On the Scattering Theory of the Laplacian with a Periodic Boundary Condition. II. Additional Channels of Scattering

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Abstract. We study spectral and scattering properties of the Laplacian $H^{(\sigma)} = -\Delta$ in $L_2(\mathbb{R}^2_+)$ corresponding to the boundary condition $\frac{\partial u}{\partial \nu} + \sigma u = 0$ for a wide class of periodic functions σ. For non-negative σ we prove that $H^{(\sigma)}$ is unitarily equivalent to the Neumann Laplacian $H^{(0)}$. In general, there appear additional channels of scattering which are analyzed in detail.

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Introduction

0.1 Setting of the problem

The present paper is a continuation of [Fr], but can be read independently. It studies the Laplacian

$$H^{(\sigma)}u = -\Delta u \qquad \text{on } \mathbb{R}^2_+ \tag{0.1}$$

together with a boundary condition of the third type

$$\frac{\partial u}{\partial \nu} + \sigma u = 0 \qquad \text{on } \mathbb{R} \times \{0\}, \tag{0.2}$$

where ν denotes the exterior unit normal and where the function $\sigma : \mathbb{R} \to \mathbb{R}$ is assumed to be 2π-periodic. Moreover, let

$$\sigma \in L_{q,loc}(\mathbb{R}) \qquad \text{for some } q > 1.$$

Under this condition $H^{(\sigma)}$ can be defined as a self-adjoint operator in $L_2(\mathbb{R}^2_+)$ by means of the lower semibounded and closed quadratic form

$$\int_{\mathbb{R}^2_+} |\nabla u(x)|^2\,dx + \int_{\mathbb{R}} \sigma(x_1)|u(x_1,0)|^2\,dx_1, \qquad u \in H^1(\mathbb{R}^2_+).$$

We analyze the spectrum of $H^{(\sigma)}$ and develop a scattering theory viewing $H^{(\sigma)}$ as a (rather singular) perturbation of $H^{(0)}$, the Neumann Laplacian on $\mathbb{R}^2_+$. (For the abstract mathematical scattering theory see, e.g., [Ya1].)
By means of the Bloch-Floquet theory we represent $H^{(\sigma)}$ as a direct integral

$$\int_{-1/2}^{1/2} \oplus H^{(\sigma)}(k)\,dk \tag{0.3}$$

with fiber operators $H^{(\sigma)}(k)$ acting in $L_2(\Pi)$ where $\Pi := (-\pi,\pi)\times\mathbb{R}_+$ is the halfstrip. Due to the relation (0.3) the investigation of the operator $H^{(\sigma)}$ reduces to the study of the operators $H^{(\sigma)}(k)$.

0.2 The main results

It was shown in [Fr] that the wave operators

$$W_\pm^{(\sigma)}(k) := W_\pm(H^{(\sigma)}(k), H^{(0)}(k))$$

on the halfstrip exist and are complete. This immediately implies the existence of the wave operators

$$W_\pm^{(\sigma)} := W_\pm(H^{(\sigma)}, H^{(0)})$$

on the halfplane and the coincidence of the ranges

$$\mathcal{R}(W_+^{(\sigma)}) = \mathcal{R}(W_-^{(\sigma)}).$$

(Of course, the existence of the wave operators can also be obtained by a modification of the Cook method, see Section 17 in [Ya2].) Moreover, it was shown in [Fr] that the singular continuous spectrum of the operators $H^{(\sigma)}(k)$ is empty.
In the present paper we will study the point spectrum of the operators $H^{(\sigma)}(k)$. In general, there will be (discrete or embedded) eigenvalues which may produce bands in the spectrum of the operator $H^{(\sigma)}$ on the halfplane. In this case, the wave operators are not complete and there appear additional channels of scattering. For the additional bands in the spectrum we give some quantitative estimates and we construct an example where a gap in the spectrum appears. Moreover, we prove that the spectrum of the operator $H^{(\sigma)}$ is purely absolutely continuous.
Under the additional assumption

$$\sigma(x_1) \geq 0, \qquad \text{a.e. } x_1 \in \mathbb{R}, \tag{0.4}$$

we prove that the operators $H^{(\sigma)}(k)$ have no eigenvalues. This implies that the wave operators $W_\pm^{(\sigma)}$ are unitary and provide a unitary equivalence between the operators $H^{(\sigma)}$ and $H^{(0)}$.

0.3 Additional channels of scattering

Additional channels of scattering were already discovered in a number of other problems that exhibit periodicity with respect to some but not all space directions. Without aiming at completeness we mention the papers [DaSi], [Sa] concerning the scattering theory of problems of this type, [GrHoMe], [Ka] concerning Schrödinger operators with periodic point interactions and [BeBrPa] concerning the case of discrete Schrödinger operators.
In the present paper, using the specific properties of the operator under consideration we are able not only to show the appearance of additional channels of scattering but also to develop a more detailed analysis of these channels. In particular, we give some sufficient conditions for existence and non-existence of additional channels and prove that the spectrum of the operator is purely absolutely continuous.
The problem of absolute continuity in a case with partial periodicity is also investigated in [FiKl], where the Schrödinger operator with an electric potential is considered.

0.4 Outline of the paper

Let us explain the structure of this paper. In Section 1 we recall the precise definition of the operators $H^{(\sigma)}$ and $H^{(\sigma)}(k)$ in terms of quadratic forms and the direct integral decomposition. In Subsection 1.2 we state the main result in the case of non-negative σ (Theorem 1.1) and the main result about absolute continuity (Theorem 1.2).
In Section 2 we transform the eigenvalue problem for $H^{(\sigma)}(k)$ and $\lambda \in \mathbb{R}$ in the spirit of the Birman-Schwinger principle to the problem whether 0 is an eigenvalue of a certain "discrete pseudo-differential operator" of order one in $L_2(\mathbb{T})$. In this way we reduce the problem of (possibly embedded) eigenvalues to the study of operators with compact resolvent. In Section 3 we prove the absence of eigenvalues of $H^{(\sigma)}(k)$ under the condition (0.4), which implies Theorem 1.1. The general case is treated in Section 4 and the proof of Theorem 1.2 is given in Subsection 4.3. We supplement this in Section 5 with a more detailed analysis in the case when σ is a trigonometric polynomial. Finally, in Section 6 we describe and discuss the additional channels of scattering that appear in the general case. In Subsection 6.2 we construct an example of an open gap.

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1 Setting of the problem. The main result

1.1 Notation

We introduce the *halfplane*

$$\mathbb{R}^2_+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : \ x_2 > 0\} = \mathbb{R} \times \mathbb{R}_+,$$

and the *halfstrip*

$$\Pi := \{x = (x_1, x_2) \in \mathbb{R}^2 : \ -\pi < x_1 < \pi, \ x_2 > 0\} = (-\pi, \pi) \times \mathbb{R}_+,$$

where $\mathbb{R}_+ := (0, +\infty)$. Moreover, we need the lattice $2\pi\mathbb{Z}$. Unless stated otherwise, periodicity conditions are understood with repect to this lattice. We think of the corresponding torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ as the interval $[-\pi, \pi]$ with endpoints identified.

We use the notation $D = (D_1, D_2) = -i\nabla$ in $\mathbb{R}^2$.

For a measurable set $\Lambda \subset \mathbb{R}$ we denote by $\operatorname{meas}\Lambda$ its Lebesgue measure.

For an open set $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, the index in the notation of the norm $\|.\|_{L_2(\Omega)}$ is usually dropped. The space $L_2(\mathbb{T})$ may be formally identified with $L_2(-\pi, \pi)$. We denote the Fourier coefficients of a function $f \in L_2(\mathbb{T})$ by $\hat{f}_n := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x_1) e^{-inx_1}\, dx_1$, $n \in \mathbb{Z}$.

Next, $H^s(\Omega)$ is the Sobolev space of order $s \in \mathbb{R}$ (with integrability index 2). By $H^s(\mathbb{T})$ we denote the closure of $C^\infty(\mathbb{T})$ in $H^s(-\pi, \pi)$. Here $C^\infty(\mathbb{T})$ is the space of functions in $C^\infty(-\pi, \pi)$ which can be extended 2π-periodically to functions in $C^\infty(\mathbb{R})$. The space $H^s(\mathbb{T})$ is endowed with the norm

$$\|f\|^2_{H^s(\mathbb{T})} := \sum_{n \in \mathbb{Z}} (1 + n^2)^s |\hat{f}_n|^2, \qquad f \in H^s(\mathbb{T}).$$

By $\tilde{H}^s(\Pi)$ we denote the closure of $\tilde{C}^\infty(\Pi) \cap H^s(\Pi)$ in $H^s(\Pi)$. Here $\tilde{C}^\infty(\Pi)$ is the space of functions in $C^\infty(\Pi)$ which can be extended 2π-periodically with respect to x_1 to functions in $C^\infty(\mathbb{R}^2_+)$.

