Let A be an f-ring and $A \subset R$ a ws extension of A.

i) There exists a unique partial ordering R^+ on R such that R, equipped with this ordering, is an f-extension of A. Moreover $Q(A)$ is an f-extension of R.
ii) R^+ is the set of all $x \in R$ such that $(A{:}\,x)^+ \cdot x \subset A^+$.
iii) R^+ is the set of all $x \in R$ such that there exists some dense subset M of A with $M \subset A^+$ and $Mx \subset A^+$.

*) Recall that "ws" abbreviates "weakly surjective" (I, §3).

iv) Every overring of A in R, which is ws over A, is an ℓ-subring of R.

Proof. Defining R^+ by $R^+ := U_R = \{x \in R \mid (A\colon x)^+ x \subset A^+\}$ we know from above (Propositions 6 and 8), that R^+ is a partial ordering of R, and that this is the only candidate such that (R, R^+) is an f-ring and $R^+ \cap A = A^+$. We further know from above (Lemma 5) that, given a dense subset M of A with $M \subset A^+$, any $x \in R$ with $Mx \subset A^+$ is an element of R^+.

Let $\mathcal{F}$ denote the filter on A consisting of the R-regular ideals of A, $\mathcal{F} := \mathcal{F}(R/A)$. As observed above (Example 10.iii), $\mathcal{F}$ is positively generated. It follows by Propositions 11 and 8 that $A_{\mathcal{F}}$, equipped with the canonical ordering induced by A^+, is an f-ring, and both $A \subset A_{\mathcal{F}}$ and $A_{\mathcal{F}} \subset R$ are f-extensions.

Clearly $R \subset A_{\mathcal{F}}$, since $(A\colon x) \in \mathcal{F}$ for every $x \in R$ (Recall Th.I.3.13.) Conversely, if $x \in A_{\mathcal{F}} \subset Q(A)$, there exists some $I \in \mathcal{F}$ with $Ix \in A$. Multiplying by R we obtain $Rx = RIx \subset R$, i.e. $x \in R$. Thus $R = A_{\mathcal{F}}$. Now claims i) – iii) are evident.

Finally, if B is an overring of A in R which is ws over A, then applying what we have proved to $A \subset B$ instead of $A \subset R$, we see that B is an ℓ-subring of $Q(A)$, hence an ℓ-subring of R. □

We continue to assume that A is an f-ring. We write down two corollaries of Theorem 12. Nothing new is needed to prove them.

Corollary 10.13. Let S be a multiplicative subset of A consisting of non-zero divisors. There is a unique partial ordering $(S^{-1}A)^+$ on $S^{-1}A$ such that $S^{-1}A$ becomes an f-extension of A. We have

$$(S^{-1}A)^+ = \left\{ \frac{a}{s^2} \mid a \in A^+, s \in S \right\} = \left\{ \frac{a}{s} \mid a \in A^+, s \in S^+ \right\}.$$

With this ordering $S^{-1}A$ is an ℓ-subring of $Q(A)$. □

Corollary 10.14. Let $A \subset R$ be a Prüfer extension. There is a unique partial ordering R^+ on R such that R becomes an f-extension of A. An element x of R lies in R^+ iff there exists an invertible (or: R-invertible) ideal I of A with $I^+ x \subset A^+$, or alternatively, with $I^{(2)} x \subset A^+$. With this ordering $S^{-1}A$ is an ℓ-subring of $Q(A)$. □

Henceforth we equip every overring R of A in $Q(A)$ with the canonical ordering R^+ induced by A^+. If $A \subset R$ is Prüfer, or more generally ws, R is an f-ring and both $A \subset R$ and $R \subset Q(A)$ are f-extensions.

It now makes sense to define an "absolute" Prüfer convexity cover of A, as announced at the end of §7.

Definition 5. Let $P_c(A)$ denote the polar $C(P(A)/A)^\circ$ of the convex hull $C(P(A)/A)$ of A in the f-ring $P(A)$ (over A, in $P(A)$). We call $P_c(A)$ the *Prüfer convexity cover of A*. $\square$

From Theorem 7.9 we read off the following fact.

Theorem 10.15. $P_c(A)$ is the unique maximal overring E of A in $Q(A)$ (thus, up to isomorphy over A, the unique maximal ring of quotients of A), such that A is Prüfer and convex in E. $\square$

Remarks 10.16. i) It follows, say, from Theorem 9.10, that every A-submodule I of $P_c(A)$, which is $P_c(A)$-regular, is *absolutely* convex in $P_c(A)$. In particular this holds for every overring of A in $P_c(A)$. Thus we may replace the word "convex" in Theorem 15 by "absolutely convex".

ii) If A has bounded inversion, it follows from Theorem 7.2 that $P_c(A) = P(A)$. Also now every overring of A in $P(A)$ has again bounded inversion (cf.Th.9.15).

iii) For R any overring of A in $Q(A)$ we obtain the Prüfer convexity cover $P_c(A, R)$ of A in R, as defined in §7, by intersecting $P_c(A)$ with R, $P_c(A, R) = R \cap P_c(A)$. Indeed, A is Prüfer and convex in $R \cap P_c(A)$, hence $R \cap P_c(A) \subset P_c(A, R)$, and A is also Prüfer and convex in $P_c(A, R)$, hence $P_c(A, R) \subset R \cap P_c(A)$.

Notice that $P_c(A, R)$ is an ℓ-subring of $Q(A)$, even if R is not. $\square$

We want to find out which ℓ-subrings of $Q(A)$ have the same Prüfer convexity cover as A.

Definition 6. The *convex holomorphy ring* of the f-ring A is the holomorphy ring $\mathrm{Hol}_{A^+}(A)$ of A with respect to its ordering A^+ (cf.§6, Def.1). We denote this subring of A more briefly by $\mathrm{Hol}_c(A)$.

We know by Theorem 6.3 that $\mathrm{Hol}_c(A)$ is the smallest subring of A which is convex in A with respect to the saturation $(A^+)^\wedge$ (cf.§5, Def.2), i.e.

$$\mathrm{Hol}_c(A) = \{f \in A \mid \exists\, n \in \mathbb{N}: \quad n \pm f \in (A^+)^\wedge\}.$$

$\mathrm{Hol}_c(A)$ is an absolutely convex subring of A, in particular an ℓ-subring of A, and thus an f-ring.

Theorem 10.17. Assume that $\mathrm{Hol}(A)$ is Prüfer in A. {N.B. This is a mild condition, cf. Theorems 2.6, 2.6$'$.} Let B be a subring of $Q(A)$. The following are equivalent.

(1) B is an ℓ-subring of $Q(A)$ and $P_c(B) = P_c(A)$.

(2) $\mathrm{Hol}_c(A) \subset B \subset P_c(A)$.

Proof. a) Let $R := P_c(A)$ and $H := \mathrm{Hol}_c(A)$. Since $\mathrm{Hol}(A) \subset H \subset A$ and $\mathrm{Hol}(A)$ is assumed to be Prüfer in A, the ring H is Prüfer in A. It is also convex in A. We conclude that H is Prüfer and convex in R.
b) It follows by Theorem 6.7 that H is $(R^+)^\wedge$-convex in R. Thus $\mathrm{Hol}_c(R) \subset H$, and we have inclusions $\mathrm{Hol}_c(R) \subset H \subset A \subset R$. It follows that $\mathrm{Hol}_c(R)$ is Prüfer and convex in A, hence is $(A^+)^\wedge$-convex in A. This implies that $H \subset \mathrm{Hol}_c(R)$, and we conclude that $\mathrm{Hol}_c(R) = H$.
c) Since H is Prüfer and convex in R, we have $R \subset P_c(H)$, hence the inclusions $H \subset A \subset R \subset P_c(H)$. It follows by Remark 16.i that A is convex in $P_c(H)$. The ring A is also Prüfer in $P_c(H)$. This implies $P_c(H) \subset R$, and we conclude that $P_c(H) = R$.
d) If now B is any overring of H in R then we learn by Remark 16.i that B is absolutely convex in R. Thus B is an ℓ-subring of R, hence an ℓ-subring of $Q(A)$. Further we conclude from $H = \mathrm{Hol}_c(R)$ and $R = P_c(H)$ by arguments as in b) and c) that $\mathrm{Hol}_c(B) = H$ and $P_c(B) = R$.
e) Finally, if B is an ℓ-subring of $Q(A)$ with $P_c(B) = R$, then B is a subring of R which is Prüfer and convex in R, hence is $(R^+)^\wedge$-convex in R. It follows that $H \subset B \subset R$. $\square$

§11 The Prüfer hull of $C(X)$

Let X be any topological space, Hausdorff or not, and let $R := C(X)$, the ring of $\mathbb{R}$-valued continuous functions on X. We equip R with the partial ordering $R^+ := \{f \in R \mid f(x) \geq 0 \text{ for every } x \in X\}$. Obviously this makes R an f-ring. We are interested in finding the Prüfer subrings of R and the overrings of R in the complete ring of quotients $Q(R)$, in which R is Prüfer.

