Separatedness in Constructive Topology

Douglas Bridges and Luminiţa Vîţă

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Abstract. We discuss three natural, classically equivalent, Hausdorff separation properties for topological spaces in constructive mathematics. Using Brouwerian examples, we show that our results are the best possible in our constructive framework.

1 Introduction

A typical feature of constructive mathematics—that is, mathematics with intuitionistic logic [1, 2, 3, 4, 10]—is that a classical property may have several constructively inequivalent counterparts. In this paper we describe such counterparts of the notion of a Hausdorff space, examine their interconnections, and, by means of Brouwerian examples, show that our results cannot be improved without some additional, nonconstructive principles.

The original impetus for our work came from the constructive theory of apartness (point–set [5, 11] and set–set [9]). However, in order to make the work below accessible to anyone familiar with only the most basic notions of topology, we have chosen to work with the usual notion of topological space. Note, however, that we require a topological space to be equipped from the outset with an inequality relation $\neq$ satisfying the following properties for all x and y:

$$x \neq y \Rightarrow \neg (x = y)\,,$$
$$x \neq y \Rightarrow y \neq x.$$

We then denote the complement of a subset S of X by

$$\sim S = \{x \in X : \forall y \in S\,(x \neq y)\}\,.$$

On the other hand, the apartness complement of S defined to be

$$-S = (\sim S)^{\circ}\,.$$

If

$$\forall x, y\,(\neg\,(x \neq y) \Rightarrow x = y)\,,$$

then we say that the inequality is TIGHT.

We note here, for future reference, that a topological space X is

- TOPOLOGICALLY COTRANSITIVE if

$$(x \in U \wedge U \in \tau) \Rightarrow \forall y \in X\,(x \neq y \vee y \in U)\,;$$

- LOCALLY DECOMPOSABLE if

$$(x \in U \wedge U \in \tau) \Rightarrow \exists V \in \tau\,(x \in V \wedge X = U \cup \sim\! V)\,.$$

Note that local decomposability implies topological cotransitivity. In the constructive theory of point–set apartness spaces, topological cotransitivity is postulated, and local decomposability is an extremely valuable property to have [6, 7]. For example, local decomposability ensures that if an apartness relation is induced by a topology, then the natural topology produced by the apartness relation coincides with the original topology. So important is local decomposability that it is actually *postulated* as a property of an apartness relation between sets [6].

We will need some basic facts about nets in constructive topology. By a DIRECTED SET we mean a nonempty set D with a preorder[1] $\succcurlyeq$ such that for all $m, n \in D$ there exists $p \in D$ with $p \succcurlyeq m$ and $p \succcurlyeq n$. If (X, τ) is a topological space, then to each x in X there corresponds a special net defined as follows. Let

$$D_x = \{(\xi, U) \in X \times \tau : x \in U \wedge \xi \in U\}\,,$$

with equality defined by

$$(\xi, U) = (\xi', U') \Leftrightarrow (\xi = \xi' \wedge U = U')\,,$$

and for each $n = (\xi, U)$ in D_x define $x_n = \xi$. It is easy to see that D is a directed set under the INCLUSION PREORDER defined by

$$(\xi, U) \succcurlyeq (\xi', U') \Leftrightarrow U \subset U',$$

so that $\mathcal{N}_x = (x_n)_{n \in D_x}$ is a net—the BASIC NEIGHBOURHOOD NET of x.

We say that a net $(x_n)_{n \in D}$ in X CONVERGES to a LIMIT x in X if

$$\forall U \in \tau\ (x \in U \Rightarrow \exists\, n_0 \in D\ \forall n \succcurlyeq n_0\ (x_n \in U))\,.$$

[1]The classical theory of nets requires a partial order. If we used a partial order in our constructive theory, we would run into difficulties which the classical theory avoids by applications of the axiom of choice.

