# Stability of Arakelov Bundles and Tensor Products without Global Sections

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Received: June 12, 2003

Communicated by Ulf Rehmann

Abstract. This paper deals with Arakelov vector bundles over an arithmetic curve, i.e. over the set of places of a number field. The main result is that for each semistable bundle E, there is a bundle F such that $E \otimes F$ has at least a certain slope, but no global sections. It is motivated by an analogous theorem of Faltings for vector bundles over algebraic curves and contains the Minkowski-Hlawka theorem on sphere packings as a special case. The proof uses an adelic version of Siegel's mean value formula.

2000 Mathematics Subject Classification: Primary 14G40; Secondary 11H31, 11R56.

Keywords and Phrases: Arakelov bundle, arithmetic curve, tensor product, lattice sphere packing, mean value formula, Minkowski-Hlawka theorem

## Introduction

G. Faltings has proved that for each semistable vector bundle $E$ over an algebraic curve of genus $g$, there is another vector bundle $F$ such that $E \otimes F$ has slope $g-1$ and no global sections. (Note that any vector bundle of slope $> g-1$ has global sections by Riemann–Roch.) See [3] and [4] where this result is interpreted in terms of theta functions and used for a new construction of moduli schemes of vector bundles.

In the present paper, an arithmetic analogue of that theorem is proposed. The algebraic curve is replaced by the set $X$ of all places of a number field $K$; we call $X$ an arithmetic curve. Vector bundles are replaced by so-called Arakelov bundles, cf. section 3. In the special case $K = \mathbb{Q}$, Arakelov bundles without

global sections are lattice sphere packings, and the slope $\mu$ measures the packing density.
We will see at the end of section 4 that the maximal slope of Arakelov bundles of rank $n$ without global sections is $d(\log n + \mathrm{O}(1))/2 + (\log \mathfrak{d})/2$ where $d$ is the degree and $\mathfrak{d}$ is the discriminant of $K$. Now the main result is:

Theorem 0.1 *Let $\mathcal{E}$ be a semistable Arakelov bundle over the arithmetic curve $X$. For each $n \gg 0$ there is an Arakelov bundle $\mathcal{F}$ of rank $n$ satisfying*

$$\mu(\mathcal{E} \otimes \mathcal{F}) > \frac{d}{2}(\log n - \log \pi - 1 - \log 2) + \frac{\log \mathfrak{d}}{2}$$

*such that $\mathcal{E} \otimes \mathcal{F}$ has no nonzero global sections.*

The proof is inspired by (and generalises) the Minkowski-Hlawka existence theorem for sphere packings; in particular, it is not constructive. The principal ingredients are integration over a space of Arakelov bundles (with respect to some Tamagawa measure) and an adelic version of Siegel's mean value formula. Section 2 explains the latter, section 3 contains all we need about Arakelov bundles, and the main results are proved and discussed in section 4.
This paper is a condensed and slightly improved part of the author's Ph. D. thesis [6]. I would like to thank my adviser G. Faltings for his suggestions; the work is based on his ideas. It was supported by a grant of the Max-Planck-Institut in Bonn.

## 1 Notation

Let $K$ be a number field of degree $d$ over $\mathbb{Q}$ and with ring of integers $\mathcal{O}_K$. Let $X = \mathrm{Spec}(\mathcal{O}_K) \cup X_\infty$ be the set of places of $K$; this might be called an 'arithmetic curve' in the sense of Arakelov geometry. $X_\infty$ consists of $r_1$ real and $r_2$ complex places with $r_1 + 2r_2 = d$. $w(K)$ is the number of roots of unity in $K$.
For every place $v \in X$, we endow the corresponding completion $K_v$ of $K$ with the map $|\,.\,|_v : K_v \to \mathbb{R}_{\geq 0}$ defined by $\mu(a \cdot S) = |a|_v \cdot \mu(S)$ for a Haar measure $\mu$ on $K_v$. This is the normalised valuation if $v$ is finite, the usual absolute value if $v$ is real and its square if $v$ is complex. The well known product formula $\prod_{v \in X} |a|_v = 1$ holds for every $0 \neq a \in K$. On the adele ring $\mathbb{A}$, we have the divisor map $\mathrm{div} : \mathbb{A} \to \mathbb{R}^X_{\geq 0}$ that maps each adele $a = (a_v)_{v \in X}$ to the collection $(|a_v|_v)_{v \in X}$ of its valuations.
Let $\mathcal{O}_v$ be the set of those $a \in K_v$ which satisfy $|a|_v \leq 1$; this is the ring of integers in $K_v$ for finite $v$ and the unit disc for infinite $v$. Let $\mathcal{O}_\mathbb{A}$ denote the product $\prod_{v \in X} \mathcal{O}_v$; this is the set of all adeles $a$ with $\mathrm{div}(a) \leq 1$. By $D \leq 1$ for an element $D = (D_v)_{v \in X}$ of $\mathbb{R}^X_{\geq 0}$, we always mean $D_v \leq 1$ for all $v$.
We fix a canonical Haar measure $\lambda_v$ on $K_v$ as follows:

- If $v$ is finite, we normalise by $\lambda_v(\mathcal{O}_v) = 1$.
- If $v$ is real, we take for $\lambda_v$ the usual Lebesgue measure on $\mathbb{R}$.

- If $v$ is complex, we let $\lambda_v$ come from the real volume form $idz \wedge d\bar{z}$ on $\mathbb{C}$. In other words, we take twice the usual Lebesgue measure.

This gives us a canonical Haar measure $\lambda := \prod_{v \in X} \lambda_v$ on $\mathbb{A}$. We have $\lambda(\mathbb{A}/K) = \sqrt{\mathfrak{d}}$ where $\mathfrak{d} = \mathfrak{d}_{K/\mathbb{Q}}$ denotes (the absolute value of) the discriminant. More details on this measure can be found in [12], section 2.1.

Let $V_n = \frac{\pi^{n/2}}{(n/2)!}$ be the volume of the unit ball in $\mathbb{R}^n$. For $v \in X_\infty$, we denote by $\mathcal{O}_v^n$ the unit ball with respect to the standard scalar product on $K_v^n$. Observe that this is *not* the $n$-fold Cartesian product of $\mathcal{O}_v \subseteq K_v$. Similarly, $\mathcal{O}_{\mathbb{A}}^n := \prod_{v \in X} \mathcal{O}_v^n$ is not the $n$-fold product of $\mathcal{O}_{\mathbb{A}} \subseteq \mathbb{A}$. Its volume $\lambda^n(\mathcal{O}_{\mathbb{A}}^n)$ is $V_n^{r_1}(2^n V_{2n})^{r_2}$.

## 2 A mean value formula

The following proposition is a generalisation of Siegel's mean value formula to an adelic setting: With real numbers and integers instead of adeles and elements of $K$, Siegel has already stated it in [10], and an elementary proof is given in [7]. (In the special case $l = 1$, a similar question is studied in [11].)

Proposition 2.1 *Let $1 \leq l < n$, and let $f$ be a nonnegative measurable function on the space $\mathrm{Mat}_{n\times l}(\mathbb{A})$ of $n \times l$ adele matrices. Then*

$$\int\limits_{\mathrm{Sl}_n(\mathbb{A})/\mathrm{Sl}_n(K)} \sum_{\substack{M \in \mathrm{Mat}_{n\times l}(K) \\ \mathrm{rk}(M) = l}} f(g \cdot M)\, d\tau(g) = \mathfrak{d}^{-nl/2} \int\limits_{\mathrm{Mat}_{n\times l}(\mathbb{A})} f\, d\lambda^{n\times l} \tag{1}$$

*where $\tau$ is the unique $\mathrm{Sl}_n(\mathbb{A})$-invariant probability measure on* $\mathrm{Sl}_n(\mathbb{A})/\mathrm{Sl}_n(K)$.

*Proof:* The case $l = 1$ is done in section 3.4 of [12], and the general case can be deduced along the same lines from earlier sections of this book. We sketch the main arguments here; more details are given in [6], section 3.2.

