Toric Hyperkähler Varieties

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Abstract. Extending work of Bielawski-Dancer [3] and Konno [14], we develop a theory of toric hyperkähler varieties, which involves toric geometry, matroid theory and convex polyhedra. The framework is a detailed study of semi-projective toric varieties, meaning GIT quotients of affine spaces by torus actions, and specifically, of Lawrence toric varieties, meaning GIT quotients of even-dimensional affine spaces by symplectic torus actions. A toric hyperkähler variety is a complete intersection in a Lawrence toric variety. Both varieties are non-compact, and they share the same cohomology ring, namely, the Stanley-Reisner ring of a matroid modulo a linear system of parameters. Familiar applications of toric geometry to combinatorics, including the Hard Lefschetz Theorem and the volume polynomials of Khovanskii-Pukhlikov [11], are extended to the hyperkähler setting. When the matroid is graphic, our construction gives the toric quiver varieties, in the sense of Nakajima [17].

1 Introduction

Hyperkähler geometry has emerged as an important new direction in differential and algebraic geometry, with numerous applications to mathematical physics and representation theory. Roughly speaking, a *hyperkähler manifold* is a Riemannian manifold of dimension $4n$, whose holonomy is in the unitary symplectic group $Sp(n) \subset SO(4n)$. The key example is the quaternionic space $\mathbb{H}^n \simeq \mathbb{C}^{2n} \simeq \mathbb{R}^{4n}$. Our aim is to relate hyperkähler geometry to the combinatorics of convex polyhedra. We believe that this connection is fruitful for both subjects. Our objects of study are the *toric hyperkähler manifolds* of Bielawski and Dancer [3]. They are obtained from $\mathbb{H}^n$ by taking the hyperkähler quotient [10] by an abelian subgroup of $Sp(n)$. Bialewski and Dancer found that the geometry and topology of toric hyperkähler manifolds is governed by hyperplane arrangements, and Konno [14] gave an explicit presentation of their cohomology rings. The present paper is self-contained and contains new proofs for the relevant results of [3] and [14].

We start out in Section 2 with a discussion of semi-projective toric varieties. A toric variety X is called *semi-projective* if X has a torus-fixed point and X is projective over its affinization $\mathrm{Spec}(H^0(X, \mathcal{O}_X))$. We show that semi-projective toric varieties are exactly the ones which arise as GIT quotients of a complex vector space by an abelian group. Then we calculate the cohomology ring of a semi-projective toric orbifold X. It coincides with the cohomology of the *core* of X, which is defined as the union of all compact torus orbit closures. This result and further properties of the core are derived in Section 3.

The lead characters in the present paper are the *Lawrence toric varieties*, to be introduced in Section 4 as the GIT quotients of symplectic torus actions on even-dimensional affine spaces. They can be regarded as the "most non-compact" among all semi-projective toric varieties. The combinatorics of Lawrence toric varieties is governed by the Lawrence construction of convex polytopes [22, §6.6] and its intriguing interplay with matroids and hyperplane arrangements.

In Section 6 we define *toric hyperkähler varieties* as subvarieties of Lawrence toric varieties cut out by certain natural bilinear equations. In the smooth case, they are shown to be biholomorphic with the toric hyperkähler manifolds of Bielawski and Dancer, whose differential-geometric construction is reviewed in Section 5 for the reader's convenience. Under this identification the core of the toric hyperkähler variety coincides with the core of the ambient Lawrence toric variety. We shall prove that these spaces have the same cohomology ring which has the following description. All terms and symbols appearing in Theorem 1.1 are defined in Sections 4 and 6.

THEOREM 1.1 *Let $A : \mathbb{Z}^n \to \mathbb{Z}^d$ be an epimorphism, defining an inclusion $\mathbb{T}^d_{\mathbb{R}} \subset \mathbb{T}^n_{\mathbb{R}}$ of compact tori, and let $\theta \in \mathbb{Z}^d$ be generic. Then the following graded $\mathbb{Q}$-algebras are isomorphic:*

1. *the cohomology ring of the toric hyperkähler variety $Y(A, \theta) = \mathbb{H}^n ////_{(\theta,0)} \mathbb{T}^d_{\mathbb{R}}$,*
2. *the cohomology ring of the Lawrence toric variety $X(A^{\pm}, \theta) = \mathbb{C}^{2n} //_{\theta} \mathbb{T}^d_{\mathbb{R}}$,*
3. *the cohomology ring of the core $C(A^{\pm}, \theta)$, which is the preimage of the origin under the affinization map of either the Lawrence toric variety or the toric hyperkähler variety,*
4. *the quotient ring $\mathbb{Q}[x_1, \ldots, x_n]/(M^*(\mathcal{A}) + \mathrm{Circ}(\mathcal{A}))$, where $M^*(\mathcal{A})$ is the matroid ideal which is generated by squarefree monomials representing cocircuits of A, and $\mathrm{Circ}(\mathcal{A})$ is the ideal generated by the linear forms that correspond to elements in the kernel of A.*

If the matrix A is unimodular then $X(A^{\pm},\theta)$ and $Y(A,\theta)$ are smooth and $\mathbb{Q}$ can be replaced by $\mathbb{Z}$.

Here is a simple example where all three spaces are manifolds: take $A : \mathbb{Z}^3 \to \mathbb{Z}$, $(u_1,u_2,u_3) \mapsto u_1+u_2+u_3$ with $\theta \neq 0$. Then $C(A^{\pm},\theta)$ is the complex projective plane $\mathbb{P}^2$. The Lawrence toric variety $X(A^{\pm},\theta)$ is the quotient of $\mathbb{C}^6 = \mathbb{C}^3 \oplus \mathbb{C}^3$ modulo the symplectic torus action $(x,y) \mapsto (t\cdot x, t^{-1}\cdot y)$. Geometrically, X is a rank 3 bundle over $\mathbb{P}^2$, visualized as an unbounded 5-dimensional polyhedron with a bounded 2-face, which is a triangle. The toric hyperkähler variety $Y(A,\theta)$ is embedded into $X(A^{\pm},\theta)$ as the hypersurface $x_1y_1+x_2y_2+x_3y_3=0$. It is isomorphic to the cotangent bundle of $\mathbb{P}^2$. Note that $Y(A,\theta)$ itself is not a toric variety.

For general matrices A, the varieties $X(A^{\pm},\theta)$ and $Y(A,\theta)$ are orbifolds, by the genericity hypothesis on θ, and they are always non-compact. The core $C(A^{\pm},\theta)$ is projective but almost always reducible. Each of its irreducible components is a projective toric orbifold.

In Section 7 we give a dual presentation, in terms of *cogenerators*, for the cohomology ring. These cogenerators are the volume polynomials of Khovanskii-Pukhlikov [11] of the bounded faces of our unbounded polyhedra. As an application we prove the injectivity part of the Hard Lefschetz Theorem for toric hyperkähler varieties (Theorem 7.4). In light of the following corollary to Theorem 1.1, this provides new inequalities for the h-numbers of rationally representable matroids.

Corollary 1.2 *The Betti numbers of the toric hyperkähler variety $Y(A,\theta)$ are the h-numbers (defined in Stanley's book [18, §III.3]) of the rank $n-d$ matroid given by the integer matrix A.*

The *quiver varieties* of Nakajima [17] are hyperkähler quotients of $\mathbb{H}^n$ by some subgroup $G \subset Sp(n)$ which is a product of unitary groups indexed by a quiver (i.e. a directed graph). In Section 8 we examine *toric quiver varieties* which arise when G is a compact torus. They are the toric hyperkähler manifolds obtained when A is the differential $\mathbb{Z}^{\mathrm{edges}} \to \mathbb{Z}^{\mathrm{vertices}}$ of a quiver. Note that our notion of toric quiver variety is not the same as that of Altmann and Hille [1]. Theirs are toric and projective: in fact, they are the irreducible components of our core $C(A^{\pm},\theta)$.

We close the paper by studying two examples in detail. First in Section 9 we illustrate the main results of this paper for a particular example of a toric quiver variety, corresponding to the complete bipartite graph $K_{2,3}$. In the final Section 10 we examine the ALE spaces of type A_n. Curiously, these manifolds are both toric and hyperkähler, and we show that they and their products are the only toric hyperkähler manifolds which are toric varieties in the usual sense.

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2 Semi-projective toric varieties

Projective toric varieties are associated with rational polytopes, that is, bounded convex polyhedra with rational vertices. This section describes toric varieties associated with (typically unbounded) rational polyhedra. The resulting class of semi-projective toric varieties will be seen to equal the GIT-quotients of affine space $\mathbb{C}^n$ modulo a subtorus of $\mathbb{T}^n_{\mathbb{C}}$.
Let $A = [a_1, \ldots, a_n]$ be a $d \times n$-integer matrix whose $d \times d$-minors are relatively prime. We choose an $n \times (n-d)$-matrix $B = [b_1, \ldots, b_n]^T$ which makes the following sequence exact:

$$0 \longrightarrow \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \longrightarrow 0. \tag{1}$$

The choice of B is equivalent to choosing a basis in $\ker(A)$. The configuration $\mathcal{B} := \{b_1, \ldots, b_n\}$ in $\mathbb{Z}^{n-d}$ is said to be a *Gale dual* of the given vector configuration $\mathcal{A} := \{a_1, \ldots, a_n\}$ in $\mathbb{Z}^d$.
We denote by $\mathbb{T}_{\mathbb{C}}$ the complex group $\mathbb{C}^*$ and by $\mathbb{T}_{\mathbb{R}}$ the circle $U(1)$. Their Lie algebras are denoted by $\mathfrak{t}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{R}}$ respectively. We apply the contravariant functor $\mathrm{Hom}(\,\cdot\,, \mathbb{T}_{\mathbb{C}})$ to the short exact sequence (1). This gives a short exact sequence of abelian groups:

$$1 \longleftarrow \mathbb{T}^{n-d}_{\mathbb{C}} \xleftarrow{B^T} \mathbb{T}^n_{\mathbb{C}} \xleftarrow{A^T} \mathbb{T}^d_{\mathbb{C}} \longleftarrow 1. \tag{2}$$

Thus $\mathbb{T}^d_{\mathbb{C}}$ is embedded as a d-dimensional subtorus of $\mathbb{T}^n_{\mathbb{C}}$. It acts on the affine space $\mathbb{C}^n$. We shall construct the quotients of this action in the sense of *geometric invariant theory* (= GIT). The ring of polynomial functions on $\mathbb{C}^n$ is graded by the semigroup $\mathbb{N}\mathcal{A} \subseteq \mathbb{Z}^d$:

$$S \;=\; \mathbb{C}[x_1, \ldots, x_n]\,, \quad \deg(x_i) \;=\; a_i \in \mathbb{N}\mathcal{A}. \tag{3}$$

A polynomial in S is homogeneous if and only if it is a $\mathbb{T}^d_{\mathbb{C}}$-eigenvector. For $\theta \in \mathbb{N}\mathcal{A}$, let S_θ denote the (typically infinite-dimensional) $\mathbb{C}$-vector space of homogeneous polynomials of degree θ. Note that S_θ is a module over the subalgebra S_0 of degree zero polynomials in $S = \bigoplus_{\theta \in \mathbb{N}\mathcal{A}} S_\theta$. The following lemma is a standard fact in combinatorial commutative algebra.

Lemma 2.1 *The $\mathbb{C}$-algebra S_0 is generated by a finite set of monomials, corresponding to the minimal generators of the semigroup $\mathbb{N}^n \cap \mathrm{im}(B)$. For any $\theta \in \mathbb{N}\mathcal{A}$, the graded component S_θ is a finitely generated S_0-module, and the ring $S_{(\theta)} = \bigoplus_{r=0}^{\infty} S_{r\theta}$ is a finitely generated S_0-algebra.*

The $\mathbb{C}$-algebra S_0 coincides with the ring of invariants $S^{\mathbb{T}^d_{\mathbb{C}}}$. The S_0-algebra $S_{(\theta)}$ is isomorphic to $\bigoplus_{r=0}^{\infty} t^r S_{r\theta}$. We regard it as $\mathbb{N}$-graded by the degree of t.

Definition 2.2 *The* affine GIT quotient *of $\mathbb{C}^n$ by the d-torus $\mathbb{T}^d_{\mathbb{C}}$ is the affine toric variety*

$$X(A,0) \;:=\; \mathbb{C}^n //_0\, \mathbb{T}^d_{\mathbb{C}} \;:=\; \mathrm{Spec}\,(S^{\mathbb{T}^d_{\mathbb{C}}}) = \mathrm{Spec}\,(S_0) = \mathrm{Spec}\,\big(\mathbb{C}[\mathbb{N}^n \cap \mathrm{im}(B)]\big). \quad (4)$$

For any $\theta \in \mathbb{N}\mathcal{A}$, the projective GIT quotient *of $\mathbb{C}^n$ by the d-torus $\mathbb{T}^d_{\mathbb{C}}$ is the toric variety*

$$X(A,\theta) \quad := \quad \mathbb{C}^n //_\theta\, \mathbb{T}^d_{\mathbb{C}} \;:=\; \mathrm{Proj}\,(S_{(\theta)}) \quad = \quad \mathrm{Proj} \bigoplus_{r=0}^{\infty} t^r \cdot S_{r\theta}. \quad (5)$$

Recall that the isomorphism class of any toric variety is given by a fan in a lattice. A toric variety is a *toric orbifold* if its fan is simplicial. We shall describe the fans of the toric varieties $X(A,0)$ and $X(A,\theta)$ using the notation in Fulton's book [8]. We write M for the lattice $\mathbb{Z}^{n-d}$ in (1) and $N = \mathrm{Hom}(M,\mathbb{Z})$ for its dual. The torus $\mathbb{T}^{n-d}_{\mathbb{C}}$ in (2) is identified with $N \otimes \mathbb{T}_{\mathbb{C}}$. The column vectors $\mathcal{B} = \{b_1,\ldots,b_n\}$ of the matrix B^T form a configuration in $N \simeq \mathbb{Z}^{n-d}$. We write $\mathrm{pos}(\mathcal{B})$ for the convex polyhedral cone spanned by $\mathcal{B}$ in the vector space $N_{\mathbb{R}} = N \otimes \mathbb{R} \simeq \mathbb{R}^{n-d}$. Note that the affine toric variety associated with the cone $\mathrm{pos}(\mathcal{B})$ equals $X(A,0)$.

A *triangulation* of the configuration $\mathcal{B}$ is a simplicial fan Σ whose rays lie in $\mathcal{B}$ and whose support equals $\mathrm{pos}(\mathcal{B})$. A *T-Cartier divisor* on Σ is a continuous function $\Psi : \mathrm{pos}(\mathcal{B}) \to \mathbb{R}$ which is linear on each cone of Σ and takes integer values on $N \cap \mathrm{pos}(\mathcal{B})$. The triangulation Σ is called *regular* if there exists a T-Cartier divisor Ψ which is *ample*, i.e. the function $\Psi : \mathrm{pos}(\mathcal{B}) \to \mathbb{R}$ is convex and restricts to a different linear function on each maximal cone of Σ. Two T-Cartier divisors Ψ_1 and Ψ_2 are *equivalent* if $\Psi_1 - \Psi_2$ is a linear map on $\mathrm{pos}(\mathcal{B})$, i.e. it is an element of M. A *divisor* on Σ is an equivalence class of T-Cartier divisors on Σ. Since Ψ_1 is ample if and only if Ψ_2 is ample, ampleness is well-defined for divisors $[\Psi]$. Finally, we define a *polarized triangulation* of $\mathcal{B}$ to be a pair consisting of a triangulation Σ of $\mathcal{B}$ and an ample divisor $[\Psi]$.

The cokernel of $M \xrightarrow{B} \mathbb{Z}^n$ is identified with $\mathbb{Z}^d$ in (1) and we call it the *Picard group*. Hence $\mathcal{A} = \{a_1,\ldots,a_n\}$ is a vector configuration in the Picard group. The *chamber complex* $\Gamma(\mathcal{A})$ of $\mathcal{A}$ is defined to be the coarsest fan with support $\mathrm{pos}(\mathcal{A})$ that refines all triangulations of $\mathcal{A}$. Experts in toric geometry will note that $\Gamma(\mathcal{A})$ equals the *secondary fan* of $\mathcal{B}$ as in [7]. We say that $\theta \in \mathbb{N}\mathcal{A}$ is *generic* if it lies in an open chamber of $\Gamma(\mathcal{A})$. Thus $\theta \in \mathbb{N}\mathcal{A}$ is generic if it is not in any lower-dimensional cone $\mathrm{pos}\{a_{i_1}, \ldots, a_{i_{d-1}}\}$ spanned by columns of A. The chamber complex $\Gamma(\mathcal{A})$ parameterizes the different combinatorial types of the convex polyhedra

$$P_\theta \quad = \quad \big\{ u \in \mathbb{R}^n \,:\, Au = \theta,\, u \geq 0 \big\}$$

as θ ranges over $\mathbb{N}\mathcal{A}$. In particular, θ is generic if and only if P_θ is $(n-d)$-dimensional and each of its vertices has exactly d non-zero coordinates (i.e. P_θ is simple). A vector θ in $\mathbb{N}\mathcal{A}$ is called an *integral degree* if every vertex of the polyhedron P_θ is a lattice point in $\mathbb{Z}^n$.

