Comparison of Abelian Categories Recollements

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Abstract. We give a necessary and sufficient condition for a morphism between recollements of abelian categories to be an equivalence.

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1 Introduction

Recollements of abelian and triangulated categories play an important role in geometry of singular spaces [3], in representation theory [4, 12], in polynomial functors theory [8, 9, 14] and in ring theory, where recollements are known as torsion, torsion-free theories [6]. A fundamental example of recollement of abelian categories is due to MacPherson and Vilonen [10]. It first appeared as an inductive step in the construction of perverse sheaves. The main motivation for our work was to understand when a recollement can be obtained through the construction of MacPherson and Vilonen.

A *recollement situation* consists of three abelian categories $\mathcal{A}'$, $\mathcal{A}$, $\mathcal{A}''$ together with additive functors:

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ \mathcal{A}' & \xrightarrow{i_*} & \mathcal{A} & \xrightarrow{j^*} & \mathcal{A}'' \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}$$

which satisfy the following conditions:

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i. $j_!$ is left adjoint to j^* and j^* is left adjoint to j_*

ii. the unit $Id_{\mathcal{A}''} \to j^* j_!$ and the counit $j^* j_* \to Id_{\mathcal{A}''}$ are isomorphisms

iii. i^* is left adjoint to i_* and i_* is left adjoint to $i^!$

iv. the unit $Id_{\mathcal{A}'} \to i^! i_*$ and the counit $i^* i_* \to Id_{\mathcal{A}'}$ are isomorphisms

v. i_* is an embedding onto the full subcategory of $\mathcal{A}$ with objects A such that $j^* A = 0$.

In this case one says that $\mathcal{A}$ is a *recollement of* $\mathcal{A}''$ *and* $\mathcal{A}'$. *These notations will be kept throughout the paper.* Thus in any recollement situation, the category $i_*\mathcal{A}'$ is a localizing and colocalizing subcategory of $\mathcal{A}$ in the sense of [5], and the category $\mathcal{A}''$ is equivalent to the quotient category of $\mathcal{A}$ by $i_*\mathcal{A}'$.

If $\mathcal{B}$ is also a recollement of $\mathcal{A}''$ and $\mathcal{A}'$, then a *comparison functor* $\mathcal{A} \to \mathcal{B}$ is an exact functor which commutes with all the structural functors $i^*, i_*, i^!, j_!, j^*, j_*$. According to [12, Theorem 2.5], a comparison functor between recollements of triangulated categories is an equivalence of categories. Our example in Section 2.2 shows that this is not necessarily the case for recollements of abelian categories.
Our main result, Theorem 7.2, characterizes which comparisons of recollements are equivalences of categories. As an application, we give a homological criterion deciding when a recollement can be obtained through the construction of MacPherson and Vilonen.
Theorem. *A recollement situation of categories with enough projectives is isomorphic to a MacPherson-Vilonen construction if and only if the following two conditions hold.*

i. There exists an exact functor $r\colon \mathcal{A} \to \mathcal{A}'$ *such that* $r \circ i_* = Id_{\mathcal{A}'}$.

ii. For any projective object V *of the category* $\mathcal{A}'$, $(\mathrm{L}_2 i^*)(i_* V) = 0$.

2 Examples

Our examples are related to polynomial functors. The relevance of this formalism to polynomial functors was stressed by N. Kuhn [8].
We let $\mathcal{A}'$ be the category of finite vector spaces over the field with two elements $\mathbb{F}_2$, and we let $\mathcal{A}''$ be the category of finite vector spaces over $\mathbb{F}_2$ with involution, or finite representations of Σ_2 over $\mathbb{F}_2$.

2.1

In the first example, the category $\mathcal{A}$ is a category of diagrams of finite vector spaces over $\mathbb{F}_2$:

$$(V_1, H, V_2, P):\ V_1 \rightleftarrows V_2\ ,$$

where H: $V_1 \to V_2$ and P: $V_2 \to V_1$ are linear maps which satisfy: $PHP = 0$ and $HPH = 0$. The category $\mathcal{A}$ is equivalent to the category of quadratic functors from finitely generated free abelian groups to vector spaces over $\mathbb{F}_2$. It is a recollement for the following functors:

$$i^*(V_1, H, V_2, P) = \text{Coker } (P), \; j_!(V, T) = (V_T, 1 + T, V, p)$$
$$i_*(V) = (V, 0, 0, 0), \qquad j^*(V_1, H, V_2, P) = (V_2, HP - 1)$$
$$i^!(V_1, H, V_2, P) = \text{Ker } (H) \ , \ \ j_*(V, T) = (V^T, h, V, 1 + T) \ ,$$

where $V^T = \text{Ker } (1 - T)$, $V_T = \text{Coker } (1 - T)$, h is the inclusion and p is the quotient map. Note that the functor i_* admits an obvious exact retraction r: $(V_1, H, V_2, P) \mapsto V_1$.

2.2 Comparison fails for abelian categories recollements

We now consider the full subcategory of the category $\mathcal{A}$ in Example 2.1, whose objects satisfy the relation: $PH = 0$. This category is equivalent to the category of quadratic functors from finite vector spaces to vector spaces over $\mathbb{F}_2$. The same formulae define a recollement as well. As a result, the inclusion of categories is a comparison functor. It is not, however, an equivalence of categories.