Let $G$ be the algebraic group $\mathrm{Sl}_n$ over the ground field $K$, and denote by $\tau_G$ the Tamagawa measure on $G(\mathbb{A})$ or any quotient by a discrete subgroup. The two measures $\tau$ and $\tau_G$ on $\mathrm{Sl}_n(\mathbb{A})/\mathrm{Sl}_n(K)$ coincide because the Tamagawa number of $G$ is one.

$G$ acts on the affine space $\mathrm{Mat}_{n\times l}$ by left multiplication. Denote the first $l$ columns of the $n \times n$ identity matrix by $E \in \mathrm{Mat}_{n\times l}(K)$, and let $H \subseteq G$ be the stabiliser of $E$. This algebraic group $H$ is a semi-direct product of $\mathrm{Sl}_{n-l}$ and $\mathrm{Mat}_{l\times(n-l)}$. Hence section 2.4 of [12] gives us a Tamagawa measure $\tau_H$ on $H(\mathbb{A})$, and the Tamagawa number of $H$ is also one.

Again by section 2.4 of [12], we have a Tamagawa measure $\tau_{G/H}$ on $G(\mathbb{A})/H(\mathbb{A})$ as well, and it satisfies $\tau_G = \tau_{G/H} \cdot \tau_H$ in the sense defined there. In particular, this implies

$$\int\limits_{G(\mathbb{A})/H(K)} f(g \cdot E)\, d\tau_G(g) = \int\limits_{G(\mathbb{A})/H(\mathbb{A})} f(g \cdot E)\, d\tau_{G/H}(g).$$

It is easy to see that the left hand sides of this equation and of (1) coincide. According to lemma 3.4.1 of [12], the right hand sides coincide, too. □

## 3 Arakelov vector bundles

Recall that a (Euclidean) lattice is a free $\mathbb{Z}$-module $\Lambda$ of finite rank together with a scalar product on $\Lambda \otimes \mathbb{R}$. This is the special case $K = \mathbb{Q}$ of the following notion:

Definition 3.1 An *Arakelov (vector) bundle* $\mathcal{E}$ over our arithmetic curve $X = \mathrm{Spec}(\mathcal{O}_K) \cup X_\infty$ is a finitely generated projective $\mathcal{O}_K$-module $\mathcal{E}_{\mathcal{O}_K}$ endowed with

- a Euclidean scalar product $\langle \_, \_ \rangle_{\mathcal{E},v}$ on the real vector space $\mathcal{E}_{K_v}$ for every real place $v \in X_\infty$ and
- a Hermitian scalar product $\langle \_, \_ \rangle_{\mathcal{E},v}$ on the complex vector space $\mathcal{E}_{K_v}$ for every complex place $v \in X_\infty$

where $\mathcal{E}_A := \mathcal{E}_{O_K} \otimes A$ for every $\mathcal{O}_K$-algebra $A$.

A first example is the trivial Arakelov line bundle $\mathcal{O}$. More generally, the trivial Arakelov vector bundle $\mathcal{O}^n$ consists of the free module $\mathcal{O}_K^n$ together with the standard scalar products at the infinite places.
We say that $\mathcal{E}'$ is a subbundle of $\mathcal{E}$ and write $\mathcal{E}' \subseteq \mathcal{E}$ if $\mathcal{E}'_{\mathcal{O}_K}$ is a direct summand in $\mathcal{E}_{\mathcal{O}_K}$ and the scalar product on $\mathcal{E}'_{K_v}$ is the restriction of the one on $\mathcal{E}_{K_v}$ for every infinite place $v$. Hence every vector subspace of $\mathcal{E}_K$ is the generic fibre of one and only one subbundle of $\mathcal{E}$.
From the data belonging to an Arakelov bundle $\mathcal{E}$, we can define a map

$$\| \,.\, \|_{\mathcal{E},v} : \mathcal{E}_{K_v} \longrightarrow \mathbb{R}_{\geq 0}$$

for every place $v \in X$:

- If $v$ is finite, let $\|e\|_{\mathcal{E},v}$ be the minimum of the valuations $|a|_v$ of those elements $a \in K_v$ for which $e$ lies in the subset $a \cdot \mathcal{E}_{\mathcal{O}_v}$ of $\mathcal{E}_{K_v}$. This is the nonarchimedean norm corresponding to $\mathcal{E}_{\mathcal{O}_v}$.
- If $v$ is real, we put $\|e\|_{\mathcal{E},v} := \sqrt{\langle e, e \rangle_v}$, so we just take the norm coming from the given scalar product.
- If $v$ is complex, we put $\|e\|_{\mathcal{E},v} := \langle e, e \rangle_v$ which is the square of the norm coming from our Hermitian scalar product.

