Diffeotopy Functors of ind-Algebras and Local Cyclic Cohomology

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Abstract.

We introduce a new bivariant cyclic theory for topological algebras, called local cyclic cohomology. It is obtained from bivariant periodic cyclic cohomology by an appropriate modification, which turns it into a deformation invariant bifunctor on the stable diffeotopy category of topological ind-algebras. We set up homological tools which allow the explicit calculation of local cyclic cohomology. The theory turns out to be well behaved for Banach- and C^*-algebras and possesses many similarities with Kasparov's bivariant operator K-theory. In particular, there exists a multiplicative bivariant Chern-Connes character from bivariant K-theory to bivariant local cyclic cohomology.

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Introduction

A central topic of noncommutative geometry is the study of topological algebras by means of homology theories. The most important of these theories (and most elementary in terms of its definition) is topological K-theory. Various other homology theories have been studied subsequently. This has mainly been done to obtain a better understanding of K-theory itself by means of

theories which either generalize the K-functor or which provide explicitely calculable approximations of it. The latter is the case for cyclic homology, which was introduced by Connes [Co1], and independently by Tsygan [FT], in order to extend the classical theory of characteristic classes to operator K-theory, respectively to algebraic K-theory. Concerning operator K-theory, which is $\mathbb{Z}/2\mathbb{Z}$-graded by the Bott periodicity theorem, one is mainly interested in periodic cyclic theories. Periodic cyclic homology HP is defined as the homology of a natural $\mathbb{Z}/2\mathbb{Z}$-graded chain complex $\widehat{CC}_*$ associated to each complex algebra [Co1]. It can be expressed in terms of derived functors, which allows (in principle) its explicit calculation [Co]. There exists a natural transformation $ch : K_* \longrightarrow HP_*$, called the Chern-character, from K-theory to periodic cyclic homology [Co1]. If this Chern-character comes close to an isomorphism (after tensoring with $\mathbb{C}$), then periodic cyclic homology provides an explicitely calculable approximation of the K-groups one is interested in.
It turns out, however, that the Chern-character is often quite degenerate for Banach- and C^*-algebras. Unfortunately, this is the class of algebras, for which the knowledge of the K-groups would be most significant. The main reason for the degeneracy of Chern characters lies in the different functorial behavior of K-theory and periodic cyclic homology: due to its algebraic nature, cyclic homology possesses the continuity properties of K-theory only in a weak sense. The essential properties of K-theory are:

- Invariance with respect to (continuous) homotopies
- Invariance under topologically nilpotent extensions (infinitesimal deformations)
- Topological Morita invariance
- Excision
- Stability under passage to dense subalgebras which are closed under holomorphic functional calculus.
- Compatibility with topological direct limits

Periodic cyclic homology verifies only a list of considerably weaker conditions:

- Invariance with respect to diffeotopies (smooth homotopies) [Co1], [Go]
- Invariance under nilpotent extensions [Go]
- Algebraic Morita invariance [Co1]
- Excision [CQ2]

In the sequel we will ignore the excision property. The lists of remaining properties will be called strong respectively weak axioms.
To illustrate some of the differences between the two theories we discuss two well known examples.

Example 1: While for the algebra of smooth functions on a compact manifold M the periodic cyclic cohomology

$$HP^*(\mathcal{C}^\infty(M)) \simeq H_*^{dR}(M)$$

coincides with the de Rham homology of M [Co1], the periodic cyclic cohomology of the C^*-algebra of continuous functions on a compact space X is given by

$$HP^*(C(X)) \simeq C(X)'$$

the space of Radon measures on X [Ha]. Thus HP is not stable under passage to dense, holomorphically closed subalgebras. Taking $X = [0,1]$, one sees moreover that periodic cyclic (co)homology cannot be invariant under (continuous) homotopies.
Example 2: The inclusion $A \hookrightarrow M_n(A)$ of an algebra into its matrix algebra gives rise to a (co)homology equivalence by the Morita-invariance of HP. In contrast the inclusion $B \hookrightarrow \overline{\lim\limits_{n\to\infty} M_n(B)} = B \otimes_{C^*} \mathcal{K}$ ($\mathcal{K}$ the algebra of compact operators on a Hilbert space) of a C^*-algebra into its stable matrix algebra induces the zero map in periodic cyclic (co)homology [Wo]. Thus HP is not topologically Morita invariant. Moreover, it does not commute with topological direct limits. Finally it is known that periodic cyclic cohomology is not stable under topologically nilpotent extensions or infinitesimal deformations.
In order to obtain a good homological approximation of K-theory one therefore has to find a new cyclic homology theory which possesses a similar functorial behavior and is still calculable by means of homological algebra.
In this paper we introduce such a theory, called local cyclic cohomology. It is defined on the category of formal inductive limits of nice Fréchet algebras (ind-Fréchet algebras). A well behaved bivariant Chern-Connes character with values in bivariant local cyclic cohomology is constructed in [Pu2].
We proceed in two steps. In the first part of the paper we study diffeotopy functors of topological ind-algebras which satisfy the weak axioms. Our main result is a simple criterion, which guarantees that such a functor even satisfies the strong axioms. In the second part of the paper we modify periodic cyclic homology so that it satisfies this criterion and discuss the cyclic homology theory thus obtained.
A new basic object that emerges here is the stable diffeotopy category of ind-algebras (formal inductive limits of algebras). Its definition is in some sense similar to that of the stable homotopy category of spectra [Ad]. We construct first a triangulated prestable diffeotopy category, which possesses the usual Puppe exact sequences, by inverting the smooth suspension functor. Then we invert the morphisms with weakly contractible mapping cone to obtain the stable diffeotopy category. The criterion mentioned before can now be formulated as follows:

Theorem 0.1. *Let F be a functor on the category of ind-Fréchet algebras with approximation property [LT], which satisfies the weak axioms. Suppose that F is invariant under infinitesimal deformations and under stable diffeotopy, i.e. that it factors through the stable diffeotopy category. Then F also satisfies the strong axioms.*

In order to understand why this result holds we have to explain the significance of infinitesimal deformations. The approach of Cuntz and Quillen to periodic cyclic homology [CQ], [CQ1] emphasizes the invariance of the theory under quasinilpotent extensions. The corresponding notion for Fréchet algebras is that of a topologically nilpotent extension or infinitesimal deformation [Pu1], see also [Me], which is defined as an extension of Fréchet algebras with bounded linear section and topologically nilpotent kernel. Here a Fréchet algebra is called topologically nilpotent if the family of its relatively compact subsets is stable under taking multiplicative closures. Among the possible infinitesimal deformations of an algebra there is an initial or universal one [Pu], provided one works in the more general context of formal inductive limits of Fréchet algebras (or ind-Fréchet algebras). The universal infinitesimal deformation functor $\mathcal{T}$, which is left adjoint to the forgetful functor from ind-Fréchet algebras to a category with the same objects but with a more general kind of morphisms. These are the "almost multiplicative maps" which were introduced and studied in [Pu1]. By its very definition, every functor of ind-algebras, which is invariant under infinitesimal deformations, will be functorial with respect to almost multiplicative maps. This additional functoriality, which played already a fundamental role in [Pu], gives us the means to verify the strong axioms. For example, the inclusion of a dense, smooth subalgebra into a Banach algebra (with approximation property) possesses an almost multiplicative inverse up to stable diffeotopy. It is given by any family of linear regularization maps into the subalgebra, which converges pointwise to the identity. Thus this inclusion is turned by the given functor into an isomorphism.

It should be noted that there is an alternative way to introduce universal infinitesimal extensions, which is based on bornological algebras [Me]. This approach appears to be simpler, but does not seem to lead to homology theories which are accessible to calculation or which possess nice continuity properties. In order to obtain the results of this paper it is indispensable to work with ind-algebras (see section three). It allows to replace a large and complicated topological algebra by a large diagram of algebras of a very simple type. We split thus the information encoded in the initial data into a purely combinatorial and an algebro-analytic part of very particular type. This is reminiscent of algebraic topology where one replaces complicated spaces by model spaces given by simple building blocks and combinatorial gluing data.

In the second part of the paper we apply the results obtained so far to the functor given by bivariant periodic cyclic cohomology [CQ1]. For a pair of Fréchet algebras (A, B) it is given in terms of the natural cyclic bicomplex by

$$HP_*(A,B) := \quad Mor^*_{\mathfrak{H}\mathfrak{o}}(\widehat{CC}(A), \widehat{CC}(B))$$

the group of chain homotopy classes of continuous chain maps of cyclic bicomplexes. Periodic cyclic (co)homology satisfies the weak axioms above, as mentioned at the beginning of the introduction. One can associate to it in a canonical way a homology theory which is invariant under infinitesimal deformations. This is analytic cyclic (co)homology

$$HC^{an}_*(A,B) := \quad Mor^*_{\mathfrak{H}\mathfrak{o}}(\widehat{CC}(\mathcal{T}A), \widehat{CC}(\mathcal{T}B))$$

which was defined in [Pu] and developed in great generality in [Me]. Then we introduce the derived ind-category $\mathcal{D}$ which is obtained by localizing the chain homotopy category of $\mathbb{Z}/2\mathbb{Z}$-graded ind-chain complexes with respect to chain maps with weakly contractible mapping cone. Finally we define local cyclic cohomology as

$$HC^{loc}_*(A,B) := \quad Mor^*_{\mathcal{D}}(\widehat{CC}(\mathcal{T}A), \widehat{CC}(\mathcal{T}B))$$

Thus, by construction, local cyclic cohomology satisfies the assumptions of theorem (0.1). In particular, the first list of axioms holds for local cyclic cohomology, which behaves therefore very much like K-theory.

The second issue which distinguishes local cyclic cohomology among most other cyclic theories is its computability in terms of homological algebra.

There is a spectral sequence calculating morphism groups in the derived ind-category $\mathcal{D}$ which can be used to compute local cyclic cohomology groups. If ($\mathcal{C} =$ "$\varinjlim\limits_{i\in I}$" C_i, $\mathcal{C}' =$ "$\varinjlim\limits_{j\in J}$" C'_j) is a pair of ind-chain complexes, then the E^2-term of the spectral sequence calculating $Mor^*_{\mathcal{D}}(\mathcal{C}, \mathcal{C}')$ is given by

$$E^2_{pq} \,=\, R^p \varprojlim_{i\in I} \varinjlim_{j\in J} Mor^*_{\mathfrak{H}\mathfrak{o}}(C_i, C'_j)$$

where $R^p \varprojlim\limits_{i\in I}$ denotes the p-th right derived functor of the inverse limit over I.

If the cardinalities of the index set I is not too large, then the spectral sequence converges. A consequence of this result is the following theorem which is at the basis of most calculations of local cyclic cohomology groups.

Theorem 0.2. (Limit theorem)
Suppose that the Banach algebra A is the topological direct limit of the countable family of Banach algebras $(A_n)_{n\in\mathbb{N}}$ and suppose that A satisfies the approximation property (see (6.16)). Then there is a natural isomorphism

$$\lim_{n\to\infty} HC^{loc}_*(A_n) \quad \xrightarrow{\simeq} \quad HC^{loc}_*(A)$$

of local cyclic homology groups and a natural short exact sequence

$$0 \longrightarrow \quad \varprojlim_n{}^1 HC^{*-1}_{loc}(A_n) \quad \longrightarrow \quad HC^*_{loc}(A) \quad \longrightarrow \quad \varprojlim_n HC^*_{loc}(A_n) \quad \longrightarrow 0$$

of local cyclic cohomology groups.

Although it might seem that nothing has been gained in this way, because one intractable cohomology group has been replaced by a limit of similarly intractable objects, the spectral sequence proves to be a surprisingly efficient tool for computations. The reason lies in the fact that, although the involved groups are mostly unknown, the transition maps in the corresponding limits often turn out to be quite accessible. We present a number of explicit calculations of local cyclic cohomology groups, which illustrate this principle.

The content of the different sections is as follows. In section one we introduce the notions of almost multiplicative morphism, topologically nilpotent extension, and universal infinitesimal deformation, which are used throughout the paper. This material is taken from [Pu1]. In section two the stable diffeotopy category of ind-algebras is introduced and section three presents various results about the stable diffeotopy type of universal infinitesimal deformations. The main theorem mentioned before is proved in section four. It is applied in section five to periodic cyclic homology. After a short review of the known cyclic homology theories we introduce local cyclic cohomology. In section six we develop the tools for computing local cyclic cohomology groups. Various natural transformations relating the different cyclic theories are discussed in section seven, see also [Me1], and in section eight we give examples of calculations of local cyclic cohomology. We present there also a partial solution of a problem posed by A. Connes in [Co3]. More detailed information can be found in the introductions to the various sections.

This paper is a completely revised and rewritten version of the preprint [Pu1]. A precursor of the theory presented here is asymptotic cyclic cohomology, which was introduced in [CM] and developed in [Pu]. While it shares the good functorial properties of local cyclic cohomology, there is no way to calculate asymptotic cyclic cohomology by homological means.

The excision property of local cyclic cohomology is a consequence of excision in analytic cyclic cohomology and is shown in [Pu2]. In that paper we construct a multiplicative bivariant Chern-Connes character

$$ch_{biv}: \quad KK_*(-,-) \quad \longrightarrow \quad HC_*^{loc}(-,-)$$

from Kasparov's bivariant K-theory [Ka] to bivariant local cyclic cohomology. This character provides a good approximation of (bivariant) K-theory. An equivariant version of the bivariant Chern-Connes character and the computational tools developed in this paper are used in [Pu4] and [Pu5] to verify the Kadison-Kaplansky idempotent conjecture in various cases. These applications show the potential power of local cyclic cohomology as a tool for solving problems in noncommutative geometry. The present paper and the articles [Pu2] and [Pu5] form the published version of the authors Habilitationsschrift presented at the Westfälische Wilhelms-Universität Münster.

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Contents

1 Topological ind-algebras and their universal infinitesimal deformations

1.1 Nice Fréchet algebras

A convenient category of algebras to work with for the purpose of this paper is the category of nice (or admissible) Fréchet algebras. These algebras are in many ways similar to Banach algebras. In addition, they are stable under a number of operations which cannot be performed in the category of Banach algebras, for example, the passage to a dense, holomorphically closed subalgebra. (We decided to replace the name "admissible Fréchet algebra" used in [Pu] and [Pu1] by that of a "nice Fréchet algebra" because the old terminology seemed us too ugly.)

Definition 1.1. [Pu] A Fréchet algebra A is called nice iff there exists an open neighborhood U of zero such that the multiplicative closure of any compact subset of U is precompact in A.

The open set U is called an "open unit ball" for A. It is by no means unique. The class of nice Fréchet algebras contains all Banach algebras and derived subalgebras of Banach algebras [BC] and many Fréchet algebras which occur as dense, holomorphically closed subalgebras of Banach algebras.
Nice Fréchet algebras share a number of properties with Banach algebras: the spectrum of an element of a nice Fréchet-algebra is compact and nonempty and holomorphic functional calculus is valid in nice Fréchet-algebras. (This is most easily seen by noting that according to (1.5) a nice Fréchet algebra is the algebraic direct limit of Banach algebras. Another proof can be found in [Pu], section 1.)
The class of nice Fréchet algebras is closed under taking projective tensor products [Pu], (1.17). If A is nice with open unit ball U and if X is a compact space then the Fréchet algebra $C(X,A)$ is nice with open unit ball $C(X,U)$.

1.2 Formal inductive limits

In the sequel we will work with certain diagrams of algebras. An appropriate language to deal with such diagrams is provided by the notion of a formal inductive limit.

Definition 1.2. Let $\mathcal{C}$ be a category. The category ind-$\mathcal{C}$ of ind-objects or formal inductive limits over $\mathcal{C}$ is defined as follows.
The objects of ind-$\mathcal{C}$ are small directed diagrams over $\mathcal{C}$:

$$\begin{aligned} \mathrm{Ob}_{\text{ind-}\mathcal{C}} &= \{\text{“}\varinjlim_{i\in I}\text{”} A_i \,\Big|\, I \text{ a partially ordered directed set }\} \\ &= \{A_i, f_{ij}: A_i \to A_j,\, i \leq j \in I \,\Big|\, f_{jk}\circ f_{ij} = f_{ik}\} \end{aligned}$$

The morphisms between two ind-objects are given by

$$\mathrm{Mor}_{\text{ind-}\mathcal{C}}(\text{“}\varinjlim_{i\in I}\text{”} A_i,\ \text{“}\varinjlim_{j\in J}\text{”} B_j) \ :=\ \varprojlim_{i\in I}\varinjlim_{j\in J}\mathrm{Mor}_{\mathcal{C}}(A_i, B_j)$$

where the limits on the right hand side are taken in the category of sets.

There exists a fully faithful functor $\iota : \mathcal{C} \to \text{ind-}\mathcal{C}$ which identifies $\mathcal{C}$ with the full subcategory of constant ind-objects.

LEMMA 1.3. *In ind-$\mathcal{C}$ there exist arbitrary inductive limits over directed index sets.*

This is [SGA], I, 8.5.1. Inductive limits in an ind-category will be denoted by $\underrightarrow{Lim}$. Even if direct limits exist in $\mathcal{C}$, they are usually different from the corresponding direct limit in ind-$\mathcal{C}$. If $(A_i)_{i\in I}$ is a small directed diagram in $\mathcal{C}$ which is viewed as diagram of constant ind-objects, then

$$\underset{i\in I}{\underrightarrow{Lim}}\, A_i \ \simeq\ \text{“}\varinjlim_{i\in I}\text{”} A_i$$

as objects of ind-$\mathcal{C}$. If $F : \mathcal{C} \to \mathcal{C}'$ is a functor to a category in which direct limits exist, then F possesses a unique extension $F' : \text{ind-}\mathcal{C} \to \mathcal{C}'$ which commutes with direct limits. One has

$$F'(\text{“}\varinjlim_{i\in I}\text{”} A_i) \ =\ \varinjlim_{i\in I} F(A_i)$$

1.3 DIAGRAMS OF COMPACTLY GENERATED ALGEBRAS

It is our aim to construct and study continuous functors on categories of topological algebras. By this we mean either functors which are determined by their values on suitable families of dense subalgebras or more generally functors which commute with topological direct limits. The first step towards the construction of such functors will be the functorial replacement of "large" algebras by infinite diagrams of "small" algebras of a particular type. Part of the structure of a "large" algebra will be encoded in the combinatorics of the diagram and one is left with the study of "small" algebras with peculiar properties. The natural choice for these "small" algebras will be the Banach algebras generated by the compact subsets of the original algebra.

DEFINITION AND LEMMA 1.4. *There exists a functor $\mathcal{B}$ from the category of nice Fréchet algebras to the category of ind-Banach algebras which assigns to a nice Fréchet algebra the diagram of minimal Banach completions of its compactly generated subalgebras.*

PROOF: Let A be a nice Fréchet algebra and let U be an open unit ball for A. Fix a compact subset $S \subset U$ and denote by $A[S]$ the subalgebra of A generated by S. There exists a largest submultiplicative seminorm on $A[S]$ which satisfies $\| S \| \leq 1$. For $x \in A[S]$ it is given by

$$\| x \| = \inf_{x=\sum a_i s_i} \sum |a_i|$$

where the infimum is taken over the set of all presentations $x = \sum a_i s_i$ such that $a_i \in \mathbb{C}$ and $s_i \in S^\infty$, the multiplicative closure of S. The completion of $A[S]$ with respect to this seminorm is a Banach algebra denoted by A_S. Any inclusion $S \subset S' \subset U$ of compact subsets of U gives rise to a bounded homomorphism of Banach algebras $A_S \to A_{S'}$ so that one obtains an ind-Banach algebra

$$\mathcal{B}(A,U) := \text{``} \varinjlim_{\substack{S \subset U \\ S\ compact}} \text{''} A_S$$

Let $f : A \to A'$ be a bounded homomorphism of nice Fréchet algebras and fix open unit balls $U \subset A$ and $U' \subset A'$. For $S \subset U$ compact let $S' \subset U'$ be a compact set which absorbs $f(S^\infty)$ (S^∞ denotes the multiplicative closure of S). This is possible because S^∞ is precompact (A is nice) and f is bounded. The map f gives then rise to a bounded homomorphism $A_S \to A'_{S'}$. The collection of all these homomorphisms defines a morphism of ind-Banach algebras $f_* : \mathcal{B}(A,U) \to \mathcal{B}(A',U')$. Applying this to the case $f = id$ shows that the ind-Banach algebra $\mathcal{B}(A,U)$ does not depend (up to unique isomorphism) on the choice of U. It will henceforth be denoted by $\mathcal{B}(A)$. The construction above shows furthermore that $\mathcal{B}(-)$ is a functor from the category of nice Fréchet algebras to the category of ind-Banach algebras. □

Lemma 1.5. *There exists a natural transformation of functors*

$$\phi : \mathcal{B} \to \iota$$

(see (1.2)). It is provided by the tautological homomorphism

$$\mathcal{B}(A) = \text{``} \varinjlim_{S \subset U} \text{''} A_S \to A$$

In fact $\varinjlim_{S \subset U} A_S = A$ *in the category of abstract algebras.*

Proof: The fact that the multiplicative closure of a compact subset S of a unit ball of A is precompact implies that the inclusion $A[S] \to A$ extends to a bounded homomorphism $A_S \to A$. These fit together to a bounded homomorphism

$$\phi_A : \mathcal{B}(A) \to \iota(A)$$

of ind-Fréchet algebras. It is clear that the homomorphisms ϕ_A define a natural transformation as claimed by the lemma. In fact

$$\varinjlim_S A[S] \xrightarrow{\simeq} \varinjlim_S A_S \xrightarrow{\simeq} A$$

where the limit is taken in the category of abstract algebras. This yields the second assertion. □

Lemma 1.6. *The functor $\mathcal{B}$ is fully faithful.*

Proof: Let $\psi : \mathcal{B}(A) \to \mathcal{B}(A')$ be a morphism of ind-Banach algebras. It gives rise to a homomorphism

$$\psi' : \quad A = \lim_{\substack{\longrightarrow \\ S}} A_S \quad \longrightarrow \quad \lim_{\substack{\longrightarrow \\ S'}} A'_{S'} = A'$$

of abstract algebras which maps precompact sets to bounded sets and is therefore bounded. This defines a map $mor_{ind-Alg}(\mathcal{B}(A), \mathcal{B}(A')) \to mor_{Alg}(A, A')$ which is clearly inverse to the map on morphism sets induced by $\mathcal{B}$. Therefore the functor $\mathcal{B}$ is fully faithful. □

The canonical extension of the functor $\mathcal{B}$ to the category of nice ind-Fréchet algebras (1.2) will be denoted by the same letter. To study it in further detail we introduce the following notion.

Definition 1.7. An ind-Banach algebra is called compact if it is isomorphic to an ind-Banach algebra "$\lim\limits_{\substack{\longrightarrow \\ i \in I}}$"$A_i$ satisfying the following condition: for every $i \in I$ there exists $i' \geq i$ such that the structure homomorphism $A_i \to A_{i'}$ is compact.

The proof of the following results is facilitated by the technical

Lemma 1.8. *Define a functor $\mathcal{B}'$ from the category of nice Fréchet algebras to the category of ind-Banach algebras by*

$$\mathcal{B}'(A) := \text{“} \lim_{\substack{\longrightarrow \\ S \subset U \\ S\, nullsequence}} \text{”} A_S$$

Then the canonical natural transformation $\mathcal{B}' \to \mathcal{B}$ is an isomorphism of functors.

Proof: This follows from the fact that every compact subset of a Fréchet space is contained in the convex hull of a nullsequence. (A proof is given for example in [Pu], (1.7).) The actual argument is however rather long and tedious and quite close to the one given in the proof of [Pu], (5.8). □

Lemma 1.9. *Let $\mathcal{A}$ be a nice ind-Fréchet algebra. Then the ind-Banach algebra $\mathcal{B}(\mathcal{A})$ is compact.*

Proof: It suffices by lemma (1.8) to verify that $\mathcal{B}'(A)$ is compact for every nice Fréchet algebra A. Let S be a nullsequence in U. As A is nice the multiplicative closure of S is a nullsequence $S^\infty = (a_n)_{n\in\mathbb{N}}$. Choose a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of strictly positive real numbers tending to infinity such that $S' := (\lambda_n a_n)_{n\in\mathbb{N}}$ is still a nullsequence and let $S'' \subset U$ be a compact set absorbing S'. The induced homomorphism $A_S \to A_{S''}$ of the Banach algebras generated by S respectively. S'' is then compact. □

Proposition 1.10. *The functor $\mathcal{B}$ is right adjoint to the forgetful functor from the category of compact ind-Banach algebras to the category of nice ind-Fréchet algebras. In fact for every compact ind-Banach algebra $\mathcal{A}$ and every nice ind-Fréchet algebra $\mathcal{A}'$ the natural transformation ϕ (1.5) induces an isomorphism*

$$Mor_{ind-alg}(\mathcal{A},\, \mathcal{B}(\mathcal{A}')) \overset{\simeq}{\longrightarrow} Mor_{ind-alg}(\mathcal{A},\, \mathcal{A}')$$

Proof: This is an immediate consequence of the definitions. □

Corollary 1.11. *For every compact ind-Banach algebra $\mathcal{A}$ the canonical homomorphism $\phi_{\mathcal{A}} : \mathcal{B}(\mathcal{A}) \overset{\simeq}{\longrightarrow} \mathcal{A}$ is an isomorphism.*

Proof: The corollary follows from (1.9) by applying twice the adjunction formula (1.10). □

Corollary 1.12. *The canonical natural transformation*

$$\phi_{\mathcal{B}} :\, \mathcal{B}^2 := \mathcal{B} \circ \mathcal{B} \overset{\simeq}{\longrightarrow} \mathcal{B}$$

is an isomorphism of functors on the category of nice ind-Fréchet algebras.

Proof: This follows from (1.9) and (1.11). □

1.4 Almost multiplicative maps

A fundamental property of operator K-theory and other homology theories for topological algebras is the invariance under quasinilpotent extensions or infinitesimal deformations. The Cuntz-Quillen approach to periodic cyclic homology [CQ], [CQ1] for example is based on the deformation invariance of the theory.

Invariance under infinitesimal deformations is equivalent to the given theory being functorial not only under homomorphisms between algebras but also under homomorphisms between their quasinilpotent extensions. In other words, a deformation invariant theory extends to a functor on the category obtained by inverting epimorphisms with linear section and quasinilpotent kernel.

It is this "extended functoriality" which makes deformation invariance relevant for us and which will play a fundamental role in the present work.

The notions of (quasi)nilpotent extensions or infinitesimal deformations are well known for abstract and adically complete algebras. We develop the corresponding notions for complete locally convex algebras in close analogy. The existence of a universal quasinilpotent extension of adically complete algebras allows to describe explicitely the morphisms in the extended category. These correspond to linear maps $f : R \longrightarrow S$, for which products of a large number of curvature terms [CQ]

$$\omega_f(a, a') :=\, f(a \cdot a') \,-\, f(a) \cdot f(a'),\;\; a, a' \in R,$$

(measuring the deviation from multiplicativity) are small. It is straightforward to define the corresponding kind of morphisms for diagrams of Fréchet algebras which leads to the notion of an almost multiplicative map. As such maps are stable under composition one obtains in this way a category. (In fact, we will introduce two different notions of almost multiplicativity depending on whether we are interested in uniform estimates or in estimates which are uniform on compact subsets only.)

We construct the universal infinitesimal deformation functors in the topological context as left adjoints of the corresponding forgetful functors to the almost multiplicative categories. This allows finally to introduce the notions of topological nilpotence and of a topologically quasifree algebra.

The basic motivation to study the class of almost multiplicative maps is that it contains a lot of interesting examples, in particular if one passes to diffeotopy categories. Essentially all the results of section three follow immediately from the existence of certain almost multiplicative morphisms in the stable diffeotopy category. We will comment on this fact in the introduction to section three.

An important class of almost multiplicative maps is provided by the (linear) asymptotic morphisms introduced by Connes and Higson [CH]. These are used by them to construct a universal bivariant K-functor for C^*-algebras. It will turn out that the stable diffeotopy category of universal infinitesimal deformations possesses a lot of similarities with the Connes-Higson category.

The reason for introducing topologically quasifree ind-algebras lies in their excellent homological behavior, which is similar to that of quasifree abstract (or adically complete) algebras exploited in [CQ]. The fact that universal infinitesimal deformations are topologically quasifree will allow us in section five to topologize the cyclic complexes of ind-Fréchet algebras in a straightforward way. The correct topologies on these complexes are not completely easy to find otherwise.

While for us the notion of topological nilpotence plays a minor role (compared to the notion of almost multiplicative morphisms), topological nilpotence is at the heart of the approach to analytic cyclic cohomology for bornological algebras presented by Meyer in his thesis [Me].

We begin by introducing a quite restrictive class of almost multiplicative maps. It will be used to facilitate the construction of universal infinitesimal deformations.

DEFINITION 1.13. For a linear map $f : A \to B$ of algebras and a subset $T \subset A$ put

$$\omega(f,T) := \{\, f(aa') - f(a)f(a') \,|\, a, a' \in T \,\} \subset B$$

a) A bounded linear map $f : A \to B$ of Banach algebras is called STRONGLY ALMOST MULTIPLICATIVE if

$$\lim_{n\to\infty} \parallel \omega(f,T)^n \parallel^{\frac{1}{n}} = 0$$

for any bounded subset T of A.

b) A bounded linear morphism $\Phi = (\phi_{ij}) : \text{“}\lim_{\overrightarrow{i\in I}}\text{”} A_i \to \text{“}\lim_{\overrightarrow{j\in J}}\text{”} B_j$ of ind-Banach algebras is called strongly almost multiplicative if for all $i \in I$

and all bounded subsets $T_i \subset A_i$

$$\lim_{\overrightarrow{j\in J}} \overline{\lim_{n\to\infty}} \parallel \omega(\phi_{ij}, T_i)^n \parallel^{\frac{1}{n}} = 0$$

This is independent of the choice of the family of homomorphisms (ϕ_{ij}) representing the morphism Φ of ind-objects.

The more basic notion of almost multiplicativity is the following.

Definition 1.14. a) A bounded linear map $f : A \to B$ of nice Fréchet algebras is called almost multiplicative if for every compact subset $K \subset A$ the multiplicative closure

$$\omega(f, K)^{\infty}$$

of $\omega(f, K)$ is relatively compact in B.

b) A bounded linear morphism $\Psi = (\psi_{ij}) : \text{“}\lim_{\overrightarrow{i\in I}}\text{”} A_i \to \text{“}\lim_{\overrightarrow{j\in J}}\text{”} B_j$ of nice ind-Fréchet algebras is called almost multiplicative if for all $i \in I$ and all compact subsets $K_i \subset A_i$ the multiplicative closure

$$\omega(\psi_{ij}, K_i)^{\infty}$$

is relatively compact for sufficiently large $j \in J$.

It follows immediately from the definitions that a bounded linear morphism $\Psi : \mathcal{A} \to \mathcal{A}'$ of nice ind-Fréchet algebras is almost multiplicative if and only if $\mathcal{B}(\Psi) : \mathcal{B}(\mathcal{A}) \to \mathcal{B}(\mathcal{A}')$ is a strongly almost multiplicative morphism of ind-Banach algebras.

Proposition 1.15. *The composition of strongly almost multiplicative morphisms of ind-Banach algebras is strongly almost multiplicative. Ind-Banach algebras therefore form a category under strongly almost multiplicative bounded linear morphisms. The same assertions hold for nice ind-Fréchet algebras and almost multiplicative maps.*

Proof: It suffices by the previous remark to verify the proposition in the case of strongly almost multiplicative linear maps. For any linear map $\varphi : R \to S$ of algebras let $\omega_\varphi(r, r') := \varphi(rr') - \varphi(r)\varphi(r')$. If $f : A \to B$ and $g : B \to C$ are linear maps of algebras then the deviation from multiplicativity of $g \circ f$ is given by

$$\omega_{g\circ f}(a, a') = g(\omega_f(a, a')) + \omega_g(f(a), f(a'))$$

If f and g are bounded linear maps of Banach algebras and if $a_0, \ldots, a_{2n}$ are elements of the unit ball U of A then the previous equation and the "Bianchi identity" [CQ], (1.2)

$$\omega_f(a, a') \cdot f(a'') = \omega_f(a, a' \cdot a'') - \omega_f(a \cdot a', a'') + f(a) \cdot \omega(a', a'')$$

allow to express elements of $\omega(g\circ f)^n$ naturally in normal form as

$$\omega_{g\circ f}(a_1,a_2)\cdot\ldots\cdot\omega_{g\circ f}(a_{2n-1},a_{2n})=\sum_j g(\alpha_0^j)\omega_g(\alpha_1^j,\alpha_2^j)\cdot\ldots\cdot\omega_g(\alpha_{2k_j-1}^j,\alpha_{2k_j}^j)$$

where each element α_i^j is of the form

$$\alpha_i=f(a'_0)\omega_f(a'_1,a'_2)\cdot\ldots\cdot\omega_f(a'_{2l_i-1},a'_{2l_i}),\ \ a'_0,\ldots,a'_{2l_i}\in U$$

Moreover

$$\sharp\omega_f:=\sum_{i=0}^{2k_j}l_i\leq n\,,\quad \sharp\omega_g:=k_j\leq n\,,\quad \sharp\omega_f+\sharp\omega_g\geq n$$

for all j and the number $\sharp j$ of summands is bounded by $\sharp j\leq 9^n$. For all this see [Pu] (5.1). An easy calculation allows to deduce from these estimates that strongly almost multiplicative maps of Banach algebras are stable under composition. □

The following example provides a large number of almost multiplicative maps. Moreover it gives a hint why the stable diffeotopy category of universal infinitesimal deformations possesses similarities with the categories related to bivariant K-theories [CH],[Hi].

Example 1.16. *Let $f_t:A\to B$ be a linear asymptotic morphism of Banach algebras (or nice Fréchet-algebras) [CH], i.e. $(f_t)_{,t\geq 0}$ is a bounded continuous family of bounded linear maps such that*

$$\lim_{t\to\infty}f_t(aa')-f_t(a)f_t(a')=0\quad\forall a,a'\in A$$

Let $\tilde f:A\to C_b(\mathbb{R}_+,B)$ be the associated linear map satisfying $\mathrm{eval}_t\circ\tilde f=f_t$. *Then $\tilde f$ defines an almost multiplicative linear map*

$$\tilde f:A\to\text{“}\lim_{t\to\infty}\text{”}\,C_b([t,\infty[,B)$$

The class of almost multiplicative maps is considerably larger than the class of asymptotic morphisms. Whereas the curvature terms $\omega_f(a,a'),a,a'\in A$ of a linear asymptotic morphism become arbitrarily small in norm, almost multiplicativity means only that products of a large number of such terms become small in norm. So in particular the spectral radius of the curvature terms has to be arbitrarily small. This will explain an important difference between the homotopy category of asymptotic morphisms, used in E-theory, and the stable diffeotopy category of universal infinitesimal deformations: in the latter one the universal deformation of a Banach algebra is often isomorphic to the universal deformations of its dense and holomorphically closed subalgebras.

1.5 Infinitesimal deformations and topologically nilpotent algebras

With the notion of almost multiplicative morphism at hand one can introduce the topological analogs of nilpotence, infinitesimal deformation and formal smoothness.
As mentioned in the introduction of this section, Meyer has introduced a much more general notion of topological nilpotence which plays a crucial role in his approach to analytic cyclic cohomology for bornological algebras. We refer the reader to [Me].
We give here a slightly less general definition than the one in [Pu1], which suffices however for our purpose.

Definition 1.17. a) A Banach algebra A is strongly topologically nilpotent if the multiplicative closure of every norm bounded subset is norm bounded.

b) A Fréchet algebra A is topologically nilpotent if the multiplicative closure of every relatively compact subset is relatively compact.

c) An ind-Banach algebra "$\underset{i\in I}{\varinjlim}$"$A_i$ is strongly topologically nilpotent if for each $i \in I$ and each bounded subset $U_i \subset A_i$ there exists $i' \geq i$ such that the image of the multiplicative closure U_i^∞ in $A_{i'}$ is bounded.

d) A nice ind-Fréchet algebra "$\underset{j\in J}{\varinjlim}$"$B_j$ is topologically nilpotent if for each $j \in J$ and each compact subset $K_j \subset B_j$ there exists $j' \geq j$ such that the image of the multiplicative closure K_j^∞ in $B_{j'}$ is relatively compact.

Note that a topologically nilpotent Fréchet algebra is necessarily nice, the algebra itself being a possible open unit ball.

Definition 1.18. Let

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{R} \xrightarrow{\pi} \mathcal{S} \rightarrow 0$$

be an extension of nice ind-Fréchet algebras (ind-Banach algebras) which possesses a bounded linear section. In particular, the ind-Fréchet space underlying $\mathcal{R}$ splits into the direct sum of $\mathcal{I}$ and $\mathcal{S}$. Then $\mathcal{R}$ is called a (strong) infinitesimal deformation of $\mathcal{S}$ iff $\mathcal{I}$ is (strongly) topologically nilpotent.

The generic example of a (strongly) almost multiplicative map is given by

Lemma 1.19. *Let*

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{R} \xrightarrow{\pi} \mathcal{S} \rightarrow 0$$

be a (strong) infinitesimal deformation of the nice ind-Fréchet algebra (ind-Banach algebra) $\mathcal{S}$. Then every bounded linear section of π is (strongly) almost multiplicative.

