Some Remarks on Morphisms between Fano Threefolds

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Abstract. Let X, Y be Fano threefolds of Picard number one and such that the ample generators of Picard groups are very ample. Let X be of index one and Y be of index two. It is shown that the only morphisms from X to Y are double coverings. In fact nearly the whole paper is the analysis of the case where Y is the linear section of the Grassmannian G(1,4), since the other cases were more or less solved in another article. This remaining case is treated with the help of Debarre's connectedness theorem for inverse images of Schubert cycles.

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Some twenty-five years ago, Iskovskih classified the smooth complex Fano threefolds with Picard number one. Apart from $\mathbb{P}^3$ and the quadric, his list includes 5 families of Fano varieties of index two and 11 families of varieties of index one (for index one threefolds, the cube of the anticanonical divisor takes all even values from 2 to 22, except 20). Recently, the author ([A]) and C. Schuhmann ([S]) made some efforts to classify the morphisms between such Fano threefolds, the starting point being a question of Peternell: let $f : X \to Y$ be a non-trivial morphism between Fano varieties with Picard number one, is it then true that the index of X does not exceed the index of Y?

In particular, Schuhmann ([S]) proved that there are no morphisms from index-two to index-one threefolds, and that any morphism between index-two threefolds is an isomorphism (under certain additional hypotheses, some of which were handled later in [A], [IS]). As for morphisms from index-one to index-two Fano threefolds, such morphisms do exist: an index-two threefold has a double covering (branched along an anticanonical divisor) which is of index one. It is

therefore natural to ask if every morphism from index-one Fano threefold X with Picard number one to index-two Fano threefold Y with Picard number one is a double covering. In [A], I proved a theorem (Theorem 3.1) indicating that the answer should be yes, however not settling the question completely. The essential problem was that the methods of [A] would never work for $Y = V_5$, the linear section of the Grassmannian $G(1,4)$ in the Plücker embedding (all smooth three-dimensional linear sections of $G(1,4)$ are isomorphic). Though there are several ways to obtain bounds for the degree of a morphism between Fano threefolds with second Betti number one ([HM], [A]), these bounds are still too rough for our purpose.
This paper is an attempt to handle this problem. The main result is the following

THEOREM *Let X be a smooth complex Fano threefold of index one and such that $Pic(X) = \mathbb{Z}$. Suppose moreover that X is anticanonically embedded. Let $f : X \to V_5$ be a non-trivial morphism. Then X is of degree 10 ("X is of type V_{10}") and f is a double covering. In other words, X is a hyperquadric section of a cone over V_5 in $\mathbb{P}^7$.*

I believe that the extra assumption made on X is purely technical and can be ruled out if one refines the arguments below. This assumption excludes two families of Fano threefolds: sextic double solids and double coverings of the quadric branched along a hyperquartic section.
Together with Theorem 3.1 of [A] and a few remarks, this theorem implies that any morphism from an index-one to an index-two threefold with cyclic Picard group is a double covering, at least under an additional assumption of the very ampleness of the generator of the Picard group of the two threefolds (see Theorem 4.1 of Section 4).
The proof, somewhat unexpectedly, relies on a connectedness theorem due to Debarre, which enables one to show that the inverse images of certain special lines on V_5 must be connected; at the same time it is well-known that, if $f : X \to V_5$ is as in our theorem, then the inverse image of a general line on V_5 is a disjoint union of conics. Starting from this, we use some Hilbert scheme manipulations to show that the connected inverse images must have very special properties, and deduce the theorem.
A smooth anticanonically embedded Fano threefold of index one and Picard number one is sometimes called *a prime Fano threefold*. We shall also call it thus throughout this paper.

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1. Preliminaries: the geometry of V_5

Let us recall some more or less classical facts on the threefold $V_5 \subset \mathbb{P}^6$, most of which can be found in [I] or [FN]. First of all, as any Fano threefold of index two and Picard number 1, it has a two-dimensional family of lines. A general line has trivial normal bundle (call it a $(0,0)$-line), whereas there is a one-dimensional subfamily of lines with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ (call them $(-1,1)$-lines). The Hilbert scheme of lines on V_5 is isomorphic to $\mathbb{P}^2$, the curve of $(-1,1)$-lines is a conic in this $\mathbb{P}^2$, and there are 3 lines through a general point of V_5. More precisely, the $(-1,1)$-lines form the tangent surface D to a rational normal sextic B on V_5 (in particular, they never intersect), and there are three lines through any point away from D, two lines through a point on D but not on B, and one line through a point of B. The surface D is of degree 10, thus a hyperquadric section of V_5.

We shall denote by U resp. Q the restriction to V_5 of the universal bundle U_G resp. the universal quotient bundle Q_G on the Grassmannian $G(1,4)$. The cohomology groups related to those bundles are computed starting from the cohomologies of vector bundles on the Grassmannian. In particular the bundles U and Q remain stable.

