Classification of Holomorphic Vector Bundles on Noncommutative Two-Tori

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Abstract. We prove that every holomorphic vector bundle on a noncommutative two-torus T can be obtained by successive extensions from standard holomorphic bundles considered in [2]. This implies that the category of holomorphic bundles on T is equivalent to the heart of a certain t-structure on the derived category of coherent sheaves on an elliptic curve.

1. Introduction

In this paper we continue the study of holomorphic bundles on noncommutative two-tori that was begun in [2]. Recall that for every $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $\tau \in \mathbb{C} \setminus \mathbb{R}$ we considered in [2] holomorphic vector bundles on a noncommutative complex torus $T = T_{\theta,\tau}$. By definition, the algebra A_θ of smooth functions on T consists of series $\sum_{(m,n)\in\mathbb{Z}^2} a_{m,n}U_1^m U_2^n$ where the coefficients $a_{m,n} \in \mathbb{C}$ decrease rapidly at infinity and the multiplication is defined using the rule

$$U_1U_2 = \exp(2\pi i\theta)U_2U_1.$$

We consider the derivation $\delta = \delta_\tau : A_\theta \to A_\theta$ defined by

$$\delta(\sum a_{m,n}U_1^mU_2^n) = 2\pi i\sum_{m,n}(m\tau + n)a_{m,n}U_1^mU_2^n$$

as an analogue of the $\overline{\partial}$-operator. A holomorphic bundle over T is a pair $(E, \overline{\nabla})$ consisting of a finitely generated projective right A_θ-module E and an operator $\overline{\nabla} : E \to E$ satisfying the Leibnitz identity

$$\overline{\nabla}(ea) = \overline{\nabla}(e)a + e\delta(a),$$

where $e \in E$, $a \in A_\theta$. There is an obvious definition of a holomorphic map between holomorphic bundles, so we can define the category $\mathcal{C}(T)$ of holomorphic bundles on T.

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For every pair of relatively prime integers (c,d) such that $c\theta+d>0$ and a complex number z we define a *standard holomorphic bundle* $(E_{d,c}(\theta),\overline{\nabla}_z)$ as follows. If $c\neq 0$ then

$$E_{d,c}(\theta)=\mathcal{S}(\mathbb{R}\times\mathbb{Z}/c\mathbb{Z})=\mathcal{S}(\mathbb{R})^{|c|},$$

where $\mathcal{S}(\mathbb{R})$ is the Schwartz space of functions on $\mathbb{R}$, with the A_θ-action defined by

$$fU_1(x,\alpha)=f(x-\frac{1}{\mu},\alpha-1),\ fU_2(x,\alpha)=\exp(2\pi i(x-\frac{\alpha d}{c}))f(x,\alpha),$$

where $x\in\mathbb{R}$, $\alpha\in\mathbb{Z}/c\mathbb{Z}$, $\mu=\frac{c}{c\theta+d}$. The operator $\overline{\nabla}_z$ on this space is given by

$$\overline{\nabla}_z(f)=\frac{\partial f}{\partial x}+2\pi i(\tau\mu x+z)f. \tag{1.1}$$

For $c=0$ and $d=1$ we set $E_{1,0}(\theta)=A_\theta$ with the natural right A_θ-action and the operator $\overline{\nabla}_z$ is given by

$$\overline{\nabla}_z(a)=\delta(a)+2\pi iza.$$

We define degree, rank and slope of a bundle $E=E_{d,c}(\theta)$ by setting $\deg(E)=c$, $\mathrm{rk}(E)=c\theta+d$ and $\mu(E)=\deg(E)/\mathrm{rk}(E)$. Note that $\mathrm{rk}(E)>0$ and $\mu=\mu(E)$ in the formulae above.

According to the theorem of Rieffel (see [5]) every finitely generated projective right A_θ-module is isomorphic to $E=E_{d,c}(\theta)^{\oplus n}$ for some (c,d) as above and $n\geq 0$. Moreover, the degree and rank defined above extend to additive functions on the category of finitely generated projective A_θ-modules.

The category of holomorphic bundles $\mathcal{C}=\mathcal{C}(T)$ has a natural structure of a $\mathbb{C}$-linear exact category. In particular, for every pair of holomorphic bundles E_1 and E_2 we can form the vector space $\mathrm{Ext}^1_{\mathcal{C}}(E_1,E_2)$ parametrizing extensions of E_1 by E_2. Sometimes we will also use the notation $\mathrm{Ext}^0_{\mathcal{C}}:=\mathrm{Hom}_{\mathcal{C}}$. Let $\mathcal{C}'\subset\mathcal{C}$ be the minimal full subcategory of $\mathcal{C}$ containing all standard holomorphic bundles and closed under extensions. Our main result is the following theorem.

Theorem 1.1. *One has $\mathcal{C}'=\mathcal{C}$.*

Combining this theorem with the study of the category $\mathcal{C}'$ in [2] we obtain the following result.

Corollary 1.2. *The category $\mathcal{C}$ is abelian. It is equivalent to the heart $\mathcal{C}^\theta$ of the t-structure on the derived category of coherent sheaves on the elliptic curve $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$, associated with θ (see section 3 of* [2] *or section 3.1 below).*

Remark. Recall that we always assume θ to be irrational. For rational θ the category $\mathcal{C}$ will not be abelian.

Corollary 1.3. *For every indecomposable holomorphic bundle E on T there exists a standard holomorphic bundle $\overline{E}$ and a filtration $0=E_0\subset E_1\subset\ldots\subset E_n=E$ by holomorphic subbundles such that all quotients E_i/E_{i-1} are isomorphic to $\overline{E}$.*

The proof of Theorem 1.1 consists of two steps. First, we develop the cohomology theory for holomorphic bundles on T and prove the analogues of the standard theorems for them (such as finiteness, Riemann-Roch and Serre duality). Then we combine these results with the techniques of [4] where the category $\mathcal{C}'$ was described in terms of coherent modules over a certain algebra.

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2. Cohomology of holomorphic bundles on noncommutative two-tori

2.1. Cohomology and Ext-spaces. Let $(E, \overline{\nabla})$ be a holomorphic bundle on $T = T_{\theta,\tau}$. Then the *cohomology* of E is defined by

$$H^i(E) = H^i(E, \overline{\nabla}) = H^i(E \xrightarrow{\overline{\nabla}} E)$$

where $i = 0$ or $i = 1$. Thus, $H^0(E) = \ker(\overline{\nabla})$, $H^1(E) = \operatorname{coker}(\overline{\nabla})$. These spaces are closely related to Ext^i-spaces in the category of holomorphic bundles (where $i = 0$ or $i = 1$). To explain this connection we have to use Morita equivalences between noncommutative tori. Recall that for every standard bundle $E_0 = E_{d,c}(\theta)$ the algebra of endomorphisms $\operatorname{End}_{A_\theta}(E_0)$ can be identified with the algebra $A_{\theta'}$ for some $\theta' \in \mathbb{R}$. In fact, $\theta' = \frac{a\theta+b}{c\theta+d}$, where a and b are chosen in such a way that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Furthermore, if E_0 is equipped with a standard holomorphic structure $\overline{\nabla}$ then the formula $\phi \mapsto [\overline{\nabla}, \phi]$ defines a derivation of $\operatorname{End}_{A_\theta}(E_0) \simeq A_{\theta'}$, hence the corresponding torus $T_{\theta'}$ is equipped with a complex structure. In fact, this derivation on $A_{\theta'}$ is equal to $\delta_\tau / \operatorname{rk}(E_0)$, where τ is the same parameter that was used to define the complex structure on T_θ (see Proposition 2.1 of [2]). Now one can define the Morita equivalence

$$\mathcal{C}(T_{\theta',\tau}) \to \mathcal{C}(T_{\theta,\tau}) : E \mapsto E \otimes_{A_{\theta'}} E_0,$$

where the tensor product is equipped with the complex structure

$$\overline{\nabla}(e \otimes e_0) = \frac{1}{\operatorname{rk}(E_0)} \overline{\nabla}_E(e) \otimes e_0 + e \otimes \overline{\nabla}_{E_0}(e_0)$$

(see Propositions 2.1 and 3.2 of [2]). This functor sends standard holomorphic bundles on $T_{\theta',\tau}$ to standard holomorphic bundles on $T_{\theta,\tau}$. The inverse functor is

$$\mathcal{C}(T_{\theta,\tau}) \to \mathcal{C}(T_{\theta',\tau}) : E \mapsto \operatorname{Hom}_{A_\theta}(E_0, E), \tag{2.1}$$

where the latter space has a natural right action of $A_{\theta'} \simeq \operatorname{End}_{A_\theta}(E_0)$. Now we can formulate the connection between the cohomology and Ext-groups. For every holomorphic bundle E and a standard holomorphic bundle E_0 on $T =$

$T_{\theta,\tau}$ one has a natural isomorphism

$$\mathrm{Ext}^i_{\mathcal{C}}(E_0, E) \simeq H^i(\mathrm{Hom}_{A_\theta}(E_0, E)), \tag{2.2}$$

where $\mathrm{Hom}_{A_\theta}(E_0, E)$ is viewed as a holomorphic bundle on $T_{\theta'}$ (the proof is similar to Proposition 2.4 of [2]). Note that for an arbitrary pair of holomorphic bundles E_1 and E_2 one can still define an operator $\overline{\nabla}$ on $\mathrm{Hom}_{A_\theta}(E_1, E_2)$ such that the analogue of isomorphism (2.2) holds. However, we have a natural interpretation of $\mathrm{Hom}_{A_\theta}(E_1, E_2)$ as a holomorphic bundle on some noncommutative two-torus only in the case when one of the bundles E_1 or E_2 is standard (see (2.3) below).

