Asymptotic Expansions for Bounded Solutions to Semilinear Fuchsian Equations

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Abstract. It is shown that bounded solutions to semilinear elliptic Fuchsian equations obey complete asymptotic expansions in terms of powers and logarithms in the distance to the boundary. For that purpose, Schulze's notion of asymptotic type for conormal asymptotic expansions near a conical point is refined. This in turn allows to perform explicit computations on asymptotic types — modulo the resolution of the spectral problem for determining the singular exponents in the asymptotic expansions.

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1 Introduction

In this paper, we study solutions $u = u(x)$ to semilinear elliptic equations of the form

$$Au = F(x, B_1u, \dots, B_Ku) \quad \text{on } X^\circ = X \setminus \partial X. \tag{1.1}$$

Here, X is a smooth compact manifold with boundary, ∂X, and of dimension $n+1$, A, $B_1, \dots, B_K$ are Fuchsian differential operators on X°, see Definition 2.1, with real-valued coefficients and of orders μ, $\mu_1, \dots, \mu_K$, respectively, where $\mu_J < \mu$ for $1 \le J \le K$, and $F = F(x, \nu)\colon X^\circ \times \mathbb{R}^K \to \mathbb{R}$ is a smooth function subject to further conditions as $x \to \partial X$. In case A is elliptic in the sense of Definition 2.2 (a) we shall prove that bounded solutions $u\colon X^\circ \to \mathbb{R}$ to Eq. (1.1) possess complete conormal asymptotic expansion of the form

$$u(t, y) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} t^{-p_j} \log^k t \, c_{jk}(y) \ \text{ as } t \to +0. \tag{1.2}$$

Here, $(t, y) \in [0, 1) \times Y$ are normal coordinates in a neighborhood $\mathcal{U}$ of ∂X, Y is diffeomorphic to ∂X, and the exponents $p_j \in \mathbb{C}$ appear in conjugated pairs, $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$, $m_j \in \mathbb{N}$, and $c_{jk}(y) \in C^\infty(Y)$. Note that such conormal asymptotic expansions are typical of solutions u to linear equations of the form (1.1), i.e., in case $F(x) = F(x, \nu)$ is independent of $\nu \in \mathbb{R}^K$.

The general form (1.2) of asymptotics was first thoroughly investigated by Kondrat'ev in his nowadays classical paper [9]. After that to assign asymptotic types to conormal asymptotic expansions of the form (1.2) has proved to be very fruitful. In its consequence, it provides a functional-analytic framework for treating singular problems, both linear and non-linear ones, of the kind (1.1). Function spaces with asymptotics will be discussed in Sections 2.4, 3.1. In its standard setting, going back to Rempel–Schulze [14] in case $n = 0$ (when Y is always assumed be a point) and Schulze [15] in the general case, an asymptotic type P for conormal asymptotic expansions of the form (1.2) is given by a sequence $\{(p_j, m_j, L_j)\}_{j=0}^{\infty}$, where $p_j \in \mathbb{C}$, $m_j \in \mathbb{N}$ are as in (1.2), and L_j is a finite-dimensional linear subspace of $C^\infty(Y)$ to which the coefficients $c_{jk}(y)$ for $0 \le k \le m_j$ are required to belong. (In case $n = 0$, the spaces $L_j = \mathbb{C}$ disappear.) A function $u(x)$ is said to have conormal asymptotics of type P as $x \to \partial X$ if $u(x)$ obeys a conormal asymptotic expansion of the form (1.2), with the data given by P.

When treating semilinear equations we shall encounter asymptotic types belonging to bounded functions $u(x)$, i.e., asymptotic types P for which

$$\begin{cases} p_0 = 0,\, m_0 = 0,\, L_0 = \operatorname{span}\{1\}, \\ \operatorname{Re} p_j < 0 \text{ for all } j \geq 1, \end{cases} \tag{1.3}$$

where $1 \in L_0$ is the function on Y being constant 1.

It turns out that this notion of asymptotic type resolves asymptotics not fine enough to suit a treatment of semilinear problems. The difficulty with it is that only the aspect of the production of asymptotics is emphasized — via the finite-dimensionality of the spaces L_j — but not the aspect of their annihilation. For semilinear problems, however, the latter affair becomes crucial. Therefore, in Section 2, we shall introduce a refined notion of asymptotic type, where additionally linear relations between the various coefficients $c_{jk}(y) \in L_j$, even for different j, are taken into account.

Let $\underline{\mathrm{As}}(Y)$ be the set of all these refined asymptotic types, while $\underline{\mathrm{As}}^\sharp(Y) \subset \underline{\mathrm{As}}(Y)$ denotes the set of asymptotic types belonging to bounded functions according to (1.3). For $R \in \underline{\mathrm{As}}(Y)$, let $C_R^\infty(X)$ be the space of smooth functions $u \in C^\infty(X^\circ)$ having conormal asymptotic expansions of type R, and $C_R^\infty(X \times \mathbb{R}^K) = C^\infty(\mathbb{R}^K; C_R^\infty(X))$, where $C_R^\infty(X)$ is equipped with its natural (nuclear) Fréchet topology. In the formulation of Theorem 1.1, below, we will assume that $F \in C_R^\infty(X \times \mathbb{R}^K)$, where

$$\omega(t) t^{\mu - \bar{\mu} - \varepsilon} C_R^\infty(X) \subset L^\infty(X) \tag{1.4}$$

for some $\varepsilon > 0$. Here, $\bar{\mu} = \max_{1 \leq J \leq K} \mu_J < \mu$ and $\omega = \omega(t)$ is a cut-off function supported in $\mathcal{U}$, i.e., $\omega \in C^\infty(X)$, $\operatorname{supp} \omega \Subset \mathcal{U}$. Here and in the sequel, we always assume that $\omega = \omega(t)$ depends only on t for $0 < t < 1$ and $\omega(t) = 1$ for $0 < t \leq 1/2$. Condition (1.4) means that, given the operator A and then compared to the operators $B_1, \ldots, B_K$, functions in $C_R^\infty(X)$ cannot be too singular as $t \to +0$.

There is a small difference between the set $\underline{\mathrm{As}}^b(Y)$ of all bounded asymptotic types and the set $\underline{\mathrm{As}}^\sharp(Y)$ of asymptotic types as described by (1.3); $\underline{\mathrm{As}}^\sharp(Y) \subsetneq \underline{\mathrm{As}}^b(Y)$. The set $\underline{\mathrm{As}}^\sharp(Y)$ actually appears as the set of multiplicatively closable asymptotic types, see Lemma 3.4. This shows up in the fact that when only boundedness is presumed asymptotic types belonging to $\underline{\mathrm{As}}^b(Y)$ — but not to $\underline{\mathrm{As}}^\sharp(Y)$ — need to be excluded from the considerations by the following non-resonance type condition (1.5), below:

Let $\mathcal{H}^{-\infty,\delta}(X) = \bigcup_{s \in \mathbb{R}} \mathcal{H}^{s,\delta}(X)$ for $\delta \in \mathbb{R}$ be the space of distributions $u = u(x)$ on X° having conormal order at least δ. (The weighted Sobolev space $\mathcal{H}^{s,\delta}(X)$, where $s \in \mathbb{R}$ is Sobolev regularity, is introduced in (2.31).) Note that $\bigcup_{\delta \in \mathbb{R}} \mathcal{H}^{-\infty,\delta}(X)$ is the space of all extendable distributions on X° that in turn is dual to the space $C_{\mathcal{O}}^\infty(X)$ of all smooth functions on X vanishing to infinite order at ∂X. Note also that the conormal order δ for $\delta \to \infty$ is the parameter in which the asymptotics (1.2) are understood.

Now, fix $\delta \in \mathbb{R}$ and suppose that a real-valued $u \in \mathcal{H}^{-\infty,\delta}(X)$ satisfying $Au \in C^\infty_{\mathcal{O}}(X)$ has an asymptotic expansion of the form

$$u(x) \sim \operatorname{Re}\left(\sum_{j=0}^{\infty}\sum_{k=0}^{m_j} t^{l+j+i\beta}\log^k t\, c_{jk}(y)\right) \quad \text{as } t \to +0,$$

where $l \in \mathbb{Z}$, $\beta \in \mathbb{R}$, $\beta \neq 0$ (and $l > \delta - 1/2$ provided that $c_{0m_0}(y) \not\equiv 0$ due to the assumption $u \in \mathcal{H}^{-\infty,\delta}(X)$). Then, for each $1 \leq J \leq K$, it is additional required that

$$B_J u = O(1) \text{ as } t \to +0 \text{ implies } B_J u = o(1) \text{ as } t \to +0, \tag{1.5}$$

where O and o are Landau's symbols. Condition (1.5) means that there is no real-valued $u \in \mathcal{H}^{-\infty,\delta}(X)$ with $Au \in C^\infty_{\mathcal{O}}(X)$ such that $B_J u$ admits an asymptotic series starting with the term $\operatorname{Re}(t^{i\beta}d(y))$ for some $\beta \in \mathbb{R} \setminus \{0\}$, $d(y) \in C^\infty(Y)$. This condition is void if $\delta \geq 1/2 + \bar{\mu}$.
Our main theorem states:

Theorem 1.1. *Let $\delta \in \mathbb{R}$ and $A \in \mathrm{Diff}^{\mu}_{\mathrm{Fuchs}}(X)$ be elliptic in the sense of* Definition 2.2 (a)*, $B_J \in \mathrm{Diff}^{\mu_J}_{\mathrm{Fuchs}}(X)$ for $1 \leq J \leq K$, where $\mu_J < \mu$, and $F \in C^\infty_R(X \times \mathbb{R}^k)$ for some asymptotic type $R \in \underline{\mathrm{As}}(Y)$ satisfying* (1.4)*. Further, let the non-resonance type condition* (1.5) *be satisfied. Then there exists an asymptotic type $P \in \underline{\mathrm{As}}(Y)$ expressible in terms of A, $B_1, \dots, B_K$, R, and δ such that each solution $u \in \mathcal{H}^{-\infty,\delta}(X)$ to* Eq. (1.1) *satisfying $B_J u \in L^\infty(X)$ for $1 \leq J \leq K$ belongs to the space $C^\infty_P(X)$.*

Under the conditions of Theorem 1.1, interior elliptic regularity already implies $u \in C^\infty(X^\circ)$. Thus, the statement concerns the fact that u possesses a complete conormal asymptotic expansion of type P near ∂X. Furthermore, the asymptotic type P can at least in principle be calculated once A, $B_1, \dots, B_K$, R, and δ are known.
Some remarks about Theorem 1.1 are in order: First, the solution u is asked to belong to the space $\mathcal{H}^{-\infty,\delta}(X)$. Thus, if the non-resonance type condition (1.5) is satisfied for all $\delta \in \mathbb{R}$ — which is generically true — then the foregoing requirement can be replaced by the requirement for u being an extendable distribution. In this case, $P_\delta \preccurlyeq P_{\delta'}$ for $\delta \geq \delta'$ in the natural ordering of asymptotic types, where P_δ denotes the asymptotic type associated with the conormal order δ. Moreover, jumps in this relation occur only for a discrete set of values of $\delta \in \mathbb{R}$ and, generically, P_δ eventually stabilizes as $\delta \to -\infty$.
Secondly, for a solution $u \in C^\infty_P(X)$ to Eq. (1.1), neither u nor the right-hand side $F(x, B_1u(x), \dots, B_Ku(x))$ need be bounded. Unboundedness of u, however, requires that, up to a certain extent, asymptotics governed by the elliptic operator A are canceled jointly by the operators $B_1, \dots, B_K$. Again, this is a non-generic situation. Furthermore, in applications one often has that one of the operators B_J, say B_1, is the identity — belonging to $\mathrm{Diff}^0_{\mathrm{Fuchs}}(X)$ — i.e., $B_1 u = u$ for all u. Then this leads to $u \in L^\infty(X)$ and explains the term "bounded solutions" in the paper's title.

Remark 1.2. Theorem 1.1 continues to hold for sectional solutions in vector bundles over X. Let E_0, E_1, E_2 be smooth vector bundles over X, $A \in \mathrm{Diff}^{\mu}_{\mathrm{Fuchs}}(X; E_0, E_1)$ be elliptic in the above sense, $B \in \mathrm{Diff}^{\mu-1}_{\mathrm{Fuchs}}(X; E_0, E_2)$, and $F \in C^{\infty}_{R}(X, E_2; E_1)$. Then, under the same technical assumptions as above, each solution u to $Au = F(x, Bu)$ in the class of extendable distributions with $Bu \in L^{\infty}(X; E_2)$ belongs to the space $C^{\infty}_{P}(X; E_0)$ for some resulting asymptotic type P.

Theorem 1.1 has actually been stated as one, though basic example for a more general method for deriving — and then justifying — conormal asymptotic expansions for solutions to semilinear elliptic Fuchsian equations. This method always works if one has boundedness assumptions as made above, but boundedness can often successfully be replaced by *structural assumptions* on the nonlinearity. An example is provided in Section 3.4. The proposed method works indeed not only for elliptic Fuchsian equations, but for other Fuchsian equations as well. In technical terms, what counts is the invertible of the complete sequence of conormal symbols in the algebra of complete Mellin symbols under the Mellin translation product, and this is equivalent to the ellipticity of the principal conormal symbol (which, in fact, is a substitute for the non-characteristic boundary in boundary problems). For elliptic Fuchsian differential operator, this latter condition is always fulfilled.

The derivation of conormal asymptotic expansions for solutions to semilinear Fuchsian equations is a purely algebraic business once the singular exponents and their multiplicities for the linear part are known. However, a strict justification of these conormal asymptotic expansions — in the generality supplied in this paper — requires the introduction of the refined notion of asymptotic type and corresponding function spaces with asymptotics. For this reason, from a technical point of view the main result of this paper is Theorem 2.42 which states the existence of a complete sequence of holomorphic Mellin symbols realizing a given proper asymptotic type in the sense of exactly annihilating asymptotics of that given type. (The term "proper" is introduced in Definition 2.22.) The construction of such Mellin symbols relies on the factorization result of Witt [21].

Remark 1.3. Behind part of the linear theory, there is Schulze's cone pseudodifferential calculus. The interested reader should consult Schulze [15, 16]. We do not go much into the details, since for most of the arguments this is not needed. Indeed, the algebra of complete Mellin symbols controls the production and annihilation of asymptotics, and it is this algebra that is detailed discussed.

The relation with conical points is as follows: A conical point leads — via blow-up, i.e., the introduction of polar coordinates — to a manifold with boundary. Vice versa, each manifold with boundary gives rise to a space with a conical point — via shrinking the boundary to a point. Since in both situations the analysis is taken place over the interior of the underlying configuration, i.e., away from the conical point and the boundary, respectively, there is no essential difference between these two situations. Thus, the geometric situation is given by the kind of degeneracy admitted for, say, differential operators. In the case considered in this paper, this degeneracy is of Fuchsian type.

The first part of this paper, Section 2, is devoted to the linear theory and the introduction of the refined notion of asymptotic type. Then, in a second part, Theorem 1.1 is proved in Section 3.

2 Asymptotic types

In this section, we introduce the notion of discrete asymptotic type. A comparison of this notion with the formerly known notions of weakly discrete asymptotic type and strongly discrete asymptotic type, respectively, can be found in Figure 1. The definition of discrete asymptotic type is modeled on part of the Gohberg-Sigal theory of the inversion of finitely meromorphic, operator-valued functions at a point, see Gohberg-Sigal [4]. See also Witt [18] for the corresponding notion of local asymptotic type, i.e., asymptotic types at one singular exponent $p \in \mathbb{C}$ in (1.2) only. Finally, in Section 2.4, function spaces with asymptotics are introduced. The definition of these function spaces relies on the existence of complete (holomorphic) Mellin symbols realizing a prescribed proper asymptotic type. The existence of such complete Mellin symbols is stated and proved in Theorem 2.42.

Added in proof. To keep this article of reasonable length, following the referee's advice, proofs of Theorems 2.6, 2.30, and 2.42 and Propositions 2.28 (b), 2.31, 2.32, 2.35, 2.36, 2.40, 2.44, 2.46, 2.47, 2.48, 2.49, and 2.52 are only sketchy or missing at all. They are available from the second author's homepage[1].

2.1 Fuchsian differential operators

Let X be a compact C^∞ manifold with boundary, ∂X. Throughout, we fix a collar neighborhood $\mathcal{U}$ of ∂X and a diffeomorphism $\chi\colon \mathcal{U} \to [0,1) \times Y$, with Y being a closed C^∞ manifold diffeomorphic to ∂X. Hence, we work in a fixed splitting of coordinates (t, y) on $\mathcal{U}$, where $t \in [0, 1)$ and $y \in Y$. Let (τ, η) be the covariables to (t, y). The compressed covariable $t\tau$ to t is denoted by $\tilde{\tau}$, i.e., $(\tilde{\tau}, \eta)$ is the linear variable in the fiber of the compressed cotangent bundle $\tilde{T}^*X\big|_{\mathcal{U}}$. Finally, let $\dim X = n + 1$.

Definition 2.1. A differential operator A with smooth coefficients of order μ on $X^\circ = X \setminus \partial X$ is called *Fuchsian* if

$$\chi_*\left(A\big|_{\mathcal{U}}\right) = t^{-\mu} \sum_{k=0}^{\mu} a_k(t, y, D_y)\big(-t\partial_t\big)^k, \tag{2.1}$$

where $a_k \in C^\infty([0, 1); \mathrm{Diff}^{\mu-k}(Y))$ for $0 \le k \le \mu$. The class of all Fuchsian differential operators of order μ on X° is denoted by $\mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X)$.

[1] `http://www.ma.imperial.ac.uk/~ifw/asymptotics.html`

Weakly discrete asymptotic types	Singular exponents with multiplicities, (p_j, m_j), are prescribed, the coefficients $c_{jk}(y) \in C^\infty(Y)$ are arbitrary. The general form of asymptotics is observed, cf., e.g., Kondrat'ev (1967), Melrose (1993), Schulze (1998).
$\downarrow$	
Strongly discrete asymptotic types	Singular exponents with multiplicities, (p_j, m_j), are prescribed, $c_{jk}(y) \in L_j \subset C^\infty(Y)$, where $\dim L_j < \infty$. The *production of asymptotics* is observed, cf. Rempel–Schulze (1989), Schulze (1991).
$\downarrow$	
Discrete asymptotic types	Linear relation between the various coefficients $c_{jk}(y) \in L_j$, even for different j, are additionally allowed. Thus the *production/annihilation of asymptotics* is observed, cf. this article.

Figure 1: Schematic overview of asymptotic types

Henceforth, we shall suppress writing the restriction $\cdot\big|_{\mathcal{U}}$ and the operator push-forward χ_* in expressions like (2.1). For $A \in \mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X)$, we denote by

$$\sigma^\mu_\psi(A)(t,y,\tau,\eta) = t^{-\mu} \sum_{k=0}^{\mu} \sigma^{\mu-k}_\psi(a_k(t))(y,\eta)(it\tau)^k$$

the principal symbol of A, by $\tilde{\sigma}^\mu_\psi(A)(t,y,\tilde{\tau},\eta)$ its compressed principal symbol related to $\sigma^\mu_\psi(A)(t,y,\tau,\eta)$ via

$$\sigma^\mu_\psi(A)(t,y,\tau,\eta) = t^{-\mu}\tilde{\sigma}^\mu_\psi(A)(t,y,t\tau,\eta)$$

in $(\tilde{T}^*X \setminus 0)\big|_{\mathcal{U}}$, and by $\sigma^\mu_M(A)(z)$ its *principal conormal symbol*,

$$\sigma^\mu_M(A)(z) = \sum_{k=0}^{\mu} a_k(0) z^k, \quad z \in \mathbb{C}.$$

Further, we introduce the *jth conormal symbol* $\sigma^{\mu-j}_M(A)(z)$ for $j = 1, 2, \dots$ by

$$\sigma^{\mu-j}_M(A)(z) = \sum_{k=0}^{\mu} \frac{1}{j!} \frac{\partial^j a_k}{\partial t^j}(0) z^k, \quad z \in \mathbb{C}.$$

Note that $\tilde{\sigma}^\mu_\psi(A)(t,y,\tilde{\tau},\eta)$ is smooth up to $t = 0$ and that $\sigma^{\mu-j}_M(z)$ for $j = 0, 1, 2, \dots$ is a holomorphic function in z taking values in $\mathrm{Diff}^\mu(Y)$. Moreover, if $A \in \mathrm{Diff}^{\mu}_{\mathrm{Fuchs}}(X)$, $B \in \mathrm{Diff}^{\nu}_{\mathrm{Fuchs}}(X)$, then $AB \in \mathrm{Diff}^{\mu+\nu}_{\mathrm{Fuchs}}(X)$,

$$\sigma_M^{\mu+\nu-l}(AB)(z) = \sum_{j+k=l} \sigma_M^{\mu-j}(A)(z+\nu-k)\sigma_M^{\nu-k}(B)(z) \tag{2.2}$$

for all $l = 0, 1, 2, \ldots$ This formula is called the *Mellin translation product* (due to the shifts of $\nu - k$ in the argument of the first factors).

Definition 2.2. (a) The operator $A \in \mathrm{Diff}^{\mu}_{\mathrm{Fuchs}}(X)$ is called *elliptic* if A is an elliptic differential operator on X° and

$$\tilde{\sigma}^{\mu}_{\psi}(A)(t, y, \tilde{\tau}, \eta) \neq 0, \quad (t, y, \tilde{\tau}, \eta) \in (\tilde{T}^*X \setminus 0)\big|_{\mathcal{U}}. \tag{2.3}$$

(b) The operator $A \in \mathrm{Diff}^{\mu}_{\mathrm{Fuchs}}(X)$ is called *elliptic with respect to the weight* $\delta \in \mathbb{R}$ if A is elliptic in the sense of (a) and, in addition,

$$\sigma^{\mu}_{M}(A)(z) \colon H^s(Y) \to H^{s-\mu}(Y), \quad z \in \Gamma_{(n+1)/2-\delta}, \tag{2.4}$$

is invertible for some $s \in \mathbb{R}$ (and then for all $s \in \mathbb{R}$). Here, $\Gamma_\beta = \{z \in \mathbb{C};\ \mathrm{Re}\, z = \beta\}$ for $\beta \in \mathbb{R}$.

