Homology Stability for Unitary Groups

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Abstract. In this paper the homology stability for unitary groups over a ring with finite unitary stable rank is established. First we develop a 'nerve theorem' on the homotopy type of a poset in terms of a cover by subposets, where the cover is itself indexed by a poset. We use the nerve theorem to show that a poset of sequences of isotropic vectors is highly connected, as conjectured by Charney in the eighties. Homology stability of symplectic groups and orthogonal groups appear as a special case of our results.

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1. Introduction

Interest in homological stability problems in algebraic K-theory started with Quillen, who used it in [15] to study the higher K-groups of a ring of integers. As a result of stability he proved that these groups are finitely generated (see also [7]). After that there has been considerable interest in homological stability for general linear groups. The most general results in this direction are due to the second author [20] and Suslin [19].

Parallel to this, similar questions for other classical groups such as orthogonal and symplectic groups have been studied. For work in this direction, see [23], [1], [5], [12], [13]. The most general result is due to Charney [5]. She proved the homology stability for orthogonal and symplectic groups over a Dedekind domain. Panin in [13] proved a similar result but with a different method and with better range of stability.

Our goal in this paper is to prove that homology stabilizes of the unitary groups over rings with finite unitary stable rank. To do so we prove that the poset of isotropic unimodular sequences is highly connected. Recall that Panin in [12] had already sketched how one can do this for a finite dimensional affine algebra over an infinite field, in the case of symplectic and orthogonal

groups. However, while the assumption about the infinite field provides a significant simplification, it excludes cases of primary interest, namely rings that are finitely generated over the integers.
Our approach is as follows. We first extend a theorem of Quillen [16, Thm 9.1] which was his main tool to prove that certain posets are highly connected. We use it to develop a quantitative analogue for posets of the nerve theorem, which expresses the homotopy type of a space in terms of the the nerve of a suitable cover. In our situation both the elements of the cover and the nerve are replaced with posets. We work with posets of ordered sequences 'satisfying the chain condition', as this is a good replacement for simplicial complexes in the presence of group actions. (Alternatively one might try to work with barycentric subdivisions of a simplicial complex.) The new nerve theorem allows us to exploit the higher connectivity of the poset of unimodular sequences due to the second author. The higher connectivity of the poset of isotropic unimodular sequences follows inductively. We conclude with the homology stability theorem.

2. Preliminaries

Recall that a topological space X is (-1)-*connected* if it is non-empty, 0-*connected* if it is non-empty and path connected, 1-*connected* if it is non-empty and simply connected. In general for $n \geq 1$, X is called n-*connected* if X is nonempty, X is 0-connected and $\pi_i(X, x) = 0$ for every base point $x \in X$ and $1 \leq i \leq n$. For $n \geq -1$ a space X is called n-*acyclic* if it is nonempty and $\tilde{H}_i(X, \mathbb{Z}) = 0$ for $0 \leq i \leq n$. For $n < -1$ the conditions of n-connectedness and n-acyclicness are vacuous.

Theorem 2.1 (Hurewicz). *For $n \geq 0$, a topological space X is n-connected if and only if the reduced homology groups $\tilde{H}_i(X, \mathbb{Z})$ are trivial for $0 \leq i \leq n$ and X is 1-connected if $n \geq 1$.*

Proof. See [25], Chap. IV, Corollaries 7.7 and 7.8. □

Let X be a partially ordered set or briefly a *poset*. Consider the simplicial complex associated to X, that is the simplicial complex where vertices or 0-simplices are the elements of X and the k-simplices are the $(k+1)$-tuples $(x_0, \dots, x_k)$ of elements of X with $x_0 < \dots < x_k$. We denote it again by X. We denote the geometric realization of X by $|X|$ and we consider it with the weak topology. It is well known that $|X|$ is a CW-complex [11]. By a *morphism* or *map* of posets $f : X \to Y$ we mean an order-preserving map i. e. if $x \leq x'$ then $f(x) \leq f(x')$. Such a map induces a continuous map $|f| : |X| \to |Y|$.