3 The construction of MacPherson and Vilonen [10]

3.1

Let $\mathcal{A}'$ and $\mathcal{A}''$ be abelian categories. Let F: $\mathcal{A}'' \to \mathcal{A}'$ be a right exact functor, let G: $\mathcal{A}'' \to \mathcal{A}'$ be a left exact functor and let ξ: $F \to G$ be a natural transformation. Define the category $\mathcal{A}(\xi)$ as follows. The objects of $\mathcal{A}(\xi)$ are tuples (X, V, α, β), where X is in $\mathcal{A}''$, V is in $\mathcal{A}'$, $\alpha : F(X) \to V$ and $\beta : V \to G(X)$ are morphisms in $\mathcal{A}'$ such that the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[rr, "\xi_X"] \arrow[dr, "\alpha"'] & & G(X) \\ & V \arrow[ur, "\beta"'] & \end{tikzcd} .$$

A morphism from (X, V, α, β) to $(X', V', \alpha', \beta')$ is a pair (f, φ), where $f : X \to X'$ is a morphism in $\mathcal{A}''$ and φ: $V \to V'$ is a morphism in $\mathcal{A}'$, such that the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "\alpha"] \arrow[d, "F(f)"] & V \arrow[r, "\beta"] \arrow[d, "\varphi"] & G(X) \arrow[d, "G(f)"] \\ F(X') \arrow[r, "\alpha'"] & V' \arrow[r, "\beta'"] & G(X') \end{tikzcd} .$$

The category $\mathcal{A}(\xi)$ comes with functors:

$$i^*(X,V,\alpha,\beta) = \text{Coker }(\alpha)\ , \qquad j_!(X) = (X, F(X), Id_{F(X)}, \xi_X)\ ,$$

$$i_*(V) = (0,V,0,0)\ , \qquad j^*(X,V,\alpha,\beta) = X\ ,$$

$$i^!(X,V,\alpha,\beta) = \text{Ker }(\beta)\ , \qquad j_*(X) = (X, G(X), \xi_X, Id_{G(X)})\ .$$

The functor i_* has a retraction functor r:

$$r(X,V,\alpha,\beta) = V\ .$$

The category $\mathcal{A}(\xi)$ is abelian in such a way that the functors r and j^* are exact. The above data define a recollement. Note that we recover the natural transformation ξ from the retraction r and the recollement data as:

$$F = rj_! \qquad G = rj_* \qquad \xi \simeq rN\ .$$

The category $\mathcal{A}$ depends only [10, Proposition 1.2] on the class of the extension

$$0 \to i^!j_! \to F \xrightarrow{\xi} G \to i^*j_* \to 0\ ,$$

image by r of the exact sequence (4).

3.2

We now consider two particular cases of this construction, already known to Grothendieck (see [1]). Let F: $\mathcal{A}'' \to \mathcal{A}'$ be a right exact functor. Take ξ: $F \to 0$ to be the transformation into the trivial functor. The corresponding construction is denoted by $\mathcal{A}' \rtimes_F \mathcal{A}''$. Thus objects of this category are triples (V,X,α), where V and X are objects of $\mathcal{A}'$ and $\mathcal{A}''$ respectively and α is a morphism α: $F(X) \to V$ of the category $\mathcal{A}'$. Note that $i^*j_* = 0$ and $i^!j_! \cong F$. Moreover, $i^!$ and j_* are exact functors.
Similarly, let $\mathcal{B}'$ and $\mathcal{B}''$ be abelian categories and let G: $\mathcal{B}'' \to \mathcal{B}'$ be a left exact functor. We take ξ: $0 \to G$ to be the natural transformation from the trivial functor. The corresponding recollement is denoted by $\mathcal{B}' \ltimes_G \mathcal{B}''$. Objects of this category are triples $(B'', B', \beta : B' \to G(B''))$. Assuming now $\mathcal{B}' = \mathcal{A}''$, $\mathcal{B}'' = \mathcal{A}'$ and $G : \mathcal{A}' \to \mathcal{A}''$ is right adjoint to F, the category $\mathcal{A}' \rtimes_F \mathcal{A}'' = \mathcal{A}'' \ltimes_G \mathcal{A}'$ fits into two different recollement situations.

4 General properties of recollements

Most of the properties in this section can probably be found in [3] or other references. We list them for convenience. Note however that, when they are not a consequence of [5], they are usually stated and proved in the context of triangulated categories. We consistently provide statements (and a few proofs) in the context of abelian categories and derived functors.

4.1 First properties

We remark as usual that taking opposite categories results in the exchange of $j_!$ and i^* with j_* and $i^!$ respectively. This is referred to as duality. For instance, the relation $j^*i_* = 0$ - a consequence of (v) - yields the dual relation $i^!j_* = 0$.