We shall also use the following result from [S]: let X be a prime Fano threelold, and let $f : X \to V_5$ be a finite morphism. Let m be such that $f^*\mathcal{O}_{V_5}(1) = \mathcal{O}_X(m)$. Then the inverse image of a general line consists of $\frac{m^2 deg(X)}{10}$ disjoint conics; in general, if one replaces V_5 by another Fano threefold Y of index two with Picard number one, the inverse image of a general line shall consist of $\frac{m^2 deg(X)}{2 deg(Y)}$ disjoint conics. Here by $deg(Y)$ we mean the self-intersection number of the ample generator of $Pic(Y)$.

Our starting point is the following fact, which shall be proved in the end of this section:

Proposition 1.1 *For l a $(-1,1)$-line on V_5, any irreducible projective variety X and a surjective morphism $f : X \to V_5$, $f^{-1}(l)$ is connected.*

To prove this proposition, we shall need some further details on the geometry of V_5.

Remark that the Schubert cycles of type $\sigma_{1,1}$, which are sets of points of $G(1,4)$ corresponding to lines lying in a fixed hyperplane, and are also caracterized as zero-loci of sections of the bundle dual to the universal, are 4-dimensional quadrics in the Plücker $G(1,4)$, so each of them intersects V_5 along a conic. Conversely, every smooth conic on V_5 is an intersection with such a Schubert cycle. Indeed, every conics on a Grassmannian is obviously contained in some $G(1,3)$; and if this conic is strictly contained in $G(1,3) \cap V_5$, then $G(1,3) \cap V_5$ is a surface, so the bundle U^* has a section vanishing along a surface; but this contradicts the stability of U^*.

The same is (by the same argument) true for pairs of intersecting lines on

V_5. Moreover, the correspondence between the Schubert cycles and the conics is one-to-one (it is induced by the restriction map on the global sections $H^0(G(1,4),U_G^*) \to H^0(V_5,U^*)$ which is an isomorphism).
Let us show that among these conics, there is a one-dimensional family of double lines.

Proposition 1.2 *Fix an embedding $V_5 \subset G(1,4) \subset \mathbb{P}^9$. There is a one-dimensional family of Schubert cycles Σ_t such that for each t, the intersection of V_5 and Σ_t is a (double) line. Moreover, lines on V_5 which are obtained as a set-theoretic intersection with a Schubert cycle of type $\sigma_{1,1}$, are exactly $(-1,1)$-lines.*

Proof: The three-dimensional linear sections of $G(1,4)$ in the Plücker embedding are parametrized by the Grassmann variety $G(6,9)$; let, for $P \in G(6,9)$, V_P denote the intersection of $G(1,4)$ with the corresponding linear subspace (which we will denote also by P). The Schubert cycles are parametrized by $G(3,4) = \mathbb{P}^4$; likewise, denote by Σ_t the Schubert cycle corresponding to $t \in \mathbb{P}^4$. Consider the following incidence subvariety $\mathcal{I} \subset G(6,9) \times \mathbb{P}^4$:

$$\mathcal{I} = \{(P,t) \in G(6,9) \times \mathbb{P}^4 | V_P \cap \Sigma_t \textit{ is a line}\}.$$

The fiber $\mathcal{I}_t$ of $\mathcal{I}$ over any $t \in \mathbb{P}^4$ parametrizes the six-dimensional subspaces P of $\mathbb{P}^9$ intersecting Σ_t along a line. Σ_t is a quadric in $\mathbb{P}^5 \subset \mathbb{P}^9$, and P intersects it along a line l if and only if the plane $H = P \cap \mathbb{P}^5$ is tangent to Σ_t along l, i.e. lies in every $T_x\Sigma_t, x \in l$. The intersection of all tangent spaces to $\Sigma_t \subset \mathbb{P}^5$ along l is a three-dimensional projective space (the tangent spaces form a pencil of hyperplanes in $\mathbb{P}^5$, because Σ_t is a quadric). This means that for every l, the planes tangent to Σ_t along l form a one-dimensional family. The family of lines on a 4-dimensional quadric ($= G(1,3)$) is a 5-dimensional flag variety, so the planes in $\mathbb{P}^5$ tangent to Σ_t along a line are parametrized by a six-dimensional irreducible variety (a $\mathbb{P}^1$-bundle over a flag variety). This implies that $\mathcal{I}_t$ is irreducible of codimension 3 in $G(6,9)$, so $\mathcal{I}$ is irreducible of codimension 3 in $G(6,9) \times \mathbb{P}^4$.
We must show that the first projection $p_1 : \mathcal{I} \to G(6,9)$ is surjective and its general fiber is of dimension one. First of all, remark that there are points P in the image of p_1 such that the corresponding V_P is smooth (so, is a V_5). Indeed, fix, as above, Σ_t, $l \subset \Sigma_t$, H a plane in $\mathbb{P}^5 =< \Sigma_t >$ such that $H \cap \Sigma_t = l$; the remark will follow if we show that for a general $\mathbb{P}^6 = P \subset \mathbb{P}^9$ containing H, $G(1,4) \cap P$ is smooth. We have $H \cap G(1,4) = H \cap \Sigma_t = l$ (because $G(1,4) \cap < \Sigma_t >= \Sigma_t$), so the smoothness away from l is obvious, and one checks, again by standart dimension count, that for $x \in l$, the set $A_x = \{P | H \subset P, G(1,4) \cap P \textit{ is singular at } x\}$ is of codimension two in the space of all P's containing H. Therefore for P general in the image of p_1, V_P is smooth.
It is clear that if a smooth $V_P = G(1,4) \cap P$ is such that $V_P \cap \Sigma_t = l$, then the corresponding plane H is tangent along l not only to Σ_t, but also to V_P.

