On the scattering theory of the Laplacian with a periodic boundary condition. I. Existence of wave operators

Rupert L. Frank

Received: December 12, 2003

Communicated by Heinz Siedentop

Abstract. We study spectral and scattering properties of the Laplacian $H^{(\sigma)} = -\Delta$ in $L_2(\mathbb{R}^2_+)$ corresponding to the boundary condition $\frac{\partial u}{\partial \nu} + \sigma u = 0$ for a wide class of periodic functions σ. The Floquet decomposition leads to problems on an unbounded cell which are analyzed in detail. We prove that the wave operators $W_\pm(H^{(\sigma)}, H^{(0)})$ exist.

2000 Mathematics Subject Classification: Primary 35J10; Secondary 35J25, 35P05, 35P25.
Keywords and Phrases: Scattering theory, periodic operator, Schrödinger operator, singular potential.

Introduction

0.1 Setting of the problem

The present paper studies the Laplacian

$$H^{(\sigma)}u = -\Delta u \qquad \text{on } \mathbb{R}^2_+ \tag{0.1}$$

on the halfplane together with a boundary condition of the third type

$$\frac{\partial u}{\partial \nu} + \sigma u = 0 \qquad \text{on } \mathbb{R} \times \{0\}, \tag{0.2}$$

where ν denotes the exterior unit normal and where the function $\sigma : \mathbb{R} \to \mathbb{R}$ is assumed to be 2π-periodic. Moreover, let

$$\sigma \in L_{q,loc}(\mathbb{R}) \qquad \text{for some } q > 1.$$

Under this condition $H^{(\sigma)}$ can be defined as a self-adjoint operator in $L_2(\mathbb{R}^2_+)$ by means of the closed and lower semibounded quadratic form

$$\int_{\mathbb{R}^2_+} |\nabla u(x)|^2\,dx + \int_{\mathbb{R}} \sigma(x_1)|u(x_1,0)|^2\,dx_1, \qquad u \in H^1(\mathbb{R}^2_+).$$

This is the first part of a paper where we analyze the spectrum of $H^{(\sigma)}$ and develop a scattering theory viewing $H^{(\sigma)}$ as a (rather singular) perturbation of $H^{(0)}$, the Neumann Laplacian on $\mathbb{R}^2_+$. (For the abstract mathematical scattering theory see, e.g., [Ya].)
The main result of the present paper is that the wave operators

$$W_\pm^{(\sigma)} := W_\pm(H^{(\sigma)}, H^{(0)})$$

exist.

0.2 Physical interpretation

In the physical interpretation, $H^{(\sigma)}$ is the Hamiltonian of a two-dimensional quantum-mechanical system which consists of a particle in the upper halfplane and a crystal that fills the lower halfplane. The particle can not enter the crystal but interacts non-trivially with the surface of the crystal, described by the function σ. The existence of the wave operators means that every particle which is described by a state $u \in \mathcal{R}\left(W_\pm^{(\sigma)}\right)$ behaves like a free particle in the distant future and the distant past. We emphasize that there may also exist particles which are described by a state $u \in \mathcal{R}\left(W_\pm^{(\sigma)}\right)^\perp$. These are surface states which propagate along the boundary and decay exponentially away from the boundary. Such surface states will be investigated in the second part [FrSh] of the paper.

0.3 Outline of the paper

Let us explain some of the mathematical ideas involved. A precise definition of the operator $H^{(\sigma)}$ in terms of a quadratic form is given in Subsection 1.4. By means of the Bloch-Floquet theory we represent $H^{(\sigma)}$ in Subsection 2.2 as a direct integral

$$\int_{-1/2}^{1/2} \bigoplus H^{(\sigma)}(k)\,dk$$

with fiber operators $H^{(\sigma)}(k)$ acting in $L_2(\Pi)$ where $\Pi := (-\pi,\pi)\times\mathbb{R}_+$ is the halfstrip. Functions in the domain of $H^{(\sigma)}(k)$ satisfy the third type condition (0.2) on $(-\pi,\pi)\times\{0\}$ (at least if σ is smooth), so $H^{(\sigma)}(k)$ differs from $H^{(0)}(k)$ by a relatively compact form perturbation. This makes a rather detailed analysis of the operators $H^{(\sigma)}(k)$ possible.
Our approach leans on a quadratic form version of the resolvent identity which

we present in Subsection 3.2 following [Ya]. A similar approach has been successfully applied to study periodic Schrödinger operators (cf. [BShSu]). In our case it allows to show that the difference of resolvents of $H^{(\sigma)}(k)$ and $H^{(0)}(k)$ belongs to the trace class, and from the Birman-Kreĭn theorem (which is sometimes called Birman-Kuroda theorem, unaware of [BKr]) we deduce in Subsection 3.4 the existence and completeness of the wave operators on the halfstrip. Using the same representation we can prove a limiting absorption principle in Subsection 3.6, which implies the absence of singular continuous spectrum.
The existence of the wave operators $W_{\pm}^{(\sigma)}$ on the halfplane is derived from the existence of the wave operators on the halfstrip.

0.4 Acknowledgements

The author is deeply grateful to Prof. Birman for the setting of the problem, useful discussions and constant attention to the work. He would like to thank Dr. R. G. Shterenberg for numerous consultations and the help in the preparation of the present paper. He is also deeply grateful to Prof. T. A. Suslina for a systematic help with the theory of embedding theorems and boundary value problems. The author appreciates the hospitality of the KTH Stockholm and the Mittag-Leffler Institute where the results were obtained. Partial financial support of the German Merit Foundation and of the European Union through the IHP network HPRN-CT-2002-00277 is gratefully acknowledged.

1 Setting of the problem. The main result

1.1 Notation

We introduce the *halfplane*

$$\mathbb{R}^2_+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : \ x_2 > 0\} = \mathbb{R} \times \mathbb{R}_+,$$

and the *halfstrip*

$$\Pi := \{x = (x_1, x_2) \in \mathbb{R}^2_+ : \ -\pi < x_1 < \pi, \ x_2 > 0\} = (-\pi, \pi) \times \mathbb{R}_+,$$

where $\mathbb{R}_+ := (0, +\infty)$. Moreover, we need the lattice $2\pi\mathbb{Z}$. Unless stated otherwise, periodicity conditions are understood with respect to this lattice. We think of the corresponding torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ as the interval $[-\pi, \pi]$ with endpoints identified.
We use the notation $D = (D_1, D_2) = -i\nabla$ in $\mathbb{R}^2$.
For an open set $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, the index in the notation of the norm $\|.\|_{L_2(\Omega)}$ is usually dropped. The space $L_2(\mathbb{T})$ may be formally identified with $L_2(-\pi, \pi)$. We define the (discrete) Fourier transformation $\mathcal{F} : L_2(\mathbb{T}) \to l_2(\mathbb{Z})$ by

$$(\mathcal{F}f)_n = \hat{f}_n := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x_1) e^{-inx_1} \, dx_1. \qquad n \in \mathbb{Z},$$

Next, for an open set $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, $H^s(\Omega)$ is the Sobolev space of order $s \in \mathbb{R}$ (with integrability index 2). By $H^s(\mathbb{T})$ we denote the closure of $C^\infty(\mathbb{T})$ in $H^s(-\pi,\pi)$. Here $C^\infty(\mathbb{T})$ is the space of functions in $C^\infty(-\pi,\pi)$ which can be extended 2π-periodically to functions in $C^\infty(\mathbb{R})$. The space $H^s(\mathbb{T})$ is endowed with the norm

$$\|f\|^2_{H^s(\mathbb{T})} := \sum_{n\in\mathbb{Z}} (1+n^2)^s |\hat{f}_n|^2, \qquad f \in H^s(\mathbb{T}).$$

By $\tilde{H}^s(\Pi)$ we denote the closure of $\tilde{C}^\infty(\Pi) \cap H^s(\Pi)$ in $H^s(\Pi)$. Here $\tilde{C}^\infty(\Pi)$ is the space of functions in $C^\infty(\Pi)$ which can be extended 2π-periodically with respect to x_1 to functions in $C^\infty(\mathbb{R}^2_+)$.
Statements and formulae which contain the double index "$\pm$" are understood as two independent assertions.