Taken together, they yield a divisor map

$$\mathrm{div}_{\mathcal{E}} : \mathcal{E}_{\mathbb{A}} \to \mathbb{R}^X_{\geq 0} \qquad\qquad e = (e_v) \mapsto (\|e_v\|_{\mathcal{E},v}).$$

Although $\mathcal{O}_{\mathbb{A}}$ is not an $\mathcal{O}_K$-algebra, we will use the notation $\mathcal{E}_{\mathcal{O}_{\mathbb{A}}}$, namely for the compact set defined by

$$\mathcal{E}_{\mathcal{O}_{\mathbb{A}}} := \{e \in \mathcal{E}_{\mathbb{A}} : \mathrm{div}_{\mathcal{E}}(e) \leq 1\}.$$

Recall that these norms are used in the definition of the Arakelov degree: If $\mathcal{L}$ is an Arakelov line bundle and $0 \neq l \in \mathcal{L}_K$ a nonzero generic section, then

$$\deg(\mathcal{L}) := -\log \prod_{v \in X} \|l\|_{\mathcal{L},v}$$

and the degree of an Arakelov vector bundle $\mathcal{E}$ is by definition the degree of the Arakelov line bundle $\det(\mathcal{E})$. $\mu(\mathcal{E}) := \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E})$ is called the slope of $\mathcal{E}$. One can form the tensor product of two Arakelov bundles in a natural manner, and it has the property $\mu(\mathcal{E} \otimes \mathcal{F}) = \mu(\mathcal{E}) + \mu(\mathcal{F})$.

Moreover, the notion of stability is based on slopes: For $1 \leq l \leq \mathrm{rk}(\mathcal{E})$, denote by $\mu_{\max}^{(l)}$ the supremum (in fact it is the maximum) of the slopes $\mu(\mathcal{E}')$ of subbundles $\mathcal{E}' \subseteq \mathcal{E}$ of rank $l$. $\mathcal{E}$ is said to be stable if $\mu_{\max}^{(l)} < \mu(\mathcal{E})$ holds for all $l < \mathrm{rk}(\mathcal{E})$, and semistable if $\mu_{\max}^{(l)} \leq \mu(\mathcal{E})$ for all $l$.

To each projective variety over $K$ endowed with a metrized line bundle, one can associate a zeta function as in [5] or [1]. We recall its definition in the special case of Grassmannians associated to Arakelov bundles:

Definition 3.2 If $\mathcal{E}$ is an Arakelov bundle over $X$ and $l \leq \mathrm{rk}(\mathcal{E})$ is a positive integer, then we define

$$\zeta_{\mathcal{E}}^{(l)}(s) := \sum_{\substack{\mathcal{E}' \subseteq \mathcal{E} \\ \mathrm{rk}(\mathcal{E}')=l}} \exp(s \cdot \deg(\mathcal{E}')).$$

The growth of these zeta functions is related to the stability of $\mathcal{E}$. More precisely, we have the following asymptotic bound:

Lemma 3.3 *There is a constant $C = C(\mathcal{E})$ such that*

$$\zeta_{\mathcal{E}}^{(l)}(s) \leq C \cdot \exp(s \cdot l\mu_{\max}^{(l)}(\mathcal{E}))$$

*for all sufficiently large real numbers $s$.*

*Proof:* Fix $\mathcal{E}$ and $l$. Denote by $N(T)$ the number of subbundles $\mathcal{E}' \subseteq \mathcal{E}$ of rank $l$ and degree at least $-T$. There are $C_1, C_2 \in \mathbb{R}$ such that

$$N(T) \leq \exp(C_1 T + C_2)$$

holds for all $T \in \mathbb{R}$. (Embedding the Grassmannian into a projective space, this follows easily from [9]. See [6], lemma 3.4.8 for more details.)