Proposition 2.3 *There is a one-to-one correspondence between generic integral degrees θ in $\mathbb{N}\mathcal{A}$ and polarized triangulations $\big(\Sigma, [\Psi]\big)$ of $\mathcal{B}$. When forgetting the polarization this correspondence gives a bijection between open chambers of $\Gamma(\mathcal{A})$ and regular triangulations Σ of $\mathcal{B}$.*

Proof: Given a generic integral degree θ, we construct the corresponding polarized triangulation $\big(\Sigma, [\Psi]\big)$. First choose any $\psi \in \mathbb{Z}^n$ such that $A\psi = -\theta$. Then consider the polyhedron

$$Q_\psi \quad := \quad \big\{ v \in M_{\mathbb{R}} \,:\, Bv \,\geq\, \psi \,\big\}.$$

The map $v \mapsto Bv - \psi$ is an affine-linear isomorphism from Q_ψ onto P_θ which identifies the set of lattice points $Q_\psi \cap M$ with the set of lattice points $P_\theta \cap \mathbb{Z}^n$. The set of linear functionals which are bounded below on Q_ψ is precisely the cone $\mathrm{pos}(\mathcal{B}) \subset N$. Finally, define the function

$$\Psi \,:\, \mathrm{pos}(\mathcal{B}) \to \mathbb{R}\,, \;\; w \mapsto \min\{\, w \cdot v \,:\, v \in Q_\psi \,\}.$$

This is the *support function* of Q_ψ, which is piecewise-linear, convex and continuous. It takes integer values on $N \cap \mathrm{pos}(\mathcal{B})$ because each vertex of Q_ψ lies in M. Since Q_ψ is a simple polyhedron, its *normal fan* is a regular triangulation Σ_θ of $\mathcal{B}$, and Ψ restricts to a different linear function on each maximal face of Σ_θ. Hence $\big(\Sigma_\theta, [\Psi]\big)$ is a polarized triangulation of $\mathcal{B}$.
Conversely, if we are given a polarized triangulation $\big(\Sigma, [\Psi]\big)$ of $\mathcal{B}$, then we define $\psi := (\Psi(b_1), \ldots, \Psi(b_n)) \in \mathbb{Z}^n$, and $\theta = -A\psi$ is the corresponding generic integral degree in $\mathbb{N}\mathcal{A}$. $\square$

Theorem 2.4 *Let $\theta \in \mathbb{N}\mathcal{A}$ be a generic integral degree. Then $X(A, \theta)$ is an orbifold and equals the toric variety $X(\Sigma_\theta)$, where Σ_θ is the regular triangulation of $\mathcal{B}$ given by θ as in Proposition 2.3.*

Proof: First note that the multigraded polynomial ring S is the *homogeneous coordinate ring* in the sense of Cox [6] of the toric variety $X(\Sigma_\theta)$. Specifically, our sequence (1) is precisely the second row in (1) on page 19 of [6]. The irrelevant ideal B_{Σ_θ} of $X(\Sigma_\theta)$ equals the radical of the ideal generated by

$\bigoplus_{r=1}^{\infty} S_{r\theta}$. Since Σ_θ is a simplicial fan, by [6, Theorem 2.1], $X(\Sigma_\theta)$ is the geometric quotient of $\mathbb{C}^n \backslash \mathcal{V}(B_{\Sigma_\theta})$ modulo $\mathbb{T}^d_{\mathbb{C}}$. The variety $\mathcal{V}(B_{\Sigma_\theta})$ consists of the points in $\mathbb{C}^n$ which are not semi-stable with respect to the $\mathbb{T}^d_{\mathbb{C}}$-action. By standard results in Geometric Invariant Theory, the geometric quotient of the semi-stable locus in $\mathbb{C}^n$ modulo $\mathbb{T}^d_{\mathbb{C}}$ coincides with $X(A,\theta) = \mathrm{Proj}\,(S_{(\theta)}) = \mathbb{C}^n /\!/_\theta\, \mathbb{T}^d_{\mathbb{C}}$. Therefore $X(A,\theta)$ is isomorphic to $X(\Sigma_\theta)$. □

Corollary 2.5 *The distinct GIT quotients* $X(A,\theta) = \mathbb{C}^n /\!/_\theta\, \mathbb{T}^d_{\mathbb{C}}$ *which are toric orbifolds are in bijection with the open chambers in* $\Gamma(\mathcal{A})$*, and hence with the regular triangulations of* $\mathcal{B}$*.*

Recall that for every scheme X there is a canonical morphism

$$\pi_X : X \mapsto X_0 \tag{6}$$

to the affine scheme $X_0 = \mathrm{Spec}(H^0(X, \mathcal{O}_X))$ of regular functions on X. We call a toric variety X *semi-projective* if X has at least one torus-fixed point and the morphism π_X is projective.

Theorem 2.6 *The following three classes of toric varieties coincide:*

1. *semi-projective toric orbifolds,*
2. *the GIT-quotients* $X(A,\theta)$ *constructed in* (5) *where* $\theta \in \mathbb{N}\mathcal{A}$ *is a generic integral degree,*
3. *toric varieties* $X(\Sigma)$ *where* Σ *is a regular triangulation of a set* $\mathcal{B}$ *which spans the lattice* N*.*

Proof: The equivalence of the classes 2 and 3 follows from Theorem 2.4. Let $X(\Sigma)$ be a toric variety in class 3. Since $\mathcal{B}$ spans the lattice, the fan Σ has a full-dimensional cone, and hence $X(\Sigma)$ has a torus-fixed point. Since Σ is simplicial, $X(\Sigma)$ is an orbifold. The morphism π_X can be described as follows. The ring of global sections $H^0(X(\Sigma), \mathcal{O}_{X(\Sigma)})$ is the semigroup algebra of the semigroup in M consisting of all linear functionals on N which are non-negative on the support $|\Sigma|$ of Σ. Its spectrum is the affine toric variety whose cone is $|\Sigma|$. The triangulation Σ supports an ample T-Cartier divisor Ψ. The morphism π_X is projective since it is induced by Ψ. Hence $X(\Sigma)$ is in class 1. Finally, let X be any semi-projective toric orbifold. It is represented by a fan Σ in a lattice N. The fan Σ is simplicial since X is an orbifold, and $|\Sigma|$ spans $N_{\mathbb{R}}$ since X has at least one fixed point. Since the morphism π_X is projective, the fan Σ is a regular triangulation of a subset $\mathcal{B}'$ of $|\Sigma|$ which includes the rays of Σ. The set $\mathcal{B}'$ need not span the lattice N. We choose any superset $\mathcal{B}$ of $\mathcal{B}'$ which is contained in $\mathrm{pos}(\mathcal{B}') = |\Sigma|$ and which spans the lattice N. Then Σ can also be regarded as a regular triangulation of $\mathcal{B}$, and we conclude that X is in class 3. □

Remark. 1. The passage from $\mathcal{B}'$ to $\mathcal{B}$ in the last step means that any GIT quotient of $\mathbb{C}^{n'}$ modulo any abelian subgroup of $\mathbb{T}_{\mathbb{C}}^{n'}$ can be rewritten as a GIT quotient of some bigger affine space $\mathbb{C}^n$ modulo a subtorus of $\mathbb{T}_{\mathbb{C}}^n$. This construction applies in particular when the given abelian group is finite, in which case the initial subset $\mathcal{B}'$ of N is linearly independent.
2. Our proof can be extended to show the following: if X is any toric variety where the morphism π_X is projective then X is the product of a semi-projective toric variety and a torus.
3. The affinization map (6) for $X(A,\theta)$ is the canonical map to $X(A,0)$.

A triangulation Σ of a subset $\mathcal{B}$ of $N \simeq \mathbb{Z}^{n-d}$ is called *unimodular* if every maximal cone of Σ is spanned by a basis of N. This property holds if and only if $X(\Sigma)$ is a toric manifold (= smooth toric variety). We say that a vector θ in $\mathbb{N}\mathcal{A}$ is a *smooth degree* if $C^{-1}\cdot\theta \geq 0$ implies $\det(C) = \pm 1$ for every non-singular $d\times d$-submatrix C of A. Equivalently, the edges at any vertex of the polyhedron P_θ generate $\ker_{\mathbb{Z}} A \cong \mathbb{Z}^{n-d}$. From Theorem 2.6 we conclude:

Corollary 2.7 *The following three classes of smooth toric varieties coincide:*

1. *semi-projective toric manifolds,*
2. *the GIT-quotients $X(A,\theta)$ constructed in* (5) *where $\theta \in \mathbb{N}\mathcal{A}$ is a generic smooth degree,*
3. *toric varieties $X(\Sigma)$ where Σ is a regular unimodular triangulation of a spanning set $\mathcal{B} \subset N$.*

Definition 2.8 *The matrix A is called* unimodular *if the following equivalent conditions hold:*

- *all non-zero $d\times d$-minors of A have the same absolute value,*
- *all $(n-d)\times(n-d)$-minors of the matrix B in (1) are -1, 0 or $+1$,*
- *every triangulation of $\mathcal{B}$ is unimodular,*
- *every vector θ in $\mathbb{N}\mathcal{A}$ is an integral degree,*
- *every vector θ in $\mathbb{N}\mathcal{A}$ is a smooth degree.*

Corollary 2.9 *For A unimodular, every GIT quotient $X(A,\theta)$ is a semi-projective toric manifold, and the distinct smooth quotients $X(A,\theta)$ are in bijection with the open chambers in $\Gamma(\mathcal{A})$.*

Every affine toric variety has a natural moment map onto a polyhedral cone, and every projective toric variety has a moment map onto a polytope. These are described in Section 4.2 of [8]. It is straightforward to extend this description to semi-projective toric varieties. Suppose that the S_0-algebra $S_{(\theta)}$ in Lemma

2.1 is generated by a set of $m+1$ monomials in S_θ, possibly after replacing θ by a multiple in the non-unimodular case. Let $\mathbb{P}^m_{\mathbb{C}}$ be the projective space whose coordinates are these monomials. Then, by definition of "Proj", the toric variety $X(A,\theta)$ is embedded as a closed subscheme in the product $\mathbb{P}^m_{\mathbb{C}} \times \mathrm{Spec}(S_0)$. We have an action of the $(n-d)$-torus $\mathbb{T}^n_{\mathbb{C}}/\mathbb{T}^d_{\mathbb{C}}$ on $\mathbb{P}^m_{\mathbb{C}}$, since S_θ is an eigenspace of $\mathbb{T}^d_{\mathbb{C}}$. This gives rise to a moment map $\mu_1 : \mathbb{P}^m_{\mathbb{C}} \to \mathbb{R}^{n-d}$, whose image is a convex polytope. Likewise, we have the affine moment map $\mu_2 : \mathrm{Spec}(S_0) \to \mathbb{R}^{n-d}$ whose image is the cone polar to $\mathrm{pos}(\mathcal{B})$. This defines the moment map

$$\mu \,:\, X(A,\theta) \,\subset\, \mathbb{P}^m_{\mathbb{C}} \times \mathrm{Spec}(S_0) \,\to\, \mathbb{R}^{n-d}, \;\; (u,v) \mapsto \mu_1(u) + \mu_2(v). \tag{7}$$

The image of $X(A,\theta)$ under the moment map μ is the polyhedron $P_\theta \simeq Q_\psi$, since the convex hull of its vertices equals the image of μ_1 and the cone $P_0 \simeq Q_0$ equals the image of μ_2.

Given an arbitrary fan Σ in N, Section 2.3 in [8] describes how a one-parameter subgroup λ_v, given by $v \in N$, acts on the toric variety $X(\Sigma)$. Consider any point x in $X(\Sigma)$ and let $\gamma \in \Sigma$ be the unique cone such that x lies in the orbit O_γ. The orbit O_γ is fixed pointwise by the one-parameter subgroup λ_v if and only if v lies in the $\mathbb{R}$-linear span $\mathbb{R}\gamma$ of γ. Thus the irreducible components F_i of the fixed point locus of the λ_v-action on $X(\Sigma)$ are the orbit closures $\overline{O}_{\sigma_i}$ where σ_i runs over all cones in Σ which are minimal with respect to the property $v \in \mathbb{R}\sigma_i$.

The closure of O_γ in $X(\Sigma)$ is the toric variety $X(\mathrm{Star}(\gamma))$ given by the quotient fan $\mathrm{Star}(\gamma)$ in $N(\gamma) = N/(N \cap \mathbb{R}\gamma)$; see [8, page 52]. From this we can derive the following lemma.

Lemma 2.10 *For $v \in N$ and $x \in O_\gamma$ the limit $\lim_{z\to 0} \lambda_v(z)\, x$ exists and lies in $F_i = \overline{O}_{\sigma_i}$ if and only if $\gamma \subseteq \sigma_i$ is a face and the image of v in $N_{\mathbb{R}}/\mathbb{R}\gamma$ is in the relative interior of $\sigma_i/\mathbb{R}\gamma$.*

The set of all faces γ of σ_i with this property is closed under taking intersections and hence this set has a unique minimal element. We denote this minimal element by τ_i. Thus if we denote

$$U_i^v \quad = \quad \big\{\, x \in X(\Sigma) \,:\, \lim_{z\to 0} \lambda_v(z)\, x \;\text{ exists and lies in } F_i \,\big\},$$

or just U_i for short, then this set decomposes as a union of orbits as follows:

$$U_i \quad = \quad \cup_{\tau_i \subseteq \gamma \subseteq \sigma_i} O_\gamma. \tag{8}$$

In what follows we further suppose $v \in |\Sigma|$. Then Lemma 2.10 implies $X(\Sigma) = \cup_i U_i$, which is the *Bialynicki-Birula decomposition* [2] of the toric variety with respect to the one-parameter subgroup λ_v.

We now apply this to our semi-projective toric variety $X(A,\theta)$ with fan $\Sigma = \Sigma_\theta$. The moment map μ_v for the circle action induced by λ_v is given by the inner product $\mu_v(x) = \langle v, \mu(x)\rangle$ with μ as in (7). We relabel the fixed components F_i according to the values of this moment map, so that

$$\mu_v(F_i) < \mu_v(F_j) \text{ implies } i < j. \tag{9}$$

Given this labeling, the distinguished faces $\tau_i \subseteq \sigma_i$ have the following important property:

$$\tau_i \subseteq \sigma_j \quad \text{implies} \quad i \leq j. \tag{10}$$

This generalizes the property $(*)$ in [8, Chapter 5.2], and it is equivalent to

$$U_j \quad \text{is closed in} \quad U_{\leq j} = \cup_{i\leq j} U_i. \tag{11}$$

This means that the Bialynicki-Birula decomposition of $X(A, \theta)$ is *filtrable* in the sense of [2]. The following is well-known in the projective case.

Proposition 2.11 *The integral cohomology of a smooth semi-projective toric variety $X(A, \theta)$ equals*

$$H^*(X(A,\theta);\mathbb{Z}) \quad \cong \quad \mathbb{Z}[x_1, x_2, \ldots, x_n]/(\mathrm{Circ}(\mathcal{A}) + I_\theta),$$

where I_θ is the Stanley-Reisner ideal of the simplicial fan Σ_θ, i.e. I_θ is generated by square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_k}$ corresponding to non-faces $\{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\}$ of Σ_θ, and Circ$(\mathcal{A})$ *is the* circuit ideal

$$\mathrm{Circ}(\mathcal{A}) \quad := \quad \langle \sum_{i=1}^{n} \lambda_i x_i \mid \lambda \in \mathbb{Z}^n, A\cdot\lambda = 0 \rangle.$$

Proof: Let $D_1, D_2, \ldots, D_n$ denote the divisors corresponding to the rays $b_1, b_2, \ldots, b_n$ in Σ_θ. The cohomology class of any torus orbit closure $\overline{O}_\sigma$ can be expressed in terms of the D_i's, namely if the rays in σ are $b_{i_1}, b_{i_2}, \ldots, b_{i_k}$, then $[\overline{O}_\sigma] = [D_{i_1}][D_{i_2}]\cdots[D_{i_k}]$. Following the reasoning in [8, Section 5.2], we first prove that certain torus orbit closures linearly span $H^*(X(A,\theta);\mathbb{Z})$ and hence the cohomology classes $[D_1], [D_2], \ldots, [D_n]$ generate $H^*(X(A,\theta);\mathbb{Z})$ as a $\mathbb{Z}$-algebra.

We choose $v \in |\Sigma|$ to be generic, so that each σ_i is $(n-d)$-dimensional and each F_i is just a point. Then (8) shows that U_i is isomorphic with the affine space $\mathbb{C}^{n-k_i}$, where $k_i = \dim(\tau_i)$.

We set $U_{\leq j} = \cup_{i\leq j} U_i$ and $U_{<j} = \cup_{i<j} U_i$. Note that U_j is closed in $U_{\leq j}$. Thus writing down the cohomology long exact sequence of the pair $(U_{\leq j}, U_{<j})$, we can show by induction on j that the cohomology classes of the closures of the cells U_i generate $H^*(X(A,\theta);\mathbb{Z})$ additively. Because the closure of a cell U_i is the closure of a torus orbit, it follows that the cohomology classes $[D_1], [D_2], \ldots, [D_n]$ generate $H^*(X(A,\theta);\mathbb{Z})$. Thus sending $x_i \mapsto [D_i]$ defines a surjective ring map $\mathbb{Z}[x_1, \ldots, x_n] \to H^*(X(A,\theta);\mathbb{Z})$, whose kernel is seen

to contain $\mathrm{Circ}(\mathcal{A}) + I_\theta$. That this is precisely the kernel follows from the "algebraic moving lemma" of [8, page 107]. □

A similar proof works with $\mathbb{Q}$-coefficients when $X(A,\theta)$ is not smooth but just an orbifold.