2 Hausdorff and unique limit properties

Let (X, τ) be a topological space. More or less as in classical topology, we say that X is HAUSDORFF, or SEPARATED, if it satisfies the following condition:

H *If* $x, y \in X$ *and* $x \neq y$, *then there exist* $U, V \in \tau$ *such that* $x \in U, y \in V$, *and* $U \subset \sim V$

In that case, $V \subset \sim U$.
Classically, being Hausdorff is equivalent to having the UNIQUE LIMITS PROPERTY:

ULP *If* $(x_n)_{n \in D}$ *is a net converging to limits* x *and* y *in* X, *then* $x = y$.

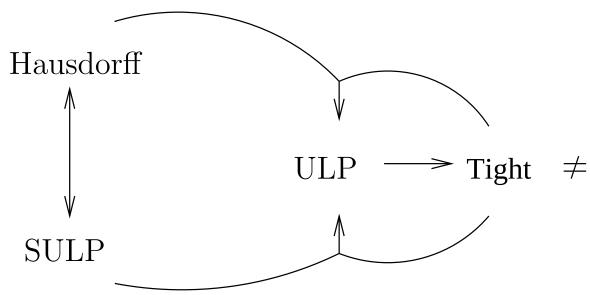
We say that a point y in X is EVENTUALLY BOUNDED AWAY FROM A NET $(x_n)_{n \in D}$ in X if there exists $n_0 \in D$ such that

$$y \in -\{x_n : n \succcurlyeq n_0\}.$$

From a constructive viewpoint, the unique limits property appears rather weak; of more likely interest is the (classically equivalent) STRONG UNIQUE LIMITS PROPERTY:

SULP *If* $(x_n)_{n \in D}$ *is a net in* X *that converges to a limit* x, *and if* $x \neq y \in X$, *then* $(x_n)_{n \in D}$ *is eventually bounded away from* y.

In this section we investigate constructively the connection between these two uniqueness properties and condition H. Specifically, we prove that the following diagram of implications occurs:



We first have an elementary, but useful, lemma.

LEMMA 1 *Let* X *be a topological space,* x *a point of* X, *and* $\nu = (\xi, U) \in D_x$. *Then*

$$U = \{x_n : n \in D_x, n \succcurlyeq \nu\}. \tag{1}$$

Proof. Consider $n \in D_x$. If $n = (x_n, W) \succcurlyeq \nu$, then $x_n \in W \subset U$. Hence $\{x_n : n \succcurlyeq \nu\} \subset U$. On the other hand, for each $y \in U$ we have $(y, U) \succcurlyeq \nu$, so $y \in \{x_n : n \succcurlyeq \nu\}$. ∎

Proposition 2 *A topological space is Hausdorff if and only if it has the strong unique limits property.*

Proof. Let (X, τ) be a topological space. Assume first that X is Hausdorff, let $(x_n)_{n \in D}$ be a net converging to a limit x in X, and let $x \neq y$ in X. Choose $U, V \in \tau$ such that $x \in U, y \in V$, and $U \subset\sim V$. There exists n_0 such that $x_n \in U$ for all $n \succcurlyeq n_0$. Then

$$y \in V \subset\sim U \subset\sim \{x_n : n \succcurlyeq n_0\},$$

so

$$y \in \left(\sim \{x_n : n \succcurlyeq n_0\}\right)^\circ = -\{x_n : n \succcurlyeq n_0\}.$$

Now suppose that X has the strong unique limits property, and let x, y be points of X with $x \neq y$. Since the net $\mathcal{N}_x$ converges to x, there exist $n_0 = (\xi, U) \in D_x$ and $V \in \tau$ such that

$$y \in V \subset\sim \{x_n : n \in D_x,\ n \succcurlyeq n_0\}.$$

By Lemma 1,

$$U = \{x_n : n \in D_x,\ n \succcurlyeq n_0\}.$$

Thus $x \in U,\ y \in V$, and $V \subset\sim U$; so X is Hausdorff. ∎

Corollary 3 *A Hausdorff space with tight inequality has the unique limits property.*

Proof. Let $(x_n)_{n \in D}$ be a net converging to limits x, y in a Hausdorff space X with tight inequality. If $x \neq y$, then we obtain a contradiction from Proposition 2. Hence $\neg (x \neq y)$, and so, by tightness, $x = y$. ∎

By a T_1–space we mean a topological space (X, τ) with the property

$$x \neq y \Rightarrow \exists U \in \tau \left(x \in U \subset \sim\{y\}\right).$$

The following lemma enables us to prove a partial converse to Corollary 3.