If we order the summands of $\zeta_{\mathcal{E}}^{(l)}$ according to their magnitude, we thus get

$$\begin{aligned} \zeta_{\mathcal{E}}^{(l)}(s) &\leq \sum_{\nu=0}^{\infty} N\left(-l\mu_{\max}^{(l)}(\mathcal{E}) + \nu + 1\right) \cdot \exp\left(s \cdot (l\mu_{\max}^{(l)}(\mathcal{E}) - \nu)\right) \\ &\leq \exp(s \cdot l\mu_{\max}^{(l)}(\mathcal{E})) \cdot \sum_{\nu=0}^{\infty} \frac{C_3}{\exp((s - C_1)\nu)}. \end{aligned}$$

But the last sum is a convergent geometric series for all $s > C_1$ and decreases as $s$ grows, so it is bounded for $s \geq C_1 + 1$. □

## 4 The main theorem

The global sections of an Arakelov bundle $\mathcal{E}$ over $X = \mathrm{Spec}(\mathcal{O}_K) \cup X_\infty$ are by definition the elements of the finite set

$$\Gamma(\mathcal{E}) := \mathcal{E}_K \cap \mathcal{E}_{\mathcal{O}_{\mathbb{A}}} \subseteq \mathcal{E}_{\mathbb{A}}.$$

Note that in the special case $K = \mathbb{Q}$, an Arakelov bundle without nonzero global sections is nothing but a (lattice) sphere packing: $\Gamma(\mathcal{E}) = 0$ means that the (closed) balls of radius $1/2$ centered at the points of the lattice $\mathcal{E}_{\mathbb{Z}}$ are disjoint. Here larger degree corresponds to denser packings.

Theorem 4.1 *Let $\mathcal{E}$ be an Arakelov bundle over the arithmetic curve $X$. If an integer $n > \mathrm{rk}(\mathcal{E})$ and an Arakelov line bundle $\mathcal{L}$ satisfy*

$$1 > \sum_{l=1}^{\mathrm{rk}(\mathcal{E})} \mathfrak{d}^{-nl/2} \cdot \lambda^{nl}\left(\frac{K^*\mathcal{O}_{\mathbb{A}}^{nl}}{K^*}\right) \cdot \zeta_{\mathcal{E}}^{(l)}(n) \exp(l \deg(\mathcal{L})),$$

*then there is an Arakelov bundle $\mathcal{F}$ of rank $n$ and determinant $\mathcal{L}$ such that*

$$\Gamma(\mathcal{E} \otimes \mathcal{F}) = 0.$$

*Proof:* Note that any global section of $\mathcal{E} \otimes \mathcal{F}$ is already a global section of $\mathcal{E}' \otimes \mathcal{F}$ for a unique minimal subbundle $\mathcal{E}' \subseteq \mathcal{E}$, namely the subbundle whose generic fibre is the image of the induced map $(\mathcal{F}_K)^{\mathrm{dual}} \to \mathcal{E}_K$. We are going to average the number of these sections (up to $K^*$) for a fixed subbundle $\mathcal{E}'$ of rank $l$.

Fix one particular Arakelov bundle $\mathcal{F}$ of rank $n$ and determinant $\mathcal{L}$. Choose linear isomorphisms $\phi_{\mathcal{E}'} : K^l \to \mathcal{E}'_K$ and $\phi_{\mathcal{F}} : K^n \to \mathcal{F}_K$ and let

$$\phi : \mathrm{Mat}_{n\times l}(K) \xrightarrow{\sim} (\mathcal{E}' \otimes \mathcal{F})_K$$

be their tensor product. Our notation will not distinguish these maps from their canonical extensions to completions or adeles.

