A Short Proof of Rost Nilpotence via Refined Correspondences

Patrick Brosnan

Received: March 15, 2003

Communicated by A. Merkurev

Abstract. I generalize the notion of composition of algebraic correspondences using the refined Gysin homorphism of Fulton-MacPherson intersection theory. Using this notion, I give a short self-contained proof of Rost's "nilpotence theorem" and a generalization of one important proposition used by Rost in his proof of the theorem.

2000 Mathematics Subject Classification: Primary 11E04; Secondary 14C25

Keywords and Phrases: quadratic forms, correspondence, Chow groups and motives

1. Introduction

In an elegant four page preprint "A shortened construction of the Rost motive" N. Karpenko (see also [4]) gives a construction of Rost's motive M_a assuming the following result of Rost widely known as the "nilpotence theorem."

Theorem 1.1. *Let Q be a smooth quadric over a field k with algebraic closure $\overline{k}$ and let $f \in \operatorname{End} M(Q)$ be an endomorphism of its integral Chow motive. Then, if $f \otimes \overline{k} = 0$ in $\operatorname{End} M(Q \otimes \overline{k})$, f is nilpotent.*

For the proof, Karpenko refers the reader to a paper of Rost which proves the theorem by invoking the fibration spectral sequence of the cycle module of a product (also due to Rost [6]). (In [7], A. Vishik gives another proof of Theorem 1.1 based on V. Voevodsky's theory of motives.)

The existence of the Rost motive and the nilpotence theorem itself are both essential to Voevodsky's proof of the Milnor conjecture. It is, therefore, desirable to have direct proofs of these fundamental results. The main goal of this paper is to provide such a proof in the spirit of Karpenko's preprint. To accomplish this, I use a generalization to singular schemes of the notion of composition of correspondences to obtain a proof of the theorem which avoids the use of cycle modules.

Both Rost's proof of Theorem 1.1 and the proof presented here involve two principal ingredients: (1) a theorem concerning nilpotent operators on $Hom(M(B), M(X))$ for B and X smooth projective varieties, (2) a decomposition theorem for the motive $M(Q)$ of a quadric Q with a k-rational point. For (1), we obtain an extension of Rost's results (Theorem 3.1) allowing the motive of B to be Tate twisted. Moreover, the method of proof can be used to extend the result to arbitrary varieties B. For (2), the theorem stated here (Theorem 4.1) is identical to Rost's, but the proof is somewhat simpler as we are able to perform computations with correspondences involving possibly singular varieties.

V. Chernousov, S. Gille and A. Merkurjev have recently generalized Theorem 1.1 to arbitrary homogeneous varieties [1]. Their approach is to write down a decomposition as in (2) for homogeneous varieties in terms of group theory and then to use the extension to (1) given here to prove a nilpotence result. I would like to thank Merkurjev for pointing out to me the usefulness of this extension.

1.1. Notation. As the main tool used in this paper is the intersection theory of Fulton-MacPherson, we use the notation of [2]. In particular, a *scheme* will be a scheme of finite type over a field and a *variety* will be an irreducible and reduced scheme. We use the notation Chow_k for k a field to denote the category of Chow motives whose definition is recalled below in Section 2. For a scheme X, $A_j X$ will denote the Chow group of dimension j cycles on X.

In section 3, we will use the notation $\mathbb{H}$ to denote the hyperbolic plane. That is, $\mathbb{H}$ is the quadratic space consisting of k^2 with quadratic form given by $q(x, y) = xy$.