Proposition 4.1 *In any recollement situation:*

$$i^*j_! = 0 \ , \quad i^!j_* = 0.$$

Proposition 4.2 *The units and counits of adjonction give rise to exact sequences of natural transformations:*

$$j_!j^* \xrightarrow{\epsilon} Id_{\mathcal{A}} \to i_*i^* \to 0 \tag{1}$$

$$0 \to i_*i^! \to Id_{\mathcal{A}} \xrightarrow{\eta} j_*j^* \ . \tag{2}$$

We now recall the definition of the norm N: $j_! \to j_*$. For any X, Y in $\mathcal{A}''$, there are natural isomorphisms:

$$\mathrm{Hom}_{\mathcal{A}}(j_!X, j_*Y) \cong \mathrm{Hom}_{\mathcal{A}''}(X, j^*j_*Y) \cong \mathrm{Hom}_{\mathcal{A}''}(X, Y).$$

For $Y = X$, let N_X: $j_!X \to j_*X$ be the map corresponding to the identity of X. It is a natural transformation [3, 1.4.6.2]. The norm N is thus defined so that: $Nj^* = \eta \circ \epsilon$. Hence:

$$N \cong N(j^*j_*) = (Nj^*)j_* \cong (\eta\circ\epsilon)j_* = \eta j_*\circ\epsilon j_* \cong \epsilon j_* \text{ and, dually } N \cong \eta j_! \ . \tag{3}$$

The image of the norm is a functor

$$j_{!*} := \mathrm{Im}\ (N : j_! \to j_*) : \ \mathcal{A}'' \to \mathcal{A} \ .$$

Proposition 4.3 *In any recollement situation:* $i^*j_{!*} = 0$, $i^!j_{!*} = 0$.

Proof. Use Proposition 4.1 and apply i^* to the epi $j_! \to j_{!*}$. □

Proposition 4.4 *In any recollement situation, there is a short exact sequence of natural transformations*

$$0 \to i_*i^!j_! \to j_! \xrightarrow{N} j_* \to i_*i^*j_* \to 0 \ . \tag{4}$$

Proof. Precompose the exact sequence (1) with j_*. Precomposition is exact, hence one gets the following exact sequence:

$$j_! \to j_* \to i_*i^*j_* \to 0 \ ,$$

where the left arrow is the norm N according to (3). Dually, there is an exact sequence:

$$0 \to i_*i^!j_! \to j_! \xrightarrow{N} j_* \ .$$

Splicing the two sequences together gives the result. □

Applying the snake lemma, one gets the following strong restriction on the functors $i^!j_!$ and i^*j_* of a recollement situation.

Corollary 4.5 *For any short exact sequence in $\mathcal{A}''$:*

$$0 \to X \to Y \to Z \to 0$$

there is an exact sequence in $\mathcal{A}'$:

$$i^! j_!(X) \to i^! j_!(Y) \to i^! j_!(Z) \to i^* j_*(X) \to i^* j_*(Y) \to i^* j_*(Z)$$

4.2 Homological properties

In this section we investigate the derived functors of the functors in a recollement situation. We use the following convention throughout this section: When mentioning left derived functors $\mathrm{L}-$, the category $\mathcal{A}$, and thus the categories $\mathcal{A}'$ and $\mathcal{A}''$, have enough projectives, and, similarly, when mentioning right derived functors $\mathrm{R}-$, the categories $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ have enough injectives. Most of the proofs consist in applying long exact sequences for derived functors to Section 4.1's exact sequences.

Proposition 4.6 *For each integer $n \geq 1$:*

$$j^*(\mathrm{L}_n j_!) = 0 \quad , \quad j^*(\mathrm{R}^n j_*) = 0 \ .$$

Proposition 4.7

$$(\mathrm{L}_1 i^*) i_* = 0 \quad , \qquad (\mathrm{R}^1 i^!) i_* = 0 \tag{5}$$

$$(\mathrm{L}_1 i^*) j_! = 0 \quad , \qquad (\mathrm{R}^1 i^!) j_* = 0 \tag{6}$$

$$(\mathrm{L}_1 i^*) j_{!*} = i^! j_! \quad , \qquad (\mathrm{R}^1 i^!) j_{!*} = i^* j_* \tag{7}$$

Proposition 4.8 *There is a natural exact sequence:*

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathcal{A}'}(i^*A, V) \to \mathrm{Ext}^1_{\mathcal{A}}(A, i_*V) \xrightarrow{\eta} \mathrm{Hom}_{\mathcal{A}'}((\mathrm{L}_1 i^*)A, V) \to \\ \to \mathrm{Ext}^2_{\mathcal{A}'}(i^*A, V) \to \mathrm{Ext}^2_{\mathcal{A}}(A, i_*V) \ . \end{aligned}$$

Proof. This follows from the spectral sequence for the derived functors of the composite functors:

$$E^2_{pq} = \mathrm{Ext}^p_{\mathcal{A}'}(\mathrm{L}_q i^*(A), V) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathcal{A}}(A, i_*V) \ . \tag{8}$$

□

Proposition 4.9 *Let A be an object in* Ker i^**. The counit ϵ_A: $j_! j^* A \to A$ is epi and its kernel is in $i_*\mathcal{A}'$. Indeed, if $\mathcal{A}$ has enough projectives, there is a short exact sequence:*

$$0 \to i_*(\mathrm{L}_1 i^*)A \to j_! j^* A \xrightarrow{\epsilon_A} A \to 0 \ . \tag{9}$$

We prove the dual statement:

PROPOSITION 4.10 *Let A be an object in* $\mathrm{Ker}\ i^!$. *The unit η_A: $A \to j_*j^*A$ is mono and its cokernel is in $i_*\mathcal{A}'$. Indeed, if $\mathcal{A}$ has enough injectives, there is a short exact sequence:*

$$0 \to A \xrightarrow{\eta_A} j_*j^*A \to i_*(\mathrm{R}^1 i^!)A \to 0\ . \tag{10}$$

Proof. When $i^!A = 0$, the exact sequence (2) simplifies to a short exact sequence:

$$0 \to A \xrightarrow{\eta_A} j_*j^*A \to \mathrm{Coker}\ \eta_A \to 0\ . \tag{11}$$

First applying the exact functor j^*, and using that $j^*\eta$ is an iso, we see that $j^*(\mathrm{Coker}\ \eta_A) = 0$. Thus $\mathrm{Coker}\ \eta_A$ is in $i_*\mathcal{A}'$. Suppose that $\mathcal{A}$ has enough injectives. Applying now the left exact functor $i^!$, the long exact sequence for right derived functors gives an exact sequence:

$$0 \to i^!A \to i^!j_*j^*A \to i^!\mathrm{Coker}\ \eta_A \to (\mathrm{R}^1 i^!)A \to (\mathrm{R}^1 i^!)j_*j^*A\ .$$

Proposition 4.1 and (6) give an isomorphism $i^!\mathrm{Coker}\ (\eta_A) \cong \mathrm{R}^1 i^!(A)$. □

4.3 DESCRIPTION OF THE IMAGE OF j_*, $j_{!*}$, $j_!$

Since $j^*j_! \cong j^*j_* \cong j^*j_{!*} \cong Id_{\mathcal{A}''}$, the functors $j_!, j_*, j_{!*}$: $\mathcal{A}'' \to \mathcal{A}$ are full embeddings. The next result describes the essential image of each of them.

PROPOSITION 4.11 *The functors $j_!, j_*, j_{!*} : \mathcal{A}'' \to \mathcal{A}$ induce the following equivalences of categories:*

$$j_{!*} : \mathcal{A}'' \to \{A \in \mathcal{A} \mid i^*(A) = 0 = i^!(A)\},$$

$$j_! : \mathcal{A}'' \to \{A \in \mathcal{A} \mid i^*(A) = 0 = \mathrm{L}_1 i^*(A)\},$$
$$j_* : \mathcal{A}'' \to \{A \in \mathcal{A} \mid i^!(A) = 0 = \mathrm{R}^1 i^!(A)\}.$$

4.4 A MONOMORPHISM ON EXT-GROUPS

Since $j^* : \mathcal{A} \to A''$ is an exact functor, it induces an homomorphism

$$\mathrm{Ext}^n_{\mathcal{A}}(A, B) \to \mathrm{Ext}^n_{\mathcal{A}''}(j^*A, j^*B),\ n \geq 0.$$

It is well-known that when A and B are simple objects, this map is injective for $n = 1$ (see for example [8, Proposition 4.12]). The following more general result holds.

PROPOSITION 4.12 *Let $A, B \in \mathcal{A}$ be objects for which $i^*A = 0$ and $i^!B = 0$. Suppose $j^*A \neq 0$ and $j^*B \neq 0$. Then*

$$\mathrm{Ext}^1_{\mathcal{A}}(A, B) \to \mathrm{Ext}^1_{\mathcal{A}''}(j^*A, j^*B)$$

is a monomorphism.

5 Description of Ker i^* and Ker $i^!$

Let Ker $i^!$ be the full subcategory of objects A of $\mathcal{A}$ such that $i^!A = 0$, and let Ker i^* be the full subcategory of objects A of $\mathcal{A}$ such that $i^*A = 0$. In this section, we describe these subcategories of $\mathcal{A}$ in terms of the categories $\mathcal{A}'$, $\mathcal{A}''$, and the functors i^*j_*, $i^!j_!$ between them, through the following construction:

Definition 5.1 *Let $T\colon \mathcal{A}'' \to \mathcal{A}'$ be an additive functor between abelian categories. The category $\mathcal{M}(T)$ has objects triples (X, V, α) where X is in $\mathcal{A}''$, V is in $\mathcal{A}'$, and $\alpha\colon V \to TX$ is a monomorphism. A map from (X, V, α) to (X', V', α') is a pair of morphisms (f, φ) such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d, "\varphi"] & T(X) \arrow[d, "T(f)"] \\ V' \arrow[r, "\alpha'"] & T(X') \end{tikzcd} .$$

The following theorem is inspired by [13].

Theorem 5.2 *In a recollement with enough projectives, the functor $A \mapsto (j^*A,\ i^*A,\ i^*\eta_A : i^*A \to i^*j_*j^*A)$ is an equivalence from the category* Ker $i^!$ *to the category $\mathcal{M}(i^*j_*)$.*

Proof. First, we show that the functor is well defined. Apply the functor i^* on the short exact sequence (11). There results an exact sequence:

$$\mathrm{L}_1 i^*(\text{Coker } \eta_A) \to i^*A \to i^*j_*j^*(A) \to i^*\text{Coker } \eta_A \to 0 .$$

whose left term cancels by Proposition 4.10 and (5). The map $i^*\eta_A$ is thus mono.