PROOF: Obvious from the definitions. □

We finally extend the notion of formal smoothness to the context of locally convex ind-algebras.

DEFINITION 1.20. A nice ind-Fréchet algebra (ind-Banach algebra) $\mathcal{A}$ is called (STRONGLY) TOPOLOGICALLY QUASIFREE if

$$Mor_{ind-alg}(\mathcal{A},\mathcal{R}) \xrightarrow{\pi_*} Mor_{ind-alg}(\mathcal{A},\mathcal{S})$$

is surjective for any (strong) infinitesimal deformation $\pi : \mathcal{R} \to \mathcal{S}$.

1.6 THE UNIVERSAL INFINITESIMAL DEFORMATION

In this section the universal infinitesimal deformation functor is introduced as the adjoint of the forgetful functor to the category of ind-algebras under almost multiplicative maps.

The construction of the universal infinitesimal deformation proceeds in two steps. The existence of a universal strong deformation of ind-Banach algebras is established first. Then the universal strong deformation of the diagram of compactly generated subalgebras of a nice ind-Fréchet algebra is identified as the universal deformation of the ind-algebra itself.

THEOREM 1.21. *The forgetful functor from the category of ind-Banach algebras to the category with the same objects and strongly almost multiplicative linear maps as morphisms possesses a left adjoint $\mathcal{T}'$, called the* STRONG UNIVERSAL INFINITESIMAL DEFORMATION *functor. This means that for all ind-Banach algebras $\mathcal{R}$, $\mathcal{S}$ there exists a natural and canonical isomorphism*

$$Mor_{\substack{ind\\ alg}}(\mathcal{T}'\mathcal{R},\mathcal{S}) \quad \xrightarrow{\simeq} \quad Mor_{\substack{str\\ alm\\ mult}}(\mathcal{R},\mathcal{S})$$

PROOF: We proceed in several steps.

- We cite from [CQ], (1.2). Let R be an algebra and let $TR := \oplus_{k=1}^{\infty} R^{\otimes k}$ be the tensor algebra over R. Let $\rho : R \to TR$ be the canonical linear inclusion and let $\pi : TR \to R$ be the canonical algebra epimorphism satisfying $\pi \circ \rho = Id_R$. The associated extension of algebras

$$0 \to IR \to TR \xrightarrow{\pi} R \to 0$$

  is the universal linear split extension of R. The kernel IR is a twosided ideal of TR and defines an adic filtration of TR. There is a canonical isomorphism of filtered vector spaces

$$(TR, \text{IR-adic filtration}) \xleftarrow{\simeq} (\Omega^{ev}R, \frac{1}{2}\,\text{degree filtration})$$

between the tensor algebra over R and the module of algebraic differential forms of even degree over R. It is given by the formulas

$$\rho(a^0)\omega(a^1,a^2)\ldots\omega(a^{2n-1},a^{2n}) \quad \longleftarrow \quad a^0da^1\ldots da^{2n}$$
$$\omega(a^1,a^2)\ldots\omega(a^{2n-1},a^{2n}) \quad \longleftarrow \quad da^1\ldots da^{2n}$$

where $\omega(a,a') := \rho(aa') - \rho(a)\rho(a') \in IR$ is the curvature of ρ.

- Let R be a Banach algebra. For $\epsilon > 0$ let $\| - \|_\epsilon$ be the largest submultiplicative seminorm on TR satisfying $\| \rho(a) \|_\epsilon \leq 2 \| a \|_R$ and $\| \omega(a,a') \|_\epsilon \leq \epsilon \| a \|_R \cdot \| a' \|_R$. Denote the completion of TR with respect to this seminorm by TR_ϵ. It is a Banach algebra. By construction $\| - \|_\epsilon \leq \| - \|_{\epsilon'}$ for $\epsilon < \epsilon'$ so that the identity on TR extends to a bounded homomorphism $TR_{\epsilon'} \to TR_\epsilon$ of Banach algebras. Put $\mathcal{T}'R := \text{"}\lim\limits_{\epsilon\to 0}\text{"}\, TR_\epsilon$. It is called the strong universal infinitesimal deformation of R.

- Let $\| - \|_{\epsilon,0}$ respectively $\| - \|_{\epsilon,1}$ be the largest seminorms on TR satisfying

$$\| \rho(a^0)\omega(a^1,a^2)\ldots\omega(a^{2n-1},a^{2n}) \|_{\epsilon,0} \leq \epsilon^n \ \| a^0 \|_R \ldots \| a^{2n} \|_R$$

respectively

$$\| \rho(a^0)\omega(a^1,a^2)\ldots\omega(a^{2n-1},a^{2n}) \|_{\epsilon,1} \leq (2+2n)\,\epsilon^n \ \| a^0 \|_R \ldots \| a^{2n} \|_R$$

It follows from the Bianchi identity

$$\omega(a,a')\,\rho(a'') = \omega(a,a'a'') - \omega(aa',a'') + \rho(a)\,\omega(a',a'')$$

that they satisfy

$$\| xy \|_{\epsilon,0} \leq \| x \|_{\epsilon,1} \| y \|_{\epsilon,0}$$

for all $x,y \in TR$. With this it is not difficult to verify the estimates

$$\| - \|_{\epsilon,0} \leq \| - \|_{4\epsilon} \leq \| - \|_{4\epsilon,1}$$

on TR.

- Let $f : R \to S$ be a bounded homomorphism of Banach algebras and let $Tf : TR \to TS$ be the induced homomorphism of tensor algebras. It is immediate from the definitions that given $\epsilon > 0$ there exist $\epsilon' > 0$ and $C > 0$ such that

$$\| Tf(x) \|_{\epsilon',1} \leq C \ \| x \|_{\epsilon,0} \quad \forall x \in TR$$

With the previous estimate this implies that Tf extends to a bounded homomorphism

$$\mathcal{T}'f : \mathcal{T}'R \longrightarrow \mathcal{T}'S$$

of ind-Banach algebras. This allows to define the strong universal infinitesimal deformation of an ind-Banach algebra as

$$\mathcal{T}'(\text{“}\varinjlim_{i\in I}\text{”} R_i) \,:=\, \underset{\overrightarrow{i\in I}}{Lim}\, \mathcal{T}' R_i$$

- Let $\varphi : R \to \mathcal{S}$ be a strongly almost multiplicative linear morphism from a Banach algebra R to some ind-Banach algebra $\mathcal{S} :=$ “$\varinjlim_{j\in J}$”S_j. Let $\varphi_j : R \to S_j$ be a bounded linear map representing φ and satisfying

$$\overline{\lim_{n\to\infty}}\, \left(\| \,\omega_{\varphi_j}(U)^n \,\|_{S_j}\right)^{\frac{1}{n}} \,\leq\, \frac{\epsilon}{8}$$

where U denotes the unit ball of R. Fix n_0 such that $\| \,\omega_{\varphi_j}(U)^n \,\|_{S_j} \leq \left(\frac{\epsilon}{4}\right)^n$ for $n \geq n_0$. Let $T\varphi_j : TR \to S_j$ be the algebra homomorphism which is characterized by the condition $T\varphi_j \circ \rho = \varphi_j$. Then for arbitrary n one has the estimate

$$\| \,T\varphi_j \,(\rho(U)\omega(U,U)^n)\, \|_{S_j} \leq\, C(n,\epsilon)\,\, \| \,\rho(U)\omega(U,U)^n \,\|_{\frac{\epsilon}{4},0}$$

with $C(n,\epsilon) = \| \,\varphi_j \,\| \cdot (\| \,\varphi_j \,\| + \| \,\varphi_j \,\|^2)^{\,n} \cdot \left(\frac{\epsilon}{4}\right)^{-n}$ whereas for $n \geq n_0$ the stronger estimate

$$\| \,T\varphi_j \,(\rho(U)\omega(U,U)^n)\, \|_{S_j} \leq \| \,\varphi_j \,\| \cdot \| \,\rho(U)\omega(U,U)^n \,\|_{\frac{\epsilon}{4},0}$$

holds. This implies

$$\| \,T\varphi_j(x)\, \|_{S_j} \leq\, C\, \;\| \,x\, \|_{\frac{\epsilon}{4},0}$$

which shows with the previous estimates that $T\varphi_j$ extends to a bounded homomorphism $T'\varphi_j : TR_\epsilon \to S_j$. Therefore φ induces a homomorphism

$$\mathcal{T}'\varphi :\, \mathcal{T}'R \,\longrightarrow\, \mathcal{S}$$

of ind-Banach algebras. If $\phi \,:\, \mathcal{R} \,\longrightarrow\, \mathcal{S}$ is a strongly almost multiplicative morphism of ind-Banach algebras then one obtains similarly a homomorphism of ind-Banach algebras

$$\mathcal{T}'\varphi :\, \mathcal{T}'\mathcal{R} \,\longrightarrow\, \mathcal{S}$$

This construction is natural and defines a canonical and natural map

$$Mor_{\substack{str\\alm\\mult}}(\mathcal{R},\mathcal{S}) \quad\longrightarrow\quad Mor_{\substack{ind\\alg}}(\mathcal{T}'\mathcal{R},\mathcal{S})$$

- Let R be a Banach algebra and consider the canonical linear embedding $\rho : R \to \mathcal{T}'(R)$. It is bounded because $\| \,\rho(U)\, \|_\epsilon \leq 2$ and strongly almost multiplicative as

$$\| \,\omega_\rho(U,U)^n \,\|_\epsilon^{\frac{1}{n}} \leq \| \,\omega(U,U)^n \,\|_\epsilon^{\frac{1}{n}} \leq\, \epsilon$$

where U denotes the unit ball of R. Similarly the canonical linear morphism $\rho : \mathcal{R} \to \mathcal{T}'\mathcal{R}$ of ind-Banach algebras is strongly almost multiplicative. In particular, composition with homomorphisms of ind-Banach algebras defines a canonical map

$$\rho^* : \quad Mor_{\substack{ind\\ alg}}(\mathcal{T}'\mathcal{R}, \mathcal{S}) \quad \longrightarrow \quad Mor_{\substack{str\\ alm\\ mult}}(\mathcal{R}, \mathcal{S})$$

that is obviously inverse to the map constructed above. Therefore ρ^* is an isomorphism.

- Finally it is easy to show that

$$\begin{array}{cccc} \mathcal{T}' : & Mor_{\substack{str\\ alm\\ mult}}(\mathcal{R}, \mathcal{S}) & \longrightarrow & Mor_{\substack{ind\\ alg}}(\mathcal{T}'\mathcal{R}, \mathcal{T}'\mathcal{S}) \\ & \phi & \to & \mathcal{T}'\phi \end{array}$$

turns the strong universal infinitesimal deformation $\mathcal{T}'$ into a covariant functor from the category of ind-Banach algebras with strongly almost multiplicative morphisms to the category of ind-Banach algebras. The previous considerations show also that it is left adjoint to the forgetful functor.

□ The proof of the previous theorem gives no explicit description of the seminorms defining the strong universal infinitesimal deformation of a Banach algebra. If one works with formal inductive limits of Fréchet algebras instead of Banach algebras such an explicit description can be given.

DEFINITION AND LEMMA 1.22. *Let R be a Banach algebra and let TR be the tensor algebra over R. Denote by $\| - \|_{\epsilon,m}$ the largest seminorm on TR satisfying*

$$\| \rho(a^0)\omega(a^1, a^2) \dots \omega(a^{2n-1}, a^{2n}) \|_{\epsilon,m} \leq (2 + 2n)^m \, \epsilon^n \; \| a^0 \| \dots \| a^{2n} \|$$

a) The seminorms $\| - \|_{\epsilon,m}$ are not submultiplicative but satisfy

$$\| xy \|_{\epsilon,m} \leq \| x \|_{\epsilon,m+1} \cdot \| y \|_{\epsilon,m}$$

b) The completion of the tensor algebra TR with respect to the seminorms $\| - \|_{\epsilon,m}$, $m \in \mathbb{N}$, is a nice Fréchet algebra TR^ϵ. An open unit ball of TR^ϵ is given by the open unit ball with respect to the seminorm $\| - \|_{\epsilon,1}$.

c) The formal inductive limit

$$\mathfrak{T}'R := \text{“}\lim_{\epsilon \to 0}\text{”} TR^\epsilon$$

is isomorphic in the category of ind-Fréchet algebras to the strong universal infinitesimal deformation $\mathcal{T}'R$ of R.

PROOF: Assertions a) and b) are shown in [Pu], (5.6). Assertion c) is a consequence of the estimates

$$\| - \|_{\epsilon,0} \leq \| - \|_{4\epsilon} \leq \| - \|_{4\epsilon,1}$$

obtained in the proof of (1.21) and the fact that $\| - \|_{\epsilon',m} \leq C_m \ \| - \|_{\epsilon,0}$ for $\epsilon' < \epsilon$. □

Using the previous results the existence of a universal infinitesimal deformation functor can be established.

THEOREM 1.23. *The forgetful functor from the category of nice ind-Fréchet algebras to the category with the same objects and almost multiplicative linear maps as morphisms possesses a left adjoint $\mathcal{T}$, which is called the* UNIVERSAL INFINITESIMAL DEFORMATION *functor. This means that for all nice ind-Fréchet algebras $\mathcal{R}$, $\mathcal{S}$ there exists a natural and canonical isomorphism*

$$Mor_{\substack{ind\\alg}}(\mathcal{T}\mathcal{R}, \mathcal{S}) \quad \xrightarrow{\simeq} \quad Mor_{\substack{alm\\mult}}(\mathcal{R}, \mathcal{S})$$

The universal infinitesimal deformation functor is given by the composition

$$\mathcal{T} = \mathcal{T}' \circ \mathcal{B}$$

of the functor $\mathcal{B}$ (1.4), associating to an algebra the diagram of its compactly generated subalgebras, and the strong universal infinitesimal deformation functor $\mathcal{T}'$.

PROOF: Put $\mathcal{T} := \mathcal{T}' \circ \mathcal{B}$. For any nice ind-Fréchet algebras $\mathcal{R}$, $\mathcal{S}$ one has a sequence of natural isomorphisms

$$Mor_{\substack{alm\\mult}}(\mathcal{R}, \mathcal{S}) \xrightarrow{\simeq} Mor_{\substack{str\\alm\\mult}}(\mathcal{B}(\mathcal{R}), \mathcal{B}(\mathcal{S}))$$

by the remark following (1.14)

$$Mor_{\substack{str\\alm\\mult}}(\mathcal{B}(\mathcal{R}), \mathcal{B}(\mathcal{S})) \xrightarrow{\simeq} Mor_{\substack{ind\\alg}}(\mathcal{T}\mathcal{R}, \mathcal{B}(\mathcal{S}))$$

by the previous theorem and

$$Mor_{\substack{ind\\alg}}(\mathcal{T}\mathcal{R}, \mathcal{B}(\mathcal{S})) \xrightarrow{\simeq} Mor_{\substack{ind\\alg}}(\mathcal{T}\mathcal{R}, \mathcal{S})$$

by (1.10) and the following lemma. □

LEMMA 1.24. *For any nice ind-Fréchet algebra $\mathcal{R}$ the ind-Banach algebra $\mathcal{T}\mathcal{R}$ is compact.*

PROOF: The ind-Banach algebra $\mathcal{B}(\mathcal{R})$ is compact by (1.9). In order to show that $\mathcal{T}\mathcal{R} = \mathcal{T}'\mathcal{B}(\mathcal{R})$ is compact it suffices therefore to prove the following. The homomorphism $\mathcal{T}'f : \mathcal{T}'A \to \mathcal{T}'B$ induced by a compact homomorphism

$f : A \to B$ of Banach algebras is compact. As the notion of compactness is stable under isomorphism one can pass to the morphism $\mathfrak{T}'f : \mathfrak{T}'A \to \mathfrak{T}'B$ of ind-Fréchet algebras (1.22). The definition of the seminorms on this ind-algebra show immediately that for given $\epsilon > 0$ the homomorphisms $Tf : TA^\epsilon \to TB^{\epsilon'}$ are compact for $\epsilon' > 0$ small enough. This establishes the lemma. $\square$

It remains to verify that the functors constructed in the previous theorems merit their names and provide in fact (strong) infinitesimal deformations.

Lemma 1.25. *Let $\mathcal{A}$ be a nice ind-Fréchet algebra. Then the canonical epimorphism $\pi : \mathcal{TA} \to \mathcal{A}$ (resp. $\pi' : \mathcal{T}'\mathcal{A} \to \mathcal{A}$) adjoint to the identity of $\mathcal{A}$ via (1.21) (resp. (1.23)) is a (strong) infinitesimal deformation in the sense of (1.18).*

Proof: We show first that $0 \to \mathcal{I}'\mathcal{A} \to \mathcal{T}'\mathcal{A} \xrightarrow{\pi} \mathcal{A} \to 0$ is a strong infinitesimal deformation. By lemma (1.22) it suffices to verify the corresponding statement for the extension $0 \to \mathfrak{I}'\mathcal{A} \to \mathfrak{T}'\mathcal{A} \xrightarrow{\pi} \mathcal{A} \to 0$. Henceforth the notations of (1.22) are used. Let $S \subset IA^\epsilon := Ker(\pi : TA^\epsilon \to A)$ be a bounded set. Then for sufficiently small $\epsilon' < \epsilon$ the (bounded) image of S in $TA^{\epsilon'}$ satisfies $\| S \|_{\epsilon',1} \leq 1$. The estimate $\| xy \|_{\epsilon',0} \leq \| x \|_{\epsilon',1} \cdot \| y \|_{\epsilon',0}$ allows then to deduce that $\| S^\infty \|_{\epsilon',0} < \infty$. It follows that the image of S^∞ in $TA^{\epsilon''}$ is bounded for $\epsilon'' < \epsilon'$ (compare the proof of (1.22)). Thus $\mathcal{T}'\mathcal{A} \xrightarrow{\pi} \mathcal{A}$ is a strong infinitesimal deformation of $\mathcal{A}$. This result and (1.24) imply finally that $\mathcal{TA} \xrightarrow{\pi} \mathcal{A}$ is an infinitesimal deformation in the sense of (1.18). $\square$

One can now make precise in which sense the almost multiplicative maps of (1.19) are generic.

Corollary 1.26. *Every almost multiplicative map $\phi : \mathcal{A} \to \mathcal{B}$ of nice ind-Fréchet algebras factorizes as $\phi = f \circ \psi$ where $\psi : \mathcal{A} \to \mathcal{A}'$ is a bounded linear section of an infinitesimal deformation $\pi : \mathcal{A}' \to \mathcal{A}$ and $f : \mathcal{A}' \to \mathcal{B}$ is a homomorphism of ind-Fréchet algebras.*

Finally the infinitesimal deformations given by the completed tensor algebras will be characterized by a universal property.

Theorem 1.27. *Let $\mathcal{A}$ be a nice ind-Fréchet algebra. The extension*

$$0 \to \mathcal{IA} \to \mathcal{TA} \xrightarrow{\pi} \mathcal{A} \to 0$$

with the canonical linear section $\rho : \mathcal{A} \to \mathcal{TA}$ adjoint to the identity of $\mathcal{TA}$ via (1.23) is the universal infinitesimal deformation of $\mathcal{A}$ in the following sense. Let

$$0 \to \mathcal{J} \to \mathcal{R} \xrightarrow{\pi'} \mathcal{S} \to 0$$

be an infinitesimal deformation of $\mathcal{S}$ with fixed bounded linear section and let $f : \mathcal{A} \to \mathcal{S}$ be a homomorphism of nice ind-Fréchet algebras. Then there exists a unique homomorphism of extensions

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{IA} & \to & \mathcal{TA} & \xrightarrow{\pi} & \mathcal{A} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow f & & \\ 0 & \to & \mathcal{J} & \to & \mathcal{R} & \xrightarrow{\pi'} & \mathcal{S} & \to & 0 \end{array}$$

compatible with the given linear sections. In particular, $\mathcal{TA}$ is topologically quasifree (1.20).
In a similar sense for a given ind-Banach algebra $\mathcal{R}$ the extension

$$0 \;\to\; \mathcal{I}'\mathcal{R} \;\to\; \mathcal{T}'\mathcal{R} \;\xrightarrow{\pi}\; \mathcal{R} \;\to\; 0$$

is the universal strong infinitesimal deformation of $\mathcal{R}$. In particular, the ind-Banach algebra $\mathcal{T}'\mathcal{R}$ is strongly topologically quasifree.

2 The stable diffeotopy category of ind-algebras

A diffeotopy is a homotopy which depends smoothly on its parameter. Diffeotopy is a finer equivalence relation than homotopy. For example, the algebra of continuous functions on a closed interval vanishing at one endpoint is nullhomotopic but not nulldiffeotopic. Following closely some well known ideas of homotopy theory (see [Ad])we set up in this chapter a stable diffeotopy category of topological ind-algebras. Its construction proceeds in several steps. One defines first an unstable diffeotopy category in a straightforward way. Then the notions of suspension and mapping cone are introduced. The diffeotopy category is stabilized by inverting the suspension functor which gives rise to a prestable category which is already triangulated, i.e. which possesses long exact Puppe sequences. The stable diffeotopy category of ind-algebras is finally obtained from the prestable one by a category theoretic localization process which is necessary to get rid of some pathologies related to weakly contractible ind-algebras. We then present a criterion for detecting isomorphisms in the stable diffeotopy category which will be frequently used in the rest of the paper.
As mentioned before we begin by introducing the relation of diffeotopy between homomorphisms of topological ind-algebras.

Definition 2.1. (Diffeotopy category)

a) Let $\mathcal{C}^\infty([0,1])$ be the nice nuclear Fréchet algebra of smooth functions on the unit interval all of whose derivatives vanish at the endpoints. For an ind-Fréchet algebra "$\underset{\overrightarrow{i\in I}}{\lim}$"$A_i$ let

$$\mathcal{C}^\infty([0,1],\,\mathcal{A}) := \text{“}\underset{\overrightarrow{i\in I}}{\lim}\text{”}\,\mathcal{C}^\infty([0,1],\,A_i) \;=\; \text{“}\underset{\overrightarrow{i\in I}}{\lim}\text{”}\,\mathcal{C}^\infty([0,1])\otimes_\pi A_i$$

It is again an ind-Fréchet algebra.

b) Two homomorphisms $\mathcal{A} \rightrightarrows \mathcal{A}'$ of ind-Fréchet algebras are called diffeotopic if they factorize as

$$\mathcal{A} \;\longrightarrow\; \mathcal{C}^\infty([0,1],\,\mathcal{A}') \;\rightrightarrows\; \mathcal{A}'$$

where the homomorphisms on the right hand side are given by evaluation at the endpoints. Diffeotopy is an equivalence relation. The equivalence classes are called diffeotopy classes of homomorphisms. The set of diffeotopy classes of homomorphisms between ind-Fréchet algebras $\mathcal{A}$ and $\mathcal{A}'$ is denoted by $[\mathcal{A},\,\mathcal{A}']$.

c) The (unstable) diffeotopy category of ind-Fréchet algebras is the category with ind-Fréchet algebras as objects and with diffeotopy classes of ind-algebra homomorphisms as morphisms.

Now suspensions and mapping cones are defined which are necessary to triangulate diffeotopy categories, i.e. to establish Puppe sequences.

DEFINITION 2.2. (SUSPENSION AND MAPPING CONE)

a) Let $\mathcal{C}^\infty(]0,1[)$ be the nice nuclear Fréchet algebra of smooth functions on the unit interval which vanish together with all their derivatives at the endpoints. If $\mathcal{A} := $ " $\varinjlim\limits_{i\in I}$ "A_i is an ind-Fréchet algebra then the ind-Fréchet algebra

$$\mathcal{SA} := \text{“}\varinjlim_{i\in I}\text{”}\,\mathcal{C}^\infty(]0,1[,\,A_i) = \text{“}\varinjlim_{i\in I}\text{”}\,\mathcal{C}^\infty(]0,1[)\otimes_\pi A_i$$

is called the suspension of $\mathcal{A}$. The suspension defines a functor of the category of ind-Fréchet algebras to itself.

b) Let $f:$ " $\varinjlim\limits_{i\in I}$ "$A_i \longrightarrow$ " $\varinjlim\limits_{j\in J}$ "A'_j be a homomorphism of ind-Fréchet algebras. Define a directed set K by

$$K := \{(i,j,f_{ij})\,|\,i\in I, j\in J,\ f_{ij}: A_i\to A'_j \text{ represents } f\}$$

and by declaring $(i,j,f_{ij})\leq(i',j',f_{i'j'})$ iff $i\leq i'$, $j\leq j'$ and the diagram

$$\begin{array}{ccc} A_{i'} & \xrightarrow{f_{i'j'}} & A'_{j'} \\ \uparrow & & \uparrow \\ A_i & \xrightarrow{f_{ij}} & A'_j \end{array}$$

commutes. The mapping cone $\mathbf{Cone}(f)$ of f is the ind-Fréchet algebra

$$\mathbf{Cone}(f) := \text{“}\varinjlim_{K}\text{”}\,Cone(f_{ij})$$

with

$$Cone(f_{ij}) := \{(a,\chi)\in A_i\times\mathcal{C}^\infty([0,1[,\,A'_j)\,|\,f_{ij}(a)=\chi(0)\}$$

Here $\mathcal{C}^\infty([0,1[)$ is the nice nuclear Fréchet subalgebra of $\mathcal{C}^\infty([0,1])$ consisting of the functions vanishing at the endpoint 1 of the unit interval.

Thus a morphism $\mathcal{B} \to \mathbf{Cone}(f)$ is given by a couple (φ, ν) consisting of a homomorphism $\varphi : \mathcal{B} \to \mathcal{A}$ and a nulldiffeotopy ν of the composed map $\mathcal{B} \xrightarrow{\varphi} \mathcal{A} \xrightarrow{f} \mathcal{A}'$.
The suspension of an ind-algebra $\mathcal{A}$ is a special case of a mapping cone as $\mathcal{SA} \simeq \mathbf{Cone}(p)$ where $p : \mathcal{C}^\infty([0,1[, \mathcal{A}) \to \mathcal{A}$ is the evaluation at 0. The mapping cone of a morphism $f : \mathcal{A} \to \mathcal{A}'$ fits into a natural sequence of homomorphisms

$$\xrightarrow{\mathcal{S}f} \quad \mathcal{SA}' \quad \xrightarrow{s} \quad \mathbf{Cone}(f) \quad \xrightarrow{p} \quad \mathcal{A} \quad \xrightarrow{f} \quad \mathcal{A}'$$

where s and p are defined on individual algebras of the formal inductive systems by $s : SA'_j \to Cone(f_{ij})$, $s(\chi) := (0, \chi)$ and $p : Cone(f_{ij}) \to A_i$, $p(a, \chi') := a$. The mapping cone functor commutes with suspensions, i.e. there exists a natural isomorphism $\mathbf{Cone}(\mathcal{S}f) \xrightarrow{\simeq} \mathcal{S}\mathbf{Cone}(f)$.
In the next step the diffeotopy category shall be stabilized so that it becomes a triangulated category with shift automorphism given by the inverse of the suspension. In order to do so the suspension has to be made an automorphism of the underlying category which leads to

Definition 2.3. (Prestable diffeotopy category) (See [Ma])
The prestable diffeotopy category of ind-Fréchet algebras is the additive category with objects given by pairs $(\mathcal{A}, n)$ consisting of an ind-Fréchet algebra $\mathcal{A}$ and an integer n and with the abelian groups

$$Mor^*((\mathcal{A}, n), (\mathcal{A}', n')) := \lim_{k \to \infty} [\mathcal{S}^{k-n}\mathcal{A}, \mathcal{S}^{k-n'}\mathcal{A}']$$

as morphisms. The transition maps in the limit are given by suspensions. The shift functor $T : T(\mathcal{A}, n) := (\mathcal{A}, n+1)$ is an automorphism of the prestable diffeotopy category and its inverse is canonically isomorphic to the suspension functor: $T^{-1} \simeq \mathcal{S}$.

The prestable diffeotopy category is in fact triangulated.

Lemma 2.4. *The prestable diffeotopy category is a triangulated category [KS] (1.5) in a natural way. The shift functor is given by the functor T of (2.3) and a triangle in the prestable diffeotopy category is distinguished iff it is isomorphic to a triangle of the form*

$$\mathcal{SA}' \quad \xrightarrow{s} \quad \mathbf{Cone}(f) \quad \xrightarrow{p} \quad \mathcal{A} \quad \xrightarrow{f} \quad \mathcal{A}'$$

Proof: A classical result asserts that the homotopy category of pointed topological spaces becomes triangulated after inverting the suspension functor. Here one declares a triangle to be distinguished iff it is isomorphic to a cofibration sequence $X \xrightarrow{f} Y \longrightarrow Cone(f) \longrightarrow \Sigma X$. A proof of this can be found in [Ma], Chapter 1 and Appendix II. Section 1.4 of [KS] might also be helpful. The present lemma is obtained from this result by the following modifications.

One restricts to locally compact spaces, considers the dual function algebras and generalizes to arbitrary Fréchet algebras. Then one passes from algebras to ind-algebras. Finally one replaces the homotopy relation by the finer diffeotopy relation. The demonstration that the various prestable categories obtained along the way are triangulated carries over through each of these steps. This yields the assertion. □

As a consequence [KS], one obtains

Corollary 2.5. *Every homomorphism of ind-Fréchet algebras induces a covariant and a contravariant long exact Puppe sequence in the prestable diffeotopy category.*

The prestable diffeotopy category turns out to be too rigid for our purposes. In fact there is a class of ind-algebras, the weakly contractible ones, which one would like to be equivalent to zero in a reasonable stable diffeotopy category.

Definition 2.6. An ind-Fréchet algebra $\mathcal{A} = \text{“}\varinjlim_{i\in I}\text{”} A_i$ is called weakly contractible if for each $i \in I$ there exists $i' \geq i$ such that the structure homomorphism $A_i \to A_{i'}$ is nulldiffeotopic. It is called stably weakly contractible if $\mathcal{S}^k\mathcal{A}$ is weakly contractible for $k >> 0$.

Every direct limit (1.2) (in the category of ind-algebras) of weakly contractible ind-algebras is weakly contractible.

Lemma 2.7. *The family of stably weakly contractible ind-Fréchet algebras forms a null system, [KS], 1.6.6, in the prestable diffeotopy category.*

Proof: It is easily shown that the family $\mathcal{N}$ of stably weakly contractible ind-algebras is closed under isomorphism in the prestable diffeotopy category. If $f : \mathcal{A} \to \mathcal{A}'$ is a stable homomorphism of weakly contractible ind-algebras then it is almost immediate that $\mathbf{Cone}(f)$ is weakly contractible, too. □

Finally we arrive at

Definition 2.8. (Stable diffeotopy category) (See [Ad], [Ma])

The smooth stable diffeotopy category of ind-Fréchet algebras is the triangulated category obtained from the prestable diffeotopy category by inverting all morphisms with stably weakly contractible mapping cone. A triangle in the stable diffeotopy category is distinguished if it is isomorphic to the image of a distinguished triangle in the prestable diffeotopy category.

It follows in particular that exact covariant and contravariant Puppe sequences exist in the stable diffeotopy category.

In the sequel we will make use of the

Proposition 2.9. (Isomorphism criterion)

Let $f : \text{“}\varinjlim_{i\in I}\text{”} A_i \longrightarrow \text{“}\varinjlim_{j\in J}\text{”} A'_j$ *be a homomorphism of ind-Fréchet algebras and suppose that the following conditions are satisfied: Every homomorphism* f_{ij} *representing the restriction of* f *to* A_i *fits into a diagram*

$$\begin{tikzcd}
A_{i'} \arrow[r, "f_{i'j'}"] & A'_{j'} \\
A_i \arrow[u] \arrow[r, "f_{ij}"] & A'_j \arrow[u] \arrow[ul, "\psi"]
\end{tikzcd}$$

such that

- *the vertical arrows are given by the structure homomorphisms of the corresponding ind-algebras.*
- *the horizontal arrows represent the restrictions of f to A_i respectively $A_{i'}$.*
- *the diagram commutes up to diffeotopy.*

Then the homomorphism f becomes an isomorphism in the stable diffeotopy category.

PROOF: A simple diagram chase shows that a morphism in the unstable diffeotopy category satisfies the criterion of the proposition iff its mapping cone is weakly contractible. Therefore the morphisms under considerations are exactly those belonging to the multiplicative system [KS], 1.6.7., associated to the null system of weakly contractible ind-algebras. In particular, these morphisms become isomorphisms in the stable diffeotopy category. $\square$

There seems to be no reason for stable diffeotopy equivalences to be preserved under direct limits (in the category of ind-algebras). There is however a partial result in this direction.

PROPOSITION 2.10. *Let I be a directed set. Let $(\mathcal{A}_i)_{i\in I}$, $(\mathcal{B}_i)_{i\in I}$ be I-diagrams of ind-Fréchet algebras and let $f = (f_i : \mathcal{A}_i \longrightarrow \mathcal{B}_i)_{i\in I}$ be a morphism of I-diagrams. Suppose that the isomorphism criterion (2.9) applies to each of the morphisms f_i, $i \in I$. Then it applies also to the morphism*

$$\underset{\overrightarrow{i\in I}}{Lim}(f_i) : \underset{\overrightarrow{i\in I}}{Lim}\,\mathcal{A}_i \longrightarrow \underset{\overrightarrow{i\in I}}{Lim}\,\mathcal{B}_i$$

of direct limits which therefore is an isomorphism in the stable diffeotopy category.

PROOF: It is a tedious but straightforward exercise to show that $\underset{\overrightarrow{i\in I}}{Lim}\,\mathbf{Cone}(f_i) \overset{\simeq}{\longrightarrow} \mathbf{Cone}(\underset{\overrightarrow{i\in I}}{Lim}(f_i))$ is an isomorphism of ind-Fréchet algebras. As noted above, a morphism of ind-algebras satisfies the isomorphism criterion iff it has a weakly contractible mapping cone. Therefore the assertion follows from the fact that direct limits of weakly contractible ind-algebras are weakly contractible. □

3 The stable diffeotopy type of universal infinitesimal deformations

In this section some of the main results of this paper are presented. They describe the behavior of the universal infinitesimal deformations of nice ind-Fréchet algebras viewed as objects of the stable diffeotopy category. Among other things we show that under some mild technical assumptions the following assertions hold:

- The inclusion $\mathfrak{A} \hookrightarrow A$ of a dense and holomorphically closed subalgebra of a nice Fréchet algebra induces a stable diffeotopy equivalence of its universal infinitesimal deformations.
- The universal infinitesimal deformation of a topological direct limit of nice Fréchet algebras is stably diffeotopy equivalent to the inductive limit of the universal infinitesimal deformations of the individual algebras.

We will comment on these results in more detail in the introductions of the corresponding subsections. Instead we want to indicate why they hold.
Consider an inclusion $i : \mathfrak{A} \hookrightarrow A$ of a dense and holomorphically closed subalgebra of a nice Fréchet algebra. Suppose that a family of bounded linear "regularization" maps $\mathfrak{s} = (s_\alpha : A \to \mathfrak{A},\ \alpha \in \Lambda)$ is given which approximate the identity on $\mathfrak{A}$ uniformly on compact subsets. The norms of the "curvature" terms $\{\, \omega_\alpha(a, a') = s_\alpha(aa') - s_\alpha(a)s_\alpha(a'),\ a,\ a' \in A,\ \alpha \in \Lambda \,\}$, which measure the deviation of $\mathfrak{s}$ from multiplicativity, might be quite large in norm but their spectral radii will be very small (as they are the same if measured in $\mathfrak{A}$ or in A). Therefore large powers of curvature terms, or in many situations even any product of a large number of curvature terms, will be arbitrarily small in norm. Consequently the family $\mathfrak{s}$ of regularization maps is almost multiplicative and defines a morphism $\mathcal{T}\mathfrak{s} : \mathcal{T}A \longrightarrow \mathcal{T}\mathfrak{A}$ of universal infinitesimal deformations in the stable diffeotopy category. It turns out that the morphism $\mathcal{T}\mathfrak{s}$ provides a stable diffeotopy inverse of the morphism $\mathcal{T}i : \mathcal{T}\mathfrak{A} \longrightarrow \mathcal{T}A$ of universal deformations induced by the inclusion of $\mathfrak{A}$ into A. Thus the inclusion of a dense and holomorphically closed subalgebra induces a stable diffeotopy equivalence of universal infinitesimal deformations, provided that a sufficiently good family of linear regularizations exists. In order to guarantee this we will make some not too restrictive assumptions on the topological vector spaces underlying the algebras under consideration.

3.1 The Grothendieck approximation property

It turns out that the majority of the results presented in this section require that the topological vector spaces underlying the considered algebras verify a regularity condition. This condition is known as Grothendieck's approximation property [LT].

Definition 3.1. (Grothendieck approximation property) [LT]
Let E be a Fréchet space. Then E has the Grothendieck approximation property if the finite rank operators are dense in $\mathcal{L}(E)$ with respect to the topology of uniform convergence on compacta. Thus E possesses the approximation property iff for each seminorm $\| \ \|$ on E, for each $\epsilon > 0$, and each compact set $K \subset E$ there exists a bounded linear selfmap $\phi \in \mathcal{L}(E)$ of finite rank such that $\sup\limits_{x \in K} \| \phi(x) - x \| < \epsilon$.