Lemma 4 *Let X be a topological space. If X is a T_1–space with tight inequality, then*

$$\forall x, y \in X \left(y \in \overline{\{x\}} \Rightarrow x = y\right). \tag{2}$$

Conversely, if (2) holds and X is topologically cotransitive, then the inequality on X is tight.

Proof. Suppose that X is a T_1–space with tight inequality, let $y \in \overline{\{x\}}$, and assume that $x \neq y$. Then there exists $U \in \tau$ such that $x \in U \subset \sim\{y\}$; whence $y \notin \overline{\{x\}}$, which is absurd. Thus $\neg\,(x \neq y)$ and therefore, by tightness, $x = y$. Conversely, suppose that (2) holds and that X is topologically cotransitive. Let $\neg\,(x \neq y)$. For each $U \in \tau$ with $x \in U$, the topological cotransitivity of X implies that either $x \neq y$ or else $y \in U$; the former alternative is ruled out, so we must have $y \in U$. Hence $y \in \overline{\{x\}}$ and so, by (2), $x = y$. ■

Proposition 5 *In a topological space with the unique limits property the inequality is tight.*

Proof. Let X be a topological space with the unique limits property, and suppose that $y \in \overline{\{x\}}$. Then every open set containing x contains y. Let

$$L = \{(z, U, V) : U, V \in \tau,\ x \in U,\ y \in V,\ z \in U \cap V\},$$

where

$$(z, U, V) = (z', U', V') \Leftrightarrow z = z' \wedge U = U' \wedge V = V'.$$

Define a binary relation $\succcurlyeq$ on L by

$$(z_1, U_1, V_1) \succcurlyeq (z_2, U_2, V_2) \Longleftrightarrow U_1 \subset U_2.$$

It is easy to show that L is directed with respect to this binary relation. For each $n = (z, U, V)$ in L define $x_n = z$. Then the net $(x_n)_{n \in L}$ converges to both x and y in X; whence $x = y$. It follows from Lemma 4 that the inequality on X is tight. ■

3 Limiting examples

In this section we show that the connections (summarised in the diagram presented earlier) we have established between the Hausdorff condition, the unique limits property, and the strong unique limits property are the best possible within our constructive framework. We begin by showing that Hausdorff is not enough to establish tightness.

Proposition 6 *If every topologically cotransitive topological space with the unique limits property has tight inequality, then the law of excluded middle holds in the weak form* $(\neg\neg P \Rightarrow P)$.

Proof. Let P be any syntactically correct statement such that $\neg\neg P$ holds, and take $X = \{0, 1, 2\}$ with equality satisfying

$$0 = 1 \Leftrightarrow P$$

and inequality given by

$$0 \neq 2,\ 1 \neq 2, \text{ and } (0 \neq 1 \Leftrightarrow \neg P)\,.$$

Define a topology τ on X by taking the basic open sets to be the complements of subsets of X. To see that X has the topological cotransitivity property, consider all possible cases that arise when $x \in U$ and $U \in \tau$. We may assume that $U = \sim S$ for some $S \subset X$. If $x = 0$, then $0 \in \sim S$. It follows that $S \subset \{2\}$: for if $s \in S$, then either $s = 1$ or $s = 2$; in the former case, $0 \neq 1$ and therefore $\neg P$, which contradicts our hypotheses. Since $1 \neq 2$, we have $1 \in \sim S$; since also $0 \neq 2$, we conclude that

$$\forall y \in X \left(0 \neq y \vee y \in \sim S = U\right).$$

The case $x = 1$ is similar, and the case $x = 2$ is even easier to handle.
We claim that X is a Hausdorff apartness space. If $x \neq y$, then without loss of generality, either $x = 0$ and $y = 2$ or else $x = 1$ and $y = 2$. Taking, for illustration, the former case, we have $0 \in \{0,1\} = \sim\{2\}$, $2 \in \{2\} = \sim\{0\}$, and $\sim\{2\} = \sim\sim\{0\}$ .Thus there exist $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \subset \sim V$.
Finally, if the inequality on X is tight, then as $\neg\left(0 \neq 1\right)$, we have $0 = 1$ and therefore P. ■

Our final proposition shows that, even when the inequality is tight and certain additional hypotheses hold, the strong unique limits property does not entail being Hausdorff. For the proof we introduce a strange lemma and a general construction. The lemma may seem obvious, but in fact we have to be careful to avoid the axiom of choice, which implies the law of excluded middle [8].