2.2. Duality and metrics. One can define the category of *left holomorphic bundles* on T by replacing right A_θ-modules with left ones and changing the Leibnitz identity appropriately. The definition of cohomology for these bundles remains the same. There is a natural duality functor $E \mapsto E^\vee$ that associates to a (right) holomorphic bundle E the left holomorphic bundle $\mathrm{Hom}_{A_\theta}(E, A_\theta)$. More generally, for every standard holomorphic bundle E_0 we can consider $\mathrm{Hom}_{A_\theta}(E, E_0)$ as a left module over $\mathrm{End}_{A_\theta}(E_0) \simeq A_{\theta'}$ equipped with an induced holomorphic structure. Then the natural isomorphism

$$\mathrm{Ext}^i_{\mathcal{C}}(E, E_0) \simeq H^i(\mathrm{Hom}_{A_\theta}(E, E_0)) \tag{2.3}$$

allows to view $\mathrm{Ext}^i_{\mathcal{C}}(E, E_0)$ as cohomology of a holomorphic bundle on $T_{\theta'}$. Using duality, the functor (2.1) for a standard holomorphic bundle E_0 can be rewritten as the usual Morita functor due to the isomorphism

$$\mathrm{Hom}_{A_\theta}(E_0, E) \simeq E \otimes_{A_\theta} E_0^\vee. \tag{2.4}$$

For a standard holomorphic bundle E_0 on $T_{\theta,\tau}$ the dual bundle $E_0^\vee$ can also be considered as a right holomorphic bundle on $T_{\theta',\tau}$, where $\mathrm{End}_{A_\theta}(E_0) \simeq A_{\theta'}$. In fact, it is again a standard holomorphic bundle (see Corollary 2.3 of [2]). More precisely, for $E_0 = E_{d,c}(\theta)$ we have $\theta' = \frac{a\theta+b}{c\theta+d}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. The left action of $A_{\theta'}$ on $E_{d,c}(\theta) = \mathcal{S}(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z})$ (where $c \neq 0$) is defined by the formulae

$$U_1 f(x, \alpha) = f(x - \frac{1}{c}, \alpha - a),\ U_2 f(x, \alpha) = \exp(2\pi i(\frac{x}{c\theta + d} - \frac{\alpha}{c}))f(x, \alpha),$$

where $x \in \mathbb{R}$, $\alpha \in \mathbb{Z}/c\mathbb{Z}$. We can identify $E_{d,c}(\theta)^\vee$ considered as a right $A_{\theta'}$-module with $E_{a,-c}(\theta')$ using the natural pairing

$$t : E_{a,-c}(\theta') \otimes E_{d,c}(\theta) \to A_\theta$$

constructed as follows (see Proposition 1.2 of [2]). First, we define the map

$$b : E_{a,-c}(\theta') \otimes E_{d,c}(\theta) \to \mathbb{C}$$

by the formula

$$b(f_1, f_2) = \sum_{\alpha \in \mathbb{Z}/c\mathbb{Z}} \int_{x \in \mathbb{R}} f_1(\frac{x}{c\theta + d}, \alpha) f_2(x, -a\alpha) dx.$$

Then t is given by

$$t(f_1,f_2)=\sum_{(m,n)\in\mathbb{Z}^2}U_1^mU_2^nb(U_2^{-n}U_1^{-m}f_1\otimes f_2).$$

The corresponding isomorphism

$$E_{d,c}(\theta)^\vee\simeq E_{a,-c}(\theta')$$

is compatible with the $A_\theta-A_{\theta'}$-bimodule structures and with holomorphic structures (see Corollary 2.3 of [2]). Note that $b=\mathrm{tr}\circ t$, where $\mathrm{tr}:A_\theta\to\mathbb{C}$ is the trace functional sending $\sum a_{m,n}U_1^mU_2^n$ to $a_{0,0}$.
On the other hand, we can define a $\mathbb{C}$-antilinear isomorphism

$$\sigma:E_{d,c}(\theta)\to E_{a,-c}(\theta')$$

by the formula

$$\sigma(f)(x,\alpha)=\overline{f((c\theta+d)x,-a\alpha)}.$$

This isomorphism satisfies

$$\sigma(bea)=a^*\sigma(e)b^* \tag{2.5}$$

for $e\in E_{d,c}(\theta)$, $a\in A_\theta$, $b\in A_{\theta'}$, where $*:A_\theta\to A_\theta$ is the $\mathbb{C}$-antilinear anti-involution sending U_i to U_i^{-1}. In view of the identification of $E_{a,-c}(\theta')$ with the dual bundle to $E_{d,c}(\theta)$ the isomorphism σ should be considered as an analogue of the Hermitian metric on $E_{d,c}(\theta)$. The corresponding analogue of the scalar product on global sections is simply the Hermitian form on $E_{d,c}(\theta)$ given by the formula

$$\langle f_1,f_2\rangle=b(\sigma(f_2),f_1)=\sum_{\alpha\in\mathbb{Z}/c\mathbb{Z}}\int_{x\in\mathbb{R}}f_1(x,\alpha)\overline{f_2(x,\alpha)}dx, \tag{2.6}$$

where $f_1,f_2\in E_{d,c}(\theta)$. We can also define the corresponding L^2-norm: $||f||_0^2=\langle f,f\rangle$. The above Hermitian form is related to the structure of $A_{\theta'}-A_\theta$-bimodule on $E_{d,c}(\theta)$ in the following way:

$$\langle f_1,af_2b\rangle=\langle a^*f_1b^*,f_2\rangle,$$

where $a\in A_{\theta'}$, $b\in A_\theta$ (this is a consequence of (2.5) and of Lemma 1.1 of [2]).
In the case of the trivial bundle $E_{1,0}(\theta)=A_\theta$ we can easily modify the above definitions. First of all, $\theta'=\theta$ and the dual bundle is still A_θ. The role of σ is played by $*:A_\theta\to A_\theta$ and the Hermitian form on A_θ is given by $\langle a,b\rangle=\mathrm{tr}(ab^*)$. The corresponding L^2-norm is

$$||\sum a_{m,n}U_1^mU_2^n||_0^2=\sum|a_{m,n}|^2.$$

Note that the operator $\overline{\nabla}_z$ on $E_{d,c}(\theta)$ admits an adjoint operator $\overline{\nabla}_z^*$ with respect to the Hermitian metrics introduced above. Namely, for $c\neq 0$ it is given by

$$\overline{\nabla}_z^*(f)=-\frac{\partial f}{\partial x}-2\pi i(\overline{\tau}\mu x+\overline{z})f,$$

while for $c = 0$ we have $\overline{\nabla}_z^* = -\delta_{\overline{\tau}} - 2\pi iz\,\mathrm{id}$ on A_θ. In either case we have

$$\overline{\nabla}_z\overline{\nabla}_z^* - \overline{\nabla}_z^*\overline{\nabla}_z = \lambda \cdot \mathrm{id} \tag{2.7}$$

for some constant $\lambda \in \mathbb{R}$.

It follows that for an arbitrary holomorphic structure $\overline{\nabla}$ on $E = E_{d,c}(\theta)^{\oplus n}$ there exists an adjoint operator $\overline{\nabla}^* : E \to E$ with respect to the above Hermitian metric. Indeed, we can write $\overline{\nabla} = \overline{\nabla}_0 + \phi$, where $\overline{\nabla}_0$ is the standard holomorphic structure and set $\overline{\nabla}^* = \overline{\nabla}_0^* + \phi^*$.

2.3. Sobolev spaces. The idea to consider Sobolev spaces for bundles on noncommutative tori is due to M. Spera (see [7], [8]). Let $(E, \overline{\nabla})$ be a standard holomorphic bundle on $T_{\theta,\tau}$. For $s \in \mathbb{Z}$, $s \geq 0$ we define the s-th Sobolev norm on E by setting

$$||e||_s^2 = \sum_{i=0}^{s} ||\overline{\nabla}^i e||_0^2,$$

where $||e||_0$ is the L^2-norm on E. We define $W_s(E)$ to be the completion of E with respect to this norm. Note that there is a natural embedding $W_{s+1}(E) \subset W_s(E)$. We can define analogous spaces for $E^{\oplus n}$ in an obvious way.