2. Refined Intersections

Let V and W be schemes over a field k, let $\{V_i\}_{i=1}^m$ be the irreducible components of V and write $d_i = \dim V_i$. The group of degree r Chow correspondences is defined as

$$\mathrm{Corr}_r(V, W) = \oplus A_{d_i - r}(V_i \times W). \tag{1}$$

If X_1, X_2, X_3 are smooth proper schemes, then it is well-known that there is a composition

$$\begin{aligned} \mathrm{Corr}_r(X_1, X_2) \otimes \mathrm{Corr}_s(X_2, X_3) &\to \mathrm{Corr}_{r+s}(X_1, X_3) \\ g \otimes f &\mapsto f \circ g \end{aligned} \tag{2}$$

given by the formula

$$f \circ g = p_{13*}(p_{12}^* g \cdot p_{23}^* f) \tag{3}$$

where the $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ are the obvious projection maps. Using this formula, the category Chow_k of Chow motives can be defined as follows ([4], see also [3]): The objects are the triples (X, p, n) where X is a smooth projective scheme over k, $p \in \mathrm{Corr}_0(X,X)$ is a projector (that is, $p^2 = p$) and n is an integer. The morphisms are defined by the formula

$$\mathrm{Hom}((X,p,n),(Y,q,m)) = q\,\mathrm{Corr}_{m-n}(X,Y)p. \tag{4}$$

To fix notation, we remind the reader that the *Tate twist* of an object $M = (X,p,n)$ is the object $M(k) = (X,p,n+k)$, and the objects $\mathbb{Z}(k) = (\mathrm{Spec}\,k, \mathrm{id}, k)$ are customarily called the *Tate objects*. It is clear from (4) that

$$\mathrm{Hom}(\mathbb{Z}(k), M(X)) = A_k X, \quad \mathrm{Hom}(M(X), \mathbb{Z}(k)) = A^k X \tag{5}$$

where $M(X)$ is the motive $(X, \mathrm{id}, 0)$ associated to the scheme X.

2.1. REFINED CORRESPONDENCES. The main observation behind this paper is that a composition generalizing that of (2) holds for arbitrary varieties X_1 and X_3 provided that X_2 is smooth and proper. To define this composition we use the the Gysin pullback through the regular embedding

$$X_1 \times X_2 \times X_3 \stackrel{\mathrm{id}\times\Delta\times\mathrm{id}}{\rightarrow} X_1 \times X_2 \times X_2 \times X_3.$$

We can then define the composition by the formula

$$f \circ g = p_{13*}((\mathrm{id}\times\Delta\times\mathrm{id})^!(g \otimes f)). \tag{6}$$

We need to verify that the definition given in (6) agrees with that of (3) and satisfies various functoriality properties needed to make it a useful extension. To state these properties in their natural generality, it is helpful to also consider (6) in a slightly different situation from that of (2). For X_2 a smooth scheme and X_1, X_3 arbitrary schemes, we define a composition

$$\begin{array}{rcl} A_r(X_1 \times X_2) \otimes \mathrm{Corr}_s(X_2, X_3) & \rightarrow & A_{r-s}(X_1 \times X_3) \\ g \otimes f & \mapsto & f \circ g \end{array} \tag{7}$$

where $f \circ g$ is defined as in (6). We consider (7) because $\oplus_i \mathrm{Corr}_i(X_1, X_2)$ is not necessarily equal to $\oplus_i A_i(X_1 \times X_2)$ unless X_1 is scheme with irreducible connected components. Therefore, in the case that X_1 does not have irreducible connected components, $\mathrm{Corr}_*(X_1, X_2)$ is not a reindexing of the Chow groups of $X_1 \times X_2$.

PROPOSITION 2.1. *Let $X_i, i \in \{1,2,3\}$ be schemes with X_2 smooth and proper.*

(a) If all X_i are smooth and X_2 is proper, then the definition of $f \circ g$ for $g \in \mathrm{Corr}_r(X_1, X_2), f \in \mathrm{Corr}_s(X_2, X_3)$ given in (6) agrees with that of (3).

(b) If $\pi : X_1' \to X_1$ is a proper morphism, then the diagram

$$\begin{tikzcd} A_r(X_1' \times X_2) \otimes \mathrm{Corr}_s(X_2, X_3) \arrow[r] \arrow[d, "\pi_*"] & A_{r-s}(X_1' \times X_3) \arrow[d, "\pi_*"] \\ A_r(X_1 \times X_2) \otimes \mathrm{Corr}_s(X_2, X_3) \arrow[r] & A_{r-s}(X_1 \times X_3) \end{tikzcd}$$

commutes. Here, for the vertical arrows, by π_* we mean the morphism induced by π_* on the first factor and the identity on the other factors.