Corollary 2.12 *The rational cohomology ring of a semi-projective toric orbifold $X(A,\theta)$ equals*

$$H^*(X(A,\theta);\mathbb{Q}) \quad \cong \quad \mathbb{Q}[u_1,u_2,\ldots,u_n]/(\mathrm{Circ}(\mathcal{A})+I_\theta).$$

In light of Corollary 2.12, the Betti numbers of $X(A,\theta)$ satisfy $b_{2i} = h_i(\Sigma_\theta)$, where $h_i(\Sigma_\theta)$ are the h-numbers of the Stanley-Reisner ideal I_θ, cf. [18, Section III.3]. This observation leads to the following result.

Corollary 2.13 *If $f_i(P_\theta^{bd})$ denotes the number of i-dimensional bounded faces of P_θ then the Betti numbers of the semi-projective toric orbifold $X(A,\theta)$ are given by the following formula:*

$$b_{2k} \quad = \quad \dim_{\mathbb{Q}} H^{2k}(X(A,\theta);\mathbb{Q}) \quad = \quad \sum_{i=k}^{n-d} (-1)^{i-k}\binom{i}{k} f_i(P_\theta^{bd}). \tag{12}$$

Proof: Lemma 2.3 of [19] implies that

$$\sum_{i=0}^{n-d} h_i(\Sigma_\theta)\cdot x^i \;=\; \sum_{\sigma\in\Sigma_\theta\backslash\partial\Sigma_\theta} (x-1)^{n-d-\dim(\sigma)}, \tag{13}$$

where $\partial\Sigma_\theta$ denotes the boundary of Σ_θ. Hence the right hand sum is over all interior cones σ of the fan Σ_θ. These cones are in order-reversing bijection with the bounded faces of P_θ. Hence (13) is the sum of $(x-1)^{\dim(F)}$ where F runs over all bounded faces of P_θ. This proves (12). □

3 The core of a toric variety

The proof of Corollary 2.13 shows the importance of interior cones of Σ_θ. They are the ones for which the closure of the corresponding torus orbit in $X(A,\theta)$ is compact. This suggests the following

Definition 3.1 *The* core *of a semi-projective toric variety $X(A,\theta)$ is $C(A,\theta) = \cup_{\sigma\in\Sigma_\theta\backslash\partial\Sigma_\theta} O_\sigma$. Thus the core $C(A,\theta)$ is the union of all compact torus orbit closures in $X(A,\theta)$.*

Theorem 3.2 *The core of a semi-projective toric orbifold $X(A,\theta)$ is the inverse image of the origin under the canonical projective morphism $X(A,\theta)\to X(A,0)$ as in (6). It also equals the inverse image of the bounded faces of the polyhedron P_θ under the moment map (7) from $X(A,\theta)$ onto P_θ. In particular, the core of $X(A,\theta)$ is a union of projective toric orbifolds.*

Proof: On the level of fans, the toric morphism $X(A,\theta) \to X(A,0)$ corresponds to forgetting the triangulation of the cone $|\Sigma| = \mathrm{pos}(\mathcal{B})$. It follows from the description of toric morphisms in Section 1.4 of [8] that the inverse image of the origin is the union of the orbit closures corresponding to interior faces of Σ. This was our first assertion. Each face of a simple polyhedron is a simple polyhedron, and each bounded face is a simple polytope. If σ is the interior cone of Σ dual to a bounded face of P_θ then the corresponding orbit closure is the projective toric orbifold $X(\mathrm{Star}(\sigma))$. The core $C(A,\theta)$ is the union of these orbifolds. $\square$

We fix a generic vector $v \in \mathrm{int}|\Sigma|$. Then the F_i above are points and lie in $C(A,\theta)$. In what follows we shall study the action of the one-parameter subgroup λ_v on the core $C(A,\theta)$. We define

$$D_i \,=\, U_i^{-v} \quad = \quad \big\{\, x \in X(A,\theta) \,:\, \lim_{z\to\infty} \lambda_v(z)\, x \ \text{ exists and equals } F_i \,\big\}.$$

Lemma 2.10 implies that this gives a decomposition of the core: $C(A,\theta) = \cup_i D_i$. The closure $\overline{D}_i$ is a projective toric orbifold, and it is the preimage of a bounded face of P_θ via the moment map (7). If we now introduce an ordering as in (9) then the counterpart of (11) is the following:

$$D_{\leq j} \;=\; \cup_{i\leq j} D_i \ \text{ is compact.} \tag{14}$$

This property of the decomposition $C(A,\theta) = \cup_i D_i$ translates into a non-trivial statement about the convex polyhedron P_θ. Let P_θ^{bd} denote the *bounded complex*, that is, the polyhedral complex consisting of all bounded faces of P_θ. Let P_j denote the bounded face of P_θ corresponding to $\overline{D}_j$, and let p_j denote the vertex of P_θ corresponding to F_j. Then $P_{\leq j} = \cup_{i\leq j} P_i$ is a subcomplex of the bounded complex P_θ^{bd}, and $P_{\leq j} \backslash P_{<j}$ consists precisely of those faces of P_j which contain p_j. This property is called *star-collapsibility*. It implies that $P_{<j}$ is a deformation retract of $P_{\leq j}$ and in turn that P_θ^{bd} is contractible. The contractibility also follows from [4, Exercise 4.27 (a)]. In summary we have proven the following result.

Theorem 3.3 *The bounded complex P_θ^{bd} of P_θ is star-collapsible; in particular, it is contractible.*

This theorem implies that the core of any semi-projective toric variety is connected, since $C(A,\theta)$ is the preimage of the bounded complex P_θ^{bd} under the continuous moment map. Moreover, since the cohomology of P_θ^{bd} vanishes, the bounded complex does not contribute to the cohomology of $C(A,\theta)$. This fact is expressed in the following proposition, which will be crucial in Section 7.

Proposition 3.4 *Let $C(A,\theta)$ be the core of a semi-projective toric orbifold and consider a class α in $H^*(C(A,\theta);\mathbb{Q})$. If α vanishes on every irreducible component of $C(A,\theta)$ then $\alpha = 0$.*

Proof: Let $v \in \mathrm{int}|\Sigma|$, F_i and D_i as above. We prove by induction on j that

$$\text{if } \alpha \in H^*(D_{\leq j};\mathbb{Q}) \text{ and } \alpha\,|_{\overline{D}_i} = 0 \text{ for } i \leq j, \text{ then } \alpha = 0. \tag{15}$$

This implies the proposition, because if α vanishes on every irreducible component of the core then it vanishes on every irreducible projective subvariety $\overline{D}_i$ of the core. The statement (15) then implies by induction that α vanishes on the core.
To prove (15) consider the Mayer-Vietoris sequence of the covering $D_{\leq j} = D_{<j} \cup \overline{D}_j$.

$$\ldots \to H^k(D_{\leq j};\mathbb{Q}) \xrightarrow{\alpha} H^k(D_{<j};\mathbb{Q}) \oplus H^k(\overline{D}_j;\mathbb{Q}) \xrightarrow{\beta} H^k(D_{<j} \cap \overline{D}_j;\mathbb{Q}) \to \ldots$$

We show that the map α is injective, which will prove our claim. For this we show that β is surjective. This follows from the surjectivity of $H^k(\overline{D}_j;\mathbb{Q}) \to H^k(\overline{D}_j \backslash D_j;\mathbb{Q})$, because clearly $D_{<j} \cap \overline{D}_j = \overline{D}_j \backslash D_j$.
To prove this we do Morse theory on the projective toric orbifold $\overline{D}_j$. First it follows from Morse theory that $H^*(\overline{D}_j;\mathbb{Q}) \to H^*(\overline{D}_j \backslash F_j;\mathbb{Q})$ surjects. Moreover we have that $\overline{D}_j \backslash D_j$ is the core of the quasi-projective variety $\overline{D}_j \backslash F_j$. This means that $\overline{D}_j \backslash D_j$ is the set of points x in $\overline{D}_j$ such that $\lim_{z\to\infty} \lambda_v(z)x$ is not in F_j. Then the proof of Theorem 3.5 shows that $H^*(\overline{D}_j \backslash F_j;\mathbb{Q})$ is isomorphic with $H^*(\overline{D}_j \backslash D_j;\mathbb{Q})$. This proves (15) and in turn our Proposition 3.4. $\square$

We finish this section with an explicit description of the cohomology ring of $C(A,\theta)$, namely, we identify it with the cohomology of the ambient semi-projective toric orbifold $X(A,\theta)$:

Theorem 3.5 *The embedding of the core $C(A,\theta)$ in $X(A,\theta)$ induces an isomorphism on cohomology with integer coefficients.*

Proof: Let $v \in \mathrm{int}|\Sigma|$, F_i, U_i and D_i as above. We clearly have an inclusion $D_{\leq j} \subset U_{\leq j}$. We show by induction on j that this inclusion induces an isomorphism on cohomology. Consider the following commutative diagram:

$$\begin{array}{ccccccc} \ldots \to & H^k(U_{\leq j}, U_{<j};\mathbb{Z}) & \to & H^k(U_{\leq j};\mathbb{Z}) & \to & H^k(U_{<j};\mathbb{Z}) & \to \ldots \\ & \downarrow & & \downarrow & & \downarrow & \\ \ldots \to & H^k(D_{\leq j}, D_{<j};\mathbb{Z}) & \to & H^k(D_{\leq j};\mathbb{Z}) & \to & H^k(D_{<j};\mathbb{Z}) & \to \ldots \end{array}.$$

The rows are the long exact sequence of the pairs $(U_{\leq j}, U_{<j})$ and $(D_{\leq j}, D_{<j})$ respectively. The vertical arrows are induced by inclusion. The last vertical arrow is an isomorphism by induction.
By excision $H^k(U_{\leq j}, U_{<j};\mathbb{Z}) \cong H^k(T(N_j), t_0; Z)$, where N_j is the normal (orbi-)bundle to U_j and $T(N_j)$ is the Thom space $N_j \cup t_0$, where t_0 is the point at infinity. Similarly $H^k(D_{\leq j}, D_{<j};\mathbb{Z}) \cong H^k(T(D_j), t_0;\mathbb{Z})$, where $T(D_j) = D_{\leq j}/D_{<j}$ is the one point compactification of D_j, which is homeomorphic to the Thom space of $N_j|_{F_j}$, the negative bundle at F_j. Because F_j is a deformation retract of U_j and because the normal bundle N_j to U_j in $U_{\leq j}$

restricts to the normal bundle of F_j in D_j, we find that $T(D_j)$ is a deformation retract of $T(N_j)$. Consequently the first vertical arrow is also an isomorphism. The Five Lemma now delivers our assertion. □

Remark. One can prove more, namely, that $C(A,\theta)$ is a deformation retract of $X(A,\theta)$. This follows from Theorem 3.5 and the analogous statement about the fundamental group, which vanishes for both spaces. Alternatively, one can use Bott-Morse theory in the spirit of the proof of [16, Theorem 3.2] to get the homotopy equivalence.

4 Lawrence toric varieties

In this section we examine an important class of toric varieties which are semi-projective but not projective. We fix an integer $d \times n$-matrix A as in (1), and we write $A^{\pm} = [A, -A]$ for the $d \times 2n$-matrix obtained by appending the negative of A to A. The corresponding vector configuration $\mathcal{A}^{\pm} = \mathcal{A} \cup -\mathcal{A}$ spans $\mathbb{Z}^d$ as a semigroup; in symbols, $\mathbb{N}\mathcal{A}^{\pm} = \mathbb{Z}\mathcal{A} = \mathbb{Z}^d$. A vector θ is *generic* with respect to $\mathcal{A}^{\pm}$ if it does not lie on any hyperplane spanned by a subset of $\mathcal{A}$.

Definition 4.1 *We call* $X(A^{\pm},\theta)$ *a* Lawrence toric variety*, for any generic vector* $\theta \in \mathbb{Z}^d$.

Our choice of name comes from the Lawrence construction in polytope theory; see e.g. Chapter 6 in [22]. The Gale dual of the centrally symmetric configuration $\mathcal{A}^{\pm}$ is denoted $\Lambda(\mathcal{B})$ and is called the *Lawrence lifting* of $\mathcal{B}$. It consists of $2n$ vectors which span $\mathbb{Z}^{2n-d}$. The cone $\mathrm{pos}(\Lambda(\mathcal{B}))$ is the cone over the $(2n-d-1)$-dimensional *Lawrence polytope* with Gale transform $\mathcal{A}^{\pm}$.
Consider the even-dimensional affine space $\mathbb{C}^{2n}$ with coordinates $z_1,\ldots,z_n,w_1,\ldots,w_n$. We call a torus action on $\mathbb{C}^{2n}$ *symplectic* if the products $z_1w_1,\ldots,z_nw_n$ are fixed under this action.

Proposition 4.2 *The following three classes of toric varieties coincide:*

1. *Lawrence toric varieties,*
2. *toric orbifolds which are GIT-quotients of a symplectic torus action on* $\mathbb{C}^{2n}$ *for some* $n \in \mathbb{N}$*,*
3. *toric varieties* $X(\Sigma)$ *where* Σ *is the cone over a regular triangulation of a Lawrence polytope.*

Proof: This follows from Theorem 2.6 using the observation that a torus action on $\mathbb{C}^{2n}$ is symplectic if and only if it arises from a matrix of the form $A^{\pm}$. This means the action looks like

$$z_i \mapsto t^{a_i} \cdot z_i \ , \qquad w_i \mapsto t^{-a_i} \cdot w_i \qquad\qquad (i = 1, 2, \ldots, n)$$

Note that a polytope is Lawrence if and only if its Gale transform is centrally symmetric. □

The matrix $A^{\pm}$ is unimodular if and only if the smaller matrix A is unimodular. Therefore unimodularity of A implies the smoothness of the Lawrence toric variety, by Corollary 2.9. An interesting feature of Lawrence toric varieties is that the converse to this statement also holds:

Proposition 4.3 *The Lawrence toric variety $X(A^{\pm},\theta)$ is smooth if and only if A is unimodular.*

Proof: The chamber complex $\Gamma(\mathcal{A}^{\pm})$ is the arrangement of hyperplanes spanned by subsets of $\mathcal{A}$. The vector θ is assumed to lie in an open cell of that arrangement. For any column basis $C = \{a_{i_1},\dots,a_{i_d}\}$ of the $d \times n$-matrix A there exists a unique linear combination

$$\lambda_1 a_{i_1} + \lambda_2 a_{i_2} + \cdots + \lambda_d a_{i_d} \quad = \quad \theta.$$

Here all the coefficients λ_j are non-zero rational numbers. We consider the polynomial ring

$$\mathbb{Z}[x,y] \quad = \quad \mathbb{Z}[x_1,\dots,x_n,y_1,\dots,y_n].$$

The $2n$ variables are used to index the elements of $\mathcal{A}^{\pm}$ and the elements of $\Lambda(\mathcal{B})$. We set

$$\sigma(C,\theta) \quad = \quad \{\, x_{i_j} \,:\, \lambda_j > 0 \,\} \,\cup\, \{\, y_{i_j} \,:\, \lambda_j < 0 \,\}.$$

Its complement $\overline{\sigma}(C,\theta) = \big\{x_1,\dots,x_n,y_1,\dots,y_n\big\}\backslash\sigma(C,\theta)$ corresponds to a subset of $\Lambda(\mathcal{B})$ which forms a basis of $\mathbb{R}^{2n-d}$. The triangulation Σ_θ of the Lawrence polytope defined by θ is identified with its set of maximal faces. This set equals

$$\Sigma_\theta \quad = \quad \big\{\,\overline{\sigma}(C,\theta) \;:\; C \text{ is any column basis of } A\,\big\}. \qquad (16)$$

Hence the Lawrence toric variety $X(A^{\pm},\theta) = X(\Sigma_\theta)$ is smooth if and only if every basis in $\Lambda(\mathcal{B})$ spans the lattice $\mathbb{Z}^{2n-d}$ if and only if every column basis C of A spans $\mathbb{Z}^d$. The latter condition is equivalent to saying that A is a unimodular matrix. □

Corollary 4.4 *The Stanley-Reisner ideal of the fan Σ_θ equals*

$$I_\theta \quad = \quad \bigcap_C \langle \sigma(C,\theta)\rangle \quad \subset \quad \mathbb{Z}[x,y], \qquad (17)$$

i.e. I_θ is the intersection of the monomial prime ideals generated by the sets $\sigma(C,\theta)$ where C runs over all column bases of A. The irrelevant ideal of the Lawrence toric variety $X(\Sigma_\theta)$ equals

$$B_\theta \quad = \quad \langle \prod \sigma(C,\theta) \,:\, C \textit{ is any column basis of } A\,\rangle \quad \subset \quad \mathbb{Z}[x,y]. \qquad (18)$$

We now compute the cohomology of a Lawrence toric variety. For simplicity of exposition we assume A is unimodular so that $X(A^{\pm},\theta)$ is smooth. The orbifold case is analogous. First note

$$\mathrm{Circ}(\mathcal{A}^{\pm}) \quad = \quad \langle x_1 + y_1, x_2 + y_2, \ldots, x_n + x_n \rangle \ + \ \mathrm{Circ}(\mathcal{A}),$$

where $\mathrm{Circ}(\mathcal{A})$ is generated by all linear forms $\sum_{i=1}^n \lambda_i x_i$ such that $\lambda = (\lambda_1,\ldots,\lambda_n)$ lies in $\ker(A) = \mathrm{im}(B)$. From Proposition 2.11, we have

$$H^*\big(X(A^{\pm},\theta) = \mathbb{Z}[x,y]\,/\bigg(\langle x_1+y_1, x_2+y_2,\ldots,x_n+y_n\rangle \ + \ \mathrm{Circ}(\mathcal{A}) \ + \ I_\theta\bigg).$$

Let ϕ denote the $\mathbb{Z}$-algebra epimorphism which collapses the variables pairwise:

$$\phi \,:\, \mathbb{Z}[x_1,\ldots,x_n,y_1,\ldots,y_n] \,\rightarrow\, \mathbb{Z}[x_1,\ldots,x_n], \quad x_i \mapsto x_i,\ y_i \mapsto -x_i.$$

Then we can rewrite the presentation of the cohomology ring as follows:

$$H^*\big(X(A^{\pm},\theta);\mathbb{Z}\big) \quad = \quad \mathbb{Z}[x_1,\ldots,x_n]\,/\bigg(\mathrm{Circ}(\mathcal{A}) \ + \ \phi(I_\theta)\bigg).$$

Clearly, the image of the ideal (17) under ϕ is the intersection of the ideals

$$\phi\,\big(\,\langle \sigma(C,\theta)\rangle\,\big) \quad = \quad \langle\, x_i \,:\, i \in C\,\rangle$$

where C runs over the column bases of A. Note that this ideal is independent of the choice of θ. It depends only on A. This ideal is called the *matroid ideal* of $\mathcal{B}$ and it is abbreviated by

$$\begin{aligned} M^*(\mathcal{A}) \quad &= \quad \bigcap \big\{ \langle x_{i_1},\ldots,x_{i_d}\rangle \,:\, \{a_{i_1},\ldots,a_{i_d}\} \subseteq \mathcal{A} \quad \text{is linearly independent}\,\big\} \\ &= \quad \langle x_{i_1}\cdots x_{i_k} : \{b_{i_1},\ldots,b_{i_k}\} \subseteq \mathcal{B} \ \text{ is linearly dependent}\ \rangle \\ &= \quad M(\mathcal{B}). \end{aligned}$$

We summarize what we have proven concerning the cohomology of a Lawrence toric variety.