Statements and formulae which contain the double index "$\pm$" are understood as two independent assertions.

1.2 The operators $H^{(\sigma)}$ on the halfplane. Main results

Before describing the main results we recall the definition of the operators $H^{(\sigma)}$ from [Fr]. Let σ be a real-valued periodic function satisfying

$$\sigma \in L_q(\mathbb{T}) \qquad \text{for some } q > 1. \tag{1.1}$$

It is easy to see (cf. [Fr]) that under this condition the quadratic form

$$\begin{aligned} &\mathcal{D}[h^{(\sigma)}] := H^1(\mathbb{R}^2_+), \\ &h^{(\sigma)}[u] := \int_{\mathbb{R}^2_+} |Du(x)|^2\,dx + \int_{\mathbb{R}} \sigma(x_1)|u(x_1,0)|^2\,dx_1 \end{aligned} \tag{1.2}$$

is lower semibounded and closed in the Hilbert space $L_2(\mathbb{R}^2_+)$, so it generates a self-adjoint operator which will be denoted by $H^{(\sigma)}$. The case $\sigma = 0$ corresponds to the *Neumann Laplacian* on the halfplane, whereas the case $\sigma \neq 0$ implements a (generalized) *boundary condition of the third type*.
The spectrum of the "unperturbed" operator $H^{(0)}$ coincides with $[0, +\infty)$ and is purely absolutely continuous of infinite multiplicity.
In [Fr] we proved the existence of the wave operators

$$W_\pm^{(\sigma)} := W_\pm(H^{(\sigma)}, H^{(0)}) = s - \lim_{t\to\pm\infty} \exp(itH^{(\sigma)}) \exp(-itH^{(0)}).$$

We state now the main results of the present part. An especially complete result can be obtained under the additional assumption

$$\sigma(x_1) \geq 0, \qquad \text{a.e. } x_1 \in \mathbb{R}. \tag{1.3}$$

Theorem 1.1. *Assume that σ satisfies* (1.1) *and* (1.3). *Then the wave operators $W_\pm^{(\sigma)}$ exist, are unitary and satisfy*

$$H^{(\sigma)} = W_\pm^{(\sigma)} H^{(0)} W_\pm^{(\sigma)*}. \tag{1.4}$$

In particular, under the condition (1.3) the spectrum of the operator $H^{(\sigma)}$ is purely absolutely continuous. This is also true for general σ.

Theorem 1.2. *Assume that σ satisfies* (1.1). *Then the operator $H^{(\sigma)}$ has purely absolutely continuous spectrum.*

However, in contrast to the case of non-negative σ now the operator $H^{(\sigma)}$ may be not unitarily equivalent to $H^{(0)}$ and then the wave operators $W_\pm^{(\sigma)}$ are not complete. This is connected with the existence of *additional channels of scattering*. The discussion of this phenomenon is conveniently postponed to Section 6.

1.3 Definition of the operators $H^{(\sigma)}(k)$ on the halfstrip. Direct Integral Decomposition

Let σ be a real-valued periodic function satisfying (1.1) and let $k \in [-\frac{1}{2}, \frac{1}{2}]$. It follows (cf. [Fr]) that the quadratic form

$$\begin{aligned}
&\mathcal{D}[h^{(\sigma)}(k)] := \tilde{H}^1(\Pi), \\
&h^{(\sigma)}(k)[u] := \int_\Pi \left(|(D_1 + k)u(x)|^2 + |D_2 u(x)|^2 \right) \, dx + \int_{-\pi}^{\pi} \sigma(x_1) |u(x_1, 0)|^2 \, dx_1
\end{aligned} \tag{1.5}$$

is lower semibounded and closed in the Hilbert space $L_2(\Pi)$, so it generates a self-adjoint operator which will be denoted by $H^{(\sigma)}(k)$. In addition to the Neumann (if $\sigma = 0$) or third type (if $\sigma \neq 0$) boundary condition at $\{x_2 = 0\}$, the functions in $\mathcal{D}(H^{(\sigma)})$ satisfy periodic boundary conditions at $\{x_1 \in \{-\pi, \pi\}\}$. The operator $H^{(\sigma)}$ on the halfplane can be partially diagonalized by means of the *Gelfand transformation.* This operator is initially defined for $u \in \mathcal{S}(\mathbb{R}^2_+)$, the Schwartz class on $\mathbb{R}^2_+$, by

$$(\mathcal{U}u)(k, x) := \sum_{n \in \mathbb{Z}} e^{-ik(x_1 + 2\pi n)} u(x_1 + 2\pi n, x_2), \qquad k \in [-\tfrac{1}{2}, \tfrac{1}{2}],\ x \in \Pi,$$

and extended by continuity to a *unitary* operator

$$\mathcal{U} : L_2(\mathbb{R}^2_+) \to \int_{-1/2}^{1/2} \oplus L_2(\Pi) \, dk. \tag{1.6}$$

One finds (cf. [Fr]) that

$$\mathcal{U} \, H^{(\sigma)} \, \mathcal{U}^* = \int_{-1/2}^{1/2} \oplus H^{(\sigma)}(k) \, dk. \tag{1.7}$$

This relation allows us to investigate the operator $H^{(\sigma)}$ by studying the fibers $H^{(\sigma)}(k)$.
In [Fr] it was shown that

$$\sigma_{ac}\left(H^{(\sigma)}(k) \right) = [k^2, +\infty), \qquad \sigma_{sc}\left(H^{(\sigma)}(k) \right) = \emptyset. \tag{1.8}$$

In the present part we give a detailed analysis of the point spectrum of $H^{(\sigma)}(k)$.

2 Characterization of eigenvalues of the operator $H^{(\sigma)}(k)$

Let σ be a real-valued periodic function satisfying (1.1) and let $k \in [-\frac{1}{2}, \frac{1}{2}]$, $\lambda \in \mathbb{R}$. In the Hilbert space $L_2(\mathbb{T})$ we consider the quadratic forms

$$\begin{aligned}
&\mathcal{D}[b^{(\sigma)}(\lambda, k)] := H^{1/2}(\mathbb{T}), \\
&b^{(\sigma)}(\lambda, k)[f] := \sum_{n \in \mathbb{Z}} \beta_n(\lambda, k) \, |\hat{f}_n|^2 + \int_{-\pi}^{\pi} \sigma(x_1) |f(x_1)|^2 \, dx_1,
\end{aligned} \tag{2.1}$$

where

$$\beta_n(\lambda,k) := \begin{cases} \sqrt{(n+k)^2-\lambda} & \text{if } (n+k)^2 > \lambda, \\ -\sqrt{\lambda-(n+k)^2} & \text{if } (n+k)^2 \le \lambda. \end{cases} \tag{2.2}$$

It follows from the Sobolev embedding theorems that the forms $b^{(\sigma)}(\lambda,k)$ are lower semibounded and closed, so they generate self-adjoint operators which will be denoted by $B^{(\sigma)}(\lambda,k)$.
The compactness of the embedding of $H^{1/2}(\mathbb{T})$ in $L_2(\mathbb{T})$ implies that the operators $B^{(\sigma)}(\lambda,k)$ have compact resolvent.
Now we characterize the eigenvalues of the operator $H^{(\sigma)}(k)$ as the values λ for which 0 is an eigenvalue of the operators $B^{(\sigma)}(\lambda,k)$. More precisely, we have

PROPOSITION 2.1. *Let $k \in [-\frac{1}{2},\frac{1}{2}]$ and $\lambda \in \mathbb{R}$.*

1. *Let $u \in \mathcal{N}(H^{(\sigma)}(k)-\lambda I)$ and define*

$$f(x_1) := u(x_1,0), \qquad x_1 \in \mathbb{T}. \tag{2.3}$$

Then $f \in \mathcal{N}(B^{(\sigma)}(\lambda,k))$, $\hat{f}_n = 0$ if $(n+k)^2 \le \lambda$ and, moreover,

$$u(x) = \frac{1}{\sqrt{2\pi}} \sum_{(n+k)^2>\lambda} \hat{f}_n\, e^{inx_1}\, e^{-\beta_n(\lambda,k)\,x_2}, \qquad x \in \Pi. \tag{2.4}$$

