Calculation of Rozansky-Witten Invariants on the Hilbert Schemes of Points on a K3 Surface and the Generalised Kummer Varieties

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Received: September 1, 2003
Revised: December 19, 2003

Communicated by Thomas Peternell

Abstract. For any holomorphic symplectic manifold (X, σ), a closed Jacobi diagram with $2k$ trivalent vertices gives rise to a Rozansky-Witten class

$$\mathrm{RW}_{X,\sigma}(\Gamma) \in \mathrm{H}^{2k}(X, \mathcal{O}_X).$$

If X is irreducible, this defines a number $\beta_\Gamma(X, \sigma)$ by $\mathrm{RW}_{X,\sigma}(\Gamma) = \beta_\Gamma(X, \sigma)[\bar\sigma]^k$.

Let $(X^{[n]}, \sigma^{[n]})$ be the Hilbert scheme of n points on a K3 surface together with a symplectic form $\sigma^{[n]}$ such that $\int_{X^{[n]}}(\sigma^{[n]}\bar\sigma^{[n]})^n = n!$. Further, let $(A^{[[n]]}, \sigma^{[[n]]})$ be the generalised Kummer variety of dimension $2n-2$ together with a symplectic form $\sigma^{[[n]]}$ such that $\int_{A^{[[n]]}}(\sigma^{[[n]]}\bar\sigma^{[[n]]})^n = n!$. J. Sawon conjectured in his doctoral thesis that for every connected Jacobi diagram, the functions $\beta_\Gamma(X^{[n]}, \sigma^{[n]})$ and $\beta_\Gamma(A^{[[n]]}, \sigma^{[[n]]})$ are linear in n.

We prove that this conjecture is true for Γ being a connected Jacobi diagram homologous to a polynomial of closed polywheels. We further show how this enables one to calculate all Rozansky-Witten invariants of $X^{[n]}$ and $A^{[[n]]}$ for closed Jacobi diagrams that are homologous to a polynomial of closed polywheels. It seems to be unknown whether every Jacobi diagram is homologous to a polynomial of closed polywheels. If indeed the closed polywheels generate the whole graph homology space as an algebrea, our methods will thus enable us to compute *all* Rozansky-Witten invariants for the Hilbert schemes and the generalised Kummer varieties using these methods.

Also discussed in this article are the definitions of the various graph homology spaces, certain operators acting on these spaces and their relations, some general facts about holomorphic symplectic manifolds and facts about the special geometry of the Hilbert schemes of points on surfaces.

[1]The author is supported by the Deutsche Forschungsgemeinschaft. This work has been written under hospitality of the Department of Pure Mathematics and Mathematical Statistics of the University of Cambridge.

2000 Mathematics Subject Classification: 53C26, 14Q15, 57M15, 05C99

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1. Introduction

A compact *hyperkähler manifold* (X, g) is a compact Riemannian manifold whose holonomy is contained in $\mathrm{Sp}(n)$. An example of such a manifold is the K3 surface together with a Ricci-flat Kähler metric (which exists by S. Yau's theorem [18]). In [15], L. Rozansky and E. Witten described how one can associate to every vertex-oriented trivalent graph Γ an invariant $b_\Gamma(X)$ to X, henceforth called a *Rozansky-Witten invariant of X associated to* Γ. In fact, this invariant only depends on the homology class of the graph, so the invariants are already defined on the level of the graph homology space $\mathcal{B}$ (see e.g. [1] and this paper for more information about graph homology).

Every hyperkähler manifold (X, g) can be given the structure of a Kähler manifold X (which is, however, not uniquely defined) whose Kähler metric is just given by g. X happens to carry a holomorphic symplectic two-form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$, whereas we shall call X a *holomorphic symplectic manifold.* Now M. Kapranov showed in [8] that one can in fact calculate $b_\Gamma(X)$ from (X, σ) by purely holomorphic methods.
The basic idea is the following: We can identify the holomorphic tangent bundle $\mathcal{T}_X$ of X with its cotangent bundle Ω_X by means of σ. Doing this, the Atiyah class α_X (see [8]) of X lies in $\mathrm{H}^1(X, \mathrm{S}_3\mathcal{T}_X)$. Now we place a copy of α_X at each trivalent vertex of the graph, take the $\cup$-product of all these copies (which gives us an element in $\mathrm{H}^{2k}(X, (\mathrm{S}_3\mathcal{T}_X)^{\otimes 2k})$ if $2k$ is the number of trivalent vertices), and finally contract $(\mathrm{S}_3\mathcal{T}_X)^{\otimes 2k})$ along the edges of the graph by means of the holomorphic symplectic form σ. Let us call the resulting element $\mathrm{RW}_{X,\sigma}(\Gamma) \in \mathrm{H}^{2k}(X, \mathcal{O}_X)$. In case $2k$ is the complex dimension of X, we can integrate this element over X after we have multiplied it with $[\sigma]^{2k}$. This gives us more or less $b_\Gamma(X)$. The orientation at the vertices of the graph is needed in the process to get a number which is not only defined up to sign.
There are two main example series of holomorphic symplectic manifolds, the Hilbert schemes $X^{[n]}$ of points on a K3 surface X and the generalised Kummer varieties $A^{[[n]]}$ (see [2]). Besides two further manifolds constructed by K. O'Grady in [13] and [12], these are the only known examples of *irreducible* holomorphic symplectic manifolds up to deformation.
Not much work was done on actual calculations of these invariants on the example series. The first extensive calculations were carried out by J. Sawon in his doctoral thesis [16]. All Chern numbers are in fact Rozansky-Witten invariants associated to certain Jacobi diagrams, called *closed polywheels.* Let $\mathcal{W}$ be the subspace spanned by these polywheels in $\mathcal{B}$. All Rozansky-Witten invariants associated to graphs lying in $\mathcal{W}$ can thus be calculated from the knowledge of the Chern numbers (which are computable in the case of $X^{[n]}$ ([3]) or $A^{[[n]]}$ ([11]). However, from complex dimension four on, there are graph homology classes that do not lie in $\mathcal{W}$. J. Sawon showed that for some of these graphs the Rozansky-Witten invariants can still be calculated from knowledge of the Chern numbers, which enables one to calculate all Rozansky-Witten invariants up to dimension five. His calculations would work for all irreducible holomorphic manifolds whose Chern numbers are known.
In this article, we will make use of the special geometry of $X^{[n]}$ and $A^{[[n]]}$. Doing this, we are able to give a method which enables us to calculate all Rozansky-Witten invariants for graphs homology classes that lie in the *algebra* $\mathcal{C}$ generated by *closed polywheels* in $\mathcal{B}$. The closed polywheels form the subspace $\mathcal{W}$ of the algebra $\mathcal{B}$ of graph homology. This is really a proper subspace. However, $\mathcal{C}$, the algebra generated by this subspace, is much larger, and, as far as the author knows, it is unknown whether $\mathcal{C} = \mathcal{B}$, i.e. whether this work enables us to calculate *all* Rozanky-Witten invariants for the main example series.
The idea to carry out this computations is the following: Let (Y, τ) be any irreducible holomorphic symplectic manifold. Then $\mathrm{H}^{2k}(Y, \mathcal{O}_Y)$ is spanned by

$[\bar{\tau}]^k$. Therefore, every graph Γ with $2k$ trivalent vertices defines a number $\beta_\Gamma(Y,\tau)$ by $\mathrm{RW}_{Y,\tau}(\Gamma) = \beta_\Gamma(Y,\tau)[\bar{\tau}]^k$. J. Sawon has already discussed how knowledge of these numbers for connected graphs is enough to deduce the values of all Rozansky-Witten invariants.

For the example series, let us fix holomorphic symplectic forms $\sigma^{[n]}$, respective $\sigma^{[[n]]}$ with $\int_{X^{[n]}}(\sigma^{[n]}\bar{\sigma}^{[n]})^n = n!$ respective $\int_{A^{[[n]]}}(\sigma^{[[n]]}\bar{\sigma}^{[[n]]})^n = n!$. J. Sawon conjectured the following:

> The functions $\beta_\Gamma(X^{[n]},\sigma^{[n]})$ and $\beta_\Gamma(X^{[[n]]},\sigma^{[[n]]})$ are linear in n for Γ being a connected graph.

The main result of this work is the proof of this conjecture for the class of connected graphs lying in $\mathcal{C}$ (see Theorem 3). We further show how one can calculate these linear functions from the knowledge of the Chern numbers and thus how to calculate all Rozansky-Witten invariants for graphs in $\mathcal{C}$.

We should note that we don't make any use of the IHX relation in our derivations, and so we could equally have worked on the level of Jacobi diagrams.

Let us finally give a short description of each section. In section 2 we collect some definitions and results which will be used later on. The next section is concerned with defining the algebra of graph homology and certain operations on this space. We define *connected polywheels* and show how they are related with the usual closed polywheels in graph homology. We further exhibit a natural $\mathfrak{sl}_2$-action on an extended graph homology space. In section 4, we first look at general holomorphic symplectic manifolds. Then we study the two example series more deeply. Section 5 defines Rozansky-Witten invariants while the last section is dedicated to the proof of our main theorem and explicit calculations.

2. Preliminaries

2.1. Some multilinear algebra. Let $\mathcal{T}$ be a tensor category (commutative and with unit). For any object V in $\mathcal{T}$, we denote by $\mathrm{S}^k V$ the coinvariants of $V^{\otimes k}$ with respect to the natural action of the symmetric group and by $\Lambda^k V$ the coinvariants with respect to the alternating action. Further, let us denote by $\mathrm{S}_k V$ and $\Lambda_k V$ the invariants of both actions.

Proposition 1. *Let I be a cyclicly ordered set of three elements. Let V be an object in $\mathcal{T}$. Then there exists a unique map $\Lambda_3 V \to V^{\otimes I}$ such that for every bijection $\phi : \{1,2,3\} \to I$ respecting the canonical cyclic ordering of $\{1,2,3\}$ and the given cyclic ordering of I the following diagram*

$$\begin{tikzcd} \Lambda_3 V \arrow[r, equal] \arrow[d] & \Lambda_3 V \arrow[d] \\ V^{\otimes 3} \arrow[r, "\phi_*"'] & V^{\otimes I} \end{tikzcd} \tag{1}$$

commutes, where the map ϕ_* is the canonical one induced by ϕ.

Proof. Let $\phi, \phi' : \{1, 2, 3\} \to I$ be two bijections respecting the cyclic ordering. Then there exist an even permutation $\alpha \in \mathfrak{A}_3$ such that the lower square of the following diagram commutes:

$$\begin{array}{ccc} \Lambda_3 V & = & \Lambda_3 V \\ \downarrow & & \downarrow \\ V^{\otimes 3} & \xrightarrow[\alpha_*]{} & V^{\otimes 3} \\ \phi\downarrow & & \downarrow\phi' \\ V^{\otimes I} & = & V^{\otimes I}. \end{array}$$

We have to show that the outer rectangle commutes. For this it suffices to show that the upper square commutes. In fact, since α is an even permutation, every element of $\Lambda_3 V$ is by definition invariant under α_*. $\square$

2.2. PARTITIONS. A partition λ of a non-negative integer $n \in \mathbb{N}_0$ is a sequence $\lambda_1, \lambda_2, \dots$ of non-negative integers such that

$$\|\lambda\| := \sum_{i=1}^{\infty} i\lambda_i = n. \tag{2}$$

Therefore almost all λ_i have to vanish. In the literature, λ is often notated by $1^{\lambda_1} 2^{\lambda_2} \dots$. The set of all partitions of n is denoted by $\mathrm{P}(n)$. The union of all $\mathrm{P}(n)$ is denoted by $\mathrm{P} := \bigcup_{n=0}^{\infty} \mathrm{P}(n)$. For every partition $\lambda \in \mathrm{P}$, we set

$$|\lambda| := \sum_{i=1}^{\infty} \lambda_i \tag{3}$$

and

$$\lambda! := \prod_{i=1}^{\infty} \lambda_i!. \tag{4}$$

Let $a_1, a_2, \dots$ be any sequence of elements of a commutative unitary ring. We set

$$a_\lambda := \prod_{i=1}^{\infty} a_i^{\lambda_i} \tag{5}$$

for any partition $\lambda \in \mathrm{P}$.