Examples of Fréchet-algebras whose underlying topological vector spaces have the approximation property are

- nuclear Fréchet-algebras
- nuclear C^*-algebras
- l^p-spaces
- separable, symmetrically normed operator ideals
- the reduced group C^*-algebra of a finitely generated free group.

The algebra of all bounded operators on an infinite dimensional Hilbert space, on the contrary, does not have the approximation property.

3.2 Approximation by ind-algebras of countable type

In order to work with universal infinitesimal deformations of nice Fréchet algebras it turns out to be indispensable to dispose of small models of their stable diffeotopy type. In particular, one is interested in models which are given by a countable formal inductive limit. Under not too restrictive assumptions, their existence is guaranteed by

Theorem 3.2. (Approximation theorem)
*Let A be a separable nice Fréchet algebra which possesses the Grothendieck approximation property. Let U be a convex open unit ball of A.
Let $0 \subset V_0 \subset V_1 \subset \ldots \subset V_n \subset \ldots$ be an increasing sequence of finite dimensional subspaces of A such that $\bigcup\limits_{n=0}^{\infty} V_n$ is a dense subalgebra of A, and let $(\lambda_n)_{n \in \mathbb{N}}$ be a strictly monotone increasing sequence of positive real numbers such that $\lim\limits_{n \to \infty} \lambda_n = 1$. Put $S_n := V_n \cap \lambda_n \overline{U}$. Then the canonical morphism*

$$\text{“}\lim_{n \to \infty}\text{”} A_{S_n} \hookrightarrow \mathcal{B}(A)$$

induces a stable diffeotopy equivalence of strong universal infinitesimal deformations. In particular, the universal infinitesimal deformation $\mathcal{T}A$ of A is stably diffeotopy equivalent to a countable formal inductive limit of Banach algebras.

PROOF:
Recall that for a nice Fréchet algebra A one has $\mathcal{T}A = \text{“}\lim_{\substack{S\subset U\\ \epsilon\to 0}}\text{”}\,(TA_S)_\epsilon$ in the notations of (1.4), (1.21) and (1.23), where S ranges over the family of compact subsets of some fixed open unit ball U of A. We want to apply the isomorphism criterion (2.9) to the morphism

$$\mathcal{T}'(\text{“}\lim_{n\to\infty}\text{”}\,A_{S_n}) = \mathop{Lim}_{n\to\infty}\,\mathcal{T}'(A_{S_n}) \longrightarrow \mathcal{T}'\mathcal{B}(A) = \mathcal{T}A$$

So it has to be shown that any structure morphism $i : (TA_{S_n})_{\epsilon_n} \longrightarrow (TA_S)_{\epsilon'}$ fits into a diagram

$$\begin{tikzcd}
(TA_{S_m})_{\epsilon_m} \arrow[r, "i'"] & (TA_{S'})_{\epsilon''} \\
(TA_{S_n})_{\epsilon_n} \arrow[u, "j"] \arrow[r, "i"'] & (TA_S)_{\epsilon'} \arrow[u, "j'"'] \arrow[ul, "T\phi"']
\end{tikzcd}$$

where the homomorphisms i, j, i', j' are given by the structure maps and which commutes up to diffeotopy.
By definition the identity homomorphism of TA_S gives rise to morphisms $(TA_S)_{\epsilon'} \longrightarrow \mathcal{T}'A_S \simeq \mathfrak{T}'A_S = \text{“}\lim_{\epsilon_1\to 0}\text{”}\,(TA_S)^{\epsilon_1}$. In order to define the diagonal morphism in the desired diagram it suffices therefore to construct a bounded homomorphism $T\phi : (TA_S)^{\epsilon_1} \longrightarrow (TA_{S_m})_{\epsilon_m}$ for ϵ_1 given and suitable $m >> 0$ large and $\epsilon_m > 0$ small enough.
Fix ϵ with $0 < 4\epsilon < \epsilon_1$. As A possesses the Grothendieck approximation property there exists a bounded finite rank selfmap ϕ of A which is close to the identity on the (relatively compact) multiplicative closure S^∞ of S. We may suppose that $\phi_t := (1-t)\cdot\phi + t\cdot Id$ satisfies $\omega_{\phi_t}(S^\infty) \subset \epsilon U$ for $t\in[0,1]$ and that $\phi(A)$ is contained in the dense subspace $\bigcup_{n=0}^{\infty} V_n$ of A. In particular $\phi(A)\subset V_m$ for some $m >> 0$.
By definition the Banach algebra A_S is the completion of the subalgebra of A generated by S with respect to the seminorm $\|a\| := \inf \sum |\lambda_i|$ where the infimum is taken over all presentations $a = \sum \lambda_i\, s_i$ of a with $\lambda_i \in \mathbb{C}$, $s_i \in S^\infty$. It follows from this that ϕ induces a bounded linear map $\phi : A_S \longrightarrow A_{S_m}$ of Banach spaces for sufficiently large m such that $\phi(A)\bigcup\phi(A)^2 \subset V_m$. Fix such m and let C_0 be the norm of the linear map ϕ.

Let $a^0, \ldots, a^{2n} \in A_S$. One finds in the notations of (1.21) and (1.22)

$$\| T\phi(\rho(a^0)\omega(a^1,a^2)\cdot\ldots\cdot\omega(a^{2n-1},a^{2n})) \|_{\epsilon_m}$$

$$\leq \| T\phi(\rho(a^0)) \|_{\epsilon_m} \cdot \prod_{i=1}^{n} \| T\phi(\omega(a^{2i-1},a^{2i})) \|_{\epsilon_m}$$

because $\| - \|_{\epsilon_m}$ is submultiplicative. The identity

$$T\phi(\omega(a,a')) = \varrho(\omega_\phi(a,a')) + \omega(\phi(a),\phi(a'))$$

shows then that

$$\| T\phi(\omega(a,a')) \|_{\epsilon_m} \leq 2 \| \omega_\phi(a,a') \|_{A_{S_m}} + \epsilon_m \| \phi(a) \|_{A_{S_m}} \cdot \| \phi(a') \|_{A_{S_m}}$$

$$\leq (2 \cdot \frac{m+1}{m} \cdot \epsilon + \epsilon_m \cdot C_0^2) \cdot \| a \|_{A_S} \cdot \| a' \|_{A_S}$$

So for ϵ_m sufficiently small

$$\| T\phi(\rho(a^0)\omega(a^1,a^2)\cdot\ldots\cdot\omega(a^{2n-1},a^{2n})) \|_{\epsilon_m} \leq 2\cdot C_0 \cdot \epsilon_1^n \cdot \| a^0 \|_{A_S} \cdot\ldots\cdot \| a^{2n} \|_{A_S}$$

from which the estimate

$$\| T\phi(\alpha) \|_{\epsilon_m} \leq C \cdot \| \alpha \|_{\epsilon_1,0}, \ \ \forall \alpha \in (TA_S)^{\epsilon_1}$$

results. This establishes the existence of the diagonal morphism in the diagram. The same kind of estimate shows that, after possibly modifying the choice of m and ϵ_m, the one parameter family $T\phi_t = T((1-t)\cdot\phi + t\cdot Id)$ defines a diffeotopy connecting the homomorphisms $T\phi \circ i$ and j from $(TA_{S_n})_{\epsilon_n}$ to $(TA_{S_m})_{\epsilon_m}$. Similarly the same family $T\phi_t$ defines a diffeotopy between the homomorphisms $i' \circ T\phi$ and j' from $(TA_S)_{\epsilon'}$ to $(TA_{S'})_{\epsilon''}$ after choosing S' and ϵ'' appropriately. This completes the proof.

□

COROLLARY 3.3. *Let* $``\varinjlim_{i\in I}"A^i$ *be a formal inductive limit of nice Fréchet algebras which possess the Grothendieck approximation property. Suppose that for each* $i \in I$ *a sequence* $(V_n^i)_{n\in\mathbb{N}}$ *of finite dimensional subspaces of* A^i *and a sequence* $(\lambda_n^i)_{n\in\mathbb{N}}$ *of real numbers has been chosen as in (3.2) and such that the structure maps* $A^i \to A^j$, $i \leq j$ *map* $\bigcup_n V_n^i$ *into* $\bigcup_n V_n^j$*. Then the countable ind-Banach algebras* $``\lim_{n\to\infty}"A^i_{S_n}$ *form an inductive system, labeled by* I*, and the natural morphism*

$$\underset{i\in I}{\underrightarrow{Lim}}\,(``\lim_{n\to\infty}"A^i_{S_n}) \longrightarrow \underset{i\in I}{\underrightarrow{Lim}}\,\mathcal{B}(A^i) = \mathcal{B}(``\varinjlim_{i\in I}"A^i)$$

induces a stable diffeotopy equivalence of strong universal infinitesimal deformations.

PROOF: The corollary follows from the proof of the previous theorem and proposition (2.10). □

3.3 Smooth subalgebras

A fundamental question in the study of functors of topological algebras is their compatibility with completions. Put differently, one asks how a functor behaves under passage to dense topological subalgebras. A prototype of such a stability phenomenon occurs in topological K-theory which is well known to be stable under passage to dense subalgebras which are closed under holomorphic functional calculus. Here we investigate stability properties of the universal infinitesimal deformation functor with values in the stable diffeotopy category. To this end we introduce a class of dense and holomorphically closed subalgebras of a nice Fréchet algebra, called smooth subalgebras. It contains in particular the domains of densely defined unbounded derivations. Under rather mild restrictions it is shown that the stable diffeotopy type of the universal infinitesimal deformation of a nice Fréchet algebra does not change under passage to smooth subalgebras. As a consequence, continuous homotopy equivalences of nice Fréchet algebras give rise to stable diffeotopy equivalences of their universal infinitesimal deformations.

Definition 3.4. Let $i : \mathfrak{A} \hookrightarrow A$ be an inclusion of Fréchet algebras with dense image and suppose that A is nice. Then $\mathfrak{A}$ is called a smooth subalgebra of A if there exists an open neighborhood U of 0 in A such that $i^{-1}(U)$ is an open unit ball of $\mathfrak{A}$ in the sense of (1.1).

In particular, smooth subalgebras of nice Fréchet algebras are nice. The condition of smoothness is quite restrictive. In fact, smooth subalgebras are closed under holomorphic functional calculus.

Lemma 3.5. *[Pu], (7.2). Let $\mathfrak{A} \subset A$ be a smooth subalgebra of the nice Fréchet algebra A. Then $\mathfrak{A}$ is closed under holomorphic functional calculus in A.*

The name "smooth subalgebra" is motivated by the following example.

Lemma 3.6. *[Pu], (7.4) Let A be a nice Fréchet algebra and let $\Delta := \{\delta_i, \, i \in I\}$ be an at most countable set of unbounded derivations on A. Suppose that there is a common dense domain $dom(\Delta)$ of all finite compositions of derivations in Δ. Then every at most countable set Σ of graph seminorms*

$$\| \, a \, \|_{k,f,m} := \sum_{J \subset \{1,\dots,k\}} \| \prod_{j \in J} \delta_{f(j)} \, (a) \, \|_m$$

defines a locally convex topology on $dom(\Delta)$, where $\| - \|_m$ ranges over a set of seminorms defining the topology of A, J runs over the ordered subsets of $\{1,\dots,k\}$ and f is a map from the finite set $\{1,\dots,k\}$ to the index set I. Denote by $\mathfrak{A}_\Sigma$ the Fréchet algebra obtained by completion of this locally convex algebra. Then $\mathfrak{A}_\Sigma$ is a smooth subalgebra of A.

Proof: We treat for simplicity the case $k = 1$, the reasoning in the general case being similar. Therefore the topology on $\mathfrak{A}$ is defined by the seminorms

$$\| \, a \, \|'_m := \| \, \partial a \, \|_m + \| \, a \, \|_m$$

Let $U \subset A$ be an open unit ball. We claim that $U' := U \cap \mathfrak{A}$ is an open unit ball in $\mathfrak{A}$. Let $K \subset U'$ be compact and choose $\lambda > 1$ such that $\lambda K \subset U'$ which is possible by the compactness of K. One finds for $a_j \in K$

$$\parallel \prod_1^n a_j \parallel'_m = \parallel \sum_{i=1}^n a_1 \cdot \ldots \cdot (\partial a_i) \cdot \ldots \cdot a_n \parallel_m + \parallel \prod_1^n a_j \parallel_m$$

$$\leq \sum_{i=1}^n \lambda^{1-n} \parallel \sum_{i=1}^n (\lambda a_1) \cdot \ldots \cdot (\partial a_i) \cdot \ldots \cdot (\lambda a_n) \parallel_m + \lambda^{-n} \parallel \prod_1^n (\lambda a_j) \parallel_m$$

By hypothesis $\lambda K \subset U$ has relatively compact multiplicative closure in A. Moreover $\partial(K) \subset A$ is compact. An estimation of the sum above yields therefore

$$\parallel \prod_1^n a_j \parallel'_m \leq (\lambda C_0 n + C_1)\lambda^{-n}$$

If one treats the case $k > 1$ one sees that the number of summands after differentiating a product of n factors k times equals n^k which is of subexponential growth in n so that the assertion holds then as well. □

Another example of smooth subalgebras is provided by

Example 3.7. *[Pu], (7.9) Let A be a separable C^*-algebra and let τ be an (unbounded), densely defined, positive trace on A. Then its domain $\ell^1(A, \tau)$ is a smooth subalgebra of A.*

The basic result about smooth subalgebras is

Theorem 3.8. (Smooth subalgebra theorem)
Let A be a nice Fréchet algebra, let $\mathfrak{A}$ be a smooth subalgebra of A and suppose that at least one of the following conditions is satisfied

- *There exists a family $(\varphi_\lambda : A \to \mathfrak{A}, \lambda \in \Lambda)$ of bounded linear maps, labeled by a directed set Λ, such that $\{i \circ \varphi_\lambda(x), \lambda \in \Lambda\}$ is bounded and $\varinjlim_{\lambda \in \Lambda} i \circ \varphi_\lambda(x) = x$ for all $x \in A$.*
- *A possesses the Grothendieck approximation property.*

Then the inclusion

$$\mathfrak{A} \longrightarrow A$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

Proof: Let $i : \mathfrak{A} \hookrightarrow A$ be the inclusion and let $\mathcal{T}i : \mathcal{T}\mathfrak{A} \to \mathcal{T}A$ be the induced homomorphism of universal infinitesimal deformations. We will apply the isomorphism criterion (2.9) to show that $\mathcal{T}i$ is an isomorphism in the stable diffeotopy category.

Fix an open unit ball U of A such that $U' := i^{-1}(U)$ is an open unit ball of $\mathfrak{A}$. This is possible because $\mathfrak{A}$ is a smooth subalgebra of A. Let $S' \subset U'$ and

$S \subset U$ be compact and let $\epsilon' > 0$, $\epsilon > 0$ be such that $Ti : (T\mathfrak{A}_{S'})_{\epsilon'} \to (TA_S)_\epsilon$ represents the restriction of Ti to $(T\mathfrak{A}_{S'})_{\epsilon'}$.
In order to verify the isomorphism criterion it suffices to show that this map fits into a diagram

$$\begin{tikzcd}
(T\mathfrak{A}_{S'_1})_{\epsilon'_1} \arrow[rr, "Ti"] & & (TA_{S_1})_{\epsilon_1} \\
(T\mathfrak{A}_{S'})_{\epsilon'} \arrow[u] \arrow[rr, "Ti"'] & & (TA_S)_\epsilon \arrow[u] \arrow[ull, "T\phi"']
\end{tikzcd}$$

with the vertical arrows given by structure maps and which commutes up to diffeotopy.
By our assumptions (and the theorem of Banach-Steinhaus in the first case mentioned above) there exist bounded linear maps $\varphi : A \to \mathfrak{A}$ such that the family $i \circ \varphi^t := (1-t) \cdot i \circ \varphi + t \cdot Id : A \to A,\ 0 \leq t \leq 1$ is arbitrarily close to the identity on S^∞ and $i(S^{'\infty})$. In particular, one can find for given $\epsilon_0 > 0$ a bounded linear map $\phi : A \to \mathfrak{A}$ such that $\omega_{\phi^t}(S^\infty) \subset \epsilon_0 U$ and $\omega_{\phi^t}(i(S^{'\infty})) \subset \epsilon_0 U$ for all $t \in [0,1]$. Consequently $\omega_\phi(S^\infty) \subset \epsilon_0 U'$ by our choice of open unit balls.
The arguments given in the proof of theorem (3.2) apply word for word and show that the homomorphism $T\phi : TA \longrightarrow T\mathfrak{A}$ defines (for a suitable choice of $S'_1 \subset U'$ compact and $\epsilon'_1 > 0$) a bounded algebra homomorphism $T\phi : (TA_S)_\epsilon \longrightarrow (T\mathfrak{A}_{S'_1})_{\epsilon'_1}$ which makes the left lower triangle of the diagram commute up to diffeotopy.
After choosing $S_1 \subset U$ and $\epsilon_1 > 0$ appropriately, the upper right triangle of the diagram will also commute up to diffeotopy by a similar reasoning.
This completes the proof of the theorem. □

COROLLARY 3.9. *Let* "$\varinjlim\limits_{i \in I}$"$A^i$ *be a nice ind-Fréchet algebra. For each* $i \in I$ *let* $\mathfrak{A}^i$ *be a smooth subalgebra of* A^i *satisfying the assumptions of (3.8). Suppose that the smooth subalgebras* $(\mathfrak{A}^i)_{i \in I}$ *form an inductive system* "$\varinjlim\limits_{i \in I}$"$\mathfrak{A}^i$ *under the structure maps of* "$\varinjlim\limits_{i \in I}$"$A^i$. *Then the canonical morphism*

$$\text{“}\varinjlim_{i \in I}\text{”}\mathfrak{A}^i \longrightarrow \text{“}\varinjlim_{i \in I}\text{”}A^i$$

induces a stable diffeotopy equivalence of universal deformations.

Proof: This follows from the proof of the previous theorem and proposition (2.10). □

3.3.1 Examples

Corollary 3.10. *Let A be a nice Fréchet algebra which possesses the Grothendieck approximation property. Let $\Delta := \{\delta_i, i \in I\}$ be an at most countable set of unbounded derivations on A and let $\mathfrak{A}_\Sigma$ be one of the completions of dom(Δ) introduced in (3.6). Then the inclusion*

$$\mathfrak{A}_\Sigma \hookrightarrow A$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

Proof: The corollary follows from (3.6) and (3.8). □

Corollary 3.11. *Let A be a nice Fréchet algebra and let $(\Phi^t)_{t\in\mathbb{R}}$ be a continuous one parameter group of automorphisms of A. Let Δ be the corresponding unbounded derivation with domain $\mathfrak{A}^\infty := \{a \in A, \Phi^t(a) \in \mathcal{C}^\infty(\mathbb{R}, A)\}$. Then the inclusion*

$$\mathfrak{A}^\infty \hookrightarrow A$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

Proof: Let u_λ, $\lambda \in \Lambda$, be a family of smooth functions with compact support on the real line which approach the delta distribution at 0. Then the family of regularization maps $\varphi_\lambda : A \longrightarrow \mathfrak{A}^\infty$, $\varphi_\lambda(a) := \int_{-\infty}^{\infty} u_\lambda(t)\, \Phi^t(a)\, dt$ satisfies the conditions of (3.8). The conclusion follows. □

Corollary 3.12. *Let $\mathcal{A}$ be a nice ind-Fréchet algebra, let M be a smooth compact manifold without boundary, and let $k \geq 0$ be an integer. Then the canonical morphisms*

$$\mathcal{C}^\infty(M, \mathcal{A}) \hookrightarrow \mathcal{C}^k(M, \mathcal{A}) \hookrightarrow C(M, \mathcal{A})$$

of nice ind-Fréchet algebras induce stable diffeotopy equivalences of universal infinitesimal deformations.

Proof: The corollary follows as before from (3.6) and (3.8) by noting that $C(M, A)$ is nice (1.1) and by using convolution with a family of smooth kernels (k_λ) on $M \times M$, approaching the delta distribution along the diagonal, as family of regularization maps (φ_λ). There is also a version for manifolds with boundary. For the definition of the appropriate function spaces see [Pu], (7.7). □

Corollary 3.13. *Let $\mathcal{A}$ be a nice ind-Fréchet algebra and let $k \geq 0$ be an integer. Then the canonical inclusions*

$$\mathcal{C}^\infty([0,1],\mathcal{A}) \hookrightarrow \mathcal{C}^k([0,1],\mathcal{A}) \hookrightarrow C([0,1],\mathcal{A})$$

of nice ind-Fréchet algebras induce stable diffeotopy equivalences of universal infinitesimal deformations.

Proof: This is the case $M = [0,1]$ of (3.12) for manifolds with boundary. □

Corollary 3.14. *Let A be a separable C^*-algebra. Let τ be a densely defined, positive, unbounded trace on A and let $\ell^1(A,\tau)$ be its domain. Then the canonical inclusion*

$$\ell^1(\tau, A) \hookrightarrow A$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

Proof: By (3.7) the domain of τ is a smooth subalgebra of A. There exists a bounded approximate unit $(u_\lambda), \lambda \in \Lambda$, for A consisting of elements of the dense twosided ideal $\ell^1(\tau, A)$. Left-multiplication with u_λ provides the regularization maps φ_λ asked for in (3.8). For details see [Pu], (7.9). □

3.4 Topological direct limits

Another fundamental question in the study of functors of topological algebras is their behavior with respect to topological direct limits. As is well known topological K-theory commutes with arbitrary topological direct limits. Under rather mild restrictions the universal infinitesimal deformation functor with values in the stable diffeotopy category possesses a similar behavior.
It turns out that, under these restrictions, the universal infinitesimal deformation of a topological direct limit is stably diffeotopy equivalent to the direct limit (in the ind-category of algebras) of the individual universal deformations. This result provides an effective tool for calculations, as will be shown in a number of examples.

Theorem 3.15. (Limit theorem)
Let “$\varinjlim_{\lambda\in\Lambda}$”$A_\lambda$ *be a directed family of nice Fréchet algebras and let*

$$f = \varprojlim f_\lambda : \text{“}\varinjlim_{\lambda\in\Lambda}\text{”} A_\lambda \longrightarrow A$$

be a homomorphism to a nice Fréchet algebra A. Suppose that the following conditions hold:

- *A is separable and possesses the Grothendieck approximation property.*
- *The image $Im(f) := \varinjlim_{\lambda\in\Lambda} f_\lambda(A_\lambda)$ is dense in A.*

- *There exist seminorms* $\| - \|_\lambda$ *on* A_λ, $\lambda \in \Lambda$, *respectively* $\| - \|$ *on* A, *and a constant* C *such that*

  i) The set of elements of length less than 1 with respect to the seminorm is an open unit ball for A_λ, $\lambda \in \Lambda$, *respectively* A.

  ii)
$$\underset{\lambda' \geq \lambda}{Sup} \parallel i_{\lambda\lambda'}(a_\lambda) \parallel_{\lambda'} < \infty$$
  for all $a_\lambda \in A_\lambda$, $\lambda \in \Lambda$.

  iii)
$$\overline{\lim_{\overrightarrow{\lambda \in \Lambda}}} \parallel a_\lambda \parallel_\lambda \leq C \parallel f(a) \parallel$$
  for all
$$a = \lim_{\overrightarrow{\lambda \in \Lambda}} a_\lambda \in \lim_{\overrightarrow{\lambda \in \Lambda}} A_\lambda$$

Then
$$f : \text{“} \lim_{\overrightarrow{\lambda \in \Lambda}} \text{”} A_\lambda \longrightarrow A$$
induces a stable diffeotopy equivalence of universal infinitesimal deformations.

During the proof we will several times make use of the following

Lemma 3.16. *Let the assumptions of the previous theorem be valid. Then for given* $K \subset A_\lambda$ *compact and given* $\epsilon > 0$ *there exists* $\lambda' \in \Lambda$ *such that* $\parallel i_{\lambda\lambda'}(a) \parallel_{\lambda'} \leq C \cdot \parallel f_\lambda(a) \parallel_A + \epsilon$ *for all* $a \in K$. *Here* C *denotes the constant of the assumption of the previous theorem.*

Proof: By the theorem of Banach-Steinhaus and our assumptions the family $\{ i_{\lambda\lambda'} : A_\lambda \to (A_{\lambda'}, \| - \|_{\lambda'}), \lambda' \geq \lambda \} \bigcup \{ f_\lambda : A_\lambda \to (A, \| - \|_A) \}$ of bounded linear maps on A_λ is equicontinuous. Accordingly there exists a seminorm $\| - \|'$ on A_λ and a constant C' such that $\parallel i_{\lambda\lambda'}(a) \parallel_{\lambda'} \leq C' \cdot \parallel a \parallel'$ and $\parallel f_\lambda(a) \parallel_A \leq C' \cdot \parallel a \parallel'$ for all $a \in A_\lambda$ and $\lambda' > \lambda$. Choose a finite subset $\{y_1, \ldots, y_k\}$ of K such that the balls with respect to $\| - \|'$ around $y_1, \ldots, y_k$ with radius $\frac{\epsilon}{2C'(1+C)}$ cover K. Choose finally $\lambda' > \lambda$ so large that one has $\parallel i_{\lambda\lambda'}(y_l) \parallel_{\lambda'} \leq C \cdot \parallel f_\lambda(y_l) \parallel_A + \frac{\epsilon}{2}$ for all y_l, $1 \leq l \leq k$, which is possible by the assumptions of the theorem. With these choices the desired estimates hold. □

Proof of the theorem:
We want to apply the isomorphism criterion (2.9) to the morphism
$$\mathcal{T}f : \mathcal{T}(\text{“} \lim_{\overrightarrow{\lambda \in \Lambda}} \text{”} A_\lambda) = \underset{\overrightarrow{\lambda \in \Lambda}}{Lim} \, \mathcal{T}A_\lambda \longrightarrow \mathcal{T}A$$
of ind-Banach algebras. So let $(\mathcal{T}A^\lambda_{S_\lambda})_{\epsilon_\lambda} \to (\mathcal{T}A_S)_{\epsilon'}$ be a homomorphism representing $\mathcal{T}f_\lambda$ where $S_\lambda \subset A_\lambda$ and $S \subset A$ are compact sets satisfying $\parallel S_\lambda \parallel_{A_\lambda} < 1$ and $\parallel S \parallel_A < 1$ and $\epsilon_\lambda > 0$, $\epsilon' > 0$. It has to be shown that this map fits into a diagram of homomorphisms

$$\begin{tikzcd}
(TA^{\lambda'}_{S_{\lambda'}})_{\epsilon_{\lambda'}} \arrow[r] & (TA_{S'})_{\epsilon''} \\
(TA^{\lambda}_{S_\lambda})_{\epsilon_\lambda} \arrow[u] \arrow[r] & (TA_S)_{\epsilon'} \arrow[u] \arrow[ul, "T\psi"']
\end{tikzcd}$$

which commutes up to diffeotopy.
Denote by $S^\infty(S^\infty_\lambda)$ the relatively compact multiplicative closures of $S(S_\lambda)$.
Step 1:
Fix $\epsilon > 0$ such that $4\epsilon < \epsilon'$. As A possesses the Grothendieck approximation property there exists a bounded linear selfmap $\phi \in \mathcal{L}(A)$ of finite rank such that $\phi(A) \subset \varinjlim_{\mu} f(A_\mu)$ and $\|\omega(\phi^t, S^\infty \cup f_\lambda(S^\infty_\lambda))\|_A < \frac{\epsilon}{2C}$ for all $t \in [0,1]$ where $\phi^t := (1-t)\cdot Id + t\cdot\phi$ and where $C \geq 1$ is a constant as in the assumption of the theorem.
Step 2:
As ϕ is of finite rank one finds a finite dimensional subspace $V \subset A_\mu$ for some $\mu \geq \lambda$ such that $f_\mu : A_\mu \to A$ maps V onto $\phi(A)$. Let $s : \phi(A) \to V$ be any linear section of $f_\mu : V \to \phi(A)$. The set $K := \omega(s\circ\phi, S^\infty)$ is then a bounded and thus relatively compact subset of the finite dimensional space $W := V + V^2 \subset A_\mu$. Similarly $K' := \bigcup_{t=0}^{1} \omega((1-t)\cdot i_{\lambda\mu} + t\cdot s\circ\phi\circ f_\lambda, S^\infty_\lambda)$ is a relatively compact subset of A_μ.
Step 3:
Choose according to the assumptions of the theorem and the previous lemma some $\lambda' \in \Lambda$ such that $\|i_{\mu\lambda'}(a)\|_{A_{\lambda'}} \leq C\cdot\|f_\mu(a)\|_A + \frac{\epsilon}{2}$ for all $a \in K \cup K' \subset A_\mu$.
Put finally

$$\Psi := i_{\mu\lambda'} \circ s \circ \phi : A \to A_{\lambda'}$$

Step 4:
We estimate the deviation of Ψ from multiplicativity on S^∞.

$$\omega(\Psi, S^\infty) = i_{\mu\lambda'}(\omega(s\circ\phi, S^\infty)) = i_{\mu\lambda'}(K)$$

so that

$$\|\,\omega(\Psi, S^\infty)\,\|_{\lambda'} = \|\,i_{\mu\lambda'}(K)\,\|_{\lambda'}$$

$$\leq C\cdot \|\,f_\mu(K)\,\|_A + \frac{\epsilon}{2} = C\cdot\|\,\omega(\phi, S^\infty)\,\|_A + \frac{\epsilon}{2} < \epsilon$$

Step 5:
We estimate the deviation of $\chi_t := (1-t)\cdot i_{\lambda\lambda'} + t\cdot\Psi\circ f_\lambda$ from multiplicativity on S_λ^∞.

$$\omega(\chi_t,\, S_\lambda^\infty) \,=\, i_{\mu\lambda'}(\omega((1-t)\cdot i_{\lambda\mu} \,+\, t\cdot s\circ\phi\circ f_\lambda,\, S_\lambda^\infty)) \,\subset\, i_{\mu\lambda'}(K')$$

so that

$$\|\,\omega(\chi_t,\, S_\lambda^\infty)\,\|_{\lambda'} \leq \|\, i_{\mu\lambda'}(K')\,\|_{\lambda'}$$
$$\leq\, C\cdot \|\, f_\mu(K')\,\|_A \,+\frac{\epsilon}{2} \,=\, C\cdot \|\,\omega(\phi^t,\, f_\lambda(S_\lambda^\infty))\,\|_A \,+\frac{\epsilon}{2} \,<\, \epsilon$$

Step 6:
We finally estimate the deviation of $\mu_t := (1-t)\cdot Id_A + t\cdot f_{\lambda'}\circ\Psi$ from multiplicativity on S^∞.

$$\omega(\mu_t,\, S^\infty) \,=\, \omega(\phi^t,\, S^\infty)$$

so that

$$\|\,\omega(\mu_t,\, S^\infty)\,\|_A < \epsilon$$

Step 7:
The arguments in the proof of (3.2) show then that for a suitable choice of $S_{\lambda'} \subset A_{\lambda'}$, $S' \subset A$ and $\epsilon_{\lambda'}$, $\epsilon'' > 0$:

- $T\psi : TA \to TA_{\lambda'}$ induces a bounded homomorphism $T\psi : (TA_S)_{\epsilon'} \longrightarrow (TA^{\lambda'}_{S_{\lambda'}})_{\epsilon_{\lambda'}}$ by the estimates of step 4.
- $T\chi_t$, $0 \leq t \leq 1$ defines a diffeotopy between $T\Psi\circ Tf_\lambda$ and $Ti_{\lambda\lambda'}$ by the estimates of step 5.
- $T\mu_t\, :\, (TA_S)_{\epsilon'} \longrightarrow (TA_{S'})_{\epsilon''}$, $0 \leq t \leq 1$, defines a diffeotopy between $Tf_{\lambda'}\circ T\Psi$ and the structure homomorphism by the estimates of step 6.

This establishes the desired diagram. The theorem is therefore proved. □

Note that for nice Fréchet algebras with Grothendieck approximation property the smooth subalgebra theorem (3.8) is a consequence of the previous direct limit theorem, applied to the constant inductive family given by the fixed smooth subalgebra.

As a special case of the limit theorem we obtain

Corollary 3.17. *Let* “ $\underset{\overrightarrow{\lambda\in\Lambda}}{\lim}$ ” A_λ *be a directed family of nice Fréchet algebras. Suppose that there exist seminorms* $\|-\|_\lambda$ *on* A_λ, $\lambda\in\Lambda$, *such that the set of elements of length less than 1 is an open unit ball for* A_λ, *and such that*

$$\underset{\overrightarrow{\lambda\in\Lambda}}{\lim}\;\|\, a\,\|_\lambda = 0$$

for all $a \in \underset{\overrightarrow{\lambda\in\Lambda}}{\lim}\, A_\lambda$. *Then the universal infinitesimal deformation*

$$\mathcal{T}(\text{“}\underset{\overrightarrow{\lambda\in\Lambda}}{\lim}\text{”} A_\lambda) \,=\, \underset{\overrightarrow{\lambda\in\Lambda}}{Lim}\,\mathcal{T}A_\lambda$$

is a weakly contractible ind-Fréchet algebra.

Proof: The given conditions are equivalent to the assertion that the constant morphism $\text{“}\varinjlim_{\lambda\in\Lambda}\text{''}\, A_\lambda \longrightarrow 0$ satisfies the assumptions of the limit theorem (3.15). The proof of this theorem shows therefore that the isomorphism criterion (2.9) applies to the constant morphism $\mathcal{T}(\text{“}\varinjlim_{\lambda\in\Lambda}\text{''}\, A_\lambda) \longrightarrow \mathcal{T}(0) = 0$. This implies our claim. □

3.4.1 Examples

The theorem allows to determine the stable diffeotopy type of universal infinitesimal deformations of numerous algebras which occur as topological direct limits. We present some examples.

Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. It is well known that every nontrivial twosided ideal $\mathcal{J}$ of $\mathcal{B}(\mathcal{H})$ satisfies $\mathcal{F} \subset \mathcal{J} \subset \mathcal{K}$, i.e. contains the smallest nonzero ideal $\mathcal{F}$ of finite rank operators and is contained in the largest ideal $\mathcal{K}$ of all compact operators. An ideal $\mathcal{J}$ is called symmetrically normed if it is complete with respect to a norm $\| \, \|_{\mathcal{J}}$ which satisfies the characteristic inequality $\|AXB\|_{\mathcal{J}} \leq \|A\|_{\mathcal{B}(\mathcal{H})} \cdot \|X\|_{\mathcal{J}} \cdot \|B\|_{\mathcal{B}(\mathcal{H})}$ for all $X \in \mathcal{J}, A, B \in \mathcal{B}(\mathcal{H})$, and $\|P\|_{\mathcal{J}} = \|P_{\mathcal{B}(\mathcal{H})}\| = 1$ for some (and therefore every) rank one projection $P \in \mathcal{B}(\mathcal{H})$. It follows easily from the definition that the inclusion $\mathcal{J} \hookrightarrow \mathcal{B}(\mathcal{H})$ is a bounded map of Banach spaces and that $\| \, \|_{\mathcal{B}(\mathcal{H})} \leq \| \, \|_{\mathcal{J}}$ on $\mathcal{J}$. This implies that $(\mathcal{J}, \| \, \|_{\mathcal{J}})$ is a nonunital Banach algebra. It is known that $\mathcal{J}$ is separable if and only if the ideal $\mathcal{F}$ of finite rank operators is dense in $\mathcal{J}$. (For all this consider [Co] and the references therein).

Corollary 3.18. *Let $\mathcal{J}$ be a separable, symmetrically normed operator ideal in $\mathcal{B}(\mathcal{H})$. Let i : $\text{“}\lim_{n\to\infty}\text{''} M_n(\mathbb{C}) \to \mathcal{J}$ be a homomorphism of ind-Banach algebras sending the matrix units (e_{kk}), $k \in \mathbb{N}$, to the orthogonal projections onto the lines spanned by the vectors of some orthonormal basis of $\mathcal{H}$. Then*

$$i : \text{“}\lim_{n\to\infty}\text{''} M_n(\mathbb{C}) \longrightarrow \mathcal{J}$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

Proof: First of all $\mathcal{J}$ possesses the Grothendieck approximation property. To see this consider the contraction with a finite rank projection P. It defines a linear selfmap of $\mathcal{J}$ of norm one because of the characteristic inequality $\|PXP\|_{\mathcal{J}} \leq \|P\|_{\mathcal{B}(\mathcal{H})}\|X\|_{\mathcal{J}}\|P\|_{\mathcal{B}(\mathcal{H})} = \|X\|_{\mathcal{J}}$. As the finite rank operators are dense in $\mathcal{J}$ ($\mathcal{J}$ is separable) it suffices to prove that for every finite set $\{A_1, .., A_k\}$ of finite rank operators and every $\epsilon > 0$ there exists a finite rank projection P satisfying $\|PA_iP - A_i\|_{\mathcal{J}} < \epsilon$ for $1 \leq i \leq k$. Every finite rank operator can be written as a product of three such operators: $A_i = B_iC_iD_i$. Therefore $\|PA_iP - A_i\|_{\mathcal{J}} \leq \|PB_i - B_i\|_{\mathcal{B}(\mathcal{H})}\|C_i\|_{\mathcal{J}}\|D_iP\|_{\mathcal{B}(\mathcal{H})}$ $+\|B_i\|_{\mathcal{B}(\mathcal{H})}\|C_i\|_{\mathcal{J}}\|D_iP - D_i\|_{\mathcal{B}(\mathcal{H})}$ so that the claim has only to be verified for the operator norm for which it is obvious. Actually this argument shows also that the image of i is dense in $\mathcal{J}$.