2. *Let $f \in \mathcal{N}(B^{(\sigma)}(\lambda,k))$ such that $\hat{f}_n = 0$ if $(n+k)^2 \le \lambda$ and define u by* (2.4).
Then $u \in \mathcal{N}(H^{(\sigma)}(k)-\lambda I)$ and, moreover, (2.3) *holds.*

For the proof of Proposition 2.1 we use the following notation. For $u \in L_2(\Pi)$ and $n \in \mathbb{Z}$ we define

$$\hat{u}_n(x_2) := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} u(x) e^{-inx_1}\, dx_1, \qquad x_2 \in \mathbb{R}_+,$$

so that, with respect to convergence in $L_2(\Pi)$,

$$u(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} e^{inx_1}\, \hat{u}_n(x_2), \qquad x \in \Pi.$$

Moreover, one finds that $u \in \tilde{H}^1(\Pi)$ iff

$$\hat{u}_n \in H^1(\mathbb{R}_+),\ n \in \mathbb{Z}, \quad \text{and} \quad \sum_{n\in\mathbb{Z}} \big((1+n^2)\|\hat{u}_n\|^2 + \|D_2\hat{u}_n\|^2\big) < \infty.$$

The proof of the following observation is straightforward.

LEMMA 2.2. *Let $k \in [-\frac{1}{2},\frac{1}{2}]$ and $\lambda \in \mathbb{R}$.*

1. *Let $u \in \tilde{H}^1(\Pi)$, then the following are equivalent:*

(i) $u \in \mathcal{N}(H^{(\sigma)}(k) - \lambda I)$,

(ii) $\int_0^\infty D_2 \hat{u}_n \overline{D_2 \varphi}\, dx_2 + \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sigma(x_1) u(x_1, 0) e^{-inx_1}\, dx_1\, \overline{\varphi(0)} =$
$= (\lambda - (n+k)^2) \int_0^\infty \hat{u}_n \overline{\varphi}\, dx_2, \qquad n \in \mathbb{Z},\ \varphi \in H^1(\mathbb{R}_+)$.

2. *Let $f \in H^{1/2}(\mathbb{T})$, then the following are equivalent:*

(i) $f \in \mathcal{N}(B^{(\sigma)}(\lambda, k))$,

(ii) $\beta_n(\lambda, k)\, \hat{f}_n + \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sigma(x_1) f(x_1) e^{-inx_1}\, dx_1 = 0, \qquad n \in \mathbb{Z}$.

Proof of Proposition 2.1. The proof follows easily from Lemma 2.2. Note that if $u \in \mathcal{N}(H^{(\sigma)}(k) - \lambda I)$, then $D_2^2 \hat{u}_n = ((\lambda - (n+k)^2)\hat{u}_n$. Therefore

$$\hat{u}_n(x_2) = \begin{cases} 0 & \text{if} \quad (n+k)^2 \le \lambda, \\ \hat{f}_n\, e^{-\beta_n(\lambda,k)\, x_2} & \text{if} \quad (n+k)^2 > \lambda, \end{cases}$$

with f defined by (2.3). □

Remark 2.3. Obviously, the statement of Proposition 2.1 does not depend on the definition of $\beta_n(\lambda, k)$ for $(n+k)^2 \le \lambda$. The reason for our choice (2.2) is of technical nature and will become clear in Subsection 4.2 below.

3 The case of non-negative σ

Proposition 2.1 allows us to deduce easily the main result if σ is non-negative. We start with the operators $H^{(\sigma)}(k)$ on the halfstrip.

Theorem 3.1. *Assume that σ satisfies* (1.1) *and* (1.3) *and let $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then the operator $H^{(\sigma)}(k)$ has purely absolutely continuous spectrum.*

Proof. In view of (1.8) it suffices to prove that $H^{(\sigma)}(k)$ has no eigenvalues. For this we use Proposition 2.1. Let $\lambda \in \mathbb{R}$ and $f \in \mathcal{N}(B^{(\sigma)}(\lambda, k))$ such that $\hat{f}_n = 0$ if $(n+k)^2 \le \lambda$. It follows that

$$b^{(\sigma)}(\lambda, k)[f] \ge \gamma \|f\|^2$$

where $\gamma := \min\{\beta_n(\lambda, k) :\ n \in \mathbb{Z},\ \hat{f}_n \ne 0\} > 0$. Together with $b^{(\sigma)}(\lambda, k)[f] = 0$ this implies $f = 0$. So by Proposition 2.1 (1), λ is not an eigenvalue of $H^{(\sigma)}(k)$. □

Concerning the operator $H^{(\sigma)}$ on the halfplane we obtain immediately the

Proof of Theorem 1.1. In [Fr] we showed that $W_\pm^{(\sigma)}$ is unitarily equivalent to the direct integral of the operators $W_\pm^{(\sigma)}(k)$, $k \in [-\frac{1}{2}, \frac{1}{2}]$. The latter were shown to be complete, and by Theorem 3.1 they are actually unitary. Thus $W_\pm^{(\sigma)}$ is unitary and (1.4) follows from the intertwining property of wave operators. □

4 The general case

4.1 The point spectrum of the operators $H^{(\sigma)}(k)$

If we impose no additional condition on σ we have the following result on the point spectrum of the operators $H^{(\sigma)}(k)$.

Theorem 4.1. *Assume that σ satisfies* (1.1) *and let $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then $\sigma_p\left(H^{(\sigma)}(k)\right)$ (if non-empty) consists of eigenvalues of finite multiplicities which may accumulate at $+\infty$ only.*

Note that the case of an infinite sequence of (embedded) eigenvalues actually occurs.

Example 4.2. Let $\sigma \equiv \sigma_0 < 0$ be a negative constant and $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then

$$\sigma_p\left(H^{(\sigma)}(k)\right) = \{-\sigma_0^2 + (n+k)^2 : \; n \in \mathbb{Z}\}.$$

This follows easily by Proposition 2.1 or directly by separation of variables.

For the proof of Theorem 4.1 we need an auxiliary result. For $k \in [-\frac{1}{2}, \frac{1}{2}]$, $\lambda \in \mathbb{R}$ we denote by $\mu_m(\lambda, k)$, $m \in \mathbb{N}$, the eigenvalues of $B^{(\sigma)}(\lambda, k)$ arranged in non-decreasing order and repeated according to their multiplicities. Then we have

Lemma 4.3. *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$, then the functions $\mu_m(., k)$, $m \in \mathbb{N}$, are continuous and strictly decreasing on $\mathbb{R}$.*

The proof (of *strict* monotonicity) uses an analyticity argument and is conveniently postponed to Subsection 4.2.

Proof of Theorem 4.1. Proposition 2.1 (1) implies for $\lambda \in \mathbb{R}$

$$\dim \mathcal{N}(H^{(\sigma)}(k) - \lambda I) \leq \dim \mathcal{N}(B^{(\sigma)}(\lambda, k)). \tag{4.1}$$

Since $B^{(\sigma)}(\lambda, k)$ has compact resolvent, it follows that eigenvalues λ of $H^{(\sigma)}(k)$ have finite multiplicities.
To prove that the only possible accumulation point of $\sigma_p\left(H^{(\sigma)}(k)\right)$ is $+\infty$, let $\Lambda = (\lambda_-, \lambda_+)$ be an open interval. It follows from (4.1) and Lemma 4.3 that

$$\begin{aligned}
&\sharp_{cm}\{\lambda \in (\lambda_-, \lambda_+) : \; \lambda \text{ is eigenvalue of } H^{(\sigma)}(k)\} \leq \\
&\quad \leq \sum_{\lambda \in (\lambda_-, \lambda_+)} \dim \mathcal{N}(B^{(\sigma)}(\lambda, k)) = \\
&\quad = \sharp\{m \in \mathbb{N} : \; \mu_m(\lambda, k) = 0 \text{ for some } \lambda \in (\lambda_-, \lambda_+)\} = \\
&\quad = \sharp\{m \in \mathbb{N} : \; \mu_m(\lambda_-, k) > 0 \text{ and } \mu_m(\lambda_+, k) < 0\} = \\
&\quad = \sharp_{cm}\{\mu < 0 : \; \mu \text{ is eigenvalue of } B^{(\sigma)}(\lambda_+, k)\} - \\
&\qquad - \sharp_{cm}\{\mu \leq 0 : \; \mu \text{ is eigenvalue of } B^{(\sigma)}(\lambda_-, k)\},
\end{aligned} \tag{4.2}$$

where $\sharp_{cm}\{...\}$ means that the cardinality of $\{...\}$ is determined according to multiplicities. The RHS of (4.2) is finite since $B^{(\sigma)}(\lambda_+,k)$, $B^{(\sigma)}(\lambda_-,k)$ are lower semibounded and have compact resolvent. This completes the proof of the theorem. $\square$