With these definitions, we can formulate the following proposition in a nice way:

PROPOSITION 2. *In $\mathbb{Q}[[a_1, a_2, \dots]]$ we have*

$$\exp\left(\sum_{i=1}^{\infty} a_i\right) = \sum_{\lambda \in \mathrm{P}} \frac{a_\lambda}{\lambda!}. \tag{6}$$

Proof. We calculate

$$\exp\left(\sum_{i=1}^\infty a_i\right) = \sum_{n=0}^\infty \frac{1}{n!}\left(\sum_{i=1}^\infty a_i\right)^n = \sum_{n=0}^\infty \frac{1}{n!}\sum_{\lambda\in\mathrm{P},|\lambda|=n} n!\prod_{i=1}^\infty \frac{a_i^{\lambda_i}}{\lambda_i!} = \sum_{\lambda\in\mathrm{P}}\frac{a_\lambda}{\lambda!}. \tag{7}$$

□

If we set

$$\frac{\partial}{\partial a_\lambda} := \prod_{i=1}^\infty \left.\frac{\partial^{\lambda_i}}{\partial a_i^{\lambda_i}}\right|_{a_i=0}, \tag{8}$$

we have due to Proposition 2:

Proposition 3. *In* $\mathbb{Q}[[s_1,s_2,\dots]][a_1,a_2,\dots]$ *we have*

$$\frac{\partial}{\partial a_\lambda}\exp\left(\sum_{i=1}^\infty a_i s_i\right) = s_\lambda. \tag{9}$$

2.3. A lemma from umbral calculus.

Lemma 1. *Let R be any $\mathbb{Q}$-algebra (commutative and with unit) and $A(t)\in R[[t]]$ and $B(t)\in tR[[t]]$ be two power series. Let the polynomial sequences $(p_n(x))$ and $(s_n(x))$ be defined by*

$$\sum_{k=0}^\infty p_k(x)\frac{t^k}{k!} = \exp(xB(t)) \tag{10}$$

and

$$\sum_{k=0}^\infty s_k(x)\frac{t^k}{k!} = A(t)\exp(xB(t)). \tag{11}$$

Let $W_B(t)\in tR[[t]]$ be defined by $W_B(t\exp(B(t)))=t$. Then we have

$$\sum_{k=0}^\infty \frac{xp_k(x-k)}{(x-k)}\frac{t^k}{k!} = \exp(xB(W_B(t))) \tag{12}$$

and

$$\sum_{k=0}^\infty \frac{s_k(x-k)}{k!}t^k = \frac{A(W_B(t))}{1+W_B(t)B'(W_B(t))}\exp(xB(W_B(t))). \tag{13}$$

Proof. It suffices to prove the result for the field $R=\mathbb{Q}(a_0,a_1,\dots,b_1,b_2,\dots)$ and $A(t)=\sum_{k=0}^\infty a_k t^k$ and $B(t)=\sum_{k=1}^\infty b_k t^k$.
So let us assume this special case for the rest of the proof. Let us denote by $f(t)$ the compositional inverse of $B(t)$, i.e. $f(B(t))=t$. We set $g(t):=A^{-1}(f(t))$. For the following we will make use of the terminology and the statements in [14]. Using this terminology, (10) states that $(p_n(x))$ is the associated sequence to $f(t)$ and (11) states that $(s_n(x))$ is the Sheffer sequence to the pair $(g(t),f(t))$ (see Theorem 2.3.4 in [14]).

Theorem 3.8.3 in [14] tells us that $(s_n(x-n))$ is the Sheffer sequence to the pair $(\tilde{g}(t), \tilde{f}(t))$ with

$$\tilde{g}(t) = g(t)(1 + f(t)/f'(t))$$

and

$$\tilde{f}(t) = f(t)\exp(t).$$

The compositional inverse of $\tilde{f}(t)$ is given by $\tilde{B}(t) := B(W_B(t))$:

$$B(W_B(\tilde{f}(t))) = B(W_B(f(t)\exp(t))) = B(W_B(f(t)\exp(B(f(t))))) = B(f(t)) = t.$$

Further, we have

$$\begin{aligned}\tilde{A}(t) &:= \tilde{g}^{-1}(\tilde{B}(t))\\ &= (g(B(t))(1 + f(B(t))/f'(B(t))))^{-1} \circ W_B(t) = \frac{A(t)}{1 + tB'(t)} \circ W_B(t),\end{aligned}$$

which proves (13) again due to Theorem 2.3.4 in [14].
It remains to prove (12), i.e. that $(\frac{xp_n(x)}{x-n})$ is the associated sequence to $\tilde{f}(t)$. We already know that $(p_n(x-n))$ is the Sheffer sequence to the pair $(1 + f(t)/f'(t), \tilde{f}(t))$. By Theorem 2.3.6 of [14] it follows that the associated sequence to $\tilde{f}(t)$ is given by $(1+f(\mathrm{d}/\mathrm{d}x)/f'(\mathrm{d}/\mathrm{d}x))p_n(x-n)$. By Theorem 2.3.7 and Corollary 3.6.6 in [14], we have

$$\begin{aligned}\left(1 + \frac{f(\mathrm{d}/\mathrm{d}x)}{f'(\mathrm{d}/\mathrm{d}x)}\right) p_n(x-n) &= p_n(x-n) + \frac{1}{f'(\mathrm{d}/\mathrm{d}x)} n p_{n-1}(x-n)\\ &= p_n(x-n) + \frac{np_n(x-n)}{x-n} = \frac{xp_n(x-n)}{x-n},\end{aligned}$$

which proves the rest of the lemma. □

3. Graph homology

This section is concerned with the space of graph homology classes of unitrivalent graphs. A very detailed discussion of this space and other graph homology spaces can be found in [1]. Further aspects of graph homology can be found in [17], and, with respect to Rozansky-Witten invariant, in [7].

3.1. The graph homology space. In this article, *graph* means a collection of vertices connected by edges, i.e. every edge connects two vertices. We want to call a half-edge (i.e. an edge together with an adjacent vertex) of a graph a *flag*. So, every edge consists of exactly two flags. Every flag belongs to exactly one vertex of the graph. On the other hand, a vertex is given by the set of its flags. It is called *univalent* if there is only one flag belonging to it, and it is called *trivalent* if there are exactly three flags belonging to it. We shall identify edges and vertices with the set of their flags. We shall also call univalent vertices *legs*. A graph is called *vertex-oriented* if, for every vertex, a cyclic ordering of its flags is fixed.

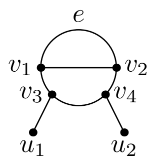

FIGURE 1. This Jacobi diagram has four trivalent vertices $v_1, \dots, v_4$, and two univalent vertices u_1 and u_2, and e is one of its 7 edges.

DEFINITION 1. A *Jacobi diagram* is a vertex-oriented graph with only uni- and trivalent vertices. A *connected Jacobi diagram* is a Jacobi diagram which is connected as a graph. A *trivalent Jacobi diagram* is a Jacobi diagram with no univalent vertices.
We define the *degree of a Jacobi diagram* to be the number of its vertices. It is always an even number.
We identify two graphs if they are isomorphic as vertex-oriented graphs in the obvious sense.

Example 1. The empty graph is a Jacobi diagram, denoted by 1. The unique Jacobi diagram consisting of two univalent vertices (which are connected by an edge) is denoted by ℓ.

FIGURE 2. The Jacobi diagram ℓ with its two univalent vertices u_1 and u_2.

Remark 1. There are different names in the literature for what we call a "Jacobi diagram", e.g. unitrivalent graphs, chord diagrams, Chinese characters, Feynman diagrams. The name chosen here is also used by D. Thurston in [17]. The name comes from the fact that the IHX relation in graph homology defined later is essentially the well-known Jacobi identity for Lie algebras.
With our definition of the degree of a Jacobi diagram, the algebra of graph homology defined later will be commutative in the graded sense. Further, the map RW that will associate to each Jacobi diagram a Rozansky-Witten class will respect this grading. But note that often the degree is defined to be *half* of the number of vertices, which still is an integer.

We can always draw a Jacobi diagram in a planar drawing so that it looks like a planar graph with vertices of valence 1, 3 or 4. Each 4-valent vertex has to be interpreted as a crossing of two non-connected edges of the drawn graph and not as one of its vertices. Further, we want the counter-clockwise ordering of the flags at each trivalent vertex in the drawing to be the same as the given cyclic ordering.

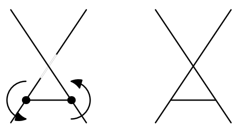

FIGURE 3. These two graphs depict the same one.

In drawn Jacobi diagrams, we also use a notation like $\cdots \overset{n}{-} \cdots$ for a part of a graph which looks like a long line with n univalent vertices ("legs") attached to it, for example $\ldots \perp\!\perp\!\perp \ldots$ for $n = 3$. The position of n indicates the placement of the legs relative to the "long line".

DEFINITION 2. Let $\mathcal{T}$ be any tensor category (commutative and with unit). Every Jacobi diagram Γ with k trivalent and l univalent vertices induces a natural transformation Ψ^Γ between the functors

$$\mathcal{T} \to \mathcal{T}, V \mapsto \mathrm{S}_k \Lambda_3 V \otimes \mathrm{S}_l V \tag{14}$$

and

$$\mathcal{T} \to \mathcal{T}, V \mapsto \mathrm{S}^e \mathrm{S}^2 V, \tag{15}$$

where $e := \frac{3k+l}{2}$ which is given by

$$\begin{aligned}\Psi^\Gamma : \mathrm{S}_k\Lambda_3 V \otimes \mathrm{S}_l V &\overset{(1)}{\to} \bigotimes_{t\in T} \Lambda_3 V \otimes \bigotimes_{f \in U} V \overset{(2)}{\to} \bigotimes_{t\in T} \bigotimes_{f\in t} V \otimes \bigotimes_{f\in U} V \\ &\overset{(3)}{\to} \bigotimes_{f\in F} V \overset{(4)}{\to} \bigotimes_{e\in E}\bigotimes_{f\in e} V \overset{(5)}{\to} \mathrm{S}^e\mathrm{S}^2 V,\end{aligned} \tag{16}$$

where T is the set of the trivalent vertices, U the set of the univalent vertices, F the set of flags, and E the set of edges of Γ. Further,

(1) is given by the natural inclusions of the invariants in the tensor products,
(2) is given by the canonical maps (see Proposition 1 and recall that the sets t are cyclicly ordered),
(3) is given by the associativity of the tensor product,
(4) is given again by the associativity of the tensor product, and finally
(5) is given by the canonical projections onto the coinvariants.

DEFINITION 3. We define $\mathcal{B}$ to be the $\mathbb{Q}$-vector space spanned by all Jacobi diagrams modulo the IHX relation

$$\mathrm{I} = \mathrm{H} - \mathrm{X} \tag{17}$$

and the anti-symmetry (AS) relation

$$\curlyvee + \curlyvee = 0, \tag{18}$$

which can be applied anywhere within a diagram. (For this definition see also [1] and [17].) Two Jacobi diagrams are said to be *homologous* if they are in the same class modulo the IHX and AS relation.
Furthermore, let $\mathcal{B}'$ be the subspace of $\mathcal{B}$ spanned by all Jacobi diagrams not containing ℓ as a component, and let ${}^{\mathrm{t}}\mathcal{B}$ be the subspace of $\mathcal{B}'$ spanned by all trivalent Jacobi diagrams. All these are graded and double-graded. The grading is induced by the degree of Jacobi diagrams, the double-grading by the number of univalent and trivalent vertices.
The completion of $\mathcal{B}$ (resp. $\mathcal{B}'$, resp. ${}^{\mathrm{t}}\mathcal{B}$) with respect to the grading will be denoted by $\hat{\mathcal{B}}$ (resp. $\hat{\mathcal{B}}'$, resp. ${}^{\mathrm{t}}\hat{\mathcal{B}}$).
We define $\mathcal{B}_{k,l}$ to be the subspace of $\hat{\mathcal{B}}$ generated by graphs with k trivalent and l univalent vertices. $\mathcal{B}'_{k,l}$ and ${}^{\mathrm{t}}\mathcal{B}_k := {}^{\mathrm{t}}\mathcal{B}_{k,0}$ are defined similarly.

All these spaces are called *graph homology spaces* and their elements are called *graph homology classes* or *graphs* for short.

Remark 2. The subspaces $\mathcal{B}_k$ of $\hat{\mathcal{B}}$ spanned by the Jacobi diagrams of degree k are always of finite dimension. The subspace $\mathcal{B}_0$ is one-dimensional and spanned by the graph homology class 1 of the empty diagram 1.

Remark 3. We have $\hat{\mathcal{B}} = \prod_{k,l\geq 0} B_{k,l}$. In view of the following Definition 4, $\hat{\mathcal{B}}'$ and $\hat{\mathcal{B}}$ are naturally $\mathbb{Q}$-algebras. As $\mathbb{Q}$-algebras, we have $\hat{\mathcal{B}} = \hat{\mathcal{B}}'[[\ell]]$. Due to the AS relation, the spaces $\mathcal{B}'_{k,l}$ are zero for $l > k$. Therefore, $\hat{\mathcal{B}}' = \prod_{k=0}^{\infty} \bigoplus_{l=0}^{k} \mathcal{B}'_{k,l}$.

Example 2. If γ is a graph which has a part looking like $\cdots \overset{n}{-} \cdots$, it will become $(-1)^n\gamma$ if we substitute the part $\cdots \overset{n}{-} \cdots$ by $\cdots \underset{n}{-} \cdots$ due to the anti-symmetry relation.