(c) If $\phi : X_1' \to X_1$ is flat of constant relative dimension e, then

$$\begin{tikzcd}
A_r(X_1 \times X_2) \otimes \mathrm{Corr}_s(X_2, X_3) \arrow[r] \arrow[d, "\phi^*"'] & A_{r-s}(X_1 \times X_3) \arrow[d, "\phi^*"] \\
A_{r+e}(X_1' \times X_2) \otimes \mathrm{Corr}_s(X_2, X_3) \arrow[r] & A_{r+e-s}(X_1' \times X_3)
\end{tikzcd}$$

commutes.

Proof. First note that it suffices to prove the proposition for X_2 irreducible of dimension d_2. This is because $\mathrm{Corr}_s(X_2, X_3)$ and $A_r(X_1 \times X_2)$ are both direct sums over the irreducible components of X_2 and all of the maps in the theorem commute with these direct sum decompositions.
(a): Another formulation of (3) is that $f \circ g$ is given by

$$p_{13*}\Delta_{123}^!(p_{12}^* g \otimes p_{23}^* f)$$

where

$$\Delta_{123} : X_1 \times X_2 \times X_3 \to (X_1 \times X_2 \times X_3) \times (X_1 \times X_2 \times X_3)$$

is the obvious diagonal.
Consider the sequence of maps

$$X_1 \times X_2 \times X_3 \stackrel{\Delta_{123}}{\to} (X_1 \times X_2 \times X_3) \times (X_1 \times X_2 \times X_3) \stackrel{p_{12} \times p_{23}}{\to} X_1 \times X_2 \times X_2 \times X_3 \tag{8}$$

with composition $\Delta_2 : X_1 \times X_2 \times X_3 \to X_1 \times X_2 \times X_2 \times X_3$.
Since all X_i are smooth, $p_{12} \times p_{23}$ is a smooth morphism. It follows from ([2] Proposition 6.5.b) that

$$\begin{aligned}
\Delta_2^!(g \otimes f) &= \Delta_{123}^!(p_{12} \times p_{23})^*(g \otimes f) \\
&= \Delta_{123}^!(p_{12}^* g \otimes p_{23}^* f).
\end{aligned}$$

(a) now follows by taking push-forwards.
(b): We have a fiber diagram

$$\begin{tikzcd}
X_1' \times X_2 \times X_3 \arrow[r, "\Delta_2'"] \arrow[d] & X_1' \times X_2 \times X_2 \times X_3 \arrow[d] \\
X_1 \times X_2 \times X_3 \arrow[r, "\Delta_2"'] & X_1 \times X_2 \times X_2 \times X_3
\end{tikzcd} \tag{9}$$

where Δ_2' and Δ_2 are both induced by the diagonal

$$X_2 \to X_2 \times X_2.$$

Since Δ_2 and Δ_2' are both regular of codimension d_2, it follows from ([2], Proposition 6.2.c) that both morphisms induce the same Gysin pullback on the top row of the diagram.

By ([2], 6.2 (a)), proper push-forward and Gysin pull-back through a regular embedding commute. Applying this fact to (9), we have

$$\begin{aligned}\pi_*(f\circ g) &= (\pi\times \mathrm{id}_3)_* p_{13*}\Delta_2^!(g\otimes f)\\ &= p_{13*}(\pi\times\mathrm{id}_2\times\mathrm{id}_3)_*\Delta_2^!(g\otimes f)\\ &= p_{13*}\Delta_2^!(\pi\times\mathrm{id}_2\times\mathrm{id}_2\times\mathrm{id}_3)_*(g\otimes f)\\ &= p_{13*}\Delta_2^!((\pi\times\mathrm{id})_*g\otimes f))\\ &= f\circ(\pi_*g).\end{aligned}$$