Remark 4.4. We emphasize the equality

$$\begin{aligned}\sharp_{cm}\{\lambda\in(-\infty,k^2):\ \lambda \text{ is eigenvalue of } H^{(\sigma)}(k)\} =\\ = \sharp_{cm}\{\mu<0:\ \mu \text{ is eigenvalue of } B^{(\sigma)}(k^2,k)\}.\end{aligned} \tag{4.3}$$

Indeed, it follows from Proposition 2.1 (2) that the estimate (4.1) becomes an equality for $\lambda<k^2$, therefore also (4.2) for $\lambda_+=k^2$, and we obtain (4.3) by choosing $-\lambda_-$ so large that $B^{(\sigma)}(\lambda_-,k)$ is positive.
The equality (4.3) can be used to obtain estimates on the number of eigenvalues of $H^{(\sigma)}(k)$ below k^2 and on its asymptotics in the limit of large coupling constant. Such calculations for the operators $B^{(\sigma)}(k^2,k)$ are rather standard, so we do not go into details.

4.2 Complexification

Now we extend the operator family $B^{(\sigma)}(\lambda,k)$ to complex values of λ and k. For this construction we fix $k_0\in[-\frac12,\frac12]$, $\lambda_0\in\mathbb{R}\setminus\{(n+k_0)^2:\ n\in\mathbb{Z}\}$. We can choose $\delta_0>0$ (depending on λ_0, k_0) such that

$$(n+\kappa)^2-z\neq 0,\qquad n\in\mathbb{Z},$$

for all $z,\kappa\in\mathbb{C}$ such that $|z-\lambda_0|<\delta_0$, $|\kappa-k_0|<\delta_0$. Therefore, if we put

$$\tilde U:=\{z\in\mathbb{C}:|z-\lambda_0|<\delta\},\qquad \tilde V:=\{\kappa\in\mathbb{C}:|\kappa-k_0|<\delta\},$$

the functions β_n, $n\in\mathbb{Z}$, admit a unique analytic continuation to $\tilde U\times\tilde V$, and we can define sectorial and closed forms $b^{(\sigma)}(z,\kappa)$ for $z\in\tilde U$, $\kappa\in\tilde V$ by (2.1) with $\beta_n(\lambda,k)$ replaced by $\beta_n(z,\kappa)$. The corresponding m-sectorial operators will be denoted by $B^{(\sigma)}(z,\kappa)$. For fixed $\kappa\in\tilde V$ ($z\in\tilde U$, respectively) they form an analytic family of type (B) with respect to $z\in\tilde U$ ($\kappa\in\tilde V$, respectively) (see, e.g., Section VII.4 in [K]).
From this construction we obtain

Lemma 4.5. *Let $k_0\in[-\frac12,\frac12]$, $\lambda_0\in\mathbb{R}\setminus\{(n+k_0)^2:\ n\in\mathbb{Z}\}$ such that 0 is an eigenvalue of $B^{(\sigma)}(\lambda_0,k_0)$. Then there exist open neighbourhoods U, $V\subset\mathbb{R}$ of λ_0, k_0 and a real-analytic function $h:U\times V\to\mathbb{C}$ such that for all $\lambda\in U$, $k\in V\cap[-\frac12,\frac12]$ one has*

$$0\in\sigma_p\left(B^{(\sigma)}(\lambda,k)\right)\qquad \textit{iff}\qquad h(\lambda,k)=0.$$

Proof. The proof is rather standard, so we only sketch the major steps. We consider the family $B^{(\sigma)}(z,\kappa)$, $z \in \tilde{U}$, $\kappa \in \tilde{V}$, constructed above. Since these operators have compact resolvent, we can use a Riesz projection to separate the eigenvalues around 0 from the rest of the spectrum. The resulting operator has finite rank and is analytic with respect to z and κ, so its determinant h has the desired properties. ☐

Our next goal is to show that for every $\lambda \in U$ the function $h(\lambda, .)$ constructed above is not identically zero. For the proof of this we need to consider quasimomenta $\kappa = k + iy$ with large imaginary part y. So fix $k \in [-\frac{1}{2}, \frac{1}{2}]$, $\lambda \in \mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\}$, then the above construction (with λ_0, k_0 replaced by λ, k) yields a $\delta > 0$ and an analytic family $B^{(\sigma)}(\lambda,\kappa)$, $|\kappa - k| < \delta$. If we assume in addition that $k \neq 0$ and choose $\delta \in (0, |k|)$, we find that

$$(n+\kappa)^2 - \lambda \neq 0, \qquad n \in \mathbb{Z},$$

holds for all $\kappa \in \mathbb{C}$ such that $|\mathrm{Re}\,\kappa - k| < \delta$. Therefore $B^{(\sigma)}(\lambda,\kappa)$ admits a further analytic extension to

$$\tilde{\tilde{V}} := \{\kappa \in \mathbb{C} : |\mathrm{Re}\,\kappa - k| < \delta\}.$$

Concerning quasimomenta with large imaginary part we have the technical

Lemma 4.6. *Let $k \in [-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}$, $\lambda \in \mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\}$ and $\delta \in (0, |k|)$ as above. Then there exist constants $y_0 = y_0(\lambda,k,\delta)$, $C = C(\lambda,k,\delta)$ such that for all $k' \in [-\frac{1}{2}, \frac{1}{2}]$, $y \in \mathbb{R}$ satisfying $|k'-k| < \delta$, $|y| > y_0$ the operator $B^{(\sigma)}(\lambda, k'+iy)$ is boundedly invertible with*

$$\left\| \left(B^{(\sigma)}(\lambda, k'+iy) \right)^{-1} \right\| \leq \frac{C}{1+|y|}.$$

Proof. It suffices to find constants y_0, $\tilde{C} = \tilde{C}(\lambda,k,\delta) > 0$ such that for all $0 \neq f \in H^{1/2}(\mathbb{T})$, $k' \in [-\frac{1}{2}, \frac{1}{2}]$, $y \in \mathbb{R}$ satisfying $|k'-k| < \delta$, $|y| > y_0$ there exists $0 \neq g \in H^{1/2}(\mathbb{T})$ such that

$$|b^{(\sigma)}(\lambda, k'+iy)[f,g]| \geq \tilde{C}\,(1+|y|)\,\|f\|\|g\|.$$

For given $0 \neq f \in H^{1/2}(\mathbb{T})$, $k' \in [-\frac{1}{2}, \frac{1}{2}] \cap (k-\delta, k+\delta)$, $y \in \mathbb{R}$ we define g by its Fourier coefficients

$$\hat{g}_n := \frac{\beta_n(\lambda, k'+iy)}{|\beta_n(\lambda, k'+iy)|}\,\hat{f}_n, \qquad n \in \mathbb{Z}.$$

(Note that $\beta_n(\lambda, k'+iy) \neq 0$ by the choice of δ.) Then we have $0 \neq g \in H^{1/2}(\mathbb{T})$, $\|g\| = \|f\|$ and

$$|b^{(\sigma)}(\lambda, k'+iy)[f,g]| \geq \sum_{n\in\mathbb{Z}} |\beta_n(\lambda, k'+iy)||\hat{f}_n|^2 - \tfrac{1}{2}\|\sqrt{|\sigma|}f\|^2 - \tfrac{1}{2}\|\sqrt{|\sigma|}g\|^2. \tag{4.4}$$

Using the elementary estimates

$$\begin{aligned} |\beta_n(\lambda, k' + iy)| &\geq c_1\,(1 + |y|), \qquad n \in \mathbb{Z},\ |k' - k| < \delta, \\ |\beta_n(\lambda, k' + iy)| &\geq c_2\,(1 + |n|), \qquad n \in \mathbb{Z},\ |k' - k| < \delta, \end{aligned} \tag{4.5}$$