Remark 2.2. If K is a simplicial complex and X the partially ordered set of simplices of K, then the space $|X|$ is the barycentric subdivision of K. Thus every simplicial complex, with weak topology, is homeomorphic to the geometric realization of some, and in fact many, posets. Furthermore since it is well known that any CW-complex is homotopy equivalent to a simplicial complex, it follows that any interesting homotopy type is realized as the geometric realization of a poset.

Proposition 2.3. *Let X and Y be posets.*
(i) *(*Segal [17]*) If $f, g : X \to Y$ are maps of posets such that $f(x) \leq g(x)$ for all $x \in X$, then $|f|$ and $|g|$ are homotopic.*
(ii) *If the poset X has a minimal or maximal element then $|X|$ is contractible.*
(iii) *If X^{op} denotes the opposite poset of X, i. e. with opposite ordering, then $|X^{op}| \simeq |X|$.*

Proof. (i) Consider the poset $I = \{0, 1 : 0 < 1\}$ and define the poset map $h : I \times X \to Y$ as $h(0, x) = f(x)$, $h(1, x) = g(x)$. Since $|I| \simeq [0, 1]$, we have $|h| : [0, 1] \times |X| \to |Y|$ with $|h|(0, x) = |f|(x)$ and $|h|(1, x) = |g|(x)$. This shows that $|f|$ and $|g|$ are homotopic.
(ii) Suppose X has a maximal element z. Consider the map $f : X \to X$ with $f(x) = z$ for every $x \in X$. Clearly for every $x \in X$, $\mathrm{id}_X(x) \leq f(x)$. This shows that id_X and the constant map f are homotopic. So X is contractible. If X has a minimal element the proof is similar.
(iii). This is natural and easy. □

The construction $X \mapsto |X|$ allows us to assign topological concepts to posets. For example we define the homology groups of a poset X to be those of $|X|$, we call X *n-connected* or *contractible* if $|X|$ is n-connected or contractible etc. Note that X is connected if and only if X is connected as a poset. By the *dimension* of a poset X, we mean the dimension of the space $|X|$, or equivalently the supremum of the integers n such that there is a chain $x_0 < \cdots < x_n$ in X. By convention the empty set has dimension -1.
Let X be a poset and $x \in X$. Define $Link^+_X(x) := \{u \in X : u > x\}$ and $Link^-_X(x) := \{u \in X : u < x\}$. Given a map $f : X \to Y$ of posets and an element $y \in Y$, define subposets f/y and $y\backslash f$ of X as follows

$$f/y := \{x \in X : f(x) \leq y\} \qquad y\backslash f := \{x \in X : f(x) \geq y\}.$$

In fact $f/y = f^{-1}(Y_{\leq y})$ and $y\backslash f = f^{-1}(Y_{\geq y})$ where $Y_{\leq y} = \{z \in Y : z \leq y\}$ and $Y_{\geq y} = \{z \in Y : z \geq y\}$. Note that by 2.3 (ii), $Y_{\leq y}$ and $Y_{\geq y}$ are contractible. If $\mathrm{id}_Y : Y \to Y$ is the identity map, then $\mathrm{id}_Y/y = Y_{\leq y}$ and $y\backslash \mathrm{id}_Y = Y_{\geq y}$.
Let $\mathcal{F} : X \to \underline{\mathrm{Ab}}$ be a functor from a poset X, regarded as a category in the usual way, to the category of abelian groups. We define the homology groups $H_i(X, \mathcal{F})$ of X with coefficient $\mathcal{F}$ to be the homology of the complex $C_*(X, \mathcal{F})$ given by

$$C_n(X, \mathcal{F}) = \bigoplus_{x_0 < \cdots < x_n} \mathcal{F}(x_0)$$

where the direct sum is taken over all n-simplices in X, with differential $\partial_n = \Sigma^n_{i=0}(-1)^i d^n_i$ where $d^n_i : C_n(X, \mathcal{F}) \to C_{n-1}(X, \mathcal{F})$ and d^n_i takes the $(x_0 < \cdots < x_n)$-component of $C_n(X, \mathcal{F})$ to the $(x_0 < \cdots < \widehat{x_i} < \cdots < x_n)$-component of $C_{n-1}(X, \mathcal{F})$ via $d^n_i = id_{\mathcal{F}(x_0)}$ if $i > 0$ and $d^n_0 : \mathcal{F}(x_0) \to \mathcal{F}(x_1)$. In particular, for the empty set we have $H_i(\varnothing, \mathcal{F}) = 0$ for $i \geq 0$.