Theorem 4.5 *The integral cohomology ring of a smooth Lawrence toric variety $X(A^{\pm},\theta)$ is independent of the choice of the generic vector θ in $\mathbb{Z}^d$. It equals*

$$H^*\big(X(A^{\pm},\theta);\mathbb{Z}\big) \quad = \quad \mathbb{Z}[x_1,\ldots,x_n]/\big(\mathrm{Circ}(\mathcal{A}) \,+\, M^*(\mathcal{A})\,\big). \tag{19}$$

The same holds for Lawrence toric orbifolds with $\mathbb{Z}$ replaced by $\mathbb{Q}$.

Remark. The independence of the cohomology ring on θ is an unusual phenomenon in the GIT-construction. Usually, the topology of the quotient changes when one crosses a wall. Theorem 4.5 says that this is not the case for symplectic torus actions. An explanation of this fact is offered through our Theorem 1.1, as there are no walls in the hyperkähler quotient construction.

The ring $\mathbb{Q}[x_1, \ldots, x_n]/M^*(\mathcal{A})$ is the Stanley-Reisner ring of the matroid complex (of linearly independent subsets) of the $(n-d)$-dimensional configuration $\mathcal{B}$. This ring is Cohen-Macaulay, and $\mathrm{Circ}(\mathcal{A})$ provides a linear system of parameters. We write $h(\mathcal{B}) = (h_0, h_1, \ldots, h_{n-d})$ for its h-vector. This is a well-studied quantity in combinatorics; see e.g. [5] and [18, Section III.3].

Corollary 4.6 *The Betti numbers of the Lawrence toric variety $X(A^{\pm}, \theta)$ are independent of θ, and they coincide with the entries in the h-vector of the rank $n-d$ matroid given by $\mathcal{B}$:*

$$\dim_{\mathbb{Q}} H^{2i}(X(A^{\pm}, \theta); \mathbb{Q}) \quad = \quad h_i(\mathcal{B}). \qquad \text{for} \quad i = 0, 1, \ldots, n-d.$$

Our second result in this section concerns the core of a Lawrence toric variety of dimension $2n-d$. We fix a generic vector θ in $\mathbb{Z}^d$. The fan Σ_θ is the normal fan of the unbounded polyhedron

$$P_\theta \quad = \quad \big\{ (u, v) \in \mathbb{R}^n \oplus \mathbb{R}^n \;:\; Au - Av \,=\, \theta,\; u, v \geq 0 \big\}.$$

As in the proof of Proposition 2.3, we chose any vector $\psi \in \mathbb{Z}^n$ such that $A\psi = -\theta$, and we consider the following full-dimensional unbounded polyhedron in $\mathbb{R}^{2n-d}$:

$$Q_\psi \quad = \quad \big\{ (w, t) \in \mathbb{R}^{n-d} \oplus \mathbb{R}^n \,:\, t \,\geq\, 0\,,\;\; Bw \,+\, t \,\geq\, \psi \,\big\}.$$

The map $(w, t) \mapsto (Bw + t - \psi, t)$ is an affine-linear isomorphism from Q_ψ onto P_θ. We define $\mathcal{H}(B, \psi)$ to be the arrangement of the following n hyperplanes in $\mathbb{R}^{n-d}$:

$$\{\, w \in \mathbb{R}^{n-d} \;:\; b_i \cdot w = \psi_i \,\} \qquad\qquad (i = 1, 2, \ldots, n).$$

The arrangement $\mathcal{H}(B, \psi)$ is regarded as a polyhedral subdivision of $\mathbb{R}^{n-d}$ into relatively open polyhedra of various dimensions. The collection of all such polyhedra which are bounded form a subcomplex, called the *bounded complex* of $\mathcal{H}(B, \psi)$ and denoted by $\mathcal{H}^{bd}(B, \psi)$.

Theorem 4.7 *The bounded complex $\mathcal{H}^{bd}(B, \psi)$ of the hyperplane arrangement $\mathcal{H}(B, \psi)$ in $\mathbb{R}^{n-d}$ is isomorphic to the complex of bounded faces of the $(2n-d)$-dimensional polyhedron $Q_\psi \simeq P_\theta$.*

Proof: We define an injective map from $\mathbb{R}^{n-d}$ into the polyhedron Q_ψ as follows

$$w \;\mapsto\; \big(w, t\big), \qquad \text{where} \;\; t_i \;=\; \max\{0, \psi_i - b_i \cdot w\}. \tag{20}$$

This map is linear on each cell of the hyperplane arrangement $\mathcal{H}(B, \psi)$, and the image of each cell is a face of Q_ψ. In particular, every bounded cell of $\mathcal{H}(B, \psi)$ is mapped to a bounded face of Q_ψ and each unbounded cell of $\mathcal{H}(B, \psi)$ is mapped to an unbounded face of Q_ψ. It remains to be shown that every bounded face of Q_ψ lies in the image of the map (20).

Now, the image of (20) is the following subcomplex in the boundary of our polyhedron:

$$\begin{array}{rc} & \{\, (w,t) \in Q_\psi \ : \ t_i \cdot (b_i \cdot w + t_i - \psi_i) \,=\, 0 \ \text{ for } i = 1,2,\ldots,n \,\} \\ \simeq & \{\, (u,v) \in P_\theta \ : \ u_i \cdot v_i \,=\, 0 \ \text{ for } i = 1,2,\ldots,n \,\} \end{array}$$

Consider any face F of P_θ which is not in this subcomplex, and let (u,v) be a point in the relative interior of F. There exists an index i with $u_i > 0$ and $v_i > 0$. Let e_i denote the i-th unit vector in $\mathbb{R}^n$. For every positive real λ, the vector $(u+\lambda e_i, v+\lambda e_i)$ lies in P_θ and has the support as (u,v). Hence $(u+\lambda e_i, v+\lambda e_i)$ lies in F for all $\lambda \geq 0$. This shows that F is unbounded. $\square$

Theorem 4.7 and Corollary 2.13 imply the following enumerative result:

Corollary 4.8 *The Betti numbers of the Lawrence toric variety $X(A^\pm,\theta)$ satisfy*

$$\dim_{\mathbb{Q}} H^{2i}(X(A^\pm,\theta);\mathbb{Q}) \quad = \quad \sum_{i=k}^{n-d} (-1)^{i-k} \binom{i}{k} f_i(\mathcal{H}^{bd}(B,\psi)),$$

where $f_i(\mathcal{H}^{bd}(B,\psi))$ denotes the number of i-dimensional bounded regions in $\mathcal{H}(B,\psi)$.

There are two natural geometric structures on any Lawrence toric variety. First the canonical bundle of $X(A^\pm,\theta)$ is trivial, because the vectors in $\mathcal{A}^\pm$ add to 0. This means that $X(A^\pm,\theta)$ is a *Calabi-Yau variety.* Moreover, since the symplectic $\mathbb{T}^d_{\mathbb{C}}$-action preserves the natural Poisson structure on $\mathbb{C}^{2n} \cong \mathbb{C}^n \oplus (\mathbb{C}^n)^*$, the GIT quotient $X(A^\pm,\theta)$ inherits a natural *holomorphic Poisson structure.* The holomorphic symplectic leaves of this Poisson structure are what we call *toric hyperkähler manifolds.* The special leaf which contains the core of $X(A^\pm,\theta)$ will be called the *toric hyperkähler variety.* We present these definitions in complete detail in the following two sections.

5 Hyperkähler quotients

Our aim is to describe an algebraic approach to the toric hyperkähler manifolds of Bielawski and Dancer [3]. In this section we sketch the original differential geometric construction in [3]. This construction is the hyperkähler analogue to the construction of toric varieties using Kähler quotients. We first briefly review the latter. Fix the standard Euclidean bilinear form on $\mathbb{C}^n$,

$$g(z,w) \quad = \quad \sum_{i=1}^{n} \left(\mathrm{re}(z_i)\mathrm{re}(w_i) + \mathrm{im}(z_i)\mathrm{im}(w_i) \right).$$

The corresponding *Kähler form* is

$$\omega(z,w) \quad = \quad g(iz,w) \quad = \quad \sum_{i=1}^{n} \left(\mathrm{re}(z_i)\mathrm{im}(w_i) - \mathrm{im}(z_i)\mathrm{re}(w_i) \right).$$

Let A be as in (1) and consider the real torus $\mathbb{T}^d_{\mathbb{R}}$ which is the maximal compact subgroup of $\mathbb{T}^d_{\mathbb{C}}$. The group $\mathbb{T}^d_{\mathbb{R}}$ acts on $\mathbb{C}^n$ preserving the Kähler structure. This action has the moment map

$$\mu_{\mathbb{R}} : \mathbb{C}^n \to (\mathfrak{t}^d_{\mathbb{R}})^* \cong \mathbb{R}^d, \quad (z_1, \ldots, z_n) \mapsto \frac{1}{2}\sum_{i=1}^{n} |z_i|^2 a_i. \tag{21}$$

Fix $\xi_{\mathbb{R}} \in \mathbb{R}^d$. The *Kähler quotient* $X(A, \xi_{\mathbb{R}}) = \mathbb{C}^n//_{\xi_{\mathbb{R}}}\mathbb{T}^d_{\mathbb{R}} = \mu^{-1}_{\mathbb{R}}(\xi_{\mathbb{R}})/\mathbb{T}^d_{\mathbb{R}}$ inherits a Kähler structure from $\mathbb{C}^n$ at its smooth points. If $\xi_{\mathbb{R}} = \theta$ lies in the lattice $\mathbb{Z}^d$ then there is a biholomorphism between the smooth loci in the GIT quotient $X(A, \theta)$ and the Kähler quotient $X(A, \xi_{\mathbb{R}})$. Hence if A is unimodular and θ generic then the complex manifolds $X(A, \theta)$ and $X(A, \xi_{\mathbb{R}})$ are biholomorphic. Now we turn to toric hyperkähler manifolds. Let $\mathbb{H}$ be the skew field of *quaternions*, the 4-dimensional real vector space with basis $1, i, j, k$ and associative algebra structure given by $i^2 = j^2 = k^2 = ijk = -1$. Left multiplication by i (resp. j and k) defines complex structures $I : \mathbb{H} \to \mathbb{H}$, with $I^2 = -\mathrm{Id}_{\mathbb{H}}$, (resp. J and K) on $\mathbb{H}$. We now put the flat metric g on $\mathbb{H}$ arising from the standard Euclidean scalar product on $\mathbb{H} \cong \mathbb{R}^4$ with $1, i, j, k$ as an orthonormal basis. This is called a *hyperkähler metric* because it is a Kähler metric with respect to all three complex structures I, J and K. It means that the differential 2-forms, the so-called *Kähler forms*, given by $\omega_I(X, Y) = g(IX, Y)$ for tangent vectors X and Y, and the analogously defined ω_J and ω_K are closed. For the reader's convenience we write down these Kähler forms in coordinates (x, y, u, v):

$$\begin{aligned}
\omega_I &= dx \wedge dy + du \wedge dv, \\
\omega_J &= dx \wedge du + dv \wedge dy, \\
\omega_K &= dx \wedge dv + dy \wedge du.
\end{aligned}$$

A special orthogonal transformation, with respect to this metric, is said to preserve the hyperkähler structure if it commutes with all three complex structures I, J and K or equivalently if it preserves the Kähler forms ω_I, ω_J and ω_K. The group of such transformations, the unitary symplectic group $Sp(1)$, is generated by multiplication by unit quaternions from the right. A maximal abelian subgroup $\mathbb{T}_{\mathbb{R}} \cong U(1) \subset Sp(1)$ is thus specified by a choice of a unit quaternion. We break the symmetry between I, J and K and choose the maximal torus generated by multiplication from the right by the unit quaternion i. Thus $U(1)$ acts on $\mathbb{H}$ by sending ξ to $\xi \exp(\phi i)$, for $exp(\phi i) \in U(1) \subset \mathbb{R} \oplus \mathbb{R}i \cong \mathbb{C}$. It follows from (21) that the moment map $\mu_I : \mathbb{H} \to \mathbb{R}$ with respect to the symplectic form ω_I is given by

$$\mu_I(x + yi + uj + vk) = \mu_I(x + yi + (-ui + v)k) = \frac{1}{2}(x^2 + y^2 - u^2 - v^2). \tag{22}$$

Similarly we obtain formulas for μ_J and μ_K by writing down the eigenspace decomposition in the respective complex structures:

$$\mu_J(x + yi + uj + vk) =$$

$$= \mu_J \left[\left(\frac{y+u}{\sqrt{2}} + \frac{-x-v}{\sqrt{2}} j \right) \frac{i+j}{\sqrt{2}} + \left(\frac{y-u}{\sqrt{2}} j + \frac{-x+v}{\sqrt{2}} \right) \frac{k-1}{\sqrt{2}} \right] = yu + xv,$$

$$\mu_K(x+yi+uj+vk) =$$

$$= \mu_K \left[\left(\frac{y+v}{\sqrt{2}} + \frac{-x+u}{\sqrt{2}} k \right) \frac{i+k}{\sqrt{2}} + \left(\frac{y-v}{\sqrt{2}} + \frac{x+u}{\sqrt{2}} k \right) \frac{i-k}{\sqrt{2}} \right] = yv - xu.$$

We now consider the map $\mu_{\mathbb{C}} = \mu_J + i\mu_K$ from $\mathbb{H}$ to $\mathbb{C}$. It can be thought of as the holomorphic moment map for the I-holomorphic action of $\mathbb{T}_{\mathbb{C}} \supset \mathbb{T}_{\mathbb{R}}$ on $\mathbb{H}$ with respect to the I-holomorphic symplectic form $\omega_{\mathbb{C}} = \omega_J + i\omega_K$. If we identify $\mathbb{H}$ with $\mathbb{C} \oplus \mathbb{C}$ by introducing two complex coordinates, $z = x + iy \in \mathbb{R} \oplus \mathbb{R}i \cong \mathbb{C}$ and $w = v - ui \in \mathbb{R} \oplus \mathbb{R}i \cong \mathbb{C}$, then the I-holomorphic moment map $\mu_{\mathbb{C}} : \mathbb{H} \to \mathbb{C}$ is given algebraically by multiplying complex numbers:

$$\mu_{\mathbb{C}}(z,w) \;=\; \mu_J(z,w) \;+\; i\mu_K(z,w) \;=\; yu + xv \;+\; i(yv - xu) \;=\; zw. \tag{23}$$

The discussion in the previous paragraph generalizes in an obvious manner to $\mathbb{H}^n$ for $n > 1$. Indeed, the n-dimensional quaternionic space $\mathbb{H}^n$ has three complex structures I,J and K, given by left multiplication with $i, j, k \in \mathbb{H}$. Putting the flat metric $g_n = g^{\oplus n}$ on $\mathbb{H}^n$ yields a hyperkähler metric, i.e. the differential 2-forms $\omega_I(X,Y) = g_n(IX,Y)$ and similarly ω_J and ω_K are Kähler (meaning closed) forms. The automorphism group of this hyperkähler structure is the unitary symplectic group $Sp(n)$. We fix the maximal torus $\mathbb{T}^n_{\mathbb{R}} = U(1)^n \subset Sp(n)$ given by the following definition. For $\lambda = (\exp(\phi_1 i), \exp(\phi_2 i), \ldots, \exp(\phi_n i)) \in \mathbb{T}^n_{\mathbb{R}}$ and $(\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{H}^n$ we set

$$\lambda(\xi_1, \xi_2, \ldots, \xi_n) \quad = \quad (\xi_1 \exp(\phi_1 i), \xi_2 \exp(\phi_2 i), \ldots, \xi_n \exp(\phi_n i)). \tag{24}$$