In this business we may assume without loss of generality that X is a *Tychonov space*, i.e. a completely regular Hausdorff space, since there exists a natural identifying continuous map $X \twoheadrightarrow X'$ onto such a space X', inducing an isomorphism of f-rings $C(X') \xrightarrow{\sim} C(X)$, cf. [GJ, §3]. But now we still refrain from the assumption that X is Tychonov. This property will become important only later in the section.

Observe that $R^+ = \{f^2 \mid f \in R\}$. Thus R^+ coincides with the smallest preordering T_0 on R. Clearly R^+ is also saturated, $R^+ = (R^+)^\wedge$. Finally $1 + R^+ \subset R^*$, i.e. R has bounded inversion. These three facts make life easier than for f-rings in general.

Since $R^+ = T_0 = \hat{T}_0$, we infer from the definitions that $\mathrm{Hol}(R) = \mathrm{Hol}_c(R)$, further from Theorem 6.3.c that $\mathrm{Hol}(R)$ coincides with the ring $C_b(X)$ of bounded continuous functions on X,

$$\mathrm{Hol}(R) = C_b(X) := \{f \in R \mid \exists\, n \in \mathbb{N}\colon\ |f| \leq n\}.$$

We had proved this by other means before (Ex.4.13).

It is clear already from Theorem 2.6 (or 2.6′) that $\mathrm{Hol}(R)$ is Prüfer in R, and it is plain that $\mathrm{Hol}(R)$ has bounded inversion.

Let $\varphi\colon S \to X$ be a continuous map from some topological space S to X. It induces a ring homomorphism $\rho := C(\varphi)$ from $C(X)$ to $C(S)$, mapping a function $f \in C(X)$ to $f \circ \varphi$. We denote the subring $\rho(C(X))$ of $C(S)$ by $C(X)|_\varphi$ and the subring $\rho(C_b(X))$ of $C_b(S)$ by $C_b(X)|_\varphi$. Since for $f, g \in C(X)$ we have $\rho(f \vee g) = \rho(f) \vee \rho(g)$ and $\rho(f \wedge g) = \rho(f) \wedge \rho(g)$, both $C(X)|_\varphi$ and $C_b(X)|_\varphi$ are ℓ-subrings of the f-ring $C(S)$.

The f-ring $A := C(X)|_\varphi$ inherits many good properties from $R = C(X)$. If $h \in A^+$, we conclude from $h = \rho(f)$ with $f \in R$, that $h = \rho(|f|) = \rho(|f|^{1/2})^2$. Thus A^+ consists of the squares of elements of A. We conclude, as above for R, that

$$\mathrm{Hol}(A) = \mathrm{Hol}_c(A) = \{h \in A \mid \exists\, n \in \mathbb{N}\colon\ |h| \leq n\}.$$

It follows that $\mathrm{Hol}(A) = C_b(X)|_\varphi$. Indeed, if $h = \rho(f)$ and $|h| \leq n$ (in A), then $h = \rho((f \wedge n) \vee (-n))$.

Since $C_b(X)$ is Prüfer in $C(X)$ and ρ maps $R = C(X)$ onto $A = C(X)|_\varphi$ and $C_b(X)$ onto $C_b(X)|_\varphi$, it follows by general principles (Prop.I.5.7) that $C_b(X)|_\varphi$ is Prüfer in $C(X)|_\varphi = A$.

Notice also that for $f \in R$ the element $1+\rho(f)^2 = \rho(1+f^2)$ is a unit of A, since $1+f^2$ is a unit of R. Thus A has bounded inversion. Theorem 2.6 (or 2.6′) tells us that $\mathrm{Hol}(A)$ is Prüfer in A. Clearly $\mathrm{Hol}(A)$ has bounded inversion. In short, A shares all the agreeable properties of R, stated above, although perhaps A is not isomorphic to a ring of continuous functions $C(Y)$.

Theorem 11.1. Let $\varphi: S \to X$ be a continuous map. The following are equivalent.
(1) $C(X)|_\varphi$ is Prüfer in $C(S)$.
(2) $C(X)|_\varphi$ is convex in $C(S)$.
(3) $C_b(X)|_\varphi = C_b(S)$.

Proof. This is a special case of Theorem 7.6, since both $A := C(X)|_\varphi$ and $B := C(S)$ have bounded inversion and $C_A = C_b(X)|_\varphi$, $C_B = C_b(S)$ in the notation used there. □

Assume now that S is a subspace of the topological space X and φ is the inclusion map $S \hookrightarrow X$. Then we write $C(X)|_S$ and $C_b(X)|_S$ for $C(X)|_\varphi$ and $C_b(X)|_\varphi$ respectively.

Definition 1 [GJ].*) S is called *C_b-embedded* (resp. *C-embedded*) *in* X if for every $h \in C_b(S)$ (resp. $h \in C(S)$) there exists some $f \in C(X)$ with $f|_S = h$.

Notice that, if h is a bounded continuous function on S which can be extended to a continuous function on X, then h can be extended to a *bounded* continuous function on X, (as has been already observed above). Thus S is C_b-embedded in X iff $C_b(X)|_S = C_b(S)$, and, of course, S is C-embedded in X iff $C(X)|_S = C(S)$.

In this terminology Theorem 1 says the following for a subspace S of X:

Corollary 11.2. $C(X)|_S$ is Prüfer in $C(S)$ iff $C(X)|_S$ is convex in $C(S)$ iff S is C_b-embedded in X. □

We now fix an element f of $C(X)$. Associated to f we have the zero set $Z(f) := \{x \in X \mid f(x) = 0\}$ and the cozero set $coz(f) := \{x \in X \mid f(x) \neq 0\}$. We are looking for relations between the ring $C(coz f)$ and the localisation $C(X)_f = f^{-\infty}C(X)$ of $C(X)$ with respect to f.

The restriction homomorphism $\rho: C(X) \to C(coz f)$ maps f to a unit of $C(coz f)$, hence induces a ring homomorphism

$$\rho_f\colon\ C(X)_f \longrightarrow C(coz f).$$

*) Gillman and Jerison write C^* instead of C_b, as is done in most of the literature on $C(X)$. Our deviation from this labelling has been motivated in 1.3.

We claim that ρ_f is injective. Indeed, let an element $\frac{g}{f^n} \in C(X)_f$ be given ($g \in C(X),\ n \in \mathbb{N}_0$), and assume that $\rho_f\left(\frac{g}{f^n}\right) = 0$. Then $\rho_f\left(\frac{g}{1}\right) = \rho(g) = g|_{cozf} = 0$. This implies $gf = 0$ and then $\frac{g}{f^n} = \frac{gf}{f^{n+1}} = 0$. Henceforth we regard $C(X)_f$ as a subring of $C(cozf)$ via ρ_f.

Lemma 11.3. $C(X)_f$ contains the subring $C_b(cozf)$ of $C(cozf)$.

Proof. Let $g \in C_b(cozf)$ be given. The function $h\colon X \to \mathbb{R}$ defined by $h(x) := f(x)g(x)$ for $x \in cozf$, $h(x) = 0$ for $x \in Z(f)$, is continuous, since g is bounded. We have $g = \rho_f\left(\frac{h}{f}\right)$. □

Theorem 11.4. For any $f \in C(X)$ the ring $C(X)_f$ is Bezout and absolutely convex in $C(cozf)$, and $C(X)_f$ has bounded inversion.

Proof. By Lemma 3 we have the inclusions $C_b(cozf) \subset C(X)_f \subset C(cozf)$. We know that $C_b(cozf)$ is Prüfer and convex in $C(cozf)$. Also both rings have bounded inversion. It follows that the extension $C_b(X) \subset C(X)_f$ is Prüfer, then by Theorem 9.15, that $C(X)_f$ has bounded inversion. Also the extension $C(X)_f \subset C(X)$ is Prüfer. We conclude by Theorem 9.15, that $C(X)_f$ is Bezout and absolutely convex in $C(X)$. □

We recall some facts about Bezout extensions from II, §10.

Definition 2 (cf.II §10, Def.6). If A is any ring, an element f of A is called a *Bezout element* of A if f is a non-zero-divisor of A and the extension $A \subset A_f$ is Bezout. The set of all Bezout elements of A is denoted by $\beta(A)$.

As has been observed in II §10, $\beta(A)$ is a saturated multiplicative subset of A. It is also clear from II §10, that for any multiplicative subset S of $\beta(A)$ the extension $A \subset S^{-1}A$ is Bezout (cf.Prop.II.10.13).Conversely any Bezout extension R of A has the shape $R = S^{-1}A$ with $S = A \cap R^*$ (cf.Prop.II.10.16).*) Thus the Bezout extensions of A in $Q(A)$ correspond uniquely with the saturated multiplicative subsets of $\beta(A)$. In particular, $\beta(A)$ itself gives us the Bezout hull $\mathrm{Bez}(A) = \beta(A)^{-1}A$ of A.