Next, we define the quasi-inverse: $\mathcal{M}(i^*j_*) \to$ Ker $i^!$. To an object (X, V, α), it associates the kernel $A(X, V, \alpha)$ of the composite of epis:

$$j_*X \overset{\epsilon j_*}{\to} i_*i^*j_*X \to \text{Coker } i_*\alpha .$$

That is, $A(X, V, \alpha)$ fits in the following map of extensions:

$$\begin{tikzcd} 0 \arrow[r] & j_{!*}X \arrow[r] & j_*X \arrow[r] & i_*i^*j_*X \arrow[r] & 0 \\ 0 \arrow[r] & j_{!*}X \arrow[r] \arrow[u, "="] & A(X,V,\alpha) \arrow[r] \arrow[u] & i_*V \arrow[r] \arrow[u, "i_*\alpha"] & 0 \end{tikzcd} .$$

To a map (f, φ), it associates the map induced by $j_*(f)$.

We leave the verifications to the reader, with the help of the isomorphism $Nj^* \cong \epsilon \circ \eta$. $\square$

The dual study of the category Ker i^* leads to the following.

Theorem 5.3 *In a recollement with enough injectives, the functor* $A \mapsto (j^*A,\ i^!\mathrm{Ker}\ \epsilon_A,\ i^!\mathrm{Ker}\ \epsilon_A \to i^! j_! j^* A)$ *is an equivalence from the category* $\mathrm{Ker}\ i^*$ *to the category* $\mathcal{M}(i^! j_!)$.

This time, the quasi-inverse fits in the following map of extensions:

$$\begin{tikzcd} 0 \arrow[r] & i_* i^! j_! X \arrow[r] \arrow[d] & j_! X \arrow[r] \arrow[d] & j_{!*} X \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & \mathrm{Coker}\ (i_*\alpha) \arrow[r] & A(X,V,\alpha) \arrow[r] & j_{!*} X \arrow[r] & 0\ . \end{tikzcd}$$

Note (Proposition 4.9) that when the recollement has enough projectives, $i^!\mathrm{Ker}\ \epsilon_A$ is nothing but $(\mathrm{L}_1 i^*)A$.

6 Recollements as linear extensions

The exact sequence (2) tells that every object A in $\mathcal{A}$ sits in a short exact sequence:

$$0 \to \mathrm{Ker}\ \eta_A \to A \xrightarrow{\eta_A} \mathrm{Im}\ \eta_A \to 0\ .$$

where $\mathrm{Ker}\ \eta_A \cong i_* i^! A$ is in $i_*\mathcal{A}'$ and $\mathrm{Im}\ \eta_A \cong A/i_* i^! A$ is in $\mathrm{Ker}\ i^!$. We denote by $\mathcal{G}$ the category encoding these data from the recollement situation. That is, objects of the category $\mathcal{G}$ are triples (A, U, e) of an object A in $\mathrm{Ker}\ i^!$, an object U in $\mathcal{A}'$ and an extension class e in the group $\mathrm{Ext}^1_{\mathcal{A}}(A, i_* U)$. A map from (A, U, e) to (A', U', e') is a pair of morphism $(\alpha : A \to A', \beta : U \to U')$ such that: $\alpha^* e' = (i_*\beta)_* e$ in the group $\mathrm{Ext}^1_{\mathcal{A}}(A', i_* U)$. It comes with a functor:

$$\mathcal{A} \to \mathcal{G} \qquad B \mapsto (\mathrm{Im}\ \eta_B,\ i^! B,\ [0 \to \mathrm{Ker}\ \eta_B \to B \xrightarrow{\eta} \mathrm{Im}\ \eta_B \to 0])\ .$$

Because of the Yoneda correspondence between extensions and elements in Ext^1, this functor induces an equivalence of categories to $\mathcal{G}$ from the following category $\mathcal{B}$. The objects of $\mathcal{B}$ are those of $\mathcal{A}$, and a map in $\mathrm{Hom}_{\mathcal{B}}(B, B')$ is a class of maps in $\mathrm{Hom}_{\mathcal{A}}(B, B')$ inducing the same map in $\mathcal{G}$.

We claim that $\mathcal{A} \to \mathcal{B}$ defines a linear extension of categories in the sense of Baues and Wirsching. For completeness, we now recall what we need from this theory (however, the following defining properties might be better understood by just looking at our example).

Definition 6.1 [2, IV.3] Let $\mathcal{B}$ be a category and let $D : \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{A}b$ be a bifunctor with abelian groups values. We say that

$$0 \longrightarrow D \longrightarrow \mathcal{C} \xrightarrow{p} \mathcal{B} \longrightarrow 0 \tag{12}$$

is a linear extension of the category $\mathcal{B}$ by D if the following conditions hold:

i. $\mathcal{C}$ is a category and p is a functor. Moreover $\mathcal{C}$ and $\mathcal{B}$ have the same objects, p is the identity on objects and p is surjective on morphisms.

ii. For any objects c and d in $\mathcal{B}$, the abelian group $D(c,d)$ acts on the set $\mathrm{Hom}_{\mathcal{C}}(c,d)$. Moreover $p(f_0) = p(g_0)$ if and only if there is unique α in $D(c,d)$ such that: $g_0 = f_0 + \alpha$. Here for each $f_0 : c \to d$ in $\mathcal{C}$ and $\alpha \in D(c,d)$ we write $f_0 + \alpha$ for the action of α on f_0.

iii. The action satisfies the linear distributivity law: for two composable maps f_0 and g_0 in $\mathcal{C}$

$$(f_0 + \alpha)(g_0 + \beta) = f_0 g_0 + f_* \beta + g^* \alpha \ ,$$

where $f = p(f_0)$ and $g = p(g_0)$.