1.2 Scattering theory

Here we summarize the definitions and basic results on scattering theory. For proofs we refer to [Ya].
Let H_0, H be self-adjoint operators in a Hilbert space $\mathfrak{H}$. The projection onto the absolutely continuous subspace of H_0 and the unitary group of H_0 are denoted by P_0 and $U_0(t) := \exp(-itH_0)$, respectively. We put $\mathfrak{H}_0^{(ac)} := \mathcal{R}(P_0)$. For the similar objects related to the operator H we omit the index "0".
In case of existence, the limit

$$W_\pm(H,H_0) := s - \lim_{t\to\pm\infty} U(-t)U_0(t)P_0$$

is called the *wave operator* for the pair H, H_0 and the sign $\pm$. Thus the elements $u = W_\pm(H,H_0)u_0^\pm \in \mathcal{R}(W_\pm(H,H_0))$, $u_0^\pm \in \mathfrak{H}_0^{(ac)}$, satisfy

$$\lim_{t\to\pm\infty} \|U(t)u - U_0(t)u_0^\pm\| = 0.$$

The wave operators are partial isometries with initial subspace $\mathfrak{H}_0^{(ac)}$. One easily establishes the *intertwining property*

$$W_\pm(H,H_0)\, H_0 = H\, W_\pm(H,H_0).$$

It follows that the subspace $\mathcal{R}(W_\pm(H,H_0))$ and its orthogonal complement are invariant under H and that the wave operator provides a unitary equivalence between the part of H on $\mathcal{R}(W_\pm(H,H_0))$ and the absolutely continuous part of H_0. In particular,

$$\mathcal{R}(W_\pm(H,H_0)) \subset \mathfrak{H}^{(ac)}. \tag{1.1}$$

The wave operator $W_\pm(H,H_0)$ is said to be *complete* if equality holds in (1.1). It is easy to see that the completeness of $W_\pm(H,H_0)$ is equivalent to the existence of $W_\pm(H_0,H)$. Thus, if the wave operator $W_\pm(H,H_0)$ exists and is

complete, then the absolutely continuous parts of H_0 and H are unitarily equivalent.
Let us conclude this brief overview with a convenient sufficient condition for the existence and completeness of the wave operators due to Birman and Kreĭn (cf. [BKr]).

Proposition 1.1. *Let H_0, H be self-adjoint operators on a Hilbert space $\mathfrak{H}$ such that $(H - zI)^{-1} - (H_0 - zI)^{-1}$ belongs to the trace class for some $z \in \rho(H_0) \cap \rho(H)$. Then the wave operators $W_\pm(H, H_0)$ exist and are complete.*

1.3 Multiplication on the boundary

Here we present auxiliary statements related to Sobolev embedding theorems. Let σ be a periodic function satisfying

$$\sigma \in L_q(\mathbb{T}) \qquad \text{for some } q > 1. \tag{1.2}$$

It follows from the compactness of the embedding $H^{1/2}(-\pi, \pi) \subset L_{2q'}(-\pi, \pi)$ (with $\frac{1}{q} + \frac{1}{q'} = 1$) that the form $\int_{-\pi}^{\pi} |\sigma(x_1)||f(x_1)|^2 \, dx_1,\ f \in H^{1/2}(-\pi, \pi)$, is compact in $H^{1/2}(-\pi, \pi)$. This implies

Lemma 1.2. *Assume* (1.2) *and let $\epsilon > 0$. Then there exists a constant $C_1(\epsilon, \sigma) > 0$ such that*

$$\int_{-\pi}^{\pi} |\sigma(x_1)||f(x_1)|^2 \, dx_1 \le \epsilon \|f\|^2_{H^{1/2}(-\pi,\pi)} + C_1(\epsilon, \sigma)\|f\|^2, \qquad f \in H^{1/2}(-\pi, \pi).$$

Now let us pass to the situation on the halfstrip and on the halfplane. The trace operator $u \mapsto u(.,0)$ is bounded from $H^1(\Pi)$ to $H^{1/2}(-\pi, \pi)$. Hence the form $\int_{-\pi}^{\pi} |\sigma(x_1)||u(x_1, 0)|^2 \, dx_1$, $u \in H^1(\Pi)$, is compact in $H^1(\Pi)$ and we obtain

Lemma 1.3. *Assume* (1.2) *and let $\epsilon > 0$. Then there exists a constant $C_2(\epsilon, \sigma) > 0$ such that*

$$\int_{-\pi}^{\pi} |\sigma(x_1)||u(x_1, 0)|^2 \, dx_1 \le \epsilon \|u\|^2_{H^1(\Pi)} + C_2(\epsilon, \sigma)\|u\|^2, \qquad u \in H^1(\Pi).$$

Now let $u \in H^1(\mathbb{R}^2_+)$. For each $n \in \mathbb{Z}$ we apply Lemma 1.3 to the function $\Pi \ni x \mapsto u(x_1 + 2\pi n, x_2)$ and then sum over all $n \in \mathbb{Z}$. This yields

Lemma 1.4. *Assume* (1.2) *and let $\epsilon > 0$. Then there exists a constant $C_2(\epsilon, \sigma) > 0$ such that*

$$\int_{\mathbb{R}} |\sigma(x_1)||u(x_1, 0)|^2 \, dx_1 \le \epsilon \|u\|^2_{H^1(\mathbb{R}^2_+)} + C_2(\epsilon, \sigma)\|u\|^2, \qquad u \in H^1(\mathbb{R}^2_+).$$