As any two norms on the finite dimensional algebras $M_n(\mathbb{C})$ are equivalent the ind-Fréchet algebra "$\lim\limits_{n\to\infty}$" $M_n(\mathbb{C})$ does not depend (up to isomorphism) on the choice of the norms on the algebras $M_n(\mathbb{C})$, $n \in \mathbb{N}$. If we choose the norms obtained from the norm on $\mathcal{J}$ by restriction to $\lim\limits_{n\to\infty} i(M_n(\mathbb{C}))$ then the corollary follows immediately from theorem (3.15). □

COROLLARY 3.19. *Let the notations of (3.18) be valid. Then for any nice ind-Fréchet algebra $\mathcal{A}$ the homomorphism*

$$Id \otimes i : \underset{n\to\infty}{Lim}\, M_n(\mathcal{A}) = \underset{n\to\infty}{Lim}\, \mathcal{A} \otimes_\pi M_n(\mathbb{C}) \longrightarrow \mathcal{A} \otimes_\pi \mathcal{J}$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

PROOF: Let $\mathcal{A} =$ "$\underset{i\in I}{\varinjlim}$"$A_i$. A reasoning similar to the proof of (3.18) shows that the isomorphism criterion (2.9) applies to the morphisms
$\mathcal{T}(\underset{n\to\infty}{Lim}\, M_n(A_i)) \longrightarrow \mathcal{T}(A_i \otimes_\pi \mathcal{J})$ for all $i \in I$. Proposition (2.10) implies then that $\mathcal{T}(\underset{n\to\infty}{Lim}\, M_n(\mathcal{A})) \longrightarrow \mathcal{T}(\mathcal{A} \otimes_\pi \mathcal{J})$ is a stable diffeotopy equivalence as well. □

COROLLARY 3.20. *Let A be a C^*-algebra. Let i : "$\lim\limits_{n\to\infty}$"$M_n(\mathbb{C}) \to \mathcal{K}(\mathcal{H})$ be an inclusion as defined in (3.18). Then the homomorphism*

$$Id \otimes i : \text{“}\lim_{n\to\infty}\text{”} M_n(A) = \text{“}\lim_{n\to\infty}\text{”} A \otimes_\pi M_n(\mathbb{C}) \longrightarrow A \otimes_{C^*} \mathcal{K}(\mathcal{H})$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

The proof is similar to that of (3.19).

COROLLARY 3.21. *Let "$\lim\limits_{n\to\infty}$"A_n be an inductive system of separable C^*-algebras and let A be the enveloping C^*-algebra of the algebraic direct limit $\lim\limits_{n\to\infty} A_n$. Suppose that A possesses the Grothendieck approximation property. Then the canonical homomorphism*

$$\text{“}\lim_{n\to\infty}\text{”} A_n \longrightarrow A$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

PROOF: Obvious from (3.15) and the fact that $\lim\limits_{n\to\infty} \parallel a_n \parallel_{A_n} = \parallel a \parallel_A$ for all $a = \lim\limits_{n\to\infty} a_n \in A$. □

4 Diffeotopy functors on categories of ind-algebras

In this section we summarize what we obtained so far concerning our original goal of improving invariance and stability properties of functors of topological algebras. The results underline the crucial role played by the stable diffeotopy category in this question. We consider functors of nice ind-Fréchet algebras, which are invariant under diffeotopy, under infinitesimal deformations, and under passage to infinite matrix algebras. Suppose in addition that the given functor is not only invariant under diffeotopy, i.e. factors through the unstable diffeotopy category, but that it factors even through the stable diffeotopy category. Then it possesses in fact a number of remarkable properties:

- Continuous homotopy invariance
- Invariance under passage to dense smooth subalgebras (in the presence of the approximation property)
- Topological Morita invariance, i.e. invariance under passage to completions of the infinite matrix algebra over the given algebra

Thus the fact that a matrix stable and deformation invariant functor factors through the stable diffeotopy category ensures already that it behaves in many ways like K-theory. It turns out that among these three required properties the factorization property is the crucial one. Suppose that F is any functor on the ind-category of nice Fréchet algebras which factors through the stable diffeotopy category. Then there is a universal matrix stable and deformation invariant functor associated to it, given by the composition

$$F' := F \circ \mathcal{T} \circ M_\infty \simeq F \circ M_\infty \circ \mathcal{T}$$

with the universal infinitesimal deformation functor $\mathcal{T}$ and the infinite matrix functor M_∞. This functor will possess all the properties listed above.
The universal example of a stable diffeotopy functor is the tautological functor from the category of nice ind-Fréchet algebras to the stable diffeotopy category. The functor $\mathcal{T} \circ M_\infty$ with values in the stable diffeotopy category has therefore a lot of similarities with the (bivariant) K-functor.
It might be interesting to compare this functor to other functors and categories that have been constructed as models of bivariant K-theory such as Higson's category [Hi],[Cu], the category of asymptotic morphisms of Connes and Higson [CH], and the bivariant theories introduced by Cuntz in [Cu1]. There is however an important difference between all these theories and the one considered in the present paper because we completely ignore the excision problem. A closer study of these questions has to be undertaken elsewhere.

Definition 4.1. Let F be a functor on the ind-category of nice Fréchet algebras.

a) F is said to factor through the stable diffeotopy category if it is isomorphic to a functor of the form $F' \circ i$ where i is the canonical functor to the stable diffeotopy category.

b) F is called invariant under infinitesimal deformations, if for every topologically nilpotent extension of nice ind-Fréchet algebras

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow 0$$

the induced morphism

$$F(\mathcal{A}) \xrightarrow{\simeq} F(\mathcal{B})$$

is an isomorphism.

c) For a nice ind-Fréchet algebra $\mathcal{A}$ let

$$M_\infty(\mathcal{A}) := \underset{n\to\infty}{\mathit{Lim}}\; M_n(\mathcal{A})$$

be its infinite matrix algebra with structure maps $M_n \hookrightarrow M_{n+1}$ given by the "inclusion of the upper left corner". The functor F is called matrix-stable if it turns the canonical morphism $\mathcal{A} \longrightarrow M_\infty(\mathcal{A})$ into an isomorphism.

Theorem 4.2. *Let F be a functor on the ind-category of nice Fréchet algebras which satisfies the following conditions:*

- *F factors through the stable diffeotopy category*
- *F is invariant under infinitesimal deformations*
- *F is matrix stable*

Then the following assertions hold

a) F is a homotopy functor, i.e. if $f, f' : \mathcal{A} \longrightarrow \mathcal{A}'$ are continuously homotopic homomorphisms of nice ind-Fréchet algebras then

$$F(f) = F(f')$$

b) If $A \hookrightarrow B$ is the inclusion of a smooth subalgebra into a nice Fréchet algebra possessing the Grothendieck approximation property, then

$$F(A) \xrightarrow{\simeq} F(B)$$

is an isomorphism.

b)' If

$$\mathcal{A} = \text{“}\varinjlim_{i\in I}\text{”} A_i \longrightarrow \text{“}\varinjlim_{i\in I}\text{”} B_i = \mathcal{B}$$

is a morphism of I-diagrams of nice Fréchet algebras such that $A_i \hookrightarrow B_i$ is the inclusion of a smooth subalgebra and such that B_i possesses the Grothendieck approximation property for all $i \in I$, then

$$F(\mathcal{A}) \overset{\simeq}{\longrightarrow} F(\mathcal{B})$$

is an isomorphism.

c) Let $\mathcal{J}$ be a separable, symmetrically normed operator ideal and let $j : \mathbb{C} \to \mathcal{J}$ be a homomorphism which maps 1 to a projection of rank one. Then

$$F(Id \otimes_\pi j) : F(\mathcal{A}) \overset{\simeq}{\longrightarrow} F(\mathcal{A} \otimes_\pi \mathcal{J})$$

is an isomorphism for every nice ind-Fréchet algebra $\mathcal{A}$.

c)' Let $\mathcal{K}(\mathcal{H})$ be the algebra of compact operators on a separable Hilbert space and let $i : \mathbb{C} \to \mathcal{K}(\mathcal{H})$ be a homomorphism which maps 1 to a projection of rank one. Then

$$F(Id \otimes_{C^*} i) : F(\mathcal{B}) \overset{\simeq}{\longrightarrow} F(\mathcal{B} \otimes_{C^*} \mathcal{K}(\mathcal{H}))$$

is an isomorphism for every ind-C^*-algebra $\mathcal{B}$.

Suppose in addition that F commutes with direct limits (Recall that direct limits exist in any ind-category). Then moreover the following is true.

d) If $(A_i)_{i\in I}$ is a directed family of nice Fréchet algebras such that the topological direct limit $\varinjlim_{i\in I} A_i$ exists in the category of nice Fréchet algebras and possesses the Grothendieck approximation property, then the natural morphism

$$\varinjlim_{i\in I} F(A_i) \overset{\simeq}{\longrightarrow} F(\varinjlim_{i\in I} A_i)$$

is an isomorphism.

It should be noted that assertions a) b) and d) do not require the matrix stability of the functor under consideration.

Proof: For a nice ind-Fréchet algebra $\mathcal{A}$ denote by $\pi : \mathcal{T}\mathcal{A} \longrightarrow \mathcal{A}$ the canonical epimorphism adjoint to the identity map of $\mathcal{A}$ (1.23). For any morphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ of nice ind-Fréchet algebras there is the commutative diagram

$$\begin{array}{ccc}
F(\mathcal{T}\mathcal{A}) & \overset{F(\mathcal{T}f)}{\longrightarrow} & F(\mathcal{T}\mathcal{B}) \\
F(\pi)\downarrow & & \downarrow F(\pi) \\
F(\mathcal{A}) & \overset{F(f)}{\longrightarrow} & F(\mathcal{B})
\end{array}$$

The invariance of F under infinitesimal deformations implies that the vertical arrows of the diagram are isomorphisms (1.25). Suppose now that the morphism f induces a stable diffeotopy equivalence of universal infinitesimal deformations, i.e. $\mathcal{T}f : \mathcal{T}\mathcal{A} \longrightarrow \mathcal{T}\mathcal{B}$ is a stable diffeotopy equivalence. Then the upper horizontal map in the diagram becomes an isomorphism, because F factors through the stable diffeotopy category. Thus one may conclude that $F(f) : F(\mathcal{A}) \longrightarrow F(\mathcal{B})$ is an isomorphism.
We now prove assertion a) of the theorem. By the smooth subalgebra theorem (3.8) the previous arguments apply to the inclusion of algebras $\mathcal{C}^\infty([0,1],\mathcal{A}) \longrightarrow C([0,1],\mathcal{A})$ so that $F(\mathcal{C}^\infty([0,1],\mathcal{A})) \xrightarrow{\simeq} F(C([0,1],\mathcal{A}))$ is an isomorphism. Any evaluation homomorphism $\mathcal{C}^\infty([0,1],\mathcal{A}) \longrightarrow \mathcal{A}$ is a diffeotopy equivalence and therefore turned by F into an isomorphism. Altogether this shows that any evaluation homomorphism $C([0,1],\mathcal{A}) \longrightarrow \mathcal{A}$ induces an isomorphism $F(C([0,1],\mathcal{A})) \xrightarrow{\simeq} F(\mathcal{A})$. This statement is equivalent to the homotopy invariance of F. Assertions b), c) and d) follow from the previous discussion and the smooth subalgebra theorem (3.8), respectively the direct limit theorem (3.15) and its corollaries (3.18) and (3.19). □
We now make some observations concerning the problem of constructing functors which satisfy the conditions of the previous theorem. It turns out that one can associate in a universal way to any functor F on the ind-category of nice Fréchet algebras, which factors through the stable diffeotopy category, a functor F', which is matrix stable and invariant under infinitesimal deformations. The modified functor F' will be shown to satisfy the assertions of theorem (4.2). This result shows the crucial role played by the stable diffeotopy category in the search for functors of topological algebras with good homotopy, stability, and continuity properties.

THEOREM 4.3. *Let F be a functor on the ind-category of nice Fréchet algebras and suppose that F factors through the stable diffeotopy category. Let*

$$F' := F \circ \mathcal{T} \circ M_\infty$$

be the functor obtained by composition with the universal infinitesimal deformation functor $\mathcal{T}$ and the infinite matrix functor M_∞. Then the functor F' is matrix stable, invariant under infinitesimal deformations, and satisfies all the assertions of Theorem (4.2).

REMARK 4.4. *If one ignores Morita invariance there is a similar statement for the functor $F'' := F \circ \mathcal{T}$. It is universal among all functors which are invariant under infinitesimal deformations and equipped with a natural transformation to F. It satisfies assertions a), b) and d) of Theorem(4.2).*

PROOF: We show first that F' is matrix stable. So let $\mathcal{A}$ be an ind-Fréchet algebra and let $\mathcal{A} \longrightarrow M_\infty(\mathcal{A})$ be the canonical inclusion. The induced homomorphism $M_\infty(\mathcal{A}) \longrightarrow M_\infty(M_\infty(\mathcal{A}))$ is known to be a diffeotopy equivalence. As the universal deformation functor preserves the relation of diffeotopy and as F is diffeotopy invariant, the conclusion follows. We verify next that F' is invariant under infinitesimal deformations. So let

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow 0$$

be an infinitesimal deformation of $\mathcal{B}$. It follows that

$$0 \longrightarrow M_\infty(\mathcal{N}) \longrightarrow M_\infty(\mathcal{A}) \longrightarrow M_\infty(\mathcal{B}) \longrightarrow 0$$

is again an infinitesimal deformation. (Note that this has only to be verified for finite matrices of fixed size.) Applying the universal infinitesimal deformation functor $\mathcal{T}$ one obtains a morphism

$$\mathcal{T}(M_\infty(\mathcal{A})) \longrightarrow \mathcal{T}(M_\infty(\mathcal{B}))$$

which is a diffeotopy equivalence by the universal properties of $\mathcal{T}$ (1.27). Applying the diffeotopy invariant functor F one concludes that

$$F'(\mathcal{A}) \longrightarrow F'(\mathcal{B})$$

is an isomorphism. Let finally $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a morphism of ind-algebras, such that for each $n \in \mathbb{N}$ the isomorphism criterion (2.9) applies to the morphism

$$\mathcal{T}(M_n(f)) : \mathcal{T}(M_n(\mathcal{A})) \longrightarrow \mathcal{T}(M_n(\mathcal{B}))$$

In particular $\mathcal{T}(M_n(f))$ is a stable diffeotopy equivalence. By (2.10) the direct limit

$$\mathcal{T}(M_\infty(f)) : \mathcal{T}(M_\infty(\mathcal{A})) \longrightarrow \mathcal{T}(M_\infty(\mathcal{B}))$$

of these morphisms is again a stable diffeotopy equivalence. Consequently the induced map

$$F'(f) : F'(\mathcal{A}) \longrightarrow F'(\mathcal{B})$$

is an isomorphism. The condition is satisfied in the following cases: For the inclusion of a smooth subalgebra (a diagram of smooth subalgebras) as in (3.8), for any of the morphisms $M_\infty(\mathcal{A}) \longrightarrow \mathcal{A} \otimes_\pi \mathcal{J}$ or $M_\infty(\mathcal{B}) \longrightarrow \mathcal{B} \otimes_{C^*} \mathcal{K}(\mathcal{H})$ considered in (3.18) and (3.19), and for the morphism $\underset{\overrightarrow{i \in I}}{Lim}\, A_i \longrightarrow A$ of a family $(A_i)_{i \in I}$ into a topological direct limit A, which possesses the Grothendieck approximation property (3.15). If, in the latter case, the functor F commutes in addition with direct limits, one deduces further from

$$F'(\underset{\overrightarrow{i \in I}}{Lim}\, A_i) = F(\mathcal{T}(M_\infty(\underset{\overrightarrow{i \in I}}{Lim}\, A_i))) = F(\underset{\overrightarrow{i \in I}}{Lim}\, \mathcal{T}(M_\infty(A_i))) = \underset{\overrightarrow{i \in I}}{Lim}\, F'(A_i)$$

that

$$\underset{\overrightarrow{i \in I}}{Lim}\, F'(A_i) \longrightarrow F'(A)$$

is an isomorphism. □

It should be noted that there is no reason for the functor F' to factor through the stable diffeotopy category in the way asked for in (4.1) although the original functor F does so. The reason lies in the fact that the suspension and universal infinitesimal deformation functors do not commute in any reasonable sense.

Corollary 4.5. *Consider the functor from the ind-category of nice Fréchet algebras to the stable diffeotopy category which associates to a nice ind-Fréchet algebra the universal infinitesimal deformation of its infinite matrix algebra. This functor is homotopy invariant, invariant under passage to smooth subalgebras (in the presence of the approximation property), and topologically Morita invariant (invariant under projective tensor products with separable symmetrically normed operator ideals or under the C^*-tensor product with the algebra of compact operators in the case of ind-C^*-algebras).*

In particular, this functor shows many similarities with a bivariant K-functor. The basic difference from a K-functor is that the present functor has no reason to satisfy excision.

5 Local cyclic cohomology

We apply now the ideas of the previous section in order to improve the homotopy, stability, and continuity properties of continuous periodic cyclic cohomology. Continuous periodic cyclic (co)homology has the following drawbacks which prevent it from being a good approximative model of K-theory. It is not invariant under continuous homotopies, it is not stable under tensorization with general operator ideals, it is not stable under passage to smooth subalgebras, and it is not compatible with topological direct limits. The considerations of the previous section suggest how to modify continuous periodic cyclic (co)homology in order to obtain a cyclic theory which does not have the mentioned disadvantages. The new cyclic theory should be invariant under infinitesimal deformations and should factor through the stable diffeotopy category of ind-algebras. There is indeed a canonical choice for a homology theory which satisfies these conditions. This is local cyclic (co)homology. The drawbacks mentioned before disappear under the passage from periodic to local cyclic cohomology. So it possesses in fact many properties which are typical for bivariant K-theory [Ka]. Besides this it will turn out to be accessible to direct computation. Local cyclic cohomology becomes thus a valuable tool for the study of problems in noncommutative geometry.

5.1 Cyclic cohomology theories

We recall some well known facts about various cyclic homology theories.

Periodic cyclic cohomology [Co1], [FT]

For a complex algebra A define the A-bimodule of algebraic differential forms by

$$\Omega^n A := \widetilde{A} \otimes A^{\otimes n}, \quad \Omega A := \bigoplus_n \Omega^n A$$

with $\widetilde{A} := A \oplus \mathbb{C}1$ the algebra obtained from A by adjoining a unit. The A-bimodule structure on ΩA is the obvious one. The Hochschild complex of A is given by

$$C_*(A) \,:=\, (\Omega^* A,\, b)$$

with Hochschild differential

$$b(a_0\otimes\ldots\otimes a_n) \,:=\, \sum_{i=0}^{n-1}(-1)^i a_0\otimes\ldots\otimes a_i a_{i+1}\otimes\ldots\otimes a_n + (-1)^n a_n a_0\otimes\ldots\otimes a_{n-1}$$

which equals

$$b(\omega da) \,=\, (-1)^{|\omega|}[\omega, a]$$

Its homology $HH_*(A,A) := H_*(C_*(A))$ is called the Hochschild homology of A. There is a canonical isomorphism

$$HH_*(A,A) \,\simeq\, Tor_*^{\widetilde{A}\otimes\widetilde{A}^{op}}(A,A)$$

Associated to the Hochschild complex there is the contractible $\mathbb{Z}/2$-graded cyclic bicomplex

$$CC_*(A) \,:=\, \left(\bigoplus_{n\in\mathbb{Z}} \Omega^{*+2n}A,\, b+B\right)$$

where the Connes differential B is given by

$$B(a_0\otimes\ldots\otimes a_n) \,:=\, \sum_{j=0}^{n}(-1)^{jn}\, 1\otimes a_j\otimes\ldots\otimes a_n\otimes a_0\otimes\ldots\otimes a_{j-1}$$

The Hodge-filtration of the cyclic bicomplex is the descending filtration defined by the subcomplexes

$$Fil^k_{Hodge}\, CC_*(A) \,:=\, \left(b\Omega^k A \bigoplus \Omega^{\geq k}A, b+B\right)$$

generated by algebraic differential forms of degree at least k.
The periodic cyclic bicomplex $\widehat{CC}_*(A)$ of a complex algebra is the completion of the cyclic bicomplex $CC_*(A)$ with respect to the Hodge filtration:

$$\widehat{CC}_*(A) \,:=\, \varprojlim_{n}\, CC_*(A)/Fil^n_{Hodge}CC_*(A)$$

Its homology $HP_*(A) := H_*(\widehat{CC}_*(A))$ is called the periodic cyclic homology of A. The cohomology $HP^*(A)$ of the dual chain complex is the periodic cyclic cohomology of A. The relation between periodic cyclic and Hochschild homology, which can be computed as a derived functor, allows the explicit calculation of periodic cyclic cohomology groups.
Cuntz and Quillen [CQ1] propose an approach to periodic cyclic (co)homology which emphasizes the $\mathbb{Z}/2\mathbb{Z}$-periodicity and the stability of the theory under nilpotent extensions. Moreover they work throughout in a bivariant setting.

For a complex algebra A consider the universal linear split extension

$$0 \to IA \to TA := \bigoplus_n A^{\otimes n} \to A \to 0$$

of A [CQ]. The completion $\widehat{TA} := \varprojlim TA/IA^n$ of the tensor algebra with respect to the corresponding adic topology is the universal topologically nilpotent extension of A in the category of adically complete algebras. It is quasifree [CQ] in the sense that every topologically nilpotent extension of it possesses a multiplicative linear section.
The X-complex [CQ1] of A is the $\mathbb{Z}/2\mathbb{Z}$-graded chain complex

$$X_*(A) := \quad \longrightarrow \quad A \quad \xrightarrow{\partial_0} \quad \Omega^1 A/[\Omega^1 A, A] \quad \xrightarrow{\partial_1} \quad A \quad \longrightarrow$$

$$\partial_0(a) \,=\, da, \quad \partial_1(a^0 da^1) \,=\, [a^0, a^1]$$

The X-complex of an adically complete algebra $\widehat{A} = \varprojlim A/I^n$ is defined as the adically complete chain complex $X_*(\widehat{A}) := \varprojlim X_*(A/I^n)$.
Cuntz and Quillen introduce the bivariant periodic cyclic cohomology of a pair of algebras (A, B) as [CQ1]

$$HP_*(A, B) := Mor_{\mathfrak{Ho}}\,(X_*(\widehat{TA}),\, X_*(\widehat{TB}))$$

where $\mathfrak{Ho}$ denotes the homotopy category of adically complete $\mathbb{Z}/2\mathbb{Z}$-graded chain complexes. This functor coincides in the case $A = \mathbb{C}$ (resp. $B = \mathbb{C}$) with the periodic cyclic homology (resp. cohomology) as defined by Connes.
Bivariant periodic cyclic cohomology is a bifunctor on the category of abstract (and more generally of adically complete) complex algebras. Its fundamental properties are

- Homotopy invariance with respect to polynomial homotopies [Co1], [Go]
- Invariance under nilpotent extensions [Go]
- Morita invariance [Co1]
- Excision [CQ2]

The invariance under nilpotent extensions implies that the projection

$$\widehat{CC}_*(\widehat{TA}) := \varprojlim_n \widehat{CC}_*(TA/IA^n) \longrightarrow \widehat{CC}_*(A)$$

is a quasiisomorphism (in fact a chain homotopy equivalence). As mentioned before, the algebra $\widehat{TA}$ is quasifree [CQ], which is equivalent to the fact that it is of projective dimension at most one as bimodule over itself. It follows that the columns and therefore the total complex of $Fil^2_{Hodge}\widehat{CC}_*(\widehat{TA})$ are contractible, so that the projection

$$\widehat{CC}_*(\widehat{TA}) \longrightarrow \widehat{CC}_*(\widehat{TA})/Fil^2_{Hodge}\widehat{CC}_*(\widehat{TA}) \simeq X_*(\widehat{TA})$$

is a quasiisomorphism (in fact a chain homotopy equivalence), too. This establishes the equivalence of the different approaches to periodic cyclic cohomology. Any reasonable definition of a cyclic cohomology theory for topological algebras has to take the topologies of the underlying algebras into account. This is usually done by topologizing the cyclic complexes and by passing then to its completions. Several such theories have been proposed, in particular the following ones.

Continuous periodic cyclic cohomology [Co1]

Let A be a locally convex algebra with jointly continuous multiplication. The A-bimodule of continuous differential forms is given by

$$\Omega^n A := \widetilde{A} \otimes_\pi A^{\otimes_\pi^n} \quad \Omega A := \bigoplus_n \Omega^n A$$

The continuous Hochschild, cyclic and periodic cyclic complexes are defined similarly to the corresponding algebraic complexes by using continuous instead of algebraic differential forms. The homology HP_* of the continuous periodic cylic bicomplex $\widehat{CC}$ is called continuous periodic cyclic homology, the cohomology HP^* of the dual complex of bounded linear functionals on $\widehat{CC}$ is called the continuous periodic cyclic cohomology. It is calculated in the same way as the cyclic groups of abstract algebras with the noteworthy difference that topologically projective resolutions [Co1] have to be used for the computation of Hochschild groups.
The Cuntz-Quillen approach in the continuous case goes as follows. One considers the universal extension

$$0 \to IA \to TA := \oplus A^{\otimes_\pi n} \to A \to 0$$

of complete, locally convex algebras with bounded linear section and denotes by $\widehat{TA} := \varprojlim TA/IA^n$ its I-adic completion.
The X-complex is given in the continuous case by

$$X_*(A) := (A \oplus \Omega^1 A/\overline{[\Omega^1 A, A]}, \partial)$$

Bivariant periodic cyclic cohomology is then defined as

$$HP_*(A,B) := Mor_{\mathfrak{Ho}}(X_*(\widehat{TA}), X_*(\widehat{TB}))$$

the group of morphisms of the X-complexes in the homotopy category of complexes of complete locally convex vector spaces. As before these bivariant

groups coincide with the ones introduced by Connes if one of the variables equals $\mathbb{C}$.
The fundamental properties of bivariant continuous periodic cyclic cohomology are

- Diffeotopy invariance [Co1],
- Invariance under nilpotent extensions [Go],
- Morita invariance [Co1]
- Excision with respect to extensions with bounded linear section [Cu2]
- Existence of a Chern-character $K_* \longrightarrow HP_*$ on topological K-theory with values in continuous periodic cyclic homology [Co1]
- Existence of a Chern-Connes character for finitely summable Fredholm modules with values in continuous periodic cyclic cohomology [Co1]

Whereas continuous periodic cyclic (co)homology is rather well behaved for nuclear Fréchet algebras it has several serious drawbacks if one intends to use it as approximation to the K-functor for Banach- or C^*-algebras.

- The continuous periodic cyclic cohomology of a nuclear C^*-algebra A equals the space of bounded traces on A [Ha]. Thus for a compact Hausdorff space X the cohomology $HP^*(C(X))$ of the C^*-algebra $C(X)$ of continuous functions on X equals the space $C(X)'$ of Radon measures on X. Consequently continuous periodic cyclic cohomology is not invariant under continuous homotopies as $HP^*(C([0,1])) = C([0,1])'$ is infinite dimensional whereas $HP^*(\mathbb{C}) = \mathbb{C}$.
- The continuous periodic cyclic (co)homology of stable C^*-algebras $A \simeq A \otimes_{C^*} \mathcal{K}(\mathcal{H})$ vanishes altogether [Wo] while K-groups remain unaffected under stabilization.
- In the cases mentioned above the Chern-character with values in continuous periodic cyclic homology is obviously far from being rationally injective.

Entire cyclic cohomology [Co2]

The search for a Chern-character in K-homology for not necessarily finitely summable Fredholm modules led Connes [Co2] to the definition of entire cyclic cohomology. For a Banach algebra A let ΩA_ϵ be the completion of the space $\Omega A = \bigoplus\limits_n \widetilde{A} \otimes_\pi A^{\otimes_\pi^n}$ with respect to the family of seminorms

$$\sum_n ([\frac{n}{2}]!)^{-1} \cdot R^{-n} \cdot \parallel - \parallel_A^{\otimes_\pi (n+1)}, \ \ R > 1$$

(For later use we introduce also the spaces $\Omega A_{\epsilon,r}$, $r<1$, as the completions of ΩA with respect to the seminorms $\sum_n ([\frac{n}{2}]!)^{-1}\cdot(1+n)^m\cdot r^n\cdot \parallel - \parallel_A^{\otimes_\pi(n+1)}$ for $m\in\mathbb{N}$.) The entire cyclic bicomplex CC_*^ϵ is defined in the usual way using the space ΩA_ϵ instead of continuous differential forms. Its (co)homology is the entire cyclic (co)homology of A. The bivariant entire cyclic cohomology groups of a pair are defined as

$$HC_*^\epsilon(A,B) := Mor_{\mathfrak{Ho}}(CC_*^\epsilon(A),\, CC_*^\epsilon(B))$$

There are similar complexes $CC_*^{\epsilon,r}$, $r<1$ based on the spaces $\Omega A_{\epsilon,r}$. Their cohomology will be denoted by $HC^*_{\epsilon,r}(B) := H^*(CC_*^{\epsilon,r}(B))$.
In the Cuntz-Quillen approach entire cyclic cohomology can be described in terms of the strong universal infinitesimal deformation functor $\mathcal{T}'$ (1.21) as

$$HC_*^\epsilon(A,\,B) \,=\, Mor_{\mathfrak{Ho}}(X_*(\mathcal{T}'A),\, X_*(\mathcal{T}'B))$$

where the morphisms are taken in the homotopy category $\mathfrak{Ho}$ of ind-complexes (5.2). This explains why Connes' definition of $CC_*^\epsilon(A)$ is natural.
The basic properties of entire cyclic (co)homology are

- Diffeotopy invariance [Co2]
- Invariance under strongly topologically nilpotent extensions [Pu1]
- Morita invariance [Co]
- Excision with respect to extensions with bounded linear section [Pu2]
- Existence of a Chern character $K_*\longrightarrow HC_*^\epsilon$ on topological K-theory with values in entire cyclic homology [Co2].
- Existence of a Chern-Connes character for Θ-summable Fredholm modules with values in entire cyclic cohomology [Co2].

Entire cyclic (co)homology can be characterized as the universal functor associated to periodic cyclic (co)homology which is invariant under strong infinitesimal deformations.
Considered as a functor on Banach- resp. C^*-algebras, entire cyclic cohomology has similar drawbacks as the continuous cyclic theory. Again the cohomology of a nuclear C^*-algebra coincides with the space of continuous traces [Kh]. Thus entire cyclic cohomology cannot be homotopy invariant. Moreover it vanishes identically on nuclear stable C^*-algebras. Finally it turns out to be very difficult to calculate entire cyclic cohomology groups directly in terms of their definition.
A basic problem, which actually motivated Connes to introduce cyclic cohomology [Co1] was the search for a Chern-character on K-homology and ultimately for a bivariant Chern-Connes character on Kasparov's bivariant K-theory [Ka]. The target theory of such a character must necessarily be invariant under continuous homotopies. Therefore there cannot exist a bivariant Chern-Connes character with values in any of the cyclic theories discussed so far.

Analytic cyclic cohomology [Pu], [Me]

Let $\mathcal{T}$ be the universal infinitesimal deformation functor (1.23) on the category of nice Fréchet algebras. The bivariant analytic cyclic cohomology of a pair of such algebras is defined [Pu], section 5, as

$$HC^*_{an}(A,\,B) := Mor^*_{\mathfrak{Ho}}(X(\mathcal{T}A),\,X(\mathcal{T}B))$$

The basic properties of analytic cyclic (co)homology are

- Diffeotopy invariance
- Invariance under topologically nilpotent extensions
- Morita invariance
- Excision with respect to extensions with bounded linear section [Pu2], [Me]
- Existence of a Chern character $K_* \longrightarrow HC^{\epsilon}_*$ on topological K-theory with values in analytic cyclic homology.
- Existence of a Chern-Connes character for arbitrary Fredholm modules with values in analytic cyclic cohomology [Me].

Analytic cyclic (co)homology can be characterized by its invariance under topologically nilpotent extensions in a similar way as the entire theory is characterized by its invariance under strong infinitesimal deformations.
In [Me] Meyer develops analytic cyclic cohomology in much greater generality for bornological algebras and shows that most of its properties continue to hold in this broader context. In the case of the precompact bornology he recovers the theory introduced in [Pu].
Analytic cyclic cohomology is quite similar to the entire cyclic theory. The main difference is the existence of a Chern-Connes character for arbitrary Fredholm modules with values in analytic cyclic cohomology. Consequently it is possible to construct interesting analytic cyclic cocycles on general Banach- and C^*-algebras. Explicit calculations of cohomology groups turn out to be quite difficult, however. It remains an open problem, whether analytic cyclic cohomology is invariant under continuous homotopies [Me]. In particular, one does not know whether there exists a bivariant Chern-Connes character with values in analytic cyclic cohomology.

Asymptotic cyclic cohomology [CM], [Pu]

Asymptotic cyclic cohomology was introduced by Connes and Moscovici in [CM] and developed further in [Pu]. The novelty is the introduction of an asymptotic parameter space with one end at "infinity". An asymptotic cocycle should be thought of as a family of densely defined cyclic cocycles indexed by

the parameter space whose domain of definition grows larger and larger as the parameter tends to infinity.
Before we give the definition of the asymptotic theory we have to recall some notation from [Pu]. A DG-object (differential graded object) will be an integer-graded object equipped with a differential d of degree one satisfying $d^2 = 0$. Any morphism of DG objects is supposed to preserve gradings and to commute with the differentials. If A is a nice Fréchet algebra, then the ind-complex $X_*(\Omega\mathcal{T}A)$ is a DG-object in an obvious way. Let M be a smooth manifold and let $\mathfrak{U} = (U_\alpha)_{\alpha\in I}$ be a cover of M by relatively compact open sets. For an ind-Fréchet space $\mathcal{V} =$ “ $\underset{\overrightarrow{i\in I}}{\lim}$”$V_i$ define an ind-DG-module by

$$\mathcal{E}(\mathfrak{U}, \mathcal{V}) := Ker(\prod_\alpha \Omega_{dR}(U_\alpha) \otimes_\pi \mathcal{V} \longrightarrow \prod_{\alpha,\beta} \Omega_{dR}(U_\alpha \cap U_\beta) \otimes_\pi \mathcal{V})$$

Up to canonical isomorphism, this formal inductive limit does not depend on the choice of $\mathfrak{U}$ and will henceforth be denoted by $\mathcal{E}(M, \mathcal{V})$. If M itself is compact, then $\mathcal{E}(M, \mathcal{V})$ is isomorphic to the space $\Omega_{dR}(M) \otimes_\pi \mathcal{V}$ of differential forms on M with coefficients in $\mathcal{V}$, but for open M this is not the case.
Let $\mathcal{U}$ be a fundamental system of neighborhoods of ∞, ordered by inclusion, in the manifold $\mathbb{R}^n_+, n >> 0$, equipped with the topology of [Pu] (1.1). Put $\mathcal{E}(\mathcal{U}, \mathcal{V}) := \underset{\overrightarrow{U\in\mathcal{U}}}{Lim}\, \mathcal{E}(U, \mathcal{V})$. If $\mathcal{V}$ is a ind-DG-complex, then so is $\mathcal{E}(\mathcal{U}, \mathcal{V})$.
The asymptotic cyclic cohomology of a pair (A, B) of nice Fréchet algebras is defined as

$$HC^\alpha_*(A, B) := Mor^{DG}_{\mathfrak{H}\mathfrak{o}}(X_*(\Omega\mathcal{T}A), \mathcal{E}(\mathcal{U}, X_*(\Omega\mathcal{T}B)))$$

Asymptotic cyclic cohomology possesses the following properties.

- Continuous homotopy invariance [Pu] (6.15)
- Invariance under topologically nilpotent extensions [Pu1]
- Invariance under passage to certain smooth subalgebras [Pu] (7.1)
- Topological Morita invariance [Pu] (7.10)
- Excision with respect to extensions with bounded linear section [Pu2]
- Existence of a multiplicative bivariant Chern-Connes character $KK_* \longrightarrow HC^\alpha_*$ on bivariant K-theory with values in bivariant asymptotic cyclic cohomology. [Pu] (10.1)

So asymptotic cyclic cohomology possesses most of the properties one would like to have for a reasonable cyclic theory of topological algebras. The main drawback of the asymptotic theory lies in the fact that, just as for entire or analytic cyclic cohomology, there are no methods to calculate it directly by homological methods.

We will pass now from algebras to ind-algebras and will generalize various cyclic theories to this context. Cyclic homology theories for ind-algebras take their value in suitable homotopy categories of ind-complexes which we introduce next.