(with some constants $c_1 = c_1(\lambda, k, \delta) > 0, c_2 = c_2(\lambda, k, \delta) > 0$) and the Sobolev embedding theorem we find that for sufficiently large y_0

$$\|\sqrt{|\sigma|}f\|^2 \leq \tfrac{1}{2} \sum_{n\in\mathbb{Z}} |\beta_n(\lambda, k' + iy)||\hat{f}_n|^2, \qquad |k' - k| < \delta,\ |y| > y_0.$$

Using a similar estimate for $\|\sqrt{|\sigma|}g\|^2$ and (4.4), (4.5) we obtain

$$|b^{(\sigma)}(\lambda, k' + iy)[f, g]| \geq \tfrac{1}{2}\,c_1\,(1 + |y|)\,\|f\|\|g\|, \qquad |k' - k| < \delta, |y| > y_0,$$

which concludes the proof. □

As announced above, we have

Lemma 4.7. *Let k_0, λ_0, h, U and V be as in Lemma 4.5. Then for all $\lambda \in U$ one has $h(\lambda, .) \not\equiv 0$.*

Proof. To arrive at a contradiction we assume that $h(\lambda, .) \equiv 0$ for some $\lambda \in U$. We choose $k \in V \setminus \{0\}$ and consider the family $B^{(\sigma)}(\lambda, \kappa)$, $\kappa \in \tilde{\tilde{V}}$ constructed above. It follows from the Analytic Fredholm Alternative (see, e.g., Theorem VII.1.10 in [K]) that all operators of this family have 0 as an eigenvalue. But this contradicts Lemma 4.6. □

As an immediate consequence of Lemmas 4.5 and 4.7 and relation (4.1) we obtain the following result which will be needed in Subsection 4.3 to prove that the spectrum of the operator $H^{(\sigma)}$ is purely absolutely continuous.

Corollary 4.8. *There exists a countable number of open intervals U_j, $V_j \subset \mathbb{R}$ and real-analytic functions $h_j : U_j \times V_j \to \mathbb{C}$ satisfying*

1. *for all $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $\lambda \in \mathbb{R} \setminus \{(n + k)^2 : n \in \mathbb{Z}\}$ such that $\lambda \in \sigma_p\left(H^{(\sigma)}(k)\right)$ there is a j such that $(\lambda, k) \in U_j \times V_j$ and $h_j(\lambda, k) = 0$, and*
2. *for all j and all $\lambda \in U_j$ one has $h_j(\lambda, .) \not\equiv 0$.*

To complete this subsection we prove Lemma 4.3 which was used in the proof of Theorem 4.1.

Proof of Lemma 4.3. That $\mu_m(., k)$ is a continuous, non-increasing function follows from the variational principle and the continuity and monotonicity of the operators $B^{(\sigma)}(\lambda, k)$ with respect to λ.
To prove the strict monotonicity we assume to the contrary that for some $m \in \mathbb{N}$ the function $\mu_m(., k)$ coincides on an interval Λ with a constant $\mu_0 \in \mathbb{R}$. We choose $\lambda_0 \in \Lambda \setminus \{(n + k)^2 : n \in \mathbb{Z}\}$ and consider the family $B^{(\sigma)}(z, k)$, $z \in \tilde{U}$,

constructed at the beginning of this subsection (with λ_0, k_0 replaced by λ_0, k). It follows from the Analytic Fredholm Alternative (see, e.g., Theorem VII.1.10 in [K]) that μ_0 is an eigenvalue of $B^{(\sigma)}(z,k)$ also for complex $z \in \tilde{U}$.
However, let $z \in \tilde{U} \cap \mathbb{C}_\pm$ and $f \in \mathcal{N}(B^{(\sigma)}(z,k) - \mu_0 I)$. We have $\mp \mathrm{Im}\, \beta_n(z,k) > 0$, $n \in \mathbb{Z}$, so $\mathrm{Im}\, b^{(\sigma)}(\lambda,k)[f] = 0$ implies that $\hat{f}_n = 0$, $n \in \mathbb{Z}$, i.e., $f = 0$. So μ_0 is not an eigenvalue of $B^{(\sigma)}(z,k)$. □

4.3 Proof of Theorem 1.2

Now we prove Theorem 1.2 following the method suggested in [FiKl]. We need the following result from Complex Analysis of Several Variables which can be proved by means of the Implicit Function Theorem (see [FiKl]).

Lemma 4.9. *Let U, $V \subset \mathbb{R}$ be open intervals and $h : U \times V \to \mathbb{C}$ be real-analytic. Let $\Lambda \subset U$ with* $\mathrm{meas}\, \Lambda = 0$ *such that for all $\lambda \in \Lambda$ one has $h(\lambda, .) \not\equiv 0$. Then*

$$\mathrm{meas}\, \{k \in V :\ h(\lambda,k) = 0 \quad \textit{for some}\ \lambda \in \Lambda\} = 0.$$

Proof of Theorem 1.2. Let $\Lambda \subset \mathbb{R}$ with $\mathrm{meas}\, \Lambda = 0$. We denote the spectral projection of $H^{(\sigma)}$ ($H^{(\sigma)}(k)$, respectively) corresponding to Λ by $E^{(\sigma)}(\Lambda)$ ($E^{(\sigma)}(\Lambda,k)$, respectively). Then it follows from (1.7) that

$$\mathcal{U}\, E^{(\sigma)}(\Lambda)\, \mathcal{U}^* = \int_{-1/2}^{1/2} \oplus E^{(\sigma)}(\Lambda,k)\, dk$$

and we have to prove that this operator is equal to 0.
For this we write $[-\frac{1}{2}, \frac{1}{2}] = K_1 \cup K_2 \cup K_3$ where

$$\begin{aligned}
K_1 &= \left\{ k \in [-\tfrac{1}{2}, \tfrac{1}{2}] :\ \sigma_p\left(H^{(\sigma)}(k)\right) \cap \Lambda = \emptyset \right\}, \\
K_2 &= \left\{ k \in [-\tfrac{1}{2}, \tfrac{1}{2}] :\ \sigma_p\left(H^{(\sigma)}(k)\right) \cap \Lambda \cap \{(n+k)^2 :\ n \in \mathbb{Z}\} \neq \emptyset \right\}, \\
K_3 &= \left\{ k \in [-\tfrac{1}{2}, \tfrac{1}{2}] :\ \emptyset \neq \sigma_p\left(H^{(\sigma)}(k)\right) \cap \Lambda \subset \left(\mathbb{R} \setminus \{(n+k)^2 :\ n \in \mathbb{Z}\}\right) \right\}.
\end{aligned}$$

Since $\sigma_{sc}\left(H^{(\sigma)}(k)\right) = \emptyset$ we immediately obtain $E^{(\sigma)}(\Lambda,k) = 0$ for $k \in K_1$. Now

$$K_2 \subset \bigcup_{n \in \mathbb{Z}} \{k \in [-\tfrac{1}{2}, \tfrac{1}{2}] :\ (n+k)^2 - \lambda = 0 \quad \text{for some}\ \lambda \in \Lambda\}, \tag{4.6}$$

and with the notation of Corollary 4.8

$$K_3 \subset \bigcup_j \{k \in V_j \cap [-\tfrac{1}{2}, \tfrac{1}{2}] :\ h_j(\lambda,k) = 0 \quad \text{for some}\ \lambda \in U_j \cap \Lambda\}. \tag{4.7}$$

It follows from Lemma 4.9 that $\mathrm{meas}\, K_2 = \mathrm{meas}\, K_3 = 0$, which concludes the proof. □

5 The case of a trigonometric polynomial σ

We have seen in Example 4.2 that the operators $H^{(\sigma)}(k)$ may have embedded eigenvalues. Let us investigate this phenomenon under the additional assumption that only finitely many Fourier coefficients of σ are non-zero. Note that in this case the operator $B^{(\sigma)}(\lambda,k)$ acts in Fourier space as a finite-diagonal matrix. This allows us to exclude the existence of large embedded eigenvalues.

Proposition 5.1. *Assume that σ is a trigonometric polynomial of degree $N > 0$ and let $k \in [-\frac{1}{2},\frac{1}{2}]$. Then*

$$\sigma_p\left(H^{(\sigma)}(k)\right) \subset \left(-\|\sigma_-\|_\infty^2 + k^2, (N-|k|)^2\right).$$

Here $\sigma_- := \frac{1}{2}(|\sigma|-\sigma)$ denotes the negative part of σ.