Lemma 7 *Let $\mathcal{C}$ be a class of subsets of a set X, and let $(S_i)_{i\in I}$ be a family of subsets of X such that for each i, if $S_i \neq \emptyset$, then S_i is a union of sets in $\mathcal{C}$. If $S = \bigcup_{i\in I} S_i \neq \emptyset$, then S is also a union of sets in $\mathcal{C}$.*

Proof. For each $x \in X$ define

$$I_x = \{i \in I : x \in S_i\}.$$

Then

$$S = \bigcup_{x\in X} \bigcup_{i\in I} S_i.$$

If $x \in S$ and $i \in I_x$, then $S_i \neq \emptyset$ and so is a union of sets in $\mathcal{C}$. Hence S itself is such a union. ■

Let X be a set with a nontrivial inequality $\neq$. We say that a subset S of X is

- finitely enumerable (respectively, finite) if there exist a natural number n and a mapping (respectively, one–one mapping) f of $\{1, \ldots, n\}$ onto S;
- cofinite if it is the complement of a finitely enumerable subset.

Note that the empty set is finitely enumerable, so X is cofinite. Also, if S is finitely enumerable, then either $S = \emptyset$ or else $S \neq \emptyset$ (that is, there exists an element of S).

We define the COFINITE TOPOLOGY on X to be

$$\tau = \{S \subset X : S \neq \emptyset \Rightarrow S \text{ is a union of cofinite sets}\}.$$

To see that this is a topology, note that (as above) $X \in \tau$ and, by *ex falso quodlibet*, $\emptyset \in \tau$. The unions axiom for a topology is an immediate consequence of Lemma 7 with $\mathcal{C} = \tau$. To verify the intersections axiom, let $(S_i)_{i\in I}$ and $(T_j)_{j\in J}$ be families of sets in τ. For all i, j choose finitely enumerable subsets A_i, B_j of X such that $S_i = \sim A_i$ and $T_j = \sim B_j$. Then

$$\begin{aligned}\left(\bigcup\nolimits_{i\in I} S_i\right) \cap \left(\bigcup\nolimits_{j\in J} T_j\right) &= \left(\bigcup\nolimits_{i\in I} \sim A_i\right) \cap \left(\bigcup\nolimits_{j\in J} \sim B_j\right) \\ &= \bigcup\nolimits_{i\in I} \bigcup\nolimits_{j\in J} (\sim A_i \cap \sim B_j) \\ &= \bigcup\nolimits_{i\in I} \bigcup\nolimits_{j\in J} \sim (A_i \cup B_j),\end{aligned}$$

where each $A_i \cup B_j$ is cofinite.

We now recall MARKOV'S PRINCIPLE,

> *For every binary sequence $(a_n)_{n=1}^{\infty}$ such that $\neg\forall n\, (a_n = 1)$, there exists n such that $a_n = 0$,*

a form of unbounded search that is well–known to be independent of Heyting arithmetic (Peano arithmetic with intuitionistic logic) and is therefore generally regarded as essentially nonconstructive.