5.2 HOMOTOPY CATEGORIES OF CHAIN-COMPLEXES

We introduce some homotopy categories of chain complexes which are similar to the diffeotopy categories of algebras treated in section 2.
Let $\mathfrak{A}$ be a fixed additive category. We let $\mathfrak{C}$ be the category of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$, i.e. the category of formal inductive limits of $\mathbb{Z}/2\mathbb{Z}$-graded chain complexes over $\mathfrak{A}$.

DEFINITION 5.1. The homotopy category $\mathfrak{Ho}$ of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$ is the category with the same objects as $\mathfrak{C}$ and with homotopy classes of chain maps (of degree $k \in \mathbb{Z}/2\mathbb{Z}$) as morphisms (of degree $k \in \mathbb{Z}/2\mathbb{Z}$):

$$\begin{aligned}
&\mathrm{Mor}^k(\text{“}\varinjlim_{i\in I}\text{”}\, C_*^{(i)}, \text{“}\varinjlim_{j\in J}\text{”}\, D_*^{(j)}) \ := \\
&= H^k(\mathrm{Hom}^*_{\mathfrak{C}}(\text{“}\varinjlim_{i\in I}\text{”}\, C^{(i)},\ \text{“}\varinjlim_{j\in J}\text{”}\, D^{(j)})) \\
&= H^k(\varprojlim_{i\in I}\varinjlim_{j\in J}\mathrm{Hom}^*_{\mathfrak{A}}(C^{(i)},\ D^{(j)})),
\end{aligned}$$

$$\mathrm{Hom}^*_{\mathfrak{A}}(C_\bullet, D_\bullet) := \left(\prod_l \mathrm{Hom}_{\mathfrak{A}}(C_l, D_{l+*}),\ \partial\right)$$

with

$$\partial(\phi) := \partial_D \circ \phi \,-\, (-1)^{deg(\phi)}\phi \circ \partial_C$$

DEFINITION 5.2. Let $f\ :\ \text{“}\varinjlim_{i\in I}\text{”}\, C_*^i \ \longrightarrow\ \text{“}\varinjlim_{j\in J}\text{”}\, C'^j_*$ be a morphism of ind-complexes. Define a directed set K of triples (i, j, f_{ij}), $i \in I$, $j \in J$, $f_{ij} : C_*^i \to C'^j_*$ in the same way as in (2.2) and define the mapping cone of f as the ind-complex

$$Cone(f) := \text{“}\varinjlim_{K}\text{”}\, Cone(f_{ij})$$

where

$$Cone(f_{ij})_* := (C_*^i[1] \oplus C'^j_*,\ \partial_{C_*^i[1]} \circ \pi_1 \oplus f_{ij} \circ \pi_1 + \partial_{C'^j_*} \circ \pi_2)$$

is the cone of the individual chain map f_{ij}. There are obvious morphisms of ind-complexes

$$\text{“}\varinjlim_{j\in J}\text{”}\, C'^j_* \longrightarrow Cone(f), \quad Cone(f) \longrightarrow \text{“}\varinjlim_{i\in I}\text{”}\, C_*^i[1]$$

LEMMA 5.3. *Call a triangle $X \to Y \to Z \to X[1]$ in $\mathfrak{Ho}$ distinguished if it is isomorphic to a triangle of the form $C \xrightarrow{f} C' \to Cone(f)$. Equipped with this family of distinguished triangles, the homotopy category $\mathfrak{Ho}$ of ind-complexes becomes a triangulated category.*

For a proof see [KS], 1.4.

DEFINITION 5.4. An ind-complex "$\varinjlim\limits_{i\in I}$" C_*^i is called weakly contractible if for each $i \in I$ there exists $i' \geq i$, such that the structure map $C_*^i \to C_*^{i'}$ is nullhomotopic.

The family of weakly contractible ind-complexes defines a nullsystem in $\mathfrak{Ho}$.

DEFINITION 5.5. (DERIVED IND-CATEGORY)
The derived ind-category $\mathfrak{D}$ of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$ is the localization of the triangulated homotopy category $\mathfrak{Ho}$ of ind-complexes obtained by inverting the morphisms with weakly contractible mapping cone. It becomes a triangulated category by declaring a triangle in $\mathfrak{D}$ distinguished if it is isomorphic to the image of a distinguished triangle in $\mathfrak{Ho}$.

The isomorphism criteria (2.9) and (2.10) apply verbatim to morphisms in the derived ind-category.

5.3 CYCLIC COHOMOLOGY THEORIES OF IND-ALGEBRAS

CONTINUOUS PERIODIC CYCLIC COHOMOLOGY

The continuous periodic cyclic bicomplex defines a functor from the category of complete, locally convex algebras with jointly continuous multiplication to the category of complexes of complete, locally convex vector spaces. We still denote by $\widehat{CC}_*$ the unique extension of this functor to the corresponding ind-categories which commutes with direct limits. Thus one has

$$\widehat{CC}_*(\text{“}\varinjlim_{i\in I}\text{”}A_i) = \text{“}\varinjlim_{i\in I}\text{”}\,\widehat{CC}_*(A_i)$$

The bivariant continuous periodic cyclic cohomology of a pair

$$(\mathcal{A}, \mathcal{B}) = (\text{“}\varinjlim_{i\in I}\text{”}A_i,\ \text{“}\varinjlim_{j\in J}\text{”}B_j)$$

of ind-algebras is then defined as

$$HP_*(\mathcal{A}, \mathcal{B}) := Mor^*_{\mathfrak{Ho}}(\widehat{CC}(\mathcal{A}), \widehat{CC}(\mathcal{B}))$$

the graded group of morphisms between the cyclic complexes in the homotopy category of ind-complexes. This group can be calculated as the cohomology of the single complex

$$\varprojlim_{i\in I}\varinjlim_{j\in J} Hom^*(\widehat{CC}(A_i), \widehat{CC}(B_j))$$

The Cuntz and Quillen approach generalizes similarly to ind-algebras and yields

$$HP_*(\mathcal{A},\,\mathcal{B})\,\simeq\,Mor^*_{\mathfrak{Ho}}(X(\widehat{T\mathcal{A}}),\,X(\widehat{T\mathcal{B}}))$$

the group of morphisms between the corresponding X-complexes of I-adic completions.
The Cartan homotopy formula [Go] shows that diffeotopic morphisms of ind-algebras induce chain homotopic maps of the corresponding cyclic ind-complexes. Therefore continuous periodic cyclic cohomology is invariant under diffeotopy, i.e. it descends to a functor from the unstable diffeotopy category of ind-algebras (2.1) to the chain homotopy category of ind-complexes.

Entire, analytic, and asymptotic cyclic cohomology

In a similar way the entire (analytic) cyclic bicomplex and entire (analytic, asymptotic) cyclic (co)homology can be naturally extended to ind-algebras. The Cuntz-Quillen approach provides a description of entire (analytic) cyclic cohomology of ind-Banach algebras (nice ind-Fréchet algebras) in terms of the (strong) universal infinitesimal deformation functor $\mathcal{T}'$ (1.21) (resp. $\mathcal{T}$ (1.23)) as follows:

$$HC^{\epsilon}_*(\mathcal{A},\,\mathcal{B})\,\simeq\,Mor^*_{\mathfrak{Ho}}(X(\mathcal{T}'\mathcal{A}),\,X(\mathcal{T}'\mathcal{B}))$$

$$HC^{an}_*(\mathcal{A},\,\mathcal{B})\,\simeq\,Mor^*_{\mathfrak{Ho}}(X(\mathcal{T}\mathcal{A}),\,X(\mathcal{T}\mathcal{B}))$$

$$HC^{\alpha}_*(\mathcal{A},\mathcal{B}):=\,Mor^{DG}_{\mathfrak{Ho}}(X_*(\Omega\mathcal{T}\mathcal{A}),\,\mathcal{E}(\mathcal{U},\,X_*(\Omega\mathcal{T}\mathcal{B})))$$

This shows the invariance of the entire (analytic) cyclic theory under strongly topologically nilpotent extensions (topologically nilpotent extensions). In fact there is the following characterization of entire (analytic) cyclic theory in terms of this invariance.

Lemma 5.6. *Let $F\,:\,ind-alg\,\longrightarrow\,\mathfrak{Ho}$ be a functor which is invariant under strongly topologically nilpotent extensions (topologically nilpotent extensions) and let $\Phi\,:\,F\to\widehat{CC}_*$ be a natural transformation to periodic cyclic homology. Then F factors uniquely through entire (analytic) cyclic homology.*

Proof: The canonical map $\widehat{CC}_*(\mathcal{T}'\mathcal{A})\;\longrightarrow\;X(\mathcal{T}'\mathcal{A})$ (respectively $\widehat{CC}_*(\mathcal{T}\mathcal{A})\;\longrightarrow\;X(\mathcal{T}\mathcal{A})$) is a deformation retraction, because $\mathcal{T}'\mathcal{A}$ (respectively $\mathcal{T}\mathcal{A}$) is strongly topologically quasifree (respectively topologically quasifree) (1.25), [CQ1]. The lemma follows then from a look at the natural commutative diagram

$$\begin{array}{ccccccc} & & & & F(\mathcal{T}'\mathcal{A}) & \overset{\simeq}{\longrightarrow} & F(\mathcal{A}) \\ & & & & \Phi\downarrow & & \downarrow\Phi \\ CC^{\epsilon}_*(\mathcal{A}) & \overset{\simeq}{\longrightarrow} & X_*(\mathcal{T}'\mathcal{A}) & \overset{\simeq}{\longleftarrow} & \widehat{CC}_*(\mathcal{T}'\mathcal{A}) & \underset{\pi_*}{\longrightarrow} & \widehat{CC}_*(\mathcal{A}) \end{array}$$

respectively

$$\begin{array}{ccccccc} & & & & F(\mathcal{T}\mathcal{A}) & \overset{\simeq}{\longrightarrow} & F(\mathcal{A}) \\ & & & & \Phi \downarrow & & \downarrow \Phi \\ CC_*^{an}(\mathcal{A}) & \overset{\simeq}{\longrightarrow} & X_*(\mathcal{T}\mathcal{A}) & \overset{\simeq}{\longleftarrow} & \widehat{CC}_*(\mathcal{T}\mathcal{A}) & \underset{\pi_*}{\longrightarrow} & \widehat{CC}_*(\mathcal{A}) \end{array}$$

and the fact that the transformations of the bottom lines coincide with the chain homotopy classes of the canonical morphisms of the cyclic complexes. □

5.4 Local cyclic cohomology

We are going to modify continuous periodic cyclic (co)homology in order to obtain a cyclic theory satisfying continuous homotopy invariance, topological Morita invariance, invariance under passage to smooth subalgebras, and compatibility with topological direct limits.
The results of section four tell us that a functor on the ind-category of nice Fréchet algebras possesses the desired properties provided

- It is invariant under infinitesimal deformations (topologically nilpotent extensions).
- It is matrix stable
- It factors through the stable diffeotopy category of ind-algebras.

Among the cyclic theories presented so far, analytic cyclic cohomology is characterized by its invariance under infinitesimal deformations. Moreover it is matrix stable. In order to obtain a cyclic theory which satisfies in addition the last condition we make the

Definition 5.7. (Local cyclic cohomology)
Let $\mathcal{T}$ be the universal infinitesimal deformation functor (1.23) on the category of nice ind-Fréchet algebras and let $\mathfrak{D}$ be the derived ind-category (5.5) of the category of complete, locally convex vector spaces. The bivariant local cyclic cohomology of a pair $(\mathcal{A}, \mathcal{B})$ of nice ind-Fréchet algebras is defined as

$$HC_*^{loc}(\mathcal{A}, \mathcal{B}) := Mor_{\mathfrak{D}}^*(X(\mathcal{T}\mathcal{A}), X(\mathcal{T}\mathcal{B}))$$

the group of morphisms in the derived ind-category between the X-complexes of the universal infinitesimal deformations of the given ind-algebras. The groups

$$HC_*^{loc}(\mathcal{A}) := HC_*^{loc}(\mathbb{C}, \mathcal{A})$$

respectively

$$HC^*_{loc}(\mathcal{A}) := HC_*^{loc}(\mathcal{A}, \mathbb{C})$$

are called the local cyclic homology, respectively local cyclic cohomology of $\mathcal{A}$.

An immediate consequence of the definition is the existence of a composition product.

Proposition 5.8. (Composition products)
Bivariant local cyclic cohomology is a bifunctor on the ind-category of nice Fréchet algebras. The composition of morphisms in the derived ind-category defines a natural associative composition product

$$\circ : HC^{loc}_*(\mathcal{A}, \mathcal{B}) \otimes HC^{loc}_*(\mathcal{B}, \mathcal{C}) \longrightarrow HC^{loc}_*(\mathcal{A}, \mathcal{C})$$

With this product the bivariant local cyclic cohomology $HC^{loc}_*(\mathcal{A}, \mathcal{A})$ becomes a unital ring, and the bivariant groups $HC^{loc}_*(\mathcal{A}, \mathcal{B})$ become $HC^{loc}_*(\mathcal{A}, \mathcal{A})$-$HC^{loc}_*(\mathcal{B}, \mathcal{B})$-bimodules. A bivariant local cyclic cohomology class is called a HC^{loc}-equivalence if the corresponding morphism of complexes in the derived ind-category is an isomorphism.

By its very definition, local cyclic cohomology satisfies the conditions mentioned before. This is shown in the following two propositions.

Proposition 5.9. *Consider the continuous periodic cyclic bicomplex as a functor on the ind-category of nice Fréchet algebras with values in the derived ind-category. Then this functor factors through the stable diffeotopy category of ind-algebras.*

Proof: The Cartan homotopy formula [Go], [CQ] shows that the functor $\widehat{CC}_*$ is invariant under diffeotopy. According to Cuntz [Cu2] continuous periodic cyclic cohomology satisfies excision for extensions with bounded linear section. In particular, a homomorphism $f : A \longrightarrow B$ of Fréchet algebras induces a natural chain homotopy equivalence $\widehat{CC}_*(Conef) \longrightarrow Cone(\widehat{CC}_*(f))[1]$. Due to the naturality of its homotopy inverse this result carries over to ind-Fréchet algebras. This proves that the continuous periodic cyclic bicomplex $\widehat{CC}_*$ defines a homological functor on the prestable diffeotopy category (2.3). It remains to verify that this functor vanishes on weakly contractible ind-algebras which is evident from the definition of the derived ind-category and the Cartan homotopy formula in periodic cyclic homology. □

The proposition shows that theorem (4.3) applies to the functor $\widehat{CC}_*$. In particular, the functor $\widehat{CC}_* \circ \mathcal{T} \circ M_\infty$ with values in the derived ind-category possesses all the properties listed in theorem (4.2). It remains to identify it with local cyclic cohomology.

Lemma 5.10. *Let $\mathcal{T}$ be the universal infinitesimal deformation functor (1.23) and let M_∞ be the infinite matrix functor (4.1) on the ind-category of nice Fréchet algebras. There is an isomorphism of functors*

$$\widehat{CC}_* \circ \mathcal{T} \circ M_\infty \quad \xrightarrow{\simeq} \quad X_* \circ \mathcal{T} \circ M_\infty \quad \xrightarrow{\simeq} \quad X_* \circ \mathcal{T}$$

with values in the homotopy category of ind-complexes. Here the transformation on the right hand side is given by the contraction with the trace [Co1].

Proof: We work in the homotopy category of ind-complexes. The universal infinitesimal deformation of any ind-algebra is topologically quasifree, which implies by [CQ1] that

$$\widehat{CC}_* \circ \mathcal{T} \circ M_\infty \overset{\simeq}{\longrightarrow} X_* \circ \mathcal{T} \circ M_\infty$$

is an isomorphism of functors. By making use of excision in analytic cyclic cohomology [Pu2], it suffices to verify that the contraction with the trace $\tau_* : X_*\mathcal{T}(M_\infty\mathcal{A}) \longrightarrow X_*\mathcal{T}\mathcal{A}$ is an isomorphism for unital $\mathcal{A}$. In this case τ_* factors as

$$X_*\mathcal{T}(M_\infty\mathcal{A}) \overset{\mu}{\longrightarrow} X_*\mathcal{T}(M_\infty\mathbb{C}) \otimes_\pi X_*\mathcal{T}\mathcal{A} \overset{\tau\otimes Id}{\longrightarrow} X_*\mathcal{T}\mathcal{A}$$

where the coproduct μ is an isomorphism by [Pu3]. It remains thus to verify that $\tau_* : X_*\mathcal{T}(M_\infty\mathbb{C}) \longrightarrow X_*\mathcal{T}\mathbb{C} \simeq \mathbb{C}$ is an isomorphism. It factorizes again as $X_*\mathcal{T}(M_\infty\mathbb{C}) \longrightarrow X_*(M_\infty\mathbb{C}) \longrightarrow \mathbb{C}$. The first map is an isomorphism because $M_\infty\mathbb{C}$ is topologically quasifree [CQ], and the second map is an isomorphism by the Morita invariance of cyclic homology. The lemma is proved. □

Corollary 5.11. *The functor $X_* \circ \mathcal{T} : ind - alg \longrightarrow \mathcal{D}$ from the ind-category of nice Fréchet algebras to the derived ind-category (5.5) is invariant under infinitesimal deformations, matrix stable, and factors through the stable diffeotopy category in the sense of (4.1).*

We are ready to list the basic properties of local cyclic (co)homology.

Theorem 5.12. (Homotopy Invariance)
Bivariant local cyclic cohomology is invariant under continuous homotopies, i.e. for a nice ind-Fréchet algebra $\mathcal{A}$ any evaluation homomorphism

$$eval : C([0,1], \mathcal{A}) \longrightarrow \mathcal{A}$$

defines a HC_{loc}-equivalence

$$eval_* \in HC_*^{loc}(C([0,1], \mathcal{A}), \mathcal{A})$$

Proof: This follows from (5.11), (5.7) and (4.2). □

Theorem 5.13. (Excision) *[Pu2]*
Every extension

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow 0$$

of nice ind-Fréchet algebras, which admits a local linear section ([Pu2],(5.12)), gives rise to natural long exact sequences

$$\begin{array}{ccccc} HC_*^{loc}(-,\mathcal{I}) & \longrightarrow & HC_*^{loc}(-,\mathcal{A}) & \longrightarrow & HC_*^{loc}(-,\mathcal{B}) \\ \partial\uparrow & & & & \downarrow\partial \\ HC_{*+1}^{loc}(-,\mathcal{B}) & \longleftarrow & HC_{*+1}^{loc}(-,\mathcal{A}) & \longleftarrow & HC_{*+1}^{loc}(-,\mathcal{I}) \end{array}$$

and

$$\begin{array}{ccccc} HC_*^{loc}(\mathcal{I}, -) & \longleftarrow & HC_*^{loc}(\mathcal{A}, -) & \longleftarrow & HC_*^{loc}(\mathcal{B}, -) \\ \partial \downarrow & & & & \uparrow \partial \\ HC_{*+1}^{loc}(\mathcal{B}, -) & \longrightarrow & HC_{*+1}^{loc}(\mathcal{A}, -) & \longrightarrow & HC_{*+1}^{loc}(\mathcal{I}, -) \end{array}$$

of local cyclic cohomology groups.

This is [Pu2], (5.12).

Theorem 5.14. (Topological Morita Invariance)
Let $\mathcal{A}$ be a nice ind-Fréchet algebra and let $\mathcal{B}$ be an ind-C^*-algebra.

a) Let

$$i : \mathcal{A} \longrightarrow M_n(\mathcal{A})$$

be a homomorphism which is given by exterior multiplication with a rank one projector in $M_n(\mathbb{C})$. Then

$$i_* \in HC_0^{loc}(\mathcal{A}, M_n(\mathcal{A}))$$

is a HC_{loc}-equivalence.

b) Let $\mathcal{J}$ be a separable, symmetrically normed operator ideal and let

$$i' : \mathcal{A} \longrightarrow \mathcal{A} \otimes_\pi \mathcal{J}$$

be a homomorphism which is given by exterior multiplication with a rank one projector in $\mathcal{J}$. Then

$$i'_* \in HC_0^{loc}(\mathcal{A}, \mathcal{A} \otimes_\pi \mathcal{J})$$

is a HC_{loc}-equivalence.

c) Let $\mathcal{K}$ be the C^*-algebra of compact operators and let

$$i'' : \mathcal{B} \longrightarrow \mathcal{B} \otimes_{C^*} \mathcal{K}$$

be a homomorphism which is given by exterior multiplication with a rank one projector in $\mathcal{K}$. Then

$$i''_* \in HC_0^{loc}(\mathcal{B}, \mathcal{B} \otimes_{C^*} \mathcal{K})$$

is a HC_{loc}-equivalence.

Proof: This follows from (5.11), (5.7) and (4.2). □

Theorem 5.15. (Invariance under passage to smooth subalgebras)
Let $\mathfrak{A}$ be a smooth subalgebra (3.4) of a nice Fréchet algebra A. If the inclusion

$$i : \mathfrak{A} \hookrightarrow A$$

satisfies the conditions of (3.8) then

$$i_* \in HC_0^{loc}(\mathfrak{A}, A)$$

is a HC^{loc}-equivalence. This is in particular the case if A possesses the Grothendieck approximation property. A similar assertion holds for the inclusion of I-diagrams of smooth subalgebras satisfying the conditions of (3.8).

Proof: This follows from (5.7), (5.11), and (4.2). □
In particular the algebra inclusions mentioned in (3.10) up to (3.14) induce HC^{loc}-equivalences.
According to [Pu3], there exists a natural exterior product on bivariant periodic, entire, analytic, and asymptotic cyclic cohomology. There is also a corresponding product in local cyclic cohomology.

Theorem 5.16. (Exterior products)
There exists a natural and associative exterior product

$$\times : \quad HC_*^{loc}(\mathcal{A}, \mathcal{C}) \otimes HC_*^{loc}(\mathcal{B}, \mathcal{D}) \quad \longrightarrow \quad HC_*^{loc}(\mathcal{A} \otimes_\pi \mathcal{B}, \mathcal{C} \otimes_\pi \mathcal{D})$$

on bivariant local cyclic cohomology of unital ind-algebras. The exterior product is compatible with the composition product in the sense that local cyclic cohomology classes α, β, α', β' satisfy

$$(\alpha \circ \beta) \times (\alpha' \circ \beta') \quad = \quad (\alpha \times \alpha') \circ (\beta \times \beta')$$

whenever these expressions are defined.

Proof: According to [Pu3] there exist natural continuous chain maps

$$\mu : \quad X\mathcal{T}(\mathcal{A} \otimes_\pi \mathcal{B}) \quad \longrightarrow \quad X(\mathcal{T}\mathcal{A}) \otimes_\pi X(\mathcal{T}\mathcal{B})$$

and

$$\nu : \quad X(\mathcal{T}\mathcal{A}) \otimes_\pi X(\mathcal{T}\mathcal{B}) \quad \longrightarrow \quad X\mathcal{T}(\mathcal{A} \otimes_\pi \mathcal{B})$$

which are naturally chain homotopy inverse to each other. On the level of chain maps of ind-complexes the exterior product is defined as

$$\alpha \times \beta := \nu_{\mathcal{A}', \mathcal{B}'} \circ (\alpha \otimes_\pi \beta) \circ \mu_{\mathcal{A}, \mathcal{B}}$$

The induced map on homology gives rise to the exterior product on analytic cyclic (co)homology. Its compatibility with the composition product follows immediately from the fact that the chain maps μ and ν are chain homotopy inverse to each other. Let $\varphi : X(\mathcal{T}\mathcal{A}) \longrightarrow X(\mathcal{T}\mathcal{B})$ be a chain map with weakly contractible mapping cone and let $\mathcal{C}$ be a nice unital ind-Fréchet algebra. The mapping cone of $\varphi \times Id_{\mathcal{C}}: X(\mathcal{T}(\mathcal{A} \otimes_\pi \mathcal{C})) \longrightarrow X(\mathcal{T}(\mathcal{B} \otimes_\pi \mathcal{C}))$ is then chain homotopy equivalent to the weakly contractible ind-complex $Cone(\varphi) \otimes_\pi X(\mathcal{T}\mathcal{C})$ and therefore weakly contractible itself. It follows that the transformation $\times$ descends to an exterior product on local cyclic (co)homology. □

Proposition 5.17. (Chern character) *[Co1]*
The Chern character map of [Co1], [CQ1], defines a natural transformation

$$ch: \quad K_* \quad \longrightarrow \quad HC_*^{loc}$$

from topological K-theory to local cyclic homology.

Proof: This follows from [Co2] and [CQ1]. □

Theorem 5.18. (Bivariant Chern-Connes character) *[Pu2]*

a) *There exists a natural transformation of bifunctors on the category of separable C^*-algebras*

$$ch_{biv}: \quad KK^*(-,-) \quad \longrightarrow \quad HC_*^{loc}(-,-)$$

from Kasparov's bivariant KK-theory to bivariant local cyclic cohomology called the bivariant Chern-Connes character.

b) *It is uniquely characterized by the following two properties:*

   – *If $f: A \to B$ is a homomorphism of C^*-algebras with associated class $[f] \in KK^0(A,B)$, then*

$$ch_{biv}([f]) = f_*$$

   – *If $\epsilon: 0 \to I \to A \to B \to 0$ is an extension of C^*-algebras with completely positive section and associated class $[\epsilon] \in KK^1(B,I)$, and if $[\delta] \in HP^1(B,I)$ denotes the boundary map in local cyclic homology then*

$$ch_{biv}([\epsilon]) = [\delta]$$

c) *The bivariant Chern-Connes character is multiplicative up to a period factor $2\pi i$.*
For any separable C^*-algebras A, B, C the diagram

$$\begin{array}{ccc}
KK^j(A,B) \otimes KK^l(B,C) & \xrightarrow{\circ} & KK^{j+l}(A,C) \\
\downarrow ch_{biv} \otimes ch_{biv} & & \downarrow ch_{biv} \\
HC_{lc}^j(A,B) \otimes HC_{lc}^l(B,C) & \xrightarrow[\frac{1}{(2\pi i)^{jl}} \circ]{} & HC_{lc}^{j+l}(A,C)
\end{array}$$

commutes, where the upper horizontal map is the Kasparov product and the lower horizontal map is given by $\frac{1}{(2\pi i)^{jl}}$ *times the composition product. (See [Pu] for an explanation of the factor* $2\pi i$*).*

d) (Grothendieck-Riemann-Roch Theorem)
Let

$$ch' := \frac{1}{(2\pi i)^j}\, ch : K_j \longrightarrow HC_j^{loc}$$

be the normalized Chern character on K*-theory, and let* $\alpha \in KK^l(A,B)$*. Then the diagram*

$$\begin{array}{ccc} K_j(A) & \stackrel{-\otimes\alpha}{\longrightarrow} & K_{j+l}(B) \\ \downarrow ch' & & \downarrow ch' \\ HC_j^{loc}(A) & \underset{-\circ \frac{1}{(2\pi i)^{jl}}\, ch_{biv}(\alpha)}{\longrightarrow} & HC_{j+l}^{loc}(B) \end{array}$$

commutes.

e) Let $0 \to I \to A \to B \to 0$ *be an extension of separable* C^**-algebras with completely positive section. Then the bivariant Chern-Connes character is compatible with long exact sequences, i.e. the diagrams*

$$\begin{array}{ccccccc} \longrightarrow & KK^j(-,B) & \stackrel{\delta}{\longrightarrow} & KK^{j-1}(-,I) & \longrightarrow \\ & \downarrow ch_{biv} & & \downarrow ch_{biv} & \\ \longrightarrow & HC^j_{loc}(-,B) & \underset{(2\pi i)^j\delta}{\longrightarrow} & HC^{j-1}_{loc}(-,I) & \longrightarrow \end{array}$$

and

$$\begin{array}{ccccccc} \longleftarrow & KK^{j+1}(B,-) & \stackrel{\delta}{\longleftarrow} & KK^j(I,-) & \longleftarrow \\ & \downarrow ch_{biv} & & \downarrow ch_{biv} & \\ \longleftarrow & HC^{j+1}_{loc}(B,-) & \underset{(2\pi i)^j\delta}{\longleftarrow} & HC^j_{loc}(I,-) & \longleftarrow \end{array}$$

commute.

Proof: This is [Pu2], (6.3). □

6 Calculation of local cyclic cohomology groups

The other issue that distinguishes local cyclic cohomology from most cyclic theories is its computability in terms of homological algebra. Striking examples of such calculations are given in [Pu4] and [Pu5]. No computational tools similar to the ones presented here are available for entire, analytic, or asymptotic cyclic cohomology. Its nice functorial properties and its computability by homological methods make local cyclic cohomology a rather accessible invariant for a large class of algebras.

6.1 CALCULATION OF MORPHISM GROUPS IN THE DERIVED IND-CATEGORY

The aim of this section is the construction of a natural spectral sequence which calculates morphism groups in the derived ind-category. The strategy for obtaining such a spectral sequence is well known [Bo], and we just have to adapt it to the setting of this paper. The idea behind it is subsumed in

DEFINITION AND LEMMA 6.1. *Let $\mathfrak{C}$ be a triangulated category, let $\mathfrak{N}$ be a nullsystem in $\mathfrak{C}$ and denote by $\mathfrak{C}/\mathfrak{N}$ the corresponding quotient triangulated category.*

a) An object $X \in ob\ \mathfrak{C}$ is called $\mathfrak{N}$-colocal if $Mor_{\mathfrak{C}}(X, N) = 0,\ \forall N \in \mathfrak{N}$.

b) Suppose that X is $\mathfrak{N}$-colocal. Then the canonical map

$$Mor_{\mathfrak{C}}(X, Y) \xrightarrow{\simeq} Mor_{\mathfrak{C}/\mathfrak{N}}(X, Y)$$

is an isomorphism for all $Y \in ob\ \mathfrak{C} = ob\ \mathfrak{C}/\mathfrak{N}$.

c) Let $X \in ob\ \mathfrak{C}$ and suppose that there exists a morphism $f : P(X) \to X$ from an $\mathfrak{N}$-colocal object $P(X)$ to X such that $Cone(f) \in \mathfrak{N}$. Then

$$(f^*) : Mor_{\mathfrak{C}/\mathfrak{N}}(X, Y) \xrightarrow{\simeq} Mor_{\mathfrak{C}}(P(X), Y)$$

is an isomorphism for all $Y \in ob\ \mathfrak{C}$.

d) If the morphisms described in c) exist for every $X \in ob\ \mathfrak{C} = ob\ \mathfrak{C}/\mathfrak{N}$, then

$$P : \mathfrak{C}/\mathfrak{N} \longrightarrow \mathfrak{C},\ \ X \longrightarrow P(X)$$

becomes a functor which is left adjoint to the forgetful functor $\mathfrak{C} \longrightarrow \mathfrak{C}/\mathfrak{N}$.

In the sequel this lemma will be applied to the nullsystem of weakly contractible ind-complexes in the triangulated homotopy category of ind-complexes over an additive category.

EXAMPLE 6.2. *Let $\mathfrak{A}$ be a fixed additive category and let $\mathfrak{C}$ be the category of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$. The associated homotopy category is denoted by $\mathfrak{Ho}$ and its derived ind-category by $\mathfrak{D}$. Let finally $\mathfrak{N}$ be the nullsystem of weakly contractible ind-complexes in $\mathfrak{Ho}$. Then*

- *Every constant ind-complex (i.e. every ordinary chain complex over $\mathfrak{A}$) is $\mathfrak{N}$-colocal.*

- *The direct limit (in $\mathfrak{C}$) of $\mathfrak{N}$-colocal ind-complexes is not necessarily $\mathfrak{N}$-colocal.*

In fact, if limits of colocal ind-complexes were colocal, then all ind-complexes would be colocal as they are limits of constant ind-complexes. In particular, weakly contractible ind-complexes would be genuinely contractible, which is not the case.
We will construct now a canonical colocal model for the direct limit of a family of colocal ind-complexes. It will serve for calculations in the derived ind-category.
The notations of the previous example will be used throughout this section.

Definition 6.3. Let $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ be a directed family of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes

$$\mathbf{C}_i := \text{“}\varinjlim_{J_i}\text{”}\, C_*^{j_i}$$

over an additive category $\mathfrak{A}$. (In other words $\mathcal{C}$ is an ind-object over $\mathfrak{C}$.)

Let $\mathcal{F}$ be the set of triples (I', φ, f) such that

1) I' is a finite directed subset of I.
2) $\varphi : I' \to \coprod_{i\in I'} J_i$ is a map such that $\varphi(i) \in J_i$ for all $i \in I'$.
3) f is a collection of morphisms $f_{ii'} : C_*^{\varphi(i)} \longrightarrow C_*^{\varphi(i')}$, $i < i' \in I'$ representing the structure maps $\mathbf{C}_i \to \mathbf{C}_{i'}$ and such that $f_{i'i''} \circ f_{ii'} = f_{ii''}$ for $i < i' < i'' \in I'$.

The set $\mathcal{F}$ is partially ordered by putting $(I', \varphi, f) \leq (I'', \varphi', f')$ iff

1) $I' \subset I''$
2) $\varphi(i) \leq \varphi'(i)$ for all $i \in I'$
3) For all $i < i' \in I'$ the diagram $\begin{array}{ccc} C_*^{\varphi'(i)} & \xrightarrow{f'_{ii'}} & C_*^{\varphi'(i')} \\ \uparrow & & \uparrow \\ C_*^{\varphi(i)} & \xrightarrow{f_{ii'}} & C_{\varphi(i')} \end{array}$ commutes, where the vertical arrows are given by the structure maps of the ind-objects $\mathbf{C}_i$ and $\mathbf{C}_{i'}$, respectively. With this order $\mathcal{F}$ becomes in fact a directed set.

For $(I', \varphi, f) \in \mathcal{F}$ define a bicomplex $P_{**}^{(I',\varphi,f)}$ with underlying bigraded object

$$P_{pq}^{(I',\varphi,f)} := \bigoplus_{\substack{i_0>\ldots>i_p \\ i_0,\ldots,i_p\in I'}} C_q^{\varphi(i_p)}, \;\; p \in \mathbb{N},\, q \in \mathbb{Z}/2\mathbb{Z},$$

and with differentials ∂', ∂'' given as follows:

$$\partial' := \sum_{k=0}^{p} (-1)^k\, \partial_k : P_{pq}^{(I',\varphi,f)} \longrightarrow P_{(p-1)q}^{(I',\varphi,f)}$$

where the face maps ∂_k, $0 \leq k \leq p$, act on the indices $(i_0, \ldots, i_p)$ by deleting the k-th element of the string and on the corresponding objects by the identity if $k < p$ respectively by the morphism
$f_{i_p i_{p-1}} : C_q{}^{\varphi(i_p)} \to C_q{}^{\varphi(i_{p-1})}$ if $k = p$.

The second differential ∂'' is given by

$$\partial'' : P_{pq}^{(I',\varphi,f)} \longrightarrow P_{p(q-1)}^{(I',\varphi,f)}$$

$$\partial'' := (-1)^p \cdot d_q$$

where $d_q : C_q{}^{\varphi(i_p)} \to C_{q-1}{}^{\varphi(i_p)}$ is the differential in the complex $C_*^{\varphi(i_p)}$.

The total $\mathbb{Z}/2\mathbb{Z}$-graded chain complexes

$$P_*^{(I',\varphi,f)} := \left(\bigoplus_p P_{p,*-p}^{(I',\varphi,f)},\, d = \partial' + \partial'' \right)$$

form an ind-complex

$$\mathbf{P}(\mathcal{C}) := \text{“} \varinjlim_{\mathcal{F}} \text{”}\, P_*^{(I',\varphi,f)}$$

It is called the canonical resolution of $\mathcal{C}$.

LEMMA 6.4. *Let I be a directed set. The canonical resolution (6.3) defines a functor*

$$\mathbf{P} : \mathfrak{C}^I \longrightarrow \mathfrak{C}$$

from the category of I-diagrams over $\mathfrak{C}$ to $\mathfrak{C}$.

The canonical resolution provides a model for the direct limit of the family $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ in the following sense:

LEMMA 6.5. *Let $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ be a directed system of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$, let $\underset{\overrightarrow{i\in I}}{Lim}\, \mathbf{C}_i$ be its direct limit in the category of ind-complexes, and let $\mathbf{P}(\mathcal{C})$ be its canonical resolution. There exists a canonical morphism*

$$\mathbf{P}(\mathcal{C}) \longrightarrow \underset{\overrightarrow{i\in I}}{Lim}\, \mathbf{C}_i$$

of ind-complexes with weakly contractible mapping cone. If $\mathcal{C} \longrightarrow \mathcal{C}'$ is a morphism of I-diagrams of ind-complexes, then the corresponding diagram

$$\begin{array}{ccc} \mathbf{P}(\mathcal{C}) & \longrightarrow & \underset{\overrightarrow{i\in I}}{Lim}\, \mathbf{C}_i \\ \downarrow & & \downarrow \\ \mathbf{P}(\mathcal{C}') & \longrightarrow & \underset{\overrightarrow{i\in I}}{Lim}\, \mathbf{C}'_i \end{array}$$

commutes.