3.2. OPERATIONS WITH GRAPHS.

DEFINITION 4. Disjoint union of Jacobi diagrams induces a bilinear map

$$\hat{\mathcal{B}} \times \hat{\mathcal{B}} \to \hat{\mathcal{B}}, (\gamma, \gamma') \mapsto \gamma \cup \gamma'. \tag{19}$$

By mapping $1 \in \mathbb{Q}$ to $1 \in \hat{\mathcal{B}}$, the space $\hat{\mathcal{B}}$ becomes a graded $\mathbb{Q}$-algebra, which has no components in odd degrees. Often, we omit the product sign "$\cup$". $\mathcal{B}$, $\mathcal{B}'$, ${}^{\mathrm{t}}\mathcal{B}$, and so on are subalgebras.

DEFINITION 5. Let $k \in \mathbb{N}$. We call the graph homology class of the Jacobi diagram $\overset{2k}{\bigcirc}$ the $2k$-*wheel* w_{2k}, i.e. $w_2 = \multimap$, $w_4 = \boxtimes$, and so on. It has $2k$ univalent and $2k$ trivalent vertices. The expression w_0 will be given a meaning later, see section 3.3.

Remark 4. The wheels w_k with k odd vanish in $\hat{\mathcal{B}}$ due to the AS relation.

Let Γ be a Jacobi diagram and u, u' be two different univalent vertices of Γ. These two should not be the two vertices of a component ℓ of Γ. Let v (resp. v') be the vertex u (resp. u') is attached to. The process of *gluing the vertices u and u'* means to remove u and u' together with the edges connecting them to v resp. v' and to add a new edge between v and v'. Thus, we arrive at a new graph $\Gamma/(u, u')$, whose number of trivalent vertices is the number of trivalent vertices of Γ and whose number of univalent vertices is the number of univalent vertices of Γ minus two. To make it a Jacobi diagram we define the cyclic orientation of the flags at v (resp. v') to be the cyclic orientation of the flags at v (resp. v') in Γ with the flag belonging to the edge connecting v (resp. v') with u (resp. u') replaced by the flag belonging to the added edge. For example,

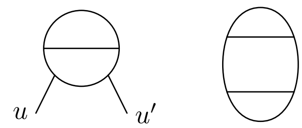

FIGURE 4. Gluing the two univalent vertices u and u' of the left graph produces the right one, denoted by $\ominus_2$.

gluing the two univalent vertices of w_2 leads to the graph $\ominus$.
If $\pi = \{\{u_1, u'_1\}, \ldots, \{u_k, u'_k\}\}$ is a set of two-element sets of legs that are pairwise disjoint and such that each pair u_k, u'_k fulfills the assumptions of the previous construction, we set

$$\Gamma/\pi := \Gamma/(u_1, u'_1)/\ldots/(u_k, u'_k). \tag{20}$$

Of course, the process of gluing two univalent vertices given above does not work if u and u' are the two univalent vertices of ℓ, thus our assumption on Γ.

DEFINITION 6. Let Γ, Γ' be two Jacobi diagrams, at least one of them without ℓ as a component and $U = \{u_1, \ldots, u_n\}$ resp. U' the sets of their univalent vertices. We define

$$\hat{\Gamma}(\Gamma') := \sum_{\substack{f: U \hookrightarrow U' \\ \text{injective}}} (\Gamma \cup \Gamma')/(u_1, f(u_1))/\ldots/(u_n, f(u_n)), \tag{21}$$

viewed as an element in $\hat{\mathcal{B}}$.
This induces for every $\gamma \in \hat{\mathcal{B}}$ a ${}^{\mathrm{t}}\hat{\mathcal{B}}$-linear map

$$\hat{\gamma} : \hat{\mathcal{B}}' \to \hat{\mathcal{B}}', \gamma' \mapsto \hat{\gamma}(\gamma'). \tag{22}$$

Example 3. Set $\partial := \frac{1}{2}\hat{\ell}$. It is is an endomorphism of $\hat{\mathcal{B}}'$ of degree -2. For example, ∂ $= \ominus$. By setting

$$\partial(\gamma, \gamma') := \partial(\gamma \cup \gamma') - \partial(\gamma) \cup \gamma' - \gamma \cup \partial(\gamma') \tag{23}$$

for $\gamma, \gamma' \in \hat{\mathcal{B}}'$, we have the following formula for all $\gamma \in \hat{\mathcal{B}}'$:

$$\partial(\gamma^n) = \binom{n}{1}\partial(\gamma)\gamma^{n-1} + \binom{n}{2}\partial(\gamma,\gamma)\gamma^{n-2}. \tag{24}$$

This shows that ∂ is a differential operator of order two acting on $\hat{\mathcal{B}}'$.
Acting by ∂ on a Jacobi diagram means to glue two of its univalent vertices in all possible ways, acting by $\partial(\cdot,\cdot)$ on two Jacobi diagrams means to connect them by gluing a univalent vertex of the first with a univalent vertex of the second in all possible ways.

Definition 7. Let Γ, Γ' be two Jacobi diagrams, at least one of them without ℓ as a component, and $U = \{u_1, \ldots, u_n\}$ resp. U' the sets of their univalent vertices. We define

$$\langle \Gamma, \Gamma' \rangle := \sum_{\substack{f: U \to U' \\ \text{bijective}}} (\Gamma \cup \Gamma')/(u_1, f(u_1))/\ldots/(u_n, f(u_n)), \tag{25}$$

viewed as an element in ${}^{\mathrm{t}}\hat{\mathcal{B}}$.
This induces a ${}^{\mathrm{t}}\hat{\mathcal{B}}$-bilinear map

$$\langle \cdot, \cdot \rangle : \hat{\mathcal{B}}' \times \hat{\mathcal{B}} \to {}^{\mathrm{t}}\hat{\mathcal{B}}, \tag{26}$$

which is symmetric on $\hat{\mathcal{B}}' \times \hat{\mathcal{B}}'$.

Note that $\langle \Gamma, \Gamma' \rangle$ is zero unless Γ and Γ' have equal numbers of univalent vertices. In this case, the expression is the sum over all possibilities to glue the univalent vertices of Γ with univalent vertices of Γ'.
Note that $\langle \Gamma, \Gamma' \rangle$ is zero unless Γ and Γ' have equal numbers of univalent vertices. In this case, the expression is the sum over all possibilities to glue the univalent vertices of Γ with univalent vertices of Γ'.

Proposition 4. *The map $\langle 1, \cdot \rangle : \hat{\mathcal{B}} \to {}^{\mathrm{t}}\hat{\mathcal{B}}$ is the canonical projection map, i.e. it removes all non-trivalent components from a graph. Furthermore, for $\gamma \in \hat{\mathcal{B}}'$ and $\gamma' \in \hat{\mathcal{B}}$, we have*

$$\left\langle \gamma, \frac{\ell}{2}\gamma' \right\rangle = \langle \partial\gamma, \gamma' \rangle. \tag{27}$$

For $\gamma, \gamma' \in \hat{\mathcal{B}}'$, we have the following (combinatorial) formula:

$$\langle \exp(\partial)(\gamma\gamma'), 1 \rangle = \langle \exp(\partial)\gamma, \exp(\partial)\gamma' \rangle. \tag{28}$$

Proof. The formula (27) should be clear from the definitions.
Let us investigate (28) a bit more. We can assume that γ and γ' are Jacobi diagrams with l resp. l' univalent vertices and $l + l' = 2n$ with $n \in \mathbb{N}_0$. So we have to prove

$$\frac{\partial^n}{n!}(\gamma\gamma') = \sum_{\substack{m,m'=0 \\ l-2m=l'-2m'}}^{\infty} \left\langle \frac{\partial^m}{m!}\gamma, \frac{\partial^{m'}}{m'!}\gamma' \right\rangle,$$

since $\langle\cdot,1\rangle:\hat{\mathcal{B}}\to {}^{\mathrm{t}}\hat{\mathcal{B}}$ means to remove the components with at least one univalent vertex. Recalling the meaning of $\langle\cdot,\cdot\rangle$, it should be clear that (28) follows from the fact that applying $\frac{\partial^k}{k!}$ on a Jacobi diagram means to glue all subsets of $2k$ of its univalent vertices to k pairs in all possible ways. $\square$

3.3. An $\mathfrak{sl}_2$-action on the space of graph homology. In this short section we want to extend the space of graph homology slightly. This is mainly due to two reasons: When we defined the expression $\hat{\Gamma}(\Gamma)$ for two Jacobi diagrams Γ and Γ', we restricted ourselves to the case that Γ or Γ' does not contain a component with an ℓ. Secondly, we have not given the *zero-wheel* w_0 a meaning yet.

We do this by adding an element $\bigcirc$ to the various spaces of graph homology.

Definition 8. *The extended space of graph homology* is the space $\hat{\mathcal{B}}[[\bigcirc]]$. Further, we set $w_0 := \bigcirc$, which, at least pictorially, is in accordance with the definition of w_k for $k>0$.

Note that this element is not depicting a Jacobi diagram as we have defined it. Nevertheless, we want to use the notion that $\bigcirc$ has no univalent and no trivalent vertices, i.e. the homogeneous component of degree zero of $\hat{\mathcal{B}}[[\bigcirc]]$ is $\mathbb{Q}[[\bigcirc]]$.

When defining $\Gamma/(u,u')$ for a Jacobi diagram Γ with two univalent vertices u and u', i.e. gluing u to u', we assumed that u and u' are not the vertices of one component ℓ of Γ. Now we extend this definition by defining $\Gamma/(u,u')$ to be the extended graph homology class we get by replacing ℓ with $\bigcirc$, whenever u and u' are the two univalent vertices of a component ℓ of Γ.

Doing so, we can give the expression $\hat{\gamma}(\gamma')\in\hat{\mathcal{B}}[[\bigcirc]]$ a meaning with no restrictions on the two graph homology classes $\gamma,\gamma'\in\hat{\mathcal{B}}$, i.e. every $\gamma\in\hat{\mathcal{B}}[[\bigcirc]]$ defines a ${}^{\mathrm{t}}\hat{\mathcal{B}}[[\bigcirc]]$-linear map

$$\hat{\gamma}:\hat{\mathcal{B}}[[\bigcirc]]\to\hat{\mathcal{B}}[[\bigcirc]]. \tag{29}$$

Example 4. We have

$$\partial\ell=\bigcirc. \tag{30}$$

Remark 5. We can similarly extend $\langle\cdot,\cdot\rangle:\hat{\mathcal{B}}'\times\hat{\mathcal{B}}\to{}^{\mathrm{t}}\hat{\mathcal{B}}$ to a ${}^{\mathrm{t}}\hat{\mathcal{B}}[[\bigcirc]]$-bilinear form

$$\langle\cdot,\cdot\rangle:\hat{\mathcal{B}}[[\bigcirc]]\times\hat{\mathcal{B}}[[\bigcirc]]\to{}^{\mathrm{t}}\hat{\mathcal{B}}[[\bigcirc]]. \tag{31}$$

Both $\ell/2$ and ∂ are two operators acting on the extended space of graph homology, the first one just multiplication with $\ell/2$. By calculating their commutator, we show that they induce a natural structure of an $\mathfrak{sl}_2$-module on $\hat{\mathcal{B}}[[\bigcirc]]$.

Proposition 5. *Let $H:\hat{\mathcal{B}}[[\bigcirc]]\to\hat{\mathcal{B}}[[\bigcirc]]$ be the linear operator which acts on $\gamma\in\hat{\mathcal{B}}_{k,l}[[\bigcirc]]$ by*

$$H\gamma=\left(\frac{1}{2}\bigcirc+l\right)\gamma. \tag{32}$$

We have the following commutator relations in $\operatorname{End}\hat{\mathcal{B}}[[\bigcirc]]$*:*

$$[\ell/2, \partial] = -H, \tag{33}$$

$$[H, \ell/2] = 2 \cdot \ell/2, \tag{34}$$

and

$$[H, \partial] = -2\partial, \tag{35}$$

i.e. the triple $(\ell/2, -\partial, H)$ *defines a* $\mathfrak{sl}_2$*-operation on* $\hat{\mathcal{B}}[[\bigcirc]]$.

Proof. Equations (34) and (35) follow from the fact that multiplying by $\bigcirc$ commutes with $\ell/2$ and ∂, and from the fact that $\ell/2$ is an operator of degree 2 with respect to the grading given by the number of univalent vertices, whereas ∂ is an operator of degree -2 with respect to the same grading.
It remains to look at (33). For $\gamma \in \hat{\mathcal{B}}_{k,l}[[\bigcirc]]$, we calculate

$$[\ell, \partial]\gamma = \ell\partial(\gamma) - \partial(\ell\gamma) = \ell\partial(\gamma) - \partial(\ell)\gamma - \ell\partial(\gamma) - \partial(\ell, \gamma) = -\bigcirc\gamma - 2l\gamma = -2H\gamma. \tag{36}$$

□

Remark 6. Since $\hat{\mathcal{B}}[[\bigcirc]]$ is infinite-dimensional, we have unfortunately difficulties to apply the standard theory of $\mathfrak{sl}_2$-representations to this $\mathfrak{sl}_2$-module. For example, there are no eigenvectors for the operator H.