Under the assumption of interior ellipticity of A, (2.3) can be reformulated as

$$\sum_{k=0}^{\mu} \sigma^{\mu-k}_{\psi}(a_k(0))(y, \eta)\big(i\tilde{\tau}\big)^k \neq 0$$

for all $(0, y, \tilde{\tau}, \eta) \in (\tilde{T}^*X \setminus 0)\big|_{\partial\mathcal{U}}$. This relation implies that $\sigma^{\mu}_{M}(A)(z)\big|_{\Gamma_{(n+1)/2-\delta}}$ is parameter-dependent elliptic as an element in $L^{\mu}_{\mathrm{cl}}\big(Y; \Gamma_{(n+1)/2-\delta}\big)$, where the latter is the space of *classical pseudodifferential operators* on Y of order μ *with parameter* z varying in $\Gamma_{(n+1)/2-\delta}$, for

$$\sigma^{\mu}_{\psi}(\sigma^{\mu}_{M}(A))(y, z, \eta)\big|_{z=(n+1)/2-\delta-\tilde{\tau}} = \tilde{\sigma}^{\mu}_{\psi}(A)(0, y, \tilde{\tau}, \eta),$$

where $\sigma^{\mu}_{\psi}(\cdot)$ on the left-hand side denotes the parameter-dependent principal symbol. Thus, if (a) is fulfilled, then it follows that $\sigma^{\mu}_{M}(A)(z)$ in (2.4) is invertible for $z \in \Gamma_{(n+1)/2-\delta}$, $|z|$ large enough.

Lemma 2.3. *If $A \in \mathrm{Diff}^{\mu}_{\mathrm{Fuchs}}(X)$ is elliptic, then there exists a discrete set $\mathcal{D} \subset \mathbb{C}$ with $\mathcal{D} \cap \{z \in \mathbb{C};\ c_0 \leq \mathrm{Re}\, z \leq c_1\}$ is finite for all $-\infty < c_0 < c_1 < \infty$ such that* (2.4) *is invertible for all $z \in \mathbb{C} \setminus \mathcal{D}$. In particular, there is a discrete set $D \subset \mathbb{R}$ such that A is elliptic with respect to the weight δ for all $\delta \in \mathbb{R} \setminus D$; $D = \mathrm{Re}\, \mathcal{D}$.*

Proof. Since $\sigma^{\mu}_{M}(A)(z)\big|_{\Gamma_\beta} \in L^{\mu}(Y; \Gamma_\beta)$ is parameter-dependent elliptic for all $\beta \in \mathbb{R}$, for each $c > 0$ there is a $C > 0$ such that $\sigma^{\mu}_{M}(A)(z) \in L^{\mu}(Y)$ is invertible for all z with $|\mathrm{Re}\, z| \leq c$, $|\mathrm{Im}\, z| \geq C$. Then the assertion follows from results on the invertibility of holomorphic operator-valued functions. See Proposition 2.5, below, or Schulze [16, Theorem 2.4.20]. $\square$

Next, we introduce the class of meromorphic functions arising in point-wise inverting parameter-dependent elliptic conormal symbols $\sigma_M^\mu(A)(z)$. The following definition is taken from Schulze [16, Definition 2.3.48]:

Definition 2.4. (a) $\mathcal{M}_{\mathcal{O}}^\mu(Y)$ for $\mu \in \mathbb{Z} \cup \{-\infty\}$ is the space of all holomorphic functions $f(z)$ on $\mathbb{C}$ taking values in $L_{\mathrm{cl}}^\mu(Y)$ such that $f(z)\big|_{z=\beta+i\tau} \in L_{\mathrm{cl}}^\mu(Y;\mathbb{R}_\tau)$ uniformly in $\beta \in [\beta_0,\beta_1]$ for all $-\infty < \beta_0 < \beta_1 < \infty$.
(b) $\mathcal{M}_{\mathrm{as}}^{-\infty}(Y)$ is the space of all meromorphic functions $f(z)$ on $\mathbb{C}$ taking values in $L^{-\infty}(Y)$ that satisfy the following conditions:
(i) The Laurent expansion around each pole $z = p$ of $f(z)$ has the form

$$f(z) = \frac{f_0}{(z-p)^\nu} + \frac{f_1}{(z-p)^{\nu-1}} + \cdots + \frac{f_{\nu-1}}{z-p} + \sum_{j\geq 0} f_{\nu+j}(z-p)^j, \qquad (2.5)$$

where $f_0, f_1, \ldots, f_{\nu-1} \in L^{-\infty}(Y)$ are finite-rank operators.
(ii) If the poles of $f(z)$ are numbered someway, $p_1, p_2, \ldots$, then $|\operatorname{Re} p_j| \to \infty$ as $j \to \infty$ if the number of poles is infinite.
(iii) For any $\bigcup_j \{p_j\}$-excision function $\chi(z) \in C^\infty(\mathbb{C})$, i.e., $\chi(z) = 0$ if $\operatorname{dist}(z, \bigcup_j \{p_j\}) \leq 1/2$ and $\chi(z) = 1$ if $\operatorname{dist}(z, \bigcup_j\{p_j\}) \geq 1$, we have $\chi(z)f(z)\big|_{z=\beta+i\tau} \in L^{-\infty}(Y;\mathbb{R}_\tau)$ uniformly in $\beta \in [\beta_0,\beta_1]$ for all $-\infty < \beta_0 < \beta_1 < \infty$.
(c) Finally, we set $\mathcal{M}_{\mathrm{as}}^\mu(Y) = \mathcal{M}_{\mathcal{O}}^\mu(Y) + \mathcal{M}_{\mathrm{as}}^{-\infty}(Y)$ for $\mu \in \mathbb{Z}$. (Note that $\mathcal{M}_{\mathcal{O}}^\mu(Y) \cap \mathcal{M}_{\mathrm{as}}^{-\infty}(Y) = \mathcal{M}_{\mathcal{O}}^{-\infty}(Y)$.)
Functions $f(z)$ belonging to $\mathcal{M}_{\mathrm{as}}^\mu(Y)$ are called *Mellin symbols* of order μ.

$\bigcup_{\mu\in\mathbb{Z}} \mathcal{M}_{\mathrm{as}}^\mu(Y)$ is a filtered algebra under pointwise multiplication.
For $f \in \mathcal{M}_{\mathrm{as}}^\mu(Y)$ for $\mu \in \mathbb{Z}$ and $f(z) = f_0(z) + f_1(z)$, where $f_0 \in \mathcal{M}_{\mathcal{O}}^\mu(Y)$, $f_1 \in \mathcal{M}_{\mathrm{as}}^{-\infty}(Y)$, the parameter-dependent principal symbol $\sigma_\psi^\mu\big(f_0(z)\big|_{z=\beta+i\tau}\big)$ is independent of the choice of the decomposition of f and also independent of $\beta \in \mathbb{R}$. It is called the *principal symbol* of f. The Mellin symbol $f \in \mathcal{M}_{\mathrm{as}}^\mu(Y)$ is called *elliptic* if its principal symbol is everywhere invertible.
For the next result, see Schulze [16, Theorem 2.4.20]:

Proposition 2.5. *The Mellin symbol $f \in \mathcal{M}_{\mathrm{as}}^\mu(Y)$ for $\mu \in \mathbb{Z}$ is invertible in the filtered algebra $\bigcup_{\mu\in\mathbb{Z}} \mathcal{M}_{\mathrm{as}}^\mu(Y)$, i.e., there is a $g \in \mathcal{M}_{\mathrm{as}}^{-\mu}(Y)$ such that $(fg)(z) = (gf)(z) = 1$ on $\mathbb{C}$, if and only if f is elliptic.*

For $f \in \mathcal{M}_{\mathrm{as}}^\mu(Y)$, $p \in \mathbb{C}$, and $N \in \mathbb{N}$, we denote by $[f(z)]_p^N$ the Laurent series of $f(z)$ around $z = p$ truncated after the term containing $(z-p)^N$, i.e.,

$$[f(z)]_p^N = \frac{f_{-\nu}}{(z-p)^\nu} + \cdots + \frac{f_{-1}}{z-p} + f_\nu + f_1(z-p) + \cdots + f_N(z-p)^N. \quad (2.6)$$

Furthermore, $[f(z)]_p^* = [f(z)]_p^{-1}$ denotes the principal part of the Laurent series of $f(z)$ around $z = p$.
In various constructions, it is important to have examples of elliptic Mellin symbols $f \in \mathcal{M}_{\mathrm{as}}^\mu(Y)$ of controlled *singularity structure*:

THEOREM 2.6. *Let $\mu \in \mathbb{Z}$ and $\{p_j\}_{j=1,2,\dots} \subset \mathbb{C}$ be a sequence obeying the property mentioned in* Definition 2.4 (b) (ii). *Let, for each $j = 1, 2, \dots$, operators $f^j_{-\nu_j}, \dots, f^j_{N_j}$ in $L^\mu_{\mathrm{cl}}(Y)$, where $\nu_j \geq 0$, $N_j + \nu_j \geq 0$, be given such that*

- *$f^j_{-\nu_j}, \dots, f^j_{\min\{N_j,0\}} \in L^{-\infty}(Y)$ are finite-rank operators,*
- *there is an elliptic $g \in \mathcal{M}^\mu_{\mathcal{O}}(Y)$ such that, for all j, $0 \leq k \leq N_j$,*

$$f^j_k - \frac{1}{k!}\, g^{(k)}(p_j) \in L^{-\infty}(Y) \tag{2.7}$$

  (*in particular, $f^j_k \in L^{\mu-k}_{\mathrm{cl}}(Y)$ for $0 \leq k \leq N_j$ and $f^j_0 \in L^\mu_{\mathrm{cl}}(Y)$ is elliptic of index zero*).

Then there is an elliptic Mellin symbol $f(z) \in \mathcal{M}^\mu_{\mathrm{as}}(Y)$ such that, for all j,

$$[f(z)]^{N_j}_{p_j} = \frac{f^j_{-\nu_j}}{(z-p_j)^{\nu_j}} + \dots + \frac{f^j_{-1}}{z-p_j} + f^j_0 + \dots + f^j_{N_j}(z-p_j)^{N_j}, \tag{2.8}$$

while $f(q) \in L^\mu_{\mathrm{cl}}(Y)$ is invertible for all $q \in \mathbb{C} \setminus \bigcup_{j=1,2,\dots}\{p_j\}$.

If $n = 0$, condition (2.7) is void. In case $n > 0$, however, this condition expresses several compatibility conditions among the $\sigma^{\mu-l}_\psi(f^j_k)$, where $j = 0, 1, 2, \dots$, $0 \leq k \leq N_j$, and $l \geq k$, and also certain topological obstructions that must be fulfilled. For instance, for any $f \in \mathcal{M}^\mu_{\mathcal{O}}(Y)$,

$$\sigma^{\mu-j}_\psi(f(z))(y,\eta) = \sum_{k=0}^{j} \frac{(z-p)^k}{k!}\, \sigma^{\mu-j}_\psi(f^{(k)}(p))(y,\eta), \quad j = 0, 1, 2, \dots$$

in local coordinates (y, η) — showing, among others, that $\sigma^{\mu-j}_\psi(f(z))$ is polynomial of degree j with respect to $z \in \mathbb{C}$. The point is that we do not assume $g(q) \in L^\mu_{\mathrm{cl}}(Y)$ be invertible for $q \in \mathbb{C} \setminus \bigcup_{j=1,2,\dots}\{p_j\}$.

Proof of Theorem 2.6. This can be proved using the results of WITT [21]. In particular, the factorization result there gives directly the existence of $f(z)$ if the sequence $\{p_j\} \subset \mathbb{C}$ is void. □

Now, we are going to introduce the basic object of study — the *algebra of complete conormal symbols.* This algebra will enable us to introduce the refined notion of asymptotic type and to study the behavior of conormal asymptotics under the action of Fuchsian differential operators.

DEFINITION 2.7. (a) For $\mu \in \mathbb{Z}$, the space $\mathrm{Symb}^\mu_M(Y)$ consists of all sequences $\mathfrak{S}^\mu = \{\mathfrak{s}^{\mu-j}(z);\, j \in \mathbb{N}\} \subset \mathcal{M}^\mu_{\mathrm{as}}(Y)$.
(b) An element $\mathfrak{S}^\mu \in \mathrm{Symb}^\mu_M(Y)$ is called holomorphic if $\mathfrak{S}^\mu = \{\mathfrak{s}^{\mu-j}(z);\, j \in \mathbb{N}\} \subset \mathcal{M}^\mu_{\mathcal{O}}(Y)$.

(c) $\bigcup_{\mu\in\mathbb{Z}} \mathrm{Symb}^\mu_M(Y)$ is a filtered algebra under the *Mellin translation product*, denoted by $\sharp_M$. Namely, for $\mathfrak{S}^\mu = \{\mathfrak{s}^{\mu-j}(z);\, j\in\mathbb{N}\} \in \mathrm{Symb}^\mu_M(Y)$, $\mathfrak{T}^\nu = \{\mathfrak{t}^{\nu-k}(z);\, k\in\mathbb{N}\} \in \mathrm{Symb}^\nu_M(Y)$, we define $\mathfrak{U}^{\mu+\nu} = \mathfrak{S}^\mu \sharp_M \mathfrak{T}^\nu \in \mathrm{Symb}^{\mu+\nu}_M(Y)$, where $\mathfrak{U}^{\mu+\nu} = \{\mathfrak{u}^{\mu+\nu-l}(z);\, l\in\mathbb{N}\}$, by

$$\mathfrak{u}^{\mu+\nu-l}(z) = \sum_{j+k=l} \mathfrak{s}^{\mu-j}(z+\nu-k)\mathfrak{t}^{\nu-k}(z) \tag{2.9}$$

for $l = 0,1,2,\ldots$ See also (2.2).

From Proposition 2.5, we immediately get:

Lemma 2.8. *$\mathfrak{S}^\mu = \{\mathfrak{s}^{\mu-j}(z);\, j\in\mathbb{N}\} \in \mathrm{Symb}^\mu_M(Y)$ is invertible in the filtered algebra $\bigcup_{\mu\in\mathbb{Z}} \mathrm{Symb}^\mu_M(Y)$ if and only if $\mathfrak{s}^\mu(z) \in \mathcal{M}^\mu_{\mathrm{as}}(Y)$ is elliptic.*

In the case of the preceding lemma, $\mathfrak{S}^\mu \in \mathrm{Symb}^\mu_M(Y)$ is called elliptic. A holomorphic elliptic $\mathfrak{S}^\mu \in \mathrm{Symb}^\mu_M(Y)$ is called elliptic with respect to the weight $\delta\in\mathbb{R}$ if the line $\Gamma_{(n+1)/2-\delta}$ is free of poles of $\mathfrak{s}^\mu(z)^{-1}$. Notice that a holomorphic elliptic $\mathfrak{S}^\mu \in \mathrm{Symb}^\mu_M(Y)$ is elliptic for all, but a discrete set of $\delta\in\mathbb{R}$. The inverse to $\mathfrak{S}^\mu$ with respect to the Mellin translation product is denoted by $(\mathfrak{S}^\mu)^{-1}$. The set of elliptic elements of $\mathrm{Symb}^\mu_M(Y)$ is denoted by $\mathrm{Ell}\,\mathrm{Symb}^\mu_M(Y)$.
There is a homomorphism of filtered algebras,

$$\bigcup_{\mu\in\mathbb{N}} \mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X) \to \bigcup_{\mu\in\mathbb{Z}} \mathrm{Symb}^\mu_M(Y), \quad A \mapsto \big\{\sigma_M^{\mu-j}(A)(z);\, j\in\mathbb{N}\big\}.$$

By the remark preceding Lemma 2.3, $\big\{\sigma_M^{\mu-j}(A)(z);\, j\in\mathbb{N}\big\} \in \mathrm{Symb}^\mu_M(Y)$ is elliptic if $A\in\mathrm{Diff}_{\mathrm{Fuchs}}(X)$ is elliptic in the sense of Definition 2.2 (a).

2.2 Definition of asymptotic types

We now start to introduce discrete asymptotic types.

2.2.1 The spaces $\mathcal{E}^\delta(Y)$ and $\mathcal{E}_V(Y)$

Here, we construct the "coefficient" space $\mathcal{E}^\delta(Y) = \bigcup_{V\in\mathcal{C}^\delta} \mathcal{E}_V(Y)$ that admits the non-canonical isomorphism (2.13), below,

$$C^{\infty,\delta}_{\mathrm{as}}(X)/C^\infty_{\mathcal{O}}(X) \xrightarrow{\cong} \mathcal{E}^\delta(Y),$$

where $C^{\infty,\delta}_{\mathrm{as}}(X)$ is the space of smooth functions on X° obeying *conormal asymptotic expansions* of the form (1.2) of conormal order at least δ, i.e., $\mathrm{Re}\, p_j < (n+1)/2-\delta$ holds for all j (with the condition that the singular exponents p_j appear in conjugated pairs dropped), and $C^\infty_{\mathcal{O}}(X)$ is the subspace of all smooth functions on X° vanishing to infinite order at ∂X.

Definition 2.9. A *carrier V of asymptotics* for distributions of conormal order δ is a discrete subset of $\mathbb{C}$ contained in the half-space $\{z \in \mathbb{C};\ \operatorname{Re} z < (n+1)/2-\delta\}$ such that, for all $\beta_0, \beta_1 \in \mathbb{R}$, $\beta_0 < \beta_1$, the intersection $V \cap \{z \in \mathbb{C};\ \beta_0 < \operatorname{Re} z < \beta_1\}$ is finite. The set of all these carriers is denoted by $\mathcal{C}^\delta$.

In particular, $V_p = p - \mathbb{N}$ for $p \in \mathbb{C}$ is such a carrier of asymptotics. Note that $V_p \in \mathcal{C}^\delta$ if and only if $\operatorname{Re} p < (n+1)/2 - \delta$. We set $T^\varrho V = \varrho + V \in \mathcal{C}^{-\varrho+\delta}$ for $\varrho \in \mathbb{R}$ and $V \in \mathcal{C}^\delta$. We further set $\mathcal{C} = \bigcup_{\delta \in \mathbb{R}} \mathcal{C}^\delta$.
Let $[C^\infty(Y)]^\infty = \bigcup_{m \in \mathbb{N}} [C^\infty(Y)]^m$ be the space of all *finite sequences* in $C^\infty(Y)$, where the sequences $(\phi_0, \dots, \phi_{m-1})$ and $(\underbrace{0, \dots, 0}_{h \text{ times}}, \phi_0, \dots, \phi_{m-1})$ for $h \in \mathbb{N}$ are identified. For $V \in \mathcal{C}^\delta$, we set $\mathcal{E}_V(Y) = \prod_{p \in V} [C^\infty(Y)]^\infty_p$, where $[C^\infty(Y)]^\infty_p$ is an isomorphic copy of $[C^\infty(Y)]^\infty$, and define $\mathcal{E}^\delta(Y)$ to be the space of all families $\Phi \in \mathcal{E}_V(Y)$ for some $V \in \mathcal{C}^\delta$ depending on Φ. Thereby, $\Phi \in \mathcal{E}_V(Y)$, $\Phi' \in \mathcal{E}_{V'}(Y)$ for possibly different $V, V' \in \mathcal{C}^\delta$ are identified if $\Phi(p) = \Phi'(p)$ for $p \in V \cap V'$, while $\Phi(p) = 0$ for $p \in V \setminus V'$, $\Phi'(p) = 0$ for $p \in V' \setminus V$. Under this identification,

$$\mathcal{E}^\delta(Y) = \bigcup_{V \in \mathcal{C}^\delta} \mathcal{E}_V(Y). \tag{2.10}$$

Moreover, $\mathcal{E}_V(Y) \cap \mathcal{E}_{V'}(Y) = \mathcal{E}_{V \cap V'}(Y)$.
On $[C^\infty(Y)]^\infty$, we define the *right shift operator T* by

$$(\phi_0, \dots, \phi_{m-2}, \phi_{m-1}) \mapsto (\phi_0, \dots, \phi_{m-2}).$$

On $\mathcal{E}^\delta(Y)$, the right shift operator T acts component-wise, i.e., $(T\Phi)(p) = T(\Phi(p))$ for $\Phi \in \mathcal{E}_V(Y)$ and all $p \in V$.

Remark 2.10. To designate different shift operators with the same symbol T, once T^ϱ for $\varrho \in \mathbb{R}$ for carriers of asymptotics, once T, T^2, etc. for vectors in $\mathcal{E}^\delta(Y)$ should not confuse the reader.

For $\Phi \in \mathcal{E}^\delta(Y)$, we define $\operatorname{c-ord}(\Phi) = (n+1)/2 - \max\{\operatorname{Re} p;\ \Phi(p) \neq 0\}$. In particular, $\operatorname{c-ord}(0) = \infty$. Note that $\operatorname{c-ord}(\Phi) > \delta$ if $\Phi \in \mathcal{E}^\delta(Y)$. For $\Phi_i \in \mathcal{E}^\delta(Y)$, $\alpha_i \in \mathbb{C}$ for $i = 1, 2, \dots$ satisfying $\operatorname{c-ord}(\Phi_i) \to \infty$ as $i \to \infty$, the sum

$$\Phi = \sum_{i=1}^\infty \alpha_i \Phi_i, \tag{2.11}$$

is defined in $\mathcal{E}^\delta(Y)$ in an obvious fashion: Let $\Phi_i \in \mathcal{E}_{V_i}(Y)$, where $V_i \in \mathcal{C}^{\delta_i}$, $\delta_i \geq \delta$, and $\delta_i \to \infty$ as $i \to \infty$. Then $V = \bigcup_i V_i \in \mathcal{C}^\delta$, and $\Phi \in \mathcal{E}_V(Y)$ is defined by $\Phi(p) = \sum_{i=1}^\infty \alpha_i \Phi_i(p)$ for $p \in V$, where, for each $p \in V$, the sum on the right-hand side is finite.