All the definitions above make sense also for rational θ. Moreover, for $\theta \in \mathbb{Z}$ the space E can be identified with the space of smooth section of a holomorphic vector bundle V on an elliptic curve $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ in such a way that $\overline{\nabla}$ corresponds to the $\overline{\partial}$-operator. Furthermore, the L^2-norm above corresponds to the L^2-norm with respect to a Hermitian metric on V that has constant curvature. This implies that in this case the Sobolev spaces $W_s(E)$ coincide with the corresponding Sobolev spaces constructed for the holomorphic bundle V. Indeed, using the equation (2.7) it is easy to see that the norm $||e||_s$ is equivalent to the norm given by

$$(||e||_s')^2 = \sum_{i=0}^{s} \langle e, (\overline{\nabla}^*\overline{\nabla})^i e\rangle$$

which is equivalent to the standard Sobolev norm.

An important observation is that the operator $\overline{\nabla}_z$ defined by (1.1) depends only on $\tau\mu$, where μ is the slope of the bundle, so it is the same for the bundle $E_{d,c}(\theta)$ on $T_{\theta,\tau}$ and the bundle $E_{d,c}(\mathrm{sign}(c)N)$ on the commutative torus $T_{\mathrm{sign}(c)N,\tau'}$, where N is a large enough integer so that $|c|N + d > 0$, $\tau' = (|c|N + d)/(c\theta + d)$. Therefore, the sequences of spaces $(W_s(E))$ in these two cases are the same. Hence, the following standard results about Sobolev spaces in the commutative case extend immediately to our situation (the first two are analogues of Rellich's lemma and Sobolev's lemma). In all these results E is a direct sum of a finite number of copies of a standard holomorphic bundle.

Lemma 2.1. *The embedding $W_s(E) \subset W_{s-1}(E)$ is a compact operator.*

Lemma 2.2. *One has $E = \cap_{s \geq 0} W_s(E)$.*

Lemma 2.3. *The operator $\overline{\nabla}$ extends to a bounded operator $W_s(E) \to W_{s-1}(E)$*

The following result is the only noncommutative contribution to the techniques of Sobolev spaces, however, it is quite easy.

Lemma 2.4. *For every $\phi \in \mathrm{End}_{A_\theta}(E)$ the operator $\phi : E \to E$ extends to a bounded operator $W_s(E) \to W_s(E)$.*

Proof. It suffices to prove that for every $s \geq 0$ one has

$$||\phi e||_s \leq C \cdot ||e||_s$$

for some constant $C > 0$. By our assumption $E \simeq E_0^{\oplus N}$ for some standard bundle E_0. Identifying $\mathrm{End}_{A_\theta}(E_0)$ with $A_{\theta'}$ for some $\theta' \in \mathbb{R}$ we can write $\phi = \sum a_{m,n} U_1^m U_2^n$ where U_1 and U_2 are unitary generators of $A_{\theta'}$, $a_{m,n}$ are complex $N \times N$ matrices. Since U_1 and U_2 act on E by unitary operators, it follows that

$$||\phi e||_0 \leq C(\phi) \cdot ||e||_0$$

for $e \in E$, where $C(\phi) = \sum_{m,n} ||a_{m,n}||$ (here $||a||$ denote the norm of a matrix a). Applying the Leibnitz rule repeatedly we derive similarly that

$$\sum_{i=0}^{s} ||\overline{\nabla}^i(\phi e)||_0^2 \leq \sum_{i=0}^{s} c_i \cdot ||\overline{\nabla}^i e||_0^2$$

for some constants $c_i > 0$ which implies the result. □

It is convenient to extend the definition of the chain $\ldots \subset W_1(E) \subset W_0(S)$ to the chain of embedded spaces

$$\ldots \subset W_1(E) \subset W_0(S) \subset W_{-1}(E) \subset \ldots$$

by setting $W_{-s}(E) = \overline{W_s(E)}^*$ (the space of $\mathbb{C}$-antilinear functionals) and using the natural Hermitian form of $W_0(E)$. It is easy to see that the results of this section hold for all integer values of s.

Lemma 2.5. *Let $\overline{\nabla} : E \to E$ be a (not necessarily standard) holomorphic structure on E. Then the operators $\overline{\nabla}$ and $\overline{\nabla}^*$ can be extended to bounded operators $W_s(E) \to W_{s-1}(E)$ for every $s \in \mathbb{Z}$.*

Proof. Let $\overline{\nabla}_0$ be a standard holomorphic structure on E. Then $\overline{\nabla} = \overline{\nabla}_0 + \phi$ for some $\phi \in \mathrm{End}_{A_\theta}(E)$. By Lemma 2.3 (resp., Lemma 2.4) there exist a continuous extension $\overline{\nabla}_0 : W_s(E) \to W_{s-1}(E)$ (resp., $\phi : W_s(E) \to W_s(E)$). Hence, $\overline{\nabla}$ extends to a family of continuous operators $\overline{\nabla}(s) : W_s(E) \to W_{s-1}(E)$ for $s \in \mathbb{Z}$. The extensions of $\overline{\nabla}^*$ are given by the adjoint operators $\overline{\nabla}(-s+1)^* : W_s(E) \to W_{s-1}(E)$. □

2.4. Applications to cohomology. We begin our study of cohomology with standard holomorphic bundles.

Proposition 2.6. *Let $(E, \overline{\nabla})$ be a direct sum of several copies of a standard holomorphic bundle on $T_{\theta,\tau}$.*

(i) The cohomology spaces $H^0(E)$ *and* $H^1(E)$ *are finite-dimensional and for* $\mathrm{Im}(\tau) < 0$ *one has*

$$\chi(E) = \dim H^0(E) - \dim H^1(E) = \deg(E).$$

(ii) There exists an operator $Q : E \to E$ *such that*

$$\mathrm{id} - Q\overline{\nabla} = \pi_{\ker \overline{\nabla}},$$

$$\mathrm{id} - \overline{\nabla} Q = \pi_{\overline{\nabla}(E)^{\perp}},$$

where $\overline{\nabla}(E)^{\perp} \subset E$ *is the orthogonal complement to* $\overline{\nabla}(E) \subset E$*, for a finite-dimensional subspace* $V \subset E$ *we denote by* $\pi_V : E \to V$ *the orthogonal projection.*
(iii) If $\mathrm{Im}(\tau) < 0$ *and* $\deg(E) > 0$ *then* $H^1(E) = 0$.
(iv) The operator $Q : E \to E$ *extends to a bounded operator* $W_s(E) \to W_{s+1}(E)$.
(v) For every $e \in W_0(E)$ *one has*

$$||Qe||_0 \leq \frac{1}{2\sqrt{\pi |\,\mathrm{Im}(\tau)\mu(E)|}}||e||_0.$$

Proof. In the commutative case the assertions (i)-(iii) are well known. For example, the operator Q is given by $\overline{\partial}^* G$, where G is the Green operator for the $\overline{\partial}$-Laplacian. The condition $\mathrm{Im}(\tau) < 0$ corresponds to the way we define the operator δ_τ on A_θ (see Proposition 3.1 of [2]). As before we can deduce (i)-(iii) in general from the commutative case. One can also prove these assertions directly in the noncommutative case (see Proposition 2.5 of [2] for the proofs of (i) and (iii)). The assertion (iv) follows immediately from the identity

$$\overline{\nabla}^n Q = \overline{\nabla}^{n-1}(\mathrm{id} - \pi_{\overline{\nabla}(E)^{\perp}}).$$

To prove (v) we can assume that $E = E_{d,c}(\theta)$, where $c \neq 0$ and $\overline{\nabla} = \overline{\nabla}_z$ for some $z \in \mathbb{C}$. Then the space $W_0(E)$ is the orthogonal sum of $|c|$ copies of $L^2(\mathbb{R})$. Moreover, the operator $\overline{\nabla}$ respects this decomposition and restricts to the operator

$$f \mapsto f' + (ax + z)f$$

on each copy, where $a = 2\pi i \tau \mu(E)$. Hence, the operator Q also respects this decomposition and it suffices to consider its restriction to one copy of $L^2(\mathbb{R})$. Since $\mathrm{Re}(a) \neq 0$, by making the unitary transformation of the form $\widetilde{f}(x) = \exp(itx)f(x + t')$ for some $t, t' \in \mathbb{R}$ we can reduce ourselves to the case $z = 0$. Furthermore, the transformation of the form $\widetilde{f} = \exp(i\,\mathrm{Im}(a)x^2/2)f$ gives a unitary equivalence with the operator $\overline{\nabla} : f \mapsto f' + \lambda x f$ where $\lambda = \mathrm{Re}(a)$. Consider the following complete orthogonal system of functions in $L^2(\mathbb{R})$:

$$f_n(x) = H_n(\sqrt{|\lambda|}x)\exp(-|\lambda|\frac{x^2}{2}),\ n = 0, 1, 2, \dots ,$$

where $H_n(x) = (-1)^n \exp(x^2)\frac{d^n}{dx^n}(\exp(-x^2))$ are Hermite polynomials ((f_n) is an eigenbasis of the operator $f \mapsto -f'' + \lambda^2 x^2$). Note that

$$||f_n||_0^2 = \frac{1}{\sqrt{\lambda}}\int_{\mathbb{R}} H_n(x)^2 \exp(-x^2)dx = \frac{2^n \cdot n! \cdot \sqrt{\pi}}{\sqrt{\lambda}}.$$