Thus the normal bundle N_{l,V_P} has a subbundle $N_{l,H}$ of degree 1, and so l is of type $(-1,1)$ on V_P. Since we have only one-dimensional family of $(-1,1)$-lines on a smooth V_P, we deduce that a fiber of p_1 over a point P such that V_P is smooth, is at most one-dimensional. The irreducibility of $\mathcal{I}$ now implies that p_1 is surjective and its general fiber is of dimension one. This proves the Proposition.

Proof of Proposition 1.1:

Let us recall the following result of Debarre ([D], partial case of Théorème 8.1, Exemple 8.2 (3)):
Let X be an irreducible projective variety, and let $f: X \to G(d,n)$ be a morphism. Let Σ be a Schubert cycle of type σ_m. If in the cohomologies of $G(d,n)$, $[f(X)] \cdot \sigma_{m+1} \neq 0$, then $f^{-1}(\Sigma)$ is connected.
Let X be an irreducible projective variety and $f: X \to V_5$ be a surjective morphism. Composing with the embedding $i: V_5 \to G(1,4)$, we can view f as a morphism to $G(1,4)$. By Proposition 1.2, each $(-1,1)$-line is the intersection of $f(X)$ with a Schubert cycle of type $\sigma_{1,1}$ on our grassmannian $G(1,4) = G(1,\mathbb{P}(U))$, where U is a five-dimensional vector space. By duality, we can view this Grassmannian as $G(2,\mathbb{P}(U^*))$, and this point of view transforms the Schubert cycles of type $\sigma_{1,1}$ into Schubert cycles of type σ_2. The condition $[f(X)] \cdot \sigma_3 \neq 0$ is obviously satisfied because $f(X) = V_5$ is cut out by three hyperplanes. Thus Proposition 1.1 follows from Proposition 1.2 and Debarre's theorem.

Remark 1.3 If we knew that the inverse image of a *general* line is always connected, this would immediately solve our problem; indeed, for a Fano threefold X of index and Picard number one, the equality $\frac{m^2 deg(X)}{10} = 1$ implies that $m = 1$, $deg(X) = 10$ and f is a double covering. However, as shows an example of Peternell and Sommese, this is false in general, even if one supposes that X is a Fano threefold. In the example of [PS], X is the universal family of lines on V_5, which turns out to be a Fano threefold (of Picard number two, of course), and f is the natural triple covering. The inverse image of a general line has two connected components.

Remark 1.4 One can ask if there is a similar connectedness statement for other Fano threefolds of Picard number one and index two. Recall that these are the following: intersection of two quadrics in $\mathbb{P}^5$; cubic in $\mathbb{P}^4$; double covering of $\mathbb{P}^3$ branched in a quartic; double covering of the cone over Veronese surface branched in a hypercubic section.
Smooth quadrics in $\mathbb{P}^5$ are Grassmannians $G(1,3)$, and a smooth intersection of two quadrics in $\mathbb{P}^5$ is a quadric line complex. It is classically known (see [GH], Chapter 6) that on a quadric line complex, there is a finite (and non-zero) number of lines obtained as set-theoretic intersection with a plane in $G(1,3)$. These lines are obviously $(-1,1)$-lines, since the corresponding plane is tangent

to the quadric line complex along this line. Our intersection of two quadrics is contained in a pencil of such Grassmannians, so there is a one-dimensional family of lines on it such that each line is the intersection with a plane lying on some Grassmannian of the pencil. The curve of $(-1,1)$-lines is irreducible (it follows from the results in [GH], Chapter 6, that it is smooth and that it is an ample divisor on the Fano surface of lines, in particular, it is connected). Thus it is just the closure of that family. So that it follows again from Debarre's paper that the inverse image of a general $(-1,1)$-line is connected.
As for the cubic, even if such a connectedness statement could hold, it would not, as far as I see, follow from any well-known general result. One can, though, remark that in the examples of Peternell-Sommese type "(universal family of lines on Y)$\to Y$", the inverse image of a $(-1,1)$-line has a tendency to be connected, whereas the inverse image of a $(0,0)$-line is certainly not connected. Indeed, it is observed in the literature that, on the threefolds as above (the cubic, the quadric line complex, V_5), a line l is in the closure of the curve $C_l = \{$*lines intersecting l but different from l*$\}$ on the Hilbert scheme if and only if l is a $(-1,1)$-line.