Lemma 2.11. *Let $\Phi_i \in \mathcal{E}^\delta(Y)$ for $i = 1, 2, \dots$, $\operatorname{c-ord}(\Phi_i) \to \infty$ as $i \to \infty$. Then* (2.11) *holds if and only if*

$$\operatorname{c-ord}\Big(\Phi - \sum_{i=1}^N \alpha_i \Phi_i\Big) \to \infty \ \text{ as } N \to \infty. \tag{2.12}$$

Note that (2.12) already implies that $\text{c-ord}(\alpha_i\Phi_i) \to \infty$ as $i \to \infty$.

Definition 2.12. Let Φ_i, $i = 1, 2, \ldots$, be a sequence in $\mathcal{E}^\delta(Y)$ with the property that $\text{c-ord}(\Phi_i) \to \infty$ as $i \to \infty$. Then this sequence is called *linearly independent* if, for all $\alpha_i \in \mathbb{C}$,

$$\sum_{i=1}^{\infty} \alpha_i \Phi_i = 0$$

implies that $\alpha_i = 0$ for all i. A linearly independent sequence Φ_i for $i = 1, 2, \ldots$ in J for a linear subspace $J \subseteq \mathcal{E}^\delta(Y)$ is called a *basis* for J if every vector $\Phi \in J$ can be represented in the form (2.11) with certain (then uniquely determined) coefficients $\alpha_i \in \mathbb{C}$.

Note that $\sum_{i=1}^{\infty} \alpha_i \Phi_i = 0$ in $\mathcal{E}^\delta(Y)$ if and only if $\text{c-ord}(\sum_{i=1}^{N} \alpha_i \Phi_i) \to \infty$ as $N \to \infty$ according to Lemma 2.11. We also obtain:

Lemma 2.13. *Let* Φ_i, $i = 1, 2, \ldots$, *be a sequence in* $\mathcal{E}^\delta(Y)$ *such that* $\text{c-ord}(\Phi_i) \to \infty$ *as* $i \to \infty$. *Further, let* $\{\delta_j\}_{j=1}^{\infty}$ *be a strictly increasing sequence such that* $\delta_j > \delta$ *for all* j *and* $\delta_j \to \infty$ *as* $j \to \infty$. *Assume that the* Φ_i *are numbered in such a way that* $\text{c-ord}(\Phi_i) \le \delta_j$ *if and only if* $1 \le i \le e_j$. *Then the sequence* Φ_i, $i = 1, 2, \ldots$, *is linearly independent provided that, for each* $j = 1, 2, \ldots$,

$$\Phi_1, \ldots, \Phi_{e_j} \text{ are linearly independent over the space } \mathcal{E}^{\delta_j}(Y).$$

We now introduce the notion of characteristic basis:

Definition 2.14. Let $J \subseteq \mathcal{E}^\delta(Y)$ be a linear subspace, $TJ \subseteq J$, and Φ_i for $i = 1, 2, \ldots$ be a sequence in J. Then Φ_i, $i = 1, 2, \ldots$, is called a *characteristic basis* of J if there are numbers $m_i \in \mathbb{N} \cup \{\infty\}$ such that $T^{m_i}\Phi_i = 0$ if $m_i < \infty$, while the sequence $\{T^k\Phi_i;\, i = 1, 2, \ldots,\, 0 \le k < m_i\}$ forms a basis for J.

Remark 2.15. This notion generalizes a notion of Witt [18]: There, given a finite-dimensional linear space J and a nilpotent operator $T\colon J \to J$, the sequence $\Phi_1, \ldots, \Phi_e$ in J has been called a *characteristic basis*, of characteristic $(m_1, \ldots, m_e)$, if

$$\Phi_1, T\Phi_1, \ldots, T^{m_1-1}\Phi_1, \ldots, \Phi_e, T\Phi_e, \ldots, T^{m_e-1}\Phi_e,$$

constitutes a Jordan basis of J. The numbers $m_1, \ldots, m_e$ appear as the sizes of Jordan blocks; $\dim J = m_1 + \cdots + m_e$. The tuple $(m_1, \ldots, m_e)$ is also called the *characteristic* of J (with respect to T), e is called the *length* of its characteristic, and $\Phi_1, \ldots, \Phi_e$ is sometimes said to be a an $(m_1, \ldots, m_e)$-*characteristic basis* of J. The space $\{0\}$ has empty characteristic of length $e = 0$.

The question of the existence of a characteristic basis obeying one more special property is taken up in Proposition 2.20.
We also need following notion:

Definition 2.16. $\Phi \in \mathcal{E}^\delta(Y)$ is called a *special vector* if $\Phi \in \mathcal{E}^\delta_{V_p}(Y)$ for some $p \in \mathbb{C}$.

Thus, $\Phi \in \mathcal{E}_V(Y)$ is a special vector if there is a $p \in \mathbb{C}$, $\operatorname{Re} p < (n+1)/2 - \delta$ such that $\Phi(p') = 0$ for all $p' \in V$, $p' \notin p - \mathbb{N}$. Obviously, if $\Phi \neq 0$, then p is uniquely determined by Φ, by the additional requirement that $\Phi(p) \neq 0$. We denote this complex number p by $\gamma(\Phi)$. In particular, $\text{c-ord}(\Phi) = (n+1)/2 - \operatorname{Re}\gamma(\Phi)$.

2.2.2 First properties of asymptotic types

In the sequel, we fix a splitting of coordinates $\mathcal{U} \to [0,1) \times Y$, $x \mapsto (t,y)$, near ∂X. Then we have the non-canonical isomorphism

$$C^{\infty,\delta}_{\mathrm{as}}(X)/C^\infty_{\mathcal{O}}(X) \xrightarrow{\cong} \mathcal{E}^\delta(Y), \tag{2.13}$$

assigning to each *formal asymptotic expansion*

$$u(x) \sim \sum_{p \in V} \sum_{k+l=m_p-1} \frac{(-1)^k}{k!}\, t^{-p} \log^k t\, \phi^{(p)}_l(y) \quad \text{as } t \to +0 \tag{2.14}$$

for some $V \in \mathcal{C}^\delta$, $m_p \in \mathbb{N}$, the vector $\Phi \in \mathcal{E}_V(Y)$ given by

$$\Phi(p) = \begin{cases} \left(\phi^{(p)}_0, \phi^{(p)}_1, \dots, \phi^{(p)}_{m_p-1}\right) & \text{if } p \in V, \\ 0 & \text{otherwise,} \end{cases}$$

see also (2.30). "*Non-canonical*" in (2.13) means that the isomorphism depends explicitly on the chosen splitting of coordinates $\mathcal{U} \to [0,1) \times Y$, $x \mapsto (t,y)$, near ∂X. Coordinate invariance is discussed in Proposition 2.32.
Note the shift from m_p to $m_p - 1$ that for notational convenience has appeared in formula (2.14) compared to formula (1.2).

Definition 2.17. An *asymptotic type*, P, for distributions as $x \to \partial X$, of conormal order at least δ, *is represented* — in the given splitting of coordinates near ∂X — *by a linear subspace* $J \subset \mathcal{E}_V(Y)$ for some $V \in \mathcal{C}^\delta$ such that the following three conditions are met:
(a) $TJ \subseteq J$.
(b) $\dim J^{\delta+j} < \infty$ for all $j \in \mathbb{N}$, where $J^{\delta+j} = J/(J \cap \mathcal{E}^{\delta+j}(Y))$.
(c) There is a sequence $\{p_j\}_{j=1}^M \subset \mathbb{C}$, where $M \in \mathbb{N} \cup \{\infty\}$, $\operatorname{Re} p_j < (n+1)/2 - \delta$, and $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$ if $M = \infty$, such that $V \subseteq \bigcup_{j=1}^M V_{p_j}$ and

$$J = \bigoplus_{j=1}^M \left(J \cap \mathcal{E}_{V_{p_j}}(Y)\right). \tag{2.15}$$

The *empty asymptotic type*, $\mathcal{O}$, is represented by the trivial subspace $\{0\} \subset \mathcal{E}^\delta(Y)$. The set of all asymptotic types of conormal order δ is denoted by $\underline{\mathrm{As}}^\delta(Y)$.

Definition 2.18. Let $u \in C^{\infty,\delta}_{\mathrm{as}}(X)$ and $P \in \underline{\mathrm{As}}^{\delta}(Y)$ be represented by $J \subset \mathcal{E}_V(Y)$. Then u is said to have *asymptotics of type* P if there is a vector $\Phi \in J$ such that

$$u(x) \sim \sum_{p\in V} \sum_{k+l=m_p-1} \frac{(-1)^k}{k!} \log^k t\, \phi_l^{(p)}(y) \text{ as } t \to +0, \tag{2.16}$$

where $\Phi(p) = (\phi_0^{(p)}, \phi_1^{(p)}, \dots, \phi_{m_p-1}^{(p)})$ for $p \in V$. The space of all these u is denoted by $C^\infty_P(X)$.

Thus, by *representation of an asymptotic type* it is meant that P that — in the philosophy of asymptotic algebras, see Witt [20] — is the same as the linear subspace $C^\infty_P(X)/C^\infty_{\mathcal{O}}(X) \subset C^{\infty,\delta}_{\mathrm{as}}(X)/C^\infty_{\mathcal{O}}(X)$, is mapped onto J by the isomorphism (2.13).
For $P \in \underline{\mathrm{As}}^{\delta}$ represented by $J \subset \mathcal{E}_V(Y)$, we introduce

$$\delta_P = \min\{\text{c-ord}(\Phi);\, \Phi \in J\}, \tag{2.17}$$

Notice that $\delta_P > \delta$ and $\delta_P = \infty$ if and only if $P = \mathcal{O}$.
Obviously, $\underline{\mathrm{As}}^{\delta}(Y) \subseteq \underline{\mathrm{As}}^{\delta'}(Y)$ if $\delta \geq \delta'$. We likewise set

$$\underline{\mathrm{As}}(Y) = \bigcup_{\delta\in\mathbb{R}} \underline{\mathrm{As}}^{\delta}(Y).$$

On asymptotic types $P \in \underline{\mathrm{As}}^{\delta}(Y)$, we have the shift operation T^ϱ for $\varrho \in \mathbb{R}$, namely $T^\varrho P$ is represented by the space

$$T^\varrho J = \big\{\Phi \in \mathcal{E}^{\varrho+\delta}_{T^\varrho V}(Y);\, \Phi(p) = \bar{\Phi}(p-\varrho),\, p \in \mathbb{C}, \text{ for some } \bar{\Phi} \in J\big\},$$

where $J \subset \mathcal{E}_V(Y)$ represents P.
Furthermore, for $J \subset \mathcal{E}_V(Y)$ as in Definition 2.17,

$$J_p = \{\Phi(p);\, \Phi \in J\} \subset [C^\infty(Y)]^\infty$$

for $p \in \mathbb{C}$ is the *localization of J at p*. Note that $TJ_p \subseteq J_p$ and $\dim J_p < \infty$; thus, J_p is a *local asymptotic type* in the sense of Witt [18].

We now investigate common properties of linear subspaces $J \subset \mathcal{E}_V(Y)$ satisfying (a) to (c) of Definition 2.17. Let $\Pi_j\colon J \to J^{\delta+j}$ be the canonical surjection. For $j' > j$, there is a natural surjective map $\Pi_{jj'}\colon J^{\delta+j'} \to J^{\delta+j}$ such that $\Pi_{jj''} = \Pi_{jj'}\Pi_{j'j''}$ for $j'' > j' > j$ and

$$\big(J, \Pi_j\big) = \operatorname*{proj\,lim}_{j\to\infty} \big(J^{\delta+j}, \Pi_{jj'}\big). \tag{2.18}$$

Note that $T\colon J^{\delta+j} \to J^{\delta+j}$ is nilpotent, where T denotes the map induced by $T\colon J \to J$. Furthermore, for $j' > j$, the diagram

$$\begin{tikzcd} J^{\delta+j'} \arrow[r, "\Pi_{jj'}"] \arrow[d, "T"'] & J^{\delta+j} \arrow[d, "T"] \\ J^{\delta+j'} \arrow[r, "\Pi_{jj'}"] & J^{\delta+j} \end{tikzcd} \tag{2.19}$$

commutes and the action of T on J is that one induced by (2.18), (2.19).

Proposition 2.19. *Let $J \subset \mathcal{E}_V(Y)$ be a linear subspace for some $V \in \mathcal{C}^\delta$. Then there is a sequence Φ_i for $i = 1, 2, \dots$ of special vectors with* c-ord$(\Phi_i) \to \infty$ *as $i \to \infty$ such that the vectors $T^k\Phi_i$ for $i = 1, 2, \dots,$ $k = 0, 1, 2\dots$ span J if and only if J fulfills conditions* (a), (b), *and* (c).

In the situation just described, we write $J = \langle \Phi_1, \Phi_2, \dots \rangle$.

Proof. Let $J \subset \mathcal{E}_V(Y)$ fulfill conditions (a) to (c). Due to (c) we may assume that $V = V_p$ for some $p \in \mathbb{C}$. Suppose that the special vectors $\Phi_1, \dots, \Phi_e \in J$ have already been chosen (where $e = 0$ is possible). Then we choose the vector Φ_{e+1} among the special vectors $\Phi \in J$ which do not belong to $\langle \Phi_1, \dots, \Phi_e \rangle$ such that $\operatorname{Re}\gamma(\Phi_{e+1})$ is minimal. We claim that $J = \langle \Phi_1, \Phi_2, \dots \rangle$. In fact, c-ord$(\Phi_i) = (n+1)/2 - \operatorname{Re}\gamma(\Phi_i) \to \infty$ as $i \to \infty$ and, if Φ is a special vector in J, then $\Phi \in \langle \Phi_1, \dots, \Phi_e \rangle$, where e is such that $\operatorname{Re}\gamma(\Phi_e) \le \operatorname{Re}\gamma(\Phi)$, while $\operatorname{Re}\gamma(\Phi_{e+1}) > \operatorname{Re}\gamma(\Phi)$. Otherwise, Φ_{e+1} would not have been chosen in the $(e+1)$th step.
The other direction is obvious. ☐

For $j \ge 1$, let $(m_1^j, \dots, m_{e_j}^j)$ denote the characteristic of the space $J^{\delta+j}$, see Remark 2.15

Proposition 2.20. *Let $J \subset \mathcal{E}_V(Y)$ be a linear subspace and assume that the special vectors Φ_i for $i = 1, 2, \dots, e$, where $e \in \mathbb{N} \cup \{\infty\}$, as constructed in* Proposition 2.19, *form a characteristic basis of J. Then the following conditions are equivalent:*
(a) *For each j, $\Pi_j\Phi_1, \dots, \Pi_j\Phi_{e_j}^j$ is an $(m_1^j, \dots, m_{e_j}^j)$-characteristic basis of $J^{\delta+j}$;*
(b) *For each j, $T^{m_1^j-1}\Phi_1, \dots, T^{m_{e_j}-1}\Phi_{e_j}$ are linearly independent over the space $\mathcal{E}^{\delta+j}(Y)$, while $T^k\Phi_i \in \mathcal{E}^{\delta+j}(Y)$ if either $1 \le i \le e_j$, $k \ge m_i^j$ or $i > e_j$.*
In particular, if (a), (b) *are fulfilled, then, for any $j' > j$, $\Pi_{jj'}\Phi_1^{j'}, \dots, \Pi_{jj'}\Phi_{e_j}^{j'}$ is a characteristic basis of $J^{\delta+j}$, while $\Pi_{jj'}\Phi_{e_j+1}^{j'} = \dots = \Pi_{jj'}\Phi_{e_j'}^{j'} = 0$. Here, $\Phi_i^{j'} = \Pi_{j'}\Phi_i$ for $1 \le i \le e_{j'}$.*

Proof. This is a consequence of Lemma 2.13 and Witt [18, Lemma 3.8]. ☐

Notice that, for a linear subspace $J \subset \mathcal{E}_V(Y)$ satisfying conditions (a) to (c) of Definition 2.17, a characteristic basis possessing the equivalent properties of Proposition 2.20 need not exist. We provide an example:

Example 2.21. Let the space $J = \langle \Phi_1, \Phi_2 \rangle \subset \mathcal{E}_{V_p}(Y)$ for some $p \in \mathbb{C}$, $\operatorname{Re} p < (n+1)/2 - \delta$, be spanned by two vectors Φ_1, Φ_2 in the sense of Proposition 2.19. We further assume that $\Phi_1(p) = (\psi_0, \star)$, $\Phi_1(p-1) = (\psi_1, \star, \star)$, $\Phi_2(p) = 0$, and $\Phi_2(p-1) = (\psi_1, \star)$, where ψ_0, $\psi_1 \in C^\infty(Y)$ are not identically zero and $\star$ stands for arbitrary entries, see Figure 2. Then, the asymptotic type represented by J is non-proper. In fact, assume that $\operatorname{Re} p \ge (n+1)/2 - \delta + 1$. Then

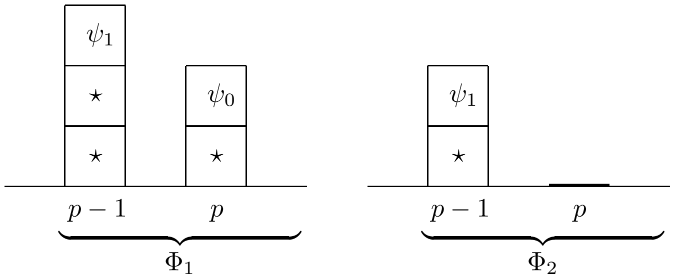

Figure 2: Example of a non-proper asymptotic type

$\Pi_2\Phi_1$, $T\Pi_2\Phi_1 - \Pi_2\Phi_2$ is a $(3,1)$-characteristic basis of $J^{\delta+2}$, and any other characteristic basis of $J^{\delta+2}$ is, up to a non-zero multiplicative constant, of the form

$$\left\{\begin{array}{l} \Pi_2\Phi_1 + \alpha_1 T\Pi_2\Phi_1 + \alpha_2 T^2\Pi_2\Phi_1 + \alpha_3\Pi_2\Phi_2, \\ \beta_1(T\Pi_2\Phi_1 - \Pi_2\Phi_2) + \beta_2 T^2\Pi_2\Phi_1, \end{array}\right. \tag{2.20}$$

where α_1, α_2, α_3, β_1, $\beta_2 \in \mathbb{C}$ and $\beta_1 \neq 0$. But then the conclusion in Proposition 2.20 is violated, since both vectors in (2.20) have non-zero image under the projection Π_{12}, while $\Pi_1\Phi_1$ is a (2)-characteristic basis of $J^{\delta+1}$.

DEFINITION 2.22. An asymptotic type $P \in \underline{\mathrm{As}}^\delta(Y)$ represented by the linear subspace $J \subset \mathcal{E}_V(Y)$ is called *proper* if J admits a characteristic basis Φ_1, $\Phi_2, \ldots$ satisfying the equivalent conditions in Proposition 2.20. The set of all proper asymptotic types is denoted by $\underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y) \subsetneq \underline{\mathrm{As}}^\delta(Y)$.

For $\Phi \in \mathcal{E}^\delta(Y)$, $p \in \mathbb{C}$, and $\Phi(p) = (\phi_0^{(p)}, \phi_1^{(p)}, \ldots, \phi_{m_p-1}^{(p)})$ we shall use, for any $q \in \mathbb{C}$, the notation

$$\Phi(p)[z-q] = \frac{\phi_0^{(p)}}{(z-q)^{m_p}} + \frac{\phi_1^{(p)}}{(z-q)^{m_p-1}} + \cdots + \frac{\phi_{m_p-1}^{(p)}}{z-q} \in \mathcal{M}_q(C^\infty(Y)),$$

where $\mathcal{M}_q(C^\infty(Y))$ is the space of germs of *meromorphic functions* at $z=q$ taking values in $C^\infty(Y)$. Analogously, $\mathcal{A}_q(C^\infty(Y))$ is the space of germs of *holomorphic functions* at $z=p$ taking values in $C^\infty(Y)$.

DEFINITION 2.23. For $\mathfrak{S}^\mu = \{\mathfrak{s}^{\mu-j}(z);\, j \in \mathbb{N}\} \in \mathrm{Symb}^\mu_M(Y)$, the linear space $L^\delta_{\mathfrak{S}^\mu} \subseteq C^{\infty,\delta}_{\mathrm{as}}(X)/C^\infty_{\mathcal{O}}(X)$ is represented by the space of $\Phi \in \mathcal{E}^\delta(Y)$ for which there are functions $\widetilde{\phi}^{(p)}(z) \in \mathcal{A}_p(C^\infty(Y))$ for $p \in \mathbb{C}$, $\operatorname{Re} p < (n+1)/2 - \delta$, such that

$$\sum_{j=0}^{[(n+1)/2-\delta+\mu-\operatorname{Re} q]^-} \mathfrak{s}^{\mu-j}(z-\mu+j)\Big(\Phi(q-\mu+j)[z-q] + \widetilde{\phi}^{(q-\mu+j)}(z-\mu+j)\Big) \in \mathcal{A}_q(C^\infty(Y)) \tag{2.21}$$

for all $q \in \mathbb{C}$, $\operatorname{Re} q < (n+1)/2 - \delta + \mu$. Here, $[a]^-$ for $a \in \mathbb{R}$ is the largest integer strictly less than a, i.e., $[a]^- \in \mathbb{Z}$ and $[a]^- < a \leq [a]^- + 1$.

Remark 2.24. (a) If $\Phi \in \mathcal{E}_V(Y)$ for $V \in \mathcal{C}^\delta$, then condition (2.21) is effective only if

$$q \in \bigcup_{j=0}^{[(n+1)/2-\delta+\mu-\operatorname{Re} q]^-} T^{\mu-j}V.$$

(b) If $\Phi \in \mathcal{E}^\delta(Y)$ belongs to the representing space of $L^\delta_{\mathfrak{S}^\mu}$, and if $u \in C^{\infty,\delta}_{\mathrm{as}}(X)$ possesses asymptotics given by the vector Φ according to (2.16), then there is a $v \in C^\infty_{\mathcal{O}}(X)$ such that

$$\sum_{j=0}^{\infty} \omega(c_j t) t^{-\mu+j} \operatorname{op}_M^{(n+1)/2-\delta} \left(\mathfrak{s}^{\mu-j}(z)\right) \tilde{\omega}(c_j t)\,(u+v) \in C^\infty_{\mathcal{O}}(X).$$

Here, the numbers $c_j > 0$ are chosen so that $c_j \to \infty$ as $j \to \infty$ sufficiently fast so that the infinite sum converges. For the notation $\operatorname{op}_M^{(n+1)/2-\delta}(\dots)$ see (2.35), below.