3.4. Closed and connected graphs, the closure of a graph. As the number of connected components of a Jacobi diagram is preserved by the IHX- and AS-relations each graph homology space inherits a grading by the number of connected components. For any $k \in \mathbb{N}_0$ we define $\mathcal{B}^k$ to be the subspace of $\mathcal{B}$ spanned by all Jacobi diagrams with exactly k connected components. Similarly, we define ${}^{\mathrm{t}}\mathcal{B}^k$, $\hat{\mathcal{B}}^k$, ${}^{\mathrm{t}}\hat{\mathcal{B}}^k$.
We have $\mathcal{B} = \bigoplus_{k=0}^{\infty} \mathcal{B}^k$ with $\mathcal{B}^0 = \mathbb{Q} \cdot 1$. Analogous results hold for ${}^{\mathrm{t}}\mathcal{B}$, $\hat{\mathcal{B}}$, ${}^{\mathrm{t}}\hat{\mathcal{B}}$.

Definition 9. A graph homology class γ is called *closed* if $\gamma \in {}^{\mathrm{t}}\hat{\mathcal{B}}$. The class γ is called *connected* if $\gamma \in \hat{\mathcal{B}}^1$. The *connected component of* γ is defined to be $\mathrm{pr}^1(\gamma)$ where $\mathrm{pr}^1 : \hat{\mathcal{B}} = \prod_{i=0}^{\infty} \hat{\mathcal{B}}^i \to \hat{\mathcal{B}}^1$ is the canonical projection. The *closure* $\langle\gamma\rangle$ *of* γ is defined by $\langle\gamma\rangle := \langle\gamma, \exp(\ell/2)\rangle$. The *connected closure* $\langle\langle\gamma\rangle\rangle$ *of* γ is defined to be the connected component of the closure $\langle\gamma\rangle$ of γ.

For every finite set L, we define $\mathrm{P}_2(L)$ to be the set of partitions of L into subsets of two elements. With this definition, we can express the closure of a Jacobi diagram Γ as

$$\langle\Gamma\rangle = \sum_{\pi \in \mathrm{P}_2(L)} \Gamma/\pi. \tag{37}$$

Example 5. We have $\langle w_2\rangle = \ominus$, $\langle\langle w_2\rangle\rangle = \ominus$, $\langle w_2^2\rangle = 2\ominus_2 + \ominus^2$, $\langle\langle w_2^2\rangle\rangle = 2\ominus_2$.

Let $L_1, \dots, L_n$ be finite and pairwise disjoint sets. We set $L := \coprod_{i=1}^{n} L_i$. Let $\pi \in \mathrm{P}_2(L)$ be a partition of L in 2-element-subsets. We say that a pair $l, l' \in L$ *is linked by* π if there is an $i \in \{1, \dots n\}$ such that $l, l' \in L_i$ or $\{l, l'\} \in \pi$. We say that π *connects the sets* $L_1, \ldots, L_n$ if and only if for each pair $l, l' \in L$ there is a chain of elements $l_1, \ldots, l_k$ such that l is linked to l_1, l_i is linked to l_{i+1} for $i \in \{1, \ldots, k-1\}$ and l_k is linked to l'. The subset of $\mathrm{P}_2(L)$ of partitions π connecting $L_1, \ldots, L_n$ is denoted by $\mathrm{P}_2(\{L_1, \ldots, L_n\})$. We have

$$\mathrm{P}_2(L) = \coprod_{\coprod \mathfrak{I} = \{1,\ldots,n\}} \left\{ \coprod_{I \in \mathfrak{I}} \pi_I : \pi_I \in P_2(\{L_i : i \in I\}) \right\}. \tag{38}$$

Here, $\coprod \mathfrak{I} = \{1, \ldots, n\}$ means that $\mathfrak{I}$ is a partition of $\{1, \ldots, n\}$ in disjoint subsets.

Let $\Gamma_1, \ldots, \Gamma_n$ be connected Jacobi diagrams. We denote by $\Gamma := \prod_{i=1}^n \Gamma_i$ the product over all these Jacobi diagrams. Let L_i be the set of legs of Γ_i and denote by $L := \coprod_{i=1}^n L_i$ the set of all legs of Γ.

For every partition $\pi \in \mathrm{P}_2(L)$ the graph Γ/π is connected if and only if $\pi \in \mathrm{P}_2(\{L_1, \ldots, L_n\})$.

Using (38) we have

$$\begin{aligned} \langle \Gamma \rangle = \sum_{\pi \in \mathrm{P}_2(L)} \Gamma/\pi &= \sum_{\coprod \mathfrak{I} = \{1,\ldots,n\}} \prod_{I \in \mathfrak{I}} \sum_{\pi \in \mathrm{P}_2(\{L_i : i \in I\})} \left(\prod_{i \in I} \Gamma_i \right) / \pi \\ &= \sum_{\coprod \mathfrak{I} = \{1,\ldots,n\}} \prod_{I \in \mathfrak{I}} \left\langle \left\langle \prod_{i \in I} \Gamma_i \right\rangle \right\rangle . \end{aligned} \tag{39}$$

With this result we can prove the following Proposition:

Proposition 6. *For any connected graph homology class γ we have*

$$\exp \langle\langle \exp \gamma \rangle\rangle = \langle \exp \gamma \rangle . \tag{40}$$

Note that both sides are well-defined in $\hat{\mathcal{B}}$ since γ and $\langle\langle \cdots \rangle\rangle$ as connected graphs have no component in degree zero.

Proof. Let Γ be any connected Jacobi diagram. By (39) we have

$$\langle \Gamma^n \rangle = \sum_{\coprod \mathfrak{I} = \{1,\ldots,n\}} \prod_{I \in \mathfrak{I}} \langle\langle \Gamma^{\# I} \rangle\rangle = \sum_{\lambda \in \mathrm{P}(n)} n! \prod_{i=1}^{\infty} \frac{1}{\lambda_i!} \left(\langle\langle \Gamma^i \rangle\rangle / i! \right)^{\lambda_i} .$$

By linearity this result holds also if we substitute Γ by the connected graph homology class γ.

Using this,

$$\begin{aligned} \langle \exp \gamma \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \gamma^n \rangle &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\lambda \in \mathrm{P}(n)} n! \prod_{i=1}^{\infty} \frac{1}{\lambda_i!} \left(\langle\langle \gamma^i \rangle\rangle / i! \right)^{\lambda_i} \\ &= \prod_{i=1}^{\infty} \sum_{\lambda=0}^{\infty} \frac{1}{\lambda!} \left(\langle\langle \gamma^i \rangle\rangle / i! \right)^{\lambda} = \prod_{i=1}^{\infty} \exp \left(\langle\langle \gamma^i \rangle\rangle / i! \right) = \exp \langle\langle \exp \gamma \rangle\rangle . \end{aligned}$$

□

3.5. Polywheels.

Definition 10. For each $n \in \mathbb{N}_0$ we set $\tilde{w}_{2n} := -w_{2n}$. Let λ be a partition of n. We set

$$\tilde{w}_{2\lambda} := \prod_{i=1}^{\infty} \tilde{w}_{2i}^{\lambda_i}. \tag{41}$$

The closure $\langle \tilde{w}_{2\lambda} \rangle$ of $\tilde{w}_{2\lambda}$ is called a *polywheel*. The subspace in ${}^{\mathrm{t}}\mathcal{B}$ spanned by all polywheels is denoted by $\mathcal{W}$ and called the *polywheel subspace*. The subalgebra in ${}^{\mathrm{t}}\mathcal{B}$ spanned by all polywheels is denoted by $\mathcal{C}$ and called the *algebra of polywheels*.
The connected closure $\langle\langle \tilde{w}_{2\lambda} \rangle\rangle$ of $\tilde{w}_{2\lambda}$ is called a *connected polywheel*.

Remark 7. As discussed by J. Sawon in his thesis [16], $\mathcal{W}$ is proper graded subspace of ${}^{\mathrm{t}}\mathcal{B}$. From degree eight on, ${}^{\mathrm{t}}\mathcal{B}_k$ is considerably larger than $\mathcal{W}_k$. On the other hand it is unknown (at least to the author) if the inclusion $\mathcal{C} \subseteq {}^{\mathrm{t}}\mathcal{B}$ is proper.

Remark 8. The subalgebra $\mathcal{C}'$ in ${}^{\mathrm{t}}\mathcal{B}$ spanned by all connected polywheels equals $\mathcal{C}$. This is since we can use (40) to express every polywheel as a polynomial of connected polywheels and vice versa.

Example 6. Using Proposition 6 we calculated the following expansions of the connected polywheels in terms of wheels:

$$\begin{aligned}
\langle\langle \tilde{w}_2 \rangle\rangle &= \langle \tilde{w}_2 \rangle \\
\langle\langle \tilde{w}_2^2 \rangle\rangle &= \langle \tilde{w}_2^2 \rangle - \langle \tilde{w}_2 \rangle^2 \\
\langle\langle \tilde{w}_4 \rangle\rangle &= \langle \tilde{w}_4 \rangle \\
\langle\langle \tilde{w}_2^3 \rangle\rangle &= \langle \tilde{w}_2^3 \rangle - 3 \langle \tilde{w}_2 \rangle \langle \tilde{w}_2^2 \rangle + 2 \langle \tilde{w}_2 \rangle^3 \\
\langle\langle \tilde{w}_2 \tilde{w}_4 \rangle\rangle &= \langle \tilde{w}_2 \tilde{w}_4 \rangle - \langle \tilde{w}_2 \rangle \langle \tilde{w}_4 \rangle \\
\langle\langle \tilde{w}_6 \rangle\rangle &= \langle \tilde{w}_6 \rangle \\
\langle\langle \tilde{w}_2^4 \rangle\rangle &= \langle \tilde{w}_2^4 \rangle - 4 \langle \tilde{w}_2 \rangle \langle \tilde{w}_2^3 \rangle - 3 \langle \tilde{w}_2^2 \rangle^2 + 12 \langle \tilde{w}_2 \rangle^2 - 6 \langle \tilde{w}_2 \rangle^4 \\
\langle\langle \tilde{w}_2^2 \tilde{w}_4 \rangle\rangle &= \langle \tilde{w}_2^2 \tilde{w}_4 \rangle - 2 \langle \tilde{w}_2 \rangle \langle \tilde{w}_2 \tilde{w}_4 \rangle - \langle \tilde{w}_2^2 \rangle \langle \tilde{w}_4 \rangle + 2 \langle \tilde{w}_2 \rangle^2 \langle \tilde{w}_4 \rangle \\
\langle\langle \tilde{w}_2 \tilde{w}_6 \rangle\rangle &= \langle \tilde{w}_2 \tilde{w}_6 \rangle - \langle \tilde{w}_2 \rangle \langle \tilde{w}_6 \rangle \\
\langle\langle \tilde{w}_4^2 \rangle\rangle &= \langle \tilde{w}_4^2 \rangle - \langle \tilde{w}_4 \rangle^2 \\
\langle\langle \tilde{w}_8 \rangle\rangle &= \langle \tilde{w}_8 \rangle .
\end{aligned} \tag{42}$$

4. Holomorphic symplectic manifolds

4.1. Definition and general properties.

Definition 11. A *holomorphic symplectic manifold* (X, σ) is a compact complex manifold X together with an everywhere non-degenerate holomorphic two-form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$. Here, we call σ *everywhere non-degenerate* if σ induces an isomorphism $\mathcal{T}_X \to \Omega_X$.

The holomorphic symplectic manifold (X, σ) is called *irreducible* if it is simply-connected and $\mathrm{H}^0(X, \Omega_X^2)$ is one-dimensional, i.e. spanned by σ.

It follows immediately that every holomorphic symplectic manifold X has trivial canonical bundle whose sections are multiples of σ^n, and, therefore, vanishing first Chern class. In fact, all odd Chern classes vanish:

Proposition 7. *Let X be a complex manifold and E a complex vector bundle on X. If E admits a symplectic two-form, i.e. there exists a section $\sigma \in \mathrm{H}^0(X, \Lambda^2 E^*)$ such that the induced morphism $E \to E^*$ is an isomorphism, all odd Chern classes of E vanish.*

Remark 9. That the odd Chern classes of E vanish up to two-torsion follows immediately from the fact $c_{2k+1}(E) = -c_{2k+1}(E^*)$ for $k \in \mathbb{N}_0$.

The following proof using the splitting principle has been suggested to me by Manfred Lehn.