2. A Hilbert scheme argument

The previous considerations show that on our Fano threefold X, a disjoint union of conics degenerates flatly to a connected l.c.i. scheme. Recall the following classical example: if one degenerates a disjoint union of two lines in the projective space into a pair of intersecting lines, the pair of intersecting lines shall have an embedded point at the intersection. So if one wants the limit to be a connected l.c.i., this limit must be a double line. This suggests to ask if a similar phenomenon can occur in our situation, that is: can it be true that a connected l.c.i. limit of disjoint conics is necessarily a multiple conic?
In any case it is easily checked that, say, a connected limit of pairs of disjoint conics does not have to have embedded points when the two conics become reducible and acquire a common component. So this is very probably false, and in any case there is no simple argument. In this paragraph we shall prove, though, that the inverse image of a sufficiently general $(-1,1)$-line is either a multiple conic, or supported on a union of lines, and in fact even slightly more (Proposition 2.5).
Let T be the Hilbert scheme of lines on V_5 and let $\mathcal{M} \subset T \times V_5$ be the universal family. We have the "universal family of the inverse images of lines under f"

$$\mathcal{S} = \mathcal{M} \times_{V_5} X \subset T \times X.$$

Since f is flat and $\mathcal{M}$ is flat over T, $\mathcal{S}$ is flat over T.
Let H' be the Hilbert scheme of conics on X. Consider the irreducible components of H' which are relevant for our problem, that is, the components such that their sufficiently general points correspond to conics which are in the inverse image of a sufficiently general line on V_5. Denote by H the union of all such components (each of them is, of course, two-dimensional).

For every point $x \in H$, the image of the corresponding conic C_x is a line. Indeed, "$f(C)$ is a line" is a closed condition on conics C because f is a finite morphism (for f arbitrary, "$f(C)$ is contained in a line" would be a closed condition on C).
This allows to construct a morphism $p : H \to T$ taking every conic to its image under f. Indeed,

$$\mathcal{L} = \{(C, f(x))|x \in C, C \in H\} \subset H \times V_5$$

is a family of lines over H; though apriori it is not clear that it is flat, this is a "well-defined family of algebraic cycles" in the sense of Kollar ([K], Chapter I) and so corresponds to a morphism from H to the Chow variety of lines on V_5, and this is the same as T.
We claim that p is finite. Indeed, it is clear that the only obstruction to the finiteness of p could be the existence of infinitely many double structures of arithmetic genus zero on some lines on X ("non-finiteness of the Hilbert-Chow morphism for the family of conics on X"). This obviously happens if one considers conics in $\mathbb{P}^3$ rather than conics on X. In our situation, however, this is impossible, and the Hilbert-Chow morphism is even one-to-one. Indeed, by [I], the normal bundle of a line in a prime Fano threefold is either $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, or $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, and there is the following

Lemma 2.1 *Let $l \subset X$ be a line on a prime Fano threefold. If $N_{l,X} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, then there is no locally Cohen-Macaulay double structure of arithmetic genus 0 on l. If $N_{l,X} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, then such a structure is unique.*

Proof: All locally Cohen-Macaulay double structures on smooth curves in a threefold are obtained by a construction due to Ferrand (see for example [BF], or else [N] for details): if $Y \subset V$ is a smooth curve on a smooth threefold, and $\tilde{Y}$ is a double structure on Y, write L for $\mathcal{I}_Y/\mathcal{I}_{\tilde{Y}}$; in fact L is a locally free rang-one $\mathcal{O}_Y$-module and $\mathcal{I}_{\tilde{Y}}$ contains $\mathcal{I}_Y^2$. The double structure is thus determined by the natural surjection from the conormal bundle of Y in V to L, up to a scalar. Now take $Y = l$, $V = X$ and let L be as above; we have an exact sequence

$$0 \to L \to \mathcal{O}_{\tilde{l}} \to \mathcal{O}_l \to 0,$$

from which it is clear that $p_a(\tilde{l}) = 0$ if and only if $L = \mathcal{O}_{\mathbb{P}^1}(-1)$. Now in the first part of our assertion, there is no non-trivial surjection from $N^*_{l,X}$ to $\mathcal{O}_{\mathbb{P}^1}(-1)$, and in the second part, such a surjection is unique up to a scalar.

Note that we do not have to consider curves which are not locally Cohen-Macaulay, since, for example, the above argument shows that there are no higher genus locally Cohen-Macaulay double structures, and an embedded point decreases the genus.

Thus, for any $t \in T$, $p^{-1}(t)$ is a finite set $\{h_1, \dots, h_k\}$, and to each h_i there corresponds one conic C_i on X, mapped to l_t by f. The next step is to show that f and p "agree with each other":

Lemma 2.2 *Let $t \in T$ be any point and $l_t \in V_5$ be the corresponding line. Let $h_1, \dots, h_k$ be the points of $p^{-1}(t)$ and $C_1, \dots, C_k$ the corresponding conics on X. Then the support of $f^{-1}(l_t)$ is $\bigcup_i C_i$.*

Proof: Indeed, for a general $t \in T$, it is true: $f^{-1}(l_t) = \bigcup_i C_i$. For a special $t \in T$, choose a curve $V \subset T$ through t, such that t is the only "non-general" point of V in the above sense, and let $U = p^{-1}(V)$. Denote by $\mathcal{C}_U \subset U \times X$ the restriction to U of the universal family of conics over H. The support of the fiber over t of $(p \times id)(\mathcal{C}_U) \subset V \times X$ is equal to $\bigcup_i C_i$. But the family $\mathcal{S}|_V$ coincides with $(p \times id)(\mathcal{C}_U)$ except at t. $\mathcal{S}|_V$ being flat, it must be the scheme-theoretic closure of $(p \times id)(\mathcal{C}_U)|_{V-\{t\}}$ in $V \times X$, and thus the support of $\mathcal{S}|_V$ is $(p \times id)(\mathcal{C}_U)$, q.e.d.