Assume first that $\lambda > 0$. Then using the formula $H_n'(x) = 2nH_{n-1}(x)$ we obtain

$$\overline{\nabla}(f_n) = 2n\sqrt{\lambda}f_{n-1}$$

for $n > 0$ and $\overline{\nabla}(f_0) = 0$. Therefore, in this case

$$Q(f_n) = \frac{1}{(2n+2)\sqrt{\lambda}}f_{n+1}$$

for all $n \geq 0$. Hence,

$$\frac{||Q(f_n)||_0}{||f_n||_0} = \frac{||f_{n+1}||_0}{(2n+2)\sqrt{\lambda}||f_n||_0} = \frac{1}{\sqrt{(2n+2)\lambda}} \leq \frac{1}{\sqrt{2\lambda}}$$

which implies (v) in this case.

Now assume that $\lambda < 0$. Then using the formula $H_n'(x) = 2xH_n(x) - H_{n+1}(x)$ we find

$$\overline{\nabla}(f_n) = -\sqrt{|\lambda|}f_{n+1}.$$

Hence,

$$Q(f_n) = -\frac{1}{\sqrt{|\lambda|}}f_{n-1}$$

for $n > 0$ and $Q(f_0) = 0$. It follows that

$$\frac{||Q(f_n)||_0}{||f_n||_0} = \frac{||f_{n-1}||_0}{\sqrt{|\lambda|}||f_n||_0} = \frac{1}{\sqrt{2n|\lambda|}} \leq \frac{1}{\sqrt{2|\lambda|}}$$

for $n \geq 1$, which again implies our statement. □

Now we are ready to prove results about cohomology of arbitrary holomorphic bundles. We will use the following well known lemma.

Lemma 2.7. *Let $L : W \to W'$ and $L' : W' \to W$ be bounded operators between Banach spaces such that $L'L = \mathrm{id} + C$, $LL' = \mathrm{id} + C'$ for some compact operators $C : W \to W$ and $C' : W' \to W'$. Then the operator L is Fredholm.*

Theorem 2.8. *(i) For every holomorphic bundle $(E, \overline{\nabla})$ on $T_{\theta,\tau}$ the spaces $H^0(E)$ and $H^1(E)$ are finite-dimensional.*
(ii) If $\operatorname{Im}\tau < 0$ then

$$\chi(E) = \dim H^0(E) - \dim H^1(E) = \deg(E).$$

(iii) Let us equip E with a metric by identifying it with the direct sum of several copies of a standard bundle. Then one has the following orthogonal decompositions

$$E = \ker(\overline{\nabla}) \oplus \overline{\nabla}^*(E),$$
$$E = \ker(\overline{\nabla}^*) \oplus \overline{\nabla}(E),$$

where $\overline{\nabla}^* : E \to E$ is the adjoint operator to $\overline{\nabla}$.

Proof. Let us write the holomorphic structure on E in the form

$$\overline{\nabla} = \overline{\nabla}_0 + \phi,$$

where $(E, \overline{\nabla}_0)$ is holomorphically isomorphic to the direct sum of several copies of a standard holomorphic bundle, $\phi \in \mathrm{End}_{A_\theta}(E)$. By Lemma 2.5, $\overline{\nabla}$ has a bounded extension to an operator $\overline{\nabla} : W_s(E) \to W_{s-1}(E)$ for every $s \in \mathbb{Z}$.
Consider the operator $Q : E \to E$ constructed in Proposition 2.6 for the holomorphic structure $\overline{\nabla}_0$. Then Q extends to a bounded operator $W_s(E) \to W_{s+1}(E)$ for every $s \in \mathbb{Z}$. We have

$$Q\overline{\nabla} = Q\overline{\nabla}_0 + Q\phi = \mathrm{id} - \pi_0 + Q\phi,$$

where π_0 is the orthogonal projection to the finite-dimensional space $\ker(\overline{\nabla}_0) \subset E$. Clearly, π_0 defines a bounded operator $W_0(E) \to \ker(\overline{\nabla}_0)$. Hence, the operator $C = Q\overline{\nabla} - \mathrm{id} : W_s(E) \to W_s(E)$ factors as a composition of some bounded operator $W_s(E) \to W_{s+1}(E)$ with the embedding $W_{s+1}(E) \to W_s(E)$. By Lemma 2.1 this implies that C is a compact operator. Similarly, the operator $C' = \overline{\nabla}Q - \mathrm{id} : W_s(E) \to W_s(E)$ is compact. Applying Lemma 2.7 we deduce that $\overline{\nabla} : W_s(E) \to W_{s-1}(E)$ is a Fredholm operator. This immediately implies that $H^0(E)$ is finite-dimensional. Moreover, we claim that

$$\ker(\overline{\nabla} : W_s(E) \to W_{s-1}(E)) = H^0(E) \subset E$$

for any $s \in \mathbb{Z}$. Indeed, it suffices to check that if $\overline{\nabla}(e) = 0$ for $e \in W_s(E)$ then $e \in E$. Let us prove by induction in $t \geq s$ that $e \in W_t(E)$. Assume that this is true for some t. Then

$$e = Q\overline{\nabla}(e) - C(e) = -C(e) \in W_{t+1}(E).$$

Since $\cap_t W_t(E) = E$ by Lemma 2.2 we conclude that $e \in E$.
Let $Q^* : W_s(E) \to W_{s+1}(E)$ be the adjoint operator to $Q : W_{-s-1}(E) \to W_{-s}(E)$, where $s \in \mathbb{Z}$. Then the operators $Q^*\overline{\nabla}^* - \mathrm{id} = (C')^*$ and $\overline{\nabla}^* Q^* - \mathrm{id} = C^*$ are compact. Thus, the same argument as before shows that for every $s \in \mathbb{Z}$ the operator $\overline{\nabla}^* : W_s(E) \to W_{s-1}(E)$ is Fredholm and one has

$$\ker(\overline{\nabla}^* : W_s(E) \to W_{s-1}(E)) \subset E.$$

Next we claim that

$$E = \ker(\overline{\nabla}^*) \oplus \overline{\nabla}(E).$$

Since the orthogonal complement to $\ker(\overline{\nabla}^*)$ in $W_0(E)$ coincides with the image of $\overline{\nabla} : W_1(E) \to W_0(E)$, it suffices to prove that $E \cap \overline{\nabla}(W_1(E)) \subset \overline{\nabla}(E)$. But if $e = \overline{\nabla}(e_1)$ for some $e \in E$, $e_1 \in W_1(E)$, then we can easily prove by induction in $s \geq 1$ that $e_1 \in W_s(E)$. Indeed, assuming that $e_1 \in W_s(E)$ we have

$$e_1 = Q\overline{\nabla}(e_1) - C(e_1) = Q(e) - C(e_1) \in W_{s+1}(E).$$

A similar argument using the operator Q^* shows that

$$E = \ker(\overline{\nabla}) \oplus \overline{\nabla}^*(E).$$

Thus, we checked that $H^0(E)$ and $H^1(E)$ are finite-dimensional and that $\chi(E)$ coincides with the index of the Fredholm operator $\overline{\nabla} = \overline{\nabla}_0 + \phi : W_1(E) \to W_0(E)$. Note that

$$\overline{\nabla}_t := \overline{\nabla}_0 + t\phi : W_1(E) \to W_0(E)$$

is a continuous family of Fredholm operators depending on $t \in [0,1]$. It follows that the index of $\overline{\nabla} = \overline{\nabla}_1$ is equal to the index of $\overline{\nabla}_0$ computed in Proposition 2.6. $\square$

Corollary 2.9. *If one of the holomorphic bundles E_1 and E_2 is standard then the spaces $\mathrm{Hom}_{\mathcal{C}}(E_1, E_2)$ and $\mathrm{Ext}^1_{\mathcal{C}}(E_1, E_2)$ are finite-dimensional and*

$$\begin{aligned} \chi(E_1, E_2) &:= \dim \mathrm{Hom}_{\mathcal{C}}(E_1, E_2) - \dim \mathrm{Ext}^1_{\mathcal{C}}(E_1, E_2) \\ &= \mathrm{rk}(E_1)\deg(E_2) - \mathrm{rk}(E_2)\deg(E_1). \end{aligned}$$

The following vanishing result will play a crucial role in the proof of Theorem 1.1.