Theorem 11.5. i) Every Prüfer extension of $C(X)$ is Bezout.
ii) The Bezout elements of $C(X)$ are the non-zero-divisors f of $C(X)$ with the property that $coz(f)$ is C_b-embedded in X.

Proof. i): We know by Theorem 10.12 that every Prüfer extension $C(X) \subset R$ is an f-extension in a natural way. Since $C(X)$ has bounded inversion we read off from Theorem 9.15 that R is Bezout over $C(X)$.

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*) Prop.II.10.16 contains a typographical error. Read "If $A \subset R$ is a Bezout extension" instead of "If A is a Bezout extension".

ii): Let f be a non-zero-divisor of $C(X)$. Then $C(X)$ embeds into $C(X)_f$. Thus we have ring extensions $C(X) \subset C(X)_f \subset C(coz f)$. We know by Theorem 4 that $C(X)_f$ is Bezout in $C(coz f)$. Thus $C(X)$ is Bezout in $C(X)_f$, i.e. f is a Bezout element, iff $C(X)$ is Bezout in $C(coz f)$ (Recall II.10.15.iii). Corollary 2 above tells us that this happens iff $coz(f)$ is C_b-embedded in X. □

Notations. We denote the set of Bezout elements $\beta(C(X))$ more briefly by $b(X)$. We further denote the set of all open subsets $coz(f)$ of X with f running through $b(X)$ by $\mathfrak{B}(X)$.

Notice that $\mathfrak{B}(X)$ is closed under finite intersections, since $coz(f_1) \cap coz(f_2) = coz(f_1 f_2)$. We have a direct system of ring extensions $(C(U) \mid U \in \mathfrak{B}(X))$ of $C(X)$. Here the index set $\mathfrak{B}(X)$ is ordered by reverse inclusion ($U \leq V$ iff $V \subset U$), and the transition maps $C(U) \to C(V)$ are the restriction homomorphisms $f \mapsto f|_V \quad (U \supset V)$. $\mathfrak{B}(X)$ has a first element $U = X = coz(1)$.

Theorems 4 and 5 lead to the following description of the Prüfer hull of $C(X)$.

Corollary 11.6. All transition maps in the system $(C(U) \mid U \in \mathfrak{B})$ are injective, and

$$P(C(X)) = \varinjlim_{U \in \mathfrak{B}(X)} C(U).$$

Proof. Each ring $C(U)$ with $U \in \mathfrak{B}(X)$ is Prüfer over $C(X)$, hence embeds into the Prüfer hull $P(C(X))$ of $C(X)$ in a unique way, which (hence) is compatible with the transition maps. It follows that all transition maps are injective. Identifying the rings $C(U)$ with their images in $P(C(X))$ we may now write

$$\varinjlim_{U \in \mathfrak{B}(X)} C(U) = \bigcup_{U \in \mathfrak{B}(X)} C(U) = \bigcup_{f \in b(X)} C(coz f). \tag{1}$$

Denoting this ring by D we have $C(X) \subset D \subset P(C(X))$. It follows that D is Prüfer over $C(X)$. {We could also have invoked I.5.14.} On the other hand, every localization $C(X)_f$, with f running through $b(X)$, can be embedded in $P(C(X))$ in a unique way over $C(X)$. Since $P(C(X))$ coincides with the Bezout hull of $C(X)$, we have

$$\bigcup_{f \in b(X)} C(X)_f \; = \; P(C(X)). \tag{2}$$

We infer from (1), (2) and $C(X)_f \subset C(coz f) \subset D$ for every $f \in b(X)$, that $D = P(C(X))$. □

Starting from now *we assume that X is a Tychonov space.* Now a function $f \in C(X)$ is a non-zero-divisor in $C(X)$ iff $coz(f)$ is dense in X. {Just observe that, if a point $p \in X \setminus \overline{coz(f)}$ is given, there exists a function $g \in C(X)$ with $g|coz(f) = 0$ and $g(p) \neq 0$. Then $fg = 0$.} Thus $\mathfrak{B}(X)$ is the set of all cozero sets U in X which are dense and C_b-embedded in X.

Let $\mathcal{D}(X)$ denote the set of all dense open subsets of X, and let $\mathcal{D}_0(X)$ denote the set of all dense cozero subsets of X. Then

$$\mathfrak{B}(X) \subset \mathcal{D}_0(X) \subset \mathcal{D}(X),$$

and these three families are all closed under finite intersections. As above we have direct systems of f-rings $\{C(U) \mid U \in \mathcal{D}(X)\}$ and $\{C(U) \mid U \in \mathcal{D}_0(X)\}$ with injective transition maps.

We introduce the ring

$$Q(X) := \varinjlim_{U \in \mathcal{D}(X)} C(U),$$

which again is an f-ring in the obvious way. Every $C(U)$, $U \in \mathcal{D}(X)$ injects into $Q(X)$ and will be regarded as a subring of $Q(X)$. We have $C(X) \subset C(U) \subset Q(X)$ for every $U \in \mathcal{D}(X)$ and

$$Q(X) = \bigcup_{U \in \mathcal{D}(X)} C(U).$$

The following has been proved by Fine, Gillman and Lambek a long time ago.

Theorem 11.7 [FGL]. $C(X)$ has the complete ring of quotients $Q(X)$ and the total ring of quotients

$$\operatorname{Quot}(C(X)) = \varinjlim_{U \in \mathcal{D}_0(X)} C(U) = \bigcup_{U \in \mathcal{D}_0(X)} C(U). \qquad \square$$

Henceforth we work in the overring $Q(C(X)) = Q(X)$ of $C(X)$. We think of the elements of $Q(X)$ as continuous functions defined on dense open subsets of X. Two such functions $g_1: U_1 \to \mathbb{R}$, $g_2: U_2 \to \mathbb{R}$ are identified if there exists a dense open set $V \subset U_1 \cap U_2$ with $g_1|V = g_2|V$. Of course, then g_1 and g_2 coincide on $U_1 \cap U_2$. Corollary 6 now reads as follows.

Scholium 11.8. A continuous function $g: U \to \mathbb{R}$ with U open and dense in X is an element of the Prüfer hull $P(C(X))$ iff there exists some $f \in C(X)$ such that $coz(f) \subset U$ and $coz(f)$ is dense and C_b-embedded in X. $\square$

Remark 11.9. Along the way we have proved that, if U_1, U_2 are dense cozero sets in $C(X)$, which both are C_b-embedded in X, then $U_1 \cap U_2$ is again C_b-embedded in X. In fact more generally the following holds: If U is an open subset of X, which is C_b-embedded in X, and T is a subspace of X, such that $U \cap T$ is dense in T, then $U \cap T$ is C_b-embedded in T, cf.[GJ, 9N]. $\square$

Already from the coincidence $P(C(X)) = \mathrm{Bez}C(X)$ (Theorem 5), we know that $P(C(X))$ is contained in $\mathrm{Quot}C(X)$. Thus we have inclusions

$$C(X) \subset P(C(X)) \subset \mathrm{Quot}C(X) \subset Q(X) = Q(C(X)).$$

We now ask for cases where $P(C(X))$ is equal to one of the other three rings. Part a) of the following theorem is due to Martinez [Mart], while Part b) is due to Dashiell, Hager and Henriksen [DHH], cf. the comments below.

THEOREM 11.10. i) $C(X)$ is Prüfer in its complete ring of quotients $Q(X)$ iff every dense open subset of X is C_b-embedded in X.
ii) $C(X)$ is Prüfer in $\mathrm{Quot}C(X)$ iff every dense cozero subset of X is C_b-embedded in X.

PROOF. a) If $\mathfrak{B}(X) = \mathcal{D}(X)$, resp. $\mathfrak{B}(X) = \mathcal{D}_0(X)$, we know by Corollary 6 and Theorem 7 that $P(C(X)) = Q(X)$, resp. $P(C(X)) \supset \mathrm{Quot}C(X)$.
b) Assume that $C(X)$ is Prüfer in $Q(X)$. Let U be a dense open subset of X. Since $C(X) \subset C(U) \subset Q(X)$, we conclude that $C(X)$ is Prüfer in $C(U)$. Now Theorem 1, more precisely Corollary 2, tells us that U is C_b-embedded in X.
c) Assume that $C(X)$ is Prüfer in $\mathrm{Quot}C(X)$. Let f be a non-zero-divisor of $C(X)$. Since $C(X) \subset C(X)_f \subset \mathrm{Quot}C(X)$, we conclude that $C(X)$ is Prüfer, hence Bezout in $C(X)_f$, i.e. f is a Bezout element of $C(X)$. Theorem 5 tells us that $coz(f)$ is C_b-embedded in $C(X)$. $\square$

COMMENTS 11.11.
a) X is called *extremally disconnected* [GJ, 1H] if every open subset of X has an open closure. It is well known that this is equivalent to the property that every open subset of X is C_b-embedded in X ([GJ, 1H.6], [PW, 6.2]). Now, if all *dense* open subsets of X are C_b-embedded in X, then this is true for *all* open subsets of X. Indeed, if U is open in X and $f \in C_b(U)$, then f can be extended by zero to a bounded continuous function on the dense open set $U \cup (X \setminus \overline{U})$ of X, and this function extends to a bounded continuous function on X. Thus Theorem 11.10.a can be coined as follows: $C(X)$ *is Prüfer in* $Q(C(X))$ *iff* X *is extremally disconnected*. {[Mart, Th.2.7]; Martinez there calls a ring A which is Prüfer in $Q(A)$,*) an "I-ring" following the terminology of Eggert [Eg]}.