Let $\mathcal{F}$ be the constant functor $\mathbb{Z}$. Then the homology groups with this coefficient coincide with the integral homology of $|X|$, that is $H_k(X,\mathbb{Z}) = H_k(|X|,\mathbb{Z})$ for all $k \in \mathbb{Z}$, [6, App. II]. Let $\tilde{H}_i(X,\mathbb{Z})$ denote the reduced integral homology of the poset X, that is $\tilde{H}_i(X,\mathbb{Z}) = \ker\{H_i(X,\mathbb{Z}) \to H_i(pt,\mathbb{Z})\}$ if $X \neq \varnothing$ and $\tilde{H}_i(\varnothing,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = -1 \\ 0 & \text{if } i \neq -1 \end{cases}$. So $\tilde{H}_i(X,\mathbb{Z}) = H_i(X,\mathbb{Z})$ for $i \geq 1$ and for $i = 0$ we have the exact sequence

$$0 \to \tilde{H}_0(X,\mathbb{Z}) \to H_0(X,\mathbb{Z}) \to \mathbb{Z} \to \tilde{H}_{-1}(X,\mathbb{Z}) \to 0$$

where $\mathbb{Z}$ is identified with the group $H_0(pt,\mathbb{Z})$. Notice that $H_0(X,\mathbb{Z})$ is identified with the free abelian group generated by the connected components of X.

A *local system* of abelian groups on a space (resp. poset) X is a functor $\mathcal{F}$ from the groupoid of X (resp. X viewed as a category), to the category of abelian groups which is morphism-inverting, i. e. such that the map $\mathcal{F}(x) \to \mathcal{F}(x')$ associated to a path from x to x' (resp. $x \leq x'$) is an isomorphism. Clearly, a local system $\mathcal{F}$ on a path connected space (resp. 0-connected poset) is determined, up to canonical isomorphism, by the following data: if $x \in X$ is a base point, it suffices to be given the group $\mathcal{F}(x)$ and an action of $\pi_1(X,x)$ on $\mathcal{F}(x)$.

The homology groups $H_k(X,\mathcal{F})$ of a space X with a local system $\mathcal{F}$ are a generalization of the ordinary homology groups. In fact if X is a 0-connected space and if $\mathcal{F}$ is a constant local system on X, then $H_k(X,\mathcal{F}) \simeq H_k(X,\mathcal{F}(x_0))$ for every $x_0 \in X$ [25, Chap. VI, 2.1].

Let X be a poset and $\mathcal{F}$ a local system on $|X|$. Then the restriction of $\mathcal{F}$ to X is a local system on X. Considering $\mathcal{F}$ as a functor from X to the category of abelian groups, we can define $H_k(X,\mathcal{F})$ as in the above. Conversely if $\mathcal{F}$ is a local system on the poset X, then there is a unique local system, up to isomorphism, on $|X|$ such that the restriction to X is $\mathcal{F}$ [25, Chap. VI, Thm 1.12], [14, I, Prop. 1]. We denote both local systems by $\mathcal{F}$.

Theorem 2.4. *Let X be a poset and $\mathcal{F}$ a local system on X. Then the homology groups $H_k(|X|,\mathcal{F})$ are isomorphic with the homology groups $H_k(X,\mathcal{F})$.*

Proof. See [25, Chap. VI, Thm. 4.8] or [14, I, p. 91]. □

Theorem 2.5. *Let X be a path connected space with a base point x and let $\mathcal{F}$ be a local system on X. Then the inclusion $\{x\} \hookrightarrow X$ induces an isomorphism $\mathcal{F}(x)/G \xrightarrow{\simeq} H_0(X,\mathcal{F})$ where G is the subgroup of $\mathcal{F}(x)$ generated by all the elements of the form $a - \beta a$ with $a \in \mathcal{F}(x)$, $\beta \in \pi_1(X,x)$.*

Proof. See [25], Chap. VI, Thm. 2.8* and Thm. 3.2. □

We need the following interesting and well known lemma about the covering spaces of the space $|X|$, where X is a poset (or more generally a simplicial set). For a definition of a covering space, useful for our purpose, and some more information, see [18, Chap. 2].