For each $g \in \mathrm{Sl}_n(\mathbb{A})$, we denote by $g\mathcal{F}$ the Arakelov bundle corresponding to the $K$-lattice $\phi_{\mathcal{F}}(gK^n) \subseteq \mathcal{F}_{\mathbb{A}}$. More precisely, $g\mathcal{F}$ is the unique Arakelov bundle satisfying $(g\mathcal{F})_{\mathbb{A}} = \mathcal{F}_{\mathbb{A}}$, $(g\mathcal{F})_{\mathcal{O}_{\mathbb{A}}} = \mathcal{F}_{\mathcal{O}_{\mathbb{A}}}$ and $(g\mathcal{F})_K = \phi_{\mathcal{F}}(gK^n)$. This gives the usual identification between $\mathrm{Sl}_n(\mathbb{A})/\mathrm{Sl}_n(K)$ and the space of Arakelov bundles of rank $n$ and fixed determinant together with local trivialisations.

Observe that the generic fibre of $\mathcal{E}' \otimes g\mathcal{F}$ is $\phi(g\mathrm{Mat}_{n\times l}(K))$. A generic section is not in $\mathcal{E}'' \otimes g\mathcal{F}$ for any $\mathcal{E}'' \subsetneq \mathcal{E}'$ if and only if the corresponding matrix has rank $l$. So according to the mean value formula of section 2, the average number of global sections

$$\int_{\mathrm{Sl}_n(\mathbb{A})/\mathrm{Sl}_n(K)} \mathrm{card}\left(\frac{K^*\Gamma(\mathcal{E}' \otimes g\mathcal{F})}{K^*} \setminus \bigcup_{\mathcal{E}'' \subsetneq \mathcal{E}'} \frac{K^*\Gamma(\mathcal{E}'' \otimes g\mathcal{F})}{K^*}\right) d\tau(g)$$

is equal to the integral

$$\mathfrak{d}^{-nl/2} \int\limits_{\mathrm{Mat}_{n\times l}(\mathbb{A})} (f_K \circ \mathrm{div}_{\mathcal{E}'\otimes\mathcal{F}} \circ \phi)\, d\lambda^{n\times l}. \tag{2}$$

Here the function $f_K : \mathbb{R}^X_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined by

$$f_K(D) := \begin{cases} 1/\mathrm{card}\{a \in K^* : \mathrm{div}(a)\cdot D \leq 1\} & \text{if } D \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

with the convention $1/\infty = 0$.
In order to compute (2), we start with the local transformation formula

$$\begin{aligned} &\lambda_v^{n\times l}\ (\{M \in \mathrm{Mat}_{n\times l}(K_v) : c_1 \leq \|\phi(M)\|_{\mathcal{E}'\otimes\mathcal{F},v} \leq c_2\}) = \\ &\lambda_v^{nl}\ (\{M \in K_v^{nl} : c_1 \leq \|M\| \leq c_2\}) \cdot \|\det(\phi)\|^{-1}_{\det(\mathcal{E}'\otimes\mathcal{F}),v} \end{aligned}$$

for all $c_1, c_2 \in \mathbb{R}_{\geq 0}$. Regarding this as a relation between measures on $\mathbb{R}_{\geq 0}$ and taking the product over all places $v \in X$, we get the equation

$$(\mathrm{div}_{\mathcal{E}'\otimes\mathcal{F}} \circ \phi)_* \lambda^{n\times l} = \exp\deg(\mathcal{E}'\otimes\mathcal{F}) \cdot (\mathrm{div}_{\mathcal{O}^{nl}})_* \lambda^{nl} \tag{3}$$

of measures on $\mathbb{R}^X_{\geq 0}$. Hence the integrals of $f_K$ with respect to these measures also coincide:

$$\int\limits_{\mathrm{Mat}_{n\times l}(\mathbb{A})} (f_K \circ \mathrm{div}_{\mathcal{E}'\otimes\mathcal{F}} \circ \phi)\, d\lambda^{n\times l} = \exp(n\deg(\mathcal{E}') + l\deg(\mathcal{F})) \cdot \lambda^{nl}\left(\frac{K^*\mathcal{O}^{nl}_{\mathbb{A}}}{K^*}\right).$$