Extremally disconnected spaces are rare but not out of the world. For example, the Stone-Čech compactification βD of any discrete space D is extremally disconnected [PW, 6.2]. There also exist extremally disconnected spaces without isolated points, cf. [PW, 6.3].

*) more precisely, a ring A such that every overring in $Q(A)$ is integrally closed in $Q(A)$, but this means the same (Th.I.5.2).

b) A Tychonov space X is an *F-space*, if every cozero-set of X is C_b-embedded in X ([GJ, 14.25]), while X is called a *quasi-F-space*, if every *dense* cozero-set of X is C_b-embedded in X [DHH], which is a truly weaker condition. Thus Theorem 10.b can be coined as follows: *$C(X)$ is Prüfer in Quot $C(X)$ iff X is a quasi-F-space* {[DHH; A ring A which is Prüfer in Quot$A$ is traditionally called a "Prüfer ring with zero divisors" [Huc]}.

Using Theorem 9.15 we may rephrase this result as follows: *$C(X)$ is convex in Quot $C(X)$ iff X is a quasi-F-space.* In this way Theorem 10.b has been stated and proved by Schwartz [Sch$_3$, Th.6.2].

F-spaces, hence quasi-F-spaces, are not so rare. Prominent examples are the spaces $\beta Y \setminus Y$ with Y locally compact and σ-compact [GJ, 14.27]. □

Concerning the case $C(X) = P(C(X))$, i.e. Prüfer closedness of $C(X)$, we have only a partial result.

Theorem 11.12. If X is a metric space then $C(X)$ is Prüfer closed.

Proof. Suppose $C(X)$ is not Prüfer closed. Then $C(X)$ has a Bezout element f which is not a unit (cf.Theorem 5.a), and this means that the set $U{:=}\, coz f$ is C_b-embedded and dense in X, but $U \neq X$ (cf.Theorem 5.b). We choose a point $p \in X \setminus U$ and then a sequence $\{x_n \mid n \in \mathbb{N}\}$ in U, consisting of pairwise different points and converging to p. The sets $Z_0{:=}\,\{x_{2n} \mid n \in \mathbb{N}\}$ and $Z_1{:=}\,\{x_{2n-1} \mid n \in \mathbb{N}\}$ are closed in U and disjoint. Let f_0 and f_1 denote the distance functions $\mathrm{dist}(-, Z_0)$ and $\mathrm{dist}(-, Z_1)$ on the metric space U. The function

$$g{:=}\, \frac{|f_0|}{|f_0| + |f_1|}$$

on U is well defined, bounded and continuous. We have $g|_{Z_0} = 0$ and $g|_{Z_1} = 1$. Thus g cannot be extended continuously to $U \cup \{p\}$. This is a contradiction and proves that $C(X) = P(C(X))$. □

We mention that Schwartz has developed general criteria for $C(X)$ to be Prüfer closed, cf.[Sch$_3$, Th.5.3]. He also gave a description of the Prüfer hull $P(C(X))$ in general, different from our Theorem 5, by use of the real spectrum of $C(X)$, cf.[Sch$_3$, Th.5.5].

§12 Valuations on f-rings

It is somewhat remarkable that in §9 and §10 we nowhere used valuations (explicitly) for gaining results about Prüfer subrings or Prüfer extensions of a given f-ring R. But, of course, in order to complete the picture, a thorough study of valuations on R is appropriate. We will experience a relation between the convex valuations on R and the prime cones $P \supset R^+$ even closer than in the general theory in §3 and §5.

In the following R is an f-ring and $v\colon R \to \Gamma \cup \infty$ is a valuation on R. For any $\gamma \in \Gamma \cup \infty$ we introduce the A_v-module

$$I_{\gamma,v} := \{x \in R \mid v(x) \geq \gamma\}. \quad ^{*)}$$

Proposition 12.1.
a) For every $x \in R$

$$v(x) = v(|x|) = \min(v(x^+), v(x^-)) ,$$

and either $v(x^+) = \infty$ or $v(x^-) = \infty$.
b) For every $\gamma \in \Gamma \cup \infty$ the set $I_{\gamma,v}$ is a sublattice of R.

Proof. a): It follows from $x^+x^- = 0$ that either $v(x^+) = \infty$ or $v(x^-) = \infty$, and then from $x = x^+ - x^-$, $|x| = x^+ + x^-$, that $v(x) = v(|x|) = \min(v(x^+), v(x^-))$.
b) It is now clear that, for every $x \in I_{\gamma,v}$, also $x^+ \in I_{\gamma,v}$ (and $x^- \in I_{\gamma,v}$). If $x, y \in I_{\gamma,v}$ are given, we conclude that

$$x \vee y = y + [(x-y) \vee 0] = y + (x-y)^+ \in I_{\gamma,v}.$$

Also $x \wedge y = -[(-x) \vee (-y)] \in I_{\gamma,v}$. Thus $I_{\gamma,v}$ is a sublattice of R. □

In the special case that v is trivial Proposition 1 reads as follows.

Corollary 12.2. Every prime ideal of R is a sublattice of R. □

Here is another consequence of Proposition 1.

Corollary 12.3. If A is a Prüfer subring of R, every R-regular A-submodule of R is a sublattice of R.

Proof. Let I be such a submodule of R. We may assume that I is finitely generated. I is the intersection of the R-regular $A_{[\mathfrak{p}]}$-submodules $I_{[\mathfrak{p}]}$ of R with $\mathfrak{p}$ running through the set $\Omega(R/A)$ of maximal R-regular ideals $\mathfrak{p}$ of A

$^{*)}$ as in Chapter III, but now allowing $\gamma \notin v(R)$ and $\gamma = \infty$. Of course, $I_{\infty,v} = \operatorname{supp} v$.

(Prop.III.1.10). To each $\mathfrak{p}$ there corresponds a non-trivial PM-valuation $v_{\mathfrak{p}}$ of R over A with $A_{v_{\mathfrak{p}}} = A_{[\mathfrak{p}]}$, and $I_{[\mathfrak{p}]}$ is a $v_{\mathfrak{p}}$-convex $A_{v_{\mathfrak{p}}}$-submodule of R (cf.Th.III.2.2). It follows from Proposition 1 that $I_{[\mathfrak{p}]}$ is a sublattice of R. Thus I is a sublattice of R. $\square$

We return to our fixed valuation $v: R \to \Gamma \cup \infty$ on R.

Proposition 12.4. For any $x, y \in R$ the set of values $\{v(x \vee y), v(x \wedge y)\}$ coincides with $\{v(x), v(y)\}$.

Proof. Let $x, y \in R$ be fixed. Without loss of generality we assume that $\gamma := v(x) \leq v(y)$. Since $I_{\gamma,v}$ is a sublattice of R, we have $\gamma \leq v(x \vee y)$ and $\gamma \leq v(x \wedge y)$.

If $\gamma = \infty$ we have $v(y) = v(x \vee y) = v(x \wedge y) = \infty$, and we are done. We now assume that $\gamma \in \Gamma$. We use the identities, stated in §9,
(F7) $\quad x + y = (x \vee y) + (x \wedge y)$,
(F8) $\quad xy = (x \vee y)(x \wedge y)$.
By F8 we have

$$(*) \qquad \gamma + v(y) \;=\; v(x \vee y) + v(x \wedge y).$$

Also, as said above, $v(x \vee y) \geq \gamma$, $v(x \wedge y) \geq \gamma$. If $v(y) = \gamma$ this forces $v(x \wedge y) = v(x \vee y) = \gamma$, and we are done in this case.

There remains the case that $v(y) > \gamma$. Now $v(x + y) = \gamma$. By (F7) we have $\gamma \geq \min(v(x \vee y), v(x \wedge y))$. Since $v(x \vee y) \geq \gamma$ and $v(x \wedge y) \geq \gamma$, this forces $\gamma = \min(v(x \vee y), v(x \wedge y))$. Now $(*)$ tells us – also in the case $v(y) = \infty$ – that $v(y) = \max(v(x \vee y), v(x \wedge y))$. $\square$

As a consequence of the proposition we have

Corollary 12.5. For any subset M of Γ the set $\{x \in R \mid v(x) \in M\}$ is either empty or a sublattice of R. In particular, A_v is an ℓ-subring of R, hence an f-ring, and both $\mathfrak{p}_v$ and $A_v \setminus \mathfrak{p}_v$ are sublattices of A_v. $\square$

Proposition 12.6. The following are equivalent.
(1) v is convex.
(2) $v(x \vee y) = \min(v(x), v(y))$ for all $x, y \in R^+$.
(3) $v(x \wedge y) = \max(v(x), v(y))$ for all $x, y \in R^+$.