Proof. The proof of $\sigma_p\left(H^{(\sigma)}(k)\right) \subset [-\|\sigma_-\|_\infty^2 + k^2, +\infty)$ is similar to the proof of Theorem 3.1. Moreover, it is easy to see that $-\|\sigma_-\|_\infty^2 + k^2 \in \sigma_p\left(H^{(\sigma)}(k)\right)$ only if σ coincides a.e. with a negative constant, which is excluded by the assumption $N>0$.
Let us prove now that $\sigma_p\left(H^{(\sigma)}(k)\right) \subset (-\infty,(N-|k|)^2)$. For this we use Proposition 2.1. Let $\lambda \geq (N-|k|)^2$ and $f \in \mathcal{N}(B^{(\sigma)}(\lambda,k))$ such that

$$\hat{f}_n = 0 \qquad \text{if } (n+k)^2 \leq \lambda. \tag{5.1}$$

In particular, we see from $B^{(\sigma)}(\lambda,k)f=0$ that

$$\sqrt{(n+k)^2-\lambda}\,\hat{f}_n + \frac{1}{\sqrt{2\pi}}\sum_{m=-N}^{N}\hat{\sigma}_m\hat{f}_{n-m} = 0 \qquad \text{if } (n+k)^2 \geq \lambda. \tag{5.2}$$

The estimate

$$\sharp\{n\in\mathbb{Z}:\ (n+k)^2\leq\lambda\} \geq \sharp\{n\in\mathbb{Z}:\ (n+k)^2 \leq (N-|k|)^2\} \geq 2N$$

and (5.1) imply that $\hat{f}_n=0$ for at least $2N$ consecutive n. Using $\hat{\sigma}_N = \overline{\hat{\sigma}_{-N}} \neq 0$ it is easy to see from (5.2) that $\hat{f}_n = 0$ for all n, i.e. $f=0$. So by Proposition 2.1 (1), λ is not an eigenvalue of $H^{(\sigma)}(k)$. $\square$

We show now that embedded eigenvalues in the interval $[(N-1+|k|)^2,(N-|k|)^2)$ can occur but are "rare".

Proposition 5.2. *Assume that σ is a trigonometric polynomial of degree $N > 0$ and let $k \in (-\frac{1}{2},\frac{1}{2})$. Then $H^{(\sigma)}(k)$ may have only simple eigenvalues in $[(N-1+|k|)^2,(N-|k|)^2)$ and the set*

$$\{(\lambda,k)\in\mathbb{R}\times(-\tfrac{1}{2},\tfrac{1}{2}):\ \lambda\in\sigma_p\left(H^{(\sigma)}(k)\right)\cap[(N-1+|k|)^2,(N-|k|)^2)\} \tag{5.3}$$

is finite.

For the proof of this proposition we introduce the following auxiliary operators in the Hilbert space $l_2(\mathbb{N})$.

$$\mathcal{D}(A^{(\sigma)}(\lambda,k)) := \{\alpha \in l_2(\mathbb{N}) : \sum_{n=1}^{\infty}(1+n^2)|\alpha_n|^2 < \infty\},$$

$$\Big(A^{(\sigma)}(\lambda,k)\,\alpha\Big)_n := \begin{cases} \beta_n(\lambda,k)\,\alpha_n + \frac{1}{\sqrt{2\pi}}\sum_{m=-N}^{n-1}\hat{\sigma}_m\,\alpha_{n-m} & \text{if } n \le N, \\ \beta_n(\lambda,k)\,\alpha_n + \frac{1}{\sqrt{2\pi}}\sum_{m=-N}^{N}\hat{\sigma}_m\,\alpha_{n-m} & \text{if } n > N. \end{cases} \tag{5.4}$$

The operators $A^{(\sigma)}(\lambda,k)$ are self-adjoint and have compact resolvent.

Lemma 5.3. *Let $k \in (-\frac{1}{2},\frac{1}{2})$ and $\lambda \in [(N-1+|k|)^2,(N-|k|)^2)$. Then λ is an eigenvalue of $H^{(\sigma)}(k)$ iff there exist $0 \ne \alpha^+$, $\alpha^- \in \mathcal{D}(A^{(\sigma)}(\lambda,k))$ such that $A^{(\sigma)}(\lambda,k)\,\alpha^+ = A^{(\sigma)}(\lambda,-k)\,\alpha^- = 0$ and $\alpha_n^+ = \alpha_n^- = 0$ for $n < N$. In this case, λ is a simple eigenvalue.*

Proof. We use Proposition 2.1. If λ is an eigenvalue of $H^{(\sigma)}(k)$, there exists a $0 \ne f \in \mathcal{N}(B^{(\sigma)}(\lambda,k))$ such that $\hat{f}_n = 0$ if $|n| < N$. We note that the only relation between the positive and the negative Fourier coefficients of f is the equation

$$\hat{\sigma}_N\hat{f}_{-N} + \hat{\sigma}_{-N}\hat{f}_N = 0.$$

Therefore f is unique up to multiples. We put

$$\alpha_n^+ := \hat{f}_n, \qquad \alpha_n^- := \overline{\hat{f}_{-n}}, \qquad n \in \mathbb{N}, \tag{5.5}$$

and find (using $\hat{\sigma}_n = \overline{\hat{\sigma}_{-n}}$, $n \in \mathbb{Z}$) that α^+, α^- are as claimed.
Conversely, let α^+, α^- have the properties of the lemma. Then $\alpha_N^+\alpha_N^- \ne 0$ and multiplying α^+ by a constant if necessary, we can assume that $\hat{\sigma}_N\overline{\alpha_N^-} + \hat{\sigma}_{-N}\alpha_N^+ = 0$. Defining f by (5.5) and $\hat{f}_n := 0$ if $|n| < N$ we find that $0 \ne f \in \mathcal{N}(B^{(\sigma)}(\lambda,k))$, so λ is an eigenvalue of $H^{(\sigma)}(k)$ by Proposition 2.1 (2). This completes the proof. ☐

The reason for introducing the operators $A^{(\sigma)}(\lambda,k)$ is that they are not only monotone with respect to λ but also with respect to k. This is essentially used in the

Proof of Proposition 5.2. It remains to prove that the set (5.3) is finite. We denote by $\nu_m(\lambda,k)$, $m \in \mathbb{N}$, the eigenvalues of the operator $A^{(\sigma)}(\lambda,k)$, arranged in non-decreasing order and repeated according to their multiplicities. By the same arguments as in the proof of Lemma 4.3 one finds that the functions $\nu_m(\lambda,.)$ ($\nu_m(.,k)$, respectively) are continuous and strictly increasing (strictly decreasing, respectively) for fixed λ (k, respectively).
Now Lemma 5.3 implies that if λ is an eigenvalue of $H^{(\sigma)}(k)$ in $[(N-1+|k|)^2,(N-|k|)^2)$ then there exist m, $m' \in \mathbb{N}$ such that

$$\nu_m(\lambda,k) = \nu_{m'}(\lambda,-k) = 0. \tag{5.6}$$

It follows easily from the monotonicity properties mentioned above that for each pair $(m, m') \in \mathbb{N} \times \mathbb{N}$ there exists at most one pair (λ, k) with $\lambda \in [(N - 1 + |k|)^2, (N - |k|)^2)$ such that (5.6) holds. Since the functions ν_m are strictly positive for sufficiently large m we conclude that the set (5.3) is finite. $\square$

Example 5.4. In the case $N = 1$ it is convenient to write σ as

$$\sigma(x_1) := -\alpha + \mathrm{Re}\beta \cos x_1 + \mathrm{Im}\beta \sin x_1, \qquad x_1 \in \mathbb{T},$$

with $\alpha \in \mathbb{R}$, $\beta \in \mathbb{C}$. Under the conditions

$$0 < \alpha < 1, \qquad 0 < |\beta| \leq 1 - \alpha, \tag{5.7}$$

one finds that

$$\nu_m(\lambda, k) > 0 \qquad \text{for } m \geq 2,\ k \in (-\tfrac{1}{2}, \tfrac{1}{2}),\ \lambda \in [k^2, (1 - |k|)^2).$$

Thus it follows from (5.6) and the strict monotonicity of $\nu_1(\lambda, .)$ that the operator $H^{(\sigma)}(k)$ has no eigenvalues in $[k^2, (1 - |k|)^2)$ for $k \neq 0$. We consider the case $k = 0$. Under condition (5.7) one easily derives the estimates

$$\begin{aligned} \nu_1(\lambda, 0) &\geq 0 \qquad \text{for } \lambda \in [0, 1 - (\alpha + |\beta|)^2], \\ \nu_1(\lambda, 0) &< 0 \qquad \text{for } \lambda \in (1 - \alpha^2, 1), \end{aligned}$$

which imply that $H^{(\sigma)}(0)$ has a (unique) embedded eigenvalue in $[0, 1)$. It can be shown (see Remark 5.5 below) that it depends real-analytically on the "coupling parameter" $|\beta| > 0$.
Let us emphasize that if $0 < \alpha < 1$ and $\beta = 0$, the operator $H^{(\sigma)}(0)$ has embedded eigenvalues $-\alpha^2 + m^2$, $m \in \mathbb{N}$, each double degenerate (see Example 4.2). As soon as the coupling is turned on (i.e., $|\beta| > 0$) all the eigenvalues above 1 as well as one of the eigenvalues in $(0, 1)$ dissolve, whereas the other one of the eigenvalues in $(0, 1)$ depends smoothly on $|\beta| \in [0, 1 - \alpha]$.