PROPOSITION 8 *If every locally decomposable T_1–space with the unique limits property and tight inequality is Hausdorff, then Markov's Principle holds.*

PROOF. We take a specific case of the foregoing construction. Let $(a_n)_{n=1}^{\infty}$ be a decreasing binary sequence such that $a_1 = 1$ and $\neg\forall n\, (a_n = 1)$. Take

$$X = \{0\} \cup \left\{\frac{a_n}{n} : n = 1, 2, 3, \ldots\right\}$$

with the discrete inequality, and let τ be the cofinite topology on X. To show that X is a T_1–space, let $x \neq y$ in X. Either one of x, y is 0 or else both are nonzero. If, for example, $x = 0$, then $y = 1/n$ for some n with $a_n = 1$. Writing

$$U = \{0\} \cup \left\{\frac{a_k}{k} : k > n\right\} = \sim\left\{\frac{a_k}{k} : k \leq n\right\},$$

we see that

$$U \in \tau \text{ and } x \in U \subset \sim\{y\}. \tag{3}$$

So we are left with the case where $x = 1/m$ and $y = 1/n$, with $a_m = a_n = 1$. In this case, without loss of generality taking $m > n$, we obtain (3) by defining

$$U = \{0\} \cup \left\{ \frac{a_k}{k} : k \geq m \right\}.$$

To show that X is locally decomposable, again consider $x \in X$ and $U \in \tau$ with $x \in U$. We may assume that $U = \sim A$ for some finitely enumerable set $A \subset X$; without loss of generality, $A \neq \emptyset$. Consider first the case $x = 0$. Let

$$K = \max \left\{ k : \frac{1}{k} \in A \right\}$$

and

$$V = \{0\} \cup \left\{ \frac{a_k}{k} : k > K \right\} = \sim \left\{ \frac{a_k}{k} : k \leq K \right\}.$$

Then V is a neighbourhood of 0. For each $y \in X$, either $y = 0 \in U$ or else $y = 1/k$ for some k with $a_k = 1$. In the latter case, if $k > K$, then $y \in \sim A = U$; whereas if $k \leq K$, then $y \in \sim V$. This deals with the case $x = 0$. Now consider the case where $x = 1/m$ for some m with $a_m = 1$. Again let $U = \sim A$ be an open neighbourhood of x, where A is finitely enumerable. If $a_{m+1} = 0$, then X is finite and hence locally decomposable; so we may assume that $a_{m+1} = 1$. Without loss of generality we may further assume that $1/(m+1) \in A$. Thus

$$L = \max \left\{ n : \frac{1}{n} \in A \right\} > m.$$

Set

$$V = \left\{ \frac{a_k}{k} : (k > L \wedge a_k = 1) \vee k = m \right\} = \sim \left(\{0\} \cup \left\{ \frac{a_k}{k} : k \leq L, k \neq m \right\} \right).$$

Then V is a neighbourhood of x. For each $y \in X$, either $y = 0$ and hence $y \in \sim V$, or else $y = a_k/k$ for some k with $a_k = 1$. If $k > L$, then $y \in \sim A = U$; if $k = m$, then $y \in U$; if $k \leq L$ and $k \neq m$, then $y \in \sim V$.

Next, we prove that X has the unique limits property. To this end, suppose that $(x_n)_{n \in D}$ is a net in X that converges to both x and y. Suppose also that $x \neq y$. For each n, if $a_n = 0$, then X is finitely enumerable and so has the unique limits property; whence $x = y$, a contradiction. Thus $a_n = 1$ for all n, which is also a contradiction. We conclude that $\neg(x \neq y)$; since we are dealing with a discrete inequality, it follows that $x = y$.

Finally, noting that $0 \neq 1$, suppose there exist U, V in τ such that $0 \in U, 1 \in V$, and $U \cap V = \emptyset$. There exist finitely enumerable sets $A, B \subset X$ such that $0 \in \sim A \subset U$ and $1 \in \sim B \subset V$. Let

$$N = \max \left\{ n : \frac{1}{n} \in A \cup B \right\}.$$

Then

$$\left\{ \frac{a_k}{k} : k > N \right\} \subset \sim A \cap \sim B \subset U \cap V.$$

If $a_{N+1} = 1$, then $U \cap V \neq \emptyset$, a contradiction. Hence $a_{N+1} = 0$. ■

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Douglas Bridges
Department of Mathematics
& Statistics
University of Canterbury
Private bag 4800
Christchurch New Zealand
d.bridges@math.canterbury.ac.nz

Luminiţa Vîţă
Department of Mathematics
& Statistics
University of Canterbury
Private bag 4800
Christchurch New Zealand
l.vita@math.canterbury.ac.nz