Proof. The equivalence (2) $\Leftrightarrow$ (3) is clear from Proposition 4.
(1) $\Rightarrow$ (2): Since v is convex it follows from $0 \leq x \leq x \vee y$ and $0 \leq y \leq x \vee y$ that $v(x) \geq v(x \vee y)$, $v(y) \geq v(x \vee y)$, hence $\min(v(x), v(y)) \geq v(x \vee y)$. Again invoking Proposition 4 we obtain equality here.

(2) $\Rightarrow$ (1): If $x, y \in R$ and $0 \leq y \leq x$ we have $x = x \vee y$, hence $v(x) = \min(v(x), v(y))$ by (2), i.e. $v(x) \leq v(y)$. Thus v is convex. $\square$

Remark. In the vein of Corollary 2 we obtain from Proposition 6 that, for A a convex Prüfer subring of R, every R-regular A-submodule of R is absolutely convex in R. But this we already proved in §9 in another way, cf.Theorem 9.10. $\square$

Theorem 12.7. Let $\mathfrak{q}$ be a convex prime ideal of R.
a) Then $P := R^+ + \mathfrak{q}$ is a prime cone of R, and $P = \{x \in R \mid x^- \in \mathfrak{q}\}$.
b) P is the unique prime cone of R containing R^+ and with support $\mathfrak{q}$.

Proof. 1) We know by Lemma 5.9 that $P := R^+ + \mathfrak{q}$ is a preordering of R and $P \cap (-P) = \mathfrak{q}$.
2) We verify that $P = \{x \in R \mid x^- \in \mathfrak{q}\}$. Let $x \in R$ be given. If $x^- \in \mathfrak{q}$, then $x = x^+ - x^- \in R^+ + \mathfrak{q} = P$. Assume now that $x \in P$. Write $x = y + z$ with $y \geq 0$ and $z \in \mathfrak{q}$. By Corollary 2 above we know that $\mathfrak{q}$ is a sublattice of R. Thus $z^- \in \mathfrak{q}$. It follows from $x = (y + z^+) - z^-$ that $0 \leq x^- \leq z^-$. Since $\mathfrak{q}$ is convex we conclude that $x^- \in \mathfrak{q}$.
3) Let $x \in R$ be given with $x \notin P$. Then $x^- \notin \mathfrak{q}$. But $x = x^+ x^- = 0 \in \mathfrak{q}$. Thus $(-x)^- = x^+ \in \mathfrak{q}$, hence $-x \in P$. This proves that $P \cup (-P) = R$. We now know that P is a prime cone of R with support $\mathfrak{q}$.
4) If P' is any prime cone of R with $P' \supset R^+$ and $\operatorname{supp} P' = \mathfrak{q}$, then $P' \supset R^+ + \mathfrak{q} = P$. Since P' and P have the same support, it follows that $P' = P$ (cf.Th.4.6). $\square$

Comment. We know for long that, if T is a proper preordering of any ring R and $\mathfrak{q}$ a T-convex prime ideal of R, there exists a prime cone $P \supset T$ with support $\mathfrak{q}$ (cf.Th.5.6 and Th.4.6). Theorem 7 states the remarkable fact that P is *unique* in the present case, where R is an f-ring and $T = R^+$. This means that we have a bijection $\mathfrak{q} \mapsto T + \mathfrak{q}$ from the set $\operatorname{Spec}_T(R)$ of all T-convex prime ideals to the set $\operatorname{Sper}_T(R)$ of prime cones $P \supset T$ of R, the inverse map being the restriction $\operatorname{Sper}_T(R) \to \operatorname{Spec}_T(R)$ of the support map $\operatorname{supp} : \operatorname{Sper}(R) \to \operatorname{Spec}(R)$.

One should view $\operatorname{Sper}_T(R)$ and $\operatorname{Spec}_T(R)$ as the real spectrum and the Zariski spectrum of the ordered ring (R, T). In the case that R is an f-ring and $T = R^+$ we leave it to the reader to verify, that our bijection $\operatorname{Sper}_T(R) \to \operatorname{Spec}_T(R)$ is a homeomorphism with respect to the subspace topologies in $\operatorname{Sper}(R)$ and $\operatorname{Spec}(R)$. $\square$

Theorem 12.8. Let U be a preordering of R containing R^+ and v a U-convex valuation on R. Then there exists a *unique* prime cone P on R such that $U \subset P$, v is P-convex, $\operatorname{supp} P = \operatorname{supp} v$. We have $P = R^+ + \operatorname{supp} v = U + \operatorname{supp} v = \{x \in R \mid v(x^-) = \infty\}$.

Proof. 1) Let $\mathfrak{q} := \operatorname{supp} v$. This prime ideal is U-convex, hence R^+-convex. We *define* $P := R^+ + \mathfrak{q}$. We know by Theorem 7 that P is a prime cone of R

with support $\mathfrak{q}$, and that P is the only candidate for a prime cone with the properties listed in Theorem 8.
2) We prove that v is P-convex. Given $x, y \in P$ it suffices to verify that $v(x+y) = \min(v(x), v(y))$, (cf. Remark 5.10.i). We have $x \equiv x^+ \mod \mathfrak{q}$, $y \equiv y^+ \mod \mathfrak{q}$, $x + y \equiv x^+ + y^+ \mod \mathfrak{q}$, hence $v(x) = v(x^+)$, $v(y) = v(y^+)$, $v(x+y) = v(x^+ + y^+)$. Since v is R^+-convex, we have $v(x^+ + y^+) = \min(v(x^+), v(y^+))$, and we conclude that indeed $v(x+y) = \min(v(x), v(y))$.
3) By Theorem 5.16 there exists a prime cone $P' \supset U$ such that v is P'-convex and $\operatorname{supp} P' = \mathfrak{q}$. The ideal $\mathfrak{q}$ then is P'-convex. By Theorem 7 this forces $P' = R^+ + \mathfrak{q} = P$. Since $R^+ \subset U \subset P'$, it follows that $P = U + \mathfrak{q}$. Since $P = R^+ + \mathfrak{q}$, we know by Theorem 7 that $P = \{x \in R \mid x^- \in \mathfrak{q}\}$. $\square$

Definition 1. If v is a convex (i.e. R^+-convex) valuation on R, we denote the unique prime cone $P \supset R^+$ such that v is P-convex and $\operatorname{supp} v = \operatorname{supp} P$ by P_v, and we call P_v the *convexity prime cone of* v.

Theorem 8 tells us that P_v is the unique *maximal preordering* U of R such that $R^+ \subset U$ and v is U-convex.

Definition 2. For v is a convex valuation on R let $v^\#$ denote the valuation v_P given by the prime cone $P := P_v$.*)

Remarks 12.9. The valuation $v^\#$ is P-convex, hence convex. We have $A_{v^\#} = A_P$ (cf.§3), further $\operatorname{supp} v^\# = \operatorname{supp} P = \operatorname{supp} v$, and $P_{v^\#} = R^+ + \operatorname{supp}(v^\#) = P$. From $v^\# = v_P$ it follows that $v^\# \leq v$ (cf.Th.5.15). Clearly $v^\# = (v^\#)^\#$. $\square$

Lemma 12.10. Assume that v and w are convex valuations on R. The following are equivalent.
(1) $P_v = P_w$,
(1′) $\operatorname{supp} v = \operatorname{supp} w$,
(2) $v^\# \leq w$,
(3) $v^\# = w^\#$.

Proof. (1) $\Leftrightarrow$ (1′): Clear, since for any convex valuation u on R we have $P_u = R^+ + \operatorname{supp} u$ and $\operatorname{supp} u = \operatorname{supp} P_u$.
(1) $\Rightarrow$ (3): Clear by Definition 2.
(3) $\Rightarrow$ (2): Clear since $w^\# \leq w$.
(2) $\Rightarrow$ (1′): We have $\operatorname{supp} v^\# = \operatorname{supp} v$. From $v^\# \leq w$ we conclude that $\operatorname{supp} v^\# = \operatorname{supp} w$. $\square$

The lemma leads us to an important result about convex valuations on R.

Definition 3. Given a prime cone P of R with $P \supset R^+$ let $\mathfrak{M}_P$ denote the set of equivalence classes of convex valuations v on R with $P_v = P$. We endow $\mathfrak{M}_P$ with the partial ordering given by the coarsening relation $v \leq w$.

*) v_P has been defined in §3.