Remark 5.5. Let us mention that the eigenvalue in the above example is due to the following symmetry. Since the operator is (up to unitary equivalence) invariant under a shift with respect to x_1 we may assume that $\beta \in \mathbb{R}$. Then σ is even with respect to $x_1 = 0$ and so for $k = 0$ the decomposition into even and odd functions reduces the operator $H^{(\sigma)}(0)$. It remains to notice that the essential spectrum of the part of the operator acting on odd functions starts at the point $\lambda = 1$.

6 Additional Channels of Scattering of the operators $H^{(\sigma)}$

6.1 Additional Channels due to discrete eigenvalues

Here we construct the additional channels of scattering of $H^{(\sigma)}$ which arise from the discrete eigenvalues of the operators $H^{(\sigma)}(k)$.
For $k \in [-\frac{1}{2}, \frac{1}{2}]$ we denote by

$$\lambda_1(k) \leq \lambda_2(k) \leq \cdots \lambda_{l(k)}(k) < k^2 \tag{6.1}$$

the discrete eigenvalues of $H^{(\sigma)}(k)$, arranged in non-decreasing order and repeated according to their multiplicities. By Theorem 4.1 $l(k)$ is a finite number, possibly equal to 0. It is convenient to set $\lambda_l(k) := k^2$ if $l > l(k)$. The functions λ_l are continuous on $[-\frac{1}{2}, \frac{1}{2}]$ for each $l \in \mathbb{N}$. Combining this with (1.7) we find

$$\sigma\left(H^{(\sigma)}\right) = \bigcup_{l\in\mathbb{N}} \lambda_l([-\tfrac{1}{2}, \tfrac{1}{2}]) \cup [0, +\infty), \tag{6.2}$$

i.e., the spectrum of $H^{(\sigma)}$ has band structure.
According to Theorem 1.2 none of the functions λ_l is constant (since this would correspond to an eigenvalue of $H^{(\sigma)}$).
To construct the additional channels of scattering we introduce some notation. We put

$$\mathcal{K}_l := \{k \in [-\tfrac{1}{2}, \tfrac{1}{2}] : \, l \le l(k)\}, \qquad l \in \mathbb{N}_0.$$

These sets are open in $[-\frac{1}{2}, \frac{1}{2}]$ and $\mathcal{K}_l = \emptyset$ for sufficiently large l. We define

$$l_0 := \max\{l \in \mathbb{N}_0 : \, \mathcal{K}_l \neq \emptyset\}.$$

Now assume $l_0 > 0$ (which means that some of the operators $H^{(\sigma)}(k)$ have discrete eigenvalues). For each $k \in [-\frac{1}{2}, \frac{1}{2}]$ we can choose orthonormal eigenfunctions $\psi_l(.,k)$, $1 \le l \le l(k)$, corresponding to the eigenvalues (6.1),

$$H^{(\sigma)}(k)\, \psi_l(.,k) = \lambda_l(k)\, \psi_l(.,k),$$

such that the mappings

$$\mathcal{K}_l \to L_2(\Pi), \; k \mapsto \psi_l(.,k), \qquad 1 \le l \le l_0,$$

are piecewise analytic. Recall that the functions $\psi_l(.,k)$ are of the form (2.4). It is convenient to define $\psi_l(.,k) := 0$ if $k \notin \mathcal{K}_l$ and to extend the functions $\psi_l(.,k)$ periodically with respect to the variable x_1 to functions on $\mathbb{R}^2_+$ for all $k \in [-\frac{1}{2}, \frac{1}{2}]$.
For $1 \le l \le l_0$ we denote by $P_l(k)$, $k \in [-\frac{1}{2}, \frac{1}{2}]$, the projection in $L_2(\Pi)$ onto the subspace spanned by $\psi_l(.,k)$. With this notation, we call the subspaces

$$\mathfrak{C}_l := \mathcal{R}\left(\mathcal{U}^*\left(\int_{-1/2}^{1/2} \oplus P_l(k)\, dk\right)\mathcal{U}\right), \qquad 1 \le l \le l_0,$$

additional channels of scattering (ACS) of the operator $H^{(\sigma)}$. Here $\mathcal{U}$ is the Gelfand transformation (1.6). Thus the functions $u \in \mathfrak{C}_l$ are precisely the functions of the form

$$u(x) = \int_{-1/2}^{1/2} f(k)\psi_l(x,k)e^{ikx_1}\, dk, \qquad x \in \mathbb{R}^2_+, \tag{6.3}$$

with $f \in L_2(\mathcal{K}_l)$ arbitrary. In particular, it follows from the form (2.4) of the eigenfunction $\psi_l(.,k)$ that functions $u \in \mathfrak{C}_l$ decay exponentially with respect to

the variable x_2 provided $\mathcal{K}_l = [-\frac{1}{2}, \frac{1}{2}]$.
Let us list some more properties of the spaces $\mathfrak{C}_l$. One has for all $1 \le l, j \le l_0$

$$\mathfrak{C}_l \perp \mathfrak{C}_j, \ \ j \neq l, \qquad \text{and} \qquad \mathfrak{C}_l \perp \mathcal{R}(W_\pm^{(\sigma)}).$$

Indeed, this follows from the fact that $\psi_l(.,k)$ is orthogonal to $\psi_j(.,k)$, $j \neq l$, and to the subspace $\mathcal{R}(W_\pm^{(\sigma)}(k))$ for all $k \in [-\frac{1}{2}, \frac{1}{2}]$. In particular, Theorem 1.2 implies that the wave operators $W_\pm^{(\sigma)}$ are not complete if there exists an ACS (i.e., $l_0 > 0$). Moreover, the spaces $\mathfrak{C}_l$ reduce the operator $H^{(\sigma)}$, and on functions $u \in \mathfrak{C}_l$ of the form (6.3) $H^{(\sigma)}$ acts by multiplying the function f with the function λ_l. Thus, the part of $H^{(\sigma)}$ on $\mathfrak{C}_l$ is unitarily equivalent to multiplication with the function λ_l on $L_2(\mathcal{K}_l)$.
Remark 1.10 of [Fr] shows that functions $u \in \mathfrak{C}_l$ correspond to states which propagate along the boundary.

6.2 Existence of ACS. Existence of gaps

It is clear from Theorem 1.1 that there are no ACS if σ is non-negative. Let us give an easy sufficient condition for the existence of ACS. It requires σ to be "negative in mean".

Proposition 6.1. *Assume that* $\hat\sigma_0 := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sigma(x_1)\, dx_1 < 0$. *Then*

$$\sigma\left(H^{(\sigma)}\right) \cap (-\infty, 0) \neq \emptyset.$$

Proof. Indeed, we prove that $H^{(\sigma)}(k)$ has an eigenvalue smaller or equal to $k^2 - \frac{1}{2\pi}\hat\sigma_0^2$ for all $k \in [-\frac{1}{2}, \frac{1}{2}]$. For this we consider the trial function defined by $u(x) := e^{\hat\sigma_0 x_2/\sqrt{2\pi}}$, $x \in \Pi$, which satisfies

$$h^{(\sigma)}(k)[u] = (k^2 - \tfrac{1}{2\pi}\hat\sigma_0^2)\|u\|^2.$$

The assertion follows now from the variational principle. □

Remark 6.2. With more elaborate techniques one can show that the conclusion of Proposition 6.1 remains valid under the assumption $\hat\sigma_0 = 0$, $\sigma \not\equiv 0$.