Definition 2.25. For $P \in \underline{\mathrm{As}}^\delta(Y)$ being represented by $J \subset \mathcal{E}_V(Y)$ and $\mathfrak{S}^\mu \in \operatorname{Symb}^\mu_M(Y)$, the *push-forward* $\mathcal{Q}^{\delta-\mu}(P;\mathfrak{S}^\mu)$ of P under $\mathfrak{S}^\mu$ is the asymptotic type in $\underline{\mathrm{As}}^{\delta-\mu}(Y)$ represented by the linear subspace $K \subset \mathcal{E}_{T^{-\mu}V}(Y)$ consisting of all vectors $\Psi \in \mathcal{E}_{T^{-\mu}V}(Y)$ such that there is a $\Phi \in J$ and there are functions $\widetilde{\phi}^{(p)}(z) \in \mathcal{A}_p(C^\infty(Y))$ for $p \in V$ such that

$$\Psi(q)[z-q] = \sum_{j=0}^{[(n+1)/2-\delta+\mu-\operatorname{Re} q]^-} \Big[\mathfrak{s}^{\mu-j}(z-\mu+j)\left(\Phi(q-\mu+j)[z-q] + \widetilde{\phi}^{(q-\mu+j)}(z-\mu+j)\right)\Big]^*_q, \tag{2.22}$$

holds for all $q \in T^\mu V$, see (2.6).

Remark 2.26. For a *holomorphic* $\mathfrak{S}^\mu \in \operatorname{Symb}^\mu_M(Y)$, one needs not to refer to the holomorphic functions $\widetilde{\phi}^{(p)}(z) \in \mathcal{A}_p(C^\infty(Y))$ for $p \in V$ in order to define the push-forward $\mathcal{Q}^{\delta-\mu}(P;\mathfrak{S}^\mu)$ in (2.22). We then also write $\mathcal{Q}(P;\mathfrak{S}^\mu)$ instead of $\mathcal{Q}^{\delta-\mu}(P;\mathfrak{S}^\mu)$.

Extending the notion of push-forward from asymptotic types to arbitrary linear subspaces of $C^{\infty,\delta}_{\mathrm{as}}(X)/C^\infty_{\mathcal{O}}(X)$, the space $L^\delta_{\mathfrak{S}^\mu} \subseteq C^{\infty,\delta}_{\mathrm{as}}(X)/C^\infty_{\mathcal{O}}(X)$ for $\mathfrak{S}^\mu \in \operatorname{Symb}^\mu_M(Y)$ appears as the largest subspace of $C^{\infty,\delta}_{\mathrm{as}}(X)/C^\infty_{\mathcal{O}}(X)$ for which

$$\mathcal{Q}^{\delta-\mu}(L^\delta_{\mathfrak{S}^\mu};\mathfrak{S}^\mu) = \mathcal{Q}^{\delta-\mu}(\mathcal{O};\mathfrak{S}^\mu). \tag{2.23}$$

In this sense, it characterizes the *amount of asymptotics* of conormal order at least δ *annihilated by* $\mathfrak{S}^\mu \in \operatorname{Symb}^\mu_M(Y)$.

Definition 2.27. A *partial ordering* on $\underline{\mathrm{As}}^\delta(Y)$ is defined by $P \preccurlyeq P'$ for $P,\, P' \in \underline{\mathrm{As}}^\delta(Y)$ if and only if $J \subseteq J'$, where $J,\, J' \subset \mathcal{E}^\delta(Y)$ are the representing spaces for P and P', respectively.

Proposition 2.28. (a) *The p.o. set* $(\underline{\mathrm{As}}^\delta(Y), \preccurlyeq)$ *is a lattice in which each non-empty subset* $\mathcal{S}$ *admits a meet,* $\bigwedge \mathcal{S}$*, represented by* $\bigcap_{P\in\mathcal{S}} J_P$*, and each bounded subset* $\mathcal{T}$ *admits a join,* $\bigvee \mathcal{T}$*, represented by* $\sum_{Q\in\mathcal{T}} J_Q$*, where* J_P *and* J_Q *represent the asymptotic types* P *and* Q*, respectively. In particular,* $\bigwedge \underline{\mathrm{As}}^\delta(Y) = \mathcal{O}$.
(b) *For* $P \in \underline{\mathrm{As}}^\delta(Y)$*,* $\mathfrak{S}^\mu \in \mathrm{Symb}^\mu_M(Y)$*, we have* $\mathcal{Q}^{\delta-\mu}(P;\mathfrak{S}^\mu) \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$.

Proof. (a) is immediate from the definition of asymptotic type and (b) can be checked directly on the level of (2.22). □

Remark 2.29. Each element $\mathfrak{S}^\mu \in \mathrm{Symb}^\mu_M(Y)$ induces a natural action $C^{\infty,\delta}_{\mathrm{as}}(X) \to C^{\infty,\delta}_{\mathrm{as}}(X)/C^\infty_{\mathcal{O}}(X)$. Its expression in the splitting of coordinates $\mathcal{U} \to [0,1) \times Y$, $x \mapsto (t,y)$, is given by (2.22).
In the language of Witt [20], this means that the quadruple $\big(\bigcup_{\mu\in\mathbb{Z}} \mathrm{Symb}^\mu_M(Y), C^{\infty,\delta}_{\mathrm{as}}(X), C^\infty_{\mathcal{O}}(X), \underline{\mathrm{As}}^\delta(Y)\big)$ is an *asymptotic algebra* that is even *reduced*; thus providing justification for the above choice of the notion of asymptotic type.

Theorem 2.30. *For a holomorphic* $\mathfrak{S}^\mu \in \mathrm{Ell}\,\mathrm{Symb}^\mu_M(Y)$*, we have* $L^\delta_{\mathfrak{S}^\mu} \in \underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$.

Proof. Let $\mathfrak{S}^\mu = \{\mathfrak{s}^{\mu-j}(z);\, j \in \mathbb{N}\} \subset \mathcal{M}^\mu_{\mathcal{O}}(Y)$. Assume that, for some $p \in \mathbb{C}$, $\mathrm{Re}\, p < (n+1)/2 - \delta$, $\Phi_0 \in L_{\mathfrak{s}^\mu(z)}$ at $z = p$, with the obvious meaning, for this see Witt [18]. (Notice that $L_{\mathfrak{s}^\mu(z)}$ at $z = p$ is contained in the space $[C^\infty(Y)]^\infty$.) We then successively calculate the sequence $\Phi_0, \Phi_1, \Phi_2, \ldots$ from the relations, at $z = p$,

$$\mathfrak{s}^\mu(z-j)\Phi_j[z-p] + \mathfrak{s}^{\mu-1}(z-j+1)\Phi_{j-1}[z-p] \\ + \cdots + \mathfrak{s}^{\mu-j}(z)\Phi_0[z-p] \in \mathcal{A}_p(C^\infty(Y)),\;\; j = 0,1,2,\ldots, \tag{2.24}$$

see (2.22) and Remark 2.26. In each step, we find $\Phi_j \in [C^\infty(Y)]^\infty$ uniquely determined modulo $L_{\mathfrak{s}^\mu(z)}$ at $z = p - j$ such that (2.24) holds. We obtain the vector $\Phi \in \mathcal{E}_{V_p}(Y)$ define by $\Phi(p-j) = \Phi_j$ that belongs to the linear subspace $J \subset \mathcal{E}^\delta(Y)$ representing $L^\delta_{\mathfrak{S}^\mu}$.
Conversely, each vector in J is a sum like in (2.11) of vectors Φ obtained in that way. Thus, upon choosing in each space $L_{\mathfrak{s}^\mu(z)}$ at $z = p$ a characteristic basis and then, for each characteristic basis vector $\Phi_0 \in [C^\infty(Y)]^\infty$, exactly one vector $\Phi \in \mathcal{E}_{V_p}(Y)$ as just constructed, we obtain a characteristic basis of J in the sense of Definition 2.14 consisting completely of special vectors (since $L_{\mathfrak{s}^\mu(z)}$ at $z = p$ equals zero for all $p \in \mathbb{C}$, $\mathrm{Re}\, p < (n+1)/2 - \delta$, but a set of p belonging to $\mathcal{C}^\delta$). In particular, $J \subset \mathcal{E}_V(Y)$ for some $V \in \mathcal{C}^\delta$ and (a) to (c) of Definition 2.17 are satisfied. By its very construction, this characteristic basis fulfills condition (b) of Proposition 2.20. Therefore, the asymptotic type $L^\delta_{\mathfrak{S}^\mu}$ represented by J is proper. □

In conclusion, we obtain:

Proposition 2.31. *Let* $\mathfrak{S}^\mu \in \operatorname{Ell}\operatorname{Symb}^\mu_M(Y)$. *Then:*
(a) $L^\delta_{\mathfrak{S}^\mu} = \mathcal{Q}^\delta(\mathcal{O}; (\mathfrak{S}^\mu)^{-1})$ *and* $L^{\delta-\mu}_{(\mathfrak{S}^\mu)^{-1}} = \mathcal{Q}^{\delta-\mu}(\mathcal{O}; \mathfrak{S}^\mu)$.
(b) *There is an order-preserving bijection*

$$\left\{P \in \underline{\mathrm{As}}^\delta(Y);\, P \succcurlyeq L^\delta_{\mathfrak{S}^\mu}\right\} \to \left\{Q \in \underline{\mathrm{As}}^{\delta-\mu}(Y);\, Q \succcurlyeq L^{\delta-\mu}_{(\mathfrak{S}^\mu)^{-1}}\right\}, \qquad (2.25)$$
$$P \mapsto \mathcal{Q}^{\delta-\mu}(P; \mathfrak{S}^\mu),$$

with the inverse given by $Q \mapsto \mathcal{Q}^\delta(Q; (\mathfrak{S}^\mu)^{-1})$.

Proof. Using Proposition 2.28 (b), the proof consists of a word-by-word repetition of the arguments given in the proof of Witt [18, Proposition 2.5]. □

In its consequence, Proposition 2.31 enables one to perform explicit calculations on asymptotic types.
We conclude this section with the following basic observation:

Proposition 2.32. *The notion of asymptotic type, as introduced above, is invariant under coordinates changes.*

Proof. Let $\kappa\colon X \to X$ be a C^∞ diffeomorphism and let $\kappa_*\colon C^\infty(X^\circ) \to C^\infty(X^\circ)$ be the corresponding push-forward on the level of functions, i.e., $(\kappa_* u)(x) = u(\kappa^{-1}(x))$ for $u \in C^\infty(X^\circ)$, where κ^{-1} denotes the inverse C^∞ diffeomorphism to κ. As is well-known, κ_* restricts to $\kappa_*\colon C^{\infty,\delta}_{\mathrm{as}}(X) \to C^{\infty,\delta}_{\mathrm{as}}(X)$ for any $\delta \in \mathbb{R}$, see, e.g., Schulze [15, Theorem 1.2.1.11].
We have to prove that, for each $P \in \underline{\mathrm{As}}^\delta(Y)$, there is a $\kappa_* P \in \underline{\mathrm{As}}^\delta(Y)$ so that the push-forward κ_* restricts further to a linear isomorphism $\kappa_*\colon C^\infty_P(X) \to C^\infty_{\kappa_* P}(X)$, i.e., we have to show that there is a $\kappa_* P \in \underline{\mathrm{As}}^\delta(Y)$ so that $\kappa_*(C^\infty_P(X)) = C^\infty_{\kappa_* P}(X)$. Using Proposition 2.19, we eventually have to prove that, for each $u \in C^{\infty,\delta}_{\mathrm{as}}(X)$ such that

$$u(x) \sim \sum_{j=0}^\infty \sum_{k+l=m_j-1} \frac{(-1)^k}{k!} \log^k t\, \phi^{(j)}_l(y) \text{ as } t \to +0, \qquad (2.26)$$

where $\Phi \in \mathcal{E}_{V_p}(Y)$ for a certain $p \in \mathbb{C}$, $\operatorname{Re} p < (n+1)/2 - \delta$, and $\Phi(p-j) = (\phi^{(j)}_0, \phi^{(j)}_1, \dots, \phi^{(j)}_{m_j-1})$ for all $j \in \mathbb{N}$, see (2.16), the push-forward $\kappa_* u$ is again of the form (2.26), with some other $\kappa_*\Phi \in \mathcal{E}_{V_p}(Y)$ in place of $\Phi \in \mathcal{E}_{V_p}(Y)$.
But this is immediate from a direct computation. □

2.2.3 Characteristics of proper asymptotic types

We introduce the notion of characteristic of a proper asymptotic type. This will be the main ingredient in the prove of Theorem 2.42.
Let $P \in \underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$ be represented by $J \subset \mathcal{E}_V(Y)$ and let $\Phi_1, \Phi_2, \dots$ by a characteristic basis of J according to Definition 2.22. As before, let $(m^j_1, \dots, m^j_{e_j})$

be the characteristic of the space $J^{\delta+j}$. From Proposition 2.20, we conclude that $e_1 \le e_2 \le \dots$ In the next lemma, we find a suitable "path through" the numbers m_i^j for $j \ge j_i$, where $j_i = \min\{j;\, e_j \ge i\}$, i.e., an appropriate re-ordering of the tuples $(m_1^j, \dots, m_{e_j}^j)$.

Lemma 2.33. *The numbering within the tuples $(m_1^j, \dots, m_{e_j}^j)$ can be chosen in such a way that, for each $j \ge 1$, there is a characteristic $(m_1^j, \dots, m_{e_j}^j)$-basis $(\Phi_1^j, \dots, \Phi_{e_j}^j)$ of $J^{\delta+j}$ such that, for all $j' > j$,*

$$\Pi_{jj'}\Phi_i^{j'} = \begin{cases} \Phi_i^j & \text{if } 1 \le i \le e_j, \\ 0 & \text{if } e_j + 1 \le i \le e_{j'} \end{cases}$$

holds.
Furthermore, the scheme

$$\begin{array}{ll} e_1 \text{ rows} \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & \begin{array}{ccccc} m_1^1 & m_1^2 & m_1^3 & m_1^4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{e_1}^1 & m_{e_1}^2 & m_{e_1}^3 & m_{e_1}^4 & \cdots \end{array} \\ e_2 - e_1 \text{ rows} \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & \begin{array}{ccccc} & m_{e_1+1}^2 & m_{e_1+1}^3 & m_{e_1+1}^4 & \cdots \\ & \cdots & \cdots & \cdots & \cdots \\ & m_{e_2}^2 & m_{e_2}^3 & m_{e_2}^4 & \cdots \end{array} \\ e_3 - e_2 \text{ rows} \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & \begin{array}{ccccc} & & m_{e_2+1}^3 & m_{e_2+1}^4 & \cdots \\ & & \cdots & \cdots & \cdots \\ & & m_{e_3}^3 & m_{e_3}^4 & \cdots \end{array} \\ \left\{ \begin{array}{c} \\ \end{array} \right. & \begin{array}{ccccc} & & & \vdots & \ddots, \end{array} \end{array} \tag{2.27}$$

where in the jth column the characteristic of the space $J^{\delta+j}$ appears, is uniquely determined up to permutation of the kth and the k'th row, where $e_j + 1 \le k,\ k' \le e_{j+1}$ for some j $(e_0 = 0)$.

Proof. This is a reformulation of Proposition 2.20 in terms of the characteristics of the spaces $J^{\delta+j}$. Notice that one can recover the characteristic basis Φ_1, $\Phi_2, \dots$ of J, that was initially given, from the property that $\Pi_j\Phi_i = \Phi_i^j$ holds for all $1 \le i \le e_j$, while $\Pi_j\Phi_i = 0$ for $i > e_j$. $\square$

Performing the constructions of the foregoing lemma for each space $J \cap \mathcal{E}_{V_{p_j}}(Y)$ in (2.15) separately, one sees that the following notion is correctly defined:

Definition 2.34. Let $P \in \underline{\mathrm{As}}^{\delta}_{\mathrm{prop}}(Y)$ and $J \subset \mathcal{E}_V(Y)$ represent P. If $\Phi_1, \Phi_2, \dots$ is a characteristic basis of J according to Definition 2.22 and if the tuples $(m_1^j, \dots, m_{e_j}^j)$ are re-ordered according to Lemma 2.33, then the sequence

$$\operatorname{char} P = \left\{ \left(\gamma(\Phi_i) \mid m_i^{j_i}, m_i^{j_i+1}, m_i^{j_i+2}, \dots \right) \right\}_{i=1}^{e} \tag{2.28}$$

is called the *characteristic* of P.

The characteristic $\operatorname{char} P$ of an asymptotic type $P \in \underline{\mathrm{As}}^{\delta}_{\mathrm{prop}}(Y)$ is unique up to permutation of the kth and the k'th entry, where $e_j + 1 \le k,\, k' \le e_{j+1}$ for some j. So far, it is an invariant associated with the representing space J; so it still depends on the splitting of coordinates. However, we have:

Proposition 2.35. *The characteristic* $\operatorname{char} P$ *of an asymptotic type* $P \in \underline{\mathrm{As}}^{\delta}_{\mathrm{prop}}(Y)$ *is independent of the chosen splitting of coordinates* $\mathcal{U} \to [0,1) \times Y$, $x \mapsto (t,y)$, *near* ∂X.

Proof. Follow the proof of Proposition 2.32 to get the assertion. $\square$

Now, let $\left\{(p_i \,|\, m_i^{j_i}, m_i^{j_i+1}, \dots)\right\}_{i=1}^{e} \subset \mathbb{C} \times \mathbb{N}^{\mathbb{N}}$ be any given sequence, where we additionally assume that $\operatorname{Re} p_i < (n+1)/2 - \delta$ for all i, $\operatorname{Re} p_i \to -\infty$ as $i \to \infty$ when $e = \infty$, the p_i are ordered so that $\operatorname{Re} p_i \ge (n+1)/2 - \delta - j$ holds if and only if $i \le e_j$ for a certain (then uniquely determined) sequence $e_1 \le e_2 \le \dots$ satisfying $e = \sup_j e_j$, and

$$1 \le m_i^{j_i} \le m_i^{j_i+1} \le m_i^{j_i+2} \le \dots,$$

where $j_i = \min\{j;\, e_j \ge i\}$ as above.

Proposition 2.36. *Let the characteristic* $\left\{\left(p_i \,\middle|\, m_i^{j_i}, m_i^{j_i+1}, \dots\right)\right\}_{i=1}^{e}$ *satisfy the properties just mentioned. If* $n = 0$, *then we assume, in addition, that* $p_i \ne p_{i'}$ *for* $i \ne i'$ *and, for all* i, $k > 0$,

$$m_i^{j_i+k} - m_i^{j_i+k-1} = a > 0 \iff p_{i'} = p_i - k \text{ for some } i' \text{ and } m_{i'}^{j_{i'}} = a$$

(*where* $j_{i'} = j_i + k$). *Then there exists a holomorphic* $\mathfrak{S}^{\mu} \in \operatorname{Symb}^{\mu}_{M}(Y)$ *that is elliptic with respect to the weight* $\delta \in \mathbb{R}$ *such that* $L^{\delta}_{\mathfrak{S}^{\mu}} \in \underline{\mathrm{As}}^{\delta}_{\mathrm{prop}}(Y)$ *has exactly this characteristic.*

Proof. Multiplying $\mathfrak{S}^{\mu}$ by an elliptic element $\mathfrak{T}^{-\mu} = \{\mathfrak{t}^{-\mu}(z), 0, 0, \dots\}$ such that $\mathfrak{t}^{-\mu}(z) \in \mathcal{M}^{-\mu}_{\mathcal{O}}$ and $\mathfrak{t}^{-\mu}(z)^{-1} \in \mathcal{M}^{\mu}_{\mathcal{O}}$, we can assume $\mu = 0$.
If $n = 0$, then we choose an elliptic $\mathfrak{s}^{0}(z) \in \mathcal{M}^{0}_{\mathcal{O}}$ that has zeros precisely at $z = p_i$ of order $m_i^{j_i}$ for $i = 1, 2, \dots$ according to Theorem 2.6.
In case $\dim Y > 0$, let $\{\phi_i\}_{i=1}^{e}$ be an orthonormal set in $C^{\infty}(Y)$ with respect to a fixed C^{∞}-density $d\mu$ on Y. Let Π_i for $i = 1, \dots, e$ be the orthogonal projection in $L^2(Y, d\mu)$ onto the subspace spanned by ϕ_i. We then choose an elliptic $\mathfrak{s}^{\mu}(z) \in \mathcal{M}^{\mu}_{\mathcal{O}}(Y)$ such that, for every $p \in V_{p_i}$ and all i,

$$[\mathfrak{s}^{\mu}(z)]_p^{N_p} = \Big(1 - \sum_{p_{i'} - k = p} \Pi_{i'}\Big) + \sum_{p_{i'} - k = p} (z - p)^{m_{i'}^{j_{i'}+k}} \Pi_{i'}$$

where the sums are extended over all i', k such that $p_{i'} - k = p$, for some N_p sufficiently large, while $\mathfrak{s}^{\mu}(q) \in L^{\mu}_{\mathrm{cl}}(Y)$ is invertible for all $q \in \mathbb{C} \setminus V$, again according to Theorem 2.6.
In both cases, we set $\mathfrak{S}^{\mu} = \{\mathfrak{s}^{\mu-j}(z)\}_{j=0}^{\infty}$ with $\mathfrak{s}^{\mu-j}(z) \equiv 0$ for $j > 0$. Then $\mathfrak{S}^{\mu} \in \operatorname{Symb}^{\mu}_{M}(Y)$ is elliptic with respect to the weight δ, and the proper asymptotic type $L^{\delta}_{\mathfrak{S}^{\mu}}$ has characteristic $\left\{(p_i \,|\, m_i^{j_i}, m_i^{j_i+1}, \dots)\right\}_{i=1}^{e}$. $\square$

2.2.4 More properties of asymptotic types

Here, we study further properties of asymptotic types. First, asymptotic types are composed of elementary building blocks:

Proposition 2.37. (a) *An asymptotic type $P \in \underline{\mathrm{As}}^\delta(Y)$ is join-irreducible, i.e., $P \neq \mathcal{O}$ and $P = P_0 \vee P_1$ for $P_0, P_1 \in \underline{\mathrm{As}}^\delta(Y)$ implies $P = P_0$ or $P = P_1$, if and only if there is a $\Phi \in \mathcal{E}^\delta(Y)$, $\Phi \neq 0$, such that the representing space, J, for P, in the given splitting of coordinates near ∂X, has characteristic basis Φ, i.e., $J = \langle \Phi \rangle$. In particular, every join-irreducible asymptotic type is proper.*
(b) *The join-irreducible asymptotic types are join-dense in $\underline{\mathrm{As}}^\delta(Y)$.*

Proof. (a) Let $P \neq \mathcal{O}$. Assume that, for some $j \geq 1$, $J^{\delta+j}$ has characteristic of length larger 1. Then $J^{\delta+j} = K_0 + K_1$ for certain linear subspaces $K_i \subsetneq J^{\delta+j}$ satisfying $TK_i \subseteq K_i$, for $i = 0, 1$. Setting $J_i = \{\Phi \in J; \Pi_j \Phi \in K_i\}$, we get that $J = J_0 + J_1$, $J_i \subsetneq J$, and $TJ_i \subseteq J_i$ for $i = 0, 1$. Since this decomposition can be chosen compatible with (2.15), we obtain that a necessary condition for P to be join-irreducible is that each space $J^{\delta+j}$ for $j \geq 1$ has characteristic of length at most 1, i.e., $J = \langle \Phi \rangle$ for some $\Phi \neq 0$. Vice versa, if $J = \langle \Phi \rangle$ for some $\Phi \neq 0$, then P is join-irreducible, since the subspace $\langle T^k \Phi \rangle \subseteq J$ for $k \in \mathbb{N}$ are the only subspaces of J that are invariant under the action of T.
(b) This follows directly from Proposition 2.19. □

Note that, by the foregoing proposition, also the proper asymptotic types are join-dense in $\underline{\mathrm{As}}^\delta(Y)$. We will utilize this fact in the definition of cone Sobolev spaces with asymptotics.
In constructing asymptotic types $P \in \underline{\mathrm{As}}^\delta(Y)$ obeying certain properties, one often encounters a situation in which P is successively constructed on strips $\{z \in \mathbb{C}; (n+1)/2 - \delta - \beta_h \leq \operatorname{Re} z < (n+1)/2 - \delta\}$ of finite width, where the sequence $\{\beta_h\}_{h=0}^\infty \subset \mathbb{R}_+$ is strictly increasing and $\beta_h \to \infty$ as $h \to \infty$. We will meet an example in Section 3.3.
To formulate the result, we need one more definition:

Definition 2.38. Let $P, P' \in \underline{\mathrm{As}}^\delta(Y)$ be represented by $J \subset \mathcal{E}_V(Y)$ and $J' \subset \mathcal{E}_V(Y)$, respectively. Then, for $\vartheta \geq 0$, the asymptotic types P and P' are said to be *equal up to the conormal order $\delta + \vartheta$* if $\Pi_\vartheta J = \Pi_\vartheta J'$, where $\Pi_\vartheta \colon J \to J\big/(J \cap \mathcal{E}^{\delta+\vartheta}(Y))$ is the canonical projection. Similarly, P and P' are said to be *equal up to the conormal order $\delta + \vartheta - 0$* if they are equal up to the conormal order $\delta + \vartheta - \epsilon$, for any $\epsilon > 0$. (Similarly for the order relation $\preccurlyeq$ instead of equality.)

Proposition 2.39. *Let $\{P_\iota\}_{\iota \in \mathcal{I}} \subset \underline{\mathrm{As}}^\delta(Y)$ be an increasing net of asymptotic types. Then the join $\bigvee_{\iota \in \mathcal{I}} P_\iota$ exists if and only if, for each $j \geq 1$, there is an $\iota_j \in \mathcal{I}$ such that $P_\iota = P_{\iota'}$ up to the conormal order $\delta + j$ for all $\iota, \iota' \geq \iota_j$.*

Proof. The condition is obviously sufficient.
Conversely, suppose that the join $\bigvee_{\iota \in \mathcal{I}} P_\iota$ exists. Let P_ι be represented by the subspace $J_\iota \subset \mathcal{E}_{V_\iota}(Y)$ for $V_\iota \in \mathcal{C}^\delta$. Since the join $\bigvee_{\iota \in \mathcal{I}} P_\iota$ exists, the carriers V_ι

can be chosen in such way that $\bigcup_{\iota\in\mathcal{I}} V_\iota \subseteq V$ for some $V \in \mathcal{C}^\delta$. Thus $J_\iota \subset \mathcal{E}_V(Y)$ for all ι. Now, for each $j \geq 1$, $\dim\bigl(\sum_{\iota\in\mathcal{I}} J_\iota^{\delta+j}\bigr) < \infty$, otherwise $\bigvee_{\iota\in\mathcal{I}} P_\iota$ does not exist. But since the net $\{J_\iota^{\delta+j}\}_{\iota\in\mathcal{I}}$ is increasing, this already implies that there is some $\iota_j \in \mathcal{I}$ such that $J_\iota^{\delta+j} = J_{\iota'}^{\delta+j}$ for ι, $\iota' \geq \iota_j$, i.e., $P_\iota = P_{\iota'}$ up to the conormal order $\delta + j$ for ι, $\iota' \geq \iota_j$. □

An equivalent condition is that the net $\{P_\iota\}_{\iota\in\mathcal{I}} \subset \underline{\mathrm{As}}^\delta(Y)$ of asymptotic types be bounded on each strip $\{z \in \mathbb{C}; (n+1)/2 - \delta - j \leq \operatorname{Re} z < (n+1)/2 - \delta\}$ of finite width.

2.3 Pseudodifferential theory

Here, we establish an analogue of Witt [18, Theorem 1.2]. We need:

Proposition 2.40. *Let P, $P_0 \in \underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$, $Q \in \underline{\mathrm{As}}^{\delta-\mu}_{\mathrm{prop}}(Y)$ for $\mu \in \mathbb{R}$. Assume that $P \wedge P_0 = \mathcal{O}$. Then there is a holomorphic $\mathfrak{S}^\mu \in \operatorname{Ell}\operatorname{Symb}^\mu_M(Y)$ that is elliptic with respect to the weight δ such that $L^\delta_{\mathfrak{S}^\mu} = P_0$ and $\mathcal{Q}(P;\mathfrak{S}^\mu) = Q$ if and only if P and Q have the same characteristic shifted by μ, i.e., we have* $\operatorname{char} P = \operatorname{char} Q - \mu$ (*with the obvious meaning of* $\operatorname{char} Q - \mu$).

Proof. It is readily seen that $P \in \underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$, $Q \in \underline{\mathrm{As}}^{\delta-\mu}_{\mathrm{prop}}(Y)$ have the same characteristic shifted by μ if there is a holomorphic $\mathfrak{S}^\mu \in \operatorname{Ell}\operatorname{Symb}^\mu_M(Y)$ such that $\mathcal{Q}(P;\mathfrak{S}^\mu) = Q$.
Suppose that $\operatorname{char} P = \operatorname{char} Q - \mu$. First, we deal with the case $P_0 = \mathcal{O}$. Let the asymptotic types P, Q be represented by $J \subset \mathcal{E}_V(Y)$ and $K \subset \mathcal{E}_{T^\mu V}(Y)$, respectively. Let $\{\Phi_i\}_{i=1}^e$ and $\{\Psi_i\}_{i=1}^e$ be characteristic bases of J and K corresponding to $\operatorname{char} P$ and $\operatorname{char} Q$, respectively.
We have to choose the sequence $\{\mathfrak{s}^{\mu-k}(z);\, k \in \mathbb{N}\} \subset \mathcal{M}^\mu_{\mathcal{O}}(Y)$. By Theorem 2.6, it suffices to construct the finite parts $[\mathfrak{s}^{\mu-k}(z)]_{p'}^{N_{p'k}}$ for $p' \in V$, $k \in \mathbb{N}$, and $N_{p'k}$ sufficiently large appropriately. Thereby, we can assume that $V = V_p$ for some $p \in \mathbb{C}$, $\operatorname{Re} p < (n+1)/2 - \delta$.
Let $e_1 \leq e_2 \leq \dots$, where $e = \sup_{j\in\mathbb{N}} e_j$, be such that $\gamma(\Phi_i) = \gamma(\Psi_i) - \mu = p - j$ for $e_{j-1} + 1 \leq i \leq e_j$ (and $e_0 = 0$). Then the finite parts $[\mathfrak{s}^{\mu-k}(z)]_{p-j}^{m^{j+k}}$ for all j, k must be chosen so that, for each $j \in \mathbb{N}$,

$$\begin{aligned}\Phi_i(p-j)^{[\mathfrak{s}^\mu(z)]_{p-j}^{m^j}} + \Phi_i(p-j+1)^{[\mathfrak{s}^{\mu-1}(z)]_{p-j+1}^{m^j}}&\\ + \dots + \Phi_i(p)^{[\mathfrak{s}^{\mu-j}(z)]_p^{m^j}} &= \Psi_i(p+\mu-j) \qquad (2.29)\end{aligned}$$

for $1 \leq i \leq e_j$, where $m^j = \sup_{1\leq i\leq e_j} m_i^j$, and $\Phi_i(p-k) = 0$ if $e_k + 1 \leq i \leq e_j$. Here, $(m_1^j, \dots, m_{e_j}^j)$ is the characteristic of $J^{\delta+j}$ and, for $\Phi = (\phi_0, \dots, \phi_{m-1})$, $\Psi = (\psi_0, \dots, \psi_{m-1}) \in [C^\infty(Y)]^\infty$, and $\mathfrak{s}(z) \in \mathcal{M}^\mu_{\mathcal{O}}(Y)$, the relation

$$\Phi^{[\mathfrak{s}(z)]_p^m} = \Psi$$

stands for the linear system

$$\begin{aligned} \mathfrak{s}(p)\phi_0 &= \psi_0, \\ \mathfrak{s}(p)\phi_1 + \frac{\mathfrak{s}'(p)}{1!}\,\phi_0 &= \psi_1, \\ &\vdots \\ \mathfrak{s}(p)\phi_{m-1} + \frac{\mathfrak{s}'(p)}{1!}\,\phi_{m-2} + \cdots + \frac{\mathfrak{s}^{(m-1)}(p)}{(m-1)!}\,\phi_0 &= \psi_{m-1}. \end{aligned}$$

System (2.29) can successively be solved for $[\mathfrak{s}^{\mu}(z-k)]^{m^j}_{p-j+k}$ for $j = 0, 1, 2, \dots$ and $0 \le k \le j$. In fact, this can be done by choosing $[\mathfrak{s}^{\mu-k}(z)]^{m^j}_{p-j+k}$ for $k > 0$ arbitrarily. In particular, we may choose $\mathfrak{s}^{\mu-k}(z) \equiv 0$ for $k > 0$.
The case $P_0 \neq \mathcal{O}$ can be reduced to the case $P_0 = \mathcal{O}$ as in the proof of Witt [18, Lemma 3.16], since the three rules from Witt [18, Lemma 2.3] applied there continues to hold in the present situation. □

Remark 2.41. (a) The proof of Proposition 2.40 shows that the holomorphic $\mathfrak{S}^{\mu} = \{\mathfrak{s}^{\mu-j};\, j \in \mathbb{N}\} \in \operatorname{Ell}\operatorname{Symb}^{\mu}_{M}(Y)$ satisfying $L^{\delta}_{\mathfrak{S}^{\mu}} = P_0$ and $\mathcal{Q}(P; \mathfrak{S}^{\mu}) = Q$ can always be chosen so that $\mathfrak{s}^{\mu-j}(z) \equiv 0$ for $j > 0$.
(b) Proposition 2.40 in connection with Theorem 2.30 also shows that $\underline{\mathrm{As}}^{\delta}_{\mathrm{prop}}(Y)$ consists precisely of those asymptotic types that are of the form $L^{\delta}_{\mathfrak{S}^{\mu}}$ for some holomorphic $\mathfrak{S}^{\mu} \in \operatorname{Ell}\operatorname{Symb}^{\mu}_{M}(Y)$ that is elliptic with respect to the weight δ. (Choose $P = Q = \mathcal{O}$ in Proposition 2.40.)

Now, we reach the final aim of this section:

Theorem 2.42. *Let $P \in \underline{\mathrm{As}}^{\delta}_{\mathrm{prop}}(Y)$ and $Q \in \underline{\mathrm{As}}^{\delta-\mu}_{\mathrm{prop}}(Y)$. Then there exists a $\mathfrak{S}^{\mu} \in \operatorname{Symb}^{\mu}_{M}(Y)$ that is elliptic with respect to the weight δ such that $L^{\delta}_{\mathfrak{S}^{\mu}} = P$ and $L^{\delta-\mu}_{(\mathfrak{S}^{\mu})^{-1}} = Q$ always when $\dim Y > 0$ and if and only if $P \wedge T^{-\mu}Q = \mathcal{O}$ when $\dim Y = 0$.*

Proof. The condition $P \wedge T^{-\mu}Q = \mathcal{O}$ is obviously necessary if $\dim Y = 0$.
In the general case, choose $P_1 \in \underline{\mathrm{As}}^{\delta}_{\mathrm{prop}}(Y)$, $Q_1 \in \underline{\mathrm{As}}^{\delta-\mu}_{\mathrm{prop}}(Y)$ having the same characteristics as P and Q, respectively, such that $P_1 \wedge T^{-\mu}Q_1 = \mathcal{O}$. As in the proof of Witt [18, Theorem 1.2], it then suffices to construct holomorphic $\mathfrak{S}^0 \in \operatorname{Ell}\operatorname{Symb}^{\mu}_{M}(Y)$, $\mathfrak{T}^0 \in \operatorname{Ell}\operatorname{Symb}^{-\mu}_{M}(Y)$ that are elliptic with respect to the weight δ such that

$$L^{\delta}_{\mathfrak{S}^0} = P_1, \;\; \mathcal{Q}^{\delta}(Q_1; \mathfrak{S}^0) = Q, \;\; L^{\delta}_{\mathfrak{T}^0} = Q_1, \;\; \mathcal{Q}^{\delta}(P_1; \mathfrak{T}^0) = P.$$

This is achieved by using Proposition 2.40. □

2.4 Function spaces with asymptotics

The definition of cone Sobolev spaces with asymptotics is based on the Mellin transformation. See Schulze [15, Sections 1.2, 2.1] for this idea and also Remark 2.45. For more details on the Mellin transformation, see Jeanquartier [5].

2.4.1 Weighted cone Sobolev spaces

Let $Mu(z) = \tilde{u}(z) = \int_0^\infty t^{z-1} u(t)\, dt$, $z \in \mathbb{C}$, be the *Mellin transformation*, first defined for $u \in C_0^\infty(\mathbb{R}_+)$ and then extended to larger distribution classes. In particular, u will be allowed to be vector-valued. Recall the following properties of M:

$$M_{t\to z}\big\{(-t\partial_t - p)u\big\}(z) = (z-p)\tilde{u}(z),$$
$$M_{t\to z}\big\{t^{-p}u\big\}(z) = \tilde{u}(z-p), \qquad p \in \mathbb{C},$$

whenever both sides are defined, $M\colon L^2(\mathbb{R}_+) \to L^2(\Gamma_{1/2}; (2\pi i)^{-1} dz)$ is an isometry, and

$$M_{t\to z}\left\{\frac{(-1)^k}{k!}\, t^{-p} \log^k t\, \chi_{(0,1)}(t)\right\}(z) = \frac{1}{(z-p)^{k+1}}, \tag{2.30}$$

where $\chi_{(0,1)}$ is the characteristic function of the interval $(0,1)$. We infer that $h(z) = M_{t\to z}\left\{(-1)^k \omega(t) t^{-p} \log^k t/k!\right\}(z) \in \mathcal{M}_{\mathrm{as}}^{-\infty}$ is a meromorphic function of z having a pole precisely at $z = p$, and the principal part of the Laurent expansion around this pole is given by the right-hand side of (2.30), i.e., $[h(z)]_p^* = (z-p)^{-(k+1)}$. Here, $\omega(t)$ is a cut-off function near $t = 0$.
For s, $\delta \in \mathbb{R}$, let $\mathcal{H}^{s,\delta}(X)$ denote the space of $u \in H^s_{\mathrm{loc}}(X^\circ)$ such that $M_{t\to z}\{\omega u\}(z) \in L^2_{\mathrm{loc}}\big(\Gamma_{(n+1)/2-\delta}; H^s(Y)\big)$ and the expression

$$\|u\|_{\mathcal{H}^{s,\delta}(X)} = \left\{\frac{1}{2\pi i}\int_{\Gamma_{(n+1)/2-\delta}} \big\|R^s(z) M_{t\to z}\{\omega u\}(z)\big\|^2_{L^2(Y)}\right\}^{1/2} \tag{2.31}$$

is finite. Here, $R^s(z) \in L^s_{\mathrm{cl}}(Y; \Gamma_{(n+1)/2-\delta})$ is an *order-reducing family*, i.e., $R^s(z)$ is parameter-dependent elliptic and $R^s(z)\colon H^r(Y) \to H^{r-s}(Y)$ is an isomorphism for some $r \in \mathbb{R}$ (and then for all $r \in \mathbb{R}$) and all $z \in \Gamma_{(n+1)/2-\delta}$. For instance, if $f(z) \in \mathcal{M}^s_{\mathcal{O}}(Y)$ is elliptic and the line $\Gamma_{(n+1)/2-\delta}$ is free of poles of $f(z)^{-1}$, then $f(z)$ is such an order-reduction. We will employ this observation in the next section when defining cone Sobolev spaces with asymptotics.

2.4.2 Cone Sobolev spaces with asymptotics

Let s, $\delta \in \mathbb{R}$, $P \in \underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$. By Theorem 2.42, there is an elliptic Mellin symbol $h^s_P(z) \in \mathcal{M}^s_{\mathcal{O}}(Y)$ such that the line $\Gamma_{(n+1)/2-\delta}$ is free of poles of $h^s_P(z)^{-1}$ and $L^\delta_{\mathfrak{S}^s} = P$ for $\mathfrak{S}^s = \big\{h^s_P(z), 0, 0, \dots\big\} \in \mathrm{Symb}^s_M(Y)$.

Definition 2.43. Let s, $\delta \in \mathbb{R}$, $\vartheta \geq 0$, and $P \in \underline{\mathrm{As}}^\delta(Y)$.
(a) For $P \in \underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$, the space $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$ consists of all functions $u \in \mathcal{H}^{s,\delta}(X)$ such that $M_{t\to z}\{\omega u\}(z)$, which is *a priori* holomorphic in $\big\{z \in \mathbb{C};\, \mathrm{Re}\, z > (n+1)/2 - \delta\big\}$ taking values in $H^s(Y)$, possesses a meromorphic continuation to the half-space $\big\{z \in \mathbb{C};\, \mathrm{Re}\, z > (n+1)/2 - \delta - \vartheta\big\}$, moreover,

$$h^s_P(z) M_{t\to z}\{\omega u\}(z) \in \mathcal{A}\big(\{z \in \mathbb{C}; \mathrm{Re}\, z > (n+1)/2 - \delta - \vartheta\}; L^2(Y)\big)$$

and the expression

$$\sup_{\delta<\delta'<\delta+\vartheta}\left\{\frac{1}{2\pi i}\int_{\Gamma_{(n+1)/2-\delta'}}\left\|h_P^s(z)M_{t\to z}\{\omega u\}(z)\right\|_{L^2(Y)}^2\,dz\right\}^{1/2} \tag{2.32}$$

is finite.
(b) For a general $P\in\underline{\mathrm{As}}^\delta(Y)$, represented as the join $P=\bigvee_{\iota\in\mathcal{I}}P_\iota$ for a bounded family $\{P_\iota\}_{\iota\in\mathcal{I}}\subset\underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$, we define $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)=\sum_{\iota\in\mathcal{I}}\mathcal{H}^{s,\delta}_{P_\iota,\vartheta}(X)$.

It is readily seen that Definition 2.43 (a) is independent of the choice of the Mellin symbol $h_P^s(z)$. Moreover, under the condition that (2.32) is finite the limit

$$h_P^s(z)M_{t\to z}\{\omega u\}(z)\big|_{z=(n+1)/2-\delta'+i\tau}\to w(\tau)\quad\text{as }\delta'\to\delta+\vartheta-0$$

exists in $L^2(\mathbb{R}_\tau;L^2(Y))$. Thus, $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$ is a Hilbert space with the norm

$$\|u\|_{\mathcal{H}^{s,\delta}_{P,\vartheta}(X)}=\left\{\|w\|^2_{L^2(\mathbb{R}_\tau;L^2(Y))}+\|u\|^2_{\mathcal{H}^{s,\delta}(X)}\right\}^{1/2}. \tag{2.33}$$

Definition 2.43 (b) is justified by Proposition 2.37 (b), since we obviously have $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)=\mathcal{H}^{s,\delta+\vartheta}(X)$ for $P\in\underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$ and $\delta_P>\delta+\vartheta$. Again, this definition is seen to be independent of the choice of the representing family $\{P_\iota\}_{\iota\in\mathcal{I}}\subset\underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$, and it also yields a Hilbert space structure for $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$.