A morphism between two linear extensions

$$\begin{tikzcd} 0 \arrow[r] & D \arrow[r] \arrow[d, "\phi_1"] & \mathcal{C} \arrow[r, "p"] \arrow[d, "\phi_0"] & \mathcal{B} \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & D' \arrow[r] & \mathcal{C}' \arrow[r, "p'"] & \mathcal{B}' \arrow[r] & 0 \end{tikzcd}$$

consists of functors ϕ and ϕ_0, such that $\phi p = p' \phi_0$, together with a natural transformation $\phi_1 : D \to D' \circ (\phi^{op} \times \phi)$ such that:

$$\phi_0(f_0 + \alpha) = \phi_0(f_0) + \phi_1(\alpha)$$

for all $f_0 : c \to d$ in $\mathcal{C}$ and α in $D(c,d)$.

The following properties of linear extensions are relevant to our problem.

i. If $\mathcal{B}$ is a small category, there is [2, IV.6] a canonical bijection

$$M(\mathcal{B}, D) \cong H^2(\mathcal{B}, D).$$

from the set of equivalence classes of linear extensions of $\mathcal{B}$ by D and the second cohomology group $H^2(\mathcal{B}, D)$ of $\mathcal{B}$ with coefficients in D.

ii. The functor p reflects isomorphisms and yields a bijection on the sets of isomorphism classes $\mathrm{Iso}(\mathcal{C}) \cong \mathrm{Iso}(\mathcal{B})$.

iii. Let (ϕ_1, ϕ_0, ϕ) be a morphism of linear extensions. Suppose that $\phi_1(c,d)$ is an isomorphism for any c and d in $\mathcal{B}$. Then ϕ is an equivalence of categories if and only if ϕ_0 is an equivalence of categories.

iv. If $\mathcal{B}$ is an additive category and D is a biadditive bifunctor, then the category $\mathcal{C}$ is additive [7, Proposition 3.4].

We now describe recollements in terms of linear extensions.

Proposition 6.2 *Let D be the bifunctor defined on $\mathcal{B}$ by:*

$$D(B, B') := \mathrm{Hom}_{\mathcal{A}}(B/i_* i^! B, i_* i^! B') \ .$$

The category $\mathcal{A}$ is a linear extension of $\mathcal{B}$ by D.

Proof. It reduces to the following. Two maps of extensions:

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "f"', "g"] & X \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & U' \arrow[r] & A' \arrow[r] & X' \arrow[r] & 0 \end{tikzcd}$$

agree on the side vertical arrows if and only if their difference $f - g$ factors through a map in the group $\mathrm{Hom}(X, U')$. □

The results of Section 5 shows that the categories $\mathcal{A}'$, $\mathcal{A}''$ and the functors i^*j_*, $i^!j_!$ of the recollement situation determine the category Ker $i^!$. We now show that it does determine the bifunctor D as well. For an object B in $\mathcal{B}$, let $((X, V, \alpha), U)$ be its image under the composite:

$$\mathcal{B} \simeq \mathcal{G} \to \mathrm{Ker}\ i^! \times \mathcal{A}' \simeq \mathcal{M}(i^*j_*) \times \mathcal{A}' \ .$$

That is: $X = j^*A$, $V = i^*A$, for $A = B/i_*i^!B$, $U = i^!B$. Then:

$$D(B, B') := \mathrm{Hom}_{\mathcal{A}}(A, i_*U') = \mathrm{Hom}_{\mathcal{A}'}(i^*A, U') = \mathrm{Hom}_{\mathcal{A}'}(V, U') \ . \tag{13}$$

7 A comparison theorem

We have seen in Section 2.2 an example of a comparison functor which is not an equivalence of categories. However, a comparison functor E indeed yields an equivalence from Ker $(i^* : \mathcal{A}_1 \to \mathcal{A}')$ to Ker $(i^* : \mathcal{A}_2 \to \mathcal{A}')$, and similarly for Ker $i^!$. If E is an equivalence of categories, then clearly E commutes with the derived functors $\mathrm{R}^\bullet i^!$ and $\mathrm{L}_\bullet i^*$. This observation leads to the following definition.