We give now an example where the first gap of $H^{(\sigma)}$ is open, i.e. where

$$\max_{k \in [-\frac{1}{2}, \frac{1}{2}]} \lambda_1(k) < \min_{k \in [-\frac{1}{2}, \frac{1}{2}]} \lambda_2(k). \tag{6.4}$$

We start with a more general construction. Let $-\pi \le c \le \pi$ be given. For $k \in [-\frac{1}{2}, \frac{1}{2}]$ we consider the self-adjoint operators $H_D^{(\sigma)}(k)$, $H_N^{(\sigma)}(k)$ in $L_2(\Pi)$ which differ from $H^{(\sigma)}(k)$ only by Dirichlet and natural boundary conditions, respectively, at $\{x_1 \in \{-\pi, c, \pi\}\}$. More precisely, the operators $H_\nu^{(\sigma)}(k)$, $\nu \in$

$\{D, N\}$, are defined by the quadratic forms $h_\nu^{(\sigma)}(k)$ given by the same formal expression (1.5) as $h^{(\sigma)}(k)$ but with domains

$$\begin{aligned}\mathcal{D}[h_D^{(\sigma)}(k)] &:= \{u \in H^1(\Pi) :\ u(.,-\pi) = u(.,c) = u(.,\pi) = 0\},\\ \mathcal{D}[h_N^{(\sigma)}(k)] &:= \{u \in L_2(\Pi) :\ u|_{(-\pi,c)\times\mathbb{R}_+} \in H^1((-\pi,c)\times\mathbb{R}_+),\\ &\qquad u|_{(c,\pi)\times\mathbb{R}_+} \in H^1((c,\pi)\times\mathbb{R}_+)\}.\end{aligned}$$

It follows that

$$H_N^{(\sigma)}(k) \le H^{(\sigma)}(k) \le H_D^{(\sigma)}(k). \tag{6.5}$$

Moreover, it is easy to see that for each ν all the operators $H_\nu^{(\sigma)}(k)$, $k \in [-\frac{1}{2}, \frac{1}{2}]$, are unitarily equivalent. Their essential spectrum starts at $(\frac{\pi}{\pi+|c|})^2$ if $\nu = D$ and at 0 if $\nu = N$. We define the numbers λ_l^ν, $l \in \mathbb{N}$, as the successive infima of the variational quotient

$$\frac{h_\nu^{(\sigma)}(k)[u]}{\|u\|^2}, \qquad 0 \ne u \in \mathcal{D}[h_\nu^{(\sigma)}(k)].$$

By the variational principle the $\lambda_l^N < 0$ coincide with the discrete eigenvalues of the operator $H_N^{(\sigma)}(k)$, and similarly for $\nu = D$. It follows from (6.5) together with the variational principle that for all $l \in \mathbb{N}$

$$\lambda_l^N \le \lambda_l(k) \le \lambda_l^D, \qquad k \in [-\tfrac{1}{2}, \tfrac{1}{2}]. \tag{6.6}$$

Let us give now an example of an open gap.

Example 6.3. Let a, $b \in \mathbb{R}$ and

$$\sigma(x_1) := \begin{cases} -a & \text{if } x_1 \in [-\pi, c],\\ b & \text{if } x_1 \in (c, \pi).\end{cases}$$

We claim that under the assumptions

$$a > \tfrac{\pi}{\pi+c}, \qquad b \ge 0, \qquad -\pi < c < \pi, \tag{6.7}$$

the inequality (6.4) holds.

Indeed, one easily finds that $\lambda_1^D = \lambda_2^N = -a^2 + (\frac{\pi}{\pi+c})^2$, so because of (6.6) and the continuity of λ_1 it suffices to prove that

$$\lambda_1(k) < -a^2 + (\tfrac{\pi}{\pi+c})^2, \qquad k \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

To arrive at a contradiction we assume that we have equality for some k. Consider the eigenfunction u of $H_D^{(\sigma)}(k)$ corresponding to the eigenvalue $-a^2 + (\frac{\pi}{\pi+c})^2$,

$$u(x) := \begin{cases} 2\sqrt{\frac{a}{\pi+c}}\, e^{-ikx_1} \sin(\frac{\pi}{\pi+c}(x_1+\pi))\, e^{-ax_2}, & x_1 \in [-\pi, c],\\ 0, & x_1 \in (c, \pi).\end{cases}$$

Then $u \in \tilde{H}^1(\Pi)$, $\|u\| = 1$ and $h^{(\sigma)}(k)[u] = \inf\{h^{(\sigma)}(k)[v] :\ v \in \tilde{H}^1(\Pi),\ \|v\| = 1\}$. It follows from general principles that $u \in \mathcal{D}(H^{(\sigma)}(k))$ and $H^{(\sigma)}(k)u = \lambda_1(k)u$. By Elliptic Regularity we must have $u \in H^2_{loc}(\Pi)$, which is obviously not true. This contradiction completes the proof of (6.4).

Remark 6.4. It follows from (6.6) that the condition $\lambda_l^D < \lambda_{l+1}^N$ for some $l \in \mathbb{N}$ is sufficient for an open gap. This can be used to construct further examples.

Remark 6.5. By an argument similar to the one in Example 6.3 we find that if there exists a non-empty connected open subset Λ of the torus such that $\sigma(x_1) \leq -\frac{\pi}{\operatorname{meas}\Lambda}$, $x_1 \in \Lambda$, then $\sigma\left(H^{(\sigma)}\right) \cap (-\infty, 0) \neq \emptyset$, so there exist ACS.

To conclude this subsection we note that the number of ACS (due to discrete eigenvalues) can be estimated using (4.3).

6.3 Additional Channels due to embedded eigenvalues

In general, the embedded eigenvalues of the operators $H^{(\sigma)}(k)$, $k \in [-\frac{1}{2}, \frac{1}{2}]$, also contribute to the spectrum of the operator $H^{(\sigma)}$. Therefore the subspace

$$\mathfrak{C}_* := \mathcal{R}\left(\mathcal{U}^*\left(\int_{-1/2}^{1/2} \oplus E^{(\sigma)}\left(\sigma_p\left(H^{(\sigma)}(k)\right) \cap \left[k^2, +\infty\right), k\right) dk\right)\mathcal{U}\right)$$

may be non-trivial and, in this case, will be called an ACS. We have

$$L_2(\mathbb{R}^2_+) = \mathcal{R}(W_\pm^{(\sigma)}) \oplus (\textstyle\sum_{l=1}^{l_0} \oplus \mathfrak{C}_l) \oplus \mathfrak{C}_*.$$

The subspace $\mathfrak{C}_*$ reduces the operator $H^{(\sigma)}$ and is orthogonal to the ACS $\mathfrak{C}_l$, $1 \leq l \leq l_0$, and to $\mathcal{R}(W_\pm^{(\sigma)})$.

Let us consider some examples. If $\sigma \equiv \sigma_0 < 0$ is a negative constant, we know from Example 4.2 that the embedded eigenvalues of $H^{(\sigma)}(k)$ depend piecewise analytically on k and all of them contribute to the spectrum of $H^{(\sigma)}$. We note that in this case the part of $H^{(\sigma)}$ on $\mathfrak{C}_*$ is an unbounded operator.

If σ is a trigonometric polynomial of degree $N > 0$, we know from Proposition 5.1 that $H^{(\sigma)}(k)$ has no embedded eigenvalues greater or equal to $(N - |k|)^2$. Moreover, we know from Proposition 5.2 that the embedded eigenvalues in the interval $[(N - 1 + |k|)^2, (N - |k|)^2)$ do not contribute to the spectrum of the operator $H^{(\sigma)}$. So the part of $H^{(\sigma)}$ on $\mathfrak{C}_*$ is a bounded operator with spectrum contained in $[0, (N - \frac{1}{2})^2]$.

In the special case when σ is a trigonometric polynomial of degree one, it follows again from Proposition 5.1 and Proposition 5.2 that $\mathfrak{C}_* = \{0\}$. We emphasize (see Example 5.4) that embedded eigenvalues of the operators $H^{(\sigma)}(k)$ actually occur in this case.

The question whether $\mathfrak{C}_*$ can be non-trivial for non-constant σ remains open.

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