Our treatment in Section 3 needs a more precise, quantitative result on the embedding $H^{1/2}(-\pi,\pi) \subset L_{2q'}(-\pi,\pi)$. We begin by recalling the definition of the weak l_p-spaces

$$l_{p,w}(\mathbb{Z}) = \{(\alpha_n)_{n\in\mathbb{Z}} : \sup_{t>0} t\rho_\alpha^{1/p}(t) < \infty\}, \qquad 0 < p < \infty,$$

where $\rho_\alpha(t) := \sharp\{n \in \mathbb{Z} : \ |\alpha_n| > t\}$ for $t > 0$. $l_{p,w}(\mathbb{N})$ is defined in a similar way. Further, recall (cf. Section 11.6 in [BS]) that $\Sigma_p(\mathfrak{H}_1,\mathfrak{H}_2)$, $0 < p < \infty$, is the class of compact operators K from a Hilbert space $\mathfrak{H}_1$ to a Hilbert space $\mathfrak{H}_2$ for which $(s_n(K))_{n\in\mathbb{N}} \in l_{p,w}(\mathbb{N})$, where $(s_n(K))_{n\in\mathbb{N}}$ is the sequence of singular numbers of K. One puts $\Sigma_p(\mathfrak{H}_1) := \Sigma_p(\mathfrak{H}_1,\mathfrak{H}_1)$. The dependence on $\mathfrak{H}_1$, $\mathfrak{H}_2$ is usually dropped in the notation if this does not lead to confusion.
The connection with the well-known Schatten class $\mathfrak{S}_p$ of order p (which consists of compact operators K for which $(s_n(K))_{n\in\mathbb{N}} \in l_p(\mathbb{N})$) can be seen from the inclusions

$$\Sigma_r \subset \mathfrak{S}_p \subset \Sigma_p, \qquad r < p.$$

We will often use the fact that $K_1 \in \Sigma_{p_1}$, $K_2 \in \Sigma_{p_2}$ implies

$$K_1K_2 \in \Sigma_{(\frac{1}{p_1}+\frac{1}{p_2})^{-1}}. \tag{1.3}$$

For a more general statement as well as for the proof of the following Cwikel-type estimate we refer to Theorem 4.8 in [BKaS].

Proposition 1.5. *Let $\beta \in L_p(\mathbb{T})$ and $\alpha \in l_{p,w}(\mathbb{Z})$ for some $p > 2$. Then $\beta\mathcal{F}^*\alpha \in \Sigma_p(l_2(\mathbb{Z}), L_2(\mathbb{T}))$.*

Let us consider the sequence α given by

$$\alpha_n := (1+n^2)^{-1/4}, \qquad n \in \mathbb{Z}, \tag{1.4}$$

and a function β as above. We write $\beta\mathcal{F}^*\alpha = \left(\beta\mathcal{F}^*\alpha^{2/p}\right)\alpha^{(p-2)/p}$. Clearly, the operator of multiplication by $\alpha^{(p-2)/p}$ belongs to $\Sigma_{2p/(p-2)}(l_2(\mathbb{Z}))$, and by Proposition 1.5 $\beta\mathcal{F}^*\alpha^{2/p} \in \Sigma_p(l_2(\mathbb{Z}), L_2(\mathbb{T}))$. Thus, taking into account (1.3), we obtain

Corollary 1.6. *Let $\beta \in L_p(\mathbb{T})$ for some $p > 2$ and α be given by* (1.4). *Then $\beta\mathcal{F}^*\alpha \in \Sigma_2(l_2(\mathbb{Z}), L_2(\mathbb{T}))$.*

This is the desired embedding result. Note that $\mathcal{F}^*\alpha\mathcal{F}$ maps $L_2(\mathbb{T})$ unitarily onto $H^{1/2}(\mathbb{T})$.

1.4 Definition of the operators $H^{(\sigma)}$ on the halfplane

Let σ be a real-valued periodic function satisfying (1.2). In the Hilbert space $L_2(\mathbb{R}^2_+)$ we consider the quadratic form

$$\begin{aligned} &\mathcal{D}[h^{(\sigma)}] := H^1(\mathbb{R}^2_+), \\ &h^{(\sigma)}[u] := \int_{\mathbb{R}^2_+} |Du(x)|^2\,dx + \int_{\mathbb{R}} \sigma(x_1)|u(x_1,0)|^2\,dx_1. \end{aligned} \tag{1.5}$$

According to Lemma 1.4 the form $h^{(\sigma)}$ is lower semibounded and closed, so it generates a self-adjoint operator which will be denoted by $H^{(\sigma)}$. By construction

$$(H^{(\sigma)}u, v) = h^{(\sigma)}[u, v], \qquad u \in \mathcal{D}(H^{(\sigma)}),\ v \in H^1(\mathbb{R}^2_+),$$

so it follows that the distributional Laplacian Δu of $u \in \mathcal{D}(H^{(\sigma)})$ belongs to $L_2(\mathbb{R}^2_+)$ and that

$$H^{(\sigma)}u = -\Delta u.$$

The case $\sigma = 0$ corresponds to the *Neumann Laplacian* on the halfplane, whereas the case $\sigma \neq 0$ implements a (generalized) *boundary condition of the third type*. More precisely, we have

Remark 1.7. Under the condition that σ is absolutely continuous with $\sigma' \in L_q(\mathbb{T})$ for some $q > 1$ it can be proved that

$$\mathcal{D}(H^{(\sigma)}) = \left\{ u \in H^2(\mathbb{R}^2_+) :\ -\frac{\partial u}{\partial x_2} + \sigma u = 0 \text{ on } \mathbb{R} \times \{0\} \right\}.$$

1.5 Main result

First we remark that the spectrum of the "unperturbed" operator $H^{(0)}$ coincides with $[0, +\infty)$ and is purely absolutely continuous of infinite multiplicity. This can be seen easily by applying a Fourier transformation with respect to the variable x_1 and a Fourier cosine transformation with respect to the variable x_2.

We turn now to the "perturbed" operator $H^{(\sigma)}$. We have recalled the abstract definition of the wave operators in Subsection 1.2. In case of existence we will use the notation

$$W_\pm^{(\sigma)} := W_\pm(H^{(\sigma)}, H^{(0)}).$$

The main result of the present paper is

Theorem 1.8. *Assume that σ satisfies* (1.2). *Then the wave operators $W_\pm^{(\sigma)}$ exist and satisfy $\mathcal{R}(W_+^{(\sigma)}) = \mathcal{R}(W_-^{(\sigma)})$.*

We note that the equality $\mathcal{R}(W_+^{(\sigma)}) = \mathcal{R}(W_-^{(\sigma)})$ implies the unitarity of the scattering matrix.

It follows from Theorem 1.8 that the part of $H^{(\sigma)}$ on $\mathcal{R}(W_\pm^{(\sigma)})$ is unitarily equivalent to $H^{(0)}$, and so $\sigma_{ac}\left(H^{(\sigma)}\right) \supset [0, +\infty)$.

In the second part [FrSh] we supplement this theorem with the following results. *The operator $H^{(\sigma)}$ has purely absolutely continuous spectrum.* In general, $\sigma\left(H^{(\sigma)}\right)$ may contain (apart from $[0 + \infty)$) additional bands, so the wave operators $W_\pm^{(\sigma)}$ may be not complete. The spectral subspaces corresponding to the additional bands of $H^{(\sigma)}$ are *additional channels of scattering*. However, *under the additional assumption*

$$\sigma \geq 0 \quad a.e. \tag{1.6}$$

the wave operators $W_\pm^{(\sigma)}$ *turn out to be complete and unitary.*

Remark 1.9. The statement of Theorem 1.8 holds with obvious changes when the role of the comparison operator H_0 is played by the Dirichlet Laplacian on the halfplane or the Laplacian on the whole plane. This follows easily from the chain rule for wave operators and the fact, that the wave operators $W_\pm(H^{(0)}, H_0)$ exist and are complete. Indeed, they can be calculated easily. (Clearly, if H_0 is the Laplacian on the whole plane one has to use an identification operator.)