Let now $t \in T$ be a point corresponding to a sufficiently general $(-1, 1)$-line. We know that $f^{-1}(l_t)$ is connected. Suppose that the number k from the Lemma is > 1, so that there are several conics in the $Supp(f^{-1}(l_t))$. Decompose the set of those conics into two disjoint non-empty subsets Σ_1 and Σ_2.

Proposition 2.3 *There exists a conic in Σ_1 which has a common component with a conic in Σ_2; in other words, $(\bigcup_{C \in \Sigma_1} C) \bigcap (\bigcup_{C \in \Sigma_2} C)$ cannot be zero-dimensional.*

Proof Choose a suitable small 1-dimensional disc $(V, 0)$ centered at t. The inverse image $p^{-1}V$ is a disjoint union of two analytic sets U_1 and U_2 (U_i consists of points corresponding to conics near those of Σ_i). Repeat the procedure of the previous lemma: consider the universal families $\mathcal{C}_i$ of conics over U_i and their images $\mathcal{S}_i = (p \times id)(\mathcal{C}_i) \subset V \times X$. Let $\mathcal{S}^0$, $\mathcal{S}_i^0$ denote the restriction of our families $\mathcal{S}$, $\mathcal{S}_i$ to the punctured disc $V^0 = V - \{0\}$. The family $\mathcal{S}^0$ is just the disjoint union of $\mathcal{S}_i^0$. Now take the closure of all those (as analytic spaces) in $V \times X$: the closure of $\mathcal{S}^0$ is just $\mathcal{S}|_V$, by flatness, and the closure $\mathcal{S}'_i$ of $\mathcal{S}_i^0$ has the same support as $\mathcal{S}_i$, is contained in $\mathcal{S}|_V$ and is flat over V. The fiber of $\mathcal{S}'_i$ over 0, denoted S_i, is contained in the fiber S of $\mathcal{S}$, since the tensor multiplication preserves the surjectivity. So $f^{-1}(l_t) = S$ contains $S_1 \cup S_2$. By construction, S_i are flat limits of disjoint unions of a_i conics and S is a flat limit of disjoint unions of $a_1 + a_2$ $(= \frac{m^2 deg(X)}{10})$ conics.
If S_1 and S_2 do not have common components, then, since by flatness $deg(S) = deg(S_1) + deg(S_2)$, this implies $S = S_1 \cup S_2$, because S is purely one-dimensional (being an inverse image of a line under a finite morphism). But then we can apply the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_{S_1} \oplus \mathcal{O}_{S_2} \to \mathcal{O}_{S_1 \cap S_2} \to 0$$

and get a contradiction, since by flatness $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{S_1}) + \chi(\mathcal{O}_{S_2})$, $S_1 \cap S_2$ is non-empty and it is zero-dimensional by assumption. Thus S_1 and S_2 must have common components, and, as S_i is supported on $\bigcup_{C\in\Sigma_i} C$, the Proposition is proved.

Corollary 2.4 *In the situation as above, $f^{-1}(l_t)$ is supported either on a single conic, or on a union of lines.*

Indeed, the proposition shows that if $f^{-1}(l_t)$ contains more than one conic, then any conic from $f^{-1}(l_t)$ must have a common component with the rest of these conics, that is, it must be singular.

Some results from commutative algebra allow to prove a stronger ("local") version of Proposition 2.3:

Proposition 2.5 *In the situation of Proposition 2.3, through each intersection point P of $\bigcup_{C\in\Sigma_1} C$ and $\bigcup_{C\in\Sigma_2} C$ passes some common component of $\bigcup_{C\in\Sigma_1} C$ and $\bigcup_{C\in\Sigma_2} C$.*

Proof: The family $\mathcal{S}$ is flat over T which is smooth, and the fibers are l.c.i., thus locally Cohen-Macaulay. It follows ([EGA], 6.3.1, 6.3.5) that $\mathcal{S}$ is locally Cohen-Macaulay, and that the same it true for the restriction of $\mathcal{S}$ to any smooth curve in T. Suppose that Proposition 2.5 is not true for some intersection point P. Let $x = (t, P) \in T \times X$ be the point corresponding to P in $\mathcal{S}$. Consider the restriction of $\mathcal{S}$ to a general curve through t, and an analytic neighbourhood of x in this restriction. Clearly, if one removes x, this neighbourhood becomes disconnected: there are at least two branches corresponding to $Supp\mathcal{S}_i$ as in Proposition 2.3. But this is impossible by Hartshorne's connectedness ([H]), which implies that a connected Cohen-Macaulay neighbourhood remains connected if one removes a subvariety of codimension at least two.