Lemma 2.6. *For a poset X the category of the covering spaces of the space $|X|$ is equivalent to the category $\mathcal{L}_S(X)$, the category of functors $\mathcal{F} : X \to \underline{\mathrm{Set}}$, where $\underline{\mathrm{Set}}$ is the category of sets, such that $\mathcal{F}(x) \to \mathcal{F}(x')$ is a bijection for every relation $x \le x'$.*

Proof. See [16, Section 7] or [14, I, p. 90]. □

3. Homology and homotopy of posets

Theorem 3.1. *Let $f : X \to Y$ be a map of posets. Then there is a first quadrant spectral sequence*

$$E^2_{p,q} = H_p(Y, y \mapsto H_q(f/y, \mathbb{Z})) \Rightarrow H_{p+q}(X, \mathbb{Z}).$$

The spectral sequence is functorial, in the sense that if there is a commutative diagram of posets

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g_X} & & \downarrow{\scriptstyle g_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

then there is a natural map from the spectral sequence arising from f' to the spectral sequence arising from f. Moreover the map $g_{X*} : H_i(X', \mathbb{Z}) \to H_i(X, \mathbb{Z})$ is compatible with this natural map.

Proof. Let $C_{*,*}(f)$ be the double complex such that $C_{p,q}(f)$ is the free abelian group generated by the set $\{(x_0 < \cdots < x_q, f(x_q) < y_0 < \cdots < y_p) : x_i \in X,\ y_i \in Y\}$. The first spectral sequence of this double complex has as E^1-term $E^1_{p,q}(\mathrm{I}) = H_q(C_{p,*}(f)) = \bigoplus_{y_0 < \cdots < y_p} H_q(f/y_0, \mathbb{Z})$. By the general theory of double complexes (see for example [24, Chap. 5]), we know that $E^2_{p,q}(\mathrm{I})$ is the homology of the chain complex $E^1_{*,q}(\mathrm{I}) = C_*(Y, \mathcal{G}_q)$ where $\mathcal{G}_q : Y \to \underline{\mathrm{Ab}}$, $\mathcal{G}_q(y) = H_q(f/y, \mathbb{Z})$ and hence $E^2_{p,q}(\mathrm{I}) = H_p(Y, \mathcal{G}_q) = H_p(Y, y \mapsto H_q(f/y, \mathbb{Z}))$. The second spectral sequence has as E^1-term $E^1_{p,q}(\mathrm{II}) = H_q(C_{*,p}(f)) = \bigoplus_{f(x_p) < y_0 < \cdots < y_q} H_q(f(x_p)\backslash \mathrm{id}_Y, \mathbb{Z})$. But by 2.3 (ii), $f(x_p)\backslash \mathrm{id}_Y = Y_{\ge f(x_p)}$ is contractible, so $E^1_{*,0}(\mathrm{II}) = C_*(X^{op}, \mathbb{Z})$ and $E^1_{*,q}(\mathrm{II}) = 0$ for $q > 0$. Hence $H_i(\mathrm{Tot}(C_{*,*}(f))) \simeq H_i(X^{op}, \mathbb{Z}) \simeq H_i(X, \mathbb{Z})$. This completes the proof of existence and convergence of the spectral sequence. The functorial behavior of the spectral sequence follows from the functorial behavior of the spectral sequence of a filtration [24, 5.5.1] and the fact that the first and the second spectral sequences of the double complex arise from some filtrations. □

Remark 3.2. The above spectral sequence is a special case of a more general Theorem [6, App. II]. The above proof is taken from [9, Chap. I] where the functorial behavior of the spectral sequence is more visible. For more details see [9].

Definition 3.3. Let X be a poset. A map $\mathrm{ht}_X : X \to \mathbb{Z}_{\ge 0}$ is called height function if it is a strictly increasing map.