Theorem 2.10. *Assume that $\mathrm{Im}(\tau) < 0$. For every holomorphic bundle E on $T = T_{\theta,\tau}$ there exists a constant $C = C(E) \in \mathbb{R}$ such that for every standard holomorphic bundle E_0 on T with $\mu(E_0) < C$ one has $\mathrm{Ext}^1_{\mathcal{C}}(E_0, E) = 0$.*

Proof. Let us choose a (non-holomorphic) isomorphism $E \simeq E_1^{\oplus N}$, where E_1 is a standard holomorphic bundle on $T_{\theta,\tau}$. Then we can write the holomorphic structure on E as

$$\overline{\nabla}_E = \overline{\nabla}_0 + \phi,$$

where $\overline{\nabla}_0$ comes from the standard holomorphic structure on E_1, $\phi \in \mathrm{End}_{A_\theta}(E)$. Then for every standard holomorphic bundle E_0 we can consider the holomorphic bundle $E' = \mathrm{Hom}_{A_\theta}(E_0, E)$ on $T_{\theta',\tau}$, where $\mathrm{End}_{A_\theta}(E_0) = A_{\theta'}$. Note that $\mathrm{Ext}^1_{\mathcal{C}}(E_0, E) \simeq H^1(E')$, so we want to prove that the latter group vanishes for $\mu(E_0) << 0$. Recall that the holomorphic structure $\overline{\nabla}'$ on E' is given by

$$\overline{\nabla}'(f)(e_0) = \mathrm{rk}(E_0) \cdot [\overline{\nabla}(f(e_0)) - f(\overline{\nabla}_{E_0}(e_0))],$$

where $e_0 \in E_0$, $f \in E'$ (see section 2.2 of [2]). The isomorphism $E \simeq E_1^{\oplus N}$ induces an isomorphism $E' \simeq (E'_1)^{\oplus N}$, where E'_1 is the standard bundle $\mathrm{Hom}_{A_\theta}(E_0, E_1)$ on $T_{\theta',\tau}$. Therefore, we have

$$\overline{\nabla}' = \overline{\nabla}'_0 + \mathrm{rk}(E_0)\phi,$$

where $\overline{\nabla}'_0$ corresponds to the standard holomorphic structure on $(E'_1)^{\oplus N}$ and ϕ is now considered as an $A_{\theta'}$-linear endomorphism of E'. Note that by Proposition 2.6(iii) we have $H^1(E', \overline{\nabla}'_0) = 0$ as long as $\mu(E') > 0$. It is easy to compute that

$$\mathrm{rk}(E') = \mathrm{rk}(E)/\mathrm{rk}(E_0),\ \deg(E') = \mathrm{rk}(E)\,\mathrm{rk}(E_0)(\mu(E) - \mu(E_0)),$$

hence

$$\mu(E') = \mathrm{rk}(E_0)^2(\mu(E) - \mu(E_0)).$$

Therefore, $\mu(E') > 0$ provided that $\mu(E_0) < \mu(E)$. In this case the operator Q on E' constructed in Proposition 2.6 for the standard holomorphic structure $\overline{\nabla}'_0$ satisfies $\overline{\nabla}'_0 Q = \mathrm{id}$. Let C_0 be a constant such that

$$||Qe'||_0 \le \frac{C_0}{\sqrt{\mu(E')}}||e'||_0$$

for $e' \in W_0(E')$ (see Proposition 2.6(v)). Also, let us write $\phi \in \mathrm{End}_{A_\theta}(E) \simeq \mathrm{End}_{A_\theta}(E_1^{\oplus N})$ in the form $\phi = \sum a_{m,n}U_1^m U_2^n$, where U_1 and U_2 are unitary generators of $\mathrm{End}_{A_\theta}(E_1)$ and $a_{m,n}$ are $N \times N$ complex matrices. Then we set $C(\phi) = \sum_{m,n} ||a_{m,n}||$ (the sum of the matrix norms of all coefficients). Now we choose the constant $C < \mu(E)$ in such a way that

$$\frac{C_0 C(\phi)}{\sqrt{\mu(E) - C}} < 1.$$

Then for $\mu(E_0) < C$ we will have

$$\mathrm{rk}(E_0) \cdot ||\phi Q e'||_0 \le \frac{\mathrm{rk}(E_0) C_0 C(\phi)}{\sqrt{\mu(E')}}||e'||_0 < \frac{C_0 C(\phi)}{\sqrt{\mu(E) - C}}||e'||_0 < r \cdot ||e'||_0$$

for some $0 < r < 1$. It follows from the above estimate that the operator $\mathrm{id} + \mathrm{rk}(E_0)\phi Q : W_0(E') \to W_0(E')$ is invertible. Therefore, we can define the operator

$$\widetilde{Q} = Q(\mathrm{id} + \mathrm{rk}(E_0)\phi Q)^{-1} : W_0(E') \to W_1(E')$$

that satisfies

$$(\overline{\nabla}'_0 + \mathrm{rk}(E_0)\phi)\widetilde{Q} = \mathrm{id}\,.$$

Hence, the operator $\overline{\nabla}' = \overline{\nabla}'_0 + \mathrm{rk}(E_0)\phi : W_1(E') \to W_0(E')$ is surjective. But $H^1(E')$ can be identified with the cokernel of this operator (see the proof of Theorem 2.8), so $H^1(E') = 0$. □

2.5. Serre duality. For every holomorphic bundle E we have a natural pairing

$$E \otimes_{A_\theta} E^\vee \to A_\theta : e \otimes f \mapsto f(e).$$

It is compatible with the $\overline{\partial}$-operators, so it induces a pairing

$$H^{1-i}(E) \otimes H^i(E^\vee) \to H^1(A_\theta) \simeq \mathbb{C} \tag{2.8}$$

for $i = 0, 1$.

Theorem 2.11. *The pairing (2.8) is perfect.*

Proof. Since we can switch E and $E^\vee$, it suffices to consider the case $i = 0$. We choose an isomorphism of E with a direct sum of several copies of a standard holomorphic bundle E_0, so that we can talk about standard metrics and Sobolev spaces. Note that the isomorphism $H^1(A_\theta)$ is induced by the trace functional $\mathrm{tr} : A_\theta \to \mathbb{C} : \sum a_{m,n}U_1^m U_2^n \mapsto a_{0,0}$. Hence, the pairing (2.8) is induced by the pairing

$$b : E \otimes E^\vee \to \mathbb{C} : e \otimes f \mapsto \mathrm{tr}(f(e))$$

that satisfies the identity

$$\mathrm{rk}(E_0)b(\overline{\nabla}_E(e), e^\vee) + b(e, \overline{\nabla}_{E^\vee}(e^\vee)) = 0, \tag{2.9}$$

where $e \in E$, $e^\vee \in E^\vee$ (see Proposition 2.2 of [2]). According to Theorem 2.8(iii) we have orthogonal decompositions

$$E^\vee = \ker(\overline{\nabla}_{E^\vee}) \oplus \overline{\nabla}^*_{E^\vee}(E^\vee),$$

$$E = \ker(\overline{\nabla}^*_E) \oplus \overline{\nabla}_E(E).$$

Therefore, it suffices to check that b induces a perfect pairing between $\ker(\overline{\nabla}^*_E)$ and $\ker(\overline{\nabla}_{E^\vee})$. Let $\sigma : E \to E^\vee$ be the $\mathbb{C}$-antilinear isomorphism defined in section 2.2. We claim that σ maps $\ker(\overline{\nabla}^*_E)$ isomorphically onto $\ker(\overline{\nabla}_{E^\vee})$. Since $b(e_1, \sigma(e_2)) = \langle e_1, e_2 \rangle$ for $e_1, e_2 \in E$, the theorem would immediately follow this. To prove the claim it is enough to check that $\overline{\nabla}_{E^\vee} = -\mathrm{rk}(E_0)\sigma\overline{\nabla}^*_E\sigma^{-1}$. To this end let us rewrite (2.9) as follows:

$$\mathrm{rk}(E_0)\langle\overline{\nabla}_E(e), \sigma^{-1}e^\vee\rangle = -\langle e, \sigma^{-1}\overline{\nabla}_{E^\vee}(e^\vee)\rangle.$$

Since the left-hand side is equal to $\mathrm{rk}(E_0)\langle e, \overline{\nabla}^*_E\sigma^{-1}e^\vee\rangle$ we conclude that $\mathrm{rk}(E_0)\overline{\nabla}^*_E\sigma^{-1} = -\sigma^{-1}\overline{\nabla}_{E^\vee}$ as required. □

Recall that we denote by $\mathcal{C}' \subset \mathcal{C}$ the full subcategory consisting of all successive extensions of standard holomorphic bundles. Theorem 3.8 of [2] implies that the derived category of $\mathcal{C}'$ is equivalent to the derived category of coherent sheaves on an elliptic curve. Therefore, the standard Serre duality gives a functorial isomorphism

$$\mathrm{Ext}^1_{\mathcal{C}'}(E_1, E_2) \simeq \mathrm{Hom}_{\mathcal{C}'}(E_2, E_1)^*$$

for $E_1, E_2 \in \mathcal{C}'$. Note that we can replace here $\mathrm{Ext}^i_{\mathcal{C}'}$ with $\mathrm{Ext}^i_{\mathcal{C}}$. Now using the above theorem we can extend this isomorphism to the case when only one of the objects E_1, E_2 belongs to $\mathcal{C}'$.