We substitute this for the integral in (2). A summation over all nonzero subbundles $\mathcal{E}' \subseteq \mathcal{E}$ yields

$$\begin{aligned} \int\limits_{\mathrm{Sl}_n(\mathbb{A})/\mathrm{Sl}_n(K)} &\mathrm{card}\left(\frac{K^*\Gamma(\mathcal{E}\otimes g\mathcal{F})\setminus 0}{K^*}\right) d\tau(g) = \\ &= \sum_{l=1}^{\mathrm{rk}(\mathcal{E})} \mathfrak{d}^{-nl/2} \cdot \zeta^{(l)}_{\mathcal{E}}(n) \exp(l\deg(\mathcal{F})) \cdot \lambda^{nl}\left(\frac{K^*\mathcal{O}^{nl}_{\mathbb{A}}}{K^*}\right). \end{aligned}$$

But the right hand side was assumed to be less than one, so there there has to be a $g \in \mathrm{Sl}_n(\mathbb{A})$ with $\Gamma(\mathcal{E}\otimes g\mathcal{F}) = 0$. □

In order to apply this theorem, one needs to compute $\lambda^N(K^*\mathcal{O}^N_{\mathbb{A}}/K^*)$ for $N \geq 2$. We start with the special case $K = \mathbb{Q}$. Here each adele $a \in \mathcal{O}^N_{\mathbb{A}}$ outside a set of measure zero has a rational multiple in $\mathcal{O}^N_{\mathbb{A}}$ with valuation one at all finite places, and this multiple is unique up to sign. Hence we conclude

$$\lambda^N\left(\frac{\mathbb{Q}^*\mathcal{O}^N_{\mathbb{A}}}{\mathbb{Q}^*}\right) = \frac{V_N}{2} \cdot \prod_{p \text{ prime}} \lambda^N_p(\mathbb{Z}^N_p \setminus p\mathbb{Z}^N_p) = \frac{V_N}{2\zeta(N)}.$$

In particular, the special case $K = \mathbb{Q}$ and $\mathcal{E} = \mathcal{O}$ of the theorem above is precisely the Minkowski-Hlawka existence theorem for sphere packings [8], §15. For a general number field $K$, we note that the roots of unity preserve $\mathcal{O}_{\mathbb{A}}^N$. Then we apply Stirling's formula to the factorials occurring via the unit ball volumes and get

$$\lambda^N\left(\frac{K^*\mathcal{O}_{\mathbb{A}}^N}{K^*}\right) \leq \frac{\lambda^N(\mathcal{O}_{\mathbb{A}}^N)}{w(K)} \leq \left(\frac{2\pi e}{N}\right)^{dN/2} \cdot \left(\frac{1}{\pi N}\right)^{(r_1+r_2)/2} \cdot \frac{1}{2^{r_2/2}w(K)}.$$

Using such a bound and the asymptotic statement 3.3 about $\zeta_{\mathcal{E}}^{(l)}$, one can deduce the following corollary of theorem 4.1.

Corollary 4.2 *Let the Arakelov bundle $\mathcal{E}$ over $X$ be given. If $n$ is a sufficiently large integer and $\mu$ is a real number satisfying*

$$\mu_{\max}^{(l)}(\mathcal{E}) + \mu \leq \frac{d}{2}(\log n + \log l - \log \pi - 1 - \log 2) + \frac{\log \mathfrak{d}}{2}$$

*for all $1 \leq l \leq \mathrm{rk}(\mathcal{E})$, then there is an Arakelov bundle $\mathcal{F}$ of rank $n$ and slope larger than $\mu$ such that $\Gamma(\mathcal{E} \otimes \mathcal{F}) = 0$.*

If $\mathcal{E}$ is semistable, this gives the theorem 0.1 stated in the introduction. Here is some evidence that these bounds are not too far from being optimal:

Proposition 4.3 *Assume given $\epsilon > 0$ and a nonzero Arakelov bundle $\mathcal{E}$. Let $n > n(\epsilon)$ be a sufficiently large integer, and let $\mu$ be a real number such that*

$$\mu_{\max}^{(l)}(\mathcal{E}) + \mu \geq \frac{d}{2}(\log n + \log l - \log \pi - 1 + \log 2 + \epsilon) + \frac{\log \mathfrak{d}}{2}$$

*holds for at least one integer $1 \leq l \leq \mathrm{rk}(\mathcal{E})$. Then there is no Arakelov bundle $\mathcal{F}$ of rank $n$ and slope $\mu$ with $\Gamma(\mathcal{E} \otimes \mathcal{F}) = 0$.*