Example 3.4. The height function $\mathrm{ht}_X(x) = 1 + \dim(Link_X^-(x))$ is the usual one considered in [16], [9] and [5].

Lemma 3.5. *Let X be a poset such that $Link_X^+(x)$ is $(n - \mathrm{ht}_X(x) - 2)$-acyclic, for every $x \in X$, where ht_X is a height function on X. Let $\mathcal{F} : X \to \underline{\mathrm{Ab}}$ be a functor such that $\mathcal{F}(x) = 0$ for all $x \in X$ with $\mathrm{ht}_X(x) \geq m$, where $m \geq 1$. Then $H_k(X, \mathcal{F}) = 0$ for $k \leq n - m$.*

Proof. First consider the case of a functor $\mathcal{F}$ such that $\mathcal{F}(x) = 0$ if $\mathrm{ht}_X(x) \neq m - 1$. Then $C_k(X, \mathcal{F}) = \bigoplus_{\substack{x_0 < \cdots < x_k \\ \mathrm{ht}_X(x_0) = m-1}} \mathcal{F}(x_0)$. Clearly $0 = d_0^k = \mathcal{F}(x_0 < x_1) = \mathcal{F}(x_0) \to \mathcal{F}(x_1)$. Thus $\partial_k = \Sigma_{i=1}^k (-1)^i d_i^k$. Define $C_{-1}(Link_X^+(x_0), \mathcal{F}(x_0)) = \mathcal{F}(x_0)$ and complete the singular complex of $Link_X^+(x_0)$ with coefficient in $\mathcal{F}(x_0)$ to

$$\cdots \to C_0(Link_X^+(x_0), \mathcal{F}(x_0)) \xrightarrow{\varepsilon} C_{-1}(Link_X^+(x_0), \mathcal{F}(x_0)) \to 0$$

where $\varepsilon((g_i)) = \Sigma_i g_i$. Then

$$\begin{aligned} C_k(X, \mathcal{F}) &= \bigoplus_{\mathrm{ht}_X(x_0) = m-1} \Big(\bigoplus_{\substack{x_1 < \cdots < x_k \\ x_0 < x_1}} \mathcal{F}(x_0)\Big) \\ &= \bigoplus_{\mathrm{ht}_X(x_0) = m-1} C_{k-1}(Link_X^+(x_0), \mathcal{F}(x_0)). \end{aligned}$$

The complex $C_{k-1}(Link_X^+(x_0), \mathcal{F}(x_0))$ is the standard complex for computing the reduced homology of $Link_X^+(x_0)$ with constant coefficient $\mathcal{F}(x_0)$. So

$$H_k(X, \mathcal{F}) = \bigoplus_{\mathrm{ht}_X(x) = m-1} \tilde{H}_{k-1}(Link_X^+(x), \mathcal{F}(x)).$$

If $\mathrm{ht}_X(x_0) = m-1$ then $Link_X^+(x_0)$ is $(n-(m-1)-2)$-acyclic, and by the universal coefficient theorem [18, Chap. 5, Thm. 8], $\tilde{H}_{k-1}(Link_X^+(x_0), \mathcal{F}(x_0)) = 0$ for $-1 \leq k-1 \leq n-(m-1)-2$. This shows that $H_k(X, \mathcal{F}) = 0$ for $0 \leq k \leq n-m$. To prove the lemma in general, we argue by induction on m. If $m = 1$ then for $\mathrm{ht}_X(x) \geq 1$, $\mathcal{F}(x) = 0$. So the lemma follows from the special case above. Suppose $m \geq 2$. Define $\mathcal{F}_0$ and $\mathcal{F}_1$ to be the functors

$$\mathcal{F}_0(x) = \begin{cases} \mathcal{F}(x) & \text{if } \mathrm{ht}_X(x) < m-1 \\ 0 & \text{if } \mathrm{ht}_X(x) \geq m-1 \end{cases}, \quad \mathcal{F}_1(x) = \begin{cases} \mathcal{F}(x) & \text{if } \mathrm{ht}_X(x) = m-1 \\ 0 & \text{if } \mathrm{ht}_X(x) \neq m-1 \end{cases}$$

respectively. Then there is a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_0 \to 0$. By the above discussion, $H_k(X, \mathcal{F}_1) = 0$ for $0 \leq k \leq n - m$ and by induction for $m - 1$, we have $H_k(X, \mathcal{F}_0) = 0$ for $k \leq n - (m - 1)$. By the long exact sequence for the above short exact sequence of functors it is easy to see that $H_k(X, \mathcal{F}) = 0$ for $0 \leq k \leq n - m$. □