(c) Here the argument is very similar to the one for (b): We have a pullback diagram

$$\begin{tikzcd} X_1'\times X_2\times X_3 \arrow[r, "\phi\times\mathrm{id}_2\times\mathrm{id}_3"] \arrow[d, "p_{13}'"'] & X_1\times X_2\times X_3 \arrow[d, "p_{13}"] \\ X_1'\times X_3 \arrow[r, "\phi\times\mathrm{id}_3"'] & X_1\times X_3. \end{tikzcd}$$

Since the vertical arrows are proper and the horizontal arrows are flat, it follows from ([2], 1.7) that

$$p'_{13*}(\phi\times\mathrm{id}_2\times\mathrm{id}_3)^* = (\phi\times\mathrm{id}_3)^*p_{13*} \tag{10}$$

We then consider the pullback

$$\begin{tikzcd} X_1'\times X_2\times X_3 \arrow[r, "\Delta_2'"] \arrow[d, "\phi"'] & X_1'\times X_2\times X_2\times X_3 \arrow[d, "\phi"'] \\ X_1\times X_2\times X_3 \arrow[r, "\Delta_2"'] & X_1\times X_2\times X_2\times X_3 \end{tikzcd}$$

in which the vertical arrow are flat and the horizontal arrow are regular embeddings both of codimension d_2. By ([2], 6.2 (c)) it follows that the flat pullbacks commutes with the Gysin pullbacks; thus,

$$\begin{aligned}\phi^*(f\circ g) &= (\phi\times\mathrm{id}_3)^*p'_{13*}\Delta_2^!(g\otimes f)\\ &= p_{13*}(\phi\times\mathrm{id}_2\times\mathrm{id}_3)^*\Delta_2^!(g\times f)\\ &= p_{13*}\Delta_2^!((\phi^*g)\otimes f))\\ &= f\circ(\phi^*g).\end{aligned}$$

$\square$

Remark 2.2. If X_1 and X_3 are taken to be schemes with irreducible connected components, then in (b) and (c), we can replace $A_*(X_1\times X_2)$ with $\mathrm{Corr}_*(X_1,X_2)$ after a shift in the indices. Then the roles of X_1 and X_3 in the theorem can also be interchanged by the symmetry of $\mathrm{Corr}_*(X,Y)$.

The fact that morphisms in $\mathrm{Corr}_*(\_,\_)$ are not in general composable is mitigated somewhat by the following result.

Proposition 2.3. *Let $\{X_i\}_{i=1}^4$ be schemes with X_2 and X_3 smooth and proper.*

(a) If $\Delta \in \mathrm{Corr}_0(X_2 \times X_2)$ *is the class of the diagonal then, the morphism* $A_r(X_1 \times X_2) \to A_r(X_1 \times X_2)$ *given by* $f \mapsto \Delta \circ f$ *is the identity.*

(b) If $f_1 \in A_r(X_1 \times X_2)$ *and* $f_i \in \mathrm{Corr}_{r_i}(X_i, X_{i+1})$ *for* $i = 2, 3$*, then*

$$(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1).$$

In other words, composition is associative.

Proof. (a) can be easily checked on the level of cycles in $Z_r(X_1 \times X_2)$. For (b), the important point is the commutativity of the diagram

$$\begin{tikzcd} X_1 \times X_2 \times X_3 \times X_4 \arrow[r] \arrow[d] \arrow[dr, "\Delta_{23}"] & X_1 \times X_2 \times X_2 \times X_3 \times X_4 \arrow[d] \\ X_1 \times X_2 \times X_3 \times X_3 \times X_4 \arrow[r] & X_1 \times X_2 \times X_2 \times X_3 \times X_3 \times X_4 \end{tikzcd} \tag{11}$$

where the arrows are the obvious diagonal morphisms. Both compositions in (b) can be computed as $p_{14*}\Delta_{23}^!(f_3 \otimes f_2 \otimes f_1)$. □

3. Rost's Correspondence Theorem

If X and Y are smooth projective varieties and $f : M(X) \to M(Y)$ is a morphism, we obtain a morphism $f_* : A_r(X) \to A_r(Y)$ induced by the composition

$$\mathbb{Z}(r) \to M(X) \xrightarrow{f} M(Y)$$

using (5). Similarly, for a smooth projective variety B and an integer a, we obtain a morphism $f_* : \mathrm{Hom}(M(B)(a), M(X)) \to \mathrm{Hom}(M(B)(a), M(Y))$ given by

$$g \mapsto f \circ g \tag{12}$$

with $g \in \mathrm{Hom}(M(B)(a), M(X)) = \mathrm{Corr}_{-a}(B, X)$.
Rost's nilpotence theorem is a consequence of the following more general theorem concerning correspondences between smooth varieties.