Proof. We prove the proposition by induction over the rank of E. For $\operatorname{rk} E = 0$, the claim is obvious.
By the splitting principle (see e.g. [5]), we can assume that E has a subbundle L of rank one. Let $L^\perp$ be the σ-orthogonal subbundle to L of E. Since σ is symplectic, $L^\perp$ is of rank $n-1$ and L is a subbundle of $L^\perp$. We have the following short exact sequences of bundles on X:

$$0 \longrightarrow L \longrightarrow E \longrightarrow E/L \longrightarrow 0$$

and

$$0 \longrightarrow L^\perp/L \longrightarrow E/L \longrightarrow E/L^\perp \longrightarrow 0.$$

Since σ induces a symplectic form on $L^\perp/L$, by induction, all odd Chern classes of this bundle of rank $\operatorname{rk} E - 2$ vanish. Furthermore, note that σ induces an isomorphism between L and $(E/L^\perp)^*$, so all odd Chern classes of $L \oplus E/L^\perp$ vanish.
Now, the two exact sequences give us $c(E) = c(L \oplus E/L^\perp) \cdot c(L^\perp/L)$. Therefore, we can conclude that all odd Chern classes of E vanish. □

Proposition 8. *For any irreducible holomorphic symplectic manifold (X, σ) of dimension $2n$ and $k \in 0, \ldots, n$ the space $\mathrm{H}^{2k}(X, \mathcal{O}_X)$ is one-dimensional and spanned by the cohomology class $[\bar{\sigma}]^k$.*

Proof. See [2]. □

4.2. A pairing on the cohomology of a holomorphic symplectic manifold. Let (X, σ) be a holomorphic symplectic manifold. There is a natural pairing of coherent sheafs

$$\Lambda_* \mathcal{T}_X \otimes \Lambda^* \Omega_X \to \mathcal{O}_X. \tag{43}$$

As the natural morphism from $\Lambda_* \mathcal{T}_X$ to $\Lambda^* \mathcal{T}_X$ is an isomorphism and $\Lambda^* \mathcal{T}_X$ can be identified with $\Lambda^* \Omega_X$ by means of the symplectic form, we therefore have a natural map

$$\Lambda^* \Omega \otimes \Lambda^* \Omega_X \to \mathcal{O}_X. \tag{44}$$

We write

$$\langle \cdot, \cdot \rangle : \mathrm{H}^p(X, \Omega^*) \otimes \mathrm{H}^q(X, \Omega^*) \to \mathrm{H}^{p+q}(X, \mathcal{O}_X), (\alpha, \beta) \mapsto \langle \alpha, \beta \rangle \tag{45}$$

for the induced map for any $p, q \in \mathbb{N}_0$.

In [10] we proved the following proposition:

Proposition 9. *For any $\alpha \in \mathrm{H}^*(X, \Omega^*)$ we have*

$$\int_X \alpha \exp \sigma = \int_X \langle \alpha, \exp \sigma \rangle \exp \sigma. \tag{46}$$

4.3. Example series. There are two main series of examples of irreducible holomorphic symplectic manifolds. Both of them are based on the Hilbert schemes of points on a surface:

Let X be any smooth projective surface over $\mathbb{C}$ and $n \in \mathbb{N}_0$. By $X^{[n]}$ we denote the Hilbert scheme of zero-dimensional subschemes of length n of X. By a result of Fogarty ([4]), $X^{[n]}$ is a smooth projective variety of dimension $2n$. The Hilbert scheme can be viewed as a resolution $\rho : X^{[n]} \to X^{(n)}$ of the n-fold symmetric product $X^{(n)} := X^n/\mathfrak{S}_n$. The morphism ρ, sending closed points, i.e. subspaces of X, to their support counting multiplicities, is called the Hilbert-Chow morphism.

Let $\alpha \in \mathrm{H}^2(X, \mathbb{C})$ be any class. The class $\sum_{i=1}^n \mathrm{pr}_i^* \alpha \in \mathrm{H}^2(X^n, \mathbb{C})$ is invariant under the action of $\mathfrak{S}_n$, where $\mathrm{pr}_i : X^n \to X$ denotes the projection on the i^{th} factor. Therefore, there exists a class $\alpha^{(n)} \in \mathrm{H}^2(X^{(n)}, \mathbb{C})$ with $\pi^* \alpha^{(n)} = \sum_{i=1}^n \mathrm{pr}_i^* \alpha$, where $\pi : X^n \to X^{(n)}$ is the canonical projection. Using ρ this induces a class $\alpha^{[n]}$ in $\mathrm{H}^2(X^{[n]}, \mathbb{C})$.

If X is a K3 surface or an abelian surface, there exists a holomorphic symplectic form $\sigma \in \mathrm{H}^{2,0}(X) \subseteq \mathrm{H}^2(X, \mathbb{C})$. It was shown by Beauville in [2] that $\sigma^{[n]}$ is again symplectic, so $(X^{[n]}, \sigma^{[n]})$ is a holomorphic symplectic manifold.

Example 7. For any K3 surface X and holomorphic symplectic form $\sigma \in \mathrm{H}^{2,0}(X)$, the pair $(X^{[n]}, \sigma^{[n]})$ is in fact an irreducible holomorphic symplectic manifold.

This has also been proven by Beauville. In the case of an abelian surface A, we have to work a little bit more as $A^{[n]}$ is not irreducible in this case:

Let A be an abelian surface and let us denote by $s : A^{[n]} \to A$ the composition of the summation morphism $A^{(n)} \to A$ with the Hilbert-Chow morphism $\rho : A^{[n]} \to A^{(n)}$.

Definition 12. For any $n \in \mathbb{N}$, the n^{th} *generalised Kummer variety* $A^{[[n]]}$ is the fibre of s over $0 \in A$. For any class $\alpha \in \mathrm{H}^2(A, \mathbb{C})$, we set $\alpha^{[[n]]} := \alpha^{[n]}|_{A^{[[n]]}}$.

Remark 10. For $n = 2$ the generalised Kummer variety coincides with the Kummer model of a K3 surface (therefore the name).

Example 8. For every abelian surface A and holomorphic symplectic form $\sigma \in \mathrm{H}^{2,0}(A)$, the pair $(A^{[[n]]}, \sigma^{[[n]]})$ is an irreducible holomorphic symplectic manifold of dimension $2n-2$.

The proof can also be found in [2].

4.4. About $\alpha^{[n]}$ and $\alpha^{[[n]]}$. Let X be any smooth projective surface and $n \in \mathbb{N}_0$.
Let $X^{[n,n+1]}$ denote the incidence variety of all pairs $(\xi, \xi') \in X^{[n]} \times X^{[n+1]}$ with $\xi \subseteq \xi'$ (see [3]). We denote by $\psi : X^{[n,n+1]} \to X^{[n+1]}$ and by $\phi : X^{[n,n+1]} \to X^{[n]}$ the canonical maps. There is a third canonical map $\chi : X^{[n,n+1]} \to X$ mapping $(\xi, \xi') \mapsto x$ if ξ' is obtained by extending ξ at the closed point $x \in X$.

Proposition 10. *For any* $\alpha \in \mathrm{H}^2(X, \mathbb{C})$ *we have*

$$\psi^* \alpha^{[n+1]} = \phi^* \alpha^{[n]} + \chi^* \alpha. \tag{47}$$

Proof. Let $p : X^{(n)} \times X \to X^{(n)}$ and $q : X^{(n)} \times X \to X$ denote the canonical projections. Let $\tau : X^{(n)} \times X \to X^{(n+1)}$ the obvious symmetrising map. The following diagram

$$\begin{tikzcd}
X^{[n,n+1]} \arrow[r, equal] \arrow[d, "(\phi,\chi)"'] & X^{[n,n+1]} \arrow[d, "\psi"] \\
X^{[n]} \times X \arrow[d, "\rho \times \mathrm{id}_X"'] & X^{[n+1]} \arrow[d, "\rho'"] \\
X^{(n)} \times X \arrow[r, "\tau"] & X^{(n+1)} \\
X^{n+1} \arrow[r, equal] \arrow[u, "\pi \times \mathrm{id}_X"] & X^{n+1} \arrow[u, "\pi'"']
\end{tikzcd}$$

is commutative. (Note that we have primed some maps to avoid name clashes.) We claim that $\tau^* \alpha^{(n+1)} = p^* \alpha^{(n)} + q^* \alpha$. In fact, since

$$(\pi \times \mathrm{id}_X)^* \tau^* \alpha^{(n+1)} = \pi'^* \alpha^{(n+1)} = \sum_{i=1}^{n+1} \mathrm{pr}_i^* \alpha,$$

this follows from the definition of $\alpha^{(n)}$. Finally, we can read off the diagram that

$$\begin{aligned}
\psi^* \alpha^{[n+1]} = \psi^* \rho'^* \alpha^{(n+1)} &= (\phi, \chi)^* (\rho \times \mathrm{id}_X)^* \tau^* \alpha^{(n+1)} \\
&= (\phi, \chi)^* (\rho \times \mathrm{id}_X)^* (p^* \alpha^{(n)} + q^* \alpha) = \phi^* \alpha^{[n]} + \chi^* \alpha.
\end{aligned}$$

□

Proposition 11. *Let $X = X_1 \sqcup X_2$ be the disjoint union of two projective smooth surfaces X_1 and X_2. We then have*

$$X^{[n]} = \coprod_{n_1+n_2=n} X_1^{[n_1]} \times X_2^{[n_2]}. \tag{48}$$

If $\alpha \in \mathrm{H}^2(X, \mathbb{C})$ decomposes as $\alpha|_{X_1} = \alpha_1$ and $\alpha|_{X_2} = \alpha_2$, then $\alpha^{[n]}$ decomposes as

$$\alpha^{[n]}|_{X_!^{[n_1]}\times X_2^{[n_2]}} = \mathrm{pr}_1^* \alpha_1^{[n_1]} + \mathrm{pr}_2^* \alpha_2^{[n_2]}. \tag{49}$$

Proof. The splitting of $X^{[n]}$ follows from the universal property of the Hilbert scheme and is a well-known fact. The statement on $\alpha^{[n]}$ is easy to prove and so we shall only give a sketch: Let us denote by $i : X_1^{[n_1]} \times X_2^{[n_2]} \to X^{[n]}$ the natural inclusion. Furthermore let $j : X_1^{(n_1)} \times X_2^{(n_2)} \to X^{(n)}$ denote the natural symmetrising map. The following diagram is commutative:

$$\begin{tikzcd} X_1^{[n_1]} \times X_2^{[n_2]} \arrow[r, "i"] \arrow[d, "\rho_1\times\rho_2"'] & X^{[n]} \arrow[d, "\rho"] \\ X_1^{(n_1)} \times X_2^{(n_2)} \arrow[r, "j"'] & X^{(n)}, \end{tikzcd} \tag{50}$$

where the $\rho_i : X_i^{[n_i]} \to X_i^{(n_i)}$ are the Hilbert-Chow morphisms. Since $j^*\alpha^{(n)} = \mathrm{pr}_1^* \alpha_1^{(n_1)} + \mathrm{pr}_2^* \alpha_2^{(n_2)}$, the commutativity of the diagram proves the statement on $\alpha^{[n]}$. □

Let A be again an abelian surface and $n \in \mathbb{N}$. Since A acts on itself by translation, there is also an induced operation of A on the Hilbert scheme $A^{[n]}$. Let us denote the restriction of this operation to the generalised Kummer variety $A^{[[n]]}$ by $\nu : A \times A^{[[n]]} \to A^{[n]}$. It fits into the following cartesian square:

$$\begin{tikzcd} A \times A^{[[n]]} \arrow[r, "\nu"] \arrow[d, "\mathrm{pr}_1"'] & A^{[n]} \arrow[d, "s"] \\ A \arrow[r, "n"'] & A, \end{tikzcd} \tag{51}$$

where s is the summation map as having been defined above and $n : A \to A, a \mapsto na$ is the (multiplication-by-n)-morphism. Since n is a Galois cover of degree n^4, the same holds true for ν.

Proposition 12. *For any $\alpha \in \mathrm{H}^2(A, \mathbb{C})$, we have*

$$\nu^*\alpha^{[n]} = n \,\mathrm{pr}_1^* \alpha + \mathrm{pr}_2^* \alpha^{[[n]]}. \tag{52}$$

Proof. By the Künneth decomposition theorem, we know that $\nu^*\alpha^{[n]}$ splits:

$$\nu^*\alpha^{[n]} = \mathrm{pr}_1^* \alpha_1 + \mathrm{pr}_2^* \alpha_2.$$

Set $\iota_1 : A \to A \times A^{[[n]]}, a \mapsto (a, \xi_0)$ and $\iota_2 : A^{[[n]]} \to A \times A^{[[n]]}, \xi \mapsto (0, \xi)$, where ξ_0 is any subscheme of length n concentrated in 0. We have

$$\alpha_1 = \iota_1^* \nu^* \alpha^{[n]} = (\rho \circ \nu \circ \iota_1)^* \alpha^{(n)} = (a \mapsto (\underbrace{a, \dots a}_{n}))^* \alpha^{(n)} = n\alpha \tag{53}$$

and

$$\alpha_2 = \iota_2^* \nu^* \alpha^{[n]} = i^* \alpha^{[n]} = \alpha^{[[n]]}, \tag{54}$$

where $i : A^{[[n]]} \to A^{[n]}$ is the natural inclusion map, thus proving the proposition. □

4.5. Complex genera of Hilbert schemes of points on surfaces. The following theorem is an adaption of Theorem 4.1 of [3] to our context.