Remark 2.6 The argument of the Proposition is more or less the following: "if we have a disjoint union of certain smooth curves A and B, which degenerates flatly into a certain connected C in such a way that A and B do not acquire common components in the limit, then C will have embedded points at the intersection points of the limits of A and B, so this is impossible if we know that C is purely one-dimensional". Examples show that one cannot say anything reasonable if one allows A and B to acquire common components. But in fact our "C", that is, $f^{-1}(l_t)$, is more than just purely one-dimensional: it is a locally complete intersection. I do not know if its being a flat limit of disjoint unions of conics can impose stronger restrictions on its geometry.

To illustrate how we shall apply this, let us handle the case when $f^{-1}(l_t)$ is supported on a single conic.

PROPOSITION 2.7 *In this case $X = V_{10}$ and f is a double covering.*

Proof: As the degree of the subscheme $f^{-1}(l_t)$ of X is $\frac{m^2 deg(X)}{5}$, this conic is of multiplicity $\frac{m^2 deg(X)}{10}$ in $f^{-1}(l_t)$. That is, the local degree of f near a general point of such a conic is also $\frac{m^2 deg(X)}{10}$. Now this is the local degree of f along a certain divisor, because we have chosen the line l_t to be "sufficiently general among the $(-1,1)$ lines": it varies in a one-dimensional family. This divisor is thus a component of the ramification divisor of f, and $\frac{m^2 deg(X)}{10} - 1$ is its ramification multiplicity.
Now the ramification divisor of f is an element of $|\mathcal{O}_X(2m-1)|$, and so the local degree of f at its general point is at most $2m$, and if it is $2m$, then the ramification divisor is the inverse image of the surface covered by the $(-1,1)$-lines and set-theoretically a hyperplane section of X. So we have:

$$\frac{m^2 deg(X)}{10} \leq 2m, \ mdeg(X) \leq 20,$$

and if the equality holds, then f is unramified outside the inverse image of the surface of $(-1,1)$-lines. Also, $\frac{m^2 deg(X)}{10}$ must be an integer. The inequality thus only holds for $deg(X) = 10$ and $m = 1$ (this is a double covering) or $m = 2$ (in this case it is an equality), and for $deg(X) = 4$ and $m = 5$ (also an equality). Let us exclude the last two cases. If f is unramified outside the inverse image of the surface of $(-1,1)$-lines, then p is $\frac{m^2 deg(X)}{10}$-to-one everywhere except over the conic parametrizing the $(-1,1)$-lines on $T = \mathbb{P}^2$. It is thus a topological covering of the complement to this conic in T. But the latter is simply-connected; so that H has $\frac{m^2 deg(X)}{10}$ irreducible components and each one maps one-to-one on T. Notice that the number $\frac{m^2 deg(X)}{10}$ is superiour to three in both cases. But this is impossible. Indeed, on V_5 one has only 3 lines through a general point; whereas, if H has k components, each component would give at least one conic through a general point of X. Those conics are mapped to *different* lines through $f(x)$, because they intersect; thus $k \leq 3$.

3. PROOF OF THE THEOREM

We have seen that the inverse image of a general $(-1,1)$-line is supported either on one conic, or on a union of lines, and settled the first case in the end of the second section. Let us now settle the remaining case, using Proposition 2.5.
The following lemma is standart (and follows e.g. from the arguments of [M], Chapter 3):

LEMMA 3.1 *Let $g : X_1 \to X_2$ be a proper morphism of complex quasiprojective varieties, which is finite of degree d. Suppose that X_2 is smooth. Then the inverse image of any point $x \in X_2$ consists of d points at most, and if there* *are exactly d points in the inverse image of all $x \in X_2$, then topologically g is a covering.*

Let H be as in the last section, and let $\mathcal{C}$ be the universal family of conics over H. Each conic of H is contained in the inverse image of some line on V_5, and set-theoretically such an inverse image is a union of conics of H. Denote by D the surface covered by $(-1,1)$-lines on V_5. Recall that through each point of the complement to D in V_5 there are three lines, that D is a tangent surface to a rational normal sextic and that there are two lines (one $(-1,1)$-line and one $(0,0)$-line) through any point of D away from this sextic and a single line through each point of the sextic. Since the inverse image of a general $(0,0)$-line is a disjoint union of conics of H, there are three conics of H through a general point of X, and at least three through any point away from $f^{-1}(D)$. The natural morphism $q : \mathcal{C} \to X$ is proper and finite of degree three. Lemma 3.1 has thus an obvious corollary:

Corollary 3.2 *There are at most three conics of H through any point of X, and exactly three conics of H through any point of X away from $f^{-1}(D)$.*

Let l be a general $(-1,1)$-line on V_5. Consider the case when $Z = f^{-1}(l)$ is a set-theoretic union of degenerate conics $C_1, ..., C_k$ of H.

Lemma 3.3 *Z contains a line which belongs to a single C_i (say C_1).*

Proof: Suppose the contrary, that is, that any component of Z is contained in at least two conics of H. Through a general point x of this component there is at least one more conic of H, coming from the inverse image of the $(0,0)$-line through $f(x)$. This implies that the morphism $q : \mathcal{C} \to X$ is three-to-one outside an algebraic subset A of codimension at least two in X. That is, $\mathcal{C} - q^{-1}(A)$ is, topologically, a covering of $X - A$. But $X - A$ is simply-connected because X is Fano and thus simply-connected. This means that $\mathcal{C}$ is reducible, consists of three components and each of them maps one-to-one to X. Since X is smooth, it must be isomorphic to each of those components (by Zariski's Main Theorem). But this is impossible because the components are fibered in conics and X has cyclic Picard group.