Theorem 3.1. *Let B and X be smooth projective varieties over a field k with $\dim B = d$. For any $b \in B$, let X_b denote the fiber of the projection $\pi : B \times X \to B$. If $f \in \mathrm{End}(M(X))$ is a morphism such that $f_* A_r(X_b) = 0$ for all b and all $0 \le r \le d + a$, then*

$$f_*^{d+1} \mathrm{Hom}(M(B)(a), M(X)) = 0. \tag{13}$$

In the case $a = 0$, the theorem is due to Rost ([5], Proposition 1). Our proof of the theorem is based on Rost's proof, but uses the results of Section 2 in place of Rost's cycle module spectral sequence. Note that, while the hypotheses of the theorem assume that B is smooth, the the proof is essentially an induction on all subvarieties (smooth or otherwise) of B. Moreover, the result holds with a slight change of notation (which we describe after the proof) for arbitrary B.

Proof of Theorem 3.1. Let $Z_k(B\times X)$ denote the group of k-dimensional cycles on $B\times X$, and let $F_pZ_k(B\times X)$ denote the subgroup of $Z_k(B\times X)$ generated by subvarieties V of dimension k such that $\dim \pi(V)\le p$. Let $F_pA_k(B\times X)$ denote the image of $F_pZ_k(B\times X)$ under the rational equivalence quotient map. To prove the theorem, it is clearly sufficient to show that

$$f_*F_pA_{d+a}(B\times X)\subset F_{p-1}A_{d+a}(B\times X) \tag{14}$$

since $F_{-1}A_{d+a}(B\times X)=0$. Therefore, it is sufficient to show that, for V a $(d+a)$-dimensional subvariety of $B\times X$ such that $\dim\pi(V)=p$, $f_*[V]\in F_{p-1}A_{d+a}(B\times X)$.

Let $Y=\pi(V)$. By the hypotheses of the theorem, there is a nonempty open set $U\subset Y$ such that $f_*[V_U]=0$. (Here we write V_U for the fiber product $V\times_Y U$.) Let $W=Y-U$, and consider the short exact sequence of Chow groups

$$A_{d+a}W\times X\xrightarrow{i_*}A_{d+a}Y\times X\xrightarrow{j^*}A_{d+a}U\times X\to 0. \tag{15}$$

By the results of Section 2, $f_*[V_U]=j^*f_*[V]$ where $f_*[V]$ is the composition $f\circ[V]$ of f with $[V]$ viewed as an element of $\mathrm{Corr}_{p-d-a}Y\times X$. It follows that $f_*[V]$ lies in the image of the first morphism in (15). Thus $f_*[V]\in F_{p-1}A_{d+a}B\times X$. □

Remark 3.2. Using the associativity of composition (Proposition 2.3), it is easy to see that the above proof generalizes to the case where B is arbitrary. The statement of the theorem remains the same, except that $Hom(M(B)(a),M(X))$ is replaced with $\mathrm{Corr}_{-a}(B,X)$.

4. Rost Nilpotence

If $M=(Y,p,n)$ is a motive in Chow_k and X is an arbitrary scheme, we define

$$\mathrm{Corr}(X,M)=p\,\mathrm{Corr}_n(X,Y).$$

Since Y is smooth and projective, this definition makes sense by what we have seen in Section 2. If $j:U\to X$ is flat we obtain a pullback $\mathrm{Corr}(X,M)\to\mathrm{Corr}(U,M)$ and, if $p:X'\to X$ is proper, we obtain a push-forward $\mathrm{Corr}(X',M)\to\mathrm{Corr}(X,M)$. This follows from Proposition 2.1. Similarly, by Remark 2.2 we can define $\mathrm{Corr}(M,X)$.