Proof: Consider the ind-complex

$$“\lim_{\overrightarrow{\mathcal{F}}}”\, E_*^{(I',\varphi,f)},\ E_*^{(I',\varphi,f)} := C_*^{\varphi(i')}$$

where i' is the largest element of the finite directed set I' (the transition morphism $E_*^{(I',\varphi,f)} \longrightarrow E_*^{(I'',\varphi',f')}$ equals $f'_{i'i''}$). An easy verification shows that this ind-complex is a direct limit of the family $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$:

$$“\lim_{\overrightarrow{\mathcal{F}}}”\, E_*^{(I',\varphi,f)} \simeq \underset{\overrightarrow{I}}{Lim}\, \mathbf{C}_i$$

Let

$$\pi : \mathbf{P}(\mathcal{C}) = “\lim_{\overrightarrow{\mathcal{F}}}”\, P_*^{(I',\varphi,f)} \longrightarrow “\lim_{\overrightarrow{\mathcal{F}}}”\, E_*^{(I',\varphi,f)}$$

be the morphism of ind-complexes which is given on the level of the individual complexes P_*^α, $\alpha = (I',\varphi,f) \in \mathcal{F}$, as follows:

$$\pi_\alpha : \bigoplus_p \bigoplus_{\substack{i_0>\ldots>i_p\\ i_0,\cdots,i_p\in I'}} C_*^{\varphi(i_p)} \longrightarrow C_*^{\varphi(i')}$$

equals zero on direct summands corresponding to strings $i_0 > \ldots > i_p$ with $p > 0$ and is otherwise given by $f_{i_0 i'} : C_*^{\varphi(i_0)} \longrightarrow C_*^{\varphi(i')}$.
The cone of this morphism equals $Cone\,\pi \simeq “\lim\limits_{\overrightarrow{\mathcal{F}}}”\, Cone\,\pi_\alpha$. We are going to show that $Cone\,\pi_\alpha$ is contractible for each $\alpha \in \mathcal{F}$ which implies that $Cone\,\pi$ is weakly contractible (it will not be genuinely contractible in general).
Let $s_\alpha : E_*^\alpha \longrightarrow P_*^\alpha$ be the morphism of $\mathbb{Z}/2\mathbb{Z}$-graded objects of $\mathfrak{A}$ which identifies $E_*^\alpha = C_*^{\varphi(i')}$ with the summand of P_*^α corresponding to the string $i_0 = i'$. Let furthermore $\chi_\alpha : P_*^\alpha \longrightarrow P_{*+1}^\alpha$ be the operator which vanishes on direct summands corresponding to strings $i_0 > \ldots > i_p$ with $i_0 = i'$ and identifies otherwise the direct summand corresponding to $i_0 > \ldots > i_p$ with the direct summand corresponding to $i' > i_0 > \ldots > i_p$. The morphism

$$h_\alpha : (Cone\,\pi_\alpha)_* = P_*^\alpha[1] \oplus E_*^\alpha \longrightarrow P_{*+1}^\alpha[1] \oplus E_{*+1}^\alpha = (Cone\,\pi_\alpha)_{*+1}$$

$$h_\alpha := \begin{pmatrix} -\chi_\alpha \circ (Id - s_\alpha \circ \pi_\alpha) & -\chi_\alpha \circ (\partial \circ s_\alpha - s_\alpha \circ \partial) + s_\alpha \\ 0 & 0 \end{pmatrix}$$

defines then a contracting homotopy of $Cone\,\pi_\alpha$. The naturality of the construction with respect to morphisms of I-diagrams is obvious. □

Lemma 6.6. *Let $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ be a directed family of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$ with canonical resolution $\mathbf{P}(\mathcal{C})$ and let $\mathbf{C}'$ be some other $\mathbb{Z}/2\mathbb{Z}$-graded ind-complex over $\mathfrak{A}$. Let $Q_*(\mathcal{C},\mathbf{C}') := \prod\limits_p Q_{p,*-p}(\mathcal{C},\mathbf{C}')$ be the*

$\mathbb{Z}/2\mathbb{Z}$-graded total complex associated to the bicomplex of abelian groups

$$Q_{pq}(\mathcal{C},\mathbf{C}') := \prod_{\substack{i_0>\ldots>i_p\\ i_0,\cdots,i_p\in I}} Hom^q_{ind\text{-}\mathfrak{A}}(\mathbf{C}_{i_p},\mathbf{C}'),\ \ p\in\mathbb{N},\, q\in\mathbb{Z}/2\mathbb{Z},$$

where $Hom^q_{ind\text{-}\mathfrak{A}}(\mathbf{C}_{i_p},\mathbf{C}')$ denotes the morphisms of degree $q\in\mathbb{Z}/2\mathbb{Z}$ of the graded ind-objects over $\mathfrak{A}$ underlying the ind-complexes $\mathbf{C}_{i_p}$ and $\mathbf{C}'$. The differentials are given on the one hand by the simplicial differential $\partial' := \sum(-1)^k\partial_k$, deleting the appropriate index from the indexing strings and acting in the straightforward manner on the corresponding direct factor, and on the other hand by the differential $\partial''(\Phi) := \Phi\circ\partial_{\mathbf{C}_{i_p}} - (-1)^{|\Phi|}\partial_{\mathbf{C}'}\circ\Phi$ of the Hom-complex $Hom^*_{ind\text{-}\mathfrak{A}}(\mathbf{C}_{i_p},\mathbf{C}')$. Then there is a natural isomorphism

$$Mor^*_{\mathfrak{H}\mathfrak{o}}(\mathbf{P}(\mathcal{C}),\mathbf{C}') \,\simeq\, H_*(Q_\bullet(\mathcal{C},\mathbf{C}'))$$

i.e. the graded group of morphisms from $\mathbf{P}(\mathcal{C})$ to $\mathbf{C}'$ in the homotopy category of ind-complexes is given by the homology of $Q_*(\mathcal{C},\mathbf{C}')$.

Proof: The graded group of morphisms between two objects $\mathbf{C}$, $\mathbf{C}'$ of the homotopy category $\mathfrak{H}\mathfrak{o}$ of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$ can be calculated as the homology of the Hom-complex $Hom^*_{\text{ind-}\mathfrak{A}}(\mathbf{C},\,\mathbf{C}')$. Therefore one finds

$$Mor^n_{\mathfrak{H}\mathfrak{o}}\,(\mathbf{P}(\mathcal{C}),\mathbf{C}')$$

$$= H^n(\,Hom^*_{\text{ind-}\mathfrak{A}}(\mathbf{P}(\mathcal{C}),\mathbf{C}')\,)$$

$$= H^n(\varprojlim_{\mathcal{F}}\varinjlim_{J} Hom^*_{\mathfrak{A}}\,(\,C^{(I',\varphi,f)},\,C'^{\,j}\,)\,)$$

$$= H^n(\varprojlim_{\mathcal{F}}\varinjlim_{J} Hom^*_{\mathfrak{A}}\,(\bigoplus_p \bigoplus_{\substack{i_0>\ldots>i_p\\ i_0,\cdots,i_p\in I'}} C^{\varphi(i_p)}\,[-p],\,C'^{\,j}\,))$$

$$= H^n(\varprojlim_{\mathcal{F}}\varinjlim_{J} \prod_p \prod_{\substack{i_0>\ldots>i_p\\ i_0,\cdots,i_p\in I'}} Hom^*_{\mathfrak{A}}\,(\,C^{\varphi(i_p)}\,[-p],\,C'^{\,j}\,)\,)$$

$$= H^n(\varprojlim_{\mathcal{F}} \prod_p \prod_{\substack{i_0>\ldots>i_p\\ i_0,\cdots,i_p\in I'}} \varinjlim_{J}\, Hom^*_{\mathfrak{A}}\,(\,C^{\varphi(i_p)}\,[-p],\,C'^{\,j}\,))$$

because direct limits and finite products commute

$$= H^n(\prod_p \prod_{\substack{i_0>\ldots>i_p\\ i_0,\cdots,i_p\in I}} \varprojlim_{J_{i_p}}\varinjlim_{J}\, Hom^*_{\mathfrak{A}}\,(\,C^{(j_{i_p})}\,[-p],\,C'^{\,j}\,)\,)$$

$$= H^n(\,\prod_p \prod_{\substack{i_0>\ldots>i_p\\ i_0,\cdots,i_p\in I}} Hom^*_{\text{ind-}\mathfrak{A}}\,(\,\mathbf{C}_{i_p}\,[-p],\,\mathbf{C}'\,))$$

$$= H^n(\prod_p Q_{p,*-p}(\mathcal{C}, \mathbf{C}'))$$

$$= H^n(Q_*(\mathcal{C}, \mathbf{C}'))$$

□

The following result justifies the introduction of the canonical resolution.

Proposition 6.7. *Let $\mathfrak{N}$ be the nullsystem of weakly contractible ind-complexes in the homotopy category $\mathfrak{Ho}$ of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$. If $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ is a directed family of $\mathfrak{N}$-colocal ind-complexes, then its canonical resolution $\mathbf{P}(\mathcal{C})$ is $\mathfrak{N}$-colocal as well.*

Proof: Let $\mathbf{C}'$ be a weakly contractible $\mathbb{Z}/2\mathbb{Z}$-graded ind-complex. According to lemma (6.6)

$$Mor^*_{\mathfrak{Ho}}(\mathbf{P}(\mathcal{C}), \mathbf{C}') \simeq H_*(Q_\bullet(\mathcal{C}, \mathbf{C}'))$$

The weak contractibility of $\mathbf{C}'$ implies that the columns of the bicomplex $Q_{**}(\mathcal{C}, \mathbf{C}')$ are acyclic. In fact their homology equals

$$\prod_{\substack{i_0>\ldots>i_p \\ i_0,\ldots,i_p\in I}} Mor^q_{\mathfrak{Ho}}(\mathbf{C}_{i_p}, \mathbf{C}') = 0$$

The total complex $Q_*(\mathcal{C}, \mathbf{C}')$ is then acyclic as well which proves the claim. □

The following theorem provides the basis for most calculations in the derived ind-category.

Theorem 6.8. *Let $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ be a directed family of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$. Suppose that the ind-complexes $(\mathbf{C}_i)_{i\in I}$ are colocal with respect to the nullsystem of weakly contractible ind-complexes and let $\mathbf{C}'$ be some ind-complex.*

a) There exists a spectral sequence (E_r^{pq}, d_r) with E_2-term

$$E_2^{pq} = R^p \varprojlim_{i\in I} Mor^q_{\mathfrak{Ho}}(\mathbf{C}_i, \mathbf{C}')$$

which is natural in $\mathcal{C} \in \mathfrak{C}^I$ and $\mathbf{C}'_* \in \mathfrak{Ho}$. Here $R^p \varprojlim_{i\in I}$ denotes the p-th right derived functor of the inverse limit functor $\varprojlim_{i\in I}$.

b) Suppose that the higher derived limits $R^p \varprojlim_{i\in I}$ vanish for $p >> 0$. Then the spectral sequence converges to

$$E_\infty^{pq} = Gr^p\, Mor^{p+q}_{\mathfrak{D}}(\underset{i\in I}{\overrightarrow{Lim}}\, \mathbf{C}_i, \mathbf{C}')$$

c) Suppose that the directed set I is countable. Then the spectral sequence collapses and gives rise to a natural short exact sequence

$$
\begin{array}{ccccccc}
0 & \to & \varprojlim\limits_{i\in I}{}^{1} Mor^{n-1}_{\mathfrak{Ho}}(\mathbf{C}_i, \mathbf{C}') & \to & Mor^n_{\mathfrak{D}}(\underset{i\in I}{\underrightarrow{Lim}}\,\mathbf{C}_i, \mathbf{C}') & \to \\
 & \to & \varprojlim\limits_{i\in I} Mor^{n}_{\mathfrak{Ho}}(\mathbf{C}_i, \mathbf{C}') & \to & 0 &
\end{array}
$$

PROOF: Consider the chain complex $Q_*(\mathcal{C}, \mathbf{C}')$ introduced in (6.6). We calculate its homology in two different ways. By lemma (6.6)

$$H_*(Q_\bullet(\mathcal{C}, \mathbf{C}')) \simeq Mor^*_{\mathfrak{Ho}}(\mathbf{P}(\mathcal{C}), \mathbf{C}')$$

As the ind-complexes $\mathbf{C}_i$, $i \in I$, are $\mathcal{N}$-colocal by assumption, the ind-complex $\mathbf{P}(\mathcal{C})$ itself is $\mathcal{N}$-colocal by proposition (6.7). Therefore lemma (6.1) applies and shows that the canonical map

$$Mor^*_{\mathfrak{Ho}}(\mathbf{P}(\mathcal{C}), \mathbf{C}') \xrightarrow{\simeq} Mor^*_{\mathfrak{D}}(\mathbf{P}(\mathcal{C}), \mathbf{C}')$$

is an isomorphism. By lemma (6.5) the canonical morphism

$$\pi : \mathbf{P}(\mathcal{C}) \longrightarrow \underset{i\in I}{\underrightarrow{Lim}}\,\mathbf{C}_i$$

defines an isomorphism in the derived ind-category $\mathfrak{D}$ so that one obtains

$$Mor^*_{\mathfrak{D}}(\mathbf{P}(\mathcal{C}), \mathbf{C}') \xleftarrow{\simeq} Mor^*_{\mathfrak{D}}(\underset{i\in I}{\underrightarrow{Lim}}\,\mathbf{C}_i, \mathbf{C}')$$

This shows finally that

$$H_*(Q_\bullet(\mathcal{C}, \mathbf{C}')) \simeq Mor^*_{\mathfrak{D}}(\underset{i\in I}{\underrightarrow{Lim}}\,\mathbf{C}_i, \mathbf{C}')$$

We now exhibit a natural filtration of the complex $Q_*(\mathcal{C}, \mathbf{C}')$ and calculate its homology by the associated spectral sequence.
The bicomplex $Q_{**}(\mathcal{C}, \mathbf{C}')$ possesses a natural descending filtration with associated graded modules given by the columns Q_{p*}, $p \geq 0$. We take (E_r^{pq}, d_r) to be the spectral sequence associated to the corresponding filtration of the total complex $Q_*(\mathcal{C}, \mathbf{C}')$. For the E_1-term one obtains

$$
\begin{aligned}
E_1^{pq} &= H^q(Q_{p*}(\mathcal{C}, \mathbf{C}'), \partial'') \\
&= H^q\Big(\prod_{\substack{i_0>\ldots>i_p \\ i_0,\ldots,i_p\in I}} Hom^*_{\text{ind-}\mathfrak{A}}(\mathbf{C}_{i_p}, \mathbf{C}')\Big) \\
&= \prod_{\substack{i_0>\ldots>i_p \\ i_0,\ldots,i_p\in I}} Mor^q_{\mathfrak{Ho}}(\mathbf{C}_{i_p}, \mathbf{C}')
\end{aligned}
$$

For the E_2-term one finds

$$E_2^{pq} \;=\; H^p\,(\prod_{\substack{i_0>\ldots>i_*\\ i_0,\cdots,i_*\in I}} Mor^q_{\mathfrak{Ho}}\,(\,\mathbf{C}_{i_*},\,\mathbf{C}'\,),\,\partial'\,)$$

This latter complex equals the standard complex calculating the higher inverse limits of the system

$$Mor^q_{\mathfrak{Ho}}\,(\,\mathbf{C}_i,\,\mathbf{C}'\,),\; i\in I$$

so that one obtains finally

$$E_2^{pq} \;=\; R^p \varprojlim_{i\in I} Mor^q_{\mathfrak{Ho}}\,(\,\mathbf{C}_i,\,\mathbf{C}'\,)$$

b) The vanishing of the higher inverse limits $R^p \varprojlim_{i\in I}$ for $p >> 0$ implies that the projective system $H_*(Q\,(\mathcal{C},\mathbf{C}'\,)/Fil^k Q\,(\mathcal{C},\mathbf{C}'\,))$, $k\in\mathbb{N}$, satisfies the Mittag-Leffler condition. In particular

$$H_*(Q\,(\mathcal{C},\mathbf{C}'\,))\;\overset{\simeq}{\longrightarrow}\;\varprojlim_k H_*(Q\,(\mathcal{C},\mathbf{C}'\,)/Fil^k\,(\mathcal{C},\mathbf{C}'\,)\,)$$

i.e. the spectral sequence converges.
c) Is an immediate consequence of a) and b) and the fact that for countable I the higher inverse limits $R^p \varprojlim_{i\in I}$ vanish in degree $p>1$. □

In all applications we will deal exclusively with countable ind-complexes and therefore will only make use of part c) of the theorem.
REMARK: In [Pu1] I erroneously claimed that the spectral sequence above converges in general. In fact there is no reason why that should be the case. I thank Ralf Meyer for pointing this out to me. However, in all situations where the spectral sequence can be calculated, the condition of b) is automatically satisfied so that no convergence problem arises.
The following consequences of the previous theorem will be particularly useful.

THEOREM 6.9. *Let* $\mathbf{C} = \text{“}\varinjlim_{i\in I}\text{”} C_i$, $\mathbf{C}' = \text{“}\varinjlim_{j\in J}\text{”} C'_j$ *be* $\mathbb{Z}/2\mathbb{Z}$*-graded ind-complexes over* $\mathfrak{A}$*. Suppose that* I *is countable. Then there exists a short exact sequence*

$$\begin{array}{ccccccc} 0 & \to & \varprojlim^1_{i\in I}\varinjlim_{j\in J} Mor^{n-1}_{\mathfrak{Ho}}\,(C_i,\,C'_j\,) & \to & Mor^n_{\mathfrak{D}}\,(\,\mathbf{C},\,\mathbf{C}'\,) & \to \\ & \to & \varprojlim_{i\in I}\varinjlim_{j\in J} Mor^n_{\mathfrak{Ho}}\,(C_i,\,C'_j\,) & \to & 0 & \end{array}$$

where $\mathfrak{Ho}$ *denotes the homotopy category of* $\mathbb{Z}/2\mathbb{Z}$*-graded chain complexes and* $\mathfrak{D}$ *denotes the derived ind-category over* $\mathfrak{A}$*.*

Proof: Identify each chain complex C_i, $i \in I$, with the associated constant ind-complex $\mathbf{C}_i$, which is $\mathfrak{N}$-colocal (6.2). The direct limit of the corresponding family of constant ind-complexes equals

$$\underset{\overrightarrow{i\in I}}{Lim}\, \mathbf{C}_i \simeq \mathbf{C}$$

Theorem (6.8) therefore applies and yields the assertion as there are natural isomorphisms

$$Mor^n_{\mathfrak{Ho}}(\mathbf{C}_i, \mathbf{C}') \simeq H^n(Hom^*_{\text{ind-}\mathfrak{A}}(\mathbf{C}_i, \mathbf{C}')) = H^n(\varinjlim_{j\in J} Hom^*_{\mathfrak{A}}(C_i, C'_j))$$

$$\simeq \varinjlim_{j\in J} H^n(Hom^*_{\mathfrak{A}}(C_i, C'_j)) \simeq \varinjlim_{j\in J} Mor^n_{\mathfrak{Ho}}(C_i, C'_j)$$

□

By a similar reasoning we obtain

Theorem 6.10. *Let $\mathbf{C} =$ "$\varinjlim_{i\in I}$"C_i be a $\mathbb{Z}/2\mathbb{Z}$-graded ind-complex and let $(\mathbf{C}'_j)$, $j \in J$, be a directed family of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$. Suppose that I is countable. Then there exists a short exact sequence*

$$0 \to \varprojlim_{i\in I}{}^1 \varinjlim_{j\in J} Mor^{n-1}_{\mathfrak{Ho}}(C_i, \mathbf{C}'_j) \to Mor^n_{\mathfrak{D}}(\mathbf{C}, \underset{\overrightarrow{j\in J}}{Lim}\, \mathbf{C}'_j) \to$$

$$\to \varprojlim_{i\in I} \varinjlim_{j\in J} Mor^n_{\mathfrak{Ho}}(C_i, \mathbf{C}'_j) \to 0$$

where $\mathfrak{Ho}$ denotes the homotopy category of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes and $\mathfrak{D}$ denotes the derived ind-category over $\mathfrak{A}$.

Whereas the previous result is needed for computations, the following one allows to treat direct limits.

Theorem 6.11. *Let $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ be a directed family of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes and let $\mathbf{C}'$ be a $\mathbb{Z}/2\mathbb{Z}$-graded ind-complex over $\mathfrak{A}$. Suppose that I is countable. Then there exists a short exact sequence*

$$0 \to \varprojlim_{i\in I}{}^1 Mor^{n-1}_{\mathfrak{D}}(\mathbf{C}_i, \mathbf{C}') \to Mor^n_{\mathfrak{D}}(\underset{\overrightarrow{i\in I}}{Lim}\, \mathbf{C}_i, \mathbf{C}') \to$$

$$\to \varprojlim_{i\in I} Mor^n_{\mathfrak{D}}(\mathbf{C}_i, \mathbf{C}') \to 0$$

For the proof of the theorem we need

Lemma 6.12. *Every countable chain $\mathcal{C}_0 \to \mathcal{C}_1 \to \dots$ of ind-objects $\mathcal{C}_i =$ "$\varinjlim_{J_i}$"$C^{(i)}_{j_i}$ is isomorphic to a chain $\mathcal{C}'_0 \to \mathcal{C}'_1 \to \dots$ of ind-objects with* *one and the same index set (equal to J) and morphisms $\mathcal{C}'_n \to \mathcal{C}'_{n+1}$ given by families ${C'}_j^{(n)} \to {C'}_j^{(n+1)}$, $j \in J$, such that the diagrams*

$$\begin{array}{ccc} {C'}_{j'}^{(n)} & \to & {C'}_{j'}^{(n+1)} \\ \uparrow & & \uparrow \\ {C'}_j^{(n)} & \to & {C'}_j^{(n+1)} \end{array}$$

commute for all $j < j' \in J$.

PROOF: Let J be the set of sequences $(f_k : C_{j_k}^{(k)} \to C_{j_{k+1}}^{(k+1)})_{k\in\mathbb{N}}$ of composable morphisms such that f_k is representing the restriction of $\mathcal{C}_k \to \mathcal{C}_{k+1}$ to $C_{j_k}^{(k)}$. The set J is partially ordered (and directed) in an obvious way. Define ind-objects $\mathcal{C}'_k$, $k \in \mathbb{N}$, with index set J, by putting $\mathcal{C}'_k :=$ " $\varinjlim_J$" $C_{j_k}^{(k)}$ and define morphisms $\mathcal{C}'_n \to \mathcal{C}'_{n+1}$, $n \in \mathbb{N}$, of ind-objects by the family $(\pi_n(\alpha))$, $\alpha \in J$, given by the n-th element $\pi_n(\alpha) = f_n : C_{j_n}^{(n)} \to C_{j_{n+1}}^{(n+1)}$ of the sequence α. There is a straightforward morphism of infinite chains of ind-objects from $(\mathcal{C}'_0 \to \mathcal{C}'_1 \to \dots)$ to $(\mathcal{C}_0 \to \mathcal{C}_1 \to \dots)$, which is easily seen to be an isomorphism. □

PROOF OF THEOREM (6.11):
Let $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ be a directed family of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes, labeled by the countable index set I. After passage to a cofinal subset, which does not affect the statement of the theorem, we may assume that $I = \mathbb{N}$. By the previous lemma, we may further assume that $\mathcal{C}$ is given by a countable directed family of J-diagrams of complexes for some large directed set J. The canonical resolution of an ind-complex (6.3) is functorial on diagrams of complexes (6.4), so that we obtain a countable directed family $\mathbf{P}(\mathbf{C}_i)$ of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes. Each of these ind-complexes is colocal with respect to the nullsystem of weakly contractible ind-complexes (6.7). Theorem (6.8) applies therefore and yields for any $\mathbb{Z}/2\mathbb{Z}$-graded ind-complex $\mathbf{C}'$ a short exact sequence

$$\begin{array}{cccccc} 0 & \to & \varprojlim_{i\in I}{}^1 Mor_{\mathfrak{H}\mathfrak{o}}^{n-1}(\mathbf{P}(\mathbf{C}_i),\, \mathbf{C}') & \to & Mor_{\mathfrak{D}}^n(\underset{i\in I}{\underrightarrow{Lim}}\,\mathbf{P}(\mathbf{C}_i),\, \mathbf{C}') & \to \\ & \to & \varprojlim_{i\in I} Mor_{\mathfrak{H}\mathfrak{o}}^n(\mathbf{P}(\mathbf{C}_i),\, \mathbf{C}') & \to & 0 & \end{array}$$

The canonical projections $\pi_i : \mathbf{P}(\mathbf{C}_i) \to \mathbf{C}_i$ are natural in the sense that they give rise to a morphism of directed families (6.5). The cone of the induced morphism $\pi : \underset{i\in I}{\underrightarrow{Lim}}\,\mathbf{P}(\mathbf{C}_i) \longrightarrow \underset{i\in I}{\underrightarrow{Lim}}\,\mathbf{C}_i$ of direct limits equals the direct limit of the cones of the morphisms π_i. Because these cones are weakly contractible (6.5), the same holds for the cone of the morphism π. This morphism is therefore an isomorphism in the derived ind-category. As the groups $Mor_{\mathfrak{H}\mathfrak{o}}^n(\mathbf{P}(\mathbf{C}_i),\, \mathbf{C}')$ equal $Mor_{\mathfrak{D}}^n(\mathbf{C}_i,\, \mathbf{C}')$ by (6.7) and (6.1), the short exact sequence finally takes the form

$$0 \;\rightarrow\; \varprojlim_{i\in I}{}^{1}\, Mor_{\mathfrak{D}}^{n-1}(\mathbf{C}_i,\, \mathbf{C}')\;\rightarrow\; Mor_{\mathfrak{D}}^{n}(\underset{i\in I}{\underrightarrow{Lim}}\,\mathbf{C}_i,\, \mathbf{C}')\;\rightarrow$$

$$\rightarrow\; \varprojlim_{i\in I} Mor_{\mathfrak{D}}^{n}(\mathbf{C}_i,\, \mathbf{C}')\;\rightarrow\; 0$$

□

6.2 Applications to local cyclic cohomology

The following results provide the basic tools for explicit calculations of local cyclic cohomology groups.

Theorem 6.13. (Approximation Theorem)
Let A be a nice separable Fréchet algebra which possesses the Grothendieck approximation property and let U be a convex open unit ball of A.
Let $V_0 \subset \ldots \subset V_n \subset \ldots$ be an increasing sequence of finite dimensional subspaces of A such that $\bigcup\limits_{n=0}^{\infty} V_n$ is a dense subalgebra of A, and let $(\lambda_n)_{n\in\mathbb{N}}$, $(r_n)_{n\in\mathbb{N}}$, be monotone decreasing sequences of positive real numbers such that $\lim\limits_{n\to\infty}\lambda_n = 1$, $\lim\limits_{n\to\infty} r_n = 0$. Denote by A_n the Banach algebra obtained by completion of the subalgebra A generated by V_n with respect to the largest submultiplicative seminorm satisfying $\parallel \lambda_n V_n \cap U \parallel \leq 1$. Let $(TA)^r$, respectively $HC^*_{\epsilon,r}$, be the completions of the tensor algebra, respectively the cyclic bicomplex, introduced in (1.22), respectively in the section about entire cyclic cohomology.
Then there exists a natural isomorphism

$$\lim_{n\to\infty} HC^{\epsilon}_*(A_n) \;\xrightarrow{\simeq}\; HC^{loc}_*(A)$$

of homology groups and a natural exact sequence

$$0 \longrightarrow \varprojlim_{n}{}^{1}\, HC^{*-1}_{\epsilon,r_n}(A_n) \longrightarrow HC^*_{loc}(A) \longrightarrow \varprojlim_{n} HC^*_{\epsilon,r_n}(A_n) \longrightarrow 0$$

or

$$0 \;\longrightarrow\; \varprojlim_{n}{}^{1}\, H^{*-1}(X((TA_n)^{r_n})) \;\longrightarrow\; HC^*_{loc}(A) \;\longrightarrow$$

$$\longrightarrow\; \varprojlim_{n} H^*(X((TA_n)^{r_n})) \;\longrightarrow\; 0$$

of cohomology groups. Thus the local cyclic (co)homology groups of a nice Fréchet algebra A with approximation property can be expressed in terms of suitable cyclic (co)homology groups of the approximating Banach algebras A_n, $n \in \mathbb{N}$. A similar statement holds if the modified entire cyclic complexes of the approximating algebras are replaced by the corresponding analytic cyclic complexes.

Corollary 6.14. *Let A be a Banach algebra which possesses the Grothendieck approximation property. Let $S \subset A$ be a finite set which generates a dense subalgebra A' of A and let for $\lambda > 1$ be A_λ the completion of A' with respect to the largest submultiplicative seminorm satisfying $\| S \| \leq \lambda$. Then there exists a natural isomorphism*

$$\lim_{\lambda \to 1} HC_*^{\epsilon}(A_\lambda) \xrightarrow{\simeq} HC_*^{loc}(A)$$

of homology groups and a natural exact sequence

$$0 \longrightarrow \varprojlim_{\lambda,\, r}{}^1 HC_{\epsilon,r}^{*-1}(A_\lambda) \longrightarrow HC_{loc}^*(A) \longrightarrow \varprojlim_{\lambda,\, r} HC_{\epsilon,r}^*(A_\lambda) \longrightarrow 0$$

or

$$0 \longrightarrow \varprojlim_{\lambda,\, r}{}^1 H^{*-1}(X((TA_\lambda)^r)) \longrightarrow HC_{loc}^*(A) \longrightarrow$$

$$\longrightarrow \varprojlim_{\lambda,\, r} H^*(X((TA_\lambda)^r)) \longrightarrow 0$$

of cohomology groups.

Remark 6.15. *It should be noted that although entire and analytic cyclic cohomology groups are usually very difficult to compute, a direct or inverse limit of such groups can be quite accessible to calculation.*

Proof: By the approximation theorem for ind-algebras (3.2) there are isomorphisms

$$\text{“} \varinjlim_n \text{”} \, (TA_n)^{r_n} \xrightarrow{\simeq} \underset{\overrightarrow{n}}{Lim}\, \mathcal{T}'(A_n) \xrightarrow{\simeq} \mathcal{T}'\mathcal{B}(A) = \mathcal{T}(A)$$

in the stable diffeotopy category. Passing to continuous cyclic bicomplexes and noting that the ind-algebra “ $\varinjlim_n$” $(TA_n)^{r_n}$ is strictly topologically quasifree, one obtains an isomorphism

$$\text{“} \varinjlim_n \text{”} \, X_*((TA_n)^{r_n}) \xrightarrow{\simeq} X_*(\mathcal{T}A)$$

in the derived ind-category. As the ind-complex $X_*(\mathcal{T}(\mathbb{C}))$ is chain homotopy equivalent to the constant ind-complex $\mathbb{C}$, which is $\mathcal{N}$-colocal by (6.2), the first assertion follows from the identity $\varinjlim H_*(C_i) \xrightarrow{\simeq} Mor^*_{\mathcal{D}}(\mathbb{C}, \text{"}\varinjlim\text{"}\, C_i)$. The second assertion is a consequence of (6.9). The equivalence of the two exact sequences follows from the comparison of the Connes and Cuntz-Quillen approach to cyclic homology [CQ1], [Pu] (5.27). □

Theorem 6.16. (Limit Theorem)
Let “ $\varinjlim_{\lambda \in \Lambda}$ ” A_λ be a countable directed family of nice Fréchet algebras and let

$$f = \varprojlim f_\lambda : \text{“} \varinjlim_{\lambda \in \Lambda} \text{”} A_\lambda \longrightarrow A$$

be a homomorphism to a nice Fréchet algebra A. Suppose that the following conditions hold:

- *A is separable and possesses the Grothendieck approximation property.*
- *The image* $Im(f) := \varinjlim_{\lambda\in\Lambda} f_\lambda(A_\lambda)$ *is dense in A.*
- *There exist seminorms* $\| - \|_\lambda$ *on* A_λ, $\lambda \in \Lambda$, *respectively* $\| - \|$ *on A, and a constant C such that*

  i) The set of elements of length less than 1 with respect to the seminorm is an open unit ball for A_λ, $\lambda \in \Lambda$, *respectively A.*

  ii)
$$\overline{\varinjlim_{\lambda\in\Lambda}} \, \| a_\lambda \|_\lambda \leq C \, \| f(a) \|$$

  for all
$$a = \varinjlim_{\lambda\in\Lambda} a_\lambda \in \varinjlim_{\lambda\in\Lambda} A_\lambda$$

Then there exists a natural isomorphism

$$\varinjlim_{\lambda\in\Lambda} HC_*^{loc}(A_\lambda) \xrightarrow{\simeq} HC_*^{loc}(A)$$

of local cyclic homology groups and for any nice ind-Fréchet algebra $\mathcal{B}$ *a natural exact sequence*

$$0 \longrightarrow \varprojlim_{\lambda\in\Lambda}{}^1 HC_{*-1}^{loc}(A_\lambda, \mathcal{B}) \longrightarrow HC_*^{loc}(A, \mathcal{B}) \longrightarrow$$
$$\longrightarrow \varprojlim_{\lambda\in\Lambda} HC_*^{loc}(A_\lambda, \mathcal{B}) \longrightarrow 0$$

of bivariant local cyclic cohomology groups.

Proof: This follows from the limit theorem for ind-algebras (3.15), theorem (6.11), and the remark about the colocality of $X_*(\mathcal{T}\mathbb{C})$ made in the proof of the previous theorem. □

7 Relations between cyclic cohomology theories

The various cyclic cohomology theories are related by a number of natural transformations. These fall into two groups: transformations of functors of one variable, i.e. of homology or cohomology, and transformations of bifunctors. All transformations preserve exterior products and in the bivariant case they preserve composition products as well. We will also comment on comparison results for the various cyclic theories.

For an ind-Banach algebra $\mathcal{R}$ the identity of its tensor algebra induces a natural bounded homomorphism $\mathcal{T}'\mathcal{R} \longrightarrow \widehat{T}\mathcal{R}$ of completed tensor algebras and thus a natural transformation $\mathcal{T}' \longrightarrow \widehat{T}$ of functors. Recall the functor $\mathcal{B}$ associating to a nice ind-Fréchet algebra the diagram of associated compactly generated Banach algebras and the transformation $\mathcal{B} \to \iota$ to the identity functor (1.5). Using these one obtains natural transformations

$$\mathcal{T} = \mathcal{T}' \circ \mathcal{B} \longrightarrow \mathcal{T}' \circ \iota = \mathcal{T}' \longrightarrow \widehat{T}$$

for ind-Banach algebras and

$$\mathcal{T} = \mathcal{T}' \circ \mathcal{B} \longrightarrow \widehat{T} \circ \mathcal{B} \longrightarrow \widehat{T}$$

for nice ind-Fréchet algebras. Passing to X-complexes and taking (co)homology groups we end up with the following

Proposition 7.1. *There exist canonical natural transformations*

$$HC_*^{an}(-) \longrightarrow HC_*^{\epsilon}(-) \longrightarrow HP_*(-)$$

of cyclic homology theories for (ind-)Banach algebras, respectively

$$HC_*^{an}(-) \longrightarrow HP_*(-)$$

of cyclic homology theories for nice (ind-)Fréchet algebras. All these transformations are compatible with exterior products and with the Chern-character from topological K-theory.

Proposition 7.2. *There exist canonical natural transformations*

$$HP^*(-) \longrightarrow HC_{\epsilon}^*(-) \longrightarrow HC_{an}^*(-)$$

of cyclic cohomology theories for (ind-)Banach algebras, respectively

$$HP^*(-) \longrightarrow HC_{an}^*(-)$$

of cyclic cohomology theories for nice (ind-)Fréchet algebras. All these transformations are compatible with exterior products.

There exist various Chern characters in K-homology [Co], which are defined for suitable classes of Fredholm modules and take values in the different cyclic cohomology theories. A detailed study of the relations between these characters will be the content of another paper.

The compatibility of the transformations with exterior products is clear because these are induced by explicit natural chain maps of cyclic complexes which are continuous with respect to all relevant topologies.

It arises the question to what extent these transformations are equivalences. The comparison problem turns out to be simpler for cohomology than for homology. In [Me1] Meyer obtains a number of results concerning this problem.

He presents examples of nice Fréchet algebras for which analytic and continuous periodic cyclic homology are different. The simplest example he provides is given by the algebra $\mathcal{S}(\mathbb{Z})$ of sequences of rapid decay. One should also expect that there exist Banach algebras for which analytic, entire, and continuous periodic cyclic homology are different from each other. However no such examples have been exhibited so far. In Meyer's example the continuous periodic and analytic cohomology groups are different as well.
In [Me] Meyer constructs a Chern character for arbitrary Fredholm modules with values in analytic cyclic cohomology. The character is compatible with the index pairing. This allows to exhibit nontrivial analytic cyclic cocycles for large classes of Banach and even C^*-algebras. However there seem to be no methods to determine the corresponding cohomology groups. On the one hand the foregoing discussion shows in particular that $HC^*_{an}(\mathcal{K}(\mathcal{H}))$ does not vanish. On the other hand the results of Haagerup [Ha] and Khalkhali [Kh] imply that the entire cyclic cohomology groups of a nuclear C^*-algebra are isomorphic to the space of continuous traces on that algebra. Therefore $HC^*_\epsilon(\mathcal{K}(\mathcal{H}))$ vanishes. Thus the natural transformation $HC^*_{an} \longrightarrow HC^*_\epsilon$ from analytic to entire cyclic cohomology cannot be an equivalence.
We come now to the transformations relating bivariant cyclic theories.