Proposition 2.44. *Let s, $\delta\in\mathbb{R}$, $\vartheta\ge 0$, and $P\in\underline{\mathrm{As}}^\delta_{\mathrm{prop}}(Y)$. Further, let $\mathfrak{S}^s=\{\mathfrak{s}^{s-j}(z)\ j=0,1,2,\dots\}\in\mathrm{Symb}^s_M(Y)$ be elliptic with respect to the weight δ and $L^\delta_{\mathfrak{S}^s}=P$, $L^{\delta-s}_{(\mathfrak{S}^s)^{-1}}=\mathcal{O}$. (Condition $L^{\delta-s}_{(\mathfrak{S}^s)^{-1}}=\mathcal{O}$ means that the Mellin symbols $\mathfrak{s}^{s-j}(z)$ are holomorphic when $\mathrm{Re}\,z>(n+1)/2-\delta$.) Then a function $u\in\mathcal{H}^{s,\delta}(X)$ belongs to the space $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$ if and only if $M_{t\to z}\{\omega u\}(z)$ possesses a meromorphic continuation to the half-space $\{z\in\mathbb{C};\ \mathrm{Re}\,z>(n+1)/2-\delta-\vartheta\}$,*

$$\sum_{j=0}^{M}\mathfrak{s}^{s-j}(z-s+j)M_{t\to z}\{\omega u\}(z-s+j)\in\mathcal{A}\big(\{z\in\mathbb{C};\mathrm{Re}\,z>(n+1)/2-\delta+s-\vartheta\};L^2(Y)\big),$$

and the expression

$$\sup_{\delta<\delta'<\delta+\vartheta}\left\{\frac{1}{2\pi i}\int_{\Gamma_{(n+1)/2-\delta'+s}}\left\|\sum_{j=0}^{M}\mathfrak{s}^{s-j}(z-s+j)M_{t\to z}\{\omega u\}(z-s+j)\right\|^2_{L^2(Y)}dz\right\}^{1/2}$$

is finite. Here, M is any integer larger than ϑ.

Proof. This is an application (of an adapted version) of Witt [18, Proposition 2.6]. Note that $\mathfrak{s}^{s-j}(z-s+j)M_{t\to z}\{\omega u\}(z-s+j) \in \mathcal{A}\big(\{z \in \mathbb{C}; \operatorname{Re} z > (n+1)/2-\delta+s-j\}; L^2(Y)\big)$ so that the condition is actually independent of the choice of the integer $M > \vartheta$. □

For s, $\delta \in \mathbb{R}$, $\vartheta > 0$, and $P \in \underline{\mathrm{As}}^\delta(Y)$, we will also employ the spaces

$$\mathcal{H}^{s,\delta}_{P,\vartheta-0}(X) = \bigcap_{\epsilon>0} \mathcal{H}^{s,\delta}_{P,\vartheta-\epsilon}(X). \tag{2.34}$$

These space $\mathcal{H}^{s,\delta}_{P,\vartheta-0}(X)$ are Fréchet-Hilbert spaces, i.e., Fréchet spaces whose topology is given by a countable family of Hilbert semi-norms. We will also use notations like

$$\mathcal{H}^{\infty,\delta}_{P,\vartheta}(X) = \bigcap_{s\in\mathbb{R}} \mathcal{H}^{s,\delta}_{P,\vartheta}(X), \qquad \mathcal{H}^{-\infty,\delta}_{P,\vartheta}(X) = \bigcup_{s\in\mathbb{R}} \mathcal{H}^{s,\delta}_{P,\vartheta}(X),$$
$$\mathcal{H}^{s,\delta}_{P,\vartheta+0}(X) = \bigcup_{\epsilon>0} \mathcal{H}^{s,\delta}_{P,\vartheta+\epsilon}(X), \qquad \text{etc.}$$

Remark 2.45. In case P is a *strongly discrete asymptotic type*, the spaces $\mathcal{H}^{s,\delta}_{P,\vartheta-0}(X)$ are the function spaces introduced by Schulze [15, Section 2.1.1]. There, the notation $\mathcal{H}^{s,\delta}_P(X)_\Delta$ with the half-open interval $\Delta = (-\vartheta, 0]$ has been used. The definition of the function spaces $\mathcal{H}^{s,\delta}_P(X)_\Delta$ refers to fixed splitting of coordinates near ∂X and is, in general, not coordinate invariant.

2.4.3 Functional-analytic properties

We list some properties of the function spaces $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$:

Proposition 2.46. *Let s, s', δ, $\delta' \in \mathbb{R}$, $\vartheta \geq 0$, $P \in \underline{\mathrm{As}}^\delta(Y)$, $P' \in \underline{\mathrm{As}}^{\delta'}(Y)$, and $\{P_\iota\}_{\iota\in\mathcal{I}} \subset \underline{\mathrm{As}}^\delta(Y)$ be a family of asymptotic types. Then*:
(a) $\mathcal{H}^{s,\delta}_{P,0}(X) = \mathcal{H}^{s,\delta}(X)$.
(b) $\mathcal{H}^{s,\delta}_{P,\vartheta}(X) = \mathcal{H}^{s,\delta-a}_{P,\vartheta+a}(X)$ *for any* $a > 0$.
(c) $\mathcal{H}^{s,\delta}_{\mathcal{O},\vartheta}(X) = \mathcal{H}^{s,\delta+\vartheta}(X)$.
(d) *We have*

$$\begin{aligned}\mathcal{H}^{s,\delta}_{P,\vartheta}(X) &= \mathcal{H}^{s,\delta}_{\mathcal{O},\vartheta}(X) \\ &\oplus \Bigg\{\omega(t) \sum_{\substack{p\in V,\\ \operatorname{Re} p>(n+1)/2-\delta-\vartheta}} \sum_{k+l=m_p-1} \frac{(-1)^k}{k!}\, t^{-p} \log^k t\, \phi^{(p)}_l(y); \\ &\qquad\qquad \Phi(p) = (\phi^{(p)}_0, \dots, \phi^{(p)}_{m_p-1}) \text{ for some } \Phi \in J\Bigg\},\end{aligned}$$

where $J \subset \mathcal{E}_V(Y)$ is the linear subspace representing the asymptotic type P, provided that $\operatorname{Re} p \neq (n+1)/2-\delta-\vartheta$ *holds for all* $p \in V$.

(e) *We have* $\mathcal{H}^{s,\delta}_{P,\vartheta}(X) \subseteq \mathcal{H}^{s',\delta'}_{P',\vartheta'}(X)$ *if and only if* $s \geq s'$, $\delta + \vartheta \geq \delta' + \vartheta'$, *and* $P \preccurlyeq P'$ *up to the conormal order* $\delta' + \vartheta'$.
(f) $\mathcal{H}^{s,\delta}_{\bigwedge_{\iota\in\mathcal{I}} P_\iota,\vartheta}(X) = \bigcap_{\iota\in\mathcal{I}} \mathcal{H}^{s,\delta}_{P_\iota,\vartheta}(X)$ *if the family* $\{P_\iota\}_{\iota\in\mathcal{I}}$ *is non-empty.*
(g) $\mathcal{H}^{s,\delta}_{\bigvee_{\iota\in\mathcal{I}} P_\iota,\vartheta}(X) = \sum_{\iota\in\mathcal{I}} \mathcal{H}^{s,\delta}_{P_\iota,\vartheta}(X)$ *if the family* $\{P_\iota\}_{\iota\in\mathcal{I}}$ *is bounded* (*where the sum sign stands for the non-direct sum of Hilbert spaces*);
(h) $C^\infty_P(X) = \bigcap_{s\in\mathbb{R},\, \vartheta\geq 0} \mathcal{H}^{s,\delta}_{P,\vartheta}(X)$.
(i) $C^\infty_P(X)$ *is dense in* $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$.

Proof. The proofs of (a) to (i) are straightforward. □

From (e) we get, in particular, $\mathcal{H}^{s,\delta}_{P,\vartheta}(X) = \mathcal{H}^{s',\delta'}_{P',\vartheta'}(X)$ if and only if $s = s'$, $\delta + \vartheta = \delta' + \vartheta'$, and $P = P'$ up to the conormal order $\delta + \vartheta$. (b) and also (c), in view of (a), are special cases.

Proposition 2.47. *For* $\delta \in \mathbb{R}$, $P \in \underline{\mathrm{As}}^\delta(Y)$, *and any* $a \in \mathbb{R}$, *the family* $\left\{\mathcal{H}^{s,\delta}_{P,s-a}(X);\, s \geq a\right\}$ *of Hilbert spaces forms an interpolation scale with respect to the complex interpolation method.*

Proof. This is immediate from the definition. □

Proposition 2.48. *The spaces* $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$ *are invariant under coordinate changes, where this has to be understood in the sense of* Proposition 2.32.

Proof. Basically, this follows from the invariance of the spaces $C^\infty_P(X)$ under coordinate changes, where the latter is just a reformulation of the fact that the asymptotic types in $\underline{\mathrm{As}}^\delta(Y)$ are coordinate invariant. □

2.4.4 Mapping properties and elliptic regularity

We finally take the step from the algebra of complete conormal symbols to elliptic Fuchsian differential operators and their parametrices. These parametrices are cone pseudodifferential operators, where for the latter we refer to Schulze [16, Chapter 2]. While for general cone pseudodifferential operators, there might be a difference between the conormal asymptotics produced on the level of complete conormal symbols and operators, respectively — due to the appearance of so-called singular Green operators — for *Fuchsian differential operators* this does not happen.
In *cone pseudodifferential calculus*, one encounters operators of the form $\omega(t)t^{-\mu}\,\mathrm{op}_M^{(n+1)/2-\delta}(h)\,\tilde\omega(t)$, where $h(t,z) \in C^\infty(\overline{\mathbb{R}}_+; \mathcal{M}^\mu_{\mathrm{as}}(Y))$. Here,

$$\mathrm{op}_M^{(n+1)/2-\delta}(h(t,z))u = \frac{1}{2\pi i}\int_{\Gamma_{(n+1)/2-\delta}} t^{-z} h(t,z)\tilde u(z)\,dz \tag{2.35}$$

is a pseudodifferential operator, whose definition is based on the Mellin transformation instead of the Fourier transformation. The mapping properties of these operators in the spaces $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$ are as follows:

Proposition 2.49. *Let $h(t,z) \in C^\infty(\overline{\mathbb{R}}_+; \mathcal{M}^\mu_{\mathrm{as}}(Y))$ and assume that the line $\Gamma_{(n+1)/2-\delta}$ is free of poles of $\partial^j h(0,z)/\partial t^j$ for all $j = 0,1,2,\dots$ Then, for all $P \in \underline{\mathrm{As}}^\delta(Y)$, $s \in \mathbb{R}$, $\vartheta \geq 0$,*

$$\omega(t)t^{-\mu}\operatorname{op}_M^{(n+1)/2-\delta}(h)\,\tilde{\omega}(t)\colon \mathcal{H}^{s,\delta}_{P,\vartheta}(X) \to \mathcal{H}^{s-\mu,\delta-\mu}_{Q,\vartheta}(X),$$

where $\omega(t)$, $\tilde{\omega}(t)$ are cut-off functions, $\mathfrak{S}^\mu = \left\{\frac{1}{j!}\,\frac{\partial^j h}{\partial t^j}(0,z);\ j = 0,1,2,\dots\right\} \in \mathrm{Symb}^\mu_M(Y)$, and $Q = \mathcal{Q}^{\delta-\mu}(P,\mathfrak{S}^\mu) \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$.

Proof. The previous definitions are made to let this result hold. □

Notation. Proposition 2.49 implies that, given a cone pseudodifferential operator A in Schulze's cone calculus $\mathcal{C}^\mu(X,(\delta,\delta-\mu,(-\infty,0]))$, see Schulze [16, Chapter 2] again, for each $P \in \underline{\mathrm{As}}^\delta(Y)$, there is a $Q \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$ such that, for all $s \in \mathbb{R}$, $\vartheta \geq 0$,

$$A\colon \mathcal{H}^{s,\delta}_{P,\vartheta}(X) \to \mathcal{H}^{s-\mu,\delta-\mu}_{Q,\vartheta}(X). \tag{2.36}$$

Given $P \in \underline{\mathrm{As}}^\delta(Y)$, the minimal such asymptotic type $Q \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$, that exists by virtue of Proposition 2.28 (a) and Proposition 2.46 (f), is denoted by $\mathcal{Q}^{\delta-\mu}(P;A)$. If A is elliptic, given $Q \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$, the minimal asymptotic type $P \in \underline{\mathrm{As}}^\delta(Y)$ such that, for all $s \in \mathbb{R}$, $\vartheta \geq 0$, $u \in \mathcal{H}^{-\infty,\delta}(X)$, $Au \in \mathcal{H}^{s-\mu,\delta-\mu}_{Q,\vartheta}(X)$ implies $u \in \mathcal{H}^{s,\delta}_{P,\vartheta}(X)$ is denoted by $\mathcal{P}^\delta(Q;A)$.
We shall employ this *push-forward* notation also if more than one operator A is involved, i.e., $\mathcal{Q}^{\delta-\mu}(P;A_1,\dots,A_m)$ denotes the minimal asymptotic type Q for which $A_j\colon \mathcal{H}^{s,\delta}_{P,\vartheta}(X) \to \mathcal{H}^{s-\mu,\delta-\mu}_{Q,\vartheta}(X)$ for $1 \leq j \leq m$.

Theorem 2.50. *For $A \in \mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X)$, $P \in \underline{\mathrm{As}}^\delta(Y)$, $Q \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$, we have $\mathcal{Q}^{\delta-\mu}(P;A) = \mathcal{Q}^{\delta-\mu}(P;\mathfrak{S}^\mu)$, where $\mathfrak{S}^\mu = \{\sigma^{\mu-j}_M(A)(z);\ j = 0,1,\dots\} \in \mathrm{Symb}^\mu_M(Y)$, as well as, in case A is elliptic, $\mathcal{P}^\delta(Q;A) = \mathcal{Q}^\delta(Q;(\mathfrak{S}^\mu)^{-1})$.*

Proof. In fact, $\mathcal{Q}^{\delta-\mu}(P;A) = \mathcal{Q}^{\delta-\mu}(P;\mathfrak{S}^\mu)$ follows from Proposition 2.49.
Furthermore, it is known that formal asymptotic solutions $u \in C^\infty_{\mathrm{as}}(X)$ to the equation $Au = f$ for $f \in C^\infty_R(X)$ and any $R \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$ can be constructed, see, e.g. Melrose [13, Lemma 5.13]. More precisely, it can be shown that there is a right parametrix B to A, $B\colon \mathcal{H}^{s-\mu,\delta-\mu}(X) \to \mathcal{H}^{s,\delta}(X)$ for all $s \in \mathbb{R}$, such that

$$AB = I + R, \quad R\colon \mathcal{H}^{-\infty,\delta-\mu}(X) \to C^\infty_{\mathcal{O}}(X),$$

i.e., R is smoothing over X° and flattening to infinite order near ∂X. In fact, $B \in \mathcal{C}^{-\mu}(X,(\delta-\mu,\delta,(-\infty,0]))$ and, in particular, $B \in L^{-\mu}_{\mathrm{cl}}(X^\circ)$.
Now let $BA = I + R_0$. Obviously, R_0 is smoothing over X° such that $R_0\colon \mathcal{H}^{s,\delta}(X) \to \mathcal{H}^{\infty,\delta-\mu}(X)$ for any $s \in \mathbb{R}$. Furthermore, $A(I+R_0) = ABA = (I+R)A$ so that

$$AR_0 = RA.$$

We conclude that $R_0\colon \mathcal{H}^{s,\delta}(X) \to C^\infty_{P_0}(X)$, where $P_0 = \mathcal{Q}^\delta(\mathcal{O}; (\mathfrak{S}^\mu)^{-1})$. Hence, for $u \in \mathcal{H}^{-\infty,\delta}(X)$, $Au = f \in \mathcal{H}^{s-\mu,\delta-\mu}_{Q,\vartheta}(X)$, we get

$$u = Bf - R_0 u \in \mathcal{H}^{s,\delta}_{P,\vartheta}(X),$$

where $P = \mathcal{Q}^\delta(Q; (\mathfrak{S}^\mu)^{-1})$. Thus $\mathcal{P}^\delta(Q; A) = \mathcal{Q}^\delta(Q; (\mathfrak{S}^\mu)^{-1})$ as claimed. See also Witt [20, Remark after Proposition 5.5]. □

Notation. For $A \in \mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X)$, $\mathcal{Q}^{\delta-\mu}(P; A)$ is even independent of $\delta \in \mathbb{R}$ in view of the holomorphy of the conormal symbols $\sigma^{\mu-j}_M(A)(z)$ for $j = 0, 1, 2, \ldots$ In this case, we simply write $\mathcal{Q}(P; A) = \mathcal{Q}^{\delta-\mu}(P; A)$.

Proposition 2.51. *Let $A \in \mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X)$ be elliptic. Then there is an order-preserving bijection*

$$\big\{P \in \underline{\mathrm{As}}^\delta(Y);\, P \succcurlyeq L^\delta_{\mathfrak{S}^\mu}\big\} \to \underline{\mathrm{As}}^{\delta-\mu}(Y), \quad P \mapsto \mathcal{Q}(P; A), \tag{2.37}$$

with its inverse given by $Q \mapsto \mathcal{P}^\delta(Q; A)$. In particular, $L^\delta_{\mathfrak{S}^\mu}$ is mapped to the empty asymptotic type, $\mathcal{O}$.

Proof. This is implied by Proposition 2.31 and Theorem 2.50. Note that $L^{\delta-\mu}_{(\mathfrak{S}^\mu)^{-1}} = \mathcal{O}$, since the $\sigma^{\mu-j}_M(A)(z)$ for $j = 0, 1, 2, \ldots$ are holomorphic. □

Eventually, we have the following *locality principle*:

Proposition 2.52. *Let $A \in \mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X)$ be elliptic, Q_0, $Q_1 \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$, and $P_0 = \mathcal{P}^\delta(Q_0; A)$, $P_1 = \mathcal{P}^\delta(Q_1; A)$. Then, for any $\vartheta > 0$, $P_0 = P_1$ up to the conormal order $\delta + \vartheta$ if $Q_0 = Q_1$ up to the conormal order $\delta - \mu + \vartheta$.*

Proof. This follows from $P_0 = \mathcal{Q}^\delta(Q_0; (\mathfrak{S}^{-\mu})^{-1})$, $P_1 = \mathcal{Q}^\delta(Q_1; (\mathfrak{S}^{-\mu})^{-1})$, where $\mathfrak{S}^\mu = \{\sigma^{\mu-j}_M(A)(z);\, j \in \mathbb{N}\} \in \mathrm{Ell\,Symb}^\mu_M(Y)$. □

Remark 2.53. Combined with Theorem 2.30, Theorem 2.50 shows that each solution $u \in C^{\infty,\delta}_{\mathrm{as}}(X)$ to the equation $Au = f \in C^\infty_{\mathcal{O}}(X)$, where $A \in \mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X)$ is elliptic, can be written over finite weight intervals as a finite sum of functions of the form (2.16) modulo the corresponding flat class, where the Φ are taken from a characteristic basis of the linear subspace of $\mathcal{E}^\delta(Y)$ representing $\mathcal{P}^\delta(\mathcal{O}; A)$. If $\Phi(p) = (\phi_0, \ldots, \phi_{m-1})$ for such a vector Φ, where $p = \gamma(\Phi)$, then we say that A admits *an asymptotic series starting with the term $t^{-p} \log^{m-1} t\, \phi_0$*. Since this is then the most singular term (when $\gamma(\Phi)$ is highest possible), if it coefficient can be shown to vanish, then the whole series must vanish, up to the next appearance of a starting term for another asymptotic series.

3 Applications to semilinear equations

In this section, Theorem 1.1 is proved. To this end, multiplicatively closable and multiplicatively closed asymptotic types are investigated in Section 3.1. This allows the derivation of results concerning the action of nonlinear superposition operators on cone Sobolev spaces with asymptotics. In Section 3.2, the general scheme for establishing results of the type of Theorem 1.1 is established. This scheme is specified to multiplicatively closable asymptotic types in Section 3.3, then completing the proof of Theorem 1.1.

3.1 Multiplicatively closed asymptotic types

Here, we investigate multiplicative properties of asymptotic types and the behavior of cone Sobolev spaces $\mathcal{H}^{s,\delta}_{P,\vartheta}(X)$ under the action of nonlinear superposition.

Notation. In connection with pointwise multiplication, it is useful to employ the following notation:

$$H^s_{P,\vartheta}(X) = \begin{cases} \mathcal{H}^{s,\delta}_{P,\delta_P-\delta+\vartheta}(X) & \text{if } \vartheta \geq 0, \\ \mathcal{H}^{s,\delta_P+\vartheta}(X) & \text{otherwise,} \end{cases}$$

where $P \in \underline{\mathrm{As}}^{\delta}(Y)$, $P \neq \mathcal{O}$, and $\delta < \delta_P$ in the first line. (Proposition 2.46 (b) shows that this definition is independent of the choice of δ.) Thus, starting from δ_P, the conormal order is improved by ϑ upon allowing asymptotics of type P. Similarly for $H^s_{P,\vartheta-0}(X)$.
Furthermore, we write $\{\vartheta\}$ if we mean either ϑ or $\vartheta - 0$. For instance, $\{\vartheta\} \geq 0$ means $\vartheta \geq 0$ if $\{\vartheta\} = \vartheta$ and $\vartheta > 0$ if $\{\vartheta\} = \vartheta - 0$.

3.1.1 Multiplication of asymptotic types

The result admitting nonlinear superposition for function spaces with asymptotics is stated first:

Lemma 3.1. *Given $P \in \underline{\mathrm{As}}(Y)$, $Q \in \underline{\mathrm{As}}(Y)$, there is a minimal asymptotic type, $P \circ Q \in \underline{\mathrm{As}}(Y)$, such that*

$$C^\infty_P(X) \times C^\infty_Q(X) \to C^\infty_{P\circ Q}(X), \quad (u,v) \mapsto uv. \tag{3.1}$$

Proof. Suppose that the asymptotic types P, Q are represented by subspaces $J \subset \mathcal{E}_V(Y)$ and $K \subset \mathcal{E}_W(Y)$, respectively, for suitable $V, W \in \mathcal{C}$. Then the asymptotic type $P \circ Q$ is carried by the set $V + W$, and it is represented by the linear subspace of $\mathcal{E}_{V+W}(Y)$ consisting of all $\Theta \in \mathcal{E}_{V+W}(Y)$ for which there are $\Phi \in J$, $\Psi \in K$ such that $\Theta(r) = \sum_{\substack{p+q=r,\\ p\in V,\, q\in W}} \Phi(p) \times \Psi(q)$ holds for all $r \in V + W$.