Using this observation, we can easily obtain a result of Rost's on the decomposition of the motive of a quadric. To state the theorem, we must first recall a fact about quadrics with points.

Suppose Q is the projective quadric corresponding to a non-degenerate quadratic form q; that is, $Q=V(q)$. As we are discussing quadrics and quadratic forms, we will assume for the remainder of the paper that the field k over which Q and q are defined has characteristic not equal to 2. Suppose further that Q has a point over k. Then the quadratic form q splits as an orthogonal direct sum $q=\mathbb{H}\perp q'$. (This is a standard fact about quadratic forms which is also an easy exercise). Let Q' denote the (clearly smooth) quadric associated to q'.

THEOREM 4.1 (Rost decomposition). *If Q has a point over k then $M(Q) = \mathbb{Z} \oplus M(Q')(1) \oplus \mathbb{Z}(d)$ where $d = \dim Q$ and Q' is the smooth quadric of the proceeding paragraph.*

Proof. For the proof, we use Rost's methods and notation ([5], Proposition 2) with some simplifications coming from our results in the previous sections. ‘
We can write $q = xy + q'(z)$ where z denotes a d-dimensional variable. Let Q_1 denote the closed subvariety $V(x)$ and let p denote the closed point on Q_1 corresponding to the locus $x = z = 0, y = 1$. Note that $U_1 := Q - Q_1$ is isomorphic to $\mathbb{A}^d$. Moreover, $Q_1 - \{p\}$ is an $\mathbb{A}^1$-bundle over Q' via the morphism $(y, z) \mapsto z$. For any motive M, we thus obtain short exact sequences

$$\mathrm{Corr}(M, Q_1) \to \mathrm{Corr}(M, Q) \twoheadrightarrow \mathrm{Corr}(M, \mathbb{A}^d), \tag{16}$$
$$\mathrm{Corr}(M, p) \to \mathrm{Corr}(M, Q_1) \twoheadrightarrow \mathrm{Corr}(M(-1), Q'). \tag{17}$$

Here Q_1 is, in general, a *singular* quadric. However, by Theorem 2.3, each of the entries of (16) and (17) can each be interpreted as presheaves on the category of Chow motives given, for example, by the association $M \rightsquigarrow \mathrm{Corr}(M, Q_1)$. Moreover, by Proposition 2.1, the morphisms in (16) and (17) induce maps of presheaves, i.e., they are functorial in M.
In fact, in both sequences the first morphism is an injection and the second morphism is a split surjection. To see this we construct splittings for the first morphism in each sequence.
For (17), let $\pi : Q_1 \to p$ denote the projection to a point. Then $\pi_* : \mathrm{Corr}(M, Q_1) \to \mathrm{Corr}(M, p)$ induces a splitting. Again, by Proposition 2.1, this map is functorial in M.
For (16), let r denote the point corresponding to $x = 1, y = z = 0$, and let U denote the open subset $Q - \{r\}$ in Q. Then there is a morphism $\phi^\circ : U \to Q_1$ given by $(x, y, z) \mapsto (y, z)$. Let ϕ denote the closure of the graph of ϕ° in $\mathrm{Corr}(Q, Q_1)$. By the results of section 2, ϕ induces a morphism $\phi_* : \mathrm{Corr}(M, Q) \to \mathrm{Corr}(M, Q_1)$. We claim that ϕ_* splits (16) and is functorial in M. (This is not hard to check on the level of cycles.)
Since the push-forward on the second factor induces an isomorphism

$$\mathrm{Corr}(M, \mathbb{A}^d) \xrightarrow{\cong} \mathrm{Hom}(M, \mathbb{Z}(d)),$$

we have a decomposition

$$\mathrm{Hom}(M, M(Q)) = \mathrm{Hom}(M, \mathbb{Z}(d)) \oplus \mathrm{Hom}(M, \mathbb{Z}) \oplus \mathrm{Hom}(M, M(Q')(-1)). \tag{18}$$

The decomposition of the theorem the follows from Yoneda's lemma which applies in this case because of the functoriality of the decomposition with respect to M.