*Proof:* Fix such an $l$ and a subbundle $\mathcal{E}' \subseteq \mathcal{E}$ of rank $l$ and slope $\mu_{\max}^{(l)}(\mathcal{E})$. For each $\mathcal{F}$ of rank $n$ and slope $\mu$, we consider the Arakelov bundle $\mathcal{F}' := \mathcal{E}' \otimes \mathcal{F}$ of rank $nl$. By Stirling's formula, the hypotheses on $n$ and $\mu$ imply

$$\exp\deg(\mathcal{F}') \cdot \lambda^{nl}(\mathcal{O}_{\mathbb{A}}^{nl}) > 2^{nld} \cdot \mathfrak{d}^{nl/2}.$$

Now choose a $K$-linear isomorphism $\phi : K^{nl} \xrightarrow{\sim} \mathcal{F}'_K$ and extend it to adeles. Applying the global transformation formula (3), we get

$$\lambda^{nl}(\phi^{-1}\mathcal{F}'_{\mathcal{O}_{\mathbb{A}}}) > 2^{nld} \cdot \lambda^{nl}(\mathbb{A}^{nl}/K^{nl}).$$

According to Minkowski's theorem on lattice points in convex sets (in an adelic version like [11], theorem 3), $\phi^{-1}(\mathcal{F}'_{\mathcal{O}_{\mathbb{A}}}) \cap K^{nl} \neq \{0\}$ follows. This means that $\mathcal{F}'$ — and hence $\mathcal{E} \otimes \mathcal{F}$ — must have a nonzero global section. □

Observe that the lower bound 4.2 and the upper bound 4.3 differ only by the constant $d \log 2$. So up to this constant, the maximal slope of such tensor

products without global sections is determined by the stability of $\mathcal{E}$, more precisely by the $\mu^{(l)}_{\max}(\mathcal{E})$.
Taking $E = \mathcal{O}$, we get lower and upper bounds for the maximal slope of Arakelov bundles without global sections, as mentioned in the introduction. In the special case $\mathcal{E} = \mathcal{O}$ and $K = \mathbb{Q}$ of lattice sphere packings, [2] states that no essential improvement of corollary 4.2 is known whereas several people have improved the other bound 4.3 by constants.

## References


[1] *V. V. Batyrev and Yu. I. Manin,* Sur le nombre des points rationnels de hauteur borné des variétés algébriques. Math. Ann. 286 (1990), 27–43.

[2] *J. H. Conway and N. J. A. Sloane,* Sphere packings, lattices and groups. 3rd edition. Springer-Verlag, New York 1999.

[3] *G. Faltings,* Stable $G$-bundles and projective connections. J. Algebraic Geom. 2 (1993), 507–568.

[4] *G. Faltings,* Moduli-stacks for bundles on semistable curves. Math. Ann. 304 (1996), 489–515.

[5] *J. Franke, Yu. I. Manin, and Yu. Tschinkel,* Rational points of bounded height on Fano varieties. Invent. Math. 95 (1989), 421–435.

[6] *N. Hoffmann,* On vector bundles over algebraic and arithmetic curves. PhD thesis, University of Bonn, 2002. Bonner Mathematische Schriften 351.

[7] *A. M. Macbeath and C. A. Rogers,* Siegel's mean value theorem in the geometry of numbers. Proc. Cambridge Philos. Soc. 54 (1958), 139–151.

[8] *H. Minkowski,* Diskontinuitätsbereich für arithmetische Äquivalenz. J. Reine Angew. Math. 129 (1905), 220–274.

[9] *S. H. Schanuel,* Heights in number fields. Bull. Soc. Math. France 107 (1979), 433–449.

[10] *C. L. Siegel,* A mean value theorem in geometry of numbers. Ann. of Math. 46 (1945), 340–347.

[11] *J. L. Thunder,* An adelic Minkowski-Hlawka theorem and an application to Siegel's lemma. J. Reine Angew. Math. 475 (1996), 167–185.

[12] *A. Weil,* Adeles and algebraic groups. Birkhäuser, Boston 1982.



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