Remark 1.10. A time-dependent characterization of the range of the wave operators and its orthogonal complement can be established by standard methods (see [DaSi], [Sa]): With the notation $U^{(\sigma)}(t) := \exp(-itH^{(\sigma)})$ for $t \in \mathbb{R}$ it follows that

$$\mathcal{R}(W_\pm^{(\sigma)}) = \{u \in L_2(\mathbb{R}^2_+) : \lim_{t\to\pm\infty} \int_{\mathbb{R}\times(0,a)} |U^{(\sigma)}(t)u(x)|^2\,dx = 0,\ a \in \mathbb{R}_+\},$$

$$\mathcal{R}(W_\pm^{(\sigma)})^\perp = \{u \in L_2(\mathbb{R}^2_+) : \lim_{a\to+\infty} \sup_{t\in\mathbb{R}} \int_{\mathbb{R}\times(a,+\infty)} |U^{(\sigma)}(t)u(x)|^2\,dx = 0\}.$$

Thus, the additional channels of scattering correspond to "surface states", i.e., states concentrated near the boundary for all time.

2 Direct integral decomposition

2.1 Definition of the operators $H^{(\sigma)}(k)$ on the halfstrip

Let σ be a real-valued periodic function satisfying (1.2) and let $k \in [-\frac{1}{2}, \frac{1}{2}]$. In the Hilbert space $L_2(\Pi)$ we consider the quadratic form

$$\begin{aligned}
&\mathcal{D}[h^{(\sigma)}(k)] := \tilde{H}^1(\Pi),\\
&h^{(\sigma)}(k)[u] := \int_\Pi \left(|(D_1+k)u(x)|^2 + |D_2 u(x)|^2\right)\,dx + \int_{-\pi}^{\pi} \sigma(x_1)|u(x_1,0)|^2\,dx_1.
\end{aligned} \tag{2.1}$$

According to Lemma 1.3 the form $h^{(\sigma)}(k)$ is lower semibounded and closed, so it generates a self-adjoint operator which will be denoted by $H^{(\sigma)}(k)$. Similarly as above one finds

$$H^{(\sigma)}u = (D_1+k)^2u + D_2^2u = -\Delta u + 2kD_1u + k^2u, \qquad u \in \mathcal{D}(H^{(\sigma)}).$$

In addition to the Neumann (if $\sigma = 0$) or third type (if $\sigma \neq 0$) boundary condition at $\{x_2 = 0\}$ the functions in $\mathcal{D}(H^{(\sigma)})$ satisfy periodic boundary conditions at $\{x_1 = \pm\pi\}$. A statement analogous to Remark 1.7 holds.

2.2 DIRECT INTEGRAL DECOMPOSITION OF THE OPERATOR $H^{(\sigma)}$

The operator $H^{(\sigma)}$ can be partially diagonalized by means of the *Gelfand transformation* $\mathcal{U}$. This operator is initially defined for $u \in \mathcal{S}(\mathbb{R}^2_+)$, the Schwartz class on $\mathbb{R}^2_+$, by

$$(\mathcal{U}u)(k,x) := \sum_{n\in\mathbb{Z}} e^{-ik(x_1+2\pi n)} u(x_1+2\pi n, x_2), \qquad k \in [-\tfrac{1}{2}, \tfrac{1}{2}],\ x \in \Pi,$$

and extended by continuity to a *unitary* operator

$$\mathcal{U} : L_2(\mathbb{R}^2_+) \to \int_{-1/2}^{1/2} \oplus L_2(\Pi)\, dk.$$

Moreover, it turns out that $u \in H^1(\mathbb{R}^2_+)$ iff $(\mathcal{U}u)(k,.) \in \tilde{H}^1(\Pi)$ for a.e. $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $\int_{-1/2}^{1/2} \|(\mathcal{U}u)(k,.)\|^2_{H^1(\Pi)}\, dk < \infty$, and in this case

$$(\mathcal{U}D_1 u)(k,.) = (D_1+k)(\mathcal{U}u)(k,.), \qquad (\mathcal{U}D_2 u)(k,.) = D_2(\mathcal{U}u)(k,.).$$

Concerning the multiplication on the boundary by a periodic function σ satisfying (1.2), one finds for $u \in H^1(\mathbb{R}^2_+)$

$$\int_{-1/2}^{1/2}\int_{-\pi}^{\pi} \sigma(x_1)|(\mathcal{U}u)(k,x_1,0)|^2\, dx_1\, dk = \int_{\mathbb{R}} \sigma(x_1)|u(x_1,0)|^2\, dx_1.$$

To summarize, the Gelfand transformation satisfies

$$\mathcal{U}\left(\mathcal{D}[h^{(\sigma)}]\right) = \Bigg\{ F \in \int_{-1/2}^{1/2} \oplus L_2(\Pi)\, dk :\ F(k) \in \tilde{H}^1(\Pi) \text{ for a.e. } k \in [-\tfrac{1}{2}, \tfrac{1}{2}],$$
$$\int_{-1/2}^{1/2} \left| h^{(\sigma)}(k)[F(k)] \right|\, dk < \infty \Bigg\},$$
$$h^{(\sigma)}[u] = \int_{-1/2}^{1/2} h^{(\sigma)}(k)[(\mathcal{U}u)(k,.)]\, dk, \qquad u \in H^1(\mathbb{R}^2_+),$$

which implies

$$\mathcal{U}\, H^{(\sigma)}\, \mathcal{U}^* = \int_{-1/2}^{1/2} \oplus H^{(\sigma)}(k)\, dk. \tag{2.2}$$

This relation allows us to investigate the operator $H^{(\sigma)}$ by studying the fibers $H^{(\sigma)}(k)$.

2.3 MAIN RESULT FOR THE OPERATORS $H^{(\sigma)}(k)$ ON THE HALFSTRIP

In case of existence we will use the notation

$$W^{(\sigma)}_{\pm}(k) := W_{\pm}(H^{(\sigma)}(k), H^{(0)}(k)), \qquad k \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right].$$

Theorem 2.1. *Assume that σ satisfies* (1.2) *and let $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then the wave operators $W_{\pm}^{(\sigma)}(k)$ exist and are complete.*

The spectrum of the "unperturbed" operator $H^{(0)}(k)$ (see Subsection 3.1) coincides with $[k^2, +\infty)$ and is purely absolutely continuous. By the remarks in Subsection 1.2 Theorem 2.1 implies that the absolutely continuous part of $H^{(\sigma)}(k)$ is unitarily equivalent to $H^{(0)}(k)$, in particular

$$\sigma_{ac}\left(H^{(\sigma)}(k)\right) = [k^2, +\infty).$$

Concerning the singular continuous spectrum of $H^{(\sigma)}(k)$ we prove

Theorem 2.2. *Assume that σ satisfies* (1.2) *and let $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then*

$$\sigma_{sc}\left(H^{(\sigma)}(k)\right) = \emptyset.$$

The point spectrum of $H^{(\sigma)}(k)$ is investigated in the second part [FrSh]. We prove there that $\sigma_p\left(H^{(\sigma)}(k)\right)$ consists of eigenvalues of finite multiplicities which may accumulate at $+\infty$ only. The situation of infinitely many (embedded) eigenvalues does actually occur. The discrete eigenvalues of $H^{(\sigma)}(k)$ produce bands in the spectrum of $H^{(\sigma)}$. In general, the same is true for the embedded eigenvalues of $H^{(\sigma)}(k)$. However, under the additional assumption (1.6) we prove $\sigma_p\left(H^{(\sigma)}(k)\right) = \emptyset$, which implies the completeness and even unitarity of the wave operators $W_{\pm}^{(\sigma)}$.