Definition 7.1 *Let $(\mathcal{A}', \mathcal{A}_1, \mathcal{A}'')$ and $(\mathcal{A}', \mathcal{A}_2, \mathcal{A}'')$ be two recollement situations. Assume that the categories $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}', \mathcal{A}''$ have enough projective objects. A comparison functor E: $\mathcal{A}_1 \to \mathcal{A}_2$ is left admissible if the following diagram commutes.*

$$\begin{tikzcd} \mathcal{A}' \arrow[d, "="'] & \mathrm{Ker}\ i^! \arrow[l, "\mathrm{L}_1 i^*"'] \arrow[d, "E"] \\ \mathcal{A}' & \mathrm{Ker}\ i^! \arrow[l, "\mathrm{L}_1 i^*"] \end{tikzcd}$$

A right admissible comparison functor is defined similarly by using the functors $\mathrm{R}^1 i^!$ and the categories Ker i^*.

Theorem 7.2 *Let E be a comparison functor between categories with enough injectives and projectives. The following conditions are equivalent*

i. E is right admissible

ii. E is left admissible

iii. E is an equivalence of categories.

Proof. It is clear that iii) implies both conditions i) and ii). We only show that ii) implies iii). A dual argument shows that i) implies iii). By Section 6, the functor E yields a commutative diagram of linear extensions

$$\begin{tikzcd} 0 \arrow[r] & D_1 \arrow[r] \arrow[d] & \mathcal{A}_1 \arrow[r] \arrow[d, "E"] & \mathcal{B}_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D_2 \arrow[r] & \mathcal{A}_2 \arrow[r] & \mathcal{B}_2 \arrow[r] & 0 \end{tikzcd}$$

First we show that E yields an equivalence of categories $\mathcal{B}_1 \to \mathcal{B}_2$. By Section 6 it suffices to show that E yields an equivalence $\mathcal{G}_1 \to \mathcal{G}_2$. When there are enough projectives, E yields an equivalence on Ker $i^!$ (Theorem 5.2). The induced map

$$\mathrm{Ext}^1_{\mathcal{A}_1}(A, i_*U) \to \mathrm{Ext}^1_{\mathcal{A}_2}(E(A), i_*U)$$

is an isomorphism for U in $\mathcal{A}'$ and A in Ker $i^!$, thanks to Proposition 4.8 and the five-lemma. Once $\mathcal{B}_1$ and $\mathcal{B}_2$ are identified, we use the computation (13) to conclude that the morphism of bifunctors $D_1 \to D_2$ is an isomorphism. The rest is a consequence of the properties of linear extensions of categories. □

8 Recollement pré-héréditaire

8.1 pre-hereditary recollement

Definition 8.1 *A recollement situation with enough projectives is pre-hereditary if for any projective object V of the category $\mathcal{A}'$:*

$$(\mathrm{L}_2 i^*)(i_* V) = 0 \ .$$

Proposition 8.2 *In a pre-hereditary recollement situation:* $(\mathrm{L}_2 i^*) i_* = 0$.

Proof. By (5) the functor $(\mathrm{L}_2 i^*) i_*$ is right exact. If it vanishes on projective objects, it vanishes on all objects. □

Lemma 8.3 *In a pre-hereditary recollement situation there is an isomorphism of functors*

$$(\mathrm{L}_1 i^*) j_* \cong i^! j_!.$$

Proof. Apply the functor i^* to the short exact sequence:

$$0 \to j_{!*} \to j_* \to i_* i^* j_* \to 0 \ .$$

By (5), $\mathrm{L}_1 i^*$ vanishes on $i_* i^* j_*$, and by hypothesis $\mathrm{L}_2 i^*$ vanishes on $i_* i^* j_*$. Hence the long exact sequence for left derived functors yields an isomorphism: $(\mathrm{L}_1 i^*) j_{!*} \cong (\mathrm{L}_1 i^*) j_*$. The result follows by (7). □

Theorem 8.4 *Let $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ and $(\mathcal{A}', \mathcal{B}, \mathcal{A}'')$ be two pre-hereditary recollement situations and let $E : \mathcal{A} \to \mathcal{B}$ be a comparison functor. Then E is admissible and hence is an equivalence of categories.*

Proof. We have to prove that $\mathrm{L}_1 i^*$ has the same value on A and EA, provided that $i^! A = 0$. For such an A, there is a short exact sequence (11). Applying the functor i^* results in an exact sequence:

$$\mathrm{L}_2 i^*(\mathrm{Coker}\ \eta_A) \to \mathrm{L}_1 i^*(A) \to \mathrm{L}_1 i^*(j_* j^* A) \to \mathrm{L}_1 i^*(\mathrm{Coker}\ \eta_A)$$

whose right term cancels by Proposition 4.10 and (5), and whose left term cancels by Proposition 8.2. This gives an isomorphism: $\mathrm{L}_1 i^*(A) \cong (\mathrm{L}_1 i^*) j_* j^*(A)$. Lemma 8.3 finishes the proof. □

8.2 MacPherson-Vilonen recollements

The following proposition is a formalized version of the construction of projectives in [11, Proposition 2.5].