Before continuing our argument, let us recall some well-known facts on lines on prime Fano threefolds ([I]). Lines on our Fano threefold X are parametrized by a curve, which may of course be reducible or non-reduced. Its being reduced or not influences the geometry of the surface covered by lines on X. Namely, if a component of the Hilbert scheme of lines on X is reduced, then the natural morphism from the corresponding component of the universal family to X is an immersion along a general line; and there is a classical computation ([I], [T]) which says that if its image M is an element of $|\mathcal{O}_X(d)|$, then a general line of M intersects $d+1$ other lines of M. If a component of the Hilbert scheme

of lines is non-reduced, then the surface M covered by the corresponding lines is either a cone (but this can happen only on a quartic), or a tangent surface to a curve. One knows only one explicit example of a Fano threefold as above such that the surface covered by lines on it is a tangent surface to a curve, it is constructed by Mukai and Umemura (having been overlooked by Iskovskih) and has degree 22. The surface itself is a hyperplane section of this threefold and its lines never intersect.
The following Proposition, due to Iliev and Schuhmann, is the main result of [IS] slightly reformulated:

Proposition 3.4 *Let X be a prime Fano threefold, $\mathcal{L}$ a complete one-dimensional family of lines on X and M the surface on X covered by lines of $\mathcal{L}$. If X is different from the Mukai-Umemura threefold, then a general line of $\mathcal{L}$ intersects at least one other line of $\mathcal{L}$.*

An outline of the proof: If not, then, by what we have said above, the surface M must be a tangent surface to a curve. Studying its singularities, Iliev and Schuhmann prove that it must be a hyperplane section of X. Then they show, by case-by-case analysis (of which certain cases appear already in [A]), that the only prime Fano threefold containing a tangent surface to a curve as a hyperplane section, is the Mukai-Umemura threefold.

"Lines contained in a single C_i" cover a divisor on X as Z varies (this is the branch divisor of q). Since $(-1,1)$-lines on V_5 never intersect, Proposition 3.4 implies that if X is not the Mukai-Umemura threefold, then in Z there are at least two lines contained in a single conic (say, $l_1 \subset C_1$ and $l_2 \subset C_2$), and that they intersect, say at the point P. Notice that C_1 is necessarily different from C_2: otherwise we get a contradiction with Proposition 2.5 by considering $\Sigma_1 = \{l_1 \cup l_2\}$, Σ_2 the set of all the other C_i and the intersection point P.

Claim 3.5 Both C_1 and C_2 are pairs of lines intersecting at the point P, and Z is supported on $C_1 \cup C_2$. Thus Z is, set-theoretically, the union of three or four lines through P.

Proof:
1) If C_1 is a double line, we get a contradiction with Proposition 2.5 by considering $\Sigma_1 = \{C_1\}$ and the point P; the same is true for C_2.
2) Let $C_1 = l_1 \cup l_1'$. If l_1' does not pass through P, we get the contradiction in the same way, thus $P \in l_1'$. Also, $P \in l_2'$, where $C_2 = l_2 \cup l_2'$.
3)There are two possibilities:
a)If $l_1' \neq l_2'$, then there must be another conic from Z through P, containing l_1'. Indeed, otherwise we again get a contradiction with Proposition 2.5. In the same way, there is a conic from Z through P which contains l_2'. In fact it is the same conic, because otherwise there are at least four conics through P, contradicting Corollary 3.2. Denote it by C_3. No other conic from Z passes

through P. So $C_3 = l'_1 \cup l'_2$, and l'_1, l'_2 are not contained in conics others than C_1, C_2, C_3.
b) If $l'_1 = l'_2$, then no other conic from Z contains this line (otherwise through its general point there will pass at least four conics from H, the fourth one coming from the inverse image of the correspondent $(0,0)$-line).
4) Now the union $C_1 \cup C_2 \cup C_3$ in the case a), resp. the union $C_1 \cup C_2$ in the case b), cannot have any points in common with the other components of Z; otherwise, taking $\Sigma_1 = \{C_1, C_2, C_3\}$, resp. $\Sigma_1 = \{C_1, C_2\}$, we obtain a contradiction with Proposition 2.5. But Z is connected, so Z is supported on the lines l_1, l'_1, l_2, l'_2, q.e.d.

We are now ready to finish the proof of the theorem stated in the introduction.