2.4 Reduction of Theorem 1.8 to Theorem 2.1

Assuming Theorem 2.1, the proof of Theorem 1.8 is easy.

Proof of Theorem 1.8. For $t \in \mathbb{R}$, $k \in [-\frac{1}{2}, \frac{1}{2}]$ put $U^{(\sigma)}(t) := \exp(-itH^{(\sigma)})$ and $U^{(\sigma)}(t,k) := \exp(-itH^{(\sigma)}(k))$, and similarly with σ replaced by 0. The wave operators $W_{\pm}^{(\sigma)}(k)$ exist by Theorem 2.1 and are measurable with respect to k, so $\int_{-1/2}^{1/2} \oplus W_{\pm}^{(\sigma)}(k)\, dk$ is well defined. Moreover, by (2.2) together with Theorem XIII.85 in [ReSi4] one has

$$\mathcal{U}\, U^{(\sigma)}(t)\, \mathcal{U}^* = \exp(-it\mathcal{U}\, H^{(\sigma)}\, \mathcal{U}^*) = \int_{-1/2}^{1/2} \oplus U^{(\sigma)}(t,k)\, dk,$$

where $\mathcal{U}$ is the Gelfand transformation from Subsection 2.2, and similarly with σ replaced by 0. It follows that for all $u \in L_2(\mathbb{R}_+^2)$

$$\left\| \mathcal{U}\, U^{(\sigma)}(-t) U^{(0)}(t) u - \left(\int_{-1/2}^{1/2} \oplus W_{\pm}^{(\sigma)}(k)\, dk \right) \mathcal{U} u \right\|^2 =$$
$$= \int_{-1/2}^{1/2} \left\| U^{(\sigma)}(-t,k) U^{(0)}(t,k)\, (\mathcal{U}u)(k,.) - W_{\pm}^{(\sigma)}(k)\, (\mathcal{U}u)(k,.) \right\|^2 dk \to 0$$
$$(t \to \pm\infty)$$

by Lebesgue's theorem. This means that the strong limits of $U^{(\sigma)}(-t)U^{(0)}(t)$ for $t \to \pm\infty$ exist and coincide with $W_\pm^{(\sigma)} = \mathcal{U}^* \left(\int_{-1/2}^{1/2} \oplus W_\pm^{(\sigma)}(k)\, dk\right)\mathcal{U}$. In particular, because of the completeness of $W_\pm^{(\sigma)}(k)$,

$$\mathcal{R}(W_\pm^{(\sigma)}) = \mathcal{U}^* \left(\int_{-1/2}^{1/2} \oplus \mathcal{R}(P_{ac}^{(\sigma)}(k))\, dk\right)\mathcal{U}.$$

□

3 The operators $H^{(\sigma)}(k)$ on the halfstrip

3.1 The unperturbed operator $H^{(0)}(k)$ on the halfstrip

We start our investigation by summarizing results on the "unperturbed" operator $H^{(0)}(k)$.
Let $k \in [-\frac{1}{2}, \frac{1}{2}]$. By separation of variables one easily finds that the spectrum of the operator $H^{(0)}(k)$ coincides with $[k^2, +\infty)$, is purely absolutely continuous and that the spectral multiplicity of $\lambda \in [k^2, +\infty)$ is $\sharp\{n \in \mathbb{Z} : (n+k)^2 \le \lambda\}$. Note that the spectral multiplicity changes at the "threshold points" $(n+k)^2$, $n \in \mathbb{Z}$.
It is also easy to verify that the resolvent

$$R^{(0)}(z,k) := \left(H^{(0)}(k) - zI\right)^{-1}, \qquad z \in \rho\left(H^{(0)}(k)\right) = \mathbb{C} \setminus [k^2, +\infty),$$

is an integral operator with kernel

$$r^{(0)}(x,y;z,k) := \frac{1}{4\pi} \sum_{n\in\mathbb{Z}} \frac{e^{in(x_1-y_1)}}{\beta_n(z,k)} \left(e^{-\beta_n(z,k)\,(x_2+y_2)} + e^{-\beta_n(z,k)\,|x_2-y_2|}\right), \qquad x,\, y \in \Pi,\ x_2 \neq y_2. \tag{3.1}$$

where

$$\beta_n(z,k) := \sqrt{(n+k)^2 - z}, \qquad n \in \mathbb{Z},\ z \in \mathbb{C} \setminus [k^2, +\infty). \tag{3.2}$$

Here and in the following we choose the canonical branch of the square root on $\mathbb{C} \setminus (-\infty, 0]$ satisfying $\mathrm{Re}\,\sqrt{\cdot} > 0$.
Note that the RHS of (3.1) converges absolutely and uniformly on compact subsets of $\{(x,y) \in \Pi \times \Pi :\ x_2 \neq y_2\}$.

3.2 A general approach to the inversion of a perturbed operator

To investigate the "perturbed" operators $H^{(\sigma)}(k)$ we use a version of the resolvent identity. The "classical" resolvent identity $(H - zI)^{-1} - (H_0 - zI)^{-1} = -(H_0 - zI)^{-1}(H - H_0)(H - zI)^{-1}$ involves the difference of H and H_0, which may be not well-defined if the operators are defined via quadratic forms. Here

we present a version of the resolvent identity that works also in the quadratic form case. The proof may be found in Section 1.9 of [Ya].
We work in a general setting: Let $\mathfrak{H}$ be a Hilbert space and H, H_0 be self-adjoint operators satisfying
$$\mathcal{D}(|H|^{1/2}) = \mathcal{D}(|H_0|^{1/2}). \tag{3.3}$$
We note that H and H_0 are not assumed to be semibounded (but they will be so in our application). Denote their resolvents by
$$R_0(z) := (H_0 - zI)^{-1},\ z \in \rho(H_0), \qquad R(z) := (H - zI)^{-1},\ z \in \rho(H).$$
Suppose that there is an "auxiliary" Hilbert space $\mathfrak{G}$ and operators
$$G_0 : \mathfrak{H} \supset \mathcal{D}(G_0) \to \mathfrak{G}, \qquad G : \mathfrak{H} \supset \mathcal{D}(G) \to \mathfrak{G},$$
such that the following is true.