As always, we do not distinguish seriously between a valuation and its equivalence class, thus speaking of the convex valuations v with $P_v = P$ as elements of $\mathfrak{M}_P$.

Theorem 12.11. Let P be a prime cone of R with $R^+ \subset P$, hence $P = R^+ + \mathfrak{q}$ with $\mathfrak{q} := \operatorname{supp} P$.
i) If v and w are convex valuations on R with $v \le w$, and if $v \in \mathfrak{M}_P$ or if $w \in \mathfrak{M}_P$, both v and w are elements of $\mathfrak{M}_P$.
ii) $\mathfrak{M}_P$ is the set of all convex valutions v on R with $v^\# = v_P$, and also the set of all valuations v of R with $v_P \le v$.
iii) $\mathfrak{M}_P$ is totally ordered by the coarsening relation and has a minimal and a maximal element. The minimal element is the valuation v_P. The maximal element is the trivial valuation with support $\mathfrak{q}$.

Proof. i): If $v \le w$ then $\operatorname{supp} v = \operatorname{supp} w$, hence $P_v = P_w$ by Lemma 10.

ii): Let $u := v_P$. For every $v \in \mathfrak{M}_P$ we have $v^\# = u$ by definition of $v^\#$. Further $\operatorname{supp} u = \operatorname{supp} P$ (cf.§3, Def.3), hence $P_u = R^+ + \operatorname{supp} P = P$. Thus $u \in \mathfrak{M}_P$. If now v is a convex valuation with $v^\# = u$, then $u \le v$ (cf.Remarks 9), hence by i), or again Remarks 9, $v \in \mathfrak{M}_P$.

Finally, if v is any valuation of R with $u \le v$, then v is convex since u is convex (cf.Remark 5.10.v), and thus $v \in \mathfrak{M}_P$ by i).

iii): If u' is any valuation on any ring R' the coarsenings of u' correspond uniquely with the convex subgroups of the valuation group of u' (cf.I §1). Thus the coarsenings of u' form a totally ordered set. Clearly u' is the minimal element of this set, and the trivial valution with the same support as u' is the maximal one. □

Later we will also need an "relative" analogue of the valuation $v^\#$ which takes into account a given subring Λ of R. In order to define this analogue we introduce the set

$$\mathfrak{M}_{P,\Lambda} := \{v \in \mathfrak{M}_P \mid \Lambda \subset A_v\}.$$

Here – as before – P is a prime cone of R containing R^+. The set $\mathfrak{M}_{P,\Lambda}$ contains the maximal element of $\mathfrak{M}_P$, hence is certainly not empty.

Proposition 12.12. i) The valuation $w := v_{P,\Lambda}$ introduced in §3, Def.5 is the minimal element of $\mathfrak{M}_{P,\Lambda}$.
ii) $A_w = C(P, R/\Lambda) = A(P, R/\Lambda) = \mathrm{Hol}_P(R/\Lambda)$.
iii) If Λ is an ℓ-subring of R then

$$A_w \;=\; \{x \in R \mid \exists\, \lambda \in \Lambda^+ \colon\ \lambda \pm x \in P\}.$$

Proof. Claims i) and ii) are covered by Theorems 3.10 and 6.2. We have

$$A(P, R/\Lambda) \;=\; \{x \in R \mid \exists\, \lambda \in \Lambda \cap P \colon\ \lambda \pm x \in P\}.$$

If Λ is an ℓ-subring of R, then $\lambda \pm x \in P$ implies $\lambda^+ \pm x \in P$, since $\lambda^+ = \lambda + \lambda^- \in \Lambda^+$ and $\lambda^- \in R^+ \subset P$. Thus

$$A(P, R/\Lambda) \; \subset \; \{x \in R \mid \exists\, \lambda \in \Lambda^+ \colon \lambda \pm x \in P\}.$$

The reverse inclusion is trivial. □

Definition 3. Let v be a convex valuation on R and $P := P_v$. Let Λ be a subring of R. We define $v_\Lambda^\# := v_{P,\Lambda}$. □

The following is evident from Theorem 11 and Proposition 12.

Scholium 12.13. Let v and w be convex valuations on R. Then $v_\Lambda^\# = w_\Lambda^\#$ iff $P_v = P_w$ iff either $v \le w$ or $w \le v$. If $\Lambda \subset A_v$ then $v_\Lambda^\# \le v$. If $\Lambda \not\subset A_v$ then $v \le v_\Lambda^\#$ but $v \not\sim v_\Lambda^\#$. □

§13 Convexity preorderings and holomorphy bases

The results on convex valuations in §12 will give us new insight about the interplay between convex Prüfer subrings of an f-ring R and preorderings $T \supset R^+$ of R. We make strong use of the convexity prime cone P_v of a convex valuation v on R (§12, Def.1) and also of the valuations $v^\#$ and $v^\#_\Lambda$ studied in §12.

In the whole section R *is an f-ring* and A *is a convex Prüfer subring of* R.

Theorem 13.1. There exists a unique maximal preordering $U \supset R^+$ of R such that A is U-convex in R. More precisely, $U \supset R^+$, A is U-convex in R, and $U \supset U'$ for every preordering $U' \supset R^+$ of R such that A is U'-convex. We have

$$U = \bigcap_{v \in \omega(R/A)} P_v \quad ,$$

where – as before (§1) – $\omega(R/A)$ denotes the maximal restricted PM-spectrum of R over A (i.e. the set of all maximal non trivial PM-valuations of R over A).

Proof. Recall that A is the intersection of the rings A_v with v running through $\omega(R/A)$. We *define* U as the intersection of prime cones P_v with v running through $\omega(R/A)$. This is a preordering of R containing R^+. Each ring A_v, $v \in \omega(R/A)$, is P_v-convex by definition of P_v, hence is U-convex in R. Thus A is U-convex.

Let now a preordering $U' \supset R^+$ of R be given such that A is U'-convex in R. Theorem 6.7 tells us that, for every $v \in \omega(R/A)$, the ring A_v $(= A_{[\mathfrak{p}]}$ with $\mathfrak{p} = A \cap \mathfrak{p}_v)$ is U'-convex in R, hence the valuation v is U'-convex (cf.Th.5.11). It follows by Theorem 12.8 that $U' \subset P_v$. Since this holds for every $v \in \omega(R/A)$, we conclude that $U' \subset U$. $\square$

Definition 1. We denote this preordering U by T^R_A, or T_A for short if R is kept fixed, and we call T_A the *convexity preordering of* A *in* R.

Remarks 13.2. i) If A is PM in R then $T_A = P_v$ with v “the” PM-valuation of R such that $A = A_v$, as is clear by Theorem 5.11.
ii) In the proof of Theorem 1 we could have worked as well with the whole restricted PM-spectrum $S(R/A)$ instead of $\omega(R/A)$. Thus also

$$T_A = \bigcap_{v \in S(R/A)} P_v \quad .$$

iii) In the case $A = R$ the set $S(R/A)$ is empty. We then should read $T_A = R$. This is the only case where the preordering T_A is improper. $\square$

Given any proper subring A of R we denote the conductor of A in R by $\mathfrak{q}_A$, or more precisely by $\mathfrak{q}_A^R$ if necessary. By definition

$$\mathfrak{q}_A = \{x \in R \mid Rx \subset A\},$$

and $\mathfrak{q}_A$ is the largest ideal of R contained in A.

Recall from Chapter I (Prop.I.2.2) that, if v is a non trivial special valuation on R, then $\mathfrak{q}_{A_v} = \operatorname{supp} v$. In the case that v is PM this leads to pleasant relations between T_A and $\mathfrak{q}_A$ if A is Prüfer and convex in R, (which we continue to assume).

Corollary 13.3. i) $\mathfrak{q}_A = \bigcap\limits_{v \in \omega(R/A)} \operatorname{supp} v = \bigcap\limits_{v \in S(R/A)} \operatorname{supp} v$.
ii) $\mathfrak{q}_A$ is a convex ideal of R and $\mathfrak{q}_A = \sqrt{\mathfrak{q}_A}$.
iii) $\operatorname{supp} T_A = \mathfrak{q}_A$.
iv) $T_A = R^+ + \mathfrak{q}_A = \{x \in R \mid x^- \in \mathfrak{q}_A\}$.