PROPOSITION 7.3. *There exist canonical natural transformations*

$$HC^{an}_*(-,-) \quad\longrightarrow\quad HC^{\alpha}_*(-,-) \quad\longrightarrow\quad HC^{loc}_*(-,-)$$

of bivariant cyclic cohomology theories of nice (ind-)Fréchet algebras. All these transformations are compatible with composition and exterior products.

PROOF: Let $\mathfrak{Ho} \longrightarrow \mathfrak{D}$ be the canonical functor from the homotopy category of ind-complexes to the derived ind-category. It induces a natural map

$$Mor_{\mathfrak{Ho}}(X_*\mathcal{T}(-), X_*\mathcal{T}(-)) \longrightarrow Mor_{\mathfrak{D}}(X_*\mathcal{T}(-), X_*\mathcal{T}(-))$$

of morphism groups which defines the desired transformation from bivariant analytic to bivariant local cyclic cohomology. The compatibility with composition products follows from the compatibility of functors with the composition of morphisms and the compatibility with exterior products is a consequence of the construction of the product in local cyclic cohomology. A bit more work is needed to construct the desired transformations of asymptotic cyclic cohomology. Let DG-$\mathfrak{Ho}$ be the homotopy category of ind-complexes of DG-modules and let DG-$\mathfrak{D}$ be the localization of this homotopy category with respect to the null system given by weakly contractible ind-complexes of DG-modules. Let

$$Mor_{\mathfrak{Ho}}(X_*(\mathcal{T}\mathcal{A}),\, X_*(\mathcal{T}\mathcal{B})) \longrightarrow Mor_{\text{DG-}\mathfrak{Ho}}(X_*(\Omega\mathcal{T}\mathcal{A}),\, \mathcal{E}(\mathcal{U},\, X_*(\mathcal{T}\mathcal{B})))$$

be the transformation which extends a given morphism of ind-complexes φ to the morphism of DG-ind-complexes that equals $\varphi\otimes 1$ in degree zero and vanishes in positive degrees. By [Pu], (4.14), the canonical projection of $X_*(\Omega\mathcal{TB})$ onto its degree zero subspace induces a natural isomorphism

$$Mor_{\text{DG-}\mathfrak{Ho}}(X_*(\Omega\mathcal{TA}),\, \mathcal{E}(\mathcal{U},\, X_*(\Omega\mathcal{TB})))$$

$$\downarrow \simeq$$

$$Mor_{\text{DG-}\mathfrak{Ho}}(X_*(\Omega\mathcal{TA}),\, \mathcal{E}(\mathcal{U},\, X_*(\mathcal{TB})))$$

By composition one obtains a natural transformation

$$Mor_{\mathfrak{Ho}}(X_*\mathcal{T}(-),\, X_*\mathcal{T}(-)) \longrightarrow Mor_{\text{DG-}\mathfrak{Ho}}(X_*(\Omega\mathcal{T}(-)),\, \mathcal{E}(\mathcal{U},\, X_*(\Omega\mathcal{T}(-))))$$

from bivariant analytic to bivariant asymptotic cyclic cohomology.
The fact that the ordered family of neighborhoods of ∞ in $\mathbb{R}^n_+$ contains a cofinal family of convex open sets, the Cartan homotopy formula [Pu], (4.11), (4.12) for the asymptotic parameter space, and the isomorphism criterion (2.9) imply that

$$Id \otimes 1 :\; X_*(\Omega\mathcal{T}(-)) \longrightarrow \mathcal{E}(\mathcal{U},\, X_*(\Omega\mathcal{T}(-)))$$

is an isomorphism in DG-$\mathfrak{D}$. Then the canonical functor DG-$\mathfrak{Ho}$ $\longrightarrow$ DG-$\mathfrak{D}$ induces a natural map

$$HC^{\alpha}_*(A,B) = Mor_{\text{DG-}\mathfrak{Ho}}(X_*(\Omega\mathcal{TA}), \mathcal{E}(\mathcal{U},\, X_*(\Omega\mathcal{TB})))$$

$$\longrightarrow Mor_{\text{DG-}\mathfrak{D}}(X_*(\Omega\mathcal{TA}), \mathcal{E}(\mathcal{U},\, X_*(\Omega\mathcal{TB})))$$

$$\simeq Mor_{\text{DG-}\mathfrak{D}}(X_*(\Omega\mathcal{TA}), X_*(\Omega\mathcal{TB})) \simeq Mor_{\text{DG-}\mathfrak{D}}(X_*(\mathcal{TA}), X_*(\mathcal{TB}))$$

by [Pu], (6.9) and (4.14)

$$\simeq Mor_{\mathfrak{D}}(X_*(\mathcal{TA}), X_*(\mathcal{TB})) = HC^{loc}_*(A,B)$$

It is obvious that this map defines a natural transformation. The composition $HC^{an}_* \longrightarrow HC^{\alpha}_* \longrightarrow HC^{loc}_*$ clearly coincides with the transformation described at the beginning of the proof. □

Corollary 7.4. (Functoriality under linear asymptotic morphisms) *Let $f_t : A \longrightarrow B, t > 0$, be a linear asymptotic morphism of nice Fréchet-algebras [CH]. Then f induces a natural element $f_* \in HC^{loc}_0(A,B)$ depending only on the continuous homotopy class of f. Moreover $(g \circ f)_* = g_* \circ f_*$ under the composition product. Consequently local cyclic cohomology of nice Fréchet-algebras is functorial under linear asymptotic morphisms.*

Proof: This follows from the corresponding statement for asymptotic cyclic cohomology [Pu], (6.11), by applying the natural transformation to bivariant local cyclic cohomology. □

Lemma 7.5. *In all the previously mentioned cyclic theories there exist canonical natural equivalences*

$$H_*(-) \xrightarrow{\simeq} H_*(\mathbb{C}, -)$$

and

$$H^*(-) \xrightarrow{\simeq} H_*(-, \mathbb{C})$$

between homology resp. cohomology groups and suitable bivariant cohomology groups. These are compatible with exterior products and with the natural transformations between the various cyclic theories.

Proof: This follows from the fact that the canonical chain map

$$\mathbb{C} = X_*(\mathbb{C}) \longrightarrow X_*(\mathcal{T}\mathbb{C}),\, 1 \to ch(e)$$

is an isomorphism in the homotopy category of ind-complexes [CQ1]. □

In particular, one obtains from (7.3) canonical natural transformations

$$HC_*^{an}(-) \longrightarrow HC_*^{\alpha}(-) \longrightarrow HC_*^{loc}(-)$$

in homology and

$$HC^*_{an}(-) \longrightarrow HC^*_{\alpha}(-) \longrightarrow HC^*_{loc}(-)$$

in cohomology.

Concerning the transformations of homology groups one finds

Proposition 7.6. *The canonical natural transformations*

$$HC_*^{an}(-) \xrightarrow{\simeq} HC_*^{\alpha}(-) \quad \textit{and} \quad HC_*^{\alpha}(-) \xrightarrow{\simeq} HC_*^{loc}(-)$$

are natural equivalences.

Proof: The first assertion is shown in [Pu], (6.9). The ind-complex $X_*(\mathcal{T}\mathbb{C})$ is isomorphic to the constant ind-complex $\mathbb{C}$ ([CQ1]) and thus $\mathfrak{N}$-colocal (6.2). Therefore $HC_*^{an}(-) \xrightarrow{\simeq} HC_*^{loc}(-)$ is an isomorphism by (6.1) which implies the second assertion. □

Not much is known about the comparison between bivariant analytic and bivariant asymptotic or local cyclic cohomology. The basic unsolved question is whether analytic cyclic cohomology is invariant under continuous homotopies [Me] as it is the case for the asymptotic and local theories.

Finally a remark about the comparison between bivariant asymptotic and local cyclic cohomology. The functorial properties of both theories are identical (with the exception of the results depending on the approximation property). Recall that local cyclic cohomology was obtained from the analytic cyclic theory by turning it into a functor which factors through the stable diffeotopy category. This latter was obtained by inverting morphisms of ind-algebras with weakly contractible mapping cone. Asymptotic cyclic cohomology can be interpreted in a similar way. It is constructed by making analytic cyclic cohomology factor through the category of ind-algebras obtained by inverting morphisms with weakly contractible mapping cone labeled by a countable index set. So there is some reason to believe that both theories coincide for certain sufficiently "small" algebras.

We finally summarize the natural transformations between the various cyclic homology and cohomology theories in the diagrams

$$HP_*(-) \longleftarrow HC_*^{\epsilon}(-) \longleftarrow HC_*^{an}(-)$$

$$HC_*^{an}(-) \xrightarrow{\simeq} HC_*^{\alpha}(-) \xrightarrow{\simeq} HC_*^{loc}(-)$$

and

$$HP^*(-) \longrightarrow HC_{\epsilon}^*(-) \longrightarrow HC_{an}^*(-)$$

$$HC_{an}^*(-) \longrightarrow HC_{\alpha}^*(-) \longrightarrow HC_{loc}^*(-)$$

All transformations are compatible with exterior products and the transformations in homology are compatible with the Chern character from K-theory.

8 Examples

In this last section we give some simple but characteristic examples of explicit calculations of local cyclic cohomology groups. They illustrate the abstract computation scheme developed in section 6. Examples of a similar but more involved nature can be found [Pu4] and [Pu5]. We finally apply local cyclic cohomology to obtain a partial solution of a problem on n-traces formulated in [Co3].

The general idea is to realize the local cyclic (co)homology $HC^{loc}(A)$ of a given algebra A as a limit of the (co)homology groups $HC^{loc}(A_n)$ of a countable directed family (A_n), $n \in \mathbb{N}$, of approximating algebras of a simpler type. Whereas it is usually not possible to compute these approximating (co)homology groups, the transition maps in this directed family often turn out to be amenable to study. And they are all one needs to determine the limit one is interested in.

In the presence of the approximation property one can try to proceed as follows.

1) One looks for a dense subalgebra $\mathfrak{A}$ of A with nice homological properties. By this we mean for example that $\mathfrak{A}$ is of finite Hochschild-homological dimension. Consequently $Fil^k_{Hodge}(\widehat{CC}_*(\mathfrak{A})$ will be contractible for $k >> 0$. If one is lucky the quotient complex $\widehat{CC}(\mathfrak{A})/Fil^k_{Hodge}(\widehat{CC}_*(\mathfrak{A})$ can be identified up to chain homotopy equivalence with a complex with known homology.

2) One chooses an increasing family $0 \subset V_1 \subset \ldots \subset V_n \subset \ldots$ of finite dimensional subspaces of $\mathfrak{A}$ such that $\bigcup V_n$ is a dense subalgebra of A and constructs the enveloping approximating Banach algebras $(A_n), n \in \mathbb{N}$, as in (3.15). By the approximation theorem (6.13) the canonical morphism

$$\text{“}\lim_{n\to\infty}\text{”}\, CC_*^{\epsilon}(A_n) \sim \text{“}\lim_{n\to\infty}\text{”}\, X_*(\mathcal{T}A_n) \longrightarrow X_*(\mathcal{T}A)$$

is then an isomorphism in the derived ind-category. If one is very lucky the vanishing result of step one carries over to the Banach completions $A_n, n \in \mathbb{N}$, so that

$$\text{“}\lim_{n\to\infty}\text{”}\, Fil^k_{Hodge} CC^\epsilon_*(A_n)$$

becomes contractible for $k >> 0$.

3) If one is able to get a good hold of the Banach algebras $A_n, n \in \mathbb{N}$, constructed in step two, one can identify the formal inductive limit

$$\text{“}\lim_{n\to\infty}\text{”}\, CC^\epsilon_*(A_n)/Fil^k_{Hodge} CC^\epsilon_*(A_n) \sim X_*(\mathcal{T}A)$$

(up to chain homotopy equivalence) with a well known small chain complex.

This is how we will proceed in our first example, the algebra of holomorphic functions on an annulus, where all three steps can be carried out without any difficulty. In the second example, the C^*-algebra of continuous functions on a compact metrizable space, there is no really good choice for a dense subalgebra of finite homological dimension. For special types of spaces like smooth manifolds or finite simplicial complexes there are many more or less natural choices of dense smooth subalgebras. But none of these possess topologically projective resolutions which allow to carry out the second step above. For a compact subset $X \subset \mathbb{R}^n$ the optimal choice seems to take the subalgebra of polynomial functions in $C(X)$ as dense subalgebra and to take the rings of bounded holomorphic functions on a sequence of smaller and smaller Grauert tubes around X as approximating Banach algebras. But the lack of a nice contracting homotopy of the acyclic Koszul complex on such a tube makes it impossible to follow the strategy outlined above. We approximate instead a given compact space by a sequence of smooth compact manifolds with boundary and use the limit theorem (6.16) to reduce to the case of the algebra of smooth functions on a manifold. The local cyclic cohomology of these algebras can be calculated by the diffeotopy invariance and excision property of the theory.

In the third example we finally treat a noncommutative algebra, the reduced group C^*-algebra $C^*_r(F_n)$ of a finitely generated free group. In this case it is easy to follow the first two steps outlined above, the dense subalgebra in question being obviously the group ring. The third step however cannot be carried out directly because one has no control of the approximating Banach algebras constructed in step two. We calculate instead the local cyclic cohomology of a smooth dense Banach subalgebra $\mathcal{A}(F_n)$ of $C^*_r(F_n)$, introduced by Haagerup [Ha1], which can be done by the strategy outlined above. We refer then to the smooth subalgebra theorem (3.8) to deduce the corresponding result for the group C^*-algebra.

We want to make a remark on the possibility of using the outlined strategy (and in particular the second step of it) in concrete calculations. Suppose that a dense subalgebra $\mathfrak{A}$ of finite homological dimension d of a Banach algebra A is given. If $d = 1$, i.e. if $\mathfrak{A}$ is quasifree [CQ], then $\text{“}\lim\limits_{n\to\infty}\text{”}\, Fil^{d+1}_{Hodge} CC^\epsilon_*(A_n)$ will be contractible for any approximating sequence A_n, $n \in \mathbb{N}$, as constructed above (8.8). We want to emphasize however that this is an exceptional phenomenon and usually does not occur in homological dimension $d > 1$. Take for example $A = \ell^1(\Gamma)$, the Banach convolution algebra of a finitely generated discrete group Γ, and choose as dense subalgebra the group ring $\mathfrak{A} = \mathbb{C}[\Gamma]$. Then the Hochschild homological dimension d of $\mathfrak{A}$ equals the homological dimension of the group Γ. The assertion that "$\lim\limits_{n\to\infty}$" $Fil^{d+1}_{Hodge} CC^{\epsilon}_*(A_n)$ is contractible for some d and some approximating sequence $(A_n), n \in \mathbb{N}$, implies however that every cohomology class in $H^*(\Gamma, \mathbb{C})$ can be represented by group cocycles which are of subexponential growth with respect to any word metric on Γ, and this is rarely the case. A notable exception, where the described strategy in fact works, is the class of hyperbolic and nonpositively curved groups [Pu4], [Pu5].

It should be noted that in the presented examples the images of the approximating Banach algebras $A_n, n \in \mathbb{N}$, are not closed under holomorphic functional calculus in the ambient Banach algebra A. Dense and holomorphically closed subalgebras play a central role in K-theory but do not seem to be relevant in questions related to cyclic cohomology.

8.1 Rings of holomorphic functions on an annulus

Let $U_R := \{z \in \mathbb{C},\, R^{-1} < |z| < R\}$, $R > 1$, be the R-annulus in the complex plane. It is known that every domain in $\mathbb{C}$ with infinite cyclic fundamental group is biholomorphically equivalent to exactly one R-annulus. We consider the algebra

$$\mathcal{O}(\overline{U})_R := \mathcal{O}(U)_R \cap C(\overline{U_R})$$

of holomorphic functions on the annulus which extend continuously to its boundary. It is a unital Banach algebra with respect to the maximum norm.

We are going to determine the local cyclic cohomology of $\mathcal{O}(\overline{U})_R$. It is well known that the Banach algebras $\mathcal{O}(\overline{U}_R)$, $R > 1$, possess the Grothendieck approximation property and contain the ring of Laurent polynomials as a dense subalgebra. The algebra $\mathcal{O}(\overline{U}_R)$ is a topological direct limit of the family $\mathcal{O}(\overline{U}_{R'})$, $R' > R$, in the sense of (3.15). Therefore we deduce from the limit theorem (6.16) that the canonical chain map

$$\underset{R'\to R}{Lim}\, X_*\mathcal{T}(\mathcal{O}(\overline{U}_{R'})) \longrightarrow X_*\mathcal{T}(\mathcal{O}(\overline{U}_R))$$

is an isomorphism in the derived ind-category. One might have the impression that nothing has been gained by this because a complex which is quite hard to analyze has been replaced by a limit of similar complexes. It turns out however that the transition maps in the above limit are quite accessible to computation. So the limit $\underset{R'\to R}{Lim}\, X_*\mathcal{T}(\mathcal{O}(\overline{U}_{R'}))$ can be calculated although one has essentially no information about the individual complexes in the underlying directed family. Phenomena of this kind often arise in calculations of local cyclic cohomology groups and show the importance of the approximation and limit

theorems in explicit computations. In fact these theorems distinguish local cyclic cohomology among the known cyclic theories.
In order to carry out the computation we need to recall the reduced tensor algebra and reduced infinitesimal deformations. The reduced tensor algebra of a unital algebra A is $RA := \widetilde{TA}/(1-\rho(1_A))$. The functor $R(-)$ is characterized as the left adjoint of the forgetful functor to the category of unital algebras with unital linear maps as morphisms. The reduced universal infinitesimal deformation $\mathcal{RA} := \widetilde{\mathcal{TA}}/(1-\rho(1_{\mathcal{A}}))$ of a nice unital ind-Fréchet algebra $\mathcal{A}$ is characterized similarly by an obvious universal property. The natural map of X-complexes of universal deformations $X_*(\mathcal{TA}) \longrightarrow X_*(\mathcal{RA})$ is a chain homotopy equivalence.

Lemma 8.1. *Let $A = \mathbb{C}[z, z^{-1}]$ be the ring of Laurent polynomials and consider the norms $\| \sum\limits_n a_n z^n \|_r := \sum\limits_n |a_n| \cdot r^{|n|}$, $r > 1$. Let $\| - \|^r_{N,m}$ be the largest seminorm on the reduced tensor algebra RA satisfying*

$$\| \rho(z^{k_0})\omega(z^{k_1}, z^{k_2}) \cdot \ldots \cdot \omega(z^{k_{2n-1}}, z^{k_{2n}}) \|^r_{N,m} \leq (2+2n)^m \cdot N^{-n} \cdot r^{k_1+\ldots+k_{2n}}$$

Let $\varphi : A \to RA$ be the algebra homomorphism which splits the canonical projection
$\pi : RA \to A$ and is characterized by

$$\varphi(z) = \rho(z), \quad \varphi(z^{-1}) = \varphi(z)^{-1} = \rho(z^{-1}) \sum_{n=0}^{\infty} \omega(z, z^{-1})^n$$

Then for given $r' > r > 1$ there exists $N_0 >> 0$ and constants $C_m, m \in \mathbb{N}$, such that

$$\| \varphi(f) \|^r_{N,m} \leq C_m \cdot \| f \|_{r'}$$

for all $f \in A$ and $N \geq N_0$.

Proof: The straightforward calculation based on the Bianchi-identity $\omega(a, a')\rho(a'') = \omega(a, a'a'') - \omega(aa', a'') + \rho(a)\omega(a', a'')$ is left to the reader.
□

Lemma 8.2. *Denote by $\mathcal{R}$ the reduced infinitesimal deformation functor. The canonical projection*

$$\pi : \underset{R' \to R}{Lim}\ \mathcal{R}(\mathcal{O}(\overline{U}_{R'})) \longrightarrow \text{“} \lim_{R' \to R} \text{”} (\mathcal{O}(\overline{U}_{R'}))$$

is a diffeotopy equivalence of ind-algebras. Consequently the projection

$$\pi_* : \underset{R' \to R}{Lim}\ X_*\mathcal{T}(\mathcal{O}(\overline{U}_{R'})) \longrightarrow \text{“} \lim_{R' \to R} \text{”} X_*(\mathcal{O}(\overline{U}_{R'}))$$

is a chain homotopy equivalence of ind-complexes.

Proof: Denote by A_r the completion of the algebra A of Laurent polynomials with respect to the norm $\| - \|_r$ introduced in (8.1). It follows from Cauchy's integral formula that the ind-algebra $\text{“}\lim\limits_{R'\to R}\text{”}\,(\mathcal{O}(\overline{U}_{R'}))$ is canonically isomorphic to $\text{“}\lim\limits_{R'\to R}\text{”}\,A_{R'}$. By the estimates of the previous lemma the reduced universal infinitesimal deformation $\pi : \underset{R'\to R}{Lim}\,\mathcal{R}(A_{R'}) \longrightarrow \text{“}\lim\limits_{R'\to R}\text{”}\,A_{R'}$ possesses a multiplicative section. Consequently the ind-algebras $\text{“}\lim\limits_{R'\to R}\text{”}\,A_{R'}$ and $\text{“}\lim\limits_{R'\to R}\text{”}\,\mathcal{O}(\overline{U}_{R'})$ are topologically quasifree. This implies the first assertion and the second assertion follows from the Cartan homotopy formula for the X-complexes of quasifree algebras [CQ1] and the fact that the canonical morphism $\underset{R'\to R}{Lim}\,X_*\mathcal{T}(\mathcal{O}(\overline{U}_{R'})) \longrightarrow \underset{R'\to R}{Lim}\,X_*\mathcal{R}(\mathcal{O}(\overline{U}_{R'}))$ is a chain homotopy equivalence. □

PROPOSITION 8.3. *Let $\mathcal{O}(\overline{U}_R)$, $R > 1$, be the Banach algebra of holomorphic functions on the annulus*

$$U_R = \{z \in \mathbb{C},\, R^{-1} < |z| < R\}$$

which extend continuously to its boundary. Then there is a canonical isomorphism

$$X_*\mathcal{T}(\mathcal{O}(\overline{U}_R)) \overset{\simeq}{\longrightarrow} \mathbb{C} \bigoplus \mathbb{C}[1]$$

in the derived ind-category. For any pair of nice ind-Fréchet algebras $(\mathcal{A}, \mathcal{B})$ there are canonical and natural isomorphisms

$$HC_*^{loc}(\mathcal{O}(\overline{U}_R) \otimes_\pi \mathcal{A},\, \mathcal{B}) \simeq HC_*^{loc}(\mathcal{A}, \mathcal{B}) \oplus HC_{*+1}^{loc}(\mathcal{A}, \mathcal{B})$$

and

$$HC_*^{loc}(\mathcal{A},\, \mathcal{O}(\overline{U}_R) \otimes_\pi \mathcal{B}) \simeq HC_*^{loc}(\mathcal{A}, \mathcal{B}) \oplus HC_{*+1}^{loc}(\mathcal{A}, \mathcal{B})$$

of bivariant local cyclic cohomology groups.

PROOF: It is not easy to determine the precise structure of the complexes $X_*(\mathcal{O}(\overline{U}_R))$, $R > 1$. Using Cauchy's integral formula and the fact that the inclusion maps $\mathcal{O}(\overline{U}_{R'}) \to \mathcal{O}(\overline{U}_{R''})$ for $R' > R'' > 1$ are nuclear, one may conclude at least that the identity map on the space of algebraic differential forms over the ring of Laurent polynomials induces an isomorphism of ind-complexes

$$\text{“}\lim_{R'\to R}\text{”}\, X_*(\mathcal{O}(\overline{U}_{R'})) \overset{\simeq}{\longrightarrow} \text{“}\lim_{R'\to R}\text{”}\, \Omega^*_{dR}(\mathcal{O}(U_{R'}))$$

where $\Omega^*_{dR}(\mathcal{O}(U_{R'}))$ denotes the analytic de Rham complex on the open annulus $\mathcal{O}(U_{R'})$. It is obvious from de Rham theory that the latter is chain homotopy equivalent to $\mathbb{C} \bigoplus \mathbb{C}[1]$. So in the end one obtains a chain of isomorphisms

$$X_*\mathcal{T}(\mathcal{O}(\overline{U}_R)) \overset{\simeq}{\longleftarrow} \underset{R'\to R}{Lim}\, X_*\mathcal{T}(\mathcal{O}(\overline{U}_{R'})) \overset{\simeq}{\longrightarrow} \text{“}\lim_{R'\to R}\text{”}\, X_*(\mathcal{O}(\overline{U}_{R'}))$$

$$\text{“}\lim_{R'\to R}\text{”}\, X_*(\mathcal{O}(\overline{U}_{R'})) \overset{\simeq}{\longrightarrow} \text{“}\lim_{R'\to R}\text{”}\, \Omega^*_{dR}(\mathcal{O}(U_{R'})) \overset{\simeq}{\longrightarrow} \mathbb{C} \bigoplus \mathbb{C}[1]$$

in the derived ind-category. □

We note that neither the periodic, nor the analytic or asymptotic cyclic cohomology groups of the algebras $\mathcal{O}(\overline{U}_R)$ seem to be known. The analytic and asymptotic cyclic homology groups on the other hand coincide of course with the local ones computed here.

Thus in the example considered above the existence of a dense subalgebra of A of finite (Hochschild)-homological dimension $d = 1$ implies the contractability of the limit $\underset{n\to\infty}{Lim}\, Fil^{d+1}_{Hodge} CC^{an}_*(A_n)$ in an approximating sequence A_n of Banach subalgebras of A. It should be noted that this phenomenon is rather exceptional and in some sense peculiar to subalgebras of homological dimension at most one (quasifree algebras).

8.2 Commutative C^*-algebras

As another example we calculate the bivariant local cyclic cohomology of separable commutative C^*-algebras (see also [Pu], Chapter 11). It would be nice to apply directly the computational methods developed in this paper. Despite serious efforts I was not able to do this and therefore we have to refer in addition to the excision property [Pu2] of local cyclic cohomology. Using excision we obtain

Proposition 8.4. *Let M be a smooth compact manifold with (possibly empty) boundary. Then there is a natural chain homotopy equivalence*

$$CC^{an}_*(\mathcal{C}^\infty(M)) \xrightarrow{\sim} H^*(M, \mathbb{C})$$

from the analytic cyclic bicomplex of $\mathcal{C}^\infty(M)$ to the $\mathbb{Z}/2\mathbb{Z}$-graded sheaf cohomology groups of M, viewed as complex with vanishing differentials.

Proof: We proceed in several steps.

- Let $(M, \partial M)$ be a smooth compact Riemannian manifold with boundary and let $\mathcal{C}^\infty(M, \partial M)$ respectively $\mathcal{C}^\infty_0(M, \partial M)$ be the algebras of smooth functions on M vanishing along ∂M, respectively vanishing of infinite order along ∂M. We claim that the inclusion

$$\mathcal{C}^\infty_0(M, \partial M) \hookrightarrow \mathcal{C}^\infty(M, \partial M)$$

is a diffeotopy equivalence and that the induced morphism

$$\Omega^*_{dR}(M, \partial M) \longrightarrow \Omega_{0,dR}(M, \partial M)$$

of the associated de Rham complexes is a chain homotopy equivalence. In fact let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly monotone increasing smooth homeomorphism of the real halfline which is a diffeomorphism outside the origin, equals the identity outside $[0, 1]$, and has vanishing Taylor series at the origin. An open tubular neighborhood W of ∂M in M can be identified with $\partial M \times \mathbb{R}_+$. One can extend the smooth homeomorphism $\mathrm{Id} \times \varphi$ of $\partial M \times \mathbb{R}_+$ to a smooth homeomorphism ψ of M by putting it equal to the identity outside W. The algebra homomorphism $\psi^*: \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ maps $\mathcal{C}^\infty(M, \partial M)$ to $\mathcal{C}_0^\infty(M, \partial M)$ and is obviously an inverse to the inclusion $\mathcal{C}_0^\infty(M, \partial M) \hookrightarrow \mathcal{C}^\infty(M, \partial M)$ up to diffeotopy. By applying the Cartan homotopy formula one obtains the corresponding statement for the de Rham complexes.

- Let $(M, \partial M)$ be a smooth compact n-dimensional manifold with (possibly empty) boundary ∂M. We assume without loss of generality that M is connected.

  Every appropriate Morse function on M provides a filtration

$$D^n = M_0 \subset M_1 \subset \ldots \subset M_j = M$$

  such that for $i = 0, \ldots, j-1$

  - M_i is a codimension 0 submanifold with corners of M which does not intersect the boundary ∂M.
  - The extension of nice nuclear Fréchet algebras

$$0 \to \mathcal{C}^\infty(M_{i+1}, M_i) \to \mathcal{C}^\infty(M_{i+1}) \to \mathcal{C}^\infty(M_i) \to 0$$

    possesses a bounded linear section.
  -

$$\begin{aligned} \mathcal{C}_0^\infty(M_{i+1}, M_i) &\simeq \mathcal{C}_0^\infty(D^k \times D^{n-k}, \partial(D^k) \times D^{n-k}) \\ &\simeq \mathcal{C}_0^\infty(D^k, \partial(D^k)) \otimes_\pi \mathcal{C}^\infty(D^{n-k}) \end{aligned}$$

- Suppose that a filtration of $(M, \partial M)$ as constructed before is given. We show by induction over i that the canonical chain map

$$CC_*^{an}(\mathcal{C}^\infty(M)) \longrightarrow \Omega_{dR}^*(M)$$

  obtained by antisymmetrization of differential forms [Co] is a chain homotopy equivalence. Consider the commutative diagram

$$\begin{array}{ccccc} CC_*^{an}(\mathcal{C}^\infty(M_{i+1}, M_i)) & \rightarrow & CC_*^{an}(\mathcal{C}^\infty(M_{i+1})) & \rightarrow & CC_*^{an}(\mathcal{C}^\infty(M_i)) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_{dR}^*(M_{i+1}, M_i) & \rightarrow & \Omega_{dR}^*(M_{i+1}) & \rightarrow & \Omega_{dR}^*(M_i) \end{array}$$

  of complexes. By the excision theorem in analytic cyclic (co)homology [Pu2] the upper line is a distinguished triangle. The lower line is an exact sequence of complexes with bounded linear section and is thus a distinguished triangle as well. By the properties of the filtration of M, the induction hypothesis, and the five lemma it suffices to verify that

$$CC_*^{an}(\mathcal{C}^\infty(D^n)) \longrightarrow \Omega^*_{dR}(D^n)$$

and

$$CC_*^{an}(\mathcal{C}_0^\infty(D^k \times D^{n-k}, \partial(D^k) \times D^{n-k}))$$

$$\downarrow$$

$$\Omega^*_{0,dR}(D^k \times D^{n-k}, \partial(D^k) \times D^{n-k})$$

are chain homotopy equivalences. By the Cartan homotopy formulas for the analytic cyclic bicomplex and the de Rham complex the last statement is equivalent to the assertion that

$$CC_*^{an}(\mathcal{C}_0^\infty(D^k, \partial(D^k))) \longrightarrow \Omega^*_{0,dR}(D^k, \partial(D^k))$$

is a chain homotopy equivalence. This follows however from a simple induction over k making use of the Cartan homotopy formulas, excision in analytic cyclic cohomology, and the arguments in the first part of this demonstration.

- The classical theorems of de Rham and Hodge imply that for a smooth compact manifold without boundary there is a chain homotopy equivalence $\Omega^*_{dR}(M) \xrightarrow{\sim} H^*(M, \mathbb{C})$ where $H^*(M, \mathbb{C})$ is viewed as complex with zero differentials. We present here the proof of A. Weil which neatly covers the case of manifolds with boundary. Choose a Riemannian metric on M and let $\mathcal{U} = (U_0, \ldots, U_k)$ be a finite open cover of M by geodesically convex balls (semiballs) such that no ball with center in the interior meets the boundary of M. Consider the bicomplex

$$\check{C}^{pq}(\mathcal{U}, \Omega^*) := \prod_{i_0 < \ldots < i_p} \Omega^q_{dR}(U_{i_0} \cap \ldots \cap U_{i_p})$$

with differentials given by the Čech-differential in the horizontal and the de Rham differential in the vertical direction. On the one hand there is a canonical embedding $\Omega^*_{dR}(M) \hookrightarrow \check{C}^*(\mathcal{U}, \Omega^*)$ into the first column of $\check{C}^*(\mathcal{U}, \Omega^*)$ given by restriction of differential forms. The fact that sheaves of differential forms are fine allows to deduce that $\Omega^*_{dR}(M)$ becomes a retract of $\check{C}^*(\mathcal{U}, \Omega^*)$. On the other hand there is a canonical embedding $\check{C}^*(\mathcal{U}, \mathbb{C}) \hookrightarrow \check{C}^*(\mathcal{U}, \Omega^*)$ of the Čech complex of $\mathcal{U}$ with coefficients in the constant sheaf $\mathbb{C}$ into the first line of $\check{C}^*(\mathcal{U}, \Omega^*)$. The fact that any intersection of the balls $U_i, 0 \leq i \leq k$, is geodesically convex and the Cartan homotopy formula show, that $\check{C}^*(\mathcal{U}, \mathbb{C})$ is a retract of $\check{C}^*(\mathcal{U}, \Omega^*)$ as well. Therefore the de Rham complex $\Omega^*_{dR}(M)$ is chain homotopy equivalent to the finite dimensional complex $\check{C}^*(\mathcal{U}, \mathbb{C})$ and in particular to the complex with vanishing differentials given by the cohomology of the latter one. As $\mathcal{U}$ is a Leray cover the cohomology of $\check{C}^*(\mathcal{U},\mathbb{C})$ coincides with $H^*(M,\mathbb{C})$. Altogether we have shown that the analytic cyclic bicomplex $CC^{an}_*(\mathcal{C}^\infty(M))$ of $\mathcal{C}^\infty(M)$ is chain homotopy equivalent to the complex with vanishing differentials $H^*(M,\mathbb{C})$ given by the $\mathbb{Z}/2\mathbb{Z}$-graded sheaf cohomology of M with coefficients in $\mathbb{C}$. The naturality of the chain map is clear.

□

Proposition 8.5. *Let X be a compact metrizable space and let $C(X)$ be the C^*-algebra of continuous functions on X.*

a) There exists a projective system $(M_n, \partial M_n), n \in \mathbb{N}$, of smooth manifolds (with boundary) and smooth maps, and a continuous map

$$\varprojlim f_n : X \longrightarrow \text{“}\varprojlim\text{”}(M_n,\, \partial M_n)$$

such that the family $\{\, f_n^{-1}(U_n),\, U_n \subset M_n \text{ open}, n \in \mathbb{N} \,\}$ forms a basis of the topology of X and such that the induced morphism

$$\text{“}\lim_{n\to\infty}\text{”}\, \mathcal{C}^\infty(M_n) \longrightarrow C(X)$$

satisfies the assumptions of the limit theorem (6.16).

b) There is an isomorphism

$$\text{“}\lim_{n\to\infty}\text{”}\, H^*(M_n,\, \mathbb{C}) \xrightarrow{\simeq} X_*\mathcal{T}(C(X))$$

in the derived ind-category. Here $H^*(M_n, \mathbb{C})$ denotes the sheaf cohomology of M_n, viewed as $\mathbb{Z}/2\mathbb{Z}$-graded complex with zero differentials.

c) If $\mathcal{A}$ is a nice ind-Fréchet algebra, then there is a similar isomorphism

$$\text{“}\lim_{n\to\infty}\text{”}\, H^*(M_n,\, \mathbb{C}) \otimes X_*\mathcal{T}(\mathcal{A}) \xrightarrow{\simeq} X_*\mathcal{T}(C(X,\, \mathcal{A}))$$

in the derived ind-category, which is natural in $\mathcal{A}$.

Proof: It is well known that the Gelfand transform, which assigns to a commutative C^*-algebra its spectrum, defines an antiequivalence between the category of commutative C^*-algebras and the category of locally compact Hausdorff spaces. Under the Gelfand transform separable algebras correspond to metrizable spaces. Let X be a compact metrizable space, let $A = C(X)$ be the separable C^*-algebra of continuous functions on X, and let $(a_n), n \in \mathbb{N}, a_0 = 1$, be a countable system of selfadjoint elements generating a dense involutive subalgebra of A. For each $n \in \mathbb{N}$ let $A_n \subset A$ be the C^*-subalgebra generated by $\{a_0, \ldots, a_n\}$. Then $A = \lim\limits_{n\to\infty} A_n$ as C^*-algebras. The map $i_n : Sp(A_n) \hookrightarrow \mathbb{R}^n$, which associates to a character $\chi \in Sp(A_n)$

the n-tuple $(\chi(a_1), \dots \chi(a_n))$ defines a faithful embedding of $Sp(A_n)$ into euclidean n-space. Denote by X_n its image. Then $\pi_{n+1}(X_{n+1}) = X_n$ where $\pi_{n+1} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n coordinates. Let finally M_n be a family of smooth manifolds with boundary satisfying the following conditions for all $n \in \mathbb{N}$:

- M_n is a smooth codimension zero submanifold with boundary of $\mathbb{R}^n$.
- $\overset{\circ}{M}_n$ is an open neighborhood of X_n
- M_n is contained in a $\frac{1}{n}$-neighborhood of X_n.
- $\pi_{n+1}(M_{n+1}) \subset M_n$.