(H1) The operators G_0, G are $|H_0|^{1/2}$-bounded, i.e.,
$$\begin{array}{ll} \mathcal{D}(|H_0|^{1/2}) \subset \mathcal{D}(G_0), & \mathcal{D}(|H_0|^{1/2}) \subset \mathcal{D}(G), \\ G_0(|H_0|^{1/2} + I)^{-1} \in \mathfrak{B}(\mathfrak{H}, \mathfrak{G}), & G(|H_0|^{1/2} + I)^{-1} \in \mathfrak{B}(\mathfrak{H}, \mathfrak{G}). \end{array}$$

(H2) The operators G_0, G satisfy
$$(G_0 f, Gg) = (Gf, G_0 g), \qquad f,\ g \in \mathcal{D}(|H_0|^{1/2}).$$

(H3) The relation $H = H_0 + G^* G_0$ holds in the sense of forms, i.e.,
$$(Hf, f_0) = (f, H_0 f_0) + (Gf, G_0 f_0), \qquad f_0 \in \mathcal{D}(H_0),\ f \in \mathcal{D}(H).$$

The assumption (H1) guarantees that the operators $GR_0(z) : \mathfrak{H} \to \mathfrak{G}$ and $G_0(GR_0(\overline{z}))^* : \mathfrak{G} \to \mathfrak{G}$ are well-defined and bounded for $z \in \rho(H_0)$. With slight abuse of notation we put
$$R_0(z)G^* := (GR_0(\overline{z}))^*, \qquad G_0 R_0(z) G^* := G_0(GR_0(\overline{z}))^*.$$

Proposition 3.1. *Let H_0, H be self-adjoint operators satisfying* (3.3) *and assume that the operators G_0, G satisfy* (H1)-(H3). *Let $z \in \rho(H_0)$, then $z \in \rho(H)$ iff $I + G_0 R_0(z) G^*$ is boundedly invertible, and in this case*
$$R(z) - R_0(z) = -R_0(z) G^* \left(I + G_0 R_0(z) G^*\right)^{-1} G_0 R_0(z). \tag{3.4}$$

3.3 Some auxiliary operators

For $k \in [-\frac{1}{2}, \frac{1}{2}]$, $z \in \mathbb{C} \setminus [k^2, +\infty)$ we consider in $L_2(\mathbb{T})$ the operator
$$\begin{aligned} &\mathcal{D}(B(z,k)) := H^1(\mathbb{T}), \\ &(B(z,k)f)(x_1) := \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \beta_n(z,k)\, \hat{f}_n\, e^{inx_1}, \qquad x_1 \in \mathbb{T}, \end{aligned} \tag{3.5}$$

with $\beta_n(z,k)$ defined in (3.2). The operator $B(z,k)$ is invertible for $z \in \mathbb{C} \setminus [k^2,+\infty)$ and its square root is well-defined (it may be considered as the square root of an m-accretive operator). Now, let σ be a periodic function satisfying (1.2) and define operators on $L_2(\mathbb{T})$ by

$$T_0^{(\sigma)}(z,k) := (\operatorname{sgn}\sigma)|\sigma|^{1/2}B(z,k)^{-1/2}, \quad T^{(\sigma)}(z,k) := |\sigma|^{1/2}B(z,k)^{-1/2}.$$

It follows from Corollary 1.6 that these are compact operators of class $\Sigma_2(L_2(\mathbb{T}))$.
Finally, for $z \in \mathbb{C} \setminus [k^2,+\infty)$ we consider the integral operator $Y(z,k)$ acting from $L_2(\Pi)$ to $L_2(\mathbb{T})$ whose kernel is given by

$$Y(x_1,y;z,k) := \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{\beta_n(z,k)}}e^{in(x_1-y_1)}e^{-\beta_n(z,k)\,y_2}, \quad x_1 \in \mathbb{T},\ y \in \Pi. \tag{3.6}$$

Writing down the singular value expansion explicitly we find that $Y(z,k)$ is a compact operator of class $\Sigma_1(L_2(\Pi),L_2(\mathbb{T}))$.

3.4 The resolvent difference

We are now ready to apply the general results from Subsection 3.2 to our situation. Denote the resolvent of the operator $H^{(\sigma)}(k)$ by

$$R^{(\sigma)}(z,k) := \left(H^{(\sigma)}(k) - zI\right)^{-1}, \qquad z \in \rho\left(H^{(\sigma)}(k)\right).$$

The following statement is of crucial importance.

PROPOSITION 3.2. *Let $k \in [-\frac{1}{2},\frac{1}{2}]$ and $z \in \mathbb{C}\setminus[k^2,+\infty)$. Then $z \in \rho\left(H^{(\sigma)}(k)\right)$ iff the operator $I+T_0^{(\sigma)}(z,k)T^{(\sigma)}(\overline{z},k)^*$ is boundedly invertible, and in this case*

$$R^{(\sigma)}(z,k) - R^{(0)}(z,k) = \\ = -Y(\overline{z},k)^*\,T^{(\sigma)}(\overline{z},k)^*\left(I + T_0^{(\sigma)}(z,k)\,T^{(\sigma)}(\overline{z},k)^*\right)^{-1}T_0^{(\sigma)}(z,k)\,Y(z,k). \tag{3.7}$$

Proof. We want to apply the results of Subsection 3.2 to the case $\mathfrak{H} = L_2(\Pi)$, $\mathfrak{G} = L_2(\mathbb{T})$, $H_0 = H^{(0)}(k)$, $H = H^{(\sigma)}(k)$ and

$$\begin{aligned}\mathcal{D}(G_0) = \mathcal{D}(G) &:= \tilde{H}^1(\Pi),\\ (G_0u)(x_1) &:= (\operatorname{sgn}\sigma(x_1))\sqrt{|\sigma(x_1)|}\,u(x_1,0), \qquad x_1 \in \mathbb{T},\\ (Gu)(x_1) &:= \sqrt{|\sigma(x_1)|}\,u(x_1,0), \qquad x_1 \in \mathbb{T}.\end{aligned}$$

According to Lemma 1.3, the operators G_0, G are well-defined and bounded from $\tilde{H}^1(\Pi)$ to $L_2(\mathbb{T})$. Since $(|H_0|^{1/2}+I)^{-1}$ is a bounded operator from $\mathfrak{H}$ to $\tilde{H}^1(\Pi)$, the assumption (H1) in Subsection 3.2 holds. The remaining (3.3),

(H2) and (H3) are obvious, so that the statement of Proposition 3.1 holds. Let us determine the products $G_0R_0(z)$, $GR_0(z)$ and $G_0R_0(z)G^*$.
The explicit form (3.1) of the free resolvent implies that for $u \in L_2(\Pi)$

$$(R_0(z)u)(x_1,0) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{1}{\beta_n(z,k)} \int_\Pi u(y)\, e^{in(x_1-y_1)}\, e^{-\beta_n(z,k)\, y_2}\, dy = \\ = \Big(B(z,k)^{-1/2}\, Y(z,k)u\Big)(x_1), \qquad x_1 \in \mathbb{T},$$

and hence

$$G_0R_0(z) = T_0^{(\sigma)}(z,k)\, Y(z,k), \qquad GR_0(z) = T^{(\sigma)}(z,k)\, Y(z,k).$$

Moreover, for $f \in L_2(\mathbb{T})$ we have

$$(Y(\overline{z},k)^* f)(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \frac{\hat{f}_n}{\sqrt{\beta_n(z,k)}}\, e^{inx_1}\, e^{-\beta_n(z,k)\; x_2}, \quad x \in \Pi,$$

so that $Y(\overline{z},k)^*f \in \tilde{H}^1(\Pi)$ and $(Y(\overline{z},k)^*f)(.,0) = B(z,k)^{-1/2}f$. It follows that $G_0Y(\overline{z},k)^* = T_0^{(\sigma)}(z,k)$ and

$$G_0R_0(z)G^* = G_0\left(GR_0(\overline{z})\right)^* = T_0^{(\sigma)}(z,k)\, T^{(\sigma)}(\overline{z},k)^*.$$

This concludes the proof of the Proposition. □

As an easy consequence of (3.7) we obtain

Corollary 3.3. *Let $k \in [-\frac{1}{2},\frac{1}{2}]$ and $z \in \rho\left(H^{(\sigma)}(k)\right)$, then*

$$R^{(\sigma)}(z,k) - R^{(0)}(z,k) \in \Sigma_{1/3}(L_2(\Pi)).$$

Proof. We have remarked in Subsection 3.3 that $T_0^{(\sigma)}(z,k), T^{(\sigma)}(z,k) \in \Sigma_2(L_2(\mathbb{T}))$ and $Y(z,k) \in \Sigma_1(L_2(\Pi), L_2(\mathbb{T}))$, so the statement follows from (1.3). □

The proof of Theorem 2.1 is now immediate.