As in the $n = 1$ case above, this fixes an isomorphism $\mathbb{H}^n \cong \mathbb{C}^n \oplus \mathbb{C}^n$ where two complex vectors $z, w \in \mathbb{C}^n \cong \mathbb{R}^n \oplus i\mathbb{R}^n$ represent the quaternionic vector $z + wk \in \mathbb{H}^n \cong \mathbb{R}^n \oplus i\mathbb{R}^n \oplus j\mathbb{R}^n \oplus k\mathbb{R}^n$. Expressing vectors in $\mathbb{H}^n$ in these complex coordinates, the torus action (24) translates into

$$\lambda(z,w) \quad = \quad (\lambda z, \lambda^{-1} w) \qquad \text{for } \lambda \in \mathbb{T}^n_{\mathbb{R}} \quad \text{and} \quad (z,w) \in \mathbb{H}^n. \tag{25}$$

The toric hyperkähler manifolds in [3] are constructed by choosing a subtorus $\mathbb{T}^d_{\mathbb{R}} \subset \mathbb{T}^n_{\mathbb{R}}$ and taking the hyperkähler quotient [10] of $\mathbb{H}^n$ by $\mathbb{T}^d_{\mathbb{R}}$. We do this by choosing integer matrices A and B as in (1) and (2). The subtorus $\mathbb{T}^d_{\mathbb{R}}$ of $\mathbb{T}^n_{\mathbb{R}}$ acts on $\mathbb{H}^n$ by (25) preserving the hyperkähler structure. The *hyperkähler moment map* of the action (25) of $\mathbb{T}^d_{\mathbb{R}}$ on $\mathbb{H}^n$ is defined by

$$\mu = (\mu_I, \mu_J, \mu_K) \,:\, \mathbb{H}^n \;\to\; (\mathfrak{t}^d_{\mathbb{R}})^* \otimes \mathbb{R}^3,$$

where μ_I, μ_J and μ_K are the Kähler moment maps with respect to ω_I, ω_J and ω_K respectively. Using the formulas (22) and (23), the components of μ are in complex coordinates as follows:

$$\mu_{\mathbb{R}}(z,w) \;:=\; \mu_I(z,w) \quad = \quad \frac{1}{2} \sum_{i=1}^{n} (|z_i|^2 - |w_i|^2) \cdot a_i \quad \in \quad (\mathfrak{t}^d_{\mathbb{R}})^*, \tag{26}$$

$$\mu_{\mathbb{C}}(z,w) := \mu_J(z,w) + i\mu_K(z,w) = \sum_{i=1}^{n} z_i w_i \cdot a_i \in (\mathfrak{t}^d_{\mathbb{R}})^* \otimes \mathbb{C} \cong (\mathfrak{t}^d_{\mathbb{C}})^*. \quad (27)$$

Here a_i is the i-th column vector of the matrix A. We can also think of $\mu_{\mathbb{C}}$ as the moment map for the I-holomorphic action of $\mathbb{T}^d_{\mathbb{C}}$ on $\mathbb{H}^n$ with respect to $\omega_{\mathbb{C}} = \omega_J + i\omega_K$. Now take

$$\xi = (\xi^1, \xi^2, \xi^3) \in (\mathfrak{t}^d_{\mathbb{R}})^* \otimes \mathbb{R}^3$$

and introduce $\xi_{\mathbb{R}} = \xi^1 \in (\mathfrak{t}^d_{\mathbb{R}})^*$ and $\xi_{\mathbb{C}} = \xi^2 + i\xi^3 \in (\mathfrak{t}^d_{\mathbb{C}})^*$ so we can write $\xi = (\xi_{\mathbb{R}}, \xi_{\mathbb{C}}) \in (\mathfrak{t}^d_{\mathbb{R}})^* \oplus (\mathfrak{t}^d_{\mathbb{C}})^*$. The *hyperkähler quotient* of $\mathbb{H}^n$ by the action (25) of the torus $\mathbb{T}^d_{\mathbb{R}}$ at level ξ is defined as

$$Y(A,\xi) := \mathbb{H}^n ////_{\xi} \mathbb{T}^d_{\mathbb{R}} := \mu^{-1}(\xi)/\mathbb{T}^d_{\mathbb{R}} = \left(\mu_{\mathbb{R}}^{-1}(\xi_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})\right)/\mathbb{T}^d_{\mathbb{R}}. \quad (28)$$

By a theorem of [10], this quotient has a canonical hyperkähler structure on its smooth locus.
Bielawski and Dancer show in [3] that if $\xi \in (\mathfrak{t}^d_{\mathbb{R}})^* \otimes \mathbb{R}^3$ is generic then $Y(A,\xi)$ is an orbifold, and it is smooth if and only if A is unimodular. Since ξ is generic outside a set of codimension three in $(\mathfrak{t}^d_{\mathbb{R}})^* \otimes \mathbb{R}^3$, they can show that the topology and therefore the cohomology of the toric hyperkähler manifold is independent on ξ. In what follows we consider vectors ξ for which $\xi_{\mathbb{C}} = 0$ in $\mathbb{C}^d$ and $\xi_{\mathbb{R}} = \theta \in \mathbb{Z}^d \subset \mathbb{R}^d \cong (\mathfrak{t}^d_{\mathbb{R}})^*$. The underlying complex manifold in complex structure I of the hyperkähler manifold $Y(A,(\theta, 0_{\mathbb{C}}))$ has a purely algebraic description as explained in the next section.

6 Algebraic construction of toric hyperkähler varieties

The $\mathbb{Z}^d$-graded polynomial ring $\mathbb{C}[z,w] = \mathbb{C}[z_1,\ldots,z_n,w_1,\ldots,w_n]$, with the grading given by $A^{\pm} = [A,-A]$, is the homogeneous coordinate ring of the Lawrence toric variety $X(A^{\pm},\theta)$. By a result of Cox [6], closed subschemes of $X(A^{\pm},\theta)$ correspond to homogeneous ideals in $\mathbb{C}[z,w]$ which are saturated with respect to the irrelevant ideal B_θ in (18). Let us now consider the ideal

$$\mathrm{Circ}(\mathcal{B}) := \langle \sum_{i=1}^{n} a_{ij} z_i w_i \mid j = 1,\ldots,d \rangle \subset \mathbb{C}[z,w], \quad (29)$$

whose generators are the components of the holomorphic moment map $\mu_{\mathbb{C}}$ of (27). The ideal $\mathrm{Circ}(\mathcal{B})$ is clearly homogeneous and it is a complete intersection. We assume that none of the row vectors of the matrix B is zero. Under this hypothesis, the ideal $\mathrm{Circ}(\mathcal{B})$ is a prime ideal.

Definition 6.1 *The* toric hyperkähler variety $Y(A,\theta)$ *is the irreducible subvariety of the Lawrence toric variety* $X(A,\theta)$ *defined by the homogeneous ideal* $\mathrm{Circ}(\mathcal{B})$ *in the coordinate ring* $\mathbb{C}[z,w]$ *of* $X(A,\theta)$.

Proposition 6.2 *If θ is generic then the toric hyperkähler variety $Y(A,\theta)$ is an orbifold. It is smooth if and only if the matrix A is unimodular.*

Proof: It follows from (27) that a point in $\mathbb{C}^{2n}$ has a finite stabilizer under the group $\mathbb{T}^d_{\mathbb{C}}$ if and only if the point is regular for $\mu_{\mathbb{C}}$ of (27), i.e. if the derivative of $\mu_{\mathbb{C}}$ is surjective there. This implies that, for θ generic, the toric hyperkähler variety $Y(A,\theta)$ is an orbifold because then the variety $X(A^{\pm},\theta)$ is an orbifold. For the second statement note that if A is unimodular then $X(A^{\pm},\theta)$ is smooth, consequently $Y(A,\theta)$ is also smooth. However, if A is not unimodular then $X(A^{\pm},\theta)$ has orbifold singularities which lie in the core. Now the core $C(A^{\pm},\theta)$ lies entirely in $Y(A,\theta)$, by Lemma 6.4 below, thus $Y(A,\theta)$ inherits singular points from $X(A^{\pm},\theta)$. □

We can now prove that our toric hyperkähler varieties are biholomorphic to the toric hyperkähler manifolds of the previous section.

Theorem 6.3 *Let $\xi_{\mathbb{R}} = \theta \in \mathbb{Z}^d \subset (\mathfrak{t}^d_{\mathbb{R}})^* \cong \mathbb{R}^d$ for generic θ. Then the toric hyperkähler manifold $Y(A,(\xi_{\mathbb{R}},0))$ with complex structure I is biholomorphic with the toric hyperkähler variety $Y(A,\theta)$.*

Proof: Suppose A is unimodular. The general theory of Kähler quotients (e.g. in [12]) implies that the Lawrence toric variety $X(A^{\pm},\theta)$ and the corresponding Kähler quotient $X(A^{\pm},\xi_{\mathbb{R}}) = \mu_{\mathbb{R}}^{-1}(\xi_{\mathbb{R}})/\mathbb{T}^d_{\mathbb{R}}$ are biholomorphic, where $\mu_{\mathbb{R}}$ is defined in (26) and $\xi_{\mathbb{R}} = \theta \in \mathbb{Z}^d \subset \mathbb{R}^d \cong (\mathfrak{t}^d_{\mathbb{R}})^*$. Now the point is that $\mu_{\mathbb{C}} : \mathbb{H}^n \to \mathbb{C}^d$ is invariant under the action of $\mathbb{T}^d_{\mathbb{R}}$ and therefore descends to a map on $X(A^{\pm},\xi_{\mathbb{R}}) = \mu_{\mathbb{R}}^{-1}(\xi_{\mathbb{R}})/\mathbb{T}^d_{\mathbb{R}}$ and similarly on $X(A^{\pm},\theta)$ making the following diagram commutative:

$$\begin{array}{rccc} \mu^{\xi}_{\mathbb{C}}: & X(A^{\pm},\xi_{\mathbb{R}}) & \rightarrow & \mathbb{C}^d \\ & \cong & & \cong \\ \mu^{\theta}_{\mathbb{C}}: & X(A^{\pm},\theta) & \rightarrow & \mathbb{C}^d \end{array}.$$

It follows that $Y(A,(\xi_{\mathbb{R}},0)) = (\mu^{\xi}_{\mathbb{C}})^{-1}(0)$ and $Y(A,\theta) = (\mu^{\theta}_{\mathbb{C}})^{-1}(0)$ are biholomorphic. The proof is similar when the spaces have orbifold singularities. □

Recall the affinization map $\pi_X : X(A^{\pm},\theta) \to X(A^{\pm},0)$ from (6), and the analogous map $\pi_Y : Y(A,\theta) \to Y(A,0)$. These fit together in the following commutative diagram:

$$\begin{array}{ccc} Y(A,\theta) & \stackrel{\pi_Y}{\to} & Y(A,0) \\ i_\theta \downarrow & & \downarrow i_0 \\ X(A^{\pm},\theta) & \stackrel{\pi_X}{\to} & X(A^{\pm},0) \\ \mu^{\theta}_{\mathbb{C}} \downarrow & & \downarrow \mu^0_{\mathbb{C}} \\ \mathbb{C}^d & \cong & \mathbb{C}^d \end{array},$$

where $i_\theta : Y(A,\theta) \to X(A^{\pm},\theta)$ denotes the natural embedding in Definition 6.1 by the preimage of $\mu^{\theta}_{\mathbb{C}}$ at $0 \in \mathbb{C}^d$. From this we deduce the following lemma:

Lemma 6.4 *The cores of the Lawrence toric variety and of the toric hyperkähler variety coincide, that is, $C(A^{\pm},\theta) = \pi_X^{-1}(0) = \pi_Y^{-1}(0)$.*

Remark. It is shown in [3] that the core of the toric hyperkähler manifold $Y(A,\theta)$ is the preimage of the bounded complex in the hyperplane arrangement $\mathcal{H}(\mathcal{B},\psi)$ by the hyperkähler moment map. We know from Theorem 3.2 that the core of the Lawrence toric variety equals the preimage of P_θ^{bd} under the Kähler moment map. Thus Theorem 4.7 is a combinatorial analogue of Lemma 6.4.

We need one last ingredient in order to prove the theorem stated in the Introduction.

Lemma 6.5 *The embedding of the core $C(A^{\pm},\theta)$ in $Y(A,\theta)$ gives an isomorphism in cohomology.*

Proof: Consider the $\mathbb{T}_{\mathbb{C}}$-action on the Lawrence toric variety $X(A^{\pm},\theta)$ defined by the vector $v=\sum_{i=1}^n b_i \in \mathbb{Z}^{n-d}$. This action comes from multiplication by non-zero complex numbers on the vector space $\mathbb{C}^{2n}$. The holomorphic moment map $\mu_{\mathbb{C}}$ of (27) is homogeneous with respect to multiplication by a non-zero complex number, and consequently $\mu_{\mathbb{C}}^\theta$ is also homogeneous with respect to the circle action λ_v. It follows that this $\mathbb{T}_{\mathbb{C}}$-action leaves the toric hyperkähler variety invariant. Moreover, since v is in the interior of $\mathrm{pos}(\mathcal{B})$, all the results in Section 3 are valid for this $\mathbb{T}_{\mathbb{C}}$-action on $X(A^{\pm},\theta)$. Now the proof of Theorem 3.5 can be repeated verbatim to show that the cohomology of $Y(A,\theta)$ agrees with the cohomology of the core. □

Proof of Theorem 1.1: 1.= 3. is a consequence of Lemma 6.4 and Lemma 6.5.
2.= 3. This is a consequence of Theorem 3.5.
1.= 4. is the content of Theorem 4.5. □

Remark. 1. In fact, we could claim more than the isomorphism of cohomology rings in Theorem 1.1. The remark after Theorem 3.5 implies that the spaces $C(A^{\pm},\theta)\subset Y(A,\theta)\subset X(A^{\pm},\theta)$ are deformation retracts in one another. A similar result appears in [3, Theorem 6.5].
2. The result 2.=4. in the smooth case was proven by Konno in [14].
3. We deduce from Theorem 1.1, Corollary 4.6 and Corollary 4.8 the following formulas for Betti numbers. The second formula is due to Bielawski and Dancer [3, Theorem 6.7].

Corollary 6.6 *The Betti numbers of the toric hyperkähler variety $Y(A,\theta)$ agree with:*

- *the h-numbers of the matroid of $\mathcal{B}$:* $b_{2k}(Y(A,\theta)) = h_k(\mathcal{B})$.
- *the following linear combination of the number of bounded regions of the affine hyperplane arrangement $\mathcal{H}(B,\psi)$:*

$$b_{2k}(Y(A,\theta)) \quad = \quad \sum_{i=k}^{n-d}(-1)^{i-k}\binom{i}{k} f_i(\mathcal{H}^{bd}(B,\psi)). \tag{30}$$

This corollary shows the importance of the combinatorics of the bounded complex $\mathcal{H}^{bd}(B,\psi)$ in the topology of $Y(A,\theta)$ and $X(A^{\pm},\theta)$. This intriguing connection will be more apparent in the next section. Before we get there we infer some important properties of the bounded complex from Corollary 9.1 and Theorem E of [21].

Proposition 6.7 *The bounded complex $\mathcal{H}^{bd}(B,\psi)$ is pure-dimensional. If ψ is generic and $\mathcal{B}$ is coloop-free then every maximal face of $\mathcal{H}^{bd}(B,\psi)$ is an $(n-d)$-dimensional simple polytope.*

A *coloop* of $\mathcal{B}$ is a vector b_i which lies in every column basis of B. This is equivalent to a_i being zero. Note that if A has a zero column then we can delete it to get A', which means that $Y(A,\theta) = Y(A',\theta)\times\mathbb{C}^2$ and similarly for the Lawrence toric variety. Therefore we will assume in the next section that none of the columns of A is zero.

7 Cogenerators of the cohomology ring

There are three natural presentations of the cohomology ring of the toric hyperkähler variety $Y(A,\theta)$ associated with a $d\times n$-matrix A and a generic vector $\theta\in\mathbb{Z}^d$. In these presentations $H^*(Y(A,\theta);\mathbb{Q})$ is expressed as a quotient of the polynomial ring $\mathbb{Q}[x,y]$ in $2n$ variables, as a quotient of the polynomial ring $\mathbb{Q}[x]$ in n variables, or as a quotient of the polynomial ring $\mathbb{Q}[\,t\,] \simeq \mathbb{Q}[x]/\mathrm{Circ}(\mathcal{A})$ in d variables, respectively. In this section we compute systems of cogenerators for $H^*(Y(A,\theta);\mathbb{Q})$ relative to each of the three presentations. As an application we show that the Hard Lefschetz Theorem holds for toric hyperkähler varieties, and we discuss some implications for the combinatorial problem of classifying the h-vectors of matroid complexes.