Here,

$$\Phi \times \Psi = \Bigg(\binom{m+n}{m} \phi_0\psi_0, \binom{m+n-1}{m} \phi_0\psi_1 + \binom{m+n-1}{m-1} \phi_1\psi_0,$$
$$\binom{m+n-2}{m} \phi_0\psi_2 + \binom{m+n-2}{m-1} \phi_1\psi_1 + \binom{m+n-2}{m-2} \phi_0\psi_2,$$
$$\ldots, \binom{1}{1} \phi_{m-1}\psi_n + \binom{1}{0} \phi_m\psi_{n-1}, \binom{0}{0} \phi_m\psi_n \Bigg)$$

for $\Phi = (\phi_0, \phi_1, \ldots, \phi_m)$, $\Psi = (\psi_0, \psi_1, \ldots, \psi_n) \in [C^\infty(Y)]^\infty$. For this, see (2.16). Note that $T(\Phi \times \Psi) = (T\Phi) \times \Psi + \Phi \times (T\Psi)$ and, for $\Phi \in \mathcal{E}_{V_p}(Y)$, $\Psi \in \mathcal{E}_{V_q}(Y)$, we have $\Phi \times \Psi \in \mathcal{E}_{V_{p+q}}(Y)$ showing that the linear subspace of $\mathcal{E}_{V+W}(Y)$ described above actually represents an asymptotic type. $\square$

The multiplication of asymptotic types possesses a unit, denoted by $\mathbf{1}$, that is represented by the space $\operatorname{span}\{(1)\} \subset \mathcal{E}_{\{0\}}(Y)$, where 1 is the function identically 1 on Y.

Definition 3.2. An asymptotic type $Q \in \underline{\mathrm{As}}(Y)$ is called *multiplicatively closed* if $Q \circ Q = Q$. An asymptotic type $Q \in \underline{\mathrm{As}}(Y)$ is called *multiplicatively closable* if it is dominated by a multiplicatively closed asymptotic type. In this case, the minimal multiplicatively closed asymptotic type dominating Q is called the *multiplicative closure* of Q and is denoted by $\widetilde{Q}$.

From the proof of Lemma 3.1,

$$\delta_{P \circ Q} \geq \delta_P + \delta_Q - (n+1)/2, \tag{3.2}$$

where equality holds if $P = Q$ or if $\dim Y = 0$. Especially, $\delta_Q = (n+1)/2$ if Q is multiplicatively closed and $\delta_Q \geq (n+1)/2$ if Q is multiplicatively closable. Furthermore, it is also seen $Q \succcurlyeq \mathbf{1}$ for any multiplicatively closed asymptotic type Q, see also Lemma 3.4 below.

3.1.2 The class $\underline{\mathrm{As}}^\sharp(Y)$ of multiplicatively closable asymptotic types

We study the class of asymptotic types that belong to bounded functions. It turns out that this class is intimately connected to the multiplication of asymptotic types.

Definition 3.3. (a) The class $\underline{\mathrm{As}}^b(Y)$ of *bounded asymptotic types* consists of all asymptotic types $Q \in \underline{\mathrm{As}}(Y)$ for which $\delta_Q \geq (n+1)/2$. Equivalently, a bounded asymptotic type Q is represented by a subspace $J \subset \mathcal{E}_V(Y)$ for some $V \in \mathcal{C}$, where $V \subset \{z \in \mathbb{C};\, \operatorname{Re} z \leq 0\}$.
(b) The class $\underline{\mathrm{As}}^\sharp(Y)$ consists of all bounded asymptotic types Q represented by a subspace $J \subset \mathcal{E}(Y)$ such that $J_0 \subseteq \operatorname{span}\{(1)\}$ and $J_p = \{0\}$ for $\operatorname{Re} p = 0$, $p \neq 0$.

Lemma 3.4. *For $Q \in \underline{\mathrm{As}}(Y)$, the following conditions are equivalent:*
(a) *Q is multiplicatively closable*;
(b) *the join $\bigvee_{k\geq 1} Q^k$ does exist, where $Q^k = \underbrace{Q \circ \cdots \circ Q}_{k\ times}$ is the k-fold product*;
(c) $Q \in \underline{\mathrm{As}}^\sharp(Y)$.
In case (a) *to* (c) *are fulfilled, we have $\widetilde{Q} = \mathbf{1} \vee \bigvee_{k\geq 1} Q^k$.*

Proof. (a) and (b) are obviously equivalent. Moreover, (c) implies (b).
It remains to show that (a) also implies (c). If Q is multiplicatively closable, then $\widetilde{Q}$ exists and $\delta_{\widetilde{Q}} = (n+1)/2$. In particular, $\widetilde{Q} \in \underline{\mathrm{As}}^b(Y)$. Let $\widetilde{Q}$ be represented by $J \subset \mathcal{E}_V(Y)$ for some $V \in \mathcal{C}$, $V \subset \{z \in \mathbb{C};\ \mathrm{Re}\, z \leq 0\}$. Suppose that $\phi \in J_p$ for $p \in \mathbb{C}$, $\mathrm{Re}\, p = 0$, where $\phi \neq 0$. We immediately get $\phi^l \in J_{lp}$ for any $l \in \mathbb{N}$, $l \geq 1$. For $p \neq 0$, we obtain the contradiction $\{lp;\ l \in \mathbb{N}\} \subseteq V \in \mathcal{C}$. For $p = 0$ and ϕ not being constant, we obtain a contradiction to the fact that $\dim J_0 < \infty$. Thus, $\widetilde{Q} \in \underline{\mathrm{As}}^\sharp(Y)$ and, therefore, $Q \in \underline{\mathrm{As}}^\sharp(Y)$. □

Lemma 3.5. *For each $Q \in \underline{\mathrm{As}}(Y)$, there are asymptotic types $Q^b \in \underline{\mathrm{As}}^b(Y)$ and $Q^\sharp \in \underline{\mathrm{As}}^\sharp(Y)$ which are maximal among all asymptotic types possessing the property*

$$Q^b \preccurlyeq Q \text{ and } Q^\sharp \preccurlyeq Q, \tag{3.3}$$

respectively. In particular, $Q^\sharp \preccurlyeq Q^b$.

Proof. The proof is straightforward. □

3.1.3 Nonlinear superposition

We investigate expressions like $F(x, v(x))$, where $F(x,\nu) \in C^\infty_R(X \times \mathbb{R})$ for some $R \in \underline{\mathrm{As}}(Y)$ and $v \in H^s_{Q,\vartheta}(X) \cap L^\infty(X)$ with $s \geq 0$, $\vartheta > 0$, and $Q \in \underline{\mathrm{As}}^\sharp(Y)$. For later reference, we recall the following facts:

Proposition 3.6. (a) *For $s > (n+1)/2$, $0 \leq s' \leq s$, $\gamma, \delta \in \mathbb{R}$, pointwise multiplication induces a bilinear continuous map*

$$\mathcal{H}^{s,\gamma}(X) \times \mathcal{H}^{s',\delta}(X) \to \mathcal{H}^{s',\gamma+\delta-(n+1)/2}(X). \tag{3.4}$$

(b) *For $s, \delta \in \mathbb{R}$, $\mathcal{H}^{s,\delta}(X) \subset L^\infty(X)$ if and only if $s > (n+1)/2$, $\delta \geq (n+1)/2$.*
(c) *For $s \geq 0$, $\gamma, \delta \geq (n+1)/2$, pointwise multiplication induces a bilinear continuous map*

$$\big(\mathcal{H}^{s,\gamma}(X) \cap L^\infty(X)\big) \times \big(\mathcal{H}^{s,\delta}(X) \cap L^\infty(X)\big) \to \mathcal{H}^{s,\gamma+\delta-(n+1)/2}(X) \cap L^\infty(X).$$

(d) *For $s \geq 0$, $\delta \in \mathbb{R}$, $p \in \mathbb{C}$, $c(y) \in C^\infty(Y)$, the multiplication operator*

$$\omega(t)t^{-p}c(y)\colon \mathcal{H}^{s,\delta}(X) \to \mathcal{H}^{s,\delta-\mathrm{Re}\, p}(X),$$

where $\omega(t)$ is a cut-off function, is continuous.

(e) *For* $s \geq 0$, $v_1, \dots, v_K \in \big(1 + \mathcal{H}^{s,(n+1)/2}(X)\big) \cap L^\infty(X)$, *and* $F \in C^\infty(\mathbb{R}^K)$, *we have*

$$F(v_1, \dots, v_K) \in \big(1 + \mathcal{H}^{s,(n+1)/2}(X)\big) \cap L^\infty(X). \tag{3.5}$$

The map $\big(\big(1 + \mathcal{H}^{s,(n+1)/2}(X)\big) \cap L^\infty(X)\big)^K \to \big(1 + \mathcal{H}^{s,(n+1)/2}(X)\big) \cap L^\infty(X)$ *induced by* (3.5) *is continuous and sends bounded sets to bounded sets.*

Proof. A proof of (3.4) in case $s' = s$ has been supplied by Witt [19, Lemma 2.7] using a result of Dauge [2, Theorem (AA.3)]. The other proofs are similar. □

Remark 3.7. Property (d) fails if logarithms appear and has to be replaced by

$$\omega(t) t^{-p} \log^k t\, c(y) \colon \mathcal{H}^{s,\delta}(X) \to \mathcal{H}^{s,\delta - \operatorname{Re} p - 0}(X)$$

is continuous when $k \in \mathbb{N}$, $k \geq 1$.

First, Lemma 3.1 is sharpened:

Proposition 3.8. *For* $s > (n+1)/2$, $0 \leq s' \leq s$, $\vartheta > 0$, *and* P, $Q \in \underline{\mathrm{As}}(Y)$, *pointwise multiplication induces a bilinear continuous map*

$$H^s_{P,\vartheta-0}(X) \times H^{s'}_{Q,\vartheta-0}(X) \to H^{s'}_{P\circ Q,\vartheta-0}(X). \tag{3.6}$$

Proof. Let $u \in H^s_{P,\vartheta-0}(X)$, $v \in H^{s'}_{Q,\vartheta-0}(X)$. Then $u = u_0 + u_1$, $v = v_0 + v_1$, where

$$u_0 = \sum_{j=0}^{M} \sum_{k=0}^{m_j} \omega(t) t^{-p_j} \log^k t\, c_{jk}(y), \quad v_0 = \sum_{j'=0}^{N} \sum_{k'=0}^{n_{j'}} \omega(t) t^{-q_{j'}} \log^{k'} t\, d_{j'k'}(y), \tag{3.7}$$

$\omega(t)$ is a cut-off function, the sequences $\{(p_j, m_j, c_{jk})\}$, $\{(q_{j'}, n_{j'}, d_{j'k'})\}$ are given by the asymptotic types P and Q, respectively, according to Definition 2.18, and M, N are chosen so that $u_1 \in \mathcal{H}^{s,\delta_P+\vartheta-0}(X)$, $v_1 \in \mathcal{H}^{s',\delta_Q+\vartheta-0}(X)$. Since $u_0 \in \mathcal{H}^{\infty,\delta_P-0}(X)$, $v_0 \in \mathcal{H}^{\infty,\delta_Q-0}(X)$, we obtain

$$uv = u_0 v_0 + u_1 v_0 + u_0 v_1 + u_1 v_1,$$

where $u_1 v_0 + u_0 v_1 + u_1 v_1 \in \mathcal{H}^{s',\delta_{P\circ Q}+\vartheta-0}(X)$ by (3.4) and

$$u_0 v_0 = \sum_{j,j'=0}^{M,N} \sum_{k,k'=0}^{m_j,n_{j'}} \omega^2(t) t^{-(p_j+q_{j'})} \log^{k+k'} t\, c_{jk}(y) d_{j'k'}(y) \in H^\infty_{P\circ Q,\vartheta-0}(X),$$

for $\omega^2(t)$ is a cut-off function and the sequence

$$\Big\{\big(r_{j''}, o_{j''}, \sum_{p_j+q_{j'}=r_{j''}} \sum_{k+k'=k''} c_{jk} d_{j'k'}\big)\Big\},$$

where $o_{j''} = \max\{m_j + n_{j'};\, p_j + q_{j'} = r_{j''}\}$, is associated with an asymptotic type that equals $P \circ Q$ up to the conormal order $\delta_{P\circ Q} + \vartheta - 0$. This immediately gives $uv \in H^{s'}_{P\circ Q,\vartheta-0}(X)$. □

The significance of the class $\underline{\mathrm{As}}^b(Y)$ is uncovered by the next result:

Proposition 3.9. *For $s \geq 0$, $\delta \in \mathbb{R}$, $\delta + \{\vartheta\} \geq (n+1)/2$, and $Q \in \underline{\mathrm{As}}^\delta(Y)$,*

$$\mathcal{H}^{s,\delta}_{Q,\{\vartheta\}}(X) \cap L^\infty(X) = \mathcal{H}^{s,\delta}_{Q^b,\{\vartheta\}}(X) \cap L^\infty(X). \tag{3.8}$$

Proof. Let $u \in \mathcal{H}^{s,\delta}_{Q,\{\vartheta\}}(X) \cap L^\infty(X)$ and write

$$u(x) = \sum_{j=0}^{M} \sum_{k=0}^{m_j} \omega(t) t^{-p_j} \log^k t\, c_{jk}(y) + u_1(x),$$

where the sequences $\{(p_j, m_j, c_{jk})\}$ is given by the asymptotic type Q and M is chosen so that $u_1 \in \mathcal{H}^{s,(n+1)/2-0}(X)$. Since $u \in L^\infty(X) \subset \mathcal{H}^{0,(n+1)/2-0}(X)$, we get that $\sum_{j=0}^{M} \sum_{k=0}^{m_j} \omega(t) t^{-p_j} \log^k t\, c_{jk}(y) \in \mathcal{H}^{0,(n+1)/2-0}(X)$ which implies $c_{jk}(y) = 0$ for $\operatorname{Re} p_j > 0$. Thus $u \in \mathcal{H}^{s,\delta}_{Q^b,\vartheta}(X)$. $\square$

Lemma 3.10. *For $s \geq 0$, $\vartheta > 0$, and P, $Q \in \underline{\mathrm{As}}^b(Y)$, pointwise multiplication induces a bilinear continuous map*

$$\left(H^s_{P,\vartheta-0}(X) \cap L^\infty(X)\right) \times \left(H^s_{Q,\vartheta-0}(X) \cap L^\infty(X)\right) \to H^s_{P\circ Q,\vartheta-0}(X) \cap L^\infty(X).$$

Proof. Represent $u = u_0 + u_1 \in H^s_{P,\vartheta-0}(X) \cap L^\infty(X)$, $v = v_0 + v_1 \in H^s_{Q,\vartheta-0}(X) \cap L^\infty(X)$ as in the proof of Proposition 3.8. Since u_0, $v_0 \in L^\infty(X)$ due to the assumption P, $Q \in \underline{\mathrm{As}}^b(Y)$, we get that $u_1 \in \mathcal{H}^{s,\delta_P+\vartheta-0}(X) \cap L^\infty(X)$, $v_1 \in \mathcal{H}^{s,\delta_Q+\vartheta-0}(X) \cap L^\infty(X)$ and, therefore, $u_1 v_0 + u_0 v_1 + u_1 v_1 \in \mathcal{H}^{s,\delta_{P\circ Q}+\vartheta-0}(X) \cap L^\infty(X)$ in view of Proposition 3.6 (c). The assertion follows. $\square$

A more precise statement is possible if P, $Q \in \underline{\mathrm{As}}^\sharp(Y)$:

Lemma 3.11. *For $s \geq 0$, $\vartheta \geq 0$, and P, $Q \in \underline{\mathrm{As}}^\sharp(Y)$ satisfying $P \succcurlyeq \mathbf{1}$, $Q \succcurlyeq \mathbf{1}$, pointwise multiplication induces a bilinear continuous map*

$$\left(H^s_{P,\vartheta}(X) \cap L^\infty(X)\right) \times \left(H^s_{Q,\vartheta}(X) \cap L^\infty(X)\right) \to H^s_{P\circ Q,\vartheta}(X) \cap L^\infty(X). \tag{3.9}$$

Especially, for $s \geq 0$, $\vartheta \geq 0$, and $Q \in \underline{\mathrm{As}}^\sharp(Y)$ being multiplicatively closed, $H^s_{Q,\vartheta}(X) \cap L^\infty(X)$ is an algebra under pointwise multiplication.

Proof. We may assume that $\vartheta > 0$. Write $u = u_0 + u_1 \in H^s_{P,\vartheta}(X) \cap L^\infty(X)$, $v = v_0 + v_1 \in H^s_{Q,\vartheta}(X) \cap L^\infty(X)$ as in the proof of Proposition 3.8, where $u_0 = u_{00} + u_{01}$, $v_0 = v_{00} + v_{01}$, $u_{00} = \omega(t) c_{00}$, and $v_{00} = \omega(t) d_{00}$ with c_{00}, d_{00} being constants and in the expressions for u_{01}, v_{01} only appear exponents with $\operatorname{Re} p_j < 0$ and $\operatorname{Re} q_{j'} < 0$, respectively. Then

$$u_1 v_{01} + u_{01} v_1 + u_1 v_1 \in \mathcal{H}^{s,(n+1)/2+\vartheta+0}(X),$$

$u_{00} v \in H^s_{Q,\vartheta}(X) \subseteq H^s_{P\circ Q,\vartheta}(X)$, $u v_{00} \in H^s_{P,\vartheta}(X) \subseteq H^s_{P\circ Q,\vartheta}(X)$, and

$$u_{01} v_{01} \in H^\infty_{P\circ Q,\vartheta+0}(X),$$

which proves the assertion. $\square$

The fact which has actually been used in the last proof is that Proposition 3.6 (d) applies to the function $\omega(t)1$ ($p = 0$, $c(y) \equiv 1$). This is also used in part (b) of the next result:

Lemma 3.12. (a) *Let $s \geq 0$, $\vartheta > 0$, and $R, Q \in \underline{\mathrm{As}}(Y)$. Then pointwise multiplication induces a continuous map*

$$C^\infty_R(X) \times H^s_{Q,\vartheta-0}(X) \to H^s_{R\circ Q,\vartheta-0}(X). \tag{3.10}$$

(b) *If, in addition, $R \in \underline{\mathrm{As}}(Y)$ is so that the multiplicities of its highest singular values are one, i.e., $J_r \subseteq [C^\infty(Y)]^1$ for each $r \in V$, $\operatorname{Re} r = (n+1)/2 - \delta_R$, where $J \subset \mathcal{E}_V(Y)$ represents R, then pointwise multiplication induces a continuous map*

$$C^\infty_R(X) \times H^s_{Q,\vartheta}(X) \to H^s_{R\circ Q,\vartheta}(X).$$

Proof. (a) is immediate from Proposition 3.8. To get (b), we argue as in the proof of Lemma 3.11. □

Proposition 3.13. *Let $s \geq 0$, $\vartheta \geq 0$, and $Q \in \underline{\mathrm{As}}^\sharp(Y)$ be multiplicatively closed. Then $v_1, \dots, v_K \in H^s_{Q,\vartheta}(X) \cap L^\infty(X)$ and $F \in C^\infty(\mathbb{R}^K)$ implies that*

$$F(v_1, \dots, v_K) \in H^s_{Q,\vartheta}(X) \cap L^\infty(X). \tag{3.11}$$

Proof. We are allowed to assume that $\vartheta > 0$. Then $v \in H^s_{Q,\vartheta}(X)$ implies that $v\big|_{\partial X}$ is a constant, where $v\big|_{\partial X}$ means the factor in front of t^0 in the asymptotic expansion (1.2) (with u replaced with v) of v as $t \to +0$. Let $\beta_J = v_J\big|_{\partial X}$ for $1 \leq J \leq K$ be these constants. Using Taylor's formula, we obtain

$$\begin{aligned} F(v_1, \dots, v_K) &= \sum_{|\alpha|<N} \frac{1}{\alpha!} (\partial^\alpha F)(\beta_1, \dots, \beta_K)(v_1 - \beta_1)^{\alpha_1} \dots (v_K - \beta_K)^{\alpha_K} \\ + N \sum_{|\alpha|=N} \int_0^1 & \frac{(1-\sigma)^{N-1}}{\alpha!} (\partial^\alpha F)(\beta_1 + \sigma(v_1 - \beta_1), \dots, \beta_K + \sigma(v_K - \beta_K))\, d\sigma \\ & \times (v_1 - \beta_1)^{\alpha_1} \dots (v_K - \beta_K)^{\alpha_K}. \end{aligned} \tag{3.12}$$

By Lemma 3.11, $(v_1 - \beta_1)^{\alpha_1} \dots (v_K - \beta_K)^{\alpha_K} \in H^s_{Q,\vartheta}(X) \cap L^\infty(X)$ for any $\alpha \in \mathbb{N}^K$, thus the first summand on the right-hand side of (3.12) belongs to $H^s_{Q,\vartheta}(X) \cap L^\infty(X)$. On the other hand, choosing N sufficiently large, we can arrange that $(v_1 - \beta_1)^{\alpha_1} \dots (v_K - \beta_K)^{\alpha_K} \in \mathcal{H}^{s,(n+1)/2+\vartheta}(X) \cap L^\infty(X)$ for $|\alpha| \geq N$, since $v_J - \beta_J \in \mathcal{H}^{s,(n+1)/2+0}(X) \cap L^\infty(X)$ for $1 \leq J \leq K$. By (3.5), $\big\{(\partial^\alpha F)(\beta_1 + \sigma(v_1 - \beta_1), \dots, \beta_K + \sigma(v_K - \beta_K))\, d\sigma;\ 0 \leq \sigma \leq 1\big\}$ is a bounded set in $\big(1 + \mathcal{H}^{s,(n+1)/2}(X)\big) \cap L^\infty(X)$ for any $\alpha \in \mathbb{N}^K$. This shows that the second summand on the right-hand side of (3.12) belongs to $\mathcal{H}^{s,(n+1)/2+\vartheta}(X) \cap L^\infty(X)$. □

Proposition 3.14. (a) *Let* $s \geq 0$, $\vartheta > 0$. *Further let* $Q \in \underline{\mathrm{As}}^\sharp(Y)$ *be multiplicatively closed and* $R \in \underline{\mathrm{As}}(Y)$. *Then* $v_1, \dots, v_K \in H^s_{Q,\vartheta-0}(X) \cap L^\infty(X)$ *and* $F \in C^\infty_R(X \times \mathbb{R}^K)$ *implies that*

$$F(x, v_1, \dots, v_K) \in H^s_{R \circ Q, \vartheta-0}(X). \tag{3.13}$$

(b) *If, in addition,* R *satisfies the assumption of* Lemma 3.12 (b), *then* $v_1, \dots, v_K \in H^s_{Q,\vartheta}(X) \cap L^\infty(X)$ *and* $F \in C^\infty_R(X \times \mathbb{R}^K)$ *implies that*

$$F(x, v_1, \dots, v_K) \in H^s_{R \circ Q, \vartheta}(X).$$

Proof. We prove (a), (b) is analogous. Since $C^\infty_R(X \times \mathbb{R}^K) = C^\infty_R(X) \hat{\otimes}_\pi C^\infty(\mathbb{R}^K)$, we can write

$$F(x, v) = \sum_{j=0}^\infty \alpha_j \, \varphi_j(x) \, F_j(v),$$

where $\{\alpha_j\}_{j=0}^\infty \in l^1$ and $\{\varphi_j\}_{j=0}^\infty \subset C^\infty_R(X)$ and $\{F_j\}_{j=0}^\infty \subset C^\infty(\mathbb{R}^K)$, respectively, are null sequences. By the preceeding proposition,

$$F_j(v_1, \dots, v_K) \to 0 \quad \text{as } j \to \infty \text{ in } H^s_{Q,\vartheta-0}(X).$$

By Lemma 3.12,

$$\varphi_j(x) F_j(v_1, \dots, v_K) \to 0 \quad \text{as } j \to \infty \text{ in } H^s_{R \circ Q,\vartheta-0}(X).$$

Thus

$$F(x, v_1, \dots, v_K) = \sum_{j=0}^\infty \alpha_j \, \varphi_j(x) F_j(v_1, \dots, v_K) \in H^s_{R \circ Q,\vartheta-0}(X),$$

where the sum on the right-hand side is absolutely convergent. ∎

3.2 The bootstrapping argument

We consider the equation

$$Au = \Pi(u), \tag{3.14}$$

where $A \in \mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X)$ is an elliptic Fuchsian differential operator. Properties of the nonlinear operator $u \mapsto \Pi(u)$ are discussed below. The method proposed for deriving elliptic regularity for solutions to (3.14) amounts to *balancing* two asymptotic types — one for the left-hand and the other one for the right-hand side of (3.14).