Corollary 2.12. *For every holomorphic bundles E and E_0 such that $E_0 \in \mathcal{C}'$ the natural pairings*

$$\mathrm{Ext}^i_{\mathcal{C}}(E_0, E) \otimes \mathrm{Ext}^{1-i}_{\mathcal{C}}(E, E_0) \to \mathrm{Ext}^1_{\mathcal{C}}(E_0, E_0) \to \mathbb{C}$$

for $i = 0, 1$ are perfect. Here the functional $\mathrm{Ext}^1_{\mathcal{C}}(E_0, E_0) \to \mathbb{C}$ is induced by the Serre duality on $\mathcal{C}'$. Therefore, we have functorial isomorphisms

$$\mathrm{Ext}^{1-i}_{\mathcal{C}}(E, E_0) \widetilde{\to} \mathrm{Ext}^i_{\mathcal{C}}(E_0, E)^* \tag{2.10}$$

for $E \in \mathcal{C}$, $E_0 \in \mathcal{C}'$.

Proof. If E_0 is standard the assertion follows from Theorem 2.11. It remains to observe that if for fixed $E \in \mathcal{C}$ the map (2.10) is an isomorphism for some $E_0, E_0' \in \mathcal{C}'$ then it is also an isomorphism for any extension of E_0 by E_0'. □

3. Ampleness

3.1. Ample sequences of standard holomorphic bundles. Let us start by recalling some basic notions concerning ample sequences in abelian categories and associated $\mathbb{Z}$-algebras. The reader can consult [3] for more details.

Definition.(see [9],[3]): Let $(E_n)_{n\in\mathbb{Z}}$ be a sequence of objects in a $\mathbb{C}$-linear abelian category $\mathcal{A}$ such that $\mathrm{Hom}(E_n, E)$ is finite-dimensional for every $E \in \mathcal{A}$ and every $n \in \mathbb{Z}$. Then (E_n) is called *ample* if the following two conditions hold:
(i) for every surjection $E \to E'$ in $\mathcal{A}$ the induced map $\mathrm{Hom}(E_n, E) \to \mathrm{Hom}(E_n, E')$ is surjective for all $n << 0$;
(ii) for every object $E \in \mathcal{A}$ and every $N \in \mathbb{Z}$ there exists a surjection $\oplus_{i=1}^{s} E_{n_i} \to E$ where $n_i < N$ for all i.

To a sequence $(E_n)_{n\in\mathbb{Z}}$ one can associate a so called $\mathbb{Z}$-algebra $A = \oplus_{i\le j} A_{ij}$, where $A_{ii} = \mathbb{C}$, $A_{ij} = \mathrm{Hom}(E_i, E_j)$ for $i < j$, the multiplications $A_{jk} \otimes A_{ij} \to A_{ik}$ are induced by the composition in $\mathcal{C}$. One can define for $\mathbb{Z}$-algebras all the standard notions associated with graded algebras (see [3]). In particular, we can talk about right A-modules: these have form $M = \oplus_{i\in\mathbb{Z}} M_i$ and the right A-action is given by the maps $M_j \otimes A_{ij} \to M_i$. The analogues of free A-modules are direct sums of the modules P_n, $n \in \mathbb{Z}$, defined by $(P_n)_i = A_{ni}$. We say that an A-module M is *finitely generated* if there exists a surjection $\oplus_{i=1}^{s} P_{n_i} \to M$. A finitely generated A-module M is called *coherent* if for every morphism $f : P \to M$, where P is a finitely generated free module, the module $\ker(f)$ is finitely generated. Finally, a $\mathbb{Z}$-algebra A is called *coherent* if all the modules P_n are coherent and in addition all one-dimensional A-modules are coherent.
The main theorem of [3] asserts that if (E_n) is ample then the $\mathbb{Z}$-algebra A is coherent and the natural functor $E \mapsto \oplus_{i<0} \mathrm{Hom}(E_i, E)$ gives an equivalence of categories

$$\mathcal{A} \simeq \mathrm{cohproj}\, A \tag{3.1}$$

where $\mathrm{cohproj}\, A$ is the quotient of the category of coherent right A-modules by the subcategory of finite-dimensional modules. We are going to apply this theorem to the category $\mathcal{C}'$ generated by standard holomorphic bundles on $T = T_{\theta,\tau}$. Recall that in [2] we identified this category with a certain abelian subcategory $\mathcal{C}^\theta$ of the derived category $D^b(X)$ of coherent sheaves on the elliptic curve $X = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. To define $\mathcal{C}^\theta$ one has to consider two full subcategories in the category $\mathrm{Coh}(X)$ of coherent sheaves on X: $\mathrm{Coh}_{<\theta}$ (resp., $\mathrm{Coh}_{>\theta}$) is the minimal subcategory of $\mathrm{Coh}(X)$ closed under extensions and containing all stable bundles of slope $< \theta$ (resp., all stable bundles of slope $> \theta$ and all torsion sheaves). Then by the definition

$$\begin{aligned}\mathcal{C}^\theta = \{K \in D^b(X) \quad : \quad & H^{>0}(K) = 0, H^0(K) \in \mathrm{Coh}_{>\theta}, \\ & H^{-1}(K) \in \mathrm{Coh}_{<\theta}, H^{<-1}(K) = 0\}.\end{aligned}$$

Thus, $\mathcal{C}^\theta$ contains $\mathrm{Coh}_{>\theta}$ and $\mathrm{Coh}_{<\theta}[1]$ and these two subcategories generate $\mathcal{C}^\theta$ in an appropriate sense. The fact that $\mathcal{C}^\theta$ is abelian follows from the torsion theory (see [1]). Note that the vectors $(\deg(K), \mathrm{rk}(K)) \in \mathbb{Z}^2$ for $K \in \mathcal{C}^\theta$ are characterized by the inequality

$$\deg(K) - \theta\,\mathrm{rk}(K) > 0.$$

In [2] we showed that a version of Fourier-Mukai transform gives an equivalence $\mathcal{S} : \mathcal{C}' \widetilde{\rightarrow} \mathcal{C}^\theta$ (this $\mathcal{S}$ differs from the transform studied in section 3.3 of [2] by the shift $K \mapsto K[1]$). Standard holomorphic bundles correspond under $\mathcal{S}$ to *stable* objects of $\mathcal{C}^\theta$: the latter are structure sheaves of points and objects of the form $V[n]$ where V is a stable bundle, $n \in \{0, 1\}$. Moreover, one has

$$\deg \mathcal{S}(E_{d,c}(\theta)) = d,\ \ \mathrm{rk}\,\mathcal{S}(E_{d,c}(\theta)) = -c.$$

It follows that

$$\mathrm{rk}(\mathcal{S}(E)) = -\deg(E),\ \ \mu(\mathcal{S}(E)) = \theta - \mu(E)^{-1}.$$

The following criterion of ampleness in $\mathcal{C}'$ is essentially contained in the proof of Theorem 3.5 of [4], where we showed the existence of ample sequences in $\mathcal{C}'$.