We begin by reviewing the definition of cogenerators of a homogeneous polynomial ideal. Consider the commutative polynomial ring generated by a basis of derivations on affine m-space:

$$\mathbb{Q}[\partial] \quad = \quad \mathbb{Q}[\partial_1,\partial_2,\ldots,\partial_m].$$

The polynomials in $\mathbb{Q}[\partial]$ act as linear differential operators with constant coefficients on

$$\mathbb{Q}[x] \quad = \quad \mathbb{Q}[x_1,x_2,\ldots,x_m].$$

If Γ is any subset of $\mathbb{Q}[x]$ then its *annihilator* $\mathrm{Ann}(\Gamma)$ is the ideal in $\mathbb{Q}[\partial]$ consisting of all linear differential operators with constant coefficients which annihilate all polynomials in Γ. If I is any zero-dimensional homogeneous ideal in $\mathbb{Q}[\partial]$ then there exists a finite set Γ of homogeneous polynomials in $\mathbb{Q}[x]$ such that $I=\mathrm{Ann}(\Gamma)$. We say that Γ is a set of *cogenerators* of I. If Γ is a singleton, say, $\Gamma=\{p\}$, then $I=\mathrm{Ann}(\Gamma)$ is a *Gorenstein ideal.* In this case, the polynomial $p=p(x)$ which cogenerates I is unique up to scaling. More generally, if all polynomials in Γ are homogeneous of the same degree then

$I = \mathrm{Ann}(\Gamma)$ is a *level ideal*. In this case, the $\mathbb{Q}$-vector space spanned by Γ is unique, and it is desirable for Γ to be a nice basis for this space.

We replace the vector $\psi = (\psi_1, \ldots, \psi_n)$ in Theorem 4.7 by an indeterminate vector $x = (x_1, \ldots, x_n)$ which ranges over a small neighborhood of ψ in $\mathbb{R}^n$. For x in this neighborhood, the polyhedron Q_x remains simple and combinatorially isomorphic to Q_ψ, and the hyperplane arrangement $\mathcal{H}(B,x)$ remains isomorphic to $\mathcal{H}(B,\psi)$. Let $\Delta_1, \ldots, \Delta_r$ denote the maximal bounded regions of $\mathcal{H}(B,x)$. These are $(n-d)$-dimensional simple polytopes, by Proposition 6.7 and our assumption that $\mathcal{B}$ is coloop-free. They can be identified with the maximal bounded faces of the $(2n-d)$-dimensional polyhedron Q_x, by Theorem 4.7. The volume of the polytope Δ_i is a homogeneous polynomial in x of degree $n-d$ denoted

$$V_i(x) \quad = \quad V_i(x_1, \ldots, x_n) \quad = \quad \mathrm{vol}(\Delta_i) \qquad (i = 1, 2, \ldots, r)$$

Theorem 7.1 *The volume polynomials $V_1, \ldots, V_r$ form a basis of cogenerators for the cohomology ring of the Lawrence toric variety $X(A^\pm, \theta)$ and of the toric hyperkähler variety $Y(A, \theta)$:*

$$H^*(Y(A,\theta);\mathbb{Q}) \quad = \quad \mathbb{Q}[\partial_1, \partial_2, \ldots, \partial_n]/\mathrm{Ann}\big(\{V_1, V_2, \ldots, V_r\}\big). \tag{31}$$

Proof: Each simple polytope Δ_i represents an $(n-d)$-dimensional projective toric variety X_i. The core $C(A^\pm, \theta)$ is glued from the toric varieties $X_1, \ldots, X_r$, and it has the same cohomology as $X(A^\pm, \theta)$ and $Y(A, \theta)$ as proved in Theorem 1.1. Hence we get a natural ring epimorphism induced from the inclusion of each toric variety X_i into the core $C(A^\pm, \theta)$:

$$\phi_i \quad : \quad H^*\big(C(A^\pm,\theta);\mathbb{Q}\big) \; \to \; H^*(X_i;\mathbb{Q}). \tag{32}$$

In terms of coordinates, the map ϕ_i is described as follows:

$$\phi_i \, : \, \mathbb{Q}[\partial_1, , \ldots, \partial_n]/\big(M(\mathcal{B}) + \mathrm{Circ}(\mathcal{A})\big) \to \mathbb{Q}[\partial_1, , \ldots, \partial_n]/\big(I_{\Delta_i} + \mathrm{Circ}(\mathcal{A})\big), \tag{33}$$

where I_{Δ_i} is the Stanley-Reisner ring of the simplicial normal fan of the polytope Δ_i. Each facet of Δ_i has the form $\{\, w \in \Delta_i \, : \, b_j \cdot w = \psi_j \,\}$ for some $j \in \{1, 2, \ldots, n\}$. The ideal I_{Δ_i} is generated by all monomials $\partial_{j_1}\partial_{j_2}\cdots\partial_{j_s}$ such that the intersection of the facets $\{\, w \in \Delta_i \, : \, b_{j_\nu} \cdot w = \psi_{j_\nu} \,\}$, for $\nu = 1, 2, \ldots, s$, is the empty set. By the genericity hypothesis on ψ, this will happen if $\{b_{j_1}, b_{j_2}, \ldots, b_{j_s}\}$ is linearly dependent, or, equivalently, if $\partial_{j_1}\partial_{j_2}\cdots\partial_{j_s}$ lies in the matroid ideal $M(\mathcal{B})$. We conclude that $M(\mathcal{B}) \subseteq I_{\Delta_i}$, and the map ϕ_i in (33) is induced by this inclusion.

Proposition 3.4 implies that

$$\ker(\phi_1) \cap \ker(\phi_2) \cap \ldots \cap \ker(\phi_r) \quad = \quad \{0\}. \tag{34}$$

Here is an alternative proof for this in the toric hyperkähler case. We first note that the top-dimensional cohomology of an equidimensional union of projective varieties equals the direct sum of the pieces:

$$H^{2n-2d}\big(C(A^\pm,\theta);\mathbb{Q}\big) \quad \simeq \quad H^{2n-2d}\big(X_1;\mathbb{Q}\big) \oplus \cdots \oplus H^{2n-2d}\big(X_r;\mathbb{Q}\big), \tag{35}$$

and the restriction of the map ϕ_i to degree $2n-2d$ is the i-th coordinate projection in this direct sum. In particular, (34) holds in the top degree. We now use a theorem of Stanley [18, Theorem III.3.4] which states that the Stanley-Reisner ring of a matroid is level. Using condition (j) in [18, Proposition III.3.2], this implies that the socle of our cohomology ring $H^*\bigl(C(A^{\pm},\theta);\mathbb{Q}\bigr)$ consists precisely of the elements of degree $2n-2d$. Suppose that (34) does not hold, and pick a non-zero element $p(\partial)$ of maximal degree in the left hand side. The cohomological degree of $p(\partial)$ is strictly less than $2n-2d$ by (35). For any generator ∂_j of $H^*\bigl(C(A^{\pm},\theta);\mathbb{Q}\bigr)$, the product $\partial_j \cdot p(\partial)$ lies in the left hand side of (34) because $\phi_i(\partial_j \cdot p(\partial)) = \phi_i(\partial_j)\cdot\phi(p(\partial)) = 0$. By the maximality hypothesis in the choice of $p(\partial)$, we conclude that $\partial_j \cdot p(\partial) = 0$ in $H^*\bigl(C(A^{\pm},\theta);\mathbb{Q}\bigr)$ for all $j = 1,2,\ldots,n$. Hence $p(\partial)$ lies in the socle of $H^*\bigl(C(A^{\pm},\theta);\mathbb{Q}\bigr)$. By Stanley's Theorem, this means that $p(\partial)$ has cohomological degree $2n-2d$. This is a contradiction and our claim follows.

The result (34) which we just proved translates into the following ideal-theoretic statement:

$$M(\mathcal{B}) + \mathrm{Circ}(\mathcal{A}) \quad = \quad \bigcap_{i=1}^{r} \bigl(I_{\Delta_i} + \mathrm{Circ}(\mathcal{A}) \bigr). \tag{36}$$

Since X_i is a projective orbifold, the ring $H^*(X_i;\mathbb{Q})$ is a Gorenstein ring. A result of Khovanskii and Pukhlikov [11] states that its cogenerator is the volume polynomial, i.e.

$$I_{\Delta_i} + \mathrm{Circ}(\mathcal{A}) \quad = \quad \mathrm{Ann}(V_i) \qquad \text{for } i = 1,2,\ldots,r.$$

We conclude that $M(\mathcal{B}) + \mathrm{Circ}(\mathcal{A}) = \mathrm{Ann}\bigl(\{V_1,\ldots,V_r\}\bigr)$, which proves the identity (31). □

Remark. We note that the above proof of (34) is reversible, i.e. Proposition 3.4 actually implies the levelness result of Stanley [18, Theorem III.3.4] for matroids representable over $\mathbb{Q}$.

We next rewrite the result of Theorem 7.1 in terms of the other two presentations of our cohomology ring. From the perspective of the Lawrence toric variety $X(A^{\pm},\theta)$, it is most natural to work in a polynomial ring in $2n$ variables, one for each torus-invariant divisor of $X(A^{\pm},\theta)$.

Corollary 7.2 *The common cohomology ring $H^*(X(A^{\pm},\theta);\mathbb{Q})$ of the Lawrence toric variety and the toric hyperkähler variety has the presentation*

$$\mathbb{Q}[\partial_{x_1},\partial_{x_2},\ldots,\partial_{x_n},\partial_{y_1},\partial_{y_2},\ldots,\partial_{y_n}]/\mathrm{Ann}\bigl(V_1(x-y),\ldots,V_r(x-y)\bigr).$$

Proof: The polynomials $V_i(x-y) = V_i(x_1-y_1, x_2-y_2,\ldots,x_n-y_n)$ are annihilated precisely by the annihilators of $V_i(x)$ and by the extra ideal generators $\partial_{x_1}+\partial_{y_1},\ldots,\partial_{x_n}+\partial_{y_n}$. □

This corollary states that the cogenerators of the Lawrence toric variety are the volume polynomials of the maximal bounded faces of the associated polyhedron $Q_\psi = P_\theta$. The same result holds for any semi-projective toric variety, even if the maximal bounded faces of its polyhedron have different dimensions. This can be proved using Proposition 3.4.
The economical presentation of our cohomology ring is as a quotient of a polynomial ring in d variables $\partial_{t_1}, \ldots, \partial_{t_d}$. The matrix A defines a surjective homomorphism of polynomial rings

$$\alpha \; : \; \mathbb{Q}[\partial_{x_1}, \ldots, \partial_{x_n}] \; \rightarrow \; \mathbb{Q}[\partial_{t_1}, \ldots, \partial_{t_d}], \quad \partial_{x_j} \; \mapsto \; \sum_{i=1}^{d} a_{ij} \partial_{t_i},$$

and a dual injective homomorphism of polynomial rings

$$\alpha^* \; : \; \mathbb{Q}[t_1, \ldots, t_d] \; \rightarrow \; \mathbb{Q}[x_1, \ldots, x_n], \quad t_i \; \mapsto \; \sum_{j=1}^{n} a_{ij} x_j.$$

The kernel of α equals $\operatorname{Circ}(\mathcal{A})$ and therefore

$$H^*\big(Y(A,\theta); \mathbb{Q}\big) \quad = \quad \mathbb{Q}[\partial_{t_1}, \ldots, \partial_{t_d}] / \alpha(M(\mathcal{B})). \tag{37}$$

We obtain cogenerators for this presentation of our cohomology ring as follows. Suppose that the indeterminate vector $t = (t_1, \ldots, t_d)$ ranges over a small neighborhood of $\theta = (\theta_1, \ldots, \theta_d)$ in $\mathbb{R}^d$. For t in this neighborhood, the polyhedron P_t remains simple and combinatorially isomorphic to P_θ. The maximal bounded faces of P_t can be identified with $\Delta_1, \ldots, \Delta_r$ as before, but now the volume of Δ_i is a homogeneous polynomial of degree $n - d$ in only d variables:

$$v_i(t) \quad = \quad v_i(t_1, \ldots, t_d) \quad = \quad \operatorname{vol}(\Delta_i) \quad \text{for } i = 1, 2, \ldots, r.$$

The polynomial $v_i(t)$ is the unique preimage of the polynomial $V_i(x)$ under the inclusion α^*.

Corollary 7.3 *The cohomology of the Lawrence toric variety and the toric hyperkähler variety equals*

$$H^*\big(Y(A,\theta); \mathbb{Q}\big) \quad = \quad \mathbb{Q}[\partial_{t_1}, \ldots, \partial_{t_d}] / \operatorname{Ann}\big(\{v_1, \ldots, v_r\}\big).$$

Proof: A differential operator $f = f(\partial_{x_1}, \ldots, \partial_{x_n})$ annihilates $\alpha^*(v)$ for some $v = v(t_1, \ldots, t_d)$ if and only if the operator $\alpha(f)$ annihilates v itself. This is the Chain Rule of Calculus. Hence

$$\begin{aligned} \operatorname{Ann}(\{v_1, \ldots, v_r\}) \quad &= \quad \alpha\big(\operatorname{Ann}(\{V_1, \ldots, V_r\})\big) \\ &= \quad \alpha\big(\operatorname{Circ}(\mathcal{A}) \, + \, M(\mathcal{B})\big) \\ &= \quad \alpha(M(\mathcal{B})). \end{aligned}$$

The claim now follows from equation (37). $\square$

Remark. Since the cohomology ring of $Y(A,\theta)$ does not depend on θ, we get the remarkable fact that the vector space generated by the volume polynomials does not depend on θ either.

We close this section by presenting an application to combinatorics. We use notation and terminology as in [18, Section III.3]. Let M be any matroid of rank $n-d$ on n elements which can be represented over the field $\mathbb{Q}$, say, by a configuration $\mathcal{B} \subset \mathbb{Z}^{n-d}$ as above, and let $h(M) = (h_0, h_1, \ldots, h_k)$ be its h-vector. A longstanding open problem is to characterize the h-vectors of matroids. For a survey see [5] or [18, Section III.3]. We wish to argue that toric hyperkähler geometry can make a valuable contribution to this problem. According to Corollary 6.6 the h-numbers of M are precisely the Betti numbers of the associated toric hyperkähler variety:

$$h_i(M) \quad = \quad \operatorname{rank} H^{2i}\big(Y(A,\theta);\mathbb{Q}\big). \tag{38}$$

As a first step, we prove the injectivity part of the Hard Lefschetz Theorem for toric hyperkähler varieties. The g-vector of the matroid is $g(M) = (g_1, g_2, \ldots, g_{\lfloor \frac{n-d}{2} \rfloor})$ where $g_i = h_i - h_{i-1}$.

Theorem 7.4 *The g-vector of a rationally represented coloop-free matroid is a Macaulay vector, i.e. there exists a graded $\mathbb{Q}$-algebra $R = R_0 \oplus R_1 \oplus \cdots \oplus R_{\lfloor \frac{n-d}{2} \rfloor}$ generated by R_1 and with $g_i = \dim_{\mathbb{Q}}(R_i)$ for all i.*

Proof: Let $[D] \in H^2(Y(A,\theta);\mathbb{Q})$ be the class of an ample divisor. The restriction $D|_{X_j}$ to any component X_j of the core is an ample divisor on the projective toric orbifold X_j. Consider the map

$$L \;:\; H^{2i-2}\big(Y(A,\theta);\mathbb{Q}\big) \;\to\; H^{2i}\big(Y(A,\theta);\mathbb{Q}\big), \tag{39}$$

given by multiplication with $[D]$. We claim that this map is injective for $i = 1, \ldots, \lfloor \frac{n-d}{2} \rfloor$. To see this, let $\alpha \in H^{2i-2}(Y(A,\theta);\mathbb{Q})$ be a nonzero cohomology class. Then according to equation (34), there exists an index $j \in \{1, 2, \ldots, r\}$ such that $\alpha|_{X_j}$ is nonzero. Then the Hard Lefschetz Theorem for the projective toric orbifold X_j implies that $\alpha|_{X_j} \cdot [D|_{X_j}]$ is a non-zero class in $H^{2i}(X_j;\mathbb{Q})$. Its preimage $\alpha \cdot [D]$ under the map ϕ_j is non-zero, and we conclude that the map (39) is injective for $2i \leq n-d$. Consider the quotient algebra $R = H^*(Y(A,\theta);\mathbb{Q})/\langle [D] \rangle$. The injectivity result just established implies that

$$g_i = h_i - h_{i-1} \;=\; \dim_{\mathbb{Q}}\big(H^{2i}(Y(A,\theta);\mathbb{Q})/\langle [D] \rangle\big) \;=\; \dim_{\mathbb{Q}}(R_i).$$

This completes the proof of Theorem 7.4. $\square$

Remark. After the submission of our paper we learned that Swartz [20] has given a different proof of Theorem 7.4 for all coloop-free matroids. The explanation of this theorem in a combinatorial context and a comparision of the two proofs will appear in a forthcoming paper [9].