Theorem 1. *Let P be a polynomial in the variables $c_1, c_2, \dots$ and α over $\mathbb{Q}$. There exists a polynomial $\tilde{P} \in \mathbb{Q}[z_1, z_2, z_3, z_4]$ such that for every smooth projective surface X, $\alpha \in \mathrm{H}^2(X, \mathbb{Q})$ and $n \in \mathbb{N}_0$ we have:*

(55)
$$\int_{X^{[n]}} P(c_*(X^{[n]}), \alpha^{[n]}) = \tilde{P}\left(\int_X \alpha^2/2, \int_X c_1(X)\alpha, \int_X c_1(X)^2/2, \int_X c_2(X)\right).$$

Proof. The proof goes along the very same lines as the proof of Proposition 0.5 in [3] (see there). The only new thing we need is Proposition 10 of this paper to be used in the induction step of the adapted proof of Proposition 3.1 of [3] to our situation. □

Let R be any $\mathbb{Q}$-algebra (commutative and with unit) and let $\phi \in R[[c_1, c_2, \dots]]$ be a non-vanishing power series in the universal Chern classes such that ϕ is multiplicative with respect to the Whitney sum of vector bundles, i.e.

$$\phi(E \oplus F) = \phi(E)\phi(F) \tag{56}$$

for all complex manifolds and complex vector bundles E and F on X. Any ϕ with this property induces a complex genus, also denoted by ϕ, by setting $\phi(X) := \int_X \phi(\mathcal{T}_X)$ for X a compact complex manifold. Let us call such a ϕ *multiplicative*.

Remark 11. By Hirzebruch's theory of multiplicative sequences and complex genera ([6]), we know that

1. each complex genus is induced by a unique multiplicative ϕ, and
2. the multiplicative elements in $R[[c_1, c_2, \dots]]$ are exactly those of the form $\exp(\sum_{k=1}^{\infty} a_k s_k)$ with $a_k \in R$.

More or less formally the following theorem follows from Theorem 1.

Theorem 2. *For each multiplicative $\phi \in R[[c_1, c_2, \dots]]$, there exist unique power series $A_\phi(p), B_\phi(p), C_\phi(p), D_\phi(p) \in pR[[p]]$ with vanishing constant coefficient such that for all smooth projective surfaces X and $\alpha \in \mathrm{H}^2(X, \mathbb{C})$ we* *have:*

$$\sum_{n=0}^{\infty}\left(\int_{X^{[n]}}\phi(X^{[n]})\exp(\alpha^{[n]})\right)p^n \tag{57}$$
$$= \exp\left(A_\phi(p)\int_X \alpha^2/2 + B_\phi(p)\int_X c_1(X)\alpha + C_\phi(p)\int_X c_1^2(X)/2 + D_\phi(p)\int_X c_2(X)\right).$$

The first terms of $A_\phi(p), B_\phi(p), C_\phi(p), D_\phi(p)$ are given by

$$A_\phi(p) = p + O(p^2),\quad B_\phi(p) = \phi_1 p + O(p^2),\quad C_\phi(p) = \phi_{11} p + O(p^2), \text{ and } D_\phi(p) = \phi_2 p + O(p^2), \tag{58}$$

where ϕ_1 is the coefficient of c_1 in ϕ, ϕ_{11} the coefficient of $c_1^2/2$ and ϕ_2 the coefficient of c_2.

Proof. This theorem is again an adaption of a theorem (Theorem 4.2) of [3] to our context. Nevertheless, let us give the proof here:
Set $K := \{(X,\alpha) : X$ is a smooth projective surface and $\alpha \in \mathrm{H}^2(X,\mathbb{C})\}$ and let $\gamma : K \to \mathbb{Q}^4$ be the map $(X,\alpha) \mapsto (\alpha^2/2, c_1(X)\alpha, c_1(X)^2/2, c_2(X))$. Here, we have supressed the integral signs $\int_X$ and interpret the expressions α^2, etc. as intersection numbers on X. The image of K spans the whole $\mathbb{Q}^4$ (for explicit generators, we refer to [3]).
Now let us assume that a $(X,\alpha) \in K$ decomposes as $(X,\alpha) = (X_1,\alpha_1) \sqcup (X_2,\alpha_2)$. By the multiplicative behaviour of ϕ and exp we see that

$$\int_{X^{[n]}}\phi(c_*(X^{[n]}))\exp(\alpha^{[n]})$$
$$= \sum_{n_1+n_2=n}\left(\int_{X_1^{[n_1]}}\phi(c_*(X^{[n_1]}))\exp(\alpha_2^{[n_1]})\right)\left(\int_{X_2^{[n_2]}}\phi(c_*(X^{[n_2]}))\exp(\alpha_2^{[n_2]})\right),$$

whereas $H_\phi(p)(X,\alpha) := \sum_{n=0}^\infty \left(\int_{X^{[n]}}\phi(X^{[n]})\exp(\alpha^{[n]})\right)p^n$ fulfills

$$H_\phi(p)(X,\alpha) = H_\phi(p)(X_1,\alpha_1)H_\phi(p)(X_2,\alpha_2). \tag{$*$}$$

Since $H_\phi(p) : K \to \mathbb{Q}^4$ factors through γ and a map $h : \mathbb{Q}^4 \to R[[p]]$ by Theorem 1 and as the image of γ is Zariski dense in $\mathbb{Q}^4$, we conclude from $(*)$ that $\log h$ is a linear function which proves the first part of the theorem.
To get the first terms of the power series, we expand both sides of (57). The left hand side expands as

$$1 + (\alpha^2/2 + \phi_1 c_1(X)\alpha + \frac{\phi_{11}}{2}c_1^2(X) + \phi_2 c_2(X))p + O(p^2), \tag{59}$$

while the right hand side expands as

$$1 + (A_1\alpha^2/2 + B_1 c_1(X)\alpha + C_1 c_1(X)^2 + D_1 c_2(X))p + O(p^2), \tag{60}$$

where A_1, B_1, C_1, D_1 are the linear coefficients of A_ϕ, B_ϕ, C_ϕ, and D_ϕ, which can therefore be read off by comparing the expansions. $\square$

Corollary 1. *Let X be any smooth projective surface, $\alpha \in \mathrm{H}^2(X,\mathbb{C})$, and $n \in \mathbb{N}_0$. Then*

$$\int_{X^{[n]}} \exp(\alpha^{[n]} + \bar{\alpha}^{[n]}) = \frac{1}{n!}\left(\int_X \alpha\bar{\alpha}\right)^n. \tag{61}$$

For $X = A$ an abelian surface and $n \in \mathbb{N}$, we get

$$\int_{A^{[[n]]}} \exp(\alpha^{[[n]]} + \bar{\alpha}^{[[n]]}) = \frac{n}{(n-1)!}\left(\int_X \alpha\bar{\alpha}\right)^{n-1}. \tag{62}$$

Proof. By Theorem 2, in $\mathbb{C}[[q]]$:

$$\sum_{n=0}^{\infty}\left(\int_{X^{[n]}} \exp(q^{\frac{1}{2}}(\alpha^{[n]} + \bar{\alpha}^{[n]}))\right) p^n = \exp(pq\int_X \alpha\bar{\alpha} + O(p^2)), \tag{63}$$

which proves the first part of the corollary by comparing coefficients of q.
For the Kummer case, we calculate

$$\begin{aligned}\int_{A^{[[n]]}} \exp(\alpha^{[[n]]} + \bar{\alpha}^{[[n]]}) &= \frac{\int_{A^{[[n]]}} \exp(\alpha^{[[n]]} + \bar{\alpha}^{[[n]]}) \int_A \exp(n\alpha + n\bar{\alpha})}{\int_A \exp(n\alpha + n\bar{\alpha})}\\ &= n^2 \frac{\int_{A^{[n]}} \exp(\alpha^{[n]} + \bar{\alpha}^{[n]})}{\int_A \exp(\alpha + \bar{\alpha})},\end{aligned}$$

which proves the rest of the corollary. $\square$

Let ch be the universal Chern character. By $s_k = (2k)!\mathrm{ch}_{2k}$ we denote its components. They span the whole algebra of characteristic classes, i.e. we have $\mathbb{Q}[s_1, s_2, \dots] = \mathbb{Q}[c_1, c_2, \dots]$.
Let us fix the power series

$$\phi := \exp(\sum_{k=1}^{\infty} a_{2k}s_{2k}t^k) \in \mathbb{Q}[a_2, a_4, \dots][t][[c_1, c_2, \dots]].$$

This multiplicative series gives rise to four power series

$$A_\phi(p), B_\phi(p), C_\phi(p), D_\phi(p) \in pR[[p]]$$

according to the previous Theorem 2. We shall set for the rest of this article

$$A(t) := A_\phi(1), \quad \text{and} \quad D(t) := D_\phi(1) \tag{64}$$

The constant terms of these power series in t are given by

$$A(t) = 1 + O(t), \quad \text{and} \quad D(t) = O(t). \tag{65}$$

5. Rozansky-Witten classes and invariants

The idea to associate to every graph Γ and every hyperkähler manifold X a cohomology class $\mathrm{RW}_X(\Gamma)$ is due to L. Rozansky and E. Witten (c.f. [15]). M. Kapranov showed in [8] that the metric structure of a hyperkähler manifold is not nessessary to define these classes. It was his idea to build the whole theory upon the Atiyah class and the symplectic structure of an irreducible holomorphic symplectic manifold. We will make use of his definition of Rozansky-Witten classes in this section. A very detailed text on defining Rozansky-Witten invariants is the thesis by J. Sawon [16].

5.1. Definition. Let (X, σ) be a holomorphic symplectic manifold. Let us work in the category of complexes of coherent sheaves on X. In this category, we have for every $n \in \mathbb{Z}$ a functor $V \mapsto V[n]$ that shifts a complex V by n to the left. Due to the Koszul sign rule (i.e. the natural map $(V[m]) \otimes (W[n]) \to (W[n]) \otimes (V[m])$ for sheaves V and W and integers n and m incorporates a sign $(-1)^{mn}$), we have $\mathrm{S}^n(V[1]) = (\Lambda^n V)[n]$ and $\mathrm{S}_n(V[1]) = (\Lambda_n V)[n]$.
Every Jacobi diagram Γ with k trivalent and l univalent vertices defines in the category of complexes of coherent sheaves on X a morphism

$$\Phi^\Gamma : \mathrm{S}_k\Lambda_3(\mathcal{T}_X[-1]) \otimes \mathrm{S}_l(\mathcal{T}_X[-1]) \to \mathrm{S}^e\mathrm{S}^2(\mathcal{T}_X[-1]), \tag{66}$$

where $\mathcal{T}_X[-1]$ is the tangent sheaf of X shifted by one and $2e = 3k + l$. By the sign rule above, this is equivalent to being given a map:

$$(\Lambda_k\mathrm{S}_3\mathcal{T}_X \otimes \Lambda_l\mathcal{T}_X)[-3k - l] \to (\mathrm{S}^e\Lambda^2\mathcal{T}_X)[-2e], \tag{67}$$

which is induced by a map

$$\Lambda_k\mathrm{S}_3\mathcal{T}_X \otimes \Lambda_l\mathcal{T}_X \to \mathrm{S}^e\Lambda^2\mathcal{T}_X \tag{68}$$

in the category of coherent sheaves on X. This gives rise to a map

$$\Psi^\Gamma : \Lambda_k\mathrm{S}_3\mathcal{T}_X \otimes \mathrm{S}_e\Lambda_2\Omega_X \to \Lambda^l\Omega_X. \tag{69}$$

Let $\tilde\alpha \in \mathrm{H}^1(X, \Omega \otimes \operatorname{End}\mathcal{T}_X)$ be the Atiyah class of X, i.e. $\tilde\alpha$ represents the extension class of the sequence

$$0 \longrightarrow \Omega_X \otimes \mathcal{T}_X \longrightarrow \mathrm{J}^1\mathcal{T}_X \longrightarrow \mathcal{T}_X \longrightarrow 0 \tag{70}$$

in $\operatorname{Ext}^1_X(\mathcal{T}_X, \Omega_X \otimes \mathcal{T}_X) = \mathrm{H}^1(X, \Omega_X \otimes \operatorname{End}\mathcal{T}_X)$. Here, $\mathrm{J}^1\mathcal{T}_X$ is the bundle of one-jets of sections of $\mathcal{T}_X$ (for more on this, see [8]). The Atiyah class can also be viewed as the obstruction for a global holomorphic connection to exist on $\mathcal{T}_X$. We set $\alpha := i/(2\pi)\tilde\alpha$.
We use σ to identify the tangent bundle $\mathcal{T}_X$ of X with its cotangent bundle Ω_X. Doing this, α can be viewed as an element of $\mathrm{H}^1(X, \mathcal{T}_X^{\otimes 3})$. Now the point is that α is not any such element. The following proposition was proven by Kapranov in [8]:

Proposition 13.

$$\alpha \in \mathrm{H}^1(X, \mathrm{S}_3\mathcal{T}_X) \subseteq \mathrm{H}^1(X, \mathcal{T}_X^{\otimes 3}). \tag{71}$$

Therefore, $\alpha^{\cup k} \cup \sigma^{\cup l} \in \mathrm{H}^k(X, \Lambda_k \mathrm{S}_3 \mathcal{T}_X \otimes \mathrm{S}_l \Lambda_2 \Omega_X)$. Applying the map Ψ^Γ on the level of cohomology eventually leads to an element

$$\mathrm{RW}_{X,\sigma}(\Gamma) := \Psi^\Gamma_*(\alpha^{\cup k} \cup \sigma^{\cup l}) \in \mathrm{H}^k(X, \Omega^l_X). \tag{72}$$

We call $\mathrm{RW}_{X,\sigma}(\Gamma)$ the *Rozansky-Witten class of* (X,σ) *associated to* Γ.
For a $\mathbb{C}$-linear combination γ of Jacobi diagrams, $\mathrm{RW}_{X,\sigma}(\gamma)$ is defined by linear extension.
In [8], Kapranov also showed the following proposition, which is crucial for the next definition. It follows from a Bianchi-identity for the Atiyah class.