□

We are now prepared to prove Rost nilpotence, Theorem 1.1. The proof is essentially identical to Rost's, but I include it for the convenience of the reader.

We first note that, due to the inductive structure of the proof, it is actually helpful to strengthen the conclusion of the theorem slightly. We therefore restate the theorem with the stronger conclusion.

Theorem 4.2 (Rost [5], Proposition 2). *For each $d \in \mathbb{N}$, there is a number $N(d)$ such that, if Q is a smooth quadric of dimension d over a field k and $f \in \mathrm{End}(M(Q))$ such that $f \otimes \overline{k} = 0$, then $f^{N(d)} = 0$.*

Proof. If $d = 0$, Q either consists either of two points defined over k or of one point defined over a quadratic extension of k. In the first case, $\mathrm{End}(M(Q)) = \mathrm{End}(\mathbb{Z} \oplus \mathbb{Z})$ and in the second $\mathrm{End}(M(Q))$ is isomorphic to the rank 2 subring of $\mathrm{End}(M(Q \otimes \overline{k}))$ consisting of matrices invariant under conjugation by $\binom{0\,1}{1\,0}$. The theorem, therefore, holds trivially with $N(0) = 1$.
We then induct on d. Suppose Q is a rank $d > 0$ quadric with a point over k. Then $M(Q)$ splits as in Rost's decomposition theorem. In fact, we also have a splitting

$$\mathrm{End}(M(Q)) = \mathrm{End}(\mathbb{Z}(d)) \oplus \mathrm{End}(\mathbb{Z}) \oplus \mathrm{End}(M(Q')). \tag{19}$$

This follows from the fact that the six cross terms (e.g. $Hom(\mathbb{Z}, \mathbb{Z}(d))$, $\mathrm{Hom}(\mathbb{Z}, M(Q'))$ and $\mathrm{Hom}(M(Q'), \mathbb{Z})$) are all zero for dimension reasons. As $\mathrm{End}(\mathbb{Z}(j)) = \mathbb{Z}$, we have

$$\mathrm{End}(M(Q)) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{End}(M(Q'))$$

and $f^{N(d-2)} = 0$ by the induction hypothesis applied to Q'.
If Q does not have a point over k, then $Q \otimes k(x)$ does have a point (trivially) over the residue field of any point x. Therefore $f^{N(d-2)} \otimes k(x) = 0$ for every such point x by the induction hypothesis. (For this to hold for $\dim Q = 1$, we have to set $N(-1) = 1$.) Now apply Theorem 3.1 to $f^{N(d-2)}$. We obtain the conclusion that $f^{(d+1)N(d-2)} = 0$. Thus we can take $N(d) = (d+1)N(d-2)$ and the theorem is proved. $\square$

Remark 4.3. The proof shows that we can take $N(d) = (d+1)!!$ in the theorem.

References

[1] Vladimir Chernousov, Stefan Gille, and Alexander Merkurjev. Motivic decomposition of isotropic projective homogeneous varieties http://www.math.ucla.edu/~merkurev/papers/nilpotence9.dvi.

[2] William Fulton. *Intersection theory.* Springer-Verlag, Berlin, second edition, 1998.

[3] Uwe Jannsen. Motives, numerical equivalence, and semi-simplicity. *Invent. Math.*, 107(3):447–452, 1992.

[4] Nikita A. Karpenko. Characterization of minimal Pfister neighbors via Rost projectors. *J. Pure Appl. Algebra*, 160(2-3):195–227, 2001.

[5] Markus Rost. The motive of a Pfister form. http://www.math.ohio-state.edu/~rost/motive.html.

[6] Markus Rost. Chow groups with coefficients. *Documenta Math.*, Vol 1., (1996) 319–393.

[7] Alexander Vishik. Integral motives of quadrics. http://www.mpim-bonn.mpg.de/html/preprints/preprints.html.

Patrick Brosnan
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
USA
pbrosnan@math.ucla.edu