8 Toric quiver varieties

In this section we discuss an important class of toric hyperkähler manifolds, namely, Nakajima's quiver varieties in the special case when the dimension vector has all coordinates equal to one. Let $Q = (V, E)$ be a directed graph (a *quiver*) with $d+1$ vertices $V = \{v_0, v_1, \ldots, v_d\}$ and n edges $\{e_{ij} : (i,j) \in E\}$. We consider the group of all $\mathbb{Z}$-linear combinations of V whose coefficients sum to zero. We fix the basis $\{v_0 - v_1, \ldots, v_0 - v_d\}$ for this group, which is hence identified with $\mathbb{Z}^d$. We also identify $\mathbb{Z}^n$ with the group of $\mathbb{Z}$-linear combinations $\sum_{ij} \lambda_{ij} e_{ij}$ of the set of edges E. The boundary map of the quiver Q is the following homomorphism of abelian groups

$$A \, : \, \mathbb{Z}^n \to \mathbb{Z}^d, \; e_{ij} \, \mapsto \, v_i - v_j. \tag{40}$$

Throughout this section we assume that the underlying graph of Q is connected. This ensures that A is an epimorphism. The kernel of A consists of all $\mathbb{Z}$-linear combinations of E which represent cycles in Q. We fix an $n \times (n-d)$-matrix B whose columns form a basis for the cycle lattice $\ker(A)$. Thus we are in the situation of (1). The following result is well-known:

Lemma 8.1 *The matrix A representing the boundary map of a quiver Q is unimodular.*

Every edge e_{ij} of Q determines one coordinate function z_{ij} on $\mathbb{C}^n$ and two coordinate functions z_{ij}, w_{ij} on $\mathbb{H}^n$. The action of the d-torus on $\mathbb{C}^n$ and $\mathbb{H}^n$ given by the matrix A equals

$$z_{ij} \mapsto t_i t_j^{-1} \cdot z_{ij}, \quad w_{ij} \mapsto t_i^{-1} t_j \cdot w_{ij}. \tag{41}$$

We are interested in the various quotients of $\mathbb{C}^n$ and $\mathbb{H}^n$ by this action. Since the matrix A represents the quiver Q, we write $X(Q, \theta)$ instead of $X(A, \theta)$, we write $X(Q^{\pm}, \theta)$ instead of $X(A^{\pm}, \theta)$, and we write $Y(Q, \theta)$ instead of $Y(A, \theta)$. From Corollary 2.9 and Lemma 8.1, we conclude that all of these quotients are manifolds when the parameter vector θ is generic:

Proposition 8.2 *Let θ be a generic vector in the lattice $\mathbb{Z}^d$. Then $X(Q, \theta)$ is a smooth projective toric variety of dimension $n - d$, $X(Q^{\pm}, \theta)$ is a non-compact smooth toric variety of dimension $2n - d$, and $Y(Q, \theta)$ is a smooth toric hyperkähler variety of dimension $2(n - d)$.*

We call $Y(Q, \theta)$ a *toric quiver variety.* These are precisely the quiver varieties of Nakajima [17] in the case when the dimension vector has all coordinates equal to one. Altmann and Hille [1] used the term "toric quiver variety" for the projective toric variety $X(Q, \theta)$, which is specified by an oriented quiver. Our toric quiver variety $Y(Q, \theta)$ and its ambient Lawrence toric variety $X(Q^{\pm}, \theta)$ incoorporate all orientations of the quiver simultaneously. In view of Theorem 3.2, the Altmann-Hille variety $X(Q, \theta)$ is an irreducible component of the common core of $Y(Q,\theta)$ and $X(Q^{\pm},\theta)$. These manifolds and their core have the same integral cohomology ring, to be described in terms of quiver data in Theorem 8.3.

Fix a vector $\theta \in \mathbb{Z}^d$ and a subset $\tau \subseteq E$ which forms a *spanning tree* in Q. Then there exists a unique linear combination with integer coefficients λ_{ij}^{τ} which represents θ as follows:

$$\theta \quad = \quad \sum_{(i,j)\in\tau} \lambda_{ij}^{\tau} \cdot (v_i - v_j).$$

Note that the vector θ is generic if λ_{ij}^{τ} is non-zero for all spanning trees τ and all $(i,j) \in \tau$.

For every spanning tree τ, we define a subset of the monomials in $T = \mathbb{C}[z_{ij}, w_{ij}]$ as follows.

$$\sigma(\tau,\theta) := \bigl\{\, z_{ij} \,:\, (i,j) \in \tau \text{ and } \lambda_{ij}^{\tau} > 0 \,\bigr\} \;\cup\; \bigl\{\, w_{ij} \,:\, (i,j) \in \tau \text{ and } \lambda_{ij}^{\tau} < 0 \,\bigr\}.$$

Recall that a *cut* of the quiver Q is a collection D of edges which traverses a partition $(W, V\backslash W)$ of the vertex set V. We regard D as a signed set by recording the directions of its edges as follows

$$\begin{aligned} D^- \quad &= \quad \bigl\{\, (i,j) \in E \;:\; i \in V\backslash W \text{ and } j \in W \,\bigr\}, \\ D^+ \quad &= \quad \bigl\{\, (i,j) \in E \;:\; i \in W \text{ and } j \in V\backslash W \,\bigr\}. \end{aligned}$$

We now state our main result regarding toric quiver varieties:

Theorem 8.3 *Let $\theta \in \mathbb{Z}^d$ be generic. The Lawrence toric variety $X(Q^{\pm},\theta)$ is the smooth $(2n-d)$-dimensional toric variety defined by the fan whose $2n$ rays are the columns of $\Lambda(\mathcal{B}) = \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{0} & B^T \end{pmatrix}$ and whose maximal cones are indexed by the sets $\sigma(\tau,\theta)$, where τ runs over all spanning trees of Q. The toric quiver variety $Y(Q,\theta)$ is the $2(n-d)$-dimensional submanifold of $X(Q^{\pm},\theta)$ defined by the equations $\sum_{(i,j)\in D^+} z_{ij}w_{ij} = \sum_{(i,j)\in D^-} z_{ij}w_{ij}$ where D runs over all cuts of Q. The common cohomology ring of these manifolds is the quotient of $\mathbb{Z}[\partial_{ij} : (i,j) \in E]$ modulo the ideal generated by the linear forms in $\partial \cdot B$ and the monomials $\prod_{(i,j)\in D} \partial_{ij}$ where D runs over all cuts of Q.*

A few comments are in place: the variables ∂_{ij}, $(i,j) \in E$, are the coordinates of the row vector ∂, so the entries of $\partial \cdot B$ are a cycle basis for Q. The equations which cut out the toric quiver variety $Y(Q,\theta)$ lie in the Cox homogeneous coordinate ring of the Lawrence toric manifold $X(Q^{\pm},\theta)$. A more compact representation is obtained if we replace "cuts" by "cocircuits". By definition, a *cocircuit* in Q is a cut which is minimal with respect to inclusion. The proof of Theorem 8.3 follows from our general results for integer matrices A.

Corollary 6.6 shows that the Betti numbers of $Y(Q,\theta)$ are the h-numbers of the matroid of B. This is not the usual graphic matroid of Q but it is the *cographic* *matroid* associated with Q. Thus the Betti numbers of the toric quiver variety $Y(Q,\theta)$ are the h-numbers of the cographic matroid of Q. The generating function for the Betti numbers, the h-polynomial of the cographic matroid, is known in combinatorics as the *reliability polynomial* of the graph Q; see [5].

Corollary 8.4 *The Poincaré polynomial of the toric quiver variety $Y(Q,\theta)$ equals the reliability polynomial of the graph Q, which is the h-polynomial of its cographic matroid. In particular, the Euler characteristic of $Y(Q,\theta)$ coincides with the number of spanning trees of Q.*

Lopez [15] gives an explicit enumerative interpretation of the coefficients of the reliability polynomial of a graph and hence of the Betti numbers of a toric quiver variety. In particular, he proves Stanley's longstanding conjecture on h-vectors of matroid complexes [18, Conjecture III.3.6] for the special case of cographic matroids.

9 An example of a toric quiver variety

We shall describe a particular toric quiver variety $Y(K_{2,3},\theta)$ of complex dimension four. Consider the quiver in Figure 1, the complete bipartite graph $K_{2,3}$ given by $d=4$, $n=6$ and $E=\{(0,2),(0,3),(0,4),(1,2),(1,3),(1,4)\}$.

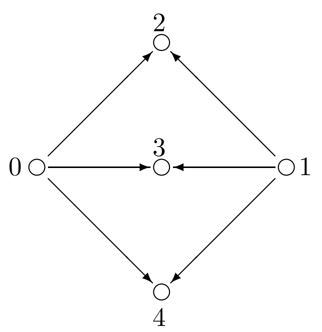

Figure 1: The quiver $K_{2,3}$

The matrix A representing the boundary map (40) is given in Figure 2. The six columns of A span the cone over a triangular prism as depicted in Figure 3. A Gale dual of this configuration is given by the six vectors in the plane in Figure 4. The rows of B^T span the cycle lattice of $K_{2,3}$.
Our manifolds are constructed algebraically from the polynomial rings

$$S \quad = \quad \mathbb{C}[z_{02},z_{03},z_{04},z_{12},z_{13},z_{14}]$$

and

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Figure 2: The matrix A

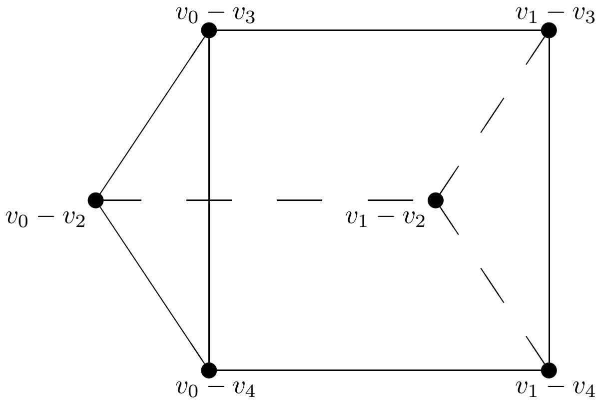

Figure 3: The column vectors of the matrix A

$$T \quad = \quad S[w_{02}, w_{03}, w_{04}, w_{12}, w_{13}, w_{14}],$$

where the degrees of the variables are given by the columns of the matrix $A^{\pm}$:

$$\text{degree}(z_{ij}) \quad = \quad -\text{degree}(w_{ij}) \quad = \quad v_i - v_j. \tag{42}$$

This grading corresponds to the torus action (41) on the polynomial rings S and T. Fix $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbb{Z}^4$. It represents the following linear combination of vertices of $K_{2,3}$:

$$(\theta_1 + \theta_2 + \theta_3 + \theta_4)v_0 - \theta_1 v_1 - \theta_2 v_2 - \theta_3 v_3 - \theta_4 v_4$$

The monomials $z_{02}^{u_{02}} z_{03}^{u_{03}} z_{04}^{u_{04}} z_{12}^{u_{12}} z_{13}^{u_{13}} z_{14}^{u_{14}}$ in the graded component S_θ correspond to the nonnegative 2×3-integer matrices $\begin{pmatrix} u_{02} & u_{03} & u_{04} \\ u_{12} & u_{13} & u_{14} \end{pmatrix}$ with column sums $\theta_2, \theta_3, \theta_4$ and row sums $\theta_1 + \theta_2 + \theta_3 + \theta_4$ and $-\theta_1$. For instance, for $\theta = (-3, 2, 2, 2)$ there are precisely seven monomials in S_θ as shown on Figure 6. Taking "Proj" of the algebra generated by these seven monomials we get

$$B^T = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix}$$

Figure 4: Transpose of the matrix B

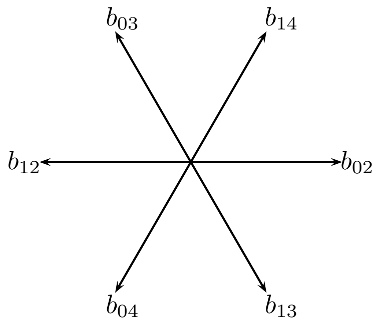

Figure 5: Rows of the matrix B

$$S_\theta = \mathbb{C}\{ \begin{array}{l} z_{02}z_{03}z_{04}z_{12}z_{13}z_{14}, \\ z_{02}^2z_{04}z_{13}^2z_{14}, \\ z_{02}^2z_{03}z_{13}z_{14}^2, \\ z_{02}z_{03}^2z_{12}z_{14}^2, \\ z_{03}^2z_{04}z_{12}^2z_{14}, \\ z_{03}z_{04}^2z_{12}^2z_{13}, \\ z_{02}z_{04}^2z_{12}z_{13}^2 \quad \} \end{array}$$

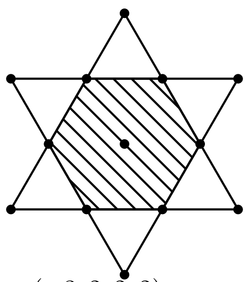

Figure 6: Monomials in multidegree $\theta = (-3, 2, 2, 2)$

a smooth toric surface $X(K_{2,3}, \theta)$ in $\mathbb{P}^6$. This surface is the blow-up of $\mathbb{P}^2$ at three points.

As θ varies, there are eighteen different types of smooth toric surfaces $X(K_{2,3}, \theta)$. They correspond to the eighteen chambers in the triangular prism, or, equivalently, to the eighteen complete fans on $\mathcal{B}$. This picture arises in the *Cremona transformation* of classical algebraic geometry, where the projective plane is blown up at three points and then the lines connecting them are blown down. The eighteen surfaces are the intermediate blow-ups and blow-downs.

We next describe the Lawrence toric varieties $X(K^{\pm}_{2,3}, \theta)$ which are the GIT quotients of $\mathbb{C}^{12}$ by the action (41). First, the (singular) affine quotient $X(K^{\pm}_{2,3}, 0)$ is the spectrum of the algebra

$$\begin{aligned} T_0 = \quad & \mathbb{C}\big[z_{02}w_{02}, z_{03}w_{03}, z_{04}w_{04}, z_{12}w_{12}, z_{13}w_{13}, z_{14}w_{14},\ z_{02}z_{13}w_{12}w_{03}, \\ & z_{03}z_{12}w_{13}w_{02}, z_{02}z_{14}w_{12}w_{04}, z_{04}z_{12}w_{14}w_{02}, z_{03}z_{14}w_{13}w_{04}, z_{04}z_{13}w_{14}w_{03}\big]. \end{aligned}$$

This is the affine toric variety whose fan is the cone over the 7-dimensional Lawrence polytope given by the matrix $\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{0} & B^T \end{pmatrix}$, where $\mathbf{I}$ is the 6×6-unit matrix. This Lawrence polytope has 160 triangulations, all of which are regular, so there are 160 different types of smooth Lawrence toric varieties $X(K_{2,3}^{\pm}, \theta)$ as θ ranges over the generic points in $\mathbb{Z}^4$. For instance, for $\theta = (-3, 2, 2, 2)$ as in Figure 6, $X(K_{2,3}^{\pm}, \theta)$ is constructed as follows. The graded component T_θ is generated as a T_0-module by 13 monomials: the seven z-monomials in S_θ and the six additional monomials:

$$\begin{array}{c} w_{02}z_{03}^2 z_{04}^2 z_{12}^3,\ w_{03}z_{02}^2 z_{04}^2 z_{13}^3,\ w_{04}z_{02}^2 z_{03}^2 z_{14}^3, \\ w_{12}z_{13}^2 z_{14}^2 z_{02}^3,\ w_{13}z_{12}^2 z_{14}^2 z_{03}^3,\ w_{14}z_{12}^2 z_{13}^2 z_{04}^3. \end{array} \tag{43}$$

The 13 monomial generators of T_θ correspond to the 13 lattice points in the star diagram in Figure 6. The toric variety $X(K_{2,3}^{\pm}, \theta) = \mathrm{Proj}(\oplus_{n \geq 0} T_{n\theta})$ is characterized by its irrelevant ideal in the Cox homogeneous coordinate ring T, which is graded by (42). The irrelevant ideal is the radical of the monomial ideal $\langle T_\theta \rangle$. It is generated by the 12 square-free monomials obtained by erasing exponents of the monomials in (43) and Figure 6. The 7-simplices in the triangulation of the Lawrence polytope are the complements of the supports of these twelve monomials,

We finally come to the toric quiver variety $Y(K_{2,3}, \theta)$, which is smooth and four-dimensional. It is the complete intersection in the Lawrence toric variety $X(K_{2,3}^{\pm}, \theta)$ defined by the equations

$$\begin{aligned} z_{02}w_{02} + z_{03}w_{03} + z_{04}w_{04} \ &= \ z_{02}w_{02} + z_{12}w_{12} \ = \ z_{03}w_{03} + z_{13}w_{13} \ = \\ &= \ z_{04}w_{04} + z_{14}w_{14} \ = \ 0. \end{aligned}$$

These equations are valid for all 160 toric quiver varieties $Y(K_{2,3}, \theta)$. The cores of the manifolds vary greatly. For instance, for $\theta = (-3, 2, 2, 2)$, the core of $Y(K_{2,3}, \theta)$ consists of six copies of the projective plane $\mathbb{P}^2$ which are glued to the blow-up of $\mathbb{P}^2$ at three points. These correspond to the six triangles which are glued to the edges of the hexagon in Figure 6.