Proposition 14. *If γ is a $\mathbb{Q}$-linear combination of Jacobi diagrams that is zero modulo the anti-symmetry and IHX relations, then* $\mathrm{RW}_{X,\sigma}(\gamma) = 0$.

Definition 13. We define a double-graded linear map

$$\mathrm{RW}_{X,\sigma} : \hat{\mathcal{B}} \to \mathrm{H}^*(X, \Omega^*_X), \tag{73}$$

which maps $\mathcal{B}_{k,l}$ into $\mathrm{H}^k(X, \Omega^l_X)$ by mapping a homology class of a Jacobi diagram Γ to $\mathrm{RW}_{X,\sigma}(\Gamma)$.

Definition 14. Let $\gamma \in \hat{\mathcal{B}}$ be any graph. The integral

$$b_\gamma(X,\sigma) := \int_X \mathrm{RW}_{X,\sigma}(\gamma) \exp(\sigma + \bar{\sigma}) \tag{74}$$

is called the *Rozansky-Witten invariant of* (X,σ) *associated to* γ.

5.2. Examples and properties of Rozansky-Witten classes. We summarise in this subsection the properties of the Rozansky-Witten classes that will be of use for us. For proofs take a look at [10], please.
Let (X,σ) again be a holomorphic symplectic manifold.

Proposition 15. *The map* $\mathrm{RW}_{X,\sigma} : \hat{\mathcal{B}} \to \mathrm{H}^{*,*}(X)$ *is a morphism of graded algebras.*

Proposition 16. *For all $\gamma \in \hat{\mathcal{B}}'$ and $\gamma' \in \hat{\mathcal{B}}$ we have*

$$\mathrm{RW}_{X,\sigma}(\langle \gamma, \gamma' \rangle) = \langle \mathrm{RW}_{X,\sigma}(\gamma), \mathrm{RW}_{X,\sigma}(\gamma') \rangle\,. \tag{75}$$

Example 9. The cohomology class $[\sigma] \in \mathrm{H}^{2,0}(X)$ is a Rozansky-Witten class; more precisely, we have

$$\mathrm{RW}_{X,\sigma}(\ell) = 2[\sigma]. \tag{76}$$

Example 10. The components of the Chern charakter are Rozansky-Witten invariants:

$$-\mathrm{RW}_{X,\sigma}(w_{2k}) = \mathrm{RW}_{X,\sigma}(\tilde{w}_{2k}) = s_{2k}. \tag{77}$$

The next two proposition actually aren't stated in [10], so we shall give ideas of their proofs here.

Proposition 17. *Let $\nu : (X, \nu^*\sigma) \to (Y,\sigma)$ be a Galois cover of holomorphic symplectic manifolds. For every graph homology class $\gamma \in \hat{\mathcal{B}}$,*

$$\mathrm{RW}_{X,\nu^*\sigma}(\gamma) = \nu^* \mathrm{RW}_{Y,\sigma}(\gamma). \tag{78}$$

Proof. As ν is a Galois cover, we can identify $\mathcal{T}_X$ with $\nu^* \mathcal{T}_Y$ and so $\tilde{\alpha}_X$ with $\nu^* \tilde{\alpha}_Y$ where $\tilde{\alpha}_X$ and $\tilde{\alpha}_Y$ are the Atiyah classes of X and Y. By definition of the Rozansky-Witten classes, (78) follows. □

Lemma 2. *Let (X, σ) and (Y, τ) be two holomorphic symplectic manifolds. If the tangent bundle of Y is trivial,*

$$\mathrm{RW}_{X \times Y, p^*\sigma + q^*\tau}(\gamma) = p^* \mathrm{RW}_{X,\sigma}(\gamma) \tag{79}$$

for all graphs $\gamma \in \hat{\mathcal{B}}'$. Here $p : X \times Y \to X$ and $q : X \times Y \to Y$ denote the canonical projections.

Proof. This lemma is a special case of the more general proposition in [16] that relates the coproduct in graph homology with the product of holomorphic symplectic manifolds. Since all Rozansky-Witten classes for graphs with at least one trivalent vertex vanish on Y, our lemma follows easily from J. Sawon's statement. □

5.3. Rozansky-Witten classes of closed graphs. Let γ be a homogeneous closed graph of degree $2k$. For every compact holomorphic symplectic manifold (X, σ), we have $\mathrm{RW}_{X,\sigma}(\gamma) \in \mathrm{H}^{0,2k}(X)$. If X is irreducible, we therefore have $\mathrm{RW}_{X,\sigma}(\gamma) = \beta_\gamma \cdot [\bar{\sigma}]^k$ for a certain $\beta_\gamma \in \mathbb{C}$. We can express β_γ as

$$\beta_\gamma = \frac{\int_X \mathrm{RW}_{X,\sigma}(\gamma) \bar{\sigma}^{n-k} \sigma^n}{\int_X (\sigma\bar{\sigma})^n} = \frac{(n-k)!}{n!} \frac{\int_X \mathrm{RW}_{X,\sigma}(\gamma) \exp(\sigma + \bar{\sigma})}{\int_X \exp(\sigma + \bar{\sigma})} \tag{80}$$

where $2n$ is the dimension of X.
This formula makes also sense for non-irreducible X, which leads us to the following definition:

Definition 15. Let (X, σ) be a compact holomorphic symplectic manifold (X, σ) of dimension $2n$. For any homogeneous closed graph homology class γ of degree $2k$ with $k \leq n$ we set

$$\beta_\gamma(X, \sigma) := \frac{(n-k)!}{n!} \frac{\int_X \mathrm{RW}_{X,\sigma}(\gamma) \exp(\sigma + \bar{\sigma})}{\int_X \exp(\sigma + \bar{\sigma})} \tag{81}$$

By linear extension, we can define $\beta_\gamma(X, \sigma)$ also for non-homogeneous closed graph homology classes γ.

Remark 12. The map ${}^{\mathrm{t}}\hat{\mathcal{B}} \to \mathbb{C}, \gamma \mapsto \beta_\gamma(X, \sigma)$ is linear. If X is irreducible, it is also a homomorphism of rings.

For polywheels $\tilde{w}_{2\lambda}$, we can express $\beta_{\langle \tilde{w}_{2\lambda} \rangle}$ in terms of characteristic classes:

Proposition 18. *Let (X, σ) be a compact holomorphic symplectic manifold of dimension $2n$ and $k \in \{1, \ldots, n\}$. Let $\lambda \in P(k)$ be any partition of k. Then*

$$\int_X \mathrm{RW}_{X,\sigma}(\langle \tilde{w}_{2\lambda} \rangle) \exp(\sigma + \bar{\sigma}) = \int_X s_{2\lambda}(X) \exp(\sigma + \bar{\sigma}). \tag{82}$$

Proof. We calculate

$$\begin{aligned}(83)\quad \int_X \mathrm{RW}_{X,\sigma}(\langle \tilde{w}_{2\lambda}\rangle)\exp(\sigma+\bar{\sigma}) &= \int_X \mathrm{RW}_{X,\sigma}(\langle \tilde{w}_{2\lambda}, \exp(\ell/2)\rangle)\exp(\sigma+\bar{\sigma}) \\ &= \int_X \langle s_{2\lambda}, \exp\sigma\rangle \exp(\sigma+\bar{\sigma}) = \int_X s_{2\lambda}\exp(\sigma+\bar{\sigma}).\end{aligned}$$

□

6. Calculation for the example series

6.1. Proof of the main theorem. Let X be a smooth projective surface that admits a holomorphic symplectic form (e.g. a K3 surface or an abelian surface). Let us fix a holomorphic symplectic form $\sigma \in H^{2,0}(X)$ that is normalised such that $\int_X \sigma\bar{\sigma} = 1$. It is known ([9]) that $X^{[n]}$ for all $n \in \mathbb{N}_0$ is a compact holomorphic symplectic manifold.
For every homogeneous closed graph homology class γ of degree $2k$ and every $n \in \mathbb{N}_0$, we set

$$(84)\qquad h^X_\gamma(n) := \beta_\gamma(X^{[k+n]}, \sigma^{[k+n]}).$$

By linear extension, we define $h^X_\gamma(n)$ for non-homogeneous closed graph homology classes γ.

Proposition 19. *For all closed graph homology classes γ, we have*

$$(85)\qquad \sum_{n=0}^\infty \frac{q^n}{n!} h^X_\gamma(n) = \sum_{l=0}^\infty \int_{X^{[l]}} \mathrm{RW}_{X^{[l]},\sigma^{[l]}}(\gamma)\exp(q^{\frac{1}{2}}(\sigma^{[l]}+\bar{\sigma}^{[l]}))$$

in $\mathbb{C}[[q]]$.

Proof. Let us assume that γ is homogeneous of degree $2k$. Then

$$\begin{aligned} h^X_\gamma(n) = \beta_\gamma(X^{[k+n]}, \sigma^{[k+n]}) = \\ \frac{n!}{(n+k)!}\frac{\int_{X^{[k+n]}} \mathrm{RW}_{X^{[k+n]},\sigma^{[k+n]}}(\gamma)\exp(\sigma^{[k+n]}+\bar{\sigma}^{[k+n]})}{\int_{X^{[k+n]}}\exp(\sigma^{[k+n]}+\bar{\sigma}^{[k+n]})} \\ = n!\int_{X^{[k+n]}} \mathrm{RW}_{X^{[k+n]},\sigma^{[k+n]}}(\gamma)\exp(\sigma+\bar{\sigma}).\end{aligned}$$

In the last equation we have used Corollary 1. Summing up and introducing the counting parameter q yields the claim. □

Proposition 20. *Let $a_2, a_4, \ldots$ be formal parameters. We set*

$$(86)\qquad \omega(t) := \sum_{k=1}^\infty a_{2k}t^k \tilde{w}_{2k} \in \hat{\mathcal{B}}^1[a_2, a_4, \ldots][t]$$

and call ω the universal wheel. *Further, we set* $W(t) := \exp(\omega(t))$ *and* $W := W(1)$. *The Rozansky-Witten classes of the universal wheel are encoded by*

$$\sum_{n=0}^{\infty} \frac{q^n}{n!} h^X_{\langle W(t)\rangle}(n) = \exp(qA(t)) \exp(c_2(X)D(t)). \tag{87}$$

Proof. Using Proposition 19 and Proposition 18 yields:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^n}{n!} h^X_{\langle W(t)\rangle}(n) &= \sum_{l=0}^{\infty} \int_{X^{[l]}} \mathrm{RW}_{X^{[l]},\sigma^{[l]}}(\langle W(t)\rangle) \exp(q^{\frac{1}{2}}(\sigma^{[l]} + \bar\sigma^{[l]})) \\ &= \sum_{l=0}^{\infty} \int_{X^{[l]}} \exp\left(\sum_{k=1}^{\infty} a_{2k} s_{2k}(X^{[l]}) t^k\right) \exp(q^{\frac{1}{2}}(\sigma^{[l]} + \bar\sigma^{[l]})) \\ &= \exp(qA(t)) \exp(c_2(X)D(t)).\end{aligned}$$

□

COROLLARY 2. *For every* $n \in \mathbb{N}_0$ *we have*

$$h^X_{\langle W(t)\rangle}(n) = \exp(c_2(X)D(t)) \exp(n \log A(t)) \tag{88}$$

Proof. Comparision of coefficients in (87) gives

$$h^X_{\langle W(t)\rangle}(n) = A(t)^n \exp(c_2(X)D(t)).$$

Lastly, note that A is a power series in t that has constant coefficient one. □

Remark 13. By equation (88) we shall extend the definition of $h^X_{\langle W(t)\rangle}(n)$ to all $n \in \mathbb{Z}$.

PROPOSITION 21. *Let* A *be an abelian surface. Let us fix a holomorphic symplectic form* $\sigma \in \mathrm{H}^{2,0}(A)$ *that is normalised such that* $\int_A \sigma\bar\sigma = 1$.
Let γ *be a homogeneous connected closed graph of degree* $2k$. *Then we have*

$$\beta_\gamma(A^{[[n]]}, \sigma^{[[n]]}) = \frac{n}{n-k} \beta_\gamma(A^{[n]}, \sigma^{[n]}) \tag{89}$$

for any $n > k$.