Proof. i): This is an immediate consequence of the facts that A is the intersection of the rings A_v, with v running through $\omega(R/A)$ or $S(R/A)$, and that $\mathfrak{q}_{A_v} = \operatorname{supp} v$.
ii): Now clear, since each ideal $\operatorname{supp} v$ is prime and convex in R.
iii): $\operatorname{supp} T_A = T_A \cap (-T_A) = \bigcap\limits_{v \in \omega(R/A)} P_v \cap \bigcap\limits_{v \in \omega(R/A)} (-P_v) =$

$$\bigcap_{v \in \omega(R/A)} (P_v \cap -P_v) = \bigcap_{v \in \omega(R/A)} \operatorname{supp} v = \mathfrak{q}_A.$$

iv): For each $v \in \omega(R/A)$ we have $P_v = R_+ + \operatorname{supp} v = \{x \in R \mid x^- \in \operatorname{supp} v\}$. Intersecting the P_v we obtain $T_A = R_+ + \mathfrak{q}_A = \{x \in R \mid x^- \in \mathfrak{q}_A\}$. □

Example 13.4. Let X be a topological space, $R := C(X)$ and $A := C_b(X)$. Assume that X is not pseudocompact, i.e. $A \neq R$. We choose on R the partial ordering $R^+ := \{f \in R \mid f(x) \geq 0 \text{ for every } x \in X\}$. Then R is an f-ring and A is an absolutely convex ℓ-subring of R. We know for long that A is Prüfer in R (even Bezout). By the corollary we have $T_A = R^+ + \mathfrak{q}_A$. It is clear that $\mathfrak{q}_A$ contains the ideal $C_c(X)$ of R consisting of all $f \in C(X)$ with compact support. If the space X is both locally compact and σ-compact (e.g. $X = \mathbb{R}^n$ for some n), then it is just an exercise to prove that $\mathfrak{q}_A = C_c(X)$. Thus in this case T_A is the set of all $f \in R$ such that $\{x \in X \mid f(x) < 0\}$ has a compact closure. □

We return to an arbitrary f-ring R and a convex Prüfer subring A of R.

Given an R-overring B of A in R we know that B is a sublattice of R, hence again an f-ring, since B is Prüfer in R (Cor.12.3). We state relations between T_A^R and T_A^B and, in case that B is also convex in R, between T_A^R and T_B^R.

Proposition 13.5. Let B be an overring of A in R.

i) $B \cap T_A^R \subset T_A^B$, $B \cap \mathfrak{q}_A^R \subset \mathfrak{q}_A^B$.
ii) $\mathfrak{q}_A^R \subset \mathfrak{q}_B^R$.
iii) If B is convex in R, then $T_A^R \subset T_B^R$ and $B \cap T_A^R \subset T_A^B \cap T_B^R$.

Proof. i): A is T_A^R-convex in R, hence $(B \cap T_A^R)$-convex in B. This implies $B \cap T_A^R \subset T_A^B$. Taking supports of these preorderings we obtain $B \cap \mathfrak{q}_A^R \subset \mathfrak{q}_A^B$. (By the way this trivially holds for any sequence of ring extensions $A \subset B \subset R$.)
ii): A trivial consequence of the definition of conductors.
iii): Assume now that B is convex in R. We obtain from ii) that

$$T_A^R = R^+ + \mathfrak{q}_A^R \subset R^+ + \mathfrak{q}_B^R = T_B^R.$$

It follows that $B \cap T_A^R \subset B \cap T_B^R \subset T_B^R$. By i) we have $B \cap T_A^R \subset T_A^B$. We conclude that $B \cap T_A^R \subset T_A^B \cap T_B^R$. □

Remark 13.6. If B is an overring of A in R which is convex in R, and U is a preordering of R with $U \supset R^+$, and A is U-convex, then it follows from $T_A^R \subset T_B^R$ that B is U-convex. Acutally we know more: If U is *any* preordering of R such that A is U-convex, then also B is U-convex. This holds by Theorem 8.7, cf. there (i) $\Rightarrow$ (iv). Indeed, since A is absolutely convex in R, A is 2-saturated in R, so the theorem applies. We could have used this fact in the proof of Proposition 4. □

Definition 2. We denote the holomorphy ring $\mathrm{Hol}_{T_A}(R)$ of the preordering T_A in R (cf.§6, Def.1) by H_A, more precisely by H_A^R if necessary. We call H_A the *holomorphy base of* A (*in* R). {Recall that we assume A to be Prüfer and convex in R.}

Since the preordering T_A is clearly saturated, we know by Theorem 6.3.c that H_A is the smallest T_A-convex subring of R,

$$H_A = C(T_A, R) = A(T_A, R).$$

In particular, $H_A \subset A$. By definition, H_A is the intersection of the rings A_v with v running through all T_A-convex valuations of R, hence H_A is a sublattice of R. It follows that H_A is absolutely convex in R.

We will often need the assumption that H_A is Prüfer in R. This certainly holds if the absolute holomorphy ring $\mathrm{Hol}(R)$ is Prüfer in R, since $\mathrm{Hol}(R) \subset H_A$. Thus it holds for example if R has positive definite inversion (Th.2.6) or if for every $x \in R$ there exists some $d \in \mathbb{N}$ with $1 + x^{2d} \in R^*$ (Th.2.6$'$).

Proposition 13.7. Assume that H_A is Prüfer in R.
i) Then T_A is also the convexity preordering of H_A.
ii) If also B is a convex Prüfer subring of R the following are equivalent.
(1) $T_A \subset T_B$,
(2) $\mathfrak{q}_A \subset \mathfrak{q}_B$,

(3) $H_B \supset H_A$,
(4) $B \supset H_A$.

Proof. i): H_A is T_A-convex in R. Thus $T_A \subset T_{H_A}$. Since $H_A \subset A$ we also have $T_{H_A} \subset T_A$ (Prop.4.iii). Thus $T_A = T_{H_A}$.
ii): (1) $\Rightarrow$ (2): Clear, since $\mathfrak{q}_A = \operatorname{supp} T_A$ and $\mathfrak{q}_B = \operatorname{supp} T_B$.
(1) $\Rightarrow$ (3): B is T_A-convex by assumption. Thus $H_B \supset H_A$.
(3) $\Rightarrow$ (4): Trivial, since $B \supset H_B$.
(4) $\Rightarrow$ (1): By Proposition 4 and i) above we have $T_B \supset T_{H_A} = T_A$. $\square$

Remark. In (ii) the implications (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold under the sole assumption that both A and B are convex and Prüfer in R. {(2) $\Rightarrow$ (1) is clear, since $T_A = R^+ + \mathfrak{q}_A$ and $T_B = R^+ + \mathfrak{q}_B$.} But for (4) $\Rightarrow$ (1) we need to know that H_A is Prüfer in R. $\square$

Corollary 13.8. We assume as before that H_A is Prüfer in R. Let C be a subring of A which is convex and Prüfer in A, hence in R. Then $T_C = T_A$ iff $H_A \subset C$. In this case $H_C = H_A$.

Proof. If $T_C = T_A$ then $H_C = H_A$ by definition of H_A and H_C. Hence $H_A \subset H_C$. {For this implication we do not need that H_A is Prüfer in R.}

Assume now that $H_A \subset C$. Proposition 7 tells us that $T_A \subset T_C$. On the other hand $T_C \subset T_A$ since $C \subset A$. Thus $T_A = T_C$. $\square$

In order to understand the amount of convexity carried by subrings of R it is helpful to have also "relative holomorphy bases" at ones disposal, to be defined now. As before we assume that A is a convex Prüfer subring of R.

Definition 3. Let Λ be any subring of A. The *holomorphy base* $H_{A/\Lambda}$ *of* A *over* Λ (*in* R) is the holomorphy ring of R over Λ of the preordering T_A,

$$H_{A/\Lambda} \colon = H^R_{A/\Lambda} \colon = \operatorname{Hol}_{T_A}(R/\Lambda). \qquad \square$$

Remarks 13.9.
i) $\operatorname{Hol}(R) \subset H_A = H_{A/\mathbb{Z}\cdot 1_R} \subset H_{A/\Lambda} \subset A$.
ii) As in the case $\Lambda = \mathbb{Z}\cdot 1_R$ we have $H_{A/\Lambda} = C(T_A, R/\Lambda) = A(T_A, R/\Lambda)$, again by Theorem 6.3.c.
iii) Assume that H_A is Prüfer in R. Then $H_{A/\Lambda} = \Lambda \cdot H_A$, as follows from Remark 7.1.iii.
iv) If $H_{A/\Lambda}$ is Prüfer in R, all statements in Proposition 7 remain true if we replace H_A and H_B there by $H_{A/\Lambda}$, $H_{B/\Lambda}$, of course assuming that Λ is a subring of both A and B. We thus also have an obvious analogue of Corollary 8 for relative holomorphy bases. $\square$

Comment. It is already here that we can see an advantage to deal with relative instead of just "absolute" holomorphy bases. If A and B are overrings of Λ in

R then we have a result as Proposition 7 under the hypothesis that $H_{A/\Lambda}$ is Prüfer in R instead of the stronger hypothesis that H_A is Prüfer in R.

Below we will study relations between the restricted PM-spectra $S(R/A)$ and $S(R/B)$ in the case that $A \subset B$ and $T_A = T_B$. For many arguments it will again suffice to assume that $H_{A/\Lambda}$ $(= H_{B/\Lambda})$ is Prüfer in R. Without invoking relative holomorphy bases we would have to assume that H_A is Prüfer in R. $\square$

Assume – as before – that A is a convex Prüfer subring of R and $\Lambda \subset A$. Let $H := H_{A/\Lambda}$. Striving for a better understanding of holomorphy bases we look for relations between the PM-valuations of R over A and over H.