The common cohomology ring of the 8-dimensional Lawrence toric varieties $X(K_{2,3}^{\pm}, \theta)$ and the 4-dimensional toric quiver varieties $Y(K_{2,3}, \theta)$ is independent of θ and equals

$$\begin{aligned} &\mathbb{Z}[\partial]/\langle \partial_{03}\partial_{04}\partial_{12}, \partial_{02}\partial_{04}\partial_{13}, \partial_{02}\partial_{03}\partial_{14}, \partial_{13}\partial_{14}\partial_{02}, \partial_{12}\partial_{14}\partial_{03}, \partial_{12}\partial_{13}\partial_{04}, \partial_{02}\partial_{03}\partial_{04}, \\ &\quad \partial_{12}\partial_{13}\partial_{14}, \partial_{02}\partial_{12}, \partial_{03}\partial_{13}, \partial_{04}\partial_{14},\ \partial_{02} - \partial_{03} - \partial_{12} + \partial_{13}, \partial_{02} - \partial_{04} - \partial_{12} + \partial_{14} \rangle. \end{aligned}$$

From this presentation we can compute the Betti numbers as follows:

$$\begin{aligned} H^*(Y(K_{2,3}); \mathbb{Z}) \quad &= \quad H^0(Y(K_{2,3}); \mathbb{Z}) \,\oplus\, H^2(Y(K_{2,3}); \mathbb{Z}) \,\oplus\, H^4(Y(K_{2,3}); \mathbb{Z}) \\ &= \quad \mathbb{Z}^1 \,\oplus\, \mathbb{Z}^4 \,\oplus\, \mathbb{Z}^7. \end{aligned}$$

The 7-dimensional space of cogenerators is spanned by the areas of the six triangles in Figure 6, e.g., $V_{\{03,04,12\}}(x) = (x_{03} + x_{04} - x_{12})^2$, together with the area polynomial of the hexagon

$$\begin{aligned} V_{hex}(x) \;=\; & 2x_{03}x_{14} + 2x_{14}x_{02} + 2x_{02}x_{13} + 2x_{13}x_{04} + 2x_{04}x_{12} + 2x_{12}x_{03} - \\ & -x_{02}^2 - x_{03}^2 - x_{04}^2 - x_{12}^2 - x_{13}^2 - x_{14}^2. \end{aligned}$$

10 Which toric varieties are hyperkähler ?

Toric hyperkähler varieties are constructed algebraically as complete intersections in Lawrence toric varieties, but they are generally not toric varieties themselves. What we mean by this is that there does not exist a subtorus of the dense torus of $X(A^{\pm}, \theta)$ such that $Y(A, \theta)$ is an orbit closure of that subtorus. The objective of this section is to characterize and study the rare exceptional cases when $Y(A, \theta)$ happens to be a toric variety. We are particularly interested in the case of manifolds, when A is unimodular. The following is the main result in this section.

Theorem 10.1 *A toric manifold is a toric hyperkähler variety if and only if it is a product of ALE spaces of type A_n if and only if it is a toric quiver variety $X(Q, \theta)$ where Q is a disjoint union of cycles.*

The ALE space of type A_n is denoted $\mathbb{C}^2//\Gamma_n$ where Γ_n is the cyclic group of order n acting on $\mathbb{C}^2$ as the matrix group $\left\{ \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} \;:\; \eta^n = 1 \right\}$. The name "ALE space" indicates the fact that these varieties are the underlying varieties of Asymptotically Locally Euclidean gravitational instantons, or in other words 4-dimensional hyperkähler manifolds (see [13] for details).
The smooth surface $\mathbb{C}^2//\Gamma_n$ is defined as the unique crepant resolution of the 2-dimensional cyclic quotient singularity

$$\mathbb{C}^2/\Gamma_n \quad = \quad \operatorname{Spec} \mathbb{C}[x, y]^{\Gamma_n} \quad = \quad \operatorname{Spec} \mathbb{C}[x^n, xy, y^n].$$

Equivalently, we can construct $\mathbb{C}^2//\Gamma_n$ as the smooth toric surface whose fan Σ_n consists of the cones $\mathbb{R}_{\geq 0}\{(1, i-1), (1, i)\}$ for $i = 1, 2, \ldots, n$ and whose lattice is the standard lattice $\mathbb{Z}^2$.
Let us start out by showing that the ALE space $\mathbb{C}^2//\Gamma_n$ is indeed a toric quiver variety. Let C_n denote the *n-cycle.* This is the quiver with vertices $V = \{0, 1, \ldots, n-1\}$ and edges

$$E \;=\; \big\{ (0,1),\, (1,2),\, (2,3),\, \ldots, (n-2, n-1),\, (n-1, 0) \big\}.$$

We prove the following well-know result to illustrate our constructions.

Lemma 10.2 *The affine quiver variety $Y(C_n, 0)$ is isomorphic to $\mathbb{C}^2/\Gamma_n$ and for any generic vector $\theta \in \mathbb{Z}^{n-1}$, the smooth quiver variety $Y(C_n, \theta)$ is isomorphic to the ALE space $\mathbb{C}^2//\Gamma_n$.*

Proof: The boundary map of the n-cycle C_n has the format $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$ and looks like

$$A \quad = \quad \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}.$$

and its Gale dual is the $1 \times n$-matrix with all entries equal to one:

$$B^T \quad = \quad (\, 1 \quad 1 \quad 1 \quad \cdots \quad 1\,). \tag{44}$$

The torus $\mathbb{T}_{\mathbb{C}}^{n-1}$ acts via $A^{\pm}$ on the polynomial ring $T = \mathbb{C}\,[\, z_{i,i+1}, w_{i,i+1} \,:\, i = 0, \ldots, n-1\,]$. The affine Lawrence toric variety $X(C_n^{\pm}, 0) = \mathbb{C}^{2n} /\!/_0 \mathbb{T}_{\mathbb{C}}^{n-1}$ is the spectrum of the invariant ring

$$T_0 \quad = \quad \mathbb{C}\,\big[\, z_{01}w_{01}\,, \,\ldots,\, z_{n-1,0}w_{n-1,0}\,,\ z_{01}z_{12}\cdots z_{n-1,0}\,,\ w_{01}w_{12}\cdots w_{n-1,0} \,\big].$$

The common defining ideal of all the quiver varieties $Y(C_n, \theta)$ is the following ideal in T:

$$\mathrm{Circ}(\mathcal{B}) \quad = \quad \langle\, z_{i-1,i}w_{i-1,i} \,-\, z_{i,i+1}w_{i,i+1} \;:\; i = 1, 2, \ldots, n \,\rangle.$$

All indices are considered modulo n. The quiver variety $Y(C_n, 0)$ is the spectrum of $T_0/(T_0 \cap \mathrm{Circ}(\mathcal{B}))$. Dividing T_0 by $T_0 \cap \mathrm{Circ}(\mathcal{B})$ means erasing the double indices of all variables:

$$T_0/(T_0 \cap \mathrm{Circ}(\mathcal{B})) \quad \simeq \quad \mathbb{C}[\, zw,\, z^n,\, w^n\,].$$

Passing to the spectra of these rings proves our first assertion: $Y(C_n, 0) \simeq \mathbb{C}^2/\Gamma_n$.
For the second assertion, we first note that $\theta = (\theta_1, \ldots, \theta_{n-1})$ is generic for $\mathcal{A}^{\pm}$ if and only if all consecutive coordinate sums $\theta_i + \theta_{i+1} + \cdots + \theta_j$ are non-zero. The associated hyperplane arrangement $\Gamma(\mathcal{A})$ is linearly isomorphic to the braid arrangement $\{u_i = u_j\}$. It has $n\,!$ chambers, and the symmetric group acts transitively on the chambers. Hence it suffices to prove $Y(C_n, \theta) \simeq \mathbb{C}^2 /\!/ \Gamma_n$ for only one vector θ which lies in the interior of any chamber.
We fix the generic vector $\theta = (1, 1, \ldots, 1)$. There are n monomials of degree θ in T, namely,

$$\prod_{j=1}^{i-1} z_{j-1,j}^{i-j} \cdot \prod_{k=i+1}^{n} w_{k-1,k}^{k-i} \qquad \text{for } i = 1, 2, \ldots, n. \tag{45}$$

The images of these monomials are minimal generators of the $T_0/(T_0 \cap \mathrm{Circ}(\mathcal{B}))$-algebra

$$\bigoplus_{r=0}^{\infty} T_{r\theta}/(T_{r\theta} \cap \mathrm{Circ}(\mathcal{A})(\mathcal{B})).$$

By definition, $Y(C_n, \theta)$ is the projective spectrum of this $\mathbb{N}$-graded algebra. Applying our isomorphism "erasing double indices", the images of our n monomials in (45) translate into

$$z^{\binom{i}{2}} \cdot w^{\binom{n-i+1}{2}} \qquad \text{for } i = 1, 2, \ldots, n. \tag{46}$$

Hence $Y(C_n, \theta)$ is the projective spectrum of the $\mathbb{C}[zw, z^n, w^n]$-algebra generated by (46). It is straightforward to see that this is the toric surface with fan Σ_n, i.e. the ALE space $\mathbb{C}^2//\Gamma_n$. □

It is instructive to write down our presentations for the cohomology ring of the ALE space $Y(C_n, \theta) = \mathbb{C}^2//\Gamma_n$. The circuit ideal of the n-cycle is the principal ideal

$$\mathrm{Circ}(\mathcal{A}) \quad = \quad \langle\, \partial_{01} + \partial_{12} + \partial_{23} + \cdots + \partial_{n-1,0} \,\rangle.$$

The matroid ideal $M(\mathcal{B})$ is generated by all quadratic squarefree monomials in $\mathbb{Z}[\partial]$. It follows that $\mathbb{Z}[\partial]/(\mathrm{Circ}(\mathcal{A}) + M(\mathcal{B}))$ is isomorphic to a polynomial ring in $n-1$ variables modulo the square of the maximal ideal generated by the variables, and hence

$$H^*(Y(C_n, \theta); \mathbb{Z}) \quad = \quad H^0(Y(C_n, \theta); \mathbb{Z}) \oplus H^2(Y(C_n, \theta); \mathbb{Z}) \quad \simeq \quad \mathbb{Z}^1 \oplus \mathbb{Z}^{n-1}.$$

On our way towards proving Theorem 10.1, let us now fix an epimorphism $A : \mathbb{Z}^n \to \mathbb{Z}^d$ and a generic vector $\theta \in \mathbb{Z}^d$. We assume that $\mathcal{A}$ is not a cone, i.e. the zero vector is not in $\mathcal{B}$. We do not assume that A is unimodular. By a *binomial* we mean a polynomial with two terms.

Proposition 10.3 *The following three statements are equivalent:*

(a) *The hyperkähler toric variety* $Y(A, \theta)$ *is a toric subvariety of* $X(A^{\pm}, \theta)$.

(b) *The ideal* $\mathrm{Circ}(\mathcal{B})$ *is generated by binomials.*

(c) *The configuration* $\mathcal{B}$ *lies on* $n-d$ *linearly independent lines through the origin in* $\mathbb{R}^{n-d}$.

Proof: The condition (b) holds if and only if the matrix A can be chosen to have two nonzero entries in each row. This defines a graph $\mathcal{G}$ on $\{1, 2, \ldots, n\}$, namely, j and k are connected by an edge if there exists $i \in \{1, \ldots, d\}$ such that $a_{ij} \neq 0$ and $a_{ik} \neq 0$. The graph $\mathcal{G}$ is a disjoint union of $n-d$ trees. Two indices j and k lie in the same connected component of $\mathcal{G}$ if and only if the vectors b_j and b_k are linearly dependent. Thus (b) is equivalent to (c).
Suppose that (b) holds. Then the prime ideal $\mathrm{Circ}(\mathcal{B})$ is generated by the quadratic binomials $a_{ij} z_j w_j + a_{ik} z_k w_k$ indexed by the edges (j,k) of $\mathcal{G}$. The corresponding coefficient-free equations

$$z_j w_j \quad = \quad z_k w_k \qquad \text{for } (j,k) \in \mathcal{G}.$$

define a subtorus $\mathbb{T}$ of the dense torus of the Lawrence toric variety $X(A^{\pm},\theta)$, and the equations

$$a_{ij}z_jw_j \;+\; a_{ik}z_kw_k \quad = \quad 0 \qquad \text{for } (j,k)\in\mathcal{G}.$$

define an orbit of $\mathbb{T}$ in the dense torus of $X(A^{\pm},\theta)$. The solution set of the same equations in $X(A^{\pm},\theta)$ has the closure of that $\mathbb{T}$-orbit as one of its irreducible components. But that solution set is our hyperkähler variety $Y(A,\theta)$. Since $Y(A,\theta)$ is irreducible, we can conclude that it coincides with the closure of the $\mathbb{T}$-orbit. Hence $Y(A,\theta)$ is a toric variety, i.e. (a) holds.
For the converse, suppose that (a) holds. The irreducible subvariety $Y(A,\theta)$ is defined by a homogeneous prime ideal J in the homogeneous coordinate ring T of $X(A^{\pm},\theta)$. Since $Y(A,\theta)$ is a torus orbit closure, the ideal J is generated by binomials. The ideal $\mathrm{Circ}(\mathcal{B})$ has the same zero set as J does, and therefore, by the Nullstellensatz and results of Cox, $\mathrm{rad}\big(\mathrm{Circ}(\mathcal{B}) : B_\theta^\infty\big) = J$. Our hypothesis $0\notin\mathcal{B}$ ensures that $\mathrm{Circ}(\mathcal{B})$ itself is a prime ideal, and therefore we conclude $\mathrm{Circ}(\mathcal{B}) = J$. In particular, this ideal is generated by binomials, i.e. (b) holds. □

Proof of Theorem 10.1: Suppose that Q is a quiver with connected components $Q_1,\dots,Q_r$. Then its boundary map is given by a matrix with block decomposition

$$A \quad = \quad A_1\oplus A_2\oplus\cdots\oplus A_r, \tag{47}$$

where A_i is the boundary map of Q_i. There is a corresponding decomposition of the Gale dual

$$B \quad = \quad B_1\oplus B_2\oplus\cdots\oplus B_r. \tag{48}$$

In this situation, the toric hyperkähler variety $Y(A,\theta)$ is the direct product of the toric hyperkähler varieties $Y(A_i,\theta)$ for $i=1,\dots,r$. For our quiver Q this means

$$Y(Q,\theta) \quad = \quad Y(Q_1,\theta)\times Y(Q_2,\theta)\times\cdots\times Y(Q_r,\theta).$$

Using Lemma 10.2, we conclude that a manifold is a product of ALE spaces of type A_n if and only if it is a toric quiver variety $Y(Q,\theta)$ where Q is a disjoint union of cycles C_{n_i}.
The matrix A in (47) is unimodular if and only if the matrices $A_1,\dots,A_r$ are unimodular. Hence a product of toric hyperkähler manifolds is a toric hyperkähler manifold. In particular, a product of ALE spaces $\mathbb{C}^2/\Gamma_{n_i}$ is a toric hyperkähler manifold which is also a toric variety.
For the converse, suppose that $Y(A,\theta)$ is a toric hyperkähler manifold which is also a toric variety, so that statement (a) in Proposition 10.3 holds. Statement (c) in Proposition 10.3 says that the matrix B has a decomposition (48) where $r=n-d$ and each B_i is a matrix with exactly one column. We may assume that none of the entries in B_i is zero. The Gale dual A_i of B_i is a unimodular matrix, and hence B_i is unimodular. For a matrix with one column this means that all entries in B_i are either $+1$ or -1. After trivial sign changes, this means $B_i^T = (\,1\,1\,\dots\,1\,)$. Now we are in the situation of (44), which means that $Y(A_i,\theta)$ is an ALE space $\mathbb{C}^2/\Gamma_{n_i}$. □

References

[1] K. Altmann and L. Hille: Strong exceptional sequences provided by quivers, *Algebras and Representation Theory* 2 (1999) 1–17.

[2] A. Bialynicki-Birula: Some properties of the decompositions of algebraic varieties determined by actions of a torus. *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* 24 (1976), no. 9, 667-674.

[3] R. Bielawski and A. Dancer: The geometry and topology of toric hyperkähler manifolds, *Comm. Anal. Geom.* 8 (2000), no. 4, 727–760.

[4] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler: Oriented Matroids, Cambridge University Press, 1999.

[5] M. Chari and C. Colbourn: Reliability polynomials: a survey. *J. Combin. Inform. System Sci.* 22 (1997) 177–193.

[6] D. Cox: The homogeneous coordinate ring of a toric variety, *Journal of Algebraic Geometry* 4 (1995) 17–50.

[7] D. Cox: Recent developments in toric geometry. *Algebraic Geometry—Santa Cruz 1995*, 389–436, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[8] W. Fulton: *Introduction to Toric Varieties*, Princeton University Press, 1993.

[9] T. Hausel: Quaternionic geometry of matroids, *in preparation.*

[10] N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček: Hyper-Kähler metrics and supersymmetry, *Comm. Math. Phys.*, 108,1987,(4), 535–589.

[11] A.G. Khovanskii and A.V. Pukhlikov: The Riemann-Roch theorem for integrals and sums of quasipolynomials on virtual polytopes. *St. Petersburg Math. J.* 4 (1993) 789–812.

[12] F.C. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Princeton University Press, Princeton, NJ, 1984.

[13] P.B. Kronheimer, The construction of ALE spaces as hyperkähler quotients, *J. Diff. Geom.* 29 (1989) no. 3, 665–683.

[14] H. Konno: Cohomology rings of toric hyperkähler manifolds, *Internat. J. Math.* 11 (2000), no. 8, 1001–1026.

[15] C.M. Lopez: Chip firing and the Tutte polynomial, *Ann. Combinatorics* 1 (1997) 253–259.

[16] J. Milnor: *Morse Theory*, Princeton University Press, Princeton, NJ, 1969.

[17] H. Nakajima: Quiver varieties and Kac-Moody algebras, *Duke Mathematical Journal* 91 (1998) 515–560.

[18] R.P. Stanley: *Combinatorics and Commutative Algebra*, 2nd ed., Birkhäuser Boston, 1996.

[19] R.P. Stanley: A monotonicity property of h-vectors and h^*-vectors, *European Journal of Combinatorics*, 14, No. 3., May 1993, 251-258.

[20] E.Swartz: g-elements of matroid complexes, *Journal of Combinatorial Theory, Series B*, to appear.

[21] T. Zaslavsky: Facing up to arrangements: face-count formulas for partitions of space by hyperplanes. *Mem. Amer. Math. Soc.* 1 (1975), issue 1, no. 154.

[22] G. Ziegler: *Lectures on Polytopes*, Springer Graduate Texts in Mathematics, 1995.

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