It is then clear that the family (M_n) satisfies the second assertion of part a) of the proposition. Let $(\mathcal{U}_n), n \in \mathbb{N}$, be a countable family of finite open covers of X such that $\bigcup_n \{U, U \in \mathcal{U}_n\}$ forms a basis of the topology of X and choose for each n a partition of unity subordinate to $\mathcal{U}_n$. If one takes as a countable generating system for $C(X)$ the family of functions occurring in these partitions of unity, then a corresponding family of manifolds will also satisfy the first claim. Assertion b) is just a special case of c) which we show now. Let $\mathcal{A} = \text{“}\varinjlim_{i \in I}\text{”}\, A_i$ be a nice ind-Fréchet algebra. Then for fixed $i \in I$ the morphism

$$\lim_{n \to \infty} \mathcal{C}^\infty(M_n, A_i) \longrightarrow C(X, A_i)$$

satisfies the conditions of theorem (6.16). Consequently, the isomorphism criterion (2.9) applies to the morphism

$$\underset{n\to\infty}{Lim}\, X_*\mathcal{T}(\mathcal{C}^\infty(M_n, A_i)) \longrightarrow X_*\mathcal{T}(C(X, A_i))$$

We deduce therefore from proposition (2.10) that the canonical morphism of ind-complexes

$$\underset{n\to\infty}{Lim} X_*\mathcal{T}(\mathcal{C}^\infty(M_n, \mathcal{A})) \simeq \underset{\overrightarrow{i \in I}}{Lim} \left(\underset{n\to\infty}{Lim}\, X_*\mathcal{T}(\mathcal{C}^\infty(M_n, A_i))\right) \longrightarrow$$

$$\longrightarrow \underset{\overrightarrow{i \in I}}{Lim}\, (X_*\mathcal{T}(C(X, A_i))) \simeq X_*\mathcal{T}(C(X, \mathcal{A}))$$

is an isomorphism in the derived ind-category. By the Eilenberg-Zilber theorem for cyclic complexes, [Pu3] and (5.16), there is a chain homotopy equivalence of ind-complexes

$$\underset{n\to\infty}{Lim}\, X_*\mathcal{T}(\mathcal{C}^\infty(M_n, \mathcal{A})) \simeq \underset{n\to\infty}{Lim}\, X_*\mathcal{T}(\mathcal{C}^\infty(M_n)) \otimes_\pi X_*\mathcal{T}(\mathcal{A})$$

The previous proposition and (2.10) show then that in the derived ind-category there are isomorphisms

$$\underset{n\to\infty}{Lim}\, X_*\mathcal{T}(\mathcal{C}^\infty(M_n)) \otimes_\pi X_*\mathcal{T}(\mathcal{A}) \overset{\simeq}{\longrightarrow}$$

$$\underset{n\to\infty}{Lim}\ (H^*(M_n,\mathbb{C})\otimes_\pi X_*\mathcal{T}(\mathcal{A}))\ \simeq\ \left(\text{“}\lim_{n\to\infty}\text{”}\ H^*(M_n,\mathbb{C})\right)\otimes X_*\mathcal{T}(\mathcal{A})$$

Altogether one obtains the desired isomorphism

$$\left(\text{“}\lim_{n\to\infty}\text{”}\ H^*(M_n,\mathbb{C})\right)\otimes X_*\mathcal{T}(\mathcal{A})\ \xrightarrow{\simeq}\ X_*\mathcal{T}(C(X,\mathcal{A}))$$

Its naturality is obvious. □

THEOREM 8.6. *Let X, Y be locally compact metrizable spaces and let $C_0(X), C_0(Y)$ be the corresponding C^*-algebras of continuous functions vanishing at infinity. For a locally compact space denote by $H_c^*(-,\mathcal{F})$ its sheaf cohomology with compact supports and coefficients in the sheaf $\mathcal{F}$.*

a) The even(odd) local cyclic homology groups of $C_0(X)$ are naturally isomorphic to the direct sum of the even(odd) sheaf cohomology groups of X with compact supports and complex coefficients

$$HC_*^{loc}(C_0(X))\quad\xrightarrow{\simeq}\quad\bigoplus_{n\in\mathbb{Z}}H_c^{*+2n}(X,\mathbb{C})$$

b) The even(odd) local cyclic cohomology groups of $C_0(X)$ are naturally isomorphic to the direct product of the even(odd) Borel-Moore homology groups of X with compact supports and complex coefficients [BM]

$$HC^*_{loc}(C_0(X))\quad\xrightarrow{\simeq}\quad\prod_{n\in\mathbb{Z}}H^c_{*+2n}(X,\mathbb{C})$$

c) The even(odd) bivariant local cyclic cohomology groups of the pair $(C_0(X), C_0(Y))$ are naturally isomorphic to the space of even(odd) linear maps from the direct sum of the sheaf cohomology groups of X with compact supports and complex coefficients to corresponding direct sum of the sheaf cohomology groups of Y

$$HC_*^{loc}(C_0(X),\,C_0(Y))\simeq Hom^*\left(\bigoplus_{n\in\mathbb{Z}}H_c^{*+2n}(X,\mathbb{C}),\bigoplus_{m\in\mathbb{Z}}H_c^{*+2m}(Y,\mathbb{C})\right)$$

d) Let $\mathcal{A}$ be a nice ind-Fréchet algebra. Then there is a natural isomorphism

$$HC_*^{loc}(C_0(X,\mathcal{A}))\quad\xrightarrow{\simeq}\quad\bigoplus_{n\in\mathbb{Z}}H_c^{*+2n}(X,HC_*^{loc}(\mathcal{A}))$$

which identifies the local cyclic homology groups of the ind-algebra of $\mathcal{A}$-valued continuous functions on X vanishing at infinity with the direct sum of the sheaf cohomology groups of X with compact supports and coefficients in the constant sheaf $HC_*^{loc}(\mathcal{A})$.

e) *Let $\mathcal{B}$ be a further nice ind-Fréchet algebra. Then there is a natural isomorphism*

$$HC_*^{loc}(C_0(X,\mathcal{A}),\mathcal{B}) \quad \xrightarrow{\simeq} \quad Hom^*\left(\bigoplus_{n\in\mathbb{Z}} H_c^{*+2n}(X,\mathbb{C}), HC_*^{loc}(\mathcal{A},\mathcal{B})\right)$$

There is a certain asymmetry in the statements concerning homology and cohomology which is due to the fact, that maps from but not into a direct limit are characterized by a universal property.
For the proof we will need the

LEMMA 8.7. *Let $\mathcal{C},\mathcal{C}'$ be $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes of complete, locally convex vector spaces. Suppose that $\mathcal{C} =$ "$\varinjlim\limits_{i\in I}$"$C_*^{(i)}$ is a formal inductive limit of finite dimensional complexes $C_*^{(i)}$ with vanishing differentials and that $\mathcal{C}'$ is colocal (see (6.1)) with respect to the nullsystem of weakly contractible ind-complexes. Then the following holds*

a) *The ind-complex $\mathcal{C}\otimes_\pi\mathcal{C}'$ is $\mathcal{N}$-colocal.*

b) *For any ind-complex $\mathcal{C}''$ there is a natural isomorphism*

$$Mor^*_{\mathfrak{Ho}}(\mathcal{C}\otimes_\pi\mathcal{C}',\mathcal{C}'') \simeq Mor^*_{Vect}(\varinjlim_{i\in I} C_*^{(i)}, Mor^*_{\mathfrak{Ho}}(\mathcal{C}',\mathcal{C}''))$$

where on the right hand side the morphisms are taken in the category $Vect$ of abstract $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces.

PROOF: Let $\mathcal{C}' =$ "$\varinjlim\limits_{j\in J}$"$\,\overline{C}_*^{(j)}$, $\mathcal{C}'' =$ "$\varinjlim\limits_{k\in K}$"$\,\overline{\overline{C}}_*^{(k)}$ be ind-complexes. Then

$$Mor^*_{\mathfrak{Ho}}(\mathcal{C}\otimes_\pi\mathcal{C}',\mathcal{C}'') = H^*(\varprojlim_{I\times J}\varinjlim_{K} Hom_{cont}(C^{(i)}\otimes\overline{C}^{(j)},\overline{\overline{C}}^{(k)}))$$

$$= H^*(\varprojlim_{I\times J}\varinjlim_{K} Hom_{Vect}(C^{(i)}, Hom_{cont}(\overline{C}^{(j)},\overline{\overline{C}}^{(k)}))$$

$$= H^*(\,Hom_{Vect}(\varinjlim_{I} C^{(i)}, \varprojlim_{J}\varinjlim_{K} Hom_{cont}(\overline{C}^{(j)},\overline{\overline{C}}^{(k)})\,)$$

because the complexes $C^{(i)}, i\in I$, are finite dimensional

$$= Hom^*_{Vect}(\varinjlim_{I} C^{(i)}, H^*(\varprojlim_{J}\varinjlim_{K} Hom_{cont}(\overline{C}^{(j)},\overline{\overline{C}}^{(k)})\,))$$

because the differentials of the complexes $C^{(i)}, i\in I$, vanish

$$= Hom^*_{Vect}(\varinjlim_{I} C^{(i)}, Mor^*_{\mathfrak{Ho}}(\mathcal{C}',\mathcal{C}''))$$

which proves the second assertion. If $\mathcal{C}'$ happens to be $\mathcal{N}$-colocal, then for weakly contractible ind-complexes $\mathcal{C}''$ one has $Mor^*_{\mathfrak{Ho}}(\mathcal{C}', \mathcal{C}'') = 0$ so that $Mor^*_{\mathfrak{Ho}}(\mathcal{C} \otimes_\pi \mathcal{C}', \mathcal{C}'') = 0$ by the previous calculation. This implies the first assertion. $\square$

Proof of theorem (8.6):

Let X and Y be compact metrizable spaces and let $\mathcal{A}, \mathcal{B}$ be nice ind-Fréchet algebras. We begin by calculating the local cyclic cohomology of the pair $(C(X, \mathcal{A}), \mathcal{B})$. Let $(M_n)_{n \in \mathbb{N}}$ be an approximating family of manifolds for X as constructed in (8.5). Then the projection maps $f_n : X \to M_n$ give rise to an isomorphism

$$\lim_{n\to\infty} H^*(M_n, \mathbb{C}) \simeq \lim_{n\to\infty} H^*(\check{C}(M_n, \mathbb{C})) \simeq H^*(\lim_{n\to\infty} \check{C}(M_n, \mathbb{C}))$$

$$\simeq H^*(\check{C}(X, \mathbb{C})) \simeq H^*(X, \mathbb{C})$$

by (8.5) a) where $\check{C}(-, \mathbb{C})$ denotes the Čech-complex calculating the cohomology of the constant sheaf $\mathbb{C}$. According to proposition (8.5) there is an isomorphism $X_* \mathcal{T}(C(X, \mathcal{A})) \simeq$ "$\lim\limits_{n\to\infty}$" $H^*(M_n, \mathbb{C}) \otimes_\pi X_*(\mathcal{T}\mathcal{A})$ in the derived ind-category. Let $\mathbf{P}(X_*(\mathcal{T}\mathcal{A}))$ be an $\mathcal{N}$-colocal model of $X_*(\mathcal{T}\mathcal{A})$. Then "$\lim\limits_{n\to\infty}$" $H^*(M_n, \mathbb{C}) \otimes_\pi \mathbf{P}(X_*(\mathcal{T}\mathcal{A}))$ is an $\mathcal{N}$-colocal model of "$\lim\limits_{n\to\infty}$" $H^*(M_n, \mathbb{C}) \otimes_\pi X_*(\mathcal{T}\mathcal{A})$ by (8.7), (6.5) and (2.10). With these remarks in mind one finds

$$HC^{loc}_*(C(X, \mathcal{A}), \mathcal{B}) = Mor_{\mathfrak{D}}(X_*(\mathcal{T}C(X, \mathcal{A})), X_*(\mathcal{T}\mathcal{B}))$$

$$= Mor_{\mathfrak{D}}(\text{“}\lim_{n\to\infty}\text{”}\, H^*(M_n, \mathbb{C}) \otimes_\pi X_*(\mathcal{T}\mathcal{A}), X_*(\mathcal{T}\mathcal{B}))$$

$$= Mor_{\mathfrak{Ho}}(\text{“}\lim_{n\to\infty}\text{”}\, H^*(M_n, \mathbb{C}) \otimes_\pi \mathbf{P}(X_*(\mathcal{T}\mathcal{A})), X_*(\mathcal{T}\mathcal{B}))$$

$$= Hom^*_{Vect}(\lim_{n\to\infty} H^*(M_n, \mathbb{C}), Mor_{\mathfrak{Ho}}(\mathbf{P}(X_*(\mathcal{T}\mathcal{A})), X_*(\mathcal{T}\mathcal{B})))$$

by lemma (8.7)

$$= Hom^*_{Vect}(\lim_{n\to\infty} H^*(M_n, \mathbb{C}), Mor_{\mathfrak{D}}(X_*(\mathcal{T}\mathcal{A}), X_*(\mathcal{T}\mathcal{B})))$$

$$= Hom^*_{Vect}(\lim_{n\to\infty} H^*(M_n, \mathbb{C}), HC^{loc}_*(\mathcal{A}, \mathcal{B}))$$

$$= Hom^*_{Vect}(H^*(X, \mathbb{C}), HC^{loc}_*(\mathcal{A}, \mathcal{B}))$$

where $H^*(X, \mathbb{C}) = \bigoplus_n H^{*+2n}(X, \mathbb{C})$ is the $\mathbb{Z}/2\mathbb{Z}$-graded sheaf cohomology of X with coefficients in $\mathbb{C}$. For d) one finds similarly

$$HC^{loc}_*(C(X, \mathcal{A})) = Mor_{\mathfrak{D}}(\mathbb{C}, X_*(\mathcal{T}C(X, \mathcal{A})))$$

$$\simeq Mor_{\mathfrak{D}}(\mathbb{C}, \text{“}\lim_{n\to\infty}\text{”}\, H^*(M_n, \mathbb{C}) \otimes_\pi X_*(\mathcal{T}\mathcal{A}))$$

$$\simeq \lim_{n\to\infty} H^*(M_n, \mathbb{C}) \otimes Mor_{\mathfrak{D}}(\mathbb{C}, X_*(\mathcal{T}\mathcal{A}))$$

$$\simeq \lim_{n\to\infty} H^*(M_n, \mathbb{C}) \otimes HC_*^{loc}(\mathcal{A}) \simeq H^*(X, HC_*^{loc}(\mathcal{A}))$$

Finally assertion c) follows from d) and e) by taking $\mathcal{A} = \mathbb{C}$, $\mathcal{B} = C(Y)$ while a) and b) are special cases of c). This establishes the theorem for compact spaces. The locally compact case follows easily by using the excision property of local cyclic cohomology.

□

8.3 Reduced C^*-algebras of free groups

We will calculate the local cyclic cohomology of the reduced group C^*-algebra of a finitely generated free group. (See also [Pu], Chapter 11.)
Let F_n be a free group on n generators. The action by left translation induces a unitary action on the Hilbert space $\mathcal{H} = \ell^2(F_n)$ of square integrable functions. Consider the corresponding representation of the group algebra $\mathbb{C}[F_n]$ on $\mathcal{H}$. The enveloping C^*-algebra of its image is the reduced group-C^*-algebra $C^*_r(F_n)$. It is well known that $C^*_r(F_n)$ possesses the Grothendieck approximation property [Ha1]. So we can make use of the approximation theorem (6.13). Moreover the dense subalgebra $\mathbb{C}[F_n]$ is quasifree [CQ]. The formal part of our calculation is the content of

Lemma 8.8. *Let R be a dense, finitely generated, unital, and quasifree subalgebra of the nice Fréchet algebra A with open unit ball U. Suppose that A possesses the Grothendieck approximation property. Let V be a finite dimensional subspace containing 1 and generating R as an algebra and denote by A_n the completion of R with respect to the largest submultiplicative seminorm satisfying $\| V^n \cap U \| \leq 1 + \frac{1}{n}$. Then the canonical morphisms of ind-complexes*

$$\text{“}\lim_{n\to\infty}\text{”} X_*(A_n) \overset{\simeq}{\longleftarrow} \underset{n\to\infty}{Lim}\, X_*(\mathcal{T}A_n) \overset{\simeq}{\longrightarrow} X_*(\mathcal{T}A)$$

are isomorphisms in the derived ind-category.

Proof: Because R is quasifree, there exists a connection ∇ in the sense of Cuntz-Quillen [CQ] on $\Omega^1 R$. It extends to a connection on $\Omega^+ R$ by the formula $\nabla(a^0 da^1 \dots da^n) := a^0 \nabla(da^1) da^2 \dots da^n$. A connection gives rise to a contracting chain homotopy of the subcomplex $Fil^2_{Hodge}\widehat{CC}_*(R)$ of the periodic cyclic bicomplex of R, which is given by the formula $h = \sum_{k=0}^{\infty} (-\nabla \circ B)^k \circ \nabla$, as well as to an explicit linear section s of the quotient map $p : \widehat{CC}_*(R) \to X_*(R)$ satisfying $s \circ p = Id - b \circ \nabla$ on forms of degree one. Under the assumptions of the lemma these linear maps extend for given n to bounded linear operators $h : Fil^2_{Hodge} CC^{an}_*(A_n) \longrightarrow Fil^2_{Hodge} CC^{an}_{*+1}(A_{n'})$ respectively $s : X_*(A_n) \longrightarrow CC^{an}_*(A_{n'})$, $n' >> n$. In order to show this one writes every element of R as a linear combination of products of a finite generating set $S \subset V \cap U$ and makes use of the formula

$$\begin{aligned} \nabla(d(s_1 \dots s_N)) &= \sum s_1 \dots s_{k-1} \nabla(ds_k) s_{k+1} \dots s_N \\ &+ \sum s_1 \dots s_{k-1} ds_k d(s_{k+1} \dots s_N) \end{aligned}$$

Note that because the elements $s_1, \dots, s_N$ belong to a finite subset of R, the differential forms $\nabla(ds_k) \in \Omega^2 R$ are finite in number. Details of the straightforward calculation can be found in [Pu], (11.22), (11.23). It follows from this result that $\underset{n\to\infty}{Lim}\, CC^{an}_*(A_n) \sim \underset{n\to\infty}{Lim}\, X_*(\mathcal{T}A_n) \longrightarrow$ "$\lim_{n\to\infty}$" $X_*(A_n)$ is a chain homotopy equivalence of ind-complexes. This establishes the first part of the assertion. The second part follows from the limit theorem (6.16). □

Unfortunately it is often difficult to apply this result directly because one has no information about the auxiliary algebras A_n used in the lemma. In the case $A = C^*_r(F_n)$, $R = \mathbb{C}[F_n]$, I do not see how to calculate the homology of the complexes $X_*(A_n)$ directly. It seems therefore to be preferable to pass first of all to a sufficiently large but well understood Banach subalgebra of $C^*_r(F_n)$ containing $\mathbb{C}[F_n]$, and to apply the previous lemma to the latter subalgebra. Such a good Banach subalgebra has been constructed by Haagerup [Ha1].

Proposition 8.9. *(Haagerup) Let F_n be a free group on n generators $s_1, \dots s_n$ and let $|-|_S$ be the corresponding word length function. Let $\mathcal{A}(F_n)$ be the completion of the group ring $\mathbb{C}[F_n]$ with respect to the seminorms*

$$\| \sum a_g u_g \|_k^2 = \sum |a_g|^2 \cdot (1 + |g|_S)^{2k}, \;\; k \in \mathbb{N},$$

Then $\mathcal{A}(F_n)$ is a nice Fréchet subalgebra of the reduced group C^*-algebra $C^*_r(F_n)$. Moreover it coincides with the domain of an unbounded derivation on $C^*_r(F_n)$.

Applying the smooth subalgebra theorem (3.8) and lemma (8.8), we deduce from Haagerup's result

Proposition 8.10. *Let F_n be a free group on n generators $s_1, \dots, s_n$. Let $\mathcal{A}(F_n)$ be the associated Haagerup algebra and let V be the linear span of $s_1^{\pm 1}, \dots, s_n^{\pm 1}$ in $\mathbb{C}[\Gamma] \subset \mathcal{A}(F_n)$. Let $\mathcal{A}_k(F_n)$ be the Banach subalgebras of $\mathcal{A}(F_n)$ introduced in lemma (8.8). Then there is an isomorphism*

$$\text{“}\lim_{k\to\infty}\text{”}\, X_*(\mathcal{A}_k(F_n)) \quad \simeq \quad X_*(\mathcal{T}C^*_r(F_n))$$

in the derived ind-category.

Lemma 8.11. *In the notations of 8.10 the continuous linear map*

$$\begin{array}{ccc} \mathcal{A}_k(F_n)^n & \longrightarrow & X_1(\mathcal{A}_k(F_n)) \\ (a_1, \dots, a_n) & \longrightarrow & a_1 ds_1 + \dots + a_n ds_n \end{array}$$

induces an isomorphism

$$\text{“}\lim_{k\to\infty}\text{”}\, \mathcal{A}_k(F_n)^n \xrightarrow{\simeq} \text{“}\lim_{k\to\infty}\text{”}\, X_1(\mathcal{A}_k(F_n))$$

of ind-Fréchet spaces.

PROOF: We use the notations of (8.8) and (8.10). Let ∇ be the unique connection on $\Omega^1(\mathbb{C}[F_n])$ satisfying $\nabla(ds_i) = 0$ for $i = 1, \ldots, n$. The image of the associated linear embedding $s : X_1(\mathbb{C}[F_n]) \longrightarrow \Omega^1(\mathbb{C}[F_n])$ coincides then with the subspace $(Id - b \circ \nabla)\Omega^1(\mathbb{C}[F_n]) = \mathbb{C}[F_n]ds_1 + \ldots + \mathbb{C}[F_n]ds_n$ of $\Omega^1(\mathbb{C}[F_n])$. The lemma follows from the fact (8.8) that s extends to a bounded morphism "$\lim\limits_{k\to\infty}$" $X_1(\mathcal{A}_k(F_n)) \longrightarrow$ "$\lim\limits_{k\to\infty}$" $\Omega^1\mathcal{A}_k(F_n)$ of ind-Fréchet spaces (8.8). □

From now on this identification of "$\lim\limits_{k\to\infty}$" $X_1(\mathcal{A}_k(F_n))$ will be understood. We determine the homotopy type of the ind-complex "$\lim\limits_{k\to\infty}$" $X_*(\mathcal{A}_k(F_n))$ in two steps.

LEMMA 8.12. *Let F_n be a free group on n generators $s_1, \ldots, s_n$.*
Let $h' : X_*(\mathbb{C}[F_n]) \to X_{*+1}(\mathbb{C}[F_n])$ be the linear operator which vanishes on X_1 and maps the element $g \in F_n \subset \mathbb{C}[F_n] = X_0(\mathbb{C}[F_n])$ to $uvd(u^{-1}) \in X_1(\mathbb{C}[F_n])$ if $g = uvu^{-1}$ is the unique reduced presentation of g in terms of the generators $s_1^{\pm 1}, \ldots, s_n^{\pm 1}$ such that the first letter of v is different from the inverse of its last letter.

a) The operator $\pi' := Id - (h' \circ \partial + \partial \circ h')$ defines a deformation retraction of $X_*(\mathbb{C}[F_n])$ onto the direct sum $X'_*(\mathbb{C}[F_n])_{hom} \bigoplus X'_*(\mathbb{C}[F_n])_{inhom}$ of the following subcomplexes. The finite dimensional subcomplex $X'_*(\mathbb{C}[F_n])_{hom}$ which is given by the linear span of $1 \in X_0(\mathbb{C}[\Gamma])$ and the finite set $\{s_i^{-1}ds_i,\ i = 1, \ldots, n\} \subset X^1(\mathbb{C}[\Gamma])$. It has vanishing differential and is thus isomorphic to the $\mathbb{Z}/2\mathbb{Z}$-graded vector space $H_*(F_n, \mathbb{C})$, viewed as trivial chain complex. The subcomplex $X'_*(\mathbb{C}[F_n])_{inhom}$ which is given by the linear span of the nontrivial elements $g \in F_n \subset X_0(\mathbb{C}[F_n])$, for which the first letter of the reduced word representing g is different from the inverse of its last letter, and of the elements of the form $g'ds_i$, $g''d(s_i^{-1}) \in X^1(\mathbb{C}[F_n])$, $i = 1, \ldots, n$ such that the first and last letter of the reduced word representing g' (respectively g'') is different from s_i^{-1} (respectively s_i).

b) The operator π' is continuous in the sense that it gives rise to a deformation retraction of completed complexes

$$\pi' : \quad \text{“}\lim_{k\to\infty}\text{”}\, X_*(\mathcal{A}_k(F_n)) \quad \longrightarrow \quad \begin{array}{c} \text{“}\lim\limits_{k\to\infty}\text{”}\, X'_*(\mathcal{A}_k(F_n))_{hom} \\ \oplus \\ \text{“}\lim\limits_{k\to\infty}\text{”}\, X'_*(\mathcal{A}_k(F_n))_{inhom} \end{array}$$

This follows from a straightforward calculation.

LEMMA 8.13. *There is an isomorphism of ind-complexes*

$$\text{“}\lim_{k\to\infty}\text{”}\, X'_*(\mathcal{A}_k(F_n))_{hom} \simeq H_*(F_n, \mathbb{C}) \simeq \mathbb{C} \oplus \mathbb{C}^n[1]$$

whereas the ind-complex "$\lim\limits_{k\to\infty}$" $X'_*(\mathcal{A}_k(F_n))_{inhom}$ is contractible.

Proof: We show the second assertion, the first being obvious from the definitions made in (8.12). Let g, $g'ds_i$, $g''ds_i^{-1}$, $(g, g', g'' \in F_n)$ be generating elements of the complex $X'_*(\mathbb{C}[F_n])_{inhom}$ and suppose that they are represented by reduced words. Then there occur no cancellations under multiplication of $(g', s_i) \to g's_i$ and $(g'', s_i^{-1}) \to g''s_i^{-1}$ and under cyclic permutations of the letters of g, $g's_i$ and $g''s_i^{-1}$. Due to this the underlying spaces of the complex $X'_*(\mathbb{C}[F_n])_{inhom}$ can be interpreted as subspaces of the tensor algebra over the vector space with basis $s_1^{\pm 1}, \ldots, s_n^{\pm 1}$ and the differentials can be described in terms of the action of the appropriate cyclic group on the tensor powers of the basis elements. Thus one finds that the differential $\partial_0 : X'_0(\mathbb{C}[F_n])_{inhom} \to X'_1(\mathbb{C}[F_n])_{inhom}$ corresponds to the cyclic averaging operator N and that the differential $\partial_1 : X'_1(\mathbb{C}[F_n])_{inhom} \to X'_0(\mathbb{C}[F_n])_{inhom}$ corresponds to the operator $1-T$ where T generates the cyclic action. This shows that $X'_*(\mathbb{C}[F_n])_{inhom}$ is acyclic, i.e. has vanishing homology. A contracting homotopy operator can be given on generating elements of length $n+1$ by the formulas $h''_0 = \frac{1}{n+1}(T^{n-1}+2T^{n-2}+\ldots+(n-1)T+1)$, respectively $h''_1 = \frac{1}{n+1}\cdot 1$. A simple calculation shows that this contracting homotopy operator is continuous with respect to the topology of the ind-complex "$\lim\limits_{k\to\infty}$" $X'_*(\mathcal{A}_k(F_n))_{inhom}$, whence the result. □

We can summarize now what we have obtained in the following

Theorem 8.14. *a) Let F_n be a free group on n generators and let $C^*_r(F_n)$ be its reduced group C^*-algebra. Then there is a canonical isomorphism*

$$X_*(\mathcal{T}C^*_r(F_n)) \quad \overset{\simeq}{\longleftarrow} \quad H_*(F_n, \mathbb{C}) \quad \simeq \quad \mathbb{C} \bigoplus \mathbb{C}^n[1]$$

in the derived ind-category.

b) Let F', F'' be finitely generated free groups and let $\mathcal{A}$, $\mathcal{B}$ be nice ind-Fréchet algebras. Then there is a canonical isomorphism between

$$HC^{loc}_*(C^*_r(F') \otimes_\pi \mathcal{A}, \, C^*_r(F'') \otimes_\pi \mathcal{B})$$

and

$$Hom^*(H_*(F', \mathbb{C}), \, H_*(F'', \mathbb{C})) \otimes HC^{loc}_*(\mathcal{A}, \, \mathcal{B})$$

which is natural in $\mathcal{A}$ and $\mathcal{B}$.

Proof: The first assertion follows from (8.10), (8.12) and (8.13). The Eilenberg-Zilber theorem for cyclic bicomplexes provides a chain homotopy equivalence (5.16)

$$X_*(\mathcal{T}(C^*_r(F) \otimes_\pi \mathcal{A})) \overset{\simeq}{\longrightarrow} X_*(\mathcal{T}(C^*_r(F)) \otimes_\pi X_*(\mathcal{T}\mathcal{A})$$

A careful look at the morphism $X_*(\mathcal{T}C^*_r(F_n)) \overset{\simeq}{\longleftarrow} H_*(F_n, \mathbb{C})$ of the first assertion shows that it is the composition of a morphism with weakly contractible mapping cone and a series of chain homotopy equivalences. Therefore its mapping cone is weakly contractible. Thus the isomorphism criterion (2.10) applies to the chain map $H_*(F_n, \mathbb{C}) \otimes X_*(\mathcal{T}\mathcal{A}) \longrightarrow X_*(\mathcal{T}(C^*_r(F)) \otimes_\pi X_*(\mathcal{T}\mathcal{A})$ showing that the latter is an isomorphism in the derived ind-category. □

8.4 n-traces and analytic traces on Banach algebras

In his monumental paper [Co3] Alain Connes introduced a special type of densely defined unbounded cyclic cocycles on Banach algebras, called n-traces. Every n-trace defines an additive functional on the K-theory of the underlying algebra, and represents thus a sensitive tool to detect nontrivial elements in K-groups. In [Co3] it is asked whether n-traces can be viewed as cocycles of a suitable cohomology theory and in particular how to define a cohomology relation between n-traces. We give here a partial answer for algebras with approximation property. This was inspired by a remark of Alain Connes.
First we recall the

Definition 8.15. Let A be a Banach algebra.

a) (A. Connes) [Co3]
An n-trace on A is a cyclic n-cocycle $\tau : \Omega^n\mathfrak{A} \to \mathbb{C}$ on a dense subalgebra $\mathfrak{A}$ of A such that for any $a^1, \ldots, a^n \in \mathfrak{A}$ there exists $C(a^1, \ldots, a^n) < \infty$ such that

$$|\tau((x^1da^1)\cdot(x^2da^2)\cdot\ldots\cdot(x^nda^n))| \leq C(a^1,\ldots,a^n)\cdot\|x^1\|_A\cdot\ldots\cdot\|x^n\|_A$$

for all $x_i \in \mathfrak{A}$.

b) An analytic trace on A is a cocycle τ' on the cyclic bicomplex $CC_*(\mathfrak{A})$ of a dense subalgebra $\mathfrak{A}$ of A, such that for every finite subset $S \subset \mathfrak{A}$ there exist constants $C_n(S), n \in \mathbb{N}$, satisfying

$$|\tau'((x^1da^1)\cdot(x^2da^2)\cdot\ldots\cdot(x^nda^n))| \leq C_n(S)\cdot(\frac{n}{2})!\cdot\|x^1\|_A\cdot\ldots\cdot\|x^n\|_A$$

for all $x^1,\ldots,x^n \in \mathfrak{A}$, $a^1,\ldots,a^n \in S$ and

$$\lim_{n\to\infty} C_n(S)^{\frac{1}{n}} = 0$$

In particular every n-trace is analytic.

Now our result is

Theorem 8.16. *Let A be a separable Banach algebra with approximation property. Then every analytic trace τ_n on A defines a unique local cyclic cohomology class*

$$[\tau_n] \in HC^n_{loc}(A)$$

The linear functional on $K_n(A)$ associated to τ_n by [Co3] coincides with the Chern character pairing (5.17) with the class $[\tau_n]$ in local cyclic cohomology.

Proof: Let τ be an analytic trace on A. Denote by $\mathfrak{A}$ its dense domain of definition and let $0 \subset V_1 \subset V_2 \subset \ldots \subset V_m \subset \ldots$ be a chain of finite dimensional subspaces of $\mathfrak{A}$ whose union is a dense subalgebra of A. Following

(6.13) choose strictly monotone decreasing sequences (λ_n), (r_n), $n \in \mathbb{N}$, such that $\lim\limits_{n\to\infty} \lambda_n = 1$ and $\lim\limits_{n\to\infty} r_n = 0$, and denote by A_n the completion of the subalgebra $A[V_n]$ of $\mathfrak{A} \subset A$ generated by V_n with respect to the largest submultiplicative seminorm satisfying $\| V_n \cap \overline{U} \| \leq \lambda_n$ (U the open unit ball of A). Denote by $(TA_n)^{(r_n)}$ the completed tensor algebras introduced in (1.22). We claim that the analytic trace τ defines a cocycle on the ind-complex

$$\text{“}\lim_{n\to\infty}\text{”}\, X_*((TA_n)^{r_n})$$

The natural inclusions of algebras $TA[V_1] \subset \ldots \subset TA[V_n] \subset \ldots \subset T\mathfrak{A}$ induce chain maps $\lim\limits_{n\to\infty} X_*(TA[V_n]) \longrightarrow \lim\limits_{n\to\infty} CC_*(A[V_n]) \longrightarrow CC_*(\mathfrak{A})$ where the first chain map is the normalized Cuntz-Quillen projection. The analytic trace yields therefore a cocycle $\tau' \in \lim\limits_{\overleftarrow{n}} X^*(TA[V_n])$ and it remains to check that τ' is continuous, i.e. extends to a functional on the completion $X_*((TA_n)^{r_n})$ of $X_*(TA_n)$. As the Cuntz-Quillen projection is continuous ([Pu], 5.25) it suffices to prove the estimates

$$|\tau(a^0 da^1 \ldots da^k)| \leq C(m) \cdot (\frac{k}{2})! \cdot (r_n)^{\frac{k}{2}}$$

for some constant $C(m)$ and all $k \in \mathbb{N}$, $a^0, \ldots, a^k \in K_n^\infty$, the multiplicative closure of $K_n := V_n \cap \lambda_n^{-1}\overline{U} \subset \mathfrak{A}$. The set $K_n \subset V_n$ being bounded, there exist finitely many elements $c_1, \ldots, c_l \in V_n$ such that K_n is contained in the circled convex hull of $S := \{c_1, \ldots, c_l\}$. The estimate $|\phi_k((x^1 da^1) \cdot \ldots \cdot (x^k da^k))| \leq C_k(S) \cdot (\frac{k}{2})! \cdot \|x^1\| \cdot \ldots \cdot \|x^k\|$ for all $x^1, \ldots, x^k \in \mathfrak{A}$ and $a^1, \ldots, a^k \in K_n$ follows. Let now $a^1, \ldots, a^k \in K_n^\infty$. This means that these elements can be written as products $a^j = b_j^1 \ldots b_j^{l_j}$ with $b_j^i \in K_n$. In particular $\mathrm{d}a^j = \Sigma_{i=1}^{l_j} b_j^1 \ldots b_j^{i-1} db_j^i b_j^{i+1} \ldots b_j^{l_j}$. As by construction $\|K_n\|_A \leq \lambda_n^{-1}$ the continuity property i) of the analytic trace implies

$$|\tau(a^0 da^1 \ldots da^k)| \leq (\Pi_{j=1}^k l_j) \cdot \lambda_n^{-(\Sigma_{j=1}^k l_j - k)} \cdot (\frac{k}{2})! \cdot C_k(S)$$

$$\leq (\frac{k}{2})! \cdot C_k(S) \cdot (\lambda_n)^k \cdot \Pi_{j=1}^k l_j (\lambda_n)^{-l_j} \;\leq\; (\frac{k}{2})! \cdot C_k(S) \cdot C(\lambda_n)^k$$

for a suitable constant $C(\lambda_n)$ and all k.
Because $(C_k(S)^{\frac{1}{k}} \cdot r_n^{-1} \cdot C(\lambda_n))^k \leq C'(n)$ by condition ii), one has

$$|\tau(a^0 da^1 \ldots da^k)| \leq C'(n) \cdot (\frac{k}{2})! \cdot r_n^k$$

for all k, and the claim follows. Thus τ defines an element of

$$Mor^*_{\mathfrak{Ho}}(\text{“}\lim_{n\to\infty}\text{”}\, X_*((TA_n)^{r_n}), \mathbb{C})$$

As the canonical morphism

$$“\lim_{n\to\infty}” X_*((TA_n)^{r_n}) \longrightarrow X_*(\mathcal{T}A)$$

is an isomorphism in the derived ind-category by the approximation theorem (6.13), the analytic trace τ defines a cohomology class

$$[\tau] \in Mor^*_{\mathfrak{D}}(“\lim_{n\to\infty}” X_*((TA_n)^{r_n}), \mathbb{C}) \simeq Mor^*_{\mathfrak{D}}(X_*(\mathcal{T}A), \mathbb{C}) = HC^*_{loc}(A)$$

This establishes the theorem.

$\square$

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