Proof of Theorem 2.1. Combine Corollary 3.3 with Proposition 1.1. □

3.5 The limiting absorption principle for the unperturbed operator

It remains to prove Theorem 2.2, the absence of singular continuous spectrum. This will be achieved by controlling the behavior of the resolvent $R^{(\sigma)}(z,k)$ as the spectral parameter z tends to the real axis. We start with the unperturbed case $\sigma \equiv 0$.

We introduce the function $\Lambda_2(x) := (1 + x_2^2)^{1/2}$, $x \in \Pi$, and note that for $k \in [-\frac{1}{2}, \frac{1}{2}]$, $z \in \mathbb{C} \setminus [k^2, +\infty)$ and $s > \frac{1}{2}$ the operator

$$\Lambda_2^{-s} R^{(0)}(z, k) \Lambda_2^{-s}$$

belongs to the Hilbert-Schmidt class. The following result is called the *limiting absorption principle* for the operator $H^{(0)}(k)$.

Proposition 3.4. *Let $s > \frac{1}{2}$ and $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then the limits*

$$\lim_{\epsilon \to 0+} \Lambda_2^{-s} R^{(0)}(\lambda \pm i\epsilon, k) \Lambda_2^{-s}, \qquad \lambda \neq (n+k)^2,\ n \in \mathbb{Z},$$

exist in the Hilbert-Schmidt norm and are uniform for λ from compact intervals of $\mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\}$.

Proof. Fix $\lambda \in \mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\}$, and define

$$r^{(0)}(x, y; \lambda \pm i0, k) := \lim_{\epsilon \to 0+} r^{(0)}(x, y; \lambda \pm i\epsilon, k), \qquad x \neq y \in \Pi,$$

as pointwise limit using formula (3.1). We have to prove that

$$\int_\Pi \int_\Pi \left| r^{(0)}(x, y; \lambda \pm i\epsilon, k) - r^{(0)}(x, y; \lambda \pm i0, k) \right|^2 \frac{dx}{(1+x_2^2)^s} \frac{dy}{(1+y_2^2)^s} \longrightarrow 0 \quad (\epsilon \to 0+). \tag{3.8}$$

We restrict ourselves to the "+"-case, the other being similar, and for simplicity of notation, we put

$$c_n(\epsilon) := \beta_n(\lambda + i\epsilon, k), \qquad c_n := \lim_{\epsilon \to 0+} c_n(\epsilon).$$

Using Parseval's identity and the triangle inequality we find

$$\begin{aligned}
&\int_\Pi \int_\Pi \left| r^{(0)}(x, y; \lambda \pm i\epsilon, k) - r^{(0)}(x, y; \lambda \pm i0, k) \right|^2 \frac{dx}{(1+x_2^2)^s} \frac{dy}{(1+y_2^2)^s} = \\
&\quad = \sum_{n \in \mathbb{Z}} \int_0^\infty \frac{dx_2}{(1+x_2^2)^s} \int_0^\infty \frac{dy_2}{(1+y_2^2)^s} \\
&\qquad \left| \frac{e^{-c_n(\epsilon)(x_2+y_2)} + e^{-c_n(\epsilon)|x_2-y_2|}}{2c_n(\epsilon)} - \frac{e^{-c_n(x_2+y_2)} + e^{-c_n|x_2-y_2|}}{2c_n} \right|^2 \le \\
&\quad \le 2 \sum_{n \in \mathbb{Z}} \int_0^\infty \int_0^\infty \left(t_{n,\epsilon}(x_2+y_2) + t_{n,\epsilon}(x_2-y_2) \right) \frac{dx_2}{(1+x_2^2)^s} \frac{dy_2}{(1+y_2^2)^s},
\end{aligned}$$

where

$$t_{n,\epsilon}(a) := \left| \frac{e^{-c_n(\epsilon)|a|}}{2c_n(\epsilon)} - \frac{e^{-c_n|a|}}{2c_n} \right|^2, \qquad a \in \mathbb{R}.$$

For $(n+k)^2 < \lambda$ it follows from Lebesgue's theorem that

$$\sum_{(n+k)^2<\lambda} \int_0^\infty \int_0^\infty (t_{n,\epsilon}(x_2+y_2) + t_{n,\epsilon}(x_2-y_2)) \frac{dx_2}{(1+x_2^2)^s} \frac{dy_2}{(1+y_2^2)^s} \longrightarrow 0,$$
$$(\epsilon \to 0+).$$

Suppose now that $(n+k)^2 > \lambda$. To control the convergence of the $t_{n,\epsilon}$ in terms of n we need the elementary estimate

$$t_{n,\epsilon}(a) \leq \epsilon^2 \frac{C}{|c_n|^6}, \qquad a \in \mathbb{R},$$

with a constant C independent of a, ϵ, λ and n. It follows that

$$\sum_{(n+k)^2>\lambda} \int_0^\infty \int_0^\infty (t_{n,\epsilon}(x_2+y_2) + t_{n,\epsilon}(x_2-y_2)) \frac{dx_2}{(1+x_2^2)^s} \frac{dy_2}{(1+y_2^2)^s} \leq$$
$$\leq 2C\epsilon^2 \left(\int_0^\infty \frac{dx_2}{(1+x_2^2)^s} \right)^2 \sum_{n\in\mathbb{Z}} \frac{1}{|c_n|^6}.$$

The RHS converges to 0 as $\epsilon \to 0+$, which completes the proof of (3.8). Finally, we remark that the limit in (3.8) is uniform in λ for λ from a compact intervall not containing any of the points $(n+k)^2$, $n \in \mathbb{Z}$. This follows from the fact, that c_n depends continuously on λ. □

3.6 The limiting absorption principle for the perturbed operator

Using the Analytic Fredholm Alternative and the resolvent identity (3.7) we derive the limiting absorption principle for the operator $H^{(\sigma)}(k)$ from Proposition 3.4.

Lemma 3.5. *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$, then the operator families*

$$T_0^{(\sigma)}(z,k),\, T^{(\sigma)}(z,k), \qquad z \in \mathbb{C} \setminus [k^2, \infty),$$

can be extended norm-continuously to the cut from above and from below with the exception of the points $z = (n+k)^2$, $n \in \mathbb{Z}$. Denoting the (upper and lower) boundary values by

$$T_0^{(\sigma)}(\lambda \pm i0, k), \qquad T^{(\sigma)}(\lambda \pm i0, k), \qquad \lambda \neq (n+k)^2,$$

the sets

$$\mathcal{N}_\pm(k) := \{\lambda \in \mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\} : \mathcal{N}(I + T_0^{(\sigma)}(\lambda \pm i0, k)\, T^{(\sigma)}(\lambda \mp i0, k)^*) \neq \{0\}\}$$

are discrete in $\mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\}$.