Proof. The proof is a straight-forward calculation:

$$\begin{aligned}\beta_\gamma(A^{[[n]]}, \sigma^{[[n]]}) &= \frac{(n-1-k)!}{(n-1)!} \frac{\int_{A^{[[n]]}} \mathrm{RW}_{A^{[[n]]},\sigma^{[[n]]}}(\gamma) \exp(\sigma^{[[n]]} + \bar\sigma^{[[n]]})}{\int_{A^{[[n]]}} \exp(\sigma^{[[n]]} + \bar\sigma^{[[n]]})} \\ &= \frac{(n-1-k)!}{(n-1)!} \frac{\int_{A^{[[n]]}} \mathrm{RW}_{A^{[[n]]},\sigma^{[[n]]}}(\gamma) \exp(\sigma^{[[n]]} + \bar\sigma^{[[n]]})}{\int_{A^{[[n]]}} \exp(\sigma^{[[n]]} + \bar\sigma^{[[n]]}} \frac{\int_A \exp(n\sigma + n\bar\sigma)}{\int_A \exp(n\sigma + n\bar\sigma)} \\ &= \frac{(n-1-k)!}{(n-1)!} \frac{\int_{A^{[n]}} \mathrm{RW}_{A^{[n]},\sigma^{[n]}}(\gamma) \exp(\sigma^{[n]} + \bar\sigma^{[n]})}{\int_{A^{[n]}} \exp(\sigma^{[n]} + \bar\sigma^{[n]})} = \frac{n}{n-k} \beta_\gamma(A^{[n]}, \sigma^{[n]}),\end{aligned}$$

where we have used Proposition 12, Proposition 17 and Lemma 2. □

Theorem 3. *For any homogenous connected closed graph of degree $2k$ lying in the algebra $\mathcal{C}$ of polywheels there exist two rational numbers a_γ, c_γ such that for each K3 surface X together with a symplectic form $\sigma \in \mathrm{H}^{2,0}(X)$ with $\int_X \sigma\bar{\sigma} = 1$ and $n \geq k$ we have*

$$\beta_\gamma(X^{[n]}, \sigma^{[n]}) = a_\gamma n + c_\gamma \tag{90}$$

and that for each abelian surface A together with a symplectic form $\sigma \in \mathrm{H}^{2,0}(X)$ with $\int_X \sigma\bar{\sigma} = 1$ and $n > k$ we have

$$\beta_\gamma(A^{[[n]]}, \sigma^{[[n]]}) = a_\gamma n. \tag{91}$$

Proof. Let (X, σ) be a K3 surface or an abelian surface together with a symplectic form with $\int_X \sigma\bar{\sigma} = 1$. Let W_{2k} be the homogeneous component of degree $2k$ of $W(1)$. Then $W(t) = \sum_{k=0}^\infty W_{2k} t^k$. Thus we have by (88):

$$h^X_{\langle W(t)\rangle}(n) = \sum_{k=0}^\infty h^X_{\langle W_{2k}\rangle}(n) t^k = U_{c_2(X)}(t) \exp(nV(t)) \tag{92}$$

with $U_{c_2(X)}(t) := \exp(c_2(X) D(t))$ and $V(t) := \log A(t)$.
Let us consider the case of a K3 surface X first. Note that $c_2(X) = 24$. By definition of $h^X_\gamma(n)$ we have

$$\beta_{\langle W_{2k}\rangle}(X^{[n]}, \sigma^{[n]}) = h^X_{\langle W_{2k}\rangle}(n-k) \tag{93}$$

for all $n \geq k$. For $n < k$ we take this equation as a definition for its left hand side. Let the power series $T(t) \in \mathbb{Q}[a_2, a_4, \ldots][[t]]$ be defined by $T(t\exp(V(t))) = t$, and set $\tilde{V}(t) := V(T(t))$ and $\tilde{U} := \frac{U_{24}(T(t))}{1+T(t)V'(T(t))}$. By Lemma 1, we have

$$\beta_{\langle W(t)\rangle}(X^{[n]}, \sigma^{[n]}) = \sum_{k=0}^\infty h^X_{\langle W_{2k}\rangle}(n-k) t^k = \tilde{U}(t) \exp(n\tilde{V}(t)).$$

Note that $W(t)$ is of the form $\exp(\gamma)$ where γ is a connected graph. By Proposition 6 and Remark 12 we therefore have

$$\beta_{\langle\langle W(t)\rangle\rangle}(X^{[n]}, \sigma^{[n]}) = \beta_{\log\langle W(t)\rangle}(X^{[n]}, \sigma^{[n]}) = \log \beta_{\langle W(t)\rangle} = n\tilde{V}(t) + \log U(t).$$

Finally, let λ be any partition. Setting

$$\partial_{2\lambda} := \left.\left(\prod_{i=1}^\infty \left.\frac{\partial^{\lambda_i}}{\partial a_i^{\lambda_i}}\right|_{a_i=0}\right)\right|_{t=0}.$$

It is

$$\beta_{\langle\langle \tilde{w}_{2\lambda}\rangle\rangle} = \partial_{2\lambda} \beta_{\langle\langle W(t)\rangle\rangle} = n\partial_{2\lambda}\tilde{V}(t) + \partial_{2\lambda} \log \tilde{U}(t),$$

so the theorem is proven for K3 surfaces and all connected graph homology classes of the form $\langle\langle \tilde{w}_{2\lambda}\rangle\rangle$ and thus for all connected graph homology classes in $\mathcal{C}$.

Let us now turn to the case of a generalised Kummer variety, i.e. let $X = A$ be an abelian surface and $n \geq 1$. Note that $c_2(A) = 0$. Here, we have due to Proposition 21:

$$\beta_{\langle W_{2k}\rangle}(A^{[[n]]}, \sigma^{[[n]]}) = \frac{n}{n-k} h^A_{\langle W_{2k}\rangle}(n-k)$$

for $n > k$. For $n \leq k$ we take this equation as a definition for its left hand side. As $U_0(t) = 1$, Lemma 1 yields in this case that

$$\beta_{\langle W(t)\rangle}(A^{[[n]]}, \sigma^{[[n]]}) = \sum_{k=0}^{\infty} \frac{n}{n-k} h^X_{\langle W_{2k}\rangle}(n-k)t^k = \exp n\tilde{V}(t).$$

We can then proceed as in the case of the Hilbert scheme of a K3 surface to finally get

$$\beta_{\langle\langle \tilde{w}_{2\lambda}\rangle\rangle} = n\partial_{2\lambda}(\tilde{V}(t)).$$

□

6.2. SOME EXPLICIT CALCULATIONS. Now, we'd like to calculate the constants a_γ and c_γ for any homogeneous connected closed graph homology class γ of degree $2k$ lying in $\mathcal{C}$. By the previous theorem, we can do this by calculating β_γ on (X, σ) for (X, σ) being the $2k$-dimensional Hilbert scheme of points on a K3 surface and the $2k$-dimensional generalised Kummer variety.
We can do this by recursion over k: Let the calculation having been done for homogeneous connected closed graph homology classes γ of degree less than $2k$ in $\mathcal{C}$ and both example series.
Let λ be any partition of k. We can express $\langle\langle \tilde{w}_{2\lambda}\rangle\rangle$ as

$$\langle\langle \tilde{w}_{2\lambda}\rangle\rangle = \langle \tilde{w}_{2\lambda}\rangle + P, \tag{94}$$

where P is a polynomial in homogeneous connected closed graph homology classes γ of degree less than $2k$ in $\mathcal{C}$ (for this see Proposition 6). Therefore, $\beta_{\langle\langle \tilde{w}_{2\lambda}\rangle\rangle}(X, \sigma)$ is given by

$$\beta_{\langle\langle \tilde{w}_{2\lambda}\rangle\rangle}(X, \sigma) = \beta_{\langle \tilde{w}_{2\lambda}\rangle}(X, \sigma) + P', \tag{95}$$

where P' is a polynomial in terms like $\beta_{\gamma'}(X, \sigma)$ with $\gamma' \in \mathcal{C}$ and $\deg \gamma' < 2k$. However, these terms have been calculated in previous recursion steps. Therefore, the only thing new we have to calculate in this recursion step is $\beta_{\langle \tilde{w}_{2\lambda}\rangle}(X, \sigma)$. We have:

$$\beta_{\langle \tilde{w}_{2\lambda}\rangle}(X, \sigma) = \frac{1}{k!} \frac{\int_X \mathrm{RW}_{X,\sigma}(\tilde{w}_{2\lambda}) \exp(\sigma + \bar{\sigma})}{\int_X \exp(\sigma + \bar{\sigma})} = \frac{\int_X s_{2\lambda}(X)}{\int_X \exp{(\sigma + \bar{\sigma})}}. \tag{96}$$

As all the Chern numbers of X can be computed with the help of Bott's residue formula (see [3] for the case of the Hilbert scheme and [11] for the case of the generalised Kummer variety), we therefore are able to calculate $\beta_{\langle \tilde{w}_{2\lambda}\rangle}(X, \sigma)$. This ends the recursion step as we have given an algorithm to compute a_γ and c_γ for any homogeneous connected closed graph homology class γ of degree $2k$ in $\mathcal{C}$.

We worked through the recursion for $k = 1, 2, 3$. Firstly, we have

$$
\begin{aligned}
\langle\langle \tilde{w}_2 \rangle\rangle &= \langle \tilde{w}_2 \rangle \\
\left\langle\left\langle \tilde{w}_2^2 \right\rangle\right\rangle &= \langle \tilde{w}_4 \rangle - \langle\langle \tilde{w}_2 \rangle\rangle^2 \\
\langle\langle \tilde{w}_4 \rangle\rangle &= \langle \tilde{w}_4 \rangle \\
\left\langle\left\langle \tilde{w}_2^3 \right\rangle\right\rangle &= \left\langle \tilde{w}_2^3 \right\rangle - 3 \langle\langle \tilde{w}_2 \rangle\rangle \left\langle\left\langle \tilde{w}_2^2 \right\rangle\right\rangle - \langle\langle \tilde{w}_2 \rangle\rangle^3 \\
\langle\langle \tilde{w}_2 \tilde{w}_4 \rangle\rangle &= \langle \tilde{w}_2 \tilde{w}_4 \rangle - \langle\langle \tilde{w}_2 \rangle\rangle \langle\langle \tilde{w}_4 \rangle\rangle \\
\langle\langle \tilde{w}_6 \rangle\rangle &= \langle \tilde{w}_6 \rangle .
\end{aligned}
\tag{97}
$$

Not let X be a K3 surface and A an abelian surface. Let us denote by σ either a holomorphic symplectic two-form on X with $\int_X \sigma\bar{\sigma} = 1$ or on A with $\int_A \sigma\bar{\sigma} = 1$. We use the following table of Chern numbers for the Hilbert scheme of points on a K3 surface:

k	s	$s[X^{[k]}]$	$s[A^{[[k+1]]}]$
1	s_2	-48	-48
2	s_2^2	3312	3024
	s_4	360	1080
3	s_2^3	-294400	-241664
	$s_2 s_4$	-29440	-66560
	s_6	-4480	-22400

Going through the recursion, we arrive at the following table:

k	γ	$\beta_\gamma(A^{[[k+1]]})$	$\beta_\gamma(X^{[k]})$	a_γ	c_γ
1	$\langle\langle \tilde{w}_2 \rangle\rangle$	-24	-48	12	-36
2	$\left\langle\left\langle \tilde{w}_2^2 \right\rangle\right\rangle$	-288	-288	-96	-96
	$\langle\langle \tilde{w}_4 \rangle\rangle$	360	360	120	120
3	$\left\langle\left\langle \tilde{w}_2^3 \right\rangle\right\rangle$	-5120	-4096	-1280	-256
	$\langle\langle \tilde{w}_2 \tilde{w}_4 \rangle\rangle$	6400	5120	1600	320
	$\langle\langle \tilde{w}_6 \rangle\rangle$	-5600	-4480	-1400	-280

Now, we would like to turn to Rozansky-Witten *invariants*: Let γ be any homogeneous closed graph homology class of degree $2k$. For any holomorphic symplectic manifold (X, σ) of dimension $2n$, the associated Rozansky-Witten invariant is given by

$$
\begin{aligned}
b_\gamma(X, \sigma) = \int_X \mathrm{RW}_{X,\sigma}(\gamma) \exp(\sigma + \bar{\sigma}) &= \frac{1}{n!(n-k)!} \beta_\gamma(X, \sigma) \int_X (\sigma\bar{\sigma})^n \\
&= \frac{n!}{(n-k)!} \beta_\gamma(X, \sigma) \int_X \exp(\sigma + \bar{\sigma}).
\end{aligned}
\tag{98}
$$

To know the Rozansky-Witten invariant associated to closed graph homology classes, we therefore have just to calculate the value of β_γ. On an irreducible holomorphic symplectic manifold, $\gamma \mapsto \beta_\gamma$ is multiplicative with respect to the disjoint union of graphs, so it is enough to calculate β_γ for connected closed graph homology classes. However, we have just done this for the Hilbert

schemes of points on a K3 surface and the generalised Kummer varieties — as long as γ is spanned by the connected polywheels.
By the procedure outlined above, Theorem 3 therefore enables us to compute all Rozansky-Witten invariants of the two example series associated to closed graph homology classes lying in $\mathcal{C}$.

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