Theorem 3.6. *Let $f : X \to Y$ be a map of posets and ht_Y a height function on Y. Assume for every $y \in Y$, that $Link_Y^+(y)$ is $(n - \mathrm{ht}_Y(y) - 2)$-acyclic and f/y*

is $(\mathrm{ht}_Y(y)-1)$*-acyclic. Then* $f_*: H_k(X,\mathbb{Z}) \to H_k(Y,\mathbb{Z})$ *is an isomorphism for* $0 \le k \le n-1$.

Proof. By theorem 3.1, we have the first quadrant spectral sequence

$$E^2_{p,q} = H_p(Y, y \mapsto H_q(f/y,\mathbb{Z})) \Rightarrow H_{p+q}(X,\mathbb{Z}).$$

Since $H_q(f/y,\mathbb{Z}) = 0$ for $0 < q \le \mathrm{ht}_Y(y) - 1$, the functor $\mathcal{G}_q : Y \to \underline{\mathrm{Ab}}$, $\mathcal{G}_q(y) = H_q(f/y,\mathbb{Z})$ is trivial for $\mathrm{ht}_Y(y) \ge q+1$, $q > 0$. By lemma 3.5, $H_p(Y,\mathcal{G}_q) = 0$ for $p \le n-(q+1)$. Hence $E^2_{p,q} = 0$ for $p+q \le n-1$, $q > 0$. If $q = 0$, by writing the long exact sequence for the short exact sequence $0 \to \tilde{H}_0(f/y,\mathbb{Z}) \to H_0(f/y,\mathbb{Z}) \to \mathbb{Z} \to 0$, valid because f/y is nonempty, we have

$$\cdots \to H_n(Y,\mathbb{Z}) \to H_{n-1}(Y, y \mapsto \tilde{H}_0(f/y,\mathbb{Z})) \to E^2_{n-1,0} \to$$
$$\cdots \to H_1(Y,\mathbb{Z}) \to H_0(Y, y \mapsto \tilde{H}_0(f/y,\mathbb{Z})) \to E^2_{0,0} \to H_0(Y,\mathbb{Z}) \to 0.$$

If $\mathrm{ht}_Y(y) \ge 1$, then $\tilde{H}_0(f/y,\mathbb{Z}) = 0$. By lemma 3.5, $H_k(Y, y \mapsto \tilde{H}_0(f/y,\mathbb{Z})) = 0$ for $0 \le k \le n-1$. Thus

$$E^2_{p,q} = \begin{cases} H_p(Y,\mathbb{Z}) & \text{if } q=0,\ 0 \le p \le n-1 \\ 0 & \text{if } p+q \le n-1,\ q>0 \end{cases}.$$

This shows that $E^2_{p,q} \simeq \cdots \simeq E^\infty_{p,q}$ for $0 \le p+q \le n-1$. Therefore $H_k(X,\mathbb{Z}) \simeq H_k(Y,\mathbb{Z})$ for $0 \le k \le n-1$. Now consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow f & & \big\downarrow \mathrm{id}_Y \\ Y & \xrightarrow{\mathrm{id}_Y} & Y \end{array}.$$

By functoriality of the spectral sequence 3.1, and the above calculation we get the diagram

$$\begin{array}{ccc} H_k(Y, y \mapsto H_0(f/y,\mathbb{Z})) & \xrightarrow{\simeq} & H_k(X,\mathbb{Z}) \\ \big\downarrow \mathrm{id}_{Y*} & & \big\downarrow f_* \\ H_k(Y, y \mapsto H_0(\mathrm{id}_Y/y,\mathbb{Z})) & \xrightarrow{\simeq} & H_k(Y,\mathbb{Z}) \end{array}.$$

Since