Theorem 3.1. *Let (E_n) be a sequence of standard holomorphic bundles on T such that $\mu(E_n) \to -\infty$ as $n \to -\infty$ and $\mathrm{rk}(E_n) > c$ for all $n << 0$ for some constant $c > 0$. Then (E_n) is an ample sequence in $\mathcal{C}'$. Moreover, for every $E \in \mathcal{C}'$ the natural morphism $\mathrm{Hom}(E_n, E) \otimes_{\mathbb{C}} E_n \to E$ in $\mathcal{C}$ is surjective for $n << 0$.*

Proof. Let $\mathcal{F}_n = \mathcal{S}(E_n)$ be the corresponding sequence of stable objects of $\mathcal{C}^\theta$. Then $\mathrm{rk}(\mathcal{F}_n) = -\deg(E_n) \to +\infty$ and $\mu(\mathcal{F}_n) = \theta - \mu(E_n)^{-1} \to \theta$ as $n \to -\infty$. Moreover, we have

$$\mu(\mathcal{F}_n) - \theta = \frac{\mathrm{rk}(E_n)}{\mathrm{rk}(\mathcal{F}_n)} > \frac{c}{\mathrm{rk}(\mathcal{F}_n)}.$$

Therefore, the same proof as in Theorem 3.5 of [4] (where we considered only the special case $c = 1$) shows that the sequence $(\mathcal{F}_n)$ is ample in $\mathcal{C}^\theta$ and that for every $\mathcal{F} \in \mathcal{C}^\theta$ the morphism $\mathrm{Hom}(\mathcal{F}_n, \mathcal{F}) \otimes_{\mathbb{C}} \mathcal{F}_n \to \mathcal{F}$ is surjective for $n << 0$. Hence, the same assertions hold for the sequence (E_n) in $\mathcal{C}'$. □

Theorem 3.2. *Let (E_n) be a sequence as in Theorem 3.1 and let $A = \oplus_{i \le j} \mathrm{Hom}_{\mathcal{C}}(E_i, E_j)$ be the corresponding $\mathbb{Z}$-algebra. Then for every holomorphic bundle E on T the A-module $M(E) = \oplus_{i<0} \mathrm{Hom}_{\mathcal{C}}(E_i, E)$ is coherent. Also, for every sufficiently small i_0 the canonical morphism of A-modules*

$$\mathrm{Hom}_{\mathcal{C}}(E_{i_0}, E) \otimes P_{i_0} \to M(E)$$

has finite-dimensional cokernel.

First, we need a criterion for finite generation of modules $M(E)$ with a weaker assumption on (E_n).

Lemma 3.3. *Let E be a holomorphic bundle on T and let $C = C(E)$ be the corresponding constant from Theorem 2.10. Let (E_i) be a sequence of standard holomorphic bundles such that $\mu(E_i) \to \mu$ as $i \to -\infty$, where $\mu \in \mathbb{R} \cup \{-\infty\}$. Assume that for some $\epsilon > 0$ one has $\operatorname{rk}(E_i)^2(\mu(E_i) - \mu) > 1 + \epsilon$ for all $i << 0$ (this condition is vacuous if $\mu = -\infty$). Then*
(i) for every sufficiently small $i_0 \in \mathbb{Z}$ there exists $i_1 = i_1(i_0)$ such that for all $i < i_1$ there exists a standard holomorphic bundle F_i fitting into the following short exact sequence in $\mathcal{C}'$:

$$0 \to E_i \overset{\mathrm{can}}{\to} \operatorname{Hom}_{\mathcal{C}}(E_i, E_{i_0})^* \otimes E_{i_0} \to F_i \to 0, \tag{3.2}$$

where can *is the canonical morphism.*
(ii) Assume in addition that $\mu < C$ and if $\mu = -\infty$ then for some $c > 0$ one has $\operatorname{rk}(E_i) > c$ for all $i << 0$. Then for all sufficiently small i_0 there exists i_1 such that for $i < i_1$ one has $\operatorname{Ext}^1(F_i, E) = 0$, where F_i is defined by (3.2). Under the same assumptions the A-module $M(E)$ is finitely generated.

Proof. (i) Let us denote $r_i = \operatorname{rk}(E_i)$, $\mu_i = \mu(E_i)$. If μ is finite then for every sufficiently small i_0 one has $r_{i_0}^2(\mu_{i_0} - \mu) > 1$. Therefore, we can find $i_1 < i_0$ such that for $i < i_1$ one has $r_{i_0}^2(\mu_{i_0} - \mu_i) > 1$. If $\mu = -\infty$ then we can take any i_0 and then still find $i_1 < i_0$ such that for $i < i_1$ the above inequality holds. We are going to construct F_i in this situation. Using the equivalence $\mathcal{S} : \mathcal{C}' \widetilde{\to} \mathcal{C}^\theta \subset D^b(X)$ we can first define $\widetilde{F}_i \in D^b(X)$ from the exact triangle

$$\mathcal{S}(E_i) \to \operatorname{Hom}_{\mathcal{C}}(E_i, E_{i_0})^* \otimes \mathcal{S}(E_{i_0}) \to \widetilde{F}_i \to \mathcal{S}(E_i)[1].$$

In other words, $\widetilde{F}_i$ is the image of E_i under the equivalence $R_{E_{i_0}} : D^b(X) \to D^b(X)$ given by the right twist with respect to E_{i_0} (see [4], sec.2.3, or [6]; our functor differs from that of [6] by a shift). It follows that $\operatorname{Hom}_{D^b(E)}(\widetilde{F}_i, \widetilde{F}_i) \simeq \operatorname{Hom}_{\mathcal{C}}(E_i, E_i) \simeq \mathbb{C}$, so $\widetilde{F}_i$ is a stable object and either $\widetilde{F}_i \in \mathcal{C}^\theta$ or $\widetilde{F}_i \in \mathcal{C}^\theta[1]$. To prove that $\widetilde{F}_i \in \mathcal{C}^\theta$ it suffices to check that $\deg(\widetilde{F}_i) - \theta \operatorname{rk}(\widetilde{F}_i) > 0$. But

$$\deg(\widetilde{F}_i) - \theta \operatorname{rk}(\widetilde{F}_i) = \chi(E_i, E_{i_0}) r_{i_0} - r_i = ((\mu_{i_0} - \mu_i) r_{i_0}^2 - 1) r_i > 0$$

by our choice of i. Hence, we have $\widetilde{F}_i \in \mathcal{C}^\theta$ and we can set $F_i = \mathcal{S}^{-1}(\widetilde{F}_i)$.
(ii) If μ is finite then we can choose i_0 such that $r_{i_0}^2(\mu_{i_0} - \mu) > 1 + \epsilon$ and $\mu_{i_0} + \epsilon^{-1}(\mu_{i_0} - \mu) < C$. If $\mu = -\infty$ then we choose i_0 such that $\mu_{i_0} + 2r_{i_0}^{-2} < C$ (here we use the assumption that $r_i > c$ for $i << 0$). In either case for sufficiently small i we have short exact sequence (3.2). Applying to it the functor $\operatorname{Hom}(?, E)$ we get the long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_{\mathcal{C}}(F_i, E) &\to \operatorname{Hom}_{\mathcal{C}}(E_{i_0}, E) \otimes \operatorname{Hom}_{\mathcal{C}}(E_i, E_{i_0}) \\ &\to \operatorname{Hom}_{\mathcal{C}}(E_i, E) \to \operatorname{Ext}^1_{\mathcal{C}}(F_i, E). \end{aligned}$$

Thus, vanishing of $\operatorname{Ext}^1_{\mathcal{C}}(F_i, E)$ for all $i << 0$ would imply that the A-module $M(E)$ is finitely generated. By the definition of the constant C this vanishing would follow from the inequality $\mu(F_i) < C$. In the case $\mu \neq -\infty$ we have for $i << 0$

$$\mu(F_i) = \mu_{i_0} + \frac{\mu_{i_0} - \mu_i}{r_{i_0}^2(\mu_{i_0} - \mu_i) - 1} < \mu_{i_0} + \epsilon^{-1}(\mu_{i_0} - \mu_i),$$

so the required inequality follows for $i << 0$ from our choice of i_0. In the case $\mu = -\infty$ we can finish the proof similarly using the inequality

$$\mu(F_i) < \mu_{i_0} + \frac{2}{r_{i_0}^2}$$

that holds for all $i << 0$. □

Proof of Theorem 3.2. By Lemma 3.3 for any sufficiently small i_0 there exists i_1 (depending on i_0) such that we have short exact sequence (3.2) and the induced sequence of A-modules

$$0 \to \oplus_{i<i_1} \operatorname{Hom}_{\mathcal{C}}(F_i, E) \to \operatorname{Hom}_{\mathcal{C}}(E_{i_0}, E) \otimes (P_{i_0})_{<i_1} \to M(E)_{<i_1} \to 0$$

is exact, where for every A-module $M = \oplus M_i$ we set $M_{<n} = \oplus_{i<n} M_i$. This immediately implies the last assertion of the theorem. Note that the structure of the A-module on $\oplus \operatorname{Hom}_{\mathcal{C}}(F_i, E)$ is defined using the natural isomorphisms $\operatorname{Hom}_{\mathcal{C}}(F_i, F_j) \simeq \operatorname{Hom}_{\mathcal{C}}(E_i, E_j)$ coming from the equality $F_i = R_{E_{i_0}}(E_i)$, where $R_{E_{i_0}}$ is the right twist with respect to E_{i_0}. It suffices to prove that the module $M'(E) := \oplus_{i<i_1} \operatorname{Hom}_{\mathcal{C}}(F_i, E)$ is finitely generated. Indeed, this would imply that the module $M(E)_{<i_1}$ is finitely presented and hence coherent (since A is coherent), therefore, the module $M(E)$ is also coherent. To check that $M'(E)$ is finitely generated we will use the criterion of Lemma 3.3 for the sequence (F_i). We have

$$\mu(F_i) \to \mu = \mu_{i_0} + 1/r_{i_0}^2$$

as $i \to -\infty$. Also,

$$\operatorname{rk}(F_i)^2(\mu(F_i) - \mu) = \frac{((\mu_{i_0} - \mu_i)r_{i_0}^2 - 1)r_i^2}{r_{i_0}^2} > \frac{((\mu_{i_0} - \mu_i)r_{i_0}^2 - 1)c^2}{r_{i_0}^2} \to +\infty$$

as $i \to -\infty$. Hence, the conditions of Lemma 3.3 will be satisfied once we show that $\mu = \mu_{i_0} + 1/r_{i_0}^2$ can be made smaller than any given constant by an appropriate choice of i_0. But this is of course true since $\mu_{i_0} + 1/r_{i_0}^2 < \mu_{i_0} + 1/c^2$ and $\mu_{i_0} \to -\infty$ as $i_0 \to -\infty$. □