Remark 3.6. One can show that the sets $\mathcal{N}_+(k)$ and $\mathcal{N}_-(k)$ coincide. Moreover, in the second part [FrSh] we will prove that the sets $\mathcal{N}_\pm(k)$ can accumulate at $+\infty$ only.

Proof. Let

$$\tilde{B}(z,k), \qquad z \in \mathbb{C} \setminus \{(n+k)^2 + iy :\ n \in \mathbb{Z},\ y \leq 0\} =: D_+(k)$$

be the analytic family of operators given by the same formal expression (3.5) as the operators $B(z,k)$, but where we choose in the definition (3.2) the branch of the square root on $\mathbb{C} \setminus \{iy :\ y \geq 0\}$ which coincides with the canonical branch on the lower halfplane. In particular,

$$\tilde{B}(z,k) = B(z,k), \qquad z \in \mathbb{C}_+. \tag{3.9}$$

It follows from Corollary 1.6 that

$$\tilde{T}^{(\sigma)}(z,k) := |\sigma|^{1/2}\tilde{B}(z,k)^{-1/2}, \qquad z \in D_+(k),$$

is an analytic family of compact operators. Because of (3.9) it is a bounded analytic (and hence norm-continuous) extension of the family $T^{(\sigma)}(z,k)$ across the cut from above. We put

$$T^{(\sigma)}(\lambda + i0, k) := \tilde{T}^{(\sigma)}(\lambda, k), \qquad \lambda \neq (n+k)^2,\ n \in \mathbb{Z}.$$

The construction of the operators $T^{(\sigma)}(\lambda - i0, k)$ is similar, replacing $D_+(k)$ by $D_-(k) := \mathbb{C} \setminus \{(n+k)^2 + iy :\ n \in \mathbb{Z},\ y \geq 0\}$, and the statement about the operators $T_0^{(\sigma)}(z,k)$ follows by multiplying $T^{(\sigma)}(z,k)$ with $\operatorname{sgn}\sigma$.
Let us prove the statement about the sets $\mathcal{N}_\pm(k)$. It follows easily that $\|\tilde{T}^{(\sigma)}(z,k)\| < 1$ if $|\operatorname{Im} z|$ is large. Now the Analytic Fredholm Alternative (cf. Theorem VII.1.9 in [K]) applied to the operators $\tilde{T}_0^{(\sigma)}(z,k)\,\tilde{T}^{(\sigma)}(\overline{z},k)^*$ yields the discreteness of the sets $\mathcal{N}_\pm(k)$. This concludes the proof. □

Proposition 3.7. *Let $s > \frac{1}{2}$ and $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then the limits*

$$\lim_{\epsilon \to 0+} \Lambda_2^{-s} R^{(\sigma)}(\lambda \pm i\epsilon, k)\Lambda_2^{-s}, \qquad \lambda \notin \{(n+k)^2 :\ n \in \mathbb{Z}\} \cup \mathcal{N}_\pm(k),$$

exist in the Hilbert-Schmidt norm and are uniform for λ from compact intervals of $\mathbb{R} \setminus \big(\{(n+k)^2 :\ n \in \mathbb{Z}\} \cup \mathcal{N}_\pm(k)\big)$.

Proof. We consider the resolvent identity (3.7). Because of Proposition 3.4 and Lemma 3.5 it suffices to prove that the limits

$$\lim_{\epsilon \to 0+} Y(\lambda \pm i\epsilon, k)\Lambda_2^{-s}, \qquad \lambda \neq (n+k)^2,\ n \in \mathbb{Z},$$

exist in the Hilbert-Schmidt norm and are uniform for λ from compact intervals of $\mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\}$.
Considering the integral kernel (3.6) of $Y(z,k)$ one can procede similar to the proof of Proposition 3.4. □

As an easy consequence of Proposition 3.7 we obtain now Theorem 2.2.

Proof of Theorem 2.2. Let $[a,b] \subset \mathbb{R} \setminus \big(\{(n+k)^2 :\ n \in \mathbb{Z}\} \cup \mathcal{N}_\pm(k)\big)$ be a compact interval. Then for every u from the dense set $\mathcal{R}(\Lambda_2^{-s})$ (with $s > \frac{1}{2}$ arbitrary) we have

$$\sup_{0<\epsilon<1} \int_a^b |(R^{(\sigma)}(\lambda \pm i\epsilon, k)u, u)|^2 \, d\lambda < \infty$$

by Proposition 3.7. It follows (cf. Proposition 1.5.2 in [Ya]) that the spectrum of $H^{(\sigma)}(k)$ is purely absolutely continuous on $[a,b]$. Therefore $\sigma_{sing}\left(H^{(\sigma)}(k)\right) \subset \{(n+k)^2 : n \in \mathbb{Z}\} \cup \mathcal{N}_\pm(k)$. Since the latter set accumulates only at $\{(n+k)^2 : n \in \mathbb{Z}\}$ and $+\infty$, we conclude $\sigma_{sc}\left(H^{(\sigma)}(k)\right) = \emptyset$, as claimed. □

References

[BKaS] M. Sh. Birman, G. E. Karadzhov, M. Z. Solomyak, *Boundedness conditions and spectrum estimates for the operators b(X)a(D) and their analogs*, Estimates and asymptotics for discrete spectra of integral and differential equations, Adv. Soviet. Math., vol. 7, Amer. Math. Soc., 1991, 85-106.

[BKr] M. Sh. Birman, M. G. Kreĭn, *On the theory of wave operators and scattering operators*, Soviet. Math. Dokl. 3 (1962), 740-744.

[BShSu] M. Sh. Birman, R. G. Shterenberg, T. A. Suslina, *Absolute continuity of the spectrum of a two-dimensional Schrödinger operator with potential supported on a periodic system of curves*, St. Petersburg Math. J. 12, no. 6 (2001), 983-1012.

[BS] M. Sh. Birman, M. Z. Solomyak, *Spectral theory of self-adjoint operators in Hilbert space*, Reidel, 1987.

[DaSi] E. B. Davies, B. Simon, *Scattering Theory for Systems with Different Spatial Asymptotics on the Left and Right*, Commun. Math. Phys. 63 (1978), 277-301.

[FrSh] R. L. Frank, R. G. Shterenberg, *On the scattering theory of the Laplacian with a periodic boundary condition. II. Additional channels of scattering*, in preparation.

[K] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1976.

[ReSi4] M. Reed, B. Simon, *Methods of Modern Mathematical Physics*, vol. 4, Academic Press, 1978.

[Sa] A. W. Saenz, *Quantum-mechanical scattering by impenetrable periodic surfaces*, J. Math. Phys. 22, no. 12 (1981), 2872-2884.

[Ya] D. R. Yafaev, *Mathematical Scattering Theory*, Amer. Math. Soc., 1992.

Rupert L. Frank
Mathematisches Institut
LMU München
Theresienstraße 39
80333 München, Germany
rfrank@mathematik.uni-muenchen.de