We assume: There are asymptotic types $\bar{P} \in \underline{\mathrm{As}}^\delta(Y)$, $\bar{Q} \in \underline{\mathrm{As}}^{\delta-\mu}(Y)$, numbers a, b, s_0, $\vartheta_0 \in \mathbb{R}$ with

$$a < \mu, \quad b < \delta_{\bar{Q}} - \delta_{\bar{P}} + \mu, \quad s_0 \geq a^+, \quad \delta_{\bar{P}} + \{\vartheta_0\} \geq \delta,$$

and a subset $\mathcal{U} \subseteq H^{s_0}_{\bar{P},\{\vartheta_0\}}(X)$ such that the following conditions are met:
(A) *A is elliptic with respect to the conormal order δ and $\bar{P} \succcurlyeq \mathcal{P}^\delta(\bar{Q}; A)$, i.e., $u \in \mathcal{H}^{-\infty,\delta}(X)$, $Au \in C^\infty_{\bar{Q}}(X)$ implies $u \in C^\infty_{\bar{P}}(X)$;*
(B) *For $s \geq s_0$, $\vartheta \geq \vartheta_0$, we have*

$$\Pi \colon \mathcal{U} \cap H^s_{\bar{P},\{\vartheta\}}(X) \to H^{s-a}_{\bar{Q},\{\vartheta\}-b}(X).$$

Note that $\{\vartheta_0\} - b + \delta_{\bar{Q}} \geq \delta - \mu$.

Proposition 3.15. *Under the conditions* (A), (B), *each solution $u \in \mathcal{U} \subseteq H^{s_0}_{\bar{P},\{\vartheta_0\}}(X)$ to* (3.14) *belongs to the space $C^\infty_{\bar{P}}(X)$.*

Proof. We prove by induction on j that

$$u \in H^{s_0+j(\mu-a)}_{\bar{P},\{\vartheta_0\}+j(\mu-b+\delta_{\bar{Q}}-\delta_{\bar{P}})}(X) \tag{3.15}$$

for all $j \in \mathbb{N}$. Since $\mu - a > 0$, $\mu - b + \delta_{\bar{Q}} - \delta_{\bar{P}} > 0$, this implies $u \in C^\infty_{\bar{P}}(X)$. By assumption, (3.15) holds for $j = 0$. Now suppose that (3.15) for some j has already been proven. From (B) we conclude that $\Pi(u) \in H^{s_0+j(\mu-a)-a}_{\bar{Q},\{\vartheta_0\}+j(\mu-b+\delta_{\bar{Q}}-\delta_{\bar{P}})-b}(X)$. In view of (A), elliptic regularity gives $u \in H^{s_0+(j+1)(\mu-a)}_{\bar{P},\{\vartheta_0\}+(j+1)(\mu-b+\delta_{\bar{Q}}-\delta_{\bar{P}})}(X)$. □

Example 3.16. Here, we provide an example for a nonlinearity Π satisfying (B). Let $\Pi(u) = K_0(u)/K_1(u)$, where K_0, K_1 are polynomials of degree m_0 and m_1, respectively. Let $u \in H^s_{P,\vartheta-0}(X)$, where $s > (n+1)/2$, $\delta_P + \vartheta > (n+1)/2$, and $\vartheta > 0$. Further, we assume that the multiplicities of the highest singular values for P are simple and the coefficient functions for these singular values nowhere vanish on Y. Then we have $K_0(u) \in H^s_{P_0,\vartheta-0}(X)$, $K_1(u) \in H^s_{P_1,\vartheta-0}(X)$ for resulting asymptotic types P_0, P_1. In particular, P_0 is dominated by $\mathbf{1} \vee \bigvee_{k=1}^{m_0} P^k$ and P_1 is dominated by $\mathbf{1} \vee \bigvee_{k=1}^{m_1} P^k$. Furthermore, it is readily seen that $v \in H^s_{P_1,\vartheta-0}(X)$ and $v \neq 0$ everywhere on X° implies that $1/v \in H^s_{Q_1,\vartheta'-0}(X)$ for some resulting asymptotic type Q_1. Hence, we are allowed to set $\bar{P} = P$, $\bar{Q} = P_0 \circ Q_1$, and

$$\mathcal{U} = \left\{u \in H^s_{P,\vartheta-0}(X);\, K_1(u) \neq 0 \text{ everywhere on } X^\circ\right\}.$$

The condition $s > (n+1)/2$ can be replaced by $s \geq 0$. Then we additionally need $u \in L^\infty_{\mathrm{loc}}(X^\circ)$.

3.3 Proof of the main theorem

The main step consists in constructing asymptotic types $\bar{P}$, $\bar{Q}$ so that Proposition 3.15 applies. Thereby, upon choosing $\delta \in \mathbb{R}$ even smaller if necessary, we can assume that

$$\delta \leq \bar{\mu} + (n+1)/2$$

and that $A \in \mathrm{Diff}^\mu_{\mathrm{Fuchs}}(X)$ is elliptic with respect to the conormal order δ. Set $\Delta = \delta_R + (\mu - \bar{\mu}) - (n+1)/2$. By assumption (1.4), $\Delta > 0$.

3.3.1 Construction of asymptotic types P, Q

We construct by induction on h sequences $\{P_h\}_{h=0}^{\infty} \subset \underline{\mathrm{As}}^{\delta}(Y)$ and $\{Q_h\}_{h=0}^{\infty} \subset \underline{\mathrm{As}}^{\sharp}(Y)$ of asymptotic types as follows: Set $P_0 = \mathcal{P}^{\delta}(\mathcal{O}; A)$. Suppose that $P_0, \dots, P_h$ and $Q_0, \dots, Q_{h-1}$ for some h have already been constructed. Then

$$Q_h = \big(\mathcal{Q}(P_h; B_1, \dots, B_K)^{\sharp}\big)^{\sim}, \tag{3.16}$$

$$P_{h+1} = \mathcal{P}^{\delta}(R \circ Q_h; A). \tag{3.17}$$

Lemma 3.17. *For each $h \geq 0$,*

$$P_h = P_{h+1} \quad \text{up to the conormal order } \delta_R + \mu + h\Delta - 0, \tag{3.18}$$

$$Q_h = Q_{h+1} \quad \text{up to the conormal order } \delta_R + (\mu - \bar{\mu}) + h\Delta - 0. \tag{3.19}$$

In particular, the joins $P = \bigvee_{h=0}^{\infty} P_h$ and $Q = \bigvee_{h=0}^{\infty} Q_h$ exist.

Proof. We set $Q_{-1} = \mathcal{O}$ and proceed by induction on h. (3.19) holds for $h = -1$, since $Q_0 \in \underline{\mathrm{As}}^{\sharp}(Y)$ and, therefore, $Q_0 = \mathcal{O}$ up to the conormal order $(n+1)/2 - 0$.
Suppose that $Q_{h-1} = Q_h$ up to the conormal order $\delta_R + (\mu - \bar{\mu}) + (h-1)\Delta - 0$ for some $h \geq 0$ has already been proved. Then $R \circ Q_{h-1} = R \circ Q_h$ up to the conormal order $\delta_R + h\Delta - 0$ and $P_h = P_{h+1}$ up to the conormal order $\delta_R + \mu + h\Delta - 0$, since $P_h = \mathcal{P}^{\delta}(R \circ Q_h; A)$, $P_{h+1} = \mathcal{P}^{\delta}(R \circ Q_{h+1}; A)$.
Now suppose that $P_h = P_{h+1}$ up to the conormal order $\delta_R + \mu + h\Delta - 0$. We obtain $\mathcal{Q}(P_h; B_1, \dots, B_K) = \mathcal{Q}(P_{h+1}; B_1, \dots, B_K)$ up to the conormal order $\delta_R + (\mu - \bar{\mu}) + h\Delta - 0$ and, therefore, $Q_h = Q_{h+1}$ up to the conormal order $\delta_R + (\mu - \bar{\mu}) + h\Delta - 0$, since $Q_h = \big(\mathcal{Q}(P_h; B_1, \dots, B_K)^{\sharp}\big)^{\sim}$, $Q_{h+1} = \big(\mathcal{Q}(P_{h+1}; B_1, \dots, B_K)^{\sharp}\big)^{\sim}$.
This completes the inductive proof. $\square$

Lemma 3.18. *The asymptotic types $P = \bigvee_{h=0}^{\infty} P_h \in \underline{\mathrm{As}}^{\delta}(Y)$, $Q = \bigvee_{h=0}^{\infty} Q_h \in \underline{\mathrm{As}}^{\sharp}(Y)$ satisfy:*
(a) $\mathcal{Q}(P; B_1, \dots, B_K)^{b} = \mathcal{Q}(P; B_1, \dots, B_K)^{\sharp}$ *and* $Q = \big(\mathcal{Q}(P; B_1, \dots, B_K)^{\sharp}\big)^{\sim}$;
(b) $P = \mathcal{P}^{\delta}(R \circ Q; A)$;
(c) *Q is multiplicatively closed.*
Furthermore, P, Q are minimal among all asymptotic types in $\underline{\mathrm{As}}^{\delta}(Y)$ and $\underline{\mathrm{As}}^{\sharp}(Y)$, respectively, satisfying (a) *to* (c).

Proof. The assertions immediately follow from the description of the asymptotic types P_h, Q_h given in the previous lemma.
Only $\mathcal{Q}(P; B_1, \dots, B_K)^{b} = \mathcal{Q}(P; B_1, \dots, B_K)^{\sharp}$ needs an argument: But $P = P_0$ up to the conormal order $\delta_R + \mu - 0$, so we get $\mathcal{Q}(P; B_1, \dots, B_K) = \mathcal{Q}(P_0; B_1, \dots, B_K)$ up to the conormal order $\delta_R + (\mu - \bar{\mu}) - 0 = (n+1)/2 + \Delta - 0 > (n+1)/2$, and $\mathcal{Q}(P_0; B_1, \dots, B_K)^{b} = \mathcal{Q}(P_0; B_1, \dots, B_K)^{\sharp}$ is exactly the non-resonance condition (1.5). $\square$

Note that, by the non-resonance condition (1.5) and Proposition 3.9,

$$
\begin{aligned}
B_J u \in \mathcal{H}^{s-\bar{\mu},\delta-\bar{\mu}}_{\mathcal{Q}(P;B_1,\dots,B_K),\vartheta-0}(X) \cap L^\infty(X) \\
\subseteq \mathcal{H}^{s-\bar{\mu},\delta-\bar{\mu}}_{\mathcal{Q}(P;B_1,\dots,B_K)^\sharp,\vartheta-0}(X) \subseteq \mathcal{H}^{s-\bar{\mu},\delta-\bar{\mu}}_{Q,\vartheta-0}(X) \quad (3.20)
\end{aligned}
$$

if $u \in \mathcal{H}^{s,\delta}_{P,\vartheta-0}(X)$, $\delta - \bar{\mu} + \vartheta > (n+1)/2$, and $B_J u \in L^\infty(X)$.

3.3.2 End of the proof of Theorem 1.1

Since $B_J u \in L^\infty(X) \subset \mathcal{H}^{0,(n+1)/2-0}(X)$ for all $1 \leq J \leq K$, we have $F(x, B_1 u, \dots, B_K u) \in \mathcal{H}^{0,\delta_R-0}(X)$ and

$$
u \in H^\mu_{P_0,\delta_R+\mu-\delta_P-0}(X) = H^\mu_{P,\delta_R+\mu-\delta_P-0}(X)
$$

by elliptic regularity.
To conclude the proof of Theorem 1.1, we apply Proposition 3.15 with $\Pi u = F(x, B_1 u, \dots, B_K u)$, $\bar{P} = P$, $\bar{Q} = R \circ Q$, where $P \in \underline{\mathrm{As}}^\delta(Y)$, $Q \in \underline{\mathrm{As}}^\sharp(Y)$ have been constructed in Lemmas 3.17, 3.18, $s_0 = \mu$, $\{\vartheta_0\} = \delta_R + \mu - \delta_P - 0$, $a = \bar{\mu}$, $b = (n+1)/2 - \delta_P + \bar{\mu}$, and

$$
\mathcal{U} = \bigl\{u \in H^\mu_{P,\delta_R+\mu-\delta_P-0}(X);\ B_J u \in L^\infty(X),\ 1 \leq J \leq K\bigr\}. \quad (3.21)
$$

Then $a < \mu$, $b < \delta_{R\circ Q} - \delta_P + \mu$ for $\delta_{R\circ Q} = \delta_R$, $\Delta > 0$, and $\delta_P + \vartheta_0 = \delta_R + \mu > \bar{\mu} + (n+1)/2 \geq \delta$, i.e., $\delta_P + \{\vartheta_0\} \geq \delta$. Moreover, condition (A) is fulfilled.
To check condition (B), note that $u \in \mathcal{U} \cap H^s_{P,\vartheta-0}(X)$ for $s \geq \mu$, $\vartheta \geq \delta_R + \mu - \delta_P$ implies

$$
F(x, B_1 u, \dots, B_K u) \in H^{s-\bar{\mu}}_{R\circ Q,\delta_P-\bar{\mu}-(n+1)/2+\vartheta-0}(X)
$$

by (3.20) and Proposition 3.14.
Thus Proposition 3.15 applies to yield $u \in C^\infty_P(X)$.

Remark 3.19. From (3.21) it is seen that the asymptotic type $P \in \underline{\mathrm{As}}^\delta(Y)$ can be taken smaller, namely instead of $P = \mathcal{P}^\delta(R \circ Q; A)$ we can choose the asymptotic type

$$
\bigvee \bigl\{P' \in \underline{\mathrm{As}}^\delta(Y);\ P' \preccurlyeq \mathcal{P}^\delta(R \circ Q; A),\ \mathcal{Q}(P'; B_1, \dots, B_K) \in \underline{\mathrm{As}}^\sharp(Y)\bigr\}.
$$

In concrete problems, the resulting asymptotic type for u can be even smaller, e.g., due to nonlinear interaction caused by the *special structure* of the nonlinearity.

3.4 Example: The equation $\Delta u = Au^2 + B(x)u$ in three space dimensions

Let Ω be a bounded, smooth domain in $\mathbb{R}^3$ containing 0. We are going to study singular solutions to the equation

$$
\Delta u = Au^2 + B(x)u \ \text{ on } \Omega \setminus \{0\}, \quad (3.22)
$$

$$
\gamma_0 u = c_0,\ \ u\big|_{\partial\Omega} = \phi, \quad (3.23)
$$

where $\gamma_0 u = \lim_{x\to 0} |x|u(x)$, $A \in \mathbb{R}$, and $B \in C^\infty(\overline{\Omega})$ is real-valued. Since the quadratic polynomial $Au^2 + B(x)u$ rather than a general nonlinearity $F(x,u)$ enters, we may admit complex-valued solutions u to (3.22). In particular, $c_0 \in \mathbb{C}$.

Remark 3.20. By results in Véron [17], one expects the limit $\lim_{x\to 0} |x|u(x)$ exist for the solutions $u = u(x)$ to (3.22).

On $\Omega \setminus \{0\}$, we introduce polar coordinates $(t,y) \in \mathbb{R}_+ \times S^2$, $t = |x|$, $y = x/|x|$. We further introduce the function spaces

$$\mathcal{X}^2 = \{c_0 t^{-1} + c_{11} \log t + u_0(x);\, c_0, c_{11} \in \mathbb{C},\, u_0 \in H^2(\Omega)\},$$
$$\mathcal{Y}^0 = \{d_0 t^{-2} + v_0(x);\, d_0 \in \mathbb{C},\, v_0 \in L^2(\Omega)\},$$

the definition of which is suggested by formal asymptotic analysis. On the space $\mathcal{X}^2$, we have the trace operators γ_0, γ_1, γ_{11}, where $\gamma_{11} u = \lim_{t\to+0}\big(u(x) - (\gamma_0 u)t^{-1}\big)/\log t$, $\gamma_1 u = \lim_{t\to+0}\big(u(x) - (\gamma_0 u)t^{-1} - (\gamma_{11}u)\log t\big)$.

Proposition 3.21. *Suppose that $B(x) \geq 0$ for all $x \in \overline{\Omega}$. Then, for all $c_0 \in \mathbb{C}$, $\phi \in H^{3/2}(\partial\Omega)$ with $|c_0| + \|\phi\|_{H^{3/2}(\partial\Omega)}$ small enough, the boundary value problem* (3.22)*,* (3.23) *admits a unique small solution $u \in \mathcal{X}^2$. This solution $u = u(x)$ obeys a complete conormal asymptotic expansion as $x \to 0$ that can successively be calculated. Especially,*

$$c_{11} = Ac_0^2, \tag{3.24}$$

where $c_{11} = \gamma_{11} u$.

Proof. Let us consider the nonlinear operator

$$\Psi\colon \mathcal{X}^2 \to \mathcal{Y}^0 \times \mathbb{C} \times H^{3/2}(\partial\Omega), \quad u \mapsto \left(\Delta u - Au^2 - B(x)u, \gamma_0 u, u\big|_{\partial\Omega}\right).$$

It is readily seen that the linearization of Ψ about $u = 0$ is an isomorphism between the indicated spaces. Thus, the existence of a unique small solution $u \in \mathcal{X}^2$ to (3.22), (3.23) is implied by the inverse function theorem. (3.24) likewise follows.

Furthermore, writing this solution in the form $u(x) = c_0 t^{-1} + c_{11}\log t + u_0(x)$, where $u_0 \in H^2(\Omega)$, we get that u_0 fulfills the equation

$$\begin{aligned} c_{11}t^{-2} + \Delta u_0 &= A\left(c_0^2 t^{-2} + 2c_0 c_{11} t^{-1}\log t + c_{11}^2 \log^2 t\right) \\ &+ 2A\left(c_0 t^{-1} + c_{11}\log t\right) u_0 + Au_0^2 + B(x)\left(c_0 t^{-1} + c_{11}\log t\right) + B(x)u_0. \end{aligned} \tag{3.25}$$

This can be brought into the form (1.1) with $A = \Delta$,

$$\begin{aligned} F(x,\nu) = \big(&2Ac_0c_{11}t^{-1}\log t + B(x)c_0 t^{-1} + Ac_{11}^2\log^2 t + B(x)c_{11}\log t\big) \\ &+ \big(2Ac_0 t^{-1} + 2Ac_{11}\log t + B(x)\big)\,\nu + A\nu^2, \end{aligned}$$

since $\Delta = t^{-2}\left((-t\partial_t)^2 - (-t\partial_t) + \Delta_{S^2}\right) \in \mathrm{Diff}^2_{\mathrm{Fuchs}}(\Omega \setminus \{0\})$, where $0 \in \Omega$ is considered as conical point with cone base $S^2 = \{x \in \mathbb{R}^3;\, |x| = 1\}$, cf. Remark 1.3, and Δ_{S^2} being the Laplace-Beltrami operator on S^2. The conditions (1.4), (1.5) are obviously satisfied.

Thus, Theorem 1.1 applies to $u_0 \in H^2(\Omega) \subset L^\infty(\Omega)$ to yield that u_0 and, therefore, u obey a complete conormal asymptotic expansion. □

Remark 3.22. (a) Taking for P the asymptotic type in $\underline{\mathrm{As}}^0(S^2)$ that comes out of the calculation of the conormal asymptotic expansion for u, i.e., we have $u \in C_P^\infty(\Omega \setminus \{0\})$, and for Q the resulting asymptotic type in $\underline{\mathrm{As}}^{-2}(S^2)$ for the right-hand side of (3.22), we are in a situation in which Proposition 3.15 directly applies without having boundedness assumptions for u.
(b) Allowing more general functions $B \in C_R^\infty(\Omega \setminus \{0\})$ for some $R \in \underline{\mathrm{As}}^{-1/2}(S^2)$ (the conormal order $-1/2$ ensures that the term $Ac_0^2t^{-2}$ dominates on the right-hand side of (3.25)) rather than $B \in C_{P_0}^\infty(\Omega \setminus \{0\})$, where P_0 is the asymptotic type for Taylor asymptotics, one can perform the same analysis as before upon replacing the space $H^2(\Omega)$ in the definition of $\mathcal{X}^2$ accordingly.

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