Proposition 8.5 *Let $\mathcal{A}(F \xrightarrow{\xi} G)$ be a MacPherson-Vilonen recollement. Assume further that the left exact functor G has a left adjoint G^*. Then the exact functor r has a left adjoint r^* defined by:*

$$r^* V = (G^* V, FG^* V \oplus V, (1, 0), \xi_{G^* V} \oplus \eta_V)$$

where in this formula η denotes the unit of adjonction: $id_{\mathcal{A}'} \to GG^*$. In particular, there is a short exact sequence:

$$0 \to j_! G^* \to r^* \to i_* \to 0\ . \tag{14}$$

Proof. Necessarily, $j^* r^* = (r j_!)^* = G^*$. Then check. □

Proposition 8.6 *Every MacPherson-Vilonen recollement with enough projectives is pre-hereditary .*

Proof. Apply the functor i^* to the short exact sequence (14). Part of the resulting long exact sequence is an exact sequence:

$$(\mathrm{L}_2 i^*) r^* \to (\mathrm{L}_2 i^*) i_* \to (\mathrm{L}_1 i^*) j_! G^*\ ,$$

whose right term cancels by (6). To conclude, if P is a projective in $\mathcal{A}'$, then $r^* P$ is a projective in $\mathcal{A}$, because r^* is left adjoint to an exact functor. □

This leads to the following characterization of MacPherson-Vilonen recollements. A special case appeared in [15, Proposition 2.6]

Theorem 8.7 *A recollement situation of categories with enough projectives is isomorphic to a MacPherson-Vilonen construction if and only if the recollement is pre-hereditary and there exists an exact functor $r\colon \mathcal{A} \to \mathcal{A}'$ such that $r \circ i_* = Id_{\mathcal{A}'}$.*

Proof. Consider a recollement with such an exact retraction functor r. The natural transformation $N\colon j_! \to j_*$ yields a transformation rN from the right exact functor $rj_!$ to the left exact functor rj_*. Thus we can form the MacPherson-Vilonen construction $\mathcal{A}(rj_! \overset{rN}{\to} rj_*)$. We define a functor $E\colon \mathcal{A} \to \mathcal{A}(rj_! \overset{rN}{\to} rj_*)$ by:

$$E(A) = (j^*(A), r(A), r(\epsilon_A), r(\eta_A)).$$

One checks with Section 3 and (3) that E is a comparison functor. By Proposition 8.6, $\mathcal{A}(rN)$ is pre-hereditary. If $\mathcal{A}$ is also pre-hereditary, Theorem 8.4 applies. □

Remark. Similarly one can define *pre-cohereditary* recollements by the condition $\mathrm{R}^2 i^!(i_* V) = 0$ for any injective V in $\mathcal{A}'$. We leave to the reader to dualize the above results.

8.3 The case when $i^* j_* = 0$ or $i^! j_! = 0$

In this section, we characterize the recollements $\mathcal{A} = \mathcal{A}' \rtimes_F \mathcal{A}''$ of Section 3.2.

Proposition 8.8 *For a recollement with enough projectives, the following are equivalent:*

i. The functor i^* is exact.

ii. $i^! j_! = 0$.

Dually, for a recollement with enough injectives, the following are equivalent:

i. The functor $i^!$ is exact.

ii. $i^* j_* = 0$.

Proof. We prove the second assertion. Assume that $i^!$ is exact. Applying $i^!$ to the epimorphism $j_* \to i_* i^* j_*$ gets an epimorphism $0 = i^! j_* \to i^! i_* i^* j_* \cong i^* j_*$. Assume conversely that $i^* j_* = 0$ and suppose that the recollement has enough injectives. We first prove that $\mathrm{R}^1 i^!(A) = 0$ when $i^! A = 0$. By Proposition 4.10, if $i^! A = 0$, there is an epimorphism $j_* j^* A \to \operatorname{Coker} \eta_A \cong i_*(\mathrm{R}^1 i^!)(A)$. Applying the right exact functor i^*, we get an epimorphism $i^* j_* j^*(A) \to (\mathrm{R}^1 i^!)(A)$. Next, we apply $i^!$ to the short exact sequence (2). It yields an exact sequence:

$$0 \to i^! \overset{\cong}{\to} i^! \to i^! \mathrm{Im}\ \eta \to (\mathrm{R}^1 i^!) i_* i^! \to \mathrm{R}^1 i^! \to (\mathrm{R}^1 i^!)\mathrm{Im}\ \eta\ .$$

By (5), $(\mathrm{R}^1 i^!) i_* i^! = 0$, so that: $i^! \mathrm{Im}\ \eta = 0$. It results that $(\mathrm{R}^1 i^!)\mathrm{Im}\ \eta = 0$ as well, and finally that $R^1 i^! = 0$. □

As an application we recover [1, Proposition 2.4].

Proposition 8.9 *Every recollement situation with enough projectives, such that: $i^! j_! = 0$, is equivalent to $\mathcal{A}' \ltimes_{i^* j_*} \mathcal{A}''$. Dually, every recollement situation with enough injectives, such that: $i^* j_* = 0$, is equivalent to $\mathcal{A}' \rtimes_{i^! j_!} \mathcal{A}''$.*

Proof. When the recollement has enough projectives, Theorem 8.7 applies for $r = i^*$. □

Corollary 8.10 *Let $\mathcal{A}', \mathcal{A}, \mathcal{A}''$ be a recollement situation with enough projective or enough injectives. If the norm $N : j_! \to j_*$ is an isomorphism, then $\mathcal{A} \cong \mathcal{A}' \times \mathcal{A}''$.*

Proof. By Proposition 4.4: $i^* j_* = i^! j_! = 0$. Then we apply Proposition 8.9. □

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