Proof of the theorem: If X is the Mukai-Umemura threefold, then the lines on X never intersect at all, so that $f^{-1}(l)$ must be supported on a single conic. Proposition 2.7 shows that a morphism from X to V_5 is impossible. (It should be, however, said at this point that the paper [HM] contains a better proof of the non-existence of morphisms from the Mukai-Umemura threefold onto any other smooth variety, besides $\mathbb{P}^3$!).
If X is not the Mukai-Umemura threefold and $f^{-1}(l)$ is not supported on a single conic, then we know by Claim 3.5 how $f^{-1}(l)$ looks. Remark that $f^{-1}(D)$ is a reducible divisor: it has two components, one swept out by the lines l_1 and l_2 as Z varies, another by l'_1 and l'_2 (the components are really different because, by construction, l_1 and l_2 are lines contained in a single conic of H, whereas l'_1 and l'_2 are not). Neither component is a hyperplane section: indeed, if a hyperplane section of X is covered by lines, then it is either a cone (impossible in our situation), or a general line intersects two other lines on the surface by the classical computation from [T] mentioned above, since a hyperplane section cannot be a tangent surface to a curve by [IS]. Let k be the multiplicity of the component corresponding to l_i and k' be the multiplicity of the component corresponding to l'_i. As $f^*(D)$ is a divisor from $|\mathcal{O}_X(2m)|$, $k + k' \leq m$. At the same time, Z must be of degree $\frac{m^2 deg(X)}{5}$, and thus $2k + 2k' = \frac{m^2 deg(X)}{5}$, so $m^2 deg(X) \leq 10$, leavng the only possibility $m = 1, deg(X) = 10$.

4. Concluding remarks

In this section, we shall make a further (minor) precision on Theorem 3.1 from [A].
In that theorem, it was proved that if X, Y are Fano threefolds with Picard number one and very ample generator of the Picard group, X is of index one, Y is of index two different from V_5 (that is, Y is a cubic or a quadric line complex), and $f: X \to Y$ is a surjective morphism, then f is a "projection", that is, $f^*\mathcal{O}_Y(1) = \mathcal{O}_X(1)$. The argument of the theorem also worked for Y a quartic double solid, whereas there were some problems (hopefully technical

ones) for Y a double Veronese cone and for X not anticanonically embedded. Even in the "good" cases, the theorem proves a little bit less than one would like; that is, we want f to be a double covering and we prove only that $f^*\mathcal{O}_Y(1) = \mathcal{O}_X(1)$. This still leaves the following additional possibilities:
(1) If Y is a cubic, X can be V_{12}, $deg(f) = 4$ (X cannot be V_{18} because of the Betti numbers: $b_3(V_{18}) < b_3(Y)$);
(2) If Y is an intersection of two quadrics, X can be V_{16}, $deg(f) = 4$ (here V_{12} is impossible since in this case the inverse image of a general line would consist of 3/2 conics).
The first possibility can be excluded by using an inequality of [ARV]: it says that for a finite morphism $f : X \to Y$ and a line bundle L on Y such that $\Omega_Y(L)$ is globally generated, $deg(f)c_{top}\Omega_Y(L) \leq c_{top}\Omega_X(f^*L)$, so, for X and Y of dimension three, $deg(f)(c_3(\Omega_Y) + c_2(\Omega_Y)L + c_1(\Omega_Y)L^2)$ must not exceed $c_3(\Omega_X) + c_2(\Omega_X)f^*L + c_1(\Omega_X)f^*L^2$.
Consider the situation of (1): we may take $L = \mathcal{O}_Y(2)$, and we know that $c_3(\Omega_Y) = 6$ and $c_3(\Omega_X) = 10$. Using the equalities $c_2(X)c_1(X) = c_2(Y)c_1(Y) = 24$, we arrive at $4(6 + 24 - 24) \leq 10 + 48 - 48$, which is false. So the case (1) cannot occur.
This inequality does not work in the case (2): indeed, now $c_3(\Omega_Y) = 0$, $c_3(\Omega_Y) = 2$ and the inequality reads as follows: $4(0 + 24 - 32) \leq 2 + 48 - 64$, so does not give a contradiction. However we can rule out this case by our connectedness argument. Indeed, the inverse image of a general $(-1, 1)$-line is connected (Remark 1.4) and the inverse image of a general $(0, 0)$-line consists of two disjoint conics. The results of Section 2 apply, of course, to our situation; it follows that the inverse image of a general $(-1, 1)$-line is either a double conic, or a union of two reducible conics which have a common component. In both cases, it is clear that the ramification locus of f projects *onto* the surface covered by $(-1, 1)$-lines. But the ramification divisor is a hyperplane section of V_{16}, and thus can project onto a surface from $|\mathcal{O}_Y(4)|$ at most. Whereas it is well-known (and follows for example from the results in [GH], Chapter 6) that the surface covered by $(-1, 1)$-lines on Y is an element of $|\mathcal{O}_Y(8)|$.
All this put together gives the following

Theorem 4.1 *Let X, Y be smooth complex Fano threefolds of Picard number one, X of index one, Y of index two. Assume further that the ample generators of $Pic(X)$ and $Pic(Y)$ are very ample. Then any morphism from X to Y is a double covering.*

I would like to mention that the verification of this statement without the very ampleness hypothesis amounts to a very small number of particular cases; for instance, if Y is a double Veronese cone, then already the formula of [ARV] combined with the knowledge of Betti numbers implies that for any morphism $f : X \to Y$ with X Fano of index one with cyclic Picard group, $deg(f) = 2$ and X is a sextic double solid. It seems that one could be able to work out the remaining cases without any essentially new ideas.

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