3.2. Proof of Theorem 1.1. Let us pick a sequence $(E_n)_{n\in\mathbb{Z}}$ of stable holomorphic bundles satisfying conditions of Theorem 3.1 (it is easy to see that such a sequence exists, see the proof of Theorem 3.5 in [4]). Let E be a holomorphic bundle on T. Then by Theorem 3.2 the module $M = \oplus_i \operatorname{Hom}_{\mathcal{C}}(E_i, E)$ is coherent, hence, we can consider the object $E' \in \mathcal{C}'$ corresponding to this module via the equivalence (3.1). By the definition this means that there is an isomorphism of A-modules

$$M(E')_{<i_0} \simeq M(E)_{<i_0} \tag{3.3}$$

for some i_0. By Theorem 3.2 assuming that i_0 is small enough we can ensure that for all $i < i_0$ the canonical morphism $M(E')_i \otimes P_i \to M(E')$ has finite-dimensional cokernel. We claim that there exists a morphism $f : E' \to E$ in $\mathcal{C}$ that induces the same isomorphism of A-modules $M(E')_{<i_1} \simeq M(E)_{<i_1}$ for some $i_1 < i_0$ as the isomorphism (3.3). Indeed, by Theorem 3.1 we can find a resolution for E' in $\mathcal{C}'$ of the form

$$\ldots \to V_1 \otimes E_{n_1} \to V_0 \otimes E_{n_0} \to E' \to 0 \tag{3.4}$$

where $V_0 = \operatorname{Hom}(E_{n_0}, E')$ and $n_j < i_0$ for all $j \geq 0$. Using this resolution we can compute $\operatorname{Hom}_{\mathcal{C}}(E', E)$:

$$\operatorname{Hom}_{\mathcal{C}}(E', E) \simeq \ker(V_0^* \otimes \operatorname{Hom}_{\mathcal{C}}(E_{n_0}, E) \to V_1^* \otimes \operatorname{Hom}_{\mathcal{C}}(E_{n_1}, E)).$$

Using isomorphism (3.3) we can identify this space with

$$\ker(V_0^* \otimes \operatorname{Hom}_{\mathcal{C}}(E_{n_0}, E') \to V_1^* \otimes \operatorname{Hom}_{\mathcal{C}}(E_{n_1}, E')) \simeq \operatorname{Hom}_{\mathcal{C}}(E', E').$$

Thus, we obtain an isomorphism $\operatorname{Hom}_{\mathcal{C}}(E', E) \simeq \operatorname{Hom}_{\mathcal{C}}(E', E')$. We define $f \in \operatorname{Hom}_{\mathcal{C}}(E', E)$ to be the element corresponding to the identity in $\operatorname{Hom}_{\mathcal{C}}(E', E')$. Let us check that f induces the same isomorphism as (3.3) on some truncations of the modules $M(E')$ and $M(E)$. The definition of f implies that the composition of the induced morphism $f_* : M(E') \to M(E)$ with the natural morphism $V_0 \otimes P_{n_0} = M(V_0 \otimes E_{n_0}) \to M(E')$ coincides with the morphism $V_0 \otimes P_{n_0} \to M(E)$ induced by the isomorphism $V_0 = \operatorname{Hom}(E_{n_0}, E') \simeq \operatorname{Hom}(E_{n_0}, E)$ induced by (3.3). Therefore, our claim follows from the fact that the above morphism $V_0 \otimes P_{n_0} \to M(E')$ induces a surjective morphism on appropriate truncations.

Thus, we can assume from the beginning that the isomorphism (3.3) is induced by a morphism $f : E' \to E$. Next, we are going to construct a morphism $g : E' \to E$ such that $g \circ f = \mathrm{id}_{E'}$. To do this we note that by Serre duality $\operatorname{Hom}_{\mathcal{C}}(E, E') \simeq \operatorname{Ext}^1_{\mathcal{C}}(E', E)^*$ (see Corollary 2.12). Let us make n_0 smaller if needed so that $\operatorname{Ext}^1_{\mathcal{C}}(E_{n_0}, E) = \operatorname{Ext}^1_{\mathcal{C}}(E_{n_0}, E') = 0$. Then the space $\operatorname{Ext}^1_{\mathcal{C}}(E', E)$ can be computed using resolution (3.4):

$$\begin{aligned} \operatorname{Ext}^1_{\mathcal{C}}(E', E) \quad \simeq \quad & H^1[V_0^* \otimes \operatorname{Hom}_{\mathcal{C}}(E_{n_0}, E) \\ & \to V_1^* \otimes \operatorname{Hom}_{\mathcal{C}}(E_{n_1}, E) \to V_2^* \otimes \operatorname{Hom}_{\mathcal{C}}(E_{n_2}, E)]. \end{aligned} \tag{3.5}$$

Indeed, let us define $K_1 \in \mathcal{C}'$ from the short exact sequence

$$0 \to K_1 \to V_0 \otimes E_{n_0} \to E' \to 0,$$

so that we have the following resolution for K_1:

$$\ldots \to V_2 \otimes E_{n_2} \to V_1 \otimes E_{n_1} \to K_1 \to 0.$$

Then the isomorphism (3.5) can be derived from the induced exact sequences

$$\operatorname{Hom}_{\mathcal{C}}(V_0 \otimes, E_{n_0}, E) \to \operatorname{Hom}_{\mathcal{C}}(K_1, E) \to \operatorname{Ext}^1_{\mathcal{C}}(E', E) \to \operatorname{Ext}^1_{\mathcal{C}}(V_0 \otimes E_{n_0}, E) = 0,$$

$$0 \to \operatorname{Hom}_{\mathcal{C}}(K_1, E) \to \operatorname{Hom}_{\mathcal{C}}(V_1 \otimes E_{n_1}, E) \to \operatorname{Hom}_{\mathcal{C}}(V_2 \otimes E_{n_2}, E).$$

Using the fact that isomorphism (3.5) is functorial in E such that $\operatorname{Ext}^1_{\mathcal{C}}(E_{n_0}, E) = 0$ we derive that the morphism $\operatorname{Ext}^1_{\mathcal{C}}(E', E') \to \operatorname{Ext}^1_{\mathcal{C}}(E', E)$

induced by f is an isomorphism. But there is a natural functional $\phi' \in \mathrm{Ext}^1_{\mathcal{C}}(E',E')^*$ given by Serre duality. Let $\phi \in \mathrm{Ext}^1_{\mathcal{C}}(E',E)^*$ be the corresponding functional. The isomorphism $\mathrm{Ext}^1_{\mathcal{C}}(E',E)^* \widetilde{\to} \mathrm{Hom}_{\mathcal{C}}(E,E')$ maps ϕ to some element $g \in \mathrm{Hom}_{\mathcal{C}}(E,E')$. By functoriality of the Serre duality the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(E,E') \arrow[r, "\alpha"] \arrow[d, "f^*"'] & \mathrm{Ext}^1_{\mathcal{C}}(E',E)^* \arrow[d, "f^*"'] \\
\mathrm{Hom}_{\mathcal{C}}(E',E') \arrow[r, "\alpha'"] & \mathrm{Ext}^1_{\mathcal{C}}(E',E')^*
\end{tikzcd}$$

where the vertical arrows are induced by f. Since $\phi' = \alpha'(\mathrm{id}_{E'})$, $f^*(\phi) = \phi'$ and $\alpha(g) = \phi$ we deduce that $f^*(g) = \mathrm{id}_{E'}$, i.e. $g \circ f = \mathrm{id}_{E'}$. Therefore, we have $E \simeq E' \oplus E''$ for some holomorphic bundle E'' such that $\mathrm{Hom}_{\mathcal{C}}(E_i, E'') = 0$ for $i < i_0$. But Theorem 2.10 implies that $\mathrm{Ext}^1_{\mathcal{C}}(E_i, E'') = 0$ for all sufficiently negative i. Together with Corollary 2.9 this implies that $E'' = 0$. □

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