Proposition 13.10. Assume that v is a non trivial Manis valuation of R over A, i.e. $v \in S(R/A)$.
a) Then $H \subset A_{v^\#_\Lambda}$.
b) Assume in addition that H is Prüfer in A. {N.B. This holds if $\mathrm{Hol}(A)$ is Prüfer in A.} Then $v^\#_\Lambda$ is a maximal PM-valuation over H, i.e. $v^\#_\Lambda \in \omega(R/H)$.

Proof. a): Let $P := P_v$ and $v' := v^\#_\Lambda$. The valuation v is T_A-convex, since A is T_A-convex in R. Thus $T_A \subset P$. {Actually we know that $T_A = \bigcap\limits_{u \in S(R/A)} P_u$.}
The valuation v' is P-convex, hence again T_A-convex. Thus $A_{v'}$ is T_A-convex in R. This implies $H \subset A_{v'}$.
b): Let $u := v'|_R$, i.e. u is the special valuation $v'|c_{v'}(\Gamma)$ associated with $v' : R \to \Gamma \cup \infty$ (cf.I,§1). We have $A_u = A_{v'} \supset H$, and we conclude that u is a PM-valuation of R over H. From $v' \le v$ we infer that $A_u \subset A_v$. Since both u and v are PM and v is not trivial, it follows that $u \le v$, and then, that $\mathrm{supp}\, u = \mathrm{supp}\, v = \mathrm{supp}\, v'$. This forces $u = v'$. The valuation u is not trivial, since $A_u \subset A_v \ne R$. Thus $v' \in S(R/H)$.

If $w \in S(R/H)$ and $w \le v'$ then it is clear that $w = v'$ since $\Lambda \subset H \subset A_w$ (cf.§12, Def.3 and Prop.12.12.i). Thus $v' \in \omega(R/H)$. $\square$

Lemma 13.11. Assume that H is Prüfer in R. For every $u \in \omega(R/H)$ we have $u = u^\#_\Lambda$.

Proof. u is T_H-convex and $T_H = T_A$. Thus $u^\#_\Lambda$ is T_A-convex. This implies $A_{u^\#_\Lambda} \supset H$. $u^\#_\Lambda$ is certainly not trivial, since $u^\#_\Lambda \le u$. Thus $u^\#_\Lambda \in S(R/H)$. Again taking into account that $u^\#_\Lambda \le u$, we conclude that $u^\#_\Lambda = u$. $\square$

Theorem 13.12. Assume that H is Prüfer in R. Let $u \in \omega(R/H)$ be given. There exists a valuation $v \in \omega(R/A)$ with $v^\#_\Lambda = u$ iff $AA_u \ne R$. In this case v is uniquely determined by u (up to equivalence). We have $A_u A = A_v$ and $v = u^\#_A$.

Proof. If $v \in S(R/A)$ and $v_\Lambda^\# = u$ then $u \leq v$, hence $A_u \subset A_v$. Since also $A \subset A_v$, we conclude that $AA_u \subset A_v$. In particular, $AA_u \neq R$.

Conversely, if $AA_u \neq R$ then, since u is PM, we have $AA_u = A_v$ with v a non trivial PM-valuation on R and $u \leq v$ (cf.Cor.III.3.2). Moreover $v \in S(R/A)$, since $A \subset A_v$. By Theorem 12.11 and Lemma 10 we infer that $v_\Lambda^\# = u_\Lambda^\# = u$. Clearly v is the minimal coarsening of u with valuation ring $A_v \supset A$. Thus $v = u_A^\#$ (cf.Prop.12.12).

If $w \in S(R/A)$ and $w \leq v$ then w is a coarsening of u, again by Theorem 12.11, hence $v = u_A^\# \leq w$, hence $v \sim w$. This proves that $v \in \omega(R/A)$.

Finally, if $w \in \omega(R/A)$ and $w_\Lambda^\# = u$ then w is again a coarsening of u. Thus $v = u_A^\# \leq w$, hence $v \sim w$. □

Corollary 13.13. Assume that H is Prüfer in R. Let $v \in S(R/A)$ be given. There exists a *unique* valuation (up to equivalence) $w \in \omega(R/A)$ with $w \leq v$. We have $A_w = AA_{v_\Lambda^\#}$ and $w = v_A^\#$.

Proof. There exists some $w \in \omega(R/A)$ with $w \leq v$. It is clear by Theorem 12.11 that w is unique, and that $v^\# = w^\#$. Theorem 12 tells us that $A_w = AA_{w^\#} = AA_{v^\#}$, and $w = w_A^\#$. From $w \leq v$ we infer that $w_A^\# = v_A^\#$ (cf.Scholium 12.13). □

The corollary generalizes readily as follows.

Proposition 13.14. Assume that H is Prüfer in R. Let C be a subring of A which is T_A-convex in A (hence in R). For every $v \in S(R/A)$ there exists a unique $w \in \omega(R/C)$ with $w \leq v$. We have $A_w = CA_{v_\Lambda^\#}$ and $w = v_C^\#$.

Proof. $H_{C/\Lambda} = H$ (cf.Corollary 8 and Remark 9.iv), and $v \in S(R/C)$. The preceding corollary gives the claim. □

As before we always assume that A *is Prüfer and convex in* R *and* Λ *is a subring of* A.

Open Problem. For which subrings Λ of A is

$$\omega(R/H_{A/\Lambda}) \;=\; \{v_\Lambda^\# \mid v \in \omega(R/A)\}\ ?$$

(Do there exist subrings for which this does not hold?)

Since this problem looks rather difficult we introduce a modification of the holomorphy base $H_{A/\Lambda}$ which seems to be more tractable.

Definition 4. The *weak holomorphy base of* A *over* Λ (*in* R) is the ring

$$H'_{A/\Lambda} := \left(H^R_{A/\Lambda}\right)' := \bigcap_{v \in \omega(R/A)} A_{v_\Lambda^\#}. \qquad \square$$

It is clear from above that $H_{A/\Lambda} \subset H'_{A/\Lambda} \subset A$, and that $H'_{A/\Lambda} = H_{A/\Lambda}$ iff the question above has a positive answer for the triple (R, A, Λ).

We fix a triple (R, A, Λ) and abbreviate $H' := H'_{A/\Lambda}$, $H := H_{A/\Lambda}$. It follows from $H \subset H' \subset A$ that $T_{H'} = T_A$ (cf.Cor.8). Moreover, quite a few results stated in Proposition 10 to Proposition 14 for H take over to H' with minor modifications.

Proposition 13.15. Assume that H' is Prüfer in R.
i) $v^{\#}_{\Lambda} \in \omega(R/H')$ for every $v \in S(R/A)$.
ii) If $v \in \omega(R/A)$ and $u := v^{\#}_{\Lambda}$ then $u^{\#}_{A} = v$ and $AA_u = A_v$.

Proof. If $v \in S(R/A)$ then $H' \subset A_{v^{\#}_{\Lambda}}$ by definition of H'. Thus $v^{\#}_{\Lambda} \in S(R/H')$. Running again through the arguments in part b) of the proof of Proposition 10, with H replaced by H', we obtain all claims. $\square$

Proposition 13.16. Assume that H is Prüfer in R. Let $u \in \omega(R/H')$ be given. The following are equivalent:
(1) There exists some $v \in \omega(R/A)$ with $v^{\#}_{\Lambda} = u$.
(2) $AA_u \neq R$.
If (1), (2) hold then $u \in \omega(R/H)$.

Proof. If (1) holds then $AA_u \subset A_v$, hence $AA_u \neq R$. Assume now (2). Let $u_0 := u^{\#}_{\Lambda}$. Applying Proposition 10 and Theorem 12 to the extension $H \subset H'$, we learn that $u_0 \in \omega(R/H)$ and $H'A_{u_0} = A_u$ and $u = (u_0)^{\#}_{A}$. We have $AA_{u_0} = AH'A_{u_0} = AA_u \neq R$, and we obtain, again by Theorem 12, that there exists a unique valuation $v \in \omega(R/A)$ with $v^{\#}_{\Lambda} = u_0$. By definition of H' we have $u_0 \in S(R/H')$. We conclude from $u_0 \leq u$ that $u_0 = u$. Thus $v^{\#}_{\Lambda} = u$ and $u \in \omega(R/H)$. $\square$

We have gained a modest insight into the restricted PM-spectra of R over the holomorphy base $H_{A/\Lambda}$ and the weak holomorphy base $H'_{A/\Lambda}$ for rings $\Lambda \subset A \subset R$ with A convex and Prüfer in R. A lot remains to be done to determine $H_{A/\Lambda}$ and $H'_{A/\Lambda}$ in more concrete terms in general and in examples.

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