Motivic Functors

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Abstract. The notion of motivic functors refers to a motivic homotopy theoretic analog of continuous functors. In this paper we lay the foundations for a homotopical study of these functors. Of particular interest is a model structure suitable for studying motivic functors which preserve motivic weak equivalences and a model structure suitable for motivic stable homotopy theory. The latter model is Quillen equivalent to the category of motivic symmetric spectra.

There is a symmetric monoidal smash product of motivic functors, and all model structures constructed are compatible with the smash product in the sense that we can do homotopical algebra on the various categories of modules and algebras. In particular, motivic cohomology is naturally described as a commutative ring in the category of motivic functors.

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1 Introduction

One of the advantages of the modern formulations of algebraic topology is that invariants can be expressed, not merely as functors into groups, but actually as functors taking values in spaces. As such, the invariants are now themselves approachable by means of standard moves in algebraic topology; they can be composed or otherwise manipulated giving structure and control which cannot be obtained when looking at isolated algebraic invariants.

Although handling much more rigid objects, Voevodsky's motivic spaces [16] are modeled on topological spaces. The power of this approach lies in that many of the techniques and results from topology turn out to work in algebraic geometry. As in topology, many of the important constructions in the theory can be viewed as functors of motivic spaces. The functor $M\mathbb{Z}$ (called L in [16]) which defines motivic cohomology is an example: it accepts motivic spaces as input and gives a motivic space as output. Given the importance of such functors and the development of algebraic topology in the 1990s, it is ripe time for a thorough study of these functors.

In this paper we initiate such a program for functors in the category of motivic spaces. The functors we shall consider are the analogs of continuous functors: *motivic functors* ($M\mathbb{Z}$ is an example; precise definitions will appear below). This involves setting up a homological – or rather homotopical – algebra for motivic functors, taking special care of how this relates to multiplicative and other algebraic properties.

A large portion of our work deals with the technicalities involved in setting up a variety of model structures on the category **MF** of motivic functors, each localizing at different aspects of motivic functors.

One of the model structures we construct on **MF** is Quillen equivalent to the stable model category of motivic spectra as defined, for instance by Jardine [10] and by Hovey [8].

Just as in the topological case, this solution comes with algebraic structure in the form of a symmetric monoidal smash product $\wedge$. Furthermore, the algebra and homotopy cooperate so that a meaningful theory paralleling that of ring spectra and modules follows. A tentative formulation is

Theorem. *There exists a monoidal model category structure* $\mathbf{MF}_{\mathrm{sph}}$ *on* **MF** *satisfying the monoid axiom, and a lax symmetric monoidal Quillen equivalence between* $\mathbf{MF}_{\mathrm{sph}}$ *and the model category of motivic symmetric spectra.*

To be slightly more concrete, a motivic space in our context is just a pointed simplicial presheaf on the category of smooth schemes over a base scheme S. There is a preferred "sphere" given by the Thom space T of the trivial line bundle $\mathbb{A}^1_S$. A motivic spectrum is a sequence of motivic spaces $E_0, E_1, \cdots$ together with structure maps

$$T \wedge E_n \longrightarrow E_{n+1}.$$

We should perhaps comment on the continuous/motivic nature of our functors, since this aspect may be new to some readers. Let $\mathcal{M}$ be the category of motivic spaces and $\mathbf{f}\mathcal{M}$ the subcategory of finitely presentable motivic spaces. A motivic functor is a functor

$$X \colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$$

which is "continuous" or "enriched" in the sense that it induces a map of internal hom objects. The enrichment implies that there is a natural map

$$A \wedge X(B) \longrightarrow X(A \wedge B).$$

As a consequence, any motivic functor X gives rise to a motivic spectrum $\mathrm{ev}(X)$ by "evaluating on spheres", that is

$$\mathrm{ev}(X)_n := X(T^{\wedge n})$$

with structure map

$$T \wedge \mathrm{ev}(X)_n = T \wedge X(T^{\wedge n}) \longrightarrow X(T \wedge T^{\wedge n}) = \mathrm{ev}(X)_{n+1}$$

given by the enrichment. The motivic functors $\mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$ form the category **MF** mentioned in the main theorem, and the evaluation on spheres induces the Quillen equivalence. The inclusion $\mathbf{f}\mathcal{M} \hookrightarrow \mathcal{M}$ is the unit in the monoidal structure and plays the rôle of the sphere spectrum.

The reader should keep in mind how simple our objects of study are: they are just functors of motivic spaces. All coherence problems one might conceive of in relation to multiplicative structure, and which are apparent if one works with e.g. motivic symmetric spectra, can safely be forgotten since they are taken care of by the coherence inherent to the category of motivic spaces. Furthermore, the smash product in our model is just like the usual tensor product in that, though it is slightly hard to picture $X \wedge Y$, it is very easy to say what the maps

$$X \wedge Y \longrightarrow Z$$

are: they are simply natural maps

$$X(A) \wedge Y(B) \longrightarrow Z(A \wedge B),$$

where the smash product is sectionwise the smash product of pointed simplicial sets; this is all we require to set up a simple motivic theory with multiplicative structure.

A *motivic ring* is a monoid in **MF**. These are the direct analogs of ring spectra. The multiplicative structure of motivic cohomology comes from the fact that $M\mathbb{Z}$ is a commutative motivic ring. This means we can consider $M\mathbb{Z}$-modules and also $M\mathbb{Z}$-algebras. Our framework allows one to do homotopical algebra. For instance:

Theorem. *The category of $M\mathbb{Z}$-modules in $\mathbf{MF}_{\mathrm{sph}}$ acquires a monoidal model category structure and the monoid axiom holds.*

The "spherewise" structure $\mathbf{MF}_{\mathrm{sph}}$ is not the only interesting model structure there is on **MF**. One aspect we shall have occasion to focus on is the fact that although most interesting motivic functors preserve weak equivalences (hence the name "homotopy functors"), categorical constructions can ruin this property. The standard way of getting around this problem is to consider only derived functors. While fully satisfying when considering one construction at the time, this soon clobbers up the global picture. A more elegant and functorially satisfying approach is to keep our category and its constructions as they are, but change our model structure. Following this idea we construct a model structure suitable for studying homotopy functors, and yet another model structure which is more suitable for setting up a theory of Goodwillie calculus for motivic spaces.

As with the stable model, these models respect the smash product and algebraic structure. The following statement gives an idea of what the homotopy functor model expresses

Theorem. *There exists a monoidal model category structure* $\mathbf{MF}_{\mathrm{hf}}$ *on* $\mathbf{MF}$ *satisfying the monoid axiom. In this structure every motivic functor is weakly equivalent to a homotopy functor, and a map of homotopy functors* $X \longrightarrow Y$ *is a weak equivalence if and only if for all finitely presentable motivic spaces* A *the evaluation* $X(A) \longrightarrow Y(A)$ *is a weak equivalence of motivic spaces.*

At this point it is interesting to compare with Lydakis' setup [11] for simplicial functors, and note how differently simplicial sets and motivic spaces behave. In the motivic case the theory fractures into many facets which coincide for simplicial sets. For instance, there is no reason why the notions of "stable" and "linear" (in Goodwillie and Waldhausen's sense) should coincide.

The paper is organized as follows. In section 2 we set up the model structures for unstable motivic homotopy theory suitable for our purposes.

In section 3 we present the four basic model structures on motivic functors. In the preprint version of this paper we allowed the source category of motivic functors to vary. This handy technical tool has been abandoned in this paper for the sake of concreteness. We thank the referee for this suggestion and other detailed comments.

All along the properties necessary for setting up a theory of rings and modules are taken care of, and the results are outlined in section 4.

Contents

2 Motivic Spaces

In this section we recall some facts about the category of *motivic spaces* and fix some notation. We briefly discuss the categorical properties, and then the homotopical properties.

For background in model category theory we refer to [7] while for enriched category theory we refer to [3] and [4].

Let S be a Noetherian scheme of finite Krull dimension. Denote by Sm/S the category of smooth S-schemes of finite type. Due to the finiteness condition, Sm/S is an essentially small category. Furthermore, it has pullbacks, a terminal object S and an initial object $\emptyset$, the empty scheme. If $U, V \in \mathrm{Ob}\,\mathrm{Sm}/S$, we denote the set of maps between U and V by $\mathbf{Set}_{\mathrm{Sm}/S}(U,V)$.

Let $\mathcal{S}$ be the closed symmetric monoidal category of pointed simplicial sets with internal hom objects $\mathcal{S}(-,-)$. Recall that the standard n-simplex Δ^n is the simplicial set represented by $[n] \in \Delta$.

Definition 2.1. A *motivic space* is a contravariant functor $A : \mathrm{Sm}/S \longrightarrow \mathcal{S}$. Let $\mathcal{M}_S$ (or just $\mathcal{M}$ if confusion is unlikely to result) denote the category of motivic spaces and natural transformations.

By reversal of priorities, $\mathcal{M}$ can alternatively be viewed as the category of pointed set-valued presheaves on $\mathrm{Sm}/S \times \Delta$. Denote by

$$\begin{aligned} \mathrm{Sm}/S &\longrightarrow \mathcal{M} \\ U &\longmapsto h_U \end{aligned}$$

the Yoneda functor $h_U(V) = \mathbf{Set}_{\mathrm{Sm}/S}(V,U)_+$ considered as a discrete pointed simplicial set (the plus denotes an added base point).

Recall the following facts about the functor category $\mathcal{M}$:

Proposition 2.2. *The category $\mathcal{M}$ is a locally finitely presentable bicomplete $\mathcal{S}$-category. The pointwise smash product gives $\mathcal{M}$ a closed symmetric monoidal structure.*

Since $\mathcal{M}$ is locally finitely presentable, it follows that finite limits commute with filtered colimits. To fix notation, we find it convenient to explicate some of this structure.

The pointwise smash $A \wedge B$ on $\mathcal{M}$ is given by

$$(A \wedge B)(U) = A(U) \wedge B(U).$$

The unit is the constant presheaf S^0. If $U \in \operatorname{Ob}\mathrm{Sm}/S$, then the evaluation functor

$$\mathrm{Ev}_U \colon \mathcal{M} \longrightarrow \mathcal{S}, \qquad \mathrm{Ev}_U(A) = A(U)$$

preserves limits and colimits. The left adjoint of Ev_U is the functor

$$\mathrm{Fr}_U \colon \mathcal{S} \longrightarrow \mathcal{M}, \qquad \mathrm{Fr}_U(K) = h_U \wedge K.$$

Note that, since $h_S(V) = S^0$, we will often write K instead of $\mathrm{Fr}_S K$. Checking the relevant conditions we easily get that the functors Fr_S and Ev_U are strict symmetric monoidal, while Fr_U is lax symmetric monoidal. The pair $(\mathrm{Fr}_U, \mathrm{Ev}_U)$ is an $\mathcal{S}$-adjoint pair.

Using Fr_S we get (co)actions ("(co)tensors") of $\mathcal{S}$ on $\mathcal{M}$: if $A \in \mathcal{M}$ and $K \in \mathcal{S}$ the functor $A \wedge K = A \wedge Fr_S K \in \mathcal{M}$ sends $U \in \operatorname{Ob}\mathrm{Sm}/S$ to $A(U) \wedge K \in \mathcal{S}$, and the functor A^K sends $U \in \operatorname{Ob}\mathrm{Sm}/S$ to $\mathcal{S}(K, A(U))$.

We let $\mathbf{Set}_{\mathcal{M}}(A, B)$ be the set of natural transformations from A to B in $\mathcal{M}$. The enrichment of $\mathcal{M}$ in $\mathcal{S}$ is defined by letting the pointed simplicial set of maps from A to B have n-simplices

$$\mathcal{S}_{\mathcal{M}}(A, B)_n := \mathbf{Set}_{\mathcal{M}}(A \wedge \Delta^n_+, B).$$

Its simplicial structure follows from functoriality of the assignment $[n] \longmapsto \Delta^n$. The internal hom object is in turn given by

$$\mathcal{M}(A, B)(U) = \mathcal{S}_{\mathcal{M}}(A \wedge h_U, B).$$

Definition 2.3. A motivic space A is *finitely presentable* if the set-valued hom functor $\mathbf{Set}_{\mathcal{M}}(A, -)$ commutes with filtered colimits. Similarly, A is $\mathcal{M}$-*finitely presentable* if the internal hom functor $\mathcal{M}(A, -)$ commutes with filtered colimits.

Recall that a pointed simplicial set is finitely presentable if and only if it is finite, that is, if it has only finitely many non-degenerate simplices. On the other hand, a pointed simplicial set K is finite if and only the $\mathcal{S}$-valued hom functor $\mathcal{S}(K, -)$ commutes with filtered colimits. The same holds for motivic spaces, as one can deduce from the following standard fact [3, 5.2.5].

Lemma 2.4. *Every motivic space is a filtered colimit of finite colimits of motivic spaces of the form $h_U \wedge \Delta^n_+$.*

Let K be a pointed simplicial set. Using Lemma 2.4, the natural isomorphism $\mathcal{M}(h_U \wedge K, A) \cong A(U \times_S -)^K$ and the fact that Ev_U commutes with colimits we get

Lemma 2.5. *Let K be a finite pointed simplicial set and $U \in \mathrm{Ob}\,\mathrm{Sm}/S$. Then $h_U \wedge K$ is $\mathcal{M}$-finitely presentable. The class of $\mathcal{M}$-finitely presentable motivic spaces is closed under retracts, finite colimits and the smash product. A motivic space is $\mathcal{M}$-finitely presentable if and only if it is finitely presentable.*

The finiteness condition imposed on objects of Sm/S implies that the full subcategory $\mathbf{f}\mathcal{M}$ of finitely presentable motivic spaces in $\mathcal{M}$ is equivalent to a small category [3, 5.3.8], cf. [3, 5.3.3] and the pointed version of [3, 5.2.2b]. Out of convenience, since $\mathbf{f}\mathcal{M}$ is the codomain of the functor category $\mathbf{MF}$ one could choose such an equivalence. This ends our discussion of categorical precursors.

2.1 Unstable homotopy theory

Summarizing this section we get a model structure $\mathcal{M}_{\mathrm{mo}}$ on $\mathcal{M}$ called the *motivic model structure* satisfying

1. $\mathcal{M}_{\mathrm{mo}}$ is weakly finitely generated.
2. $\mathcal{M}_{\mathrm{mo}}$ is proper.
3. The identity on $\mathcal{M}_{\mathrm{mo}}$ is a left Quillen equivalence to the Goerss-Jardine $\mathbb{A}^1$-model structure [10].
4. The smash product gives $\mathcal{M}_{\mathrm{mo}}$ a monoidal model structure.
5. The smash product preserves weak equivalences.
6. $\mathcal{M}_{\mathrm{mo}}$ satisfies the monoid axiom.

For the convenience of the reader we repeat briefly for $\mathcal{M}$ the definitions of the notions *weakly finitely generated, monoidal model structure* and the *monoid axiom*; for details, see for example [5, 3.4, 3.7, 3.8].

Weakly finitely generated means in particular that the cofibrations and acyclic cofibrations in $\mathcal{M}$ are generated by sets I and J, respectively [7, 2.1.7]. In addition, we require that I has finitely presented domains and codomains, the domains of J are small and that there exists a subset J' of J with finitely presented domains and codomains such that a map $A \longrightarrow B$ of motivic spaces with fibrant codomain is a fibration if and only if it has the right lifting property with respect to all objects of J'.

Let $f\colon A \longrightarrow B$ and $g\colon C \longrightarrow D$ be two maps in $\mathcal{M}$. The *pushout product* of f and g is the canonical map

$$f \square g\colon A \wedge D \coprod_{A \wedge C} B \wedge C \longrightarrow C \wedge D.$$

That $\mathcal{M}$ is a *monoidal model category* means that the pushout product of two cofibrations in $\mathcal{M}$ is a cofibration, and an acyclic cofibration if either one of the two cofibrations is so. It implies that the smash product descends to the homotopy category of $\mathcal{M}$. If $\mathrm{aCof}(\mathcal{M})$ denotes the acyclic cofibrations of $\mathcal{M}$, then the *monoid axiom* means that all the maps in $\mathrm{aCof}(\mathcal{M}) \wedge \mathcal{M}$-cell are weak equivalences. Among other nice consequences mentioned below, the monoid axiom allows to lift model structures to categories of monoids and modules over a fixed monoid [14].

Definition 2.6. A map $A \longrightarrow B$ in $\mathcal{M}$ is a *schemewise weak equivalence* if, for all $U \in \operatorname{Ob}\mathrm{Sm}/S$, $A(U) \longrightarrow B(U)$ is a weak equivalence in $\mathcal{S}$. *Schemewise fibrations* and *schemewise cofibrations* are defined similarly. A *cofibration* is a map having the left lifting property with respect to all schemewise acyclic fibrations.

Note that the schemewise cofibrations are simply the monomorphisms. We get the following basic model structure.

Theorem 2.7. *The schemewise weak equivalences, schemewise fibrations and cofibrations equip $\mathcal{M}$ with the structure of a proper monoidal $\mathcal{S}$-model category. The sets*

$$\{h_U \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+\}_{n \geq 0,\, U \in \operatorname{Ob}\mathrm{Sm}/S}$$

$$\{h_U \wedge (\Lambda^n_i \hookrightarrow \Delta^n)_+\}_{0 \leq i \leq n,\, U \in \operatorname{Ob}\mathrm{Sm}/S}$$

induced up from the corresponding maps in $\mathcal{S}$ are sets of generating cofibrations and acyclic cofibrations, respectively. The domains and codomains of the maps in these generating sets are finitely presentable. For any $U \in \operatorname{Ob}\mathcal{M}$, the pair $(\mathrm{Fr}_U, \mathrm{Ev}_U)$ is a Quillen pair.

Proof. The existence of the model structure follows from [7, 2.1.19], using the generating cofibrations and generating acyclic cofibrations described above. The properties which have to be checked are either straightforward or follow from 2.5 and properties of the standard model structure on simplicial sets. Properness follows from properness in $\mathcal{S}$, where we use that a cofibration is in particular a schemewise cofibration.

Clearly, Fr_U is a left Quillen functor for all $U \in \operatorname{Ob}\mathrm{Sm}/S$. Using the natural isomorphism

$$(h_U \wedge K) \wedge (h_V \wedge L) \cong h_{U \times_S V} \wedge (K \wedge L),$$

we see that for $f_j\colon K_j \longrightarrow L_j \in \mathcal{S}$ and $U_j \in \operatorname{Ob}\mathrm{Sm}/S$, $j = 1,2$, we may identify the pushout product of $h_{U_1} \wedge f_1$ and $h_{U_2} \wedge f_2$ with the map

$$h_{U_1 \times_S U_2} \wedge \left((K_1 \wedge L_2) \coprod_{(K_1 \wedge K_2)} (L_1 \wedge K_2) \right) \longrightarrow h_{U_1 \times_S U_2} \wedge (L_1 \wedge L_2).$$

Hence the pushout product axiom in $\mathcal{S}$ implies the pushout product axiom for $\mathcal{M}$. It follows that $\mathcal{M}$ is a monoidal $\mathcal{S}$-model category via the functor Fr_S. □

Notation 2.8. We let $\mathcal{M}_{\mathrm{sc}}$ denote the model structure of 2.7 on $\mathcal{M}$. Schemewise weak equivalences will be written $\xrightarrow{\sim\mathrm{sc}}$ and schemewise fibrations $\overset{\mathrm{sc}}{\twoheadrightarrow}$. Cofibrations are denoted by $\rightarrowtail$ (since not all schemewise cofibrations are cofibrations in $\mathcal{M}_{\mathrm{sc}}$). Choose a cofibrant replacement functor $(-)^c \longrightarrow \mathrm{Id}_{\mathcal{M}}$ in $\mathcal{M}_{\mathrm{sc}}$ so that for any motivic space A, there is a schemewise acyclic fibration $A^c \overset{\sim\mathrm{sc}}{\twoheadrightarrow} A$ with cofibrant domain. We note that every representable motivic space is cofibrant.

The following statements are easily verified.

Lemma 2.9. *Taking the smash product $-\wedge A$ or a cobase change along a schemewise cofibration preserves schemewise weak equivalences for all $A \in \operatorname{Ob}\mathcal{M}$. The monoid axiom holds in $\mathcal{M}_{\mathrm{sc}}$.*

It turns out that the properties in 2.7 and 2.9 hold in the model for motivic homotopy theory. The latter is obtained by considering Sm/S in its Nisnevich topology and by inverting the affine line $\mathbb{A}^1_S$. The following allows to incorporate Bousfield localization [6] in the motivic homotopy theory.

Recall that the Nisnevich topology is generated by *elementary distinguished squares* [12]. These are pullback squares of the form

$$Q \quad = \quad \begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow \phi \\ U & \xrightarrow{\psi} & X \end{array}$$

where ϕ is étale, ψ is an open embedding and $\phi^{-1}(X-U) \longrightarrow (X-U)$ is an isomorphism of schemes (with the reduced structure).

Definition 2.10. A schemewise fibrant motivic space A is *motivically fibrant* if the following conditions hold.

- $A(\emptyset)$ is contractible.
- If Q is an elementary distinguished square, then $A(Q)$ is a homotopy pullback square of pointed simplicial sets.

- If $U \in \operatorname{Ob} \mathrm{Sm}/S$, the canonically induced map $A(U) \longrightarrow A(U \times_S \mathbb{A}^1_S)$ is a weak equivalence of pointed simplicial sets.

The first two conditions imply that A is a sheaf up to homotopy in the Nisnevich topology. The third condition implies that $\mathbb{A}^1_S \longrightarrow S$ is a weak equivalence in the following sense (where $(-)^c$ is the cofibrant replacement functor in $\mathcal{M}_{\mathrm{sc}}$ chosen in 2.8):

Definition 2.11. A map $f : A \longrightarrow B$ of motivic spaces is a *motivic weak equivalence* if, for every motivically fibrant Z, the map

$$\mathcal{S}_{\mathcal{M}}(f^c, Z) : \mathcal{S}_{\mathcal{M}}(B^c, Z) \longrightarrow \mathcal{S}_{\mathcal{M}}(A^c, Z)$$

is a weak equivalence of pointed simplicial sets.

In 2.17 we shall note that 2.11 agrees with the corresponding notion in [12].

Using either Smith's work on combinatorial model categories or by Blander's [1, 3.1], we have

Theorem 2.12. *The motivic weak equivalences and the cofibrations define a cofibrantly generated model structure on $\mathcal{M}$.*

Notation 2.13. We refer to the model structure in 2.12 as the *motivic model structure* and make use of the notation $\mathcal{M}_{\mathrm{mo}}$. Its weak equivalences will be denoted by $\xrightarrow{\sim}$ and its fibrations by $\twoheadrightarrow$. In accordance with 2.10, we refer to the fibrations as *motivic fibrations*, since a motivic space A is motivically fibrant if and only if $A \longrightarrow *$ is a motivic fibration.

Alas, this notation conflicts slightly with [10]. See 2.17.

Next we shall derive some additional properties of the motivic model structure, starting with a characterization of motivic fibrations with motivically fibrant codomain. As above, consider an elementary distinguished square:

$$Q \quad = \quad \begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ U & \xrightarrow{\psi} & X \end{array}$$

Using the simplicial mapping cylinder we factor the induced map $h_P \longrightarrow h_Y$ as a cofibration $h_P \rightarrowtail C = (h_P \wedge \Delta^1_+) \coprod_{h_P} h_Y$ followed by a simplicial homotopy equivalence $C \xrightarrow{\simeq} h_Y$. Similarly we factor the canonical map $sq = h_U \coprod_{h_P} C \longrightarrow h_X$ as $sq \xrightarrow{q} tq \xrightarrow{\simeq} h_X$. Finally, we consider $h_{U \times_S \mathbb{A}^1_S} \longrightarrow h_U$ and the factorization $h_{U \times_S \mathbb{A}^1_S} \xrightarrow{u} C_u \xrightarrow{\simeq} h_U$.

Definition 2.14. Let $\mathcal{Q}$ denote the collection of all elementary distinguished squares in Sm/S. Since Sm/S is essentially small, we may consider a skeleton and form the set $\tilde{J}$ of maps

$$\{* \rightarrowtail h_\emptyset\} \cup \{q : sq \rightarrowtail tq\}_{Q \in \mathcal{Q}} \cup \{u : h_{U \times_S \mathbb{A}^1_S} \rightarrowtail C_u\}_{U \in \operatorname{Ob} \mathrm{Sm}/S}.$$

Let J' be the set of pushout product maps $f \Box g$ where $f \in \tilde{J}$ and $g \in \{\partial\Delta^n_+ \hookrightarrow \Delta^n_+\}$.

Lemma 2.15. *A schemewise fibration with motivically fibrant codomain is a motivic fibration if and only if it has the right lifting property with respect to the set J' of 2.14.*

Proof. We note that the (simplicial) functor $\mathcal{S}_{\mathcal{M}}(B, -)$ preserves simplicial homotopy equivalences, which in particular are schemewise weak equivalences. From the definitions, it then follows that a schemewise fibrant motivic space A is motivically fibrant if and only if the canonical map $A \xrightarrow{\text{sc}} *$ enjoys the right lifting property with respect to J'. The statement follows using properties of Bousfield localizations [6, 3.3.16]. $\square$

Corollary 2.16. *The model category $\mathcal{M}_{\text{mo}}$ is weakly finitely generated. In particular, motivic weak equivalences and motivic fibrations with motivically fibrant codomains are closed under filtered colimits.*

In the symmetric spectrum approach due to Jardine [10] one employs a slightly different model structure on motivic spaces. The cofibrations in this model structure are the schemewise cofibrations, i.e. the monomorphisms, while the weak equivalences are defined by localizing the so-called Nisnevich local weak equivalences [9] with respect to a rational point $h_S \longrightarrow h_{\mathbb{A}^1_S}$. Let us denote this model structure by $\mathcal{M}_{\text{GJ}}$. Corollary 2.16 shows an advantage of working with $\mathcal{M}_{\text{mo}}$. On the other hand, in $\mathcal{M}_{\text{GJ}}$ every motivic space is schemewise cofibrant. We compare these two model structures in

Theorem 2.17. *The weak equivalences in the model structures $\mathcal{M}_{\text{mo}}$ and $\mathcal{M}_{\text{GJ}}$ coincide. In particular, the identity $\text{Id}_{\mathcal{M}} : \mathcal{M}_{\text{mo}} \longrightarrow \mathcal{M}_{\text{GJ}}$ is the left adjoint of a Quillen equivalence.*

Proof. The fibrations in the pointed version of the model structure in [9] are called *global fibrations*. A weak equivalence in this model structure is a local weak equivalence, and a cofibration is a schemewise cofibration. We say that a globally fibrant presheaf Z is i_0*-fibrant* if the map $\mathcal{M}(h_S, Z) \longrightarrow \mathcal{M}(h_{\mathbb{A}^1_S}, Z)$ induced by the zero-section $i_0 : S \longrightarrow \mathbb{A}^1_S$ is an acyclic global fibration. Since h_{i_0} is a monomorphism, this is equivalent to the pointed version of h_{i_0}-local simplicial presheaves in [9, §1.2].

A map $f \colon A \longrightarrow B$ is an i_0*-equivalence* if for all i_0-fibrant presheaf Z, the induced map of pointed simplicial sets $\mathcal{S}_{\mathcal{M}}(B, Z) \longrightarrow \mathcal{S}_{\mathcal{M}}(A, Z)$ is a weak equivalence. The i_0-equivalences are the weak equivalences in $\mathcal{M}_{GJ}$.

First we prove that any motivic weak equivalence is an i_0-equivalence. Suppose that $f \colon A \xrightarrow{\sim} B$ and Z is i_0-fibrant. Then Z is motivically fibrant, and thus $\mathcal{S}_{\mathcal{M}}(f^c, Z)$ is a weak equivalence. Since f^c is related to f via schemewise weak equivalences, it follows that f is an i_0-equivalence. This proves that motivic weak equivalences are i_0-equivalences.

Choose a motivically fibrant Z and suppose $f : A \longrightarrow B$ is an i_0-equivalence. According to [9] there exists a map $Z \xrightarrow{\sim \mathrm{sc}} Z'$ where Z' is globally fibrant. Since the domain and codomain of h_{i_0} are cofibrant, 2.7 implies that Z' is i_0-fibrant. Using the fact that $\mathcal{M}_{\mathrm{mo}}$ is an $\mathcal{S}$-model category, we get the following commutative diagram:

$$\begin{tikzcd} \mathcal{S}_{\mathcal{M}}(B^c, Z) \arrow[r, "\mathcal{S}_{\mathcal{M}}(f^c, Z)"] \arrow[d, "\sim"'] & \mathcal{S}_{\mathcal{M}}(A^c, Z) \arrow[d, "\sim"] \\ \mathcal{S}_{\mathcal{M}}(B^c, Z') \arrow[r, "\mathcal{S}_{\mathcal{M}}(f^c, Z')"] & \mathcal{S}_{\mathcal{M}}(A^c, Z') \end{tikzcd}$$

The map $\mathcal{S}_{\mathcal{M}}(f^c, Z')$ is a weak equivalence of spaces since f^c is an i_0-equivalence, i.e. f is a motivic weak equivalence. The Quillen equivalence follows. $\square$

Lemma 2.18. *Smashing with a cofibrant motivic space preserves motivic weak equivalences.*

Proof. Suppose Z is motivically fibrant, that is, the canonical map $Z \longrightarrow\!\!\!\!\!\rightarrow *$ is a schemewise fibration having the right lifting properties with respect to J'. If C is cofibrant, then $\mathcal{M}(C, Z)$ is schemewise fibrant according to 2.7. We claim $\mathcal{M}(C, Z)$ is motivically fibrant. For this, it suffices to prove for every generating cofibration

$$i := h_U \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+,$$

the induced map $\mathcal{M}(i, Z)$ has the right lifting property with respect to J'. By adjointness, it suffices to prove that the pushout product of i and any map in J' is a composition of cobase changes of maps in J'. This holds by the following facts.

- $h_\emptyset \wedge h_U \cong h_\emptyset$
- Taking the product of an elementary distinguished square with any object $U \in \mathrm{Ob}\,\mathrm{Sm}/S$ yields an elementary distinguished square.
- $(h_{V \times_S \mathbb{A}^1} \longrightarrow h_V) \wedge h_U \cong h_{U \times_S V \times_S \mathbb{A}^1} \longrightarrow h_{U \times_S V}$
- The pushout product of $\partial\Delta^m \hookrightarrow \Delta^m$ and $\partial\Delta^n \hookrightarrow \Delta^n$ is an inclusion of simplicial sets, hence can be formed by attaching cells.

To conclude, it remains to note that for every motivically fibrant Z and every $f : A \xrightarrow{\sim} B$, the induced map $\mathcal{S}_{\mathcal{M}}((f \wedge C)^c, Z)$ is a weak equivalence. First, note that by the argument above, the map $\mathcal{S}_{\mathcal{M}}(f^c \wedge C, Z) \cong \mathcal{S}_{\mathcal{M}}(f^c, \mathcal{M}(C, Z))$

is a weak equivalence. This means that $f^c \wedge C$ is a motivic weak equivalence. But 2.9 and the commutative diagram

$$\begin{tikzcd}
(A \wedge C)^c \arrow[r, two heads, "\sim \mathrm{sc}"] \arrow[d, "(f\wedge C)^c"'] & A \wedge C \arrow[d, "f \wedge C"'] & A^c \wedge C \arrow[l, "\sim \mathrm{sc}"'] \arrow[d, "f^c \wedge C"'] \\
(B \wedge C)^c \arrow[r, two heads, "\sim \mathrm{sc}"] & B \wedge C & B^c \wedge C \arrow[l, "\sim \mathrm{sc}"']
\end{tikzcd}$$

show that $(f \wedge C)^c$ is a motivic weak equivalence if and only $f^c \wedge C$ is so. □

Corollary 2.19. *$\mathcal{M}_{\mathrm{mo}}$ is a monoidal $\mathcal{M}_{\mathrm{sc}}$-model category.*

Proof. We have to check that the pushout product of $h_U \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+$ and a generating acyclic cofibration in $\mathcal{M}_{\mathrm{mo}}$ is a motivic weak equivalence for all $U \in \mathrm{Ob}\,\mathrm{Sm}/S$ and $n \geq 0$. Since h_U is cofibrant, the result follows from 2.18 and left properness of $\mathcal{M}_{\mathrm{mo}}$. □

We can now extend 2.18 to all motivic spaces.

Lemma 2.20. *Taking the smash product $- \wedge A$ or a cobase change along a schemewise cofibration preserves motivic weak equivalences for all $A \in \mathrm{Ob}\,\mathcal{M}$.*

Proof. For the first claim: we may replace A by A^c using 2.9 and hence conclude using 2.18. The second claim follows by factoring any motivic weak equivalence as a motivic acyclic cofibration followed by a schemewise acyclic fibration, and quoting 2.9 for the schemewise acyclic fibration. □

Lemma 2.21. *The monoid axiom holds in $\mathcal{M}_{\mathrm{mo}}$.*

Proof. Let f be an acyclic cofibration in $\mathcal{M}_{\mathrm{mo}}$ and let C be any motivic space. By 2.20, $f \wedge C$ is a schemewise cofibration and a motivic weak equivalence. It suffices to prove that the class of such maps is closed under cobase changes and sequential compositions. For this we use 2.20 and 2.16, respectively. □

Lemma 2.22. *The model category $\mathcal{M}_{\mathrm{mo}}$ is proper.*

Proof. Left properness of $\mathcal{M}_{\mathrm{mo}}$ is obvious since the cofibrations are not altered. To see that the model structure is right proper, one can either employ [1, 3.1], or mimic Jardine's proof of [10, A.5]. □

Remark 2.23. It is worth noticing that all of the results above hold more generally. One may replace Sm/S by any site with interval, see [12], in which the Grothendieck topology is generated by a bounded, complete and regular cd-structure [17]. An interesting example is the cdh-topology on the category Sch/S of schemes of finite type over S and representing interval the affine line.

2.2 Stable homotopy theory

The model category $\mathcal{M}_{\mathrm{mo}}$ has all the properties required to apply the results of [8, Section 4]. On the one hand, $\mathcal{M}_{\mathrm{mo}}$ is a cellular model category by [1], so Hirschhorn's localization methods work. On the other hand, one can also use Smith's combinatorial model categories for Bousfield localization. In any case, the category $\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, A)$ of spectra of motivic spaces (with respect to some cofibrant finitely presentable motivic space A) has a stable model structure. For precise statements consult [8, 4.12 and 4.14].

We are interested in special motivic spaces A. The basic "sphere" in motivic homotopy theory is obtained in the same way as the circle in classical homotopy theory. It is defined as the Thom space $\mathbb{A}^1_S/\mathbb{A}^1_S - \{0\}$ of the trivial line bundle. Since $\mathbb{A}^1_S$ is contractible, the pushout $(\mathbb{A}^1_S - \{0\}, 1) \wedge S^1$ of the diagram

$$* \longleftarrow h_S \wedge S^1 \xrightarrow{h_{i_1} \wedge S^1} h_{\mathbb{A}^1_S - \{0\}} \wedge S^1$$

is weakly equivalent to $\mathbb{A}^1_S/\mathbb{A}^1_S - \{0\}$ [12, 3.2.2]. In the diagram, the map $i_1 : S \longrightarrow \mathbb{A}^1_S - \{0\}$ is induced by the closed point $1 \in \mathbb{A}^1_S(S)$. Note that although $h_{i_1} \wedge S^1$ is a schemewise cofibration (i.e. monomorphism), it need not be a cofibration in the motivic model structure $\mathcal{M}_{\mathrm{mo}}$.

Since the domain and codomain of $h_{i_1} \wedge S^1$ are cofibrant, we may factor this map using the simplicial mapping cylinder as a cofibration $h_S \wedge S^1 \rightarrowtail C$ and a simplicial homotopy equivalence. The quotient $T := C/h_S \wedge S^1$ is then cofibrant and a finitely presentable motivic space, schemewise weakly equivalent to the smash product $(\mathbb{A}^1_S - \{0\}, 1) \wedge S^1$. Up to motivic weak equivalence the choice of T is irrelevant. See [8, 5.7] and cp. 2.20. Now the identity $\mathrm{Id}_{\mathcal{M}}$ is a left Quillen equivalence from $\mathcal{M}_{\mathrm{mo}}$ to the pointed version of Jardine's model structure on $\mathcal{M}$ by 2.17. So that by [8, 5.7] the stable model structure on the category of motivic spectra $\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T)$ is Quillen equivalent to Jardine's model for the motivic stable homotopy category. Using Voevodsky's observation about cyclic permutations, we get

Lemma 2.24. *The functor $- \wedge T \colon \mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T) \longrightarrow \mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T)$ is a Quillen equivalence.*

Proof. The identity $\mathrm{id}_{\mathcal{M}}$ induces a commutative diagram of left Quillen functors

$$\begin{tikzcd}
\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T) \arrow[r] \arrow[d, "-\wedge T"'] & \mathrm{Sp}(\mathcal{M}_{\mathrm{GJ}}, T) \arrow[d, "-\wedge T"] \\
\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T) \arrow[r] & \mathrm{Sp}(\mathcal{M}_{\mathrm{GJ}}, T)
\end{tikzcd}$$

where the two horizontal arrows are Quillen equivalences. Here $\mathcal{M}_{GJ}$ denotes the pointed version of the Goerss-Jardine model structure on $\mathcal{M}$. In $\mathcal{M}_{GJ}$, the cofibrations are the schemewise cofibrations. Hence every presheaf is cofibrant. By [8, 10.3] it suffices to establish that T is weakly equivalent to a symmetric presheaf A, so that the next diagram commutates where cyc: $A^{\wedge 3} \longrightarrow A^{\wedge 3}$ is the cyclic permutation map and H is a homotopy from the cyclic permutation to the identity; for details we refer to [8, 10.2].

$$\begin{tikzcd} A^{\wedge 3} \wedge h_S \arrow[r, "A^{\wedge 3} \wedge h_{i_0}"] \arrow[dr, "\mathrm{id}_{A^{\wedge 3} \wedge h_S}"'] & A^{\wedge 3} \wedge h_{\mathbb{A}^1} \arrow[d, "H"] & A^{\wedge 3} \wedge h_S \arrow[l, "A^{\wedge 3} \wedge h_{i_1}"'] \arrow[dl, "\mathrm{cyc} \wedge h_S"] \\ & A^{\wedge 3} \wedge h_S & \end{tikzcd}$$

The presheaf $\mathbb{A}^1_S/\mathbb{A}^1_S - \{0\}$ is weakly equivalent to T, and symmetric according to [10, 3.13]. Hence $- \wedge T$ on the right hand side is a Quillen equivalence, which implies the same statement for the functor $- \wedge T$ on the left hand side. $\square$

3 Motivic functors

In this section we shall introduce the category of motivic functors, describe its monoidal structure and display some of its useful homotopy properties. We do this in four steps. Each step involves giving a monoidal model structure to the category of motivic functors.

The first step is defining the pointwise model, which is of little practical value, but it serves as a building block for all the other models. The second step deals with the homotopy functor model. We advocate this as a tool for doing motivic homotopy theory on a functorial basis, mimicking the grand success in algebraic topology. The most interesting functors are homotopy invariant, but many natural constructions will take to functors which do not preserve weak equivalences. The homotopy functor model structure is a convenient way of handling these problems.

Thirdly we have the stable structure, which from our point of view is the natural generalization of stable homotopy theory from algebraic topology, but which unfortunately does not automatically agree with the other proposed models for stable motivic homotopy theory. Hence we are forced to park this theory in our technical garage for time being and introduce the fourth and final model structure: the spherewise model structure. Although technically not as nice as the stable model, the spherewise model is Quillen equivalent to the other models for motivic stable homotopy theory.

Many of the results in this section can be justified by inferring references to [5]. For the convenience of the reader we will indicate most proofs of these results.

3.1 The category of motivic functors

Recall the category of motivic spaces $\mathcal{M} = \mathcal{M}_S = [(\mathrm{Sm}/S)^{\mathrm{op}}, \mathcal{S}]$ discussed in the previous section. As a closed symmetric monoidal category, it is enriched over itself, hence an $\mathcal{M}$-category. Let $\mathbf{f}\mathcal{M}$ be the full sub-$\mathcal{M}$-category of finitely presentable motivic spaces.

Definition 3.1. A *motivic functor* is an $\mathcal{M}$-functor $X\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$. That is, X assigns to any finitely presentable motivic space A a motivic space XA together with maps of motivic spaces $\hom^X_{A,B}\colon \mathcal{M}(A,B) \longrightarrow \mathcal{M}(XA,XB)$ compatible with the enriched composition and identities. We let $\mathbf{MF}$ be the category of motivic functors and $\mathcal{M}$-natural transformations.

Since $\mathbf{MF}$ is a category of functors with bicomplete codomain, it is bicomplete and enriched over $\mathcal{M}$. If X and Y are motivic functors, let $\mathcal{M}_{\mathbf{MF}}(X,Y)$ be the motivic space of maps from X to Y. If A is a finitely presentable motivic space, then the motivic functor represented by A is given as

$$\mathcal{M}(A,-)\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}, \qquad \mathcal{M}(A,-)(B) = \mathcal{M}(A,B)$$

The enriched Yoneda lemma holds, and every motivic functor can be expressed in a canonical way as a colimit of representable functors.

Theorem 3.2 (Day). *The category of motivic functors is closed symmetric monoidal with unit the inclusion* $\mathbb{I}\colon \mathbf{f}\mathcal{M} \hookrightarrow \mathcal{M}$.

This theorem is a special case of [4]; it is simple enough to sketch the basic idea. Denote the monoidal product of two motivic functors X and Y by $X \wedge Y$. Since every motivic functor is a colimit of representables, it suffices to fix the monoidal product on representable functors

$$\mathcal{M}(A,-) \wedge \mathcal{M}(B,-) := \mathcal{M}(A \wedge B,-).$$

The internal hom is defined by setting

$$\mathbf{MF}(X,Y)(A) = \mathcal{M}_{\mathbf{MF}}(X, Y(- \wedge A)).$$

Let us describe a special feature of the category of motivic functors, which makes the monoidal product more transparent. The point is just that motivic functors can be composed. Note that any motivic functor $X\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$ can be extended – via enriched left Kan extension along the full inclusion $\mathbb{I}\colon \mathbf{f}\mathcal{M} \hookrightarrow \mathcal{M}$ – to an $\mathcal{M}$-functor $\mathbb{I}_*X\mathcal{M} \longrightarrow \mathcal{M}$ satisfying $\mathbb{I}_*X \circ \mathbb{I} \cong X$. Since the category of motivic spaces is locally finitely presentable 2.2, this defines an equivalence between $\mathbf{MF}$ and the category of $\mathcal{M}$-functors $\mathcal{M} \longrightarrow \mathcal{M}$ that preserve filtered colimits. Given motivic functors X and Y, one defines their composition by setting

$$X \circ Y := \mathbb{I}_*X \circ Y.$$

Moreover, there is the natural *assembly map* $X \wedge Y \longrightarrow X \circ Y$ which is an isomorphism provided Y is representable [5, 2.8]. In fact, if both X and Y are representable, then the assembly map is the natural adjointness isomorphism

$$\mathcal{M}(A,-) \wedge \mathcal{M}(B,-) = \mathcal{M}(A \wedge B,-) \cong \mathcal{M}(A, \mathcal{M}(B,-)).$$

Remark 3.3. A *motivic ring* is a monoid in the category of motivic functors. Given the simple nature of the smash product in **MF** motivic rings can be described quite explicitly. Running through the definitions we see that a map $X \wedge X \longrightarrow X$ of motivic functors is the same as an $\mathcal{M}$-natural transformation of two variables $XA \wedge XB \longrightarrow X(A \wedge B)$, and so a motivic ring is a motivic functor X together with natural transformations $XA \wedge XB \longrightarrow X(A \wedge B)$ and $A \longrightarrow XA$ such that the relevant diagrams commute. Hence motivic rings are analogous to Bökstedt's *functors with smash product* [2].

Example 3.4. Let SmCor/S be the category of smooth correspondences over S. The special case $S = \mathrm{Spec}(k)$ is described in [18]. A *motivic space with transfers* is an additive functor, or an **Ab**-functor, $F\colon (\mathrm{SmCor}/S)^{\mathrm{op}} \longrightarrow \mathbf{sAb}$ to the category of simplicial abelian groups. Let $\mathcal{M}^{\mathrm{tr}}$ be the category of motivic spaces with transfers. By forgetting the extra structure of having transfers and composing with the opposite of the graph functor $\Gamma\colon \mathrm{Sm}/S \longrightarrow \mathrm{SmCor}/S$ it results a forgetful functor $u\colon \mathcal{M}^{\mathrm{tr}} \longrightarrow \mathcal{M}$ with left adjoint $\mathbb{Z}_{\mathrm{tr}}\colon \mathcal{M} \longrightarrow \mathcal{M}^{\mathrm{tr}}$. The functor $\mathbb{Z}_{\mathrm{tr}}$ is determined by the property that $\mathbb{Z}_{\mathrm{tr}}(h_U \wedge \Delta^n_+) = \mathrm{Hom}_{\mathrm{SmCor}/S}(-,U) \otimes \mathbb{Z}(\Delta^n)$.

Let $M\mathbb{Z} \in \mathbf{MF}$ be the composite functor

$$\mathbf{f}\mathcal{M} \hookrightarrow \mathcal{M} \xrightarrow{\mathbb{Z}_{\mathrm{tr}}} \mathcal{M}^{\mathrm{tr}} \xrightarrow{u} \mathcal{M}.$$

We claim that $M\mathbb{Z}$ is a commutative monoid in **MF**. First, the unit $\mathbb{I} \longrightarrow M\mathbb{Z}$ is the unit of the adjunction between $\mathcal{M}$ and $\mathcal{M}^{\mathrm{tr}}$. To define a multiplication, we note using [4] and [15] that $\mathcal{M}^{\mathrm{tr}}$ is closed symmetric monoidal. Since the graph functor is strict symmetric monoidal and forgetting the addition is lax symmetric monoidal, general category theory implies $\mathbb{Z}_{\mathrm{tr}}$ is strict symmetric monoidal and u is lax symmetric monoidal. In particular, we get the natural multiplication map μ on $M\mathbb{Z}$, given by

$$u(\mathbb{Z}_{\mathrm{tr}}(A)) \wedge u(\mathbb{Z}_{\mathrm{tr}}(B)) \longrightarrow u(\mathbb{Z}_{\mathrm{tr}}(A) \otimes \mathbb{Z}_{\mathrm{tr}}(B)) \longrightarrow u(\mathbb{Z}_{\mathrm{tr}}(A \wedge B)).$$

To see that $M\mathbb{Z}$ is a motivic functor, consider the composition

$$\mathcal{M}(A,B) \wedge u\mathbb{Z}_{\mathrm{tr}}A \longrightarrow u\mathbb{Z}_{\mathrm{tr}}\mathcal{M}(A,B) \wedge u\mathbb{Z}_{\mathrm{tr}}A \longrightarrow u\mathbb{Z}_{\mathrm{tr}}(\mathcal{M}(A,B) \wedge A),$$

and note that $u\mathbb{Z}_{\mathrm{tr}}(\mathcal{M}(A,B) \wedge A)$ maps naturally to $u\mathbb{Z}_{\mathrm{tr}}B$. In 4.6 we show $M\mathbb{Z}$ represents Voevodsky's motivic Eilenberg-MacLane spectrum [16].

3.2 Evaluation on spheres

As explained in [5, Section 2.5], the category $\mathrm{Sp}(\mathcal{M},T)$ of motivic spectra with respect to the T of 2.2 can be described as a category of $\mathcal{M}$-functors. Let $T\mathrm{Sph}$ be the sub-$\mathcal{M}$-category of $\mathcal{M}$ with objects the smash powers $T^0 = S^0, T, T^{\wedge 2} := T \wedge T, T^{\wedge 3} := T \wedge (T^{\wedge 2}), \cdots$ of T. If $k \geq 0$ the motivic space of morphisms in $T\mathrm{Sph}$ from $T^{\wedge n}$ to $T^{\wedge n+k}$ is $T^{\wedge k}$ considered by adjointness as a subobject of $\mathcal{M}(T^{\wedge n}, T^{\wedge n+k})$. If $k < 0$ the morphism space is trivial. Let $i: T\mathrm{Sph} \hookrightarrow \mathbf{f}\mathcal{M}$ be the inclusion. Hence every motivic functor X gives rise to a motivic spectrum $\mathrm{ev}(X) := X \circ i$.

Similarly, the category $\mathrm{Sp}^{\Sigma}(\mathcal{M}, T)$ of motivic symmetric spectra is isomorphic to the category of $\mathcal{M}$-functors (with values in $\mathcal{M}$) from a slightly larger sub-$\mathcal{M}$-category $j: T\mathrm{Sph}^{\Sigma} \hookrightarrow \mathbf{f}\mathcal{M}$, which is determined by the property that it is the smallest sub-$\mathcal{M}$-category containing $T\mathrm{Sph}$ and the symmetric group $\Sigma(n)_+ \subseteq \mathcal{M}(T^{\wedge n}, T^{\wedge n})$ for all n. Hence, if U denotes the forgetful functor, then the evaluation map $\mathrm{ev}: \mathbf{MF} \longrightarrow \mathrm{Sp}(\mathcal{M}, T)$ factors as

$$\mathbf{MF} \xrightarrow{\mathrm{ev}'} \mathrm{Sp}^{\Sigma}(\mathcal{M}, T) \xrightarrow{U} \mathrm{Sp}(\mathcal{M}, T).$$

Moreover ev' is lax symmetric monoidal and its left adjoint is strict symmetric monoidal. For further details we refer the reader to [5, Section 2.6].

3.3 The pointwise structure

We first define the pointwise model structure on $\mathbf{MF}$. As earlier commented, the pointwise structure is of no direct use for applications, but it is vital for the constructions of the useful structures to come.

Definition 3.5. A map $f: X \longrightarrow Y$ in $\mathbf{MF}$ is a

- *Pointwise weak equivalence* if for every object A in $\mathbf{f}\mathcal{M}$ the induced map $f(A): X(A) \longrightarrow Y(A)$ is a weak equivalence in $\mathcal{M}_{\mathrm{mo}}$.
- *Pointwise fibration* if for every object A in $\mathbf{f}\mathcal{M}$ the induced map $f(A): X(A) \longrightarrow Y(A)$ is a fibration in $\mathcal{M}_{\mathrm{mo}}$.
- *Cofibration* if f has the left lifting property with respect to all pointwise acyclic fibrations.

The category $\mathbf{MF}$, together with these classes of morphisms, is denoted $\mathbf{MF}_{\mathrm{pt}}$ and referred to as the *pointwise structure* on $\mathbf{MF}$.

Theorem 3.6. *The pointwise structure* $\mathbf{MF}_{\mathrm{pt}}$ *is a cofibrantly generated proper monoidal model category satisfying the monoid axiom.*

Proof. The model structure follows from [7, 2.1.19], where the monoid axiom for $\mathcal{M}_{\mathrm{mo}}$ is used to ensure that the generating acyclic cofibrations listed in 3.7, as well as sequential compositions of cobase changes of these, are pointwise weak equivalences. The form of the generating (acyclic) cofibrations, together with the behavior of $\wedge$ on representables, ensures that $\mathbf{MF}$ is a monoidal model category [7, 4.2.5]. Right properness follows at once from the fact that $\mathcal{M}_{\mathrm{mo}}$ is right proper 2.22. Left properness requires more than $\mathcal{M}_{\mathrm{mo}}$ being left proper, but follows from 2.20.

To prove the monoid axiom, let X be a motivic functor and consider the smash product

$$X \wedge \mathcal{M}(A,-) \wedge sj \xrightarrow{X \wedge \mathcal{M}(A,-) \wedge j} X \wedge \mathcal{M}(A,-) \wedge tj$$

with a generating acyclic cofibration, where j is a generating acyclic cofibration for $\mathcal{M}_{\mathrm{mo}}$. It is a pointwise weak equivalence by 2.20, and also pointwise a schemewise cofibration. In particular, any sequential composition of cobase changes of maps like these is a pointwise weak equivalence, which concludes the proof. □

Remark 3.7. If A varies over the set of isomorphism classes in $\mathbf{f}\mathcal{M}$ and $i : si \rightarrowtail ti$ varies over the generating (acyclic) cofibrations in $\mathcal{M}_{\mathrm{mo}}$, then the maps $\mathcal{M}(A,-) \wedge i : \mathcal{M}(A,-) \wedge si \longrightarrow \mathcal{M}(A,-) \wedge ti$ form a set of generating (acyclic) cofibrations for $\mathbf{MF}_{\mathrm{pt}}$. In particular, all representable motivic functors (for example the unit) are cofibrant.

The following theorem will help us to deduce the monoid axiom for some other model structures on motivic functors.

Theorem 3.8. *Smashing with a cofibrant object in* $\mathbf{MF}_{\mathrm{pt}}$ *preserves pointwise equivalences.*

Proof. If X is representable, say $X = \mathcal{M}(A,-)$ and $f : Y \longrightarrow Z$ is a pointwise weak equivalence, then the assembly map is an isomorphism

$$f \wedge \mathcal{M}(A,-) \cong f \circ \mathcal{M}(A,-) = \mathbb{I}_* f \circ \mathcal{M}(A,-).$$

Since $\mathbb{I}_* f$ commutes with filtered colimits and every motivic space is a filtered colimit of finitely presentable motivic spaces, 2.16 implies that $\mathbb{I}_* f(B)$ is a motivic weak equivalence for every motivic space B, e.g. for $B = \mathcal{M}(A, C)$.

For an arbitrary cofibrant motivic functor, the result follows from the previous case using induction on the attaching cells and the fact that cobase change along monomorphisms preserves motivic weak equivalences 2.20. □

3.4 The homotopy functor structure

The major caveat concerning the pointwise model structure is that a motivic weak equivalence $A \xrightarrow{\sim} B$ of finitely presentable motivic spaces does not necessarily induce a pointwise weak equivalence $\mathcal{M}(B,-) \longrightarrow \mathcal{M}(A,-)$ of representable motivic functors. To remedy this problem, we introduce a model structure in which every motivic functor is a homotopy functor up to weak equivalence. A homotopy functor is a functor preserving weak equivalences.

Recall that the pointwise structure is defined entirely in terms of the weakly finitely generated model structure $\mathcal{M}_{\mathrm{mo}}$. However, to define the homotopy functor structure it is also useful to consider the Quillen equivalent model structure $\mathcal{M}_{\mathrm{GJ}}$ in which all motivic spaces are cofibrant. The slogan is: “use

$\mathcal{M}_{\mathrm{GJ}}$ on the source and $\mathcal{M}_{\mathrm{mo}}$ on the target". This is the main difference from the general homotopy functor setup presented in [5].

Definition 3.9. Let M be the set of acyclic monomorphisms (i.e. maps that are both monomorphisms and motivic weak equivalences) of finitely presentable motivic spaces. For a motivic space A, let $\mathrm{ac}(A)$ be the following category. The objects of $\mathrm{ac}(A)$ are the maps $A \longrightarrow B \in \mathcal{M}$ that can be obtained by attaching finitely many cells from M. The set of morphisms from an object $\beta\colon A \longrightarrow B$ to another $\gamma\colon A \longrightarrow C$ is the set of maps $\tau\colon B \longrightarrow C$ that can be obtained by attaching finitely many cells from M such that $\tau\beta = \gamma$. Set

$$\Phi(A) := \operatorname*{colim}_{A \to B \in \mathrm{ac}(A)} B.$$

Note that the objects in $\mathrm{ac}(A)$ are acyclic cofibrations in $\mathcal{M}_{\mathrm{GJ}}$.

The techniques from [5, Section 3.3] ensure the following properties of this construction, see [5, 3.24]

Lemma 3.10. *For every motivic space A, the map $\Phi(A) \longrightarrow *$ has the right lifting property with respect to the maps in M. In particular, $\Phi(A)$ is fibrant in $\mathcal{M}_{\mathrm{mo}}$. Moreover, Φ is a functor and there exists a natural transformation $\varphi_A\colon A \longrightarrow \Phi(A)$ which is an acyclic monomorphism. If the motivic space A is finitely presentable, then $\Phi(A)$ is isomorphic to a filtered colimit of finitely presentable motivic spaces weakly equivalent to A.*

There are occasions where it is more convenient to employ M instead of the set J' introduced in 2.15. For example, every motivic weak equivalence of finitely presentable motivic spaces can be factored as a map in M, followed by a simplicial homotopy equivalence. Adjointness and 2.7 imply:

Lemma 3.11. *Suppose A is a motivic space such that $A \longrightarrow *$ has the right lifting property with respect to the maps in M. If $f\colon B \longrightarrow C$ is an acyclic monomorphism of finitely presentable motivic spaces, then the induced map $\mathcal{M}(C, A) \longrightarrow \mathcal{M}(B, A)$ is an acyclic fibration in $\mathcal{M}_{\mathrm{mo}}$.*

We define the (not necessarily motivic) functor $\hbar(X)\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$ by the composition

$$\hbar(X)(A) := \mathbb{I}_* X(\Phi(A)).$$

Note that $\varphi\colon \mathrm{Id}_{\mathcal{M}} \longrightarrow \Phi$ induces a natural transformations of functors $\mathrm{Id}_{\mathbf{MF}} \longrightarrow \hbar$.

Definition 3.12. A map $f\colon X \longrightarrow Y$ in $\mathbf{MF}$ is an

- hf-*weak equivalence* if the map $\hbar(X)(A)$ is a weak equivalence in $\mathcal{M}_{\mathrm{mo}}$ for all $A \in \mathrm{Ob}\,\mathbf{f}\mathcal{M}$.

- hf-*fibration* if f is a pointwise fibration and for all acyclic monomorphisms $\phi\colon A \overset{\sim}{\hookrightarrow} B \in \mathbf{f}\mathcal{M}$ the diagram

$$\begin{tikzcd} X(A) \arrow[r, "X(\phi)"] \arrow[d, "f(A)"'] & X(B) \arrow[d, "f(B)"] \\ Y(A) \arrow[r, "Y(\phi)"] & Y(B) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{M}_{\mathrm{mo}}$.

In the following, the hf-weak equivalences and hf-fibrations together with the class of cofibrations, will be referred to as the *homotopy functor structure* $\mathbf{MF}_{\mathrm{hf}}$ on $\mathbf{MF}$.

Lemma 3.13. *A map in* $\mathbf{MF}$ *is both an* hf*-fibration and an* hf*-equivalence if and only if it is a pointwise acyclic fibration.*

Proof. One implication is clear.

If $f\colon X \longrightarrow Y$ is an hf-fibration and an hf-equivalence, choose $A \in \mathbf{f}\mathcal{M}$ and consider the induced diagram:

$$\begin{tikzcd} X(A) \arrow[r] \arrow[d, "f(A)"'] & \mathbb{I}_*X(\Phi(A)) \arrow[d, "\mathbb{I}_*f(\Phi(A))"] \\ Y(A) \arrow[r] & \mathbb{I}_*Y(\Phi(A)) \end{tikzcd}$$

It remains to prove that $f(A)$ is a motivic weak equivalence. The right vertical map is a motivic weak equivalence by assumption, so it suffices to prove that the diagram is a homotopy pullback square. Since f is an hf-fibration and $\mathbb{I}_*Z$ commutes with filtered colimits for any motivic functor Z, 3.10 shows the square is a filtered colimit of homotopy pullback squares. By 2.16, homotopy pullback squares in $\mathcal{M}_{\mathrm{mo}}$ are closed under filtered colimits, which finishes the proof. □

Theorem 3.14. *The homotopy functor structure is a cofibrantly generated and proper monoidal model category.*

Proof. First we establish the weakly finitely generated model structure. This follows from [7, 2.1.19], where 3.13 and 3.11 are needed to check the relevant conditions. More precisely, 3.11 shows that the generating acyclic cofibrations listed in 3.17 below are hf-equivalences. By arguments which can be found in the proof of [5, 5.9], any sequential composition of cobase changes of the generating acyclic cofibrations is an hf-equivalence.

Concerning the monoidal part, the crucial observation is that if $f\colon A \longrightarrow B$ is an acyclic monomorphism in $\mathbf{f}\mathcal{M}$ and C is finitely presentable, then the map $f \wedge C\colon A \wedge C \longrightarrow B \wedge C$ is an acyclic monomorphism in $\mathbf{f}\mathcal{M}$. For details and also right properness, see [5, 5.12 and 5.13]. Left properness is clear. □

Theorem 3.15. *Smashing with a cofibrant motivic functor preserves* hf-*equivalences and* $\mathbf{MF}_{\mathrm{hf}}$ *satisfies the monoid axiom.*

Proof. We factor the hf-equivalence into an hf-acyclic cofibration followed by an hf-acyclic fibration. Now 3.13 shows that hf-acyclic fibrations are pointwise acyclic fibrations, and 3.8 shows smashing with a cofibrant object preserves pointwise weak equivalences. Hence we may assume the hf-equivalence is a cofibration. Since the model structure $\mathbf{MF}_{\mathrm{hf}}$ is monoidal, smashing with a cofibrant object preserves hf-acyclic cofibrations. This proves our first claim.

The monoid axiom is shown to hold as follows. Suppose that $X \xrightarrow{\sim \mathrm{hf}} Y$ is a generating hf-acyclic cofibration, and Z is an object of $\mathbf{MF}$ with cofibrant replacement $Z^c \xrightarrow{\sim \mathrm{pt}} Z$. Since X and Y are cofibrant, there is the diagram:

$$\begin{tikzcd} X \wedge Z^c \arrow[r, "\sim \mathrm{hf}"] \arrow[d, "\sim \mathrm{pt}"'] & Y \wedge Z^c \arrow[d, "\sim \mathrm{pt}"] \\ X \wedge Z \arrow[r] & Y \wedge Z \end{tikzcd}$$

This implies $X \wedge Z \xrightarrow{\sim \mathrm{hf}} Y \wedge Z$. The full monoid axiom follows as indicated in [5, 6.30]. □

Remark 3.16. Every motivic functor is an $\mathcal{S}$-functor since $\mathcal{M}_{\mathrm{mo}}$ is a monoidal $\mathcal{S}$-model category. As such, they preserve simplicial homotopy equivalences, see [5, 2.11]. Any motivic weak equivalence can be factored as the composition of an acyclic monomorphism and a simplicial homotopy equivalence. It follows that a pointwise fibration $f\colon X \xrightarrow{\mathrm{pt}} Y$ is an hf-fibration if and only if for every motivic weak equivalence $\phi\colon A \xrightarrow{\sim} B$ in $\mathbf{f}\mathcal{M}$ the following diagram is a homotopy pullback square in the motivic model structure:

$$\begin{tikzcd} X(A) \arrow[r, "X(\phi)"] \arrow[d, "f(A)"'] & X(B) \arrow[d, "f(B)"] \\ Y(A) \arrow[r, "Y(\phi)"] & Y(B) \end{tikzcd}$$

In particular, the fibrant functors in $\mathbf{MF}_{\mathrm{hf}}$ are the pointwise fibrant homotopy functors. On the other hand, we could have constructed the homotopy functor structure as a Bousfield localization with respect to the homotopy functors, avoiding $\hbar$ in 3.12. However, note that we have a characterization of arbitrary fibrations, as opposed to the situation for a general Bousfield localization.

Remark 3.17. The generating cofibrations for the pointwise and homotopy functor structures coincide. The generating acyclic cofibrations for $\mathbf{MF}_{\mathrm{hf}}$ may be chosen as follows. Consider an acyclic monomorphism $\phi\colon A \longrightarrow B \in \mathbf{f}\mathcal{M}$ and its associated factorization $\mathcal{M}(B,-) \xrightarrow{c_\phi} C_\phi \xrightarrow{\simeq} \mathcal{M}(A,-)$ obtained using the simplicial mapping cylinder. The hf-acyclic cofibrations are generated by the pointwise acyclic cofibrations of 3.7, together with the pushout product maps

$$c_\phi \Box i \colon \mathcal{M}(B,-)\wedge ti \coprod_{\mathcal{M}(B,-)\wedge si} C_\phi \wedge si \longrightarrow C_\phi \wedge ti,$$

where ϕ varies over the (isomorphism classes of) acyclic monomorphisms in $\mathbf{f}\mathcal{M}$ and $i\colon si \rightarrowtail ti \in I$ varies over the generating cofibrations in $\mathcal{M}_{\mathrm{mo}}$. The domains and codomains of these pushout product maps are finitely presentable in $\mathbf{MF}$.

To end this section, we indicate why $\hbar(X)(A)$ has the correct homotopy type.

Lemma 3.18. *Let $X \xrightarrow{\sim \mathrm{hf}} X^{\mathrm{hf}}$ be a fibrant replacement in $\mathbf{MF}_{\mathrm{hf}}$. Then we have natural motivic weak equivalences*

$$\hbar(X)(A) \xrightarrow{\sim} \hbar(X^{\mathrm{hf}})(A) \xleftarrow{\sim} X^{\mathrm{hf}}(A).$$

Proof. The first map is a motivic weak equivalence by definition. The second map is a motivic weak equivalence because $\hbar(X^{\mathrm{hf}})(A) \cong \operatorname{colim}_{A\xrightarrow{\sim}B} X^{\mathrm{hf}}(B)$ and X^{hf} preserves motivic weak equivalences. □

3.5 The stable structure

We start with the hf-model structure and define the stable model structure more or less as for the general case in [5, Section 6]. The stable equivalences are the maps which become pointwise weak equivalences after a stabilization process, and the stably fibrant objects are morally the "Ω-spectra".

Let us repeat the stabilization process in the case of $\mathbf{MF}$ and the motivic space T of 2.2, weakly equivalent to $\mathbb{A}^1_S/(\mathbb{A}^1_S - \{0\})$. If X is a motivic functor and A is a finitely presentable motivic space, there is a map

$$t_X(A)\colon X(A) \longrightarrow \mathbb{T}(X)(A) := \mathcal{M}(T, X(T\wedge A))$$

natural in both X and A. It is adjoint to the map $X(A)\wedge T \longrightarrow X(T\wedge A)$ which in turn is adjoint to the composition

$$T \longrightarrow \mathcal{M}(A, T\wedge A) \xrightarrow{\hom^X_{A,T\wedge A}} \mathcal{M}(XA, X(T\wedge A)).$$

Let $\mathbb{T}^\infty(X)$ be the colimit of the sequence

$$X \xrightarrow{t_X} \mathbb{T}(X) \xrightarrow{\mathbb{T}(t_X)} \mathbb{T}(\mathbb{T}(X)) \longrightarrow \cdots,$$

and let $t_X^\infty : X \longrightarrow \mathbb{T}^\infty(X)$ be the canonically induced map.

We fix a fibrant replacement $X \xrightarrow{\sim \mathrm{hf}} X^{\mathrm{hf}}$ in $\mathbf{MF}_{\mathrm{hf}}$.

Definition 3.19. A morphism $f\colon X \longrightarrow Y$ in $\mathbf{MF}$ is a

- *Stable equivalence* if the induced map $\mathbb{T}^\infty(f^{\mathrm{hf}})\colon \mathbb{T}^\infty(X^{\mathrm{hf}}) \longrightarrow \mathbb{T}^\infty(Y^{\mathrm{hf}})$ is a pointwise weak equivalence.
- *Stable fibration* if f is an hf-fibration and the diagram

$$\begin{tikzcd} X(A) \arrow[r, "t_X(A)"] \arrow[d] & \mathbb{T}(X)(A) \arrow[d] \\ Y(A) \arrow[r, "t_Y(A)"] & \mathbb{T}(Y)(A) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{M}_{\mathrm{mo}}$ for all $A \in \mathbf{f}\mathcal{M}$.

We denote by $\mathbf{MF}_{\mathrm{st}}$ the *stable structure* on $\mathbf{MF}$, i.e. the category $\mathbf{MF}$ together with the classes of stable equivalences and stable fibrations.

Remark 3.20. The definition of stable equivalences in the general setting of [5, 6.2] involves the functor $\hbar(-)$ instead of $(-)^{\mathrm{hf}}$. By 3.18, this does not make any difference. In particular, the class of stable equivalences does not depend on the choice of $(-)^{\mathrm{hf}}$.

Lemma 3.21. *A map is a stable fibration and a stable equivalence if and only if it is a pointwise acyclic fibration.*

Proof. One implication is obvious.
If f is a stable fibration and a stable equivalence, then f^{hf} is also a stable equivalence. In general, f^{hf} will not be a pointwise fibration, but – as one can prove by comparing with $\hbar(f)$ – this is the only obstruction preventing f^{hf} from being a stable fibration. That is, the relevant squares appearing in the definition of an hf-fibration 3.12 and in the definition of a stable fibration 3.19 are homotopy pullback squares for f^{hf}. Details can be found in [5, Section 6.2]. Since homotopy pullback squares are closed under filtered colimits (like $\mathbb{T}^\infty$), the statement follows. □

To prove that the stable structure is in fact a model structure, we will introduce generating stable acyclic cofibrations.

Definition 3.22. For a finitely presentable motivic space A, let τ_A be the composition

$$\mathcal{M}(T \wedge A, -) \wedge T \xrightarrow{\cong} \mathcal{M}(T, \mathcal{M}(A, -)) \wedge T \xrightarrow{\epsilon_T \mathcal{M}(A,-)} \mathcal{M}(A, -),$$

where ϵ_T is the counit of the adjunction $(- \wedge T, \mathcal{M}(T, -))$ on $\mathbf{MF}$. There exists a factorization $d_A\colon \mathcal{M}(T \wedge A, -) \wedge T \rightarrowtail D_A$ followed by a simplicial homotopy equivalence. Let $\mathcal{D}$ be the set of pushout product maps $d_A \square i$, where $i\colon si \rightarrowtail ti$ is a generating cofibration in $\mathcal{M}_{\mathrm{mo}}$.

To deduce that the stable structure is a model structure, we need to know that the maps in $\mathcal{D}$-cell are stable equivalences. For this purpose, we compare with the stable model structure on $\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T)$ which exists by [8]. If X is a motivic functor and $A \in \mathbf{f}\mathcal{M}$, we can form the composition $X \circ (- \wedge A) \in \mathbf{MF}$.

Lemma 3.23. *Let $f\colon X \longrightarrow Y$ be a map of motivic functors. Then f is a stable equivalence if and only if* $\mathrm{ev}(f^{\mathrm{hf}} \circ (- \wedge B))$ *is a stable equivalence of motivic spectra for every $B \in \mathbf{f}\mathcal{M}$.*

Proof. Although the stabilizations in $\mathbf{MF}$ and $\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T)$ do not coincide under ev, they can be compared at each $B \in \mathbf{f}\mathcal{M}$ and shown to yield motivic weak equivalences

$$\mathbb{T}^{\infty}(f^{\mathrm{hf}})(B) \xrightarrow{\sim} (\Theta^{\infty}\mathrm{ev}(f^{\mathrm{hf}}(- \wedge B)))_0.$$

Here Θ^{∞} is the stabilization defined in [8, 4.4]. Details are recorded in [5, Section 6.3]. This proves the claim. □

Lemma 3.24. *The maps in $\mathcal{D}$-cell are stable equivalences.*

Proof. Our strategy is to note that 2.24 and 3.23 imply the maps in $\mathcal{D}$ are stable equivalences. To this end, it suffices to show – using 2-out-of-3 and 2.9 – that $\epsilon_T\mathcal{M}(A,-)$ is a stable equivalence for all $A \in \mathrm{Ob}\,\mathbf{f}\mathcal{M}$. Equivalently, according to 3.23, we may consider the map of motivic spectra $\mathrm{ev}((\epsilon_T\mathcal{M}(A,-))^{\mathrm{hf}} \circ (- \wedge B))$ for $B \in \mathrm{Ob}\,\mathbf{f}\mathcal{M}$. Write $X := \mathcal{M}(A,-)$. There is a zig-zag of pointwise weak equivalences connecting $(\epsilon_T X)^{\mathrm{hf}} \circ (- \wedge B)$ and $\epsilon_T(X^{\mathrm{hf}} \circ (- \wedge B))$. It can be constructed as follows. By naturality, the diagram

$$\begin{tikzcd}
 & \mathcal{M}(T,X) \wedge T \arrow[dl] \arrow[d, "\epsilon_T X"] \arrow[dr, tail, "\sim_{hf}"] & \\
\mathcal{M}(T,X^{\mathrm{hf}}) \wedge T \arrow[dr, "\epsilon_T(X^{hf})"'] & X \arrow[d, tail, "\sim\mathrm{hf}"] & (\mathcal{M}(T,X) \wedge T)^{\mathrm{hf}} \arrow[dl, "(\epsilon_T X)^{\mathrm{hf}}"] \\
 & X^{\mathrm{hf}} &
\end{tikzcd}$$

commutes. Factor the map $\epsilon_T(X^{\mathrm{hf}})$ as a pointwise acyclic cofibration, followed by a pointwise fibration $Z \xrightarrow{\mathrm{pt}} X^{\mathrm{hf}}$. Then $Z \xrightarrow{\mathrm{pt}} X^{\mathrm{hf}}$ is in fact an hf-fibration. The reason is that X^{hf} is a pointwise fibrant homotopy functor, so $M(T, X^{\mathrm{hf}})$ is also a (pointwise fibrant) homotopy functor, since T is cofibrant. By 2.18, $\mathcal{M}(T, X^{\mathrm{hf}}) \wedge T$ is then a homotopy functor, hence the pointwise weak equivalence $\mathcal{M}(T, X^{\mathrm{hf}}) \wedge T \xrightarrow{\sim\mathrm{pt}} Z$ implies that Z is a homotopy functor. Any pointwise fibration of homotopy functors is an hf-fibration, thus $Z \xrightarrow{\mathrm{pt}} X^{\mathrm{hf}}$ is an hf-fibration. Hence there exists a lift $f\colon (\mathcal{M}(T,X) \wedge T)^{\mathrm{hf}} \longrightarrow Z$ in the diagram:

$$\begin{tikzcd}
\mathcal{M}(T,X)\wedge T \arrow[r] \arrow[d, tail, "\sim\mathrm{hf}"'] & \mathcal{M}(T,X^{\mathrm{hf}})\wedge T \arrow[r, tail, "\sim\mathrm{pt}"] & Z \arrow[d, "\mathrm{hf}"] \\
(\mathcal{M}(T,X)\wedge T)^{\mathrm{hf}} \arrow[urr, dashed, "\mathfrak{f}"] \arrow[rr, "(\epsilon_T X)^{\mathrm{hf}}"] & & X^{\mathrm{hf}}
\end{tikzcd}$$

We will prove that f is a pointwise weak equivalence. It suffices to prove that f is an hf-equivalence because both the domain and the codomain of f are homotopy functors. Hence by the 2-out-of-3 property it suffices to prove that $\mathcal{M}(T,X)\wedge T \longrightarrow \mathcal{M}(T,X^{\mathrm{hf}})\wedge T$ is an hf-equivalence. Since $-\wedge T$ preserves hf-equivalences, let us consider $\mathcal{M}(T,X) \longrightarrow \mathcal{M}(T,X^{\mathrm{hf}})$. We have to prove that for every finitely presentable motivic space C, $\hbar(\mathcal{M}(T,X) \longrightarrow \mathcal{M}(T,X^{\mathrm{hf}}))(C)$ is a motivic weak equivalence. Since T is finitely presentable and $\hbar$ can be described as a filtered colimit, the map in question is isomorphic to the map $\mathcal{M}(T,\hbar(X \xrightarrow{\sim\mathrm{hf}} X^{\mathrm{hf}})(C))$. The map $\hbar(X \xrightarrow{\sim\mathrm{hf}} X^{\mathrm{hf}})(C)$ is a motivic weak equivalence by definition, so it remains to observe that the domain and the codomain are both fibrant in $\mathcal{M}_{\mathrm{mo}}$. Now $X = \mathcal{M}(A,-)$ where A is finitely presentable, so the domain $\hbar(\mathcal{M}(A,-))(C) = \mathcal{M}(A,\Phi(C))$ is fibrant in $\mathcal{M}_{\mathrm{mo}}$. The codomain is isomorphic to a filtered colimit of fibrant objects, hence it is fibrant in $\mathcal{M}_{\mathrm{mo}}$.

We have constructed the diagram:

$$\begin{tikzcd}
\mathcal{M}(T,\mathcal{M}(A,-)^{\mathrm{hf}})\wedge T \arrow[r, "\sim\mathrm{pt}"] \arrow[dr, "\epsilon_T(\mathcal{M}(A,-)^{hf})"'] & Z \arrow[d] & (\mathcal{M}(T,\mathcal{M}(A,-))\wedge T)^{\mathrm{hf}} \arrow[l, "\sim\mathrm{pt}"'] \arrow[dl, "(\epsilon_T\mathcal{M}(A,-))^{\mathrm{hf}}"] \\
& \mathcal{M}(A,-)^{\mathrm{hf}} &
\end{tikzcd}$$

Pre-composing with $-\wedge B$ preserves pointwise weak equivalences so that we get the desired zig-zag of pointwise weak equivalences connecting the two maps $\epsilon_T(\mathcal{M}(A,-)^{\mathrm{hf}}\circ(-\wedge B))$ and $(\epsilon_T\mathcal{M}(A,-))^{\mathrm{hf}}\circ(-\wedge B)$. Since ev preserves pointwise weak equivalences, it suffices to check that

$$\mathrm{ev}(\epsilon_T(\mathcal{M}(A,-)^{\mathrm{hf}}\circ(-\wedge B))) = \epsilon_T(\mathrm{ev}(\mathcal{M}(A,-)^{\mathrm{hf}})\circ(-\wedge B))$$

is a stable equivalence. In what follows, let us abbreviate by E the pointwise fibrant motivic spectrum $\mathrm{ev}(\mathcal{M}(A,-)^{\mathrm{hf}}\circ(-\wedge B))$. Then $\iota_E\colon E \longrightarrow \Theta^\infty E$ is a stable equivalence whose codomain is a stably fibrant motivic spectrum [8, 4.12]. Moreover, since T is finitely presentable and cofibrant, the map $\mathcal{M}(T,\iota_E)\colon \mathcal{M}(T,E) \longrightarrow \mathcal{M}(T,\Theta^\infty E)$ is also a stable equivalence with stably fibrant codomain. Choose a cofibrant replacement $\mathcal{M}(T,\Theta^\infty E)^c \xrightarrow{\sim\mathrm{pt}} \mathcal{M}(T,E)$

and consider the induced commutative diagram:

$$\begin{tikzcd}
\mathcal{M}(T,E)^c \wedge T \arrow[r, "\sim"] \arrow[d, "\mathcal{M}(T,\iota_E)^c\wedge T"', "\sim"] & \mathcal{M}(T,E)\wedge T \arrow[r, "\epsilon_T E", "\sim"'] \arrow[d, "\mathcal{M}(T,\iota_E)\wedge T"', "\sim"] & E \arrow[d, "\iota_E"] \\
\mathcal{M}(T,\Theta^\infty E)^c \wedge T \arrow[r, "\sim"] & \mathcal{M}(T,\Theta^\infty E)\wedge T \arrow[r, "\epsilon_T \Theta^\infty E", "\sim"'] & \Theta^\infty E
\end{tikzcd}$$

Since $-\wedge T$ is a Quillen equivalence 2.24, the lower horizontal composition is a stable equivalence. Since $-\wedge T$ preserves pointwise weak equivalences 2.18, both horizontal maps on the left hand side are pointwise weak equivalences. The right vertical map is a stable equivalence by construction. By factoring a stable equivalence as a stable acyclic cofibration, followed by a pointwise acyclic fibration, one can see that $-\wedge T$ preserves all stable equivalences. Hence also the other two vertical maps are stable equivalences. It follows that the map in question is a stable equivalence. □

Theorem 3.25. *The stable structure* $\mathbf{MF}_{\mathrm{st}}$ *is a cofibrantly generated, proper and monoidal model category.*

Proof. The model structure follows easily from [7, 2.1.19], using 3.21 and 3.24. The smash product of $\mathcal{M}(T\wedge A,-)\wedge T \longrightarrow \mathcal{M}(A,-)$ and $\mathcal{M}(B,-)$ is isomorphic to the map $\mathcal{M}(T\wedge(A\wedge B),-)\wedge T \longrightarrow \mathcal{M}(A\wedge B,-)$. This implies that the pushout product map of a generating cofibration $\mathcal{M}(B,-)\wedge h_U \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+$ and a generating stable acyclic cofibration is again a stable acyclic cofibration, which proves that the model structure is monoidal. Left properness is clear, for right properness we refer to [5, 6.28]. □

Remark 3.26. In the pointwise and stable model structures, the generating cofibrations coincide. The set of generating acyclic cofibrations for the stable structure is the union of the set of generating hf-acyclic cofibrations in 3.17, together with the set D described above. Note that all of the maps have cofibrant domains and codomains. Furthermore, the domains and codomains of the maps in D are finitely presentable.

Remark 3.27. In fact, by the proofs of [5, 5.13 and 6.28] stable equivalences are closed under base change along pointwise fibrations.

By a verbatim copy of the argument in the hf-structure 3.15, we get the monoid axiom for the stable structure.

Theorem 3.28. *Smashing with a cofibrant object in* $\mathbf{MF}_{\mathrm{st}}$ *preserves stable equivalences, and* $\mathbf{MF}_{\mathrm{st}}$ *satisfies the monoid axiom.*

Our goal now is to compare the stable model structure on motivic functors with the stable model structure on motivic spectra.

It is clear that $\mathrm{ev}\colon \mathbf{MF} \longrightarrow \mathrm{Sp}(\mathcal{M},T)$ preserves acyclic fibrations, and from Hovey's results [8, Section 4], ev preserves stable fibrations. Hence ev is a right Quillen functor, with left adjoint i_* defined by left Kan extension along the inclusion $i\colon T\mathrm{Sph} \hookrightarrow \mathbf{f}\mathcal{M}$. (In fact, ev preserves stable equivalences of motivic homotopy functors by 3.23.) We would like ev to be a Quillen equivalence, which according to [7, 1.3.16] is equivalent to the following two conditions.

- ev detects stable equivalences of stably fibrant motivic functors.
- If E is a cofibrant motivic spectrum and $(-)^{\mathrm{st}}$ denotes a stably fibrant replacement functor for motivic spectra, then the canonical map

$$E \longrightarrow \mathrm{ev}((i_*E)^{\mathrm{st}})$$

  is a stable equivalence.

Here is a proof of the second condition.

Lemma 3.29. *Let E be a cofibrant motivic spectrum. Then $E \longrightarrow \mathrm{ev}((i_*E)^{\mathrm{st}})$ is a stable equivalence of motivic spectra.*

Proof. Let us start by observing that, by 3.23, it is sufficient to show that the map $E \longrightarrow \mathrm{ev}((i_*E)^{\mathrm{hf}})$ is a stable equivalence. To describe $(-)^{\mathrm{hf}}$ in convenient terms, we will employ the enriched fibrant replacement functor $\mathrm{Id}_{\mathcal{M}_{\mathrm{mo}}} \longrightarrow R$ [5, 3.3.2]. Its construction uses an enriched small object argument. For our notations concerning spectra see [8].

First, consider the case $E = F_0T^0$. Then $i_*F_0T^0 \cong \mathcal{M}(T^0,-) \cong \mathbb{I}$, and we can choose $\mathbb{I}^{\mathrm{hf}} = R \circ \mathbb{I}$. The map $F_0T^0 \longrightarrow \mathrm{ev}(R \circ \mathbb{I})$ in degree n is the canonical motivic weak equivalence $T^{\wedge n} \xrightarrow{\sim} R(T^{\wedge n})$, hence a pointwise weak equivalence.

To proceed in the slightly more general case when $E = F_nT^0$, note that $i_*F_nT^0 \cong \mathcal{M}(T^{\wedge n},-)$. Since $T^{\wedge n}$ is cofibrant, we may choose $\mathcal{M}(T^{\wedge n},-)^{\mathrm{hf}} = \mathcal{M}(T^{\wedge n},R(-))$, cp. 3.18. Hence $\mathrm{ev}\mathcal{M}(T^{\wedge n},R(-)) = \mathcal{M}(T^{\wedge n},\mathrm{ev}R(-))$. The map $F_nT^0 \longrightarrow \mathcal{M}(T^{\wedge n},\mathrm{ev}R(-))$ has an adjoint $F_nT^0 \wedge T^{\wedge n} \longrightarrow \mathrm{ev}R(-)$ which is $* \longrightarrow R(T^{\wedge k})$ in degree $k < n$ and the canonical motivic weak equivalence $T^{\wedge m} \xrightarrow{\sim} R(T^{\wedge m})$ in degree $m \geq n$. In particular, it is a stable equivalence. Similarly for

$$F_nT^0 \wedge T^{\wedge n} \xrightarrow{\sim} \mathrm{ev}R(-) \xrightarrow{\sim} \Theta^\infty \mathrm{ev}R(-).$$

¿From the proof of 3.24, one can see that $\mathcal{M}(T^{\wedge n},-)$ applied to the second map is a stable equivalence with a stably fibrant codomain. Since $-\wedge T$ is a Quillen equivalence on $\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}},T)$, this proves the slightly more general case.

The case $E = F_nA$, where A is any motivic space, follows since

$$F_nA \longrightarrow \mathrm{ev}((i_*F_nA)^{\mathrm{hf}}) \cong (F_nT^0 \longrightarrow \mathrm{ev}\mathcal{M}(T^{\wedge n},R(-))) \wedge A$$

and tensoring with any motivic space preserves stable equivalences of motivic spectra. The latter follows from 2.20. This includes the domains and codomains of the generating cofibrations in $\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T)$.

The general case of any cofibrant motivic spectrum E follows, since E is a retract of a motivic spectrum E' such that $* \rightarrowtail E'$ is obtained by attaching cells. That is, we can assume $E = E'$. We proceed by transfinite induction on the cells, with the successor ordinal case first. Suppose $E_{\alpha+1}$ is the pushout of

$$F_n tj \xleftarrow{F_n j} F_n sj \longrightarrow E_\alpha$$

where j is a generating cofibration in $\mathcal{M}_{\mathrm{mo}}$. Then $(i_* E_{\alpha+1}) \circ R \circ \mathbb{I}$ is the pushout of the diagram

$$\mathcal{M}(T^{\wedge n}, R(-)) \wedge tj \xleftarrow{\mathcal{M}(T^{\wedge n}, R(-)) \wedge j} \mathcal{M}(T^{\wedge n}, R(-)) \wedge sj \longrightarrow i_* E_\alpha \circ R \circ \mathbb{I}.$$

The left horizontal map is pointwise a monomorphism. All the motivic functors in this diagram are homotopy functors, so up to pointwise weak equivalence, they coincide with their fibrant replacement in $\mathbf{MF}_{\mathrm{hf}}$. The induction step follows, since ev preserves pushouts, pointwise weak equivalences and pointwise monomorphisms, by applying the gluing lemma to the diagram:

$$\begin{tikzcd}
F_n tj \arrow[d, "\sim"] & F_n sj \arrow[l, "F_n j"'] \arrow[r] \arrow[d, "\sim"] & E_\alpha \arrow[d, "\sim"] \\
\mathrm{ev}\mathcal{M}(T^{\wedge n}, R(-)) \wedge tj & \mathrm{ev}\mathcal{M}(T^{\wedge n}, R(-)) \arrow[l] \arrow[r] & \mathrm{ev}(i_* E_\alpha \circ R \circ \mathbb{I})
\end{tikzcd}$$

The limit ordinal case follows similarly; we leave the details to the reader. $\square$

For a general S, it is not known whether ev detects stable equivalences of stably fibrant motivic functors. In order to obtain the "correct" homotopy theory of motivic functors we modify the stable model structure.

3.6 The spherewise structure

Definition 3.30. A map $f\colon X \longrightarrow Y$ of motivic functors is a *spherewise equivalence* if the induced map $\mathrm{ev}(f^{\mathrm{hf}})$ is a stable equivalence of motivic spectra. The map f is a *spherewise fibration* if the following three conditions hold for every $A \in \mathbf{f}\mathcal{M}$ such that there exists an acyclic monomorphism $T^{\wedge n} \xhookrightarrow{\sim} A$ for some $n \geq 0$:

- $f(A)\colon X(A) \longrightarrow Y(A)$ is a motivic fibration.
- For every motivic weak equivalence $A \xrightarrow{\sim} B$ in $\mathbf{f}\mathcal{M}$,

$$\begin{tikzcd}
XA \arrow[r] \arrow[d, "f(A)"'] & XB \arrow[d, "f(B)"] \\
YA \arrow[r] & YB
\end{tikzcd}$$

is a homotopy pullback square in $\mathcal{M}_{\mathrm{mo}}$.

- The diagram

$$\begin{tikzcd} XA \arrow[r] \arrow[d, "f(A)"'] & \mathcal{M}(T, X(T \wedge A)) \arrow[d, "\mathcal{M}(T, f(T\wedge A))"] \\ YA \arrow[r] & \mathcal{M}(T, Y(T \wedge A)) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{M}_{\mathrm{mo}}$.

A map is a *spherewise cofibration* if it has the left lifting property with respect to the maps which are both spherewise equivalences and spherewise fibrations.

We shall refer to these classes as the *spherewise structure* on $\mathbf{MF}$ and use the notations $\mathbf{MF}_{\mathrm{sph}}$, $X \xrightarrow{\sim \mathrm{sph}} Y$, $X \xrightarrow{\mathrm{sph}} Y$ and $X \xrightarrow{\mathrm{sph}} Y$. Now every stable equivalence is a spherewise equivalence by 3.23, and stable fibrations are spherewise fibrations. Hence the identity is a left Quillen functor $\mathbf{MF}_{\mathrm{sph}} \longrightarrow \mathbf{MF}_{\mathrm{st}}$ provided the spherewise structure is a model structure.

Theorem 3.31. *The spherewise structure is a cofibrantly generated proper monoidal model structure on* $\mathbf{MF}$. *The monoid axiom holds. Furthermore, the evaluation functor*

$$\mathrm{ev}\colon \mathbf{MF}_{\mathrm{sph}} \longrightarrow \mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T)$$

is the right adjoint in a Quillen equivalence.

Proof. Let us denote by $\mathbf{t}\mathcal{M}$ the full sub-$\mathcal{M}$-category given by the finitely presentable motivic spaces A such that there exists an acyclic monomorphism $T^{\wedge n} \xrightarrow{\sim} A$ for some $n \geq 0$. It is possible to apply the general machinery from [5] to the category $[\mathbf{t}\mathcal{M}, \mathcal{M}]$ of $\mathcal{M}$-functors from $\mathbf{t}\mathcal{M}$ to $\mathcal{M}$ and get a cofibrantly generated proper model structure. We may then lift this model structure using [6, 11.3.2] from $[\mathbf{t}\mathcal{M}, \mathcal{M}]$ to $\mathbf{MF}$ via the left Kan extension along the full inclusion $\mathbf{t}\mathcal{M} \hookrightarrow \mathbf{f}\mathcal{M}$.

We follow a direct approach. By the proof of 3.21, a spherewise acyclic fibration $f\colon X \xrightarrow{\sim \mathrm{sph}} Y$ is characterized by the property that the map $f(A)\colon XA \longrightarrow YA$ is an acyclic fibration in $\mathcal{M}_{\mathrm{mo}}$ for every $A \in \mathbf{t}\mathcal{M}$. This gives us the set of generating spherewise cofibrations

$$\{\mathcal{M}(A, -) \wedge h_U \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+\}_{A \in \mathbf{t}\mathcal{M},\, U \in \mathrm{Ob}\,\mathrm{Sm}/S, n \geq 0}.$$

This set is simply the restriction of the set of generating cofibrations for the model structures on the motivic spaces in $\mathbf{t}\mathcal{M}$. Similarly, one can restrict the generating acyclic cofibrations in 3.26 to the motivic spaces in $\mathbf{t}\mathcal{M}$. This gives a set of generating spherewise acyclic cofibrations. Theorem [7, 2.1.19] implies the existence of the cofibrantly generated model structure. In fact, the conditions required to apply this theorem have been checked before without the restriction that A be in $\mathbf{t}\mathcal{M}$. For example, sequential compositions of cobase changes of the generating spherewise acyclic cofibrations are even stable equivalences by 3.24, hence in particular spherewise equivalences.

Note that $\mathbf{t}\mathcal{M}$ is closed under the smash product in $\mathcal{M}$. In fact, if the maps $T^{\wedge m} \xhookrightarrow{\sim} A$ and $T^{\wedge n} \xhookrightarrow{\sim} B$ are acyclic monomorphisms, then their smash product $T^{\wedge m+n} \longrightarrow A \wedge B$ is an acyclic monomorphism. This is the crux observation leading to the conclusion that the model structure is monoidal. We claim that the monoid axiom holds. If X is an arbitrary motivic functor and j is a generating spherewise acyclic cofibration, then j is in particular a generating stable acyclic cofibration. The monoid axiom for the stable model structure 3.28 implies that $X \wedge j$-cell consists of stable equivalences, which are in particular spherewise equivalences. Our claim follows.

Finally, since $T^{\wedge n} \in \operatorname{Ob} \mathbf{t}\mathcal{M}$ for every $n \geq 0$, the evaluation functor ev preserves spherewise fibrations and spherewise acyclic fibrations. Hence ev is a right Quillen functor. By definition, ev reflects spherewise equivalences of motivic homotopy functors. This implies ev also reflects spherewise equivalences of motivic functors which are spherewise fibrant (A spherewise fibrant motivic functor does not necessarily preserve all of the motivic weak equivalences in $\mathbf{f}\mathcal{M}$, only those in $\mathbf{t}\mathcal{M}$. However, this is sufficient.). If E is a cofibrant motivic spectrum and $i_*E \xrightarrow{\sim \mathrm{sph}} (i_*E)^{\mathrm{sph}}$ is a spherewise fibrant replacement, there is a spherewise equivalence $(i_*E)^{\mathrm{sph}} \xrightarrow{\sim \mathrm{sph}} (i_*E)^{\mathrm{st}}$.

Using 3.29 above we conclude that $\mathrm{ev}\colon \mathbf{MF}_{\mathrm{sph}} \longrightarrow \mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T)$ is a Quillen equivalence. □

Note that we do not claim that smashing with a spherewise cofibrant motivic functor preserves spherewise equivalences.

3.7 Comparison with motivic symmetric spectra

We extend the result about the Quillen equivalence 3.31 to Jardine's category of motivic symmetric spectra [10]. As mentioned above, if U is the functor induced by the inclusion $T\mathrm{Sph} \hookrightarrow T\mathrm{Sph}^{\Sigma}$, and ev' is the inclusion $T\mathrm{Sph}^{\Sigma} \hookrightarrow \mathbf{f}\mathcal{M}$, then $\mathrm{ev}\colon \mathbf{MF} \longrightarrow \mathrm{Sp}(\mathcal{M}, T)$ allows the factorization

$$\mathbf{MF} \xrightarrow{\mathrm{ev}'} \mathrm{Sp}^{\Sigma}(\mathcal{M}, T) \xrightarrow{U} \mathrm{Sp}(\mathcal{M}, T).$$

The functor ev' is lax symmetric monoidal and has a strict symmetric monoidal left adjoint. Hovey's work [8, 8.7] yields a stable model structure on $\mathrm{Sp}^{\Sigma}(\mathcal{M}_{\mathrm{mo}}, T)$, slightly different from the stable model structure on motivic symmetric spectra constructed in [10], that is, $\mathrm{Sp}^{\Sigma}(\mathcal{M}_{\mathrm{GJ}}, T)$. The latter uses as input the model category M_{GJ} in 2.17. The right adjoint of the Quillen equivalence $\mathcal{M}_{\mathrm{GJ}} \longrightarrow \mathcal{M}_{\mathrm{mo}}$ given by $\mathrm{Id}_{\mathcal{M}}$ induces the commutative square

$$\begin{array}{ccc} \mathrm{Sp}^{\Sigma}(\mathcal{M}_{\mathrm{GJ}},T) & \xrightarrow{U} & \mathrm{Sp}(\mathcal{M}_{\mathrm{GJ}},T) \\ \downarrow & & \downarrow \\ \mathrm{Sp}^{\Sigma}(\mathcal{M}_{\mathrm{mo}},T) & \xrightarrow{U} & \mathrm{Sp}(\mathcal{M}_{\mathrm{mo}},T) \end{array}$$

where the vertical functors are Quillen equivalences [8, 5.7, 9.3]. To apply Hovey's results one needs to check that $\mathcal{M}_{\mathrm{GJ}}$ is a cellular model structure. An approach is to apply Smith's work on combinatorial model categories, or one can proceed directly. Indeed, using 2.17 one can show that the stable equivalences coincide in both model structures. The upper forgetful functor in the above displayed diagram is a Quillen equivalence by [10, 4.31], hence so is the lower U. Since the evaluation $\mathrm{ev}\colon \mathbf{MF}_{\mathrm{sph}} \longrightarrow \mathrm{Sp}(\mathcal{M}_{\mathrm{mo}},T)$ is a Quillen equivalence 3.31, it suffices to prove the following result.

Theorem 3.32. *The lax symmetric monoidal functor*

$$\mathrm{ev}'\colon \mathbf{MF}_{\mathrm{sph}} \longrightarrow \mathrm{Sp}^{\Sigma}(\mathcal{M}_{\mathrm{mo}},T)$$

is the right adjoint in a Quillen equivalence. Its left adjoint is strict symmetric monoidal. The induced pair on homotopy categories is a monoidal equivalence.

Proof. If ev' is a right Quillen functor, then the monoidality statements follow from [5, 2.16] and [7, 4.3.3]. The Quillen equivalence then follows by 2-out-of-3, as explained prior to the statement of the theorem.

Since the spherewise acyclic fibrations are the maps f such that $f(A)$ is an acyclic fibration in $\mathcal{M}_{\mathrm{mo}}$ for every A weakly equivalent to some T^n, we get that ev' preserves stable acyclic fibrations. Similarly, any spherewise fibration gets mapped to a stable fibration, because its evaluation on some T^n is a fibration and the square

$$\begin{array}{ccc} XT^n & \longrightarrow & \mathcal{M}(T, XT^{n+1}) \\ {\scriptstyle F(T^{\wedge n})}\downarrow & & \downarrow{\scriptstyle \mathcal{M}(T,f(T^{n+1}))} \\ YT^n & \longrightarrow & \mathcal{M}(T, YT^{n+1}) \end{array}$$

is a homotopy pullback square in $\mathcal{M}_{\mathrm{mo}}$ for every $n \geq 0$. ¿From the definition of stable fibrations of symmetric T-spectra [10, 4.2], which also applies to $\mathcal{M}_{\mathrm{mo}}$ instead of $\mathcal{M}_{\mathrm{GJ}}$, it follows that ev' preserves stable fibrations. □

4 Algebraic structure

In the paper so far, we have set up models for doing homotopical algebra over the initial motivic ring $\mathbb{I}$, which was simply the inclusion $\mathbb{I}\colon \mathbf{f}\mathcal{M} \subseteq \mathcal{M}$.

However, the structure we have developed is sufficient to do homotopical algebra in module categories, as well as in categories of algebras over commutative ring functors.

In this section we use the results in [14] (for which many of the previous formulations were custom-built), to outline how this can be done. The spherewise structure $\mathbf{MF}_{\mathrm{sph}}$ is slightly different from the other ones, but deserves special attention due to its Quillen equivalence to motivic symmetric spectra.

The reader's attention should perhaps be drawn to corollary 4.5, where our setup gives less than one should hope for: in order for a map of motivic rings $f\colon A \longrightarrow B$ to induce a Quillen equivalence of module categories in the spherewise structure, we must assume that f is a stable equivalence. We would of course have preferred that our setup immediately gave the conclusion for spherewise equivalences, but apart from this deficiency the section can be summed up by saying that each of the model structures given in the previous section give rise to a natural homotopy theory for modules and algebras satisfying all expected properties, where the weak equivalences and fibrations are the same as in the underlying structure on $\mathbf{MF}$.

4.1 Motivic rings and modules

Recall that a motivic ring is the same as a monoid in $\mathbf{MF}$, i.e. a motivic functor A together with a "unit" $\mathbb{I} \longrightarrow A$ and a unital and associative "multiplication" $A \wedge A \longrightarrow A$. We use the same language for modules and algebras as e.g. [14]. A left A-module is a motivic functor M together with a unital and associative action $A \wedge M \longrightarrow M$. If M is a left A-module and N is a right A-module, then $N \wedge_A M$ is defined as the coequalizer of the two obvious action maps from $N \wedge A \wedge M$ to $N \wedge M$. The category mod_A of left A-modules is enriched over $\mathbf{MF}$ by a similar equalizer.

If k is a commutative motivic ring, then left and right modules can be identified and the category of k-modules becomes a closed symmetric monoidal category. The monoids therein are called k-algebras (which means that we have a third legitimate name – "$\mathbb{I}$-algebra"– for a motivic ring).

Definition 4.1. Let A be a motivic ring and k a commutative motivic ring. Let mod_A be the category of left A-modules and alg_k the category of k-algebras. A map in mod_A or alg_k is called a weak equivalence resp. fibration if it is so when considered in $\mathbf{MF}$. Cofibrations are defined by the left lifting property.

Theorem 4.2. *Let A be a motivic ring, let k be a commutative motivic ring and let $\mathbf{MF}$ be equipped with either of the model structures of section 3.*

- *The category mod_A of left A-modules is a cofibrantly generated model category.*
- *The category of k-modules is a cofibrantly generated monoidal model category satisfying the monoid axiom.*

- *The category* alg_k *of* k*-algebras is a cofibrantly generated model category.*

Proof. This follows immediately from [14, 4.1] and the results in section 3. □

By the argument for [5, 8.4], we have

Lemma 4.3. *Let* $\mathbf{MF}$ *be equipped with the pointwise structure, the homotopy functor structure or the stable structure. Let* A *be a motivic ring. Then for any cofibrant* A*-module* N*, the functor* $- \wedge_A N$ *takes weak equivalences in* $\mathrm{mod}_{A^{\mathrm{op}}}$ *to weak equivalences in* $\mathbf{MF}$*.*

Corollary 4.4. *Let* $\mathbf{MF}$ *be equipped with the pointwise structure, the homotopy functor structure or the stable structure. Let* $f\colon A \xrightarrow{\sim} B$ *be a weak equivalence of motivic rings. Then extension and restriction of scalars define the Quillen equivalence*

$$\mathrm{mod}_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} \mathrm{mod}_B.$$

If A *and* B *are commutative, there is the Quillen equivalence*

$$\mathrm{alg}_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} \mathrm{alg}_B.$$

Proof. This is a consequence of [14, 4.3 and 4.4] according to 4.3. □

In the case of the spherewise structure, we have the following result.

Corollary 4.5. *Suppose* $f\colon A \xrightarrow{\sim} B$ *is a stable equivalence of motivic rings and choose* $\mathbf{MF}_{\mathrm{sph}}$ *as our basis for model structures on modules and algebras. Then extension and restriction of scalars define the Quillen equivalence*

$$\mathrm{mod}_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} \mathrm{mod}_B.$$

If A *and* B *are commutative, there is the Quillen equivalence*

$$\mathrm{alg}_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} \mathrm{alg}_B.$$

Proof. Follows from 4.3, cf. [14, 4.3 and 4.4]. □

4.2 Motivic cohomology

Recall the commutative motivic ring $M\mathbb{Z}$ of example 3.4. We show:

Lemma 4.6. *The evaluation* $\mathrm{ev}(M\mathbb{Z})$ *of* $M\mathbb{Z}$ *represents motivic cohomology with integer coefficients.*

Proof. Let us repeat Voevodsky's construction of the spectrum representing motivic cohomology in [16]. His motivic spaces (simply called *spaces*) are pointed Nisnevich sheaves on Sm/S, equipped with a model structure in which the cofibrations are the monomorphisms. Let us denote this model category by $\mathcal{V}$. Note that $\mathcal{V}$ is closed symmetric monoidal. There is the standard cosimplicial object $\Delta_S \colon \Delta \longrightarrow \mathrm{Sm}/S$ which maps $[n]$ to the scheme $\mathbb{A}^{n+1}_S/(\Sigma_{i=0}^n X_i = 1)$. The right Quillen functor

$$\mathrm{Sing} : \mathcal{V} \longrightarrow \mathcal{M}_{\mathrm{mo}}, \qquad A \longmapsto ((U,n) \longmapsto A(\Delta^n_S \times U))$$

is a Quillen equivalence by [10, B.4, B.6] and 2.17.

Its left adjoint maps a motivic space A to the coend

$$|A|_S = \int^{n\in\Delta} \mathrm{Nis}(A_n \wedge h_{\Delta^n_S})$$

where $\mathrm{Nis}(B)$ is the Nisnevich sheafification of the presheaf B. The functor $|-|_S$ is strict symmetric monoidal. As a special case, if $A \in \mathcal{M}$ is a discrete Nisnevich sheaf (for example $A = h_U$ for some $U \in \mathrm{Sm}/S$), then $|A|_S \cong A$.

The spectrum $H\mathbb{Z}$ defined by Voevodsky is an object in $\mathrm{Sp}(\mathcal{V}, |(\mathbb{P}^1_S,\infty)|_S)$, where $|(\mathbb{P}^1_S,\infty)|_S := |h_{\mathbb{P}^1_S}/h_S|_S$. Here $h_S \longrightarrow h_{\mathbb{P}^1_S}$ corresponds to the rational point $\infty \in \mathbb{P}^1_S(S)$. Its nth term is

$$H\mathbb{Z}_n = |M\mathbb{Z}((\mathbb{P}^1_S,\infty)^{\wedge n})|_S$$

with structure map given by the composition

$$\begin{array}{c} |M\mathbb{Z}((\mathbb{P}^1_S,\infty)^{\wedge n}) \wedge (\mathbb{P}^1_S,\infty)|_S \\ \big\downarrow \\ |M\mathbb{Z}((\mathbb{P}^1_S,\infty)^{\wedge n}) \wedge M\mathbb{Z}(\mathbb{P}^1_S,\infty)|_S \\ \big\downarrow \\ |M\mathbb{Z}((\mathbb{P}^1_S,\infty)^{\wedge n+1})|_S \end{array}$$

which involves the unit and the multiplication of the motivic ring $M\mathbb{Z}$. The lemma follows now, essentially because $(\mathbb{P}^1_S,\infty)$ and T are connected via a zigzag of motivic weak equivalences, which both $|-|_S$ and $M\mathbb{Z}$ respect. For $|-|_S$ this is clear, since it is a left Quillen functor on $\mathcal{M}_{\mathrm{GJ}}$. For $M\mathbb{Z}$ the claim is not so clear, so we discuss this case in some details.

As a motivic functor, $M\mathbb{Z}$ preserves simplicial homotopy equivalences. One can equip the category $\mathcal{M}^{\mathrm{tr}}$ of motivic spaces with transfers with a whole host of model structures. In the motivic model structure on $\mathcal{M}^{\mathrm{tr}}$, a map f of motivic spaces with transfers is a weak equivalence resp. fibration if and only if $u(f) \in \mathcal{M}$ is a motivic weak equivalence resp. motivic fibration [13]. By definition, it follows that u is a right Quillen functor, so that $\mathbb{Z}_{\mathrm{tr}}$ is a left Quillen functor. Consequently, the composition $u \circ \mathbb{Z}_{\mathrm{tr}}$ maps motivic weak equivalences of cofibrant motivic spaces to motivic weak equivalences.

The zig-zag of motivic weak equivalences between $(\mathbb{P}^1_S, \infty)$ and the Tate object T involves only homotopy pushouts of representable motivic spaces and their simplicial suspensions. By repeatedly applying the simplicial mapping cylinder one can replace this zig-zag by a zig-zag of motivic weak equivalences involving only cofibrant motivic spaces, except for the weak equivalence $T' \xrightarrow{\sim} (\mathbb{P}^1_S, \infty)$. Here $T' = C/h_S$ where C denotes the simplicial mapping cylinder of the map $h_S \longrightarrow h_{\mathbb{P}^1_S}$. However, we claim the following map is a weak equivalence

$$\mathbb{Z}_{\mathrm{tr}}(C/h_S) \longrightarrow \mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1_S, \infty).$$

Our claim holds because the following map of chain complexes of motivic spaces with transfers is schemewise a quasi-isomorphism:

$$\begin{tikzcd}
\mathbb{Z}_{\mathrm{tr}}(h_{\mathbb{P}^1_S}) \arrow[d] & \mathbb{Z}_{\mathrm{tr}}(h_S) \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\mathbb{Z}_{\mathrm{tr}}(h_{\mathbb{P}^1_S})/\mathbb{Z}_{\mathrm{tr}}(h_S) & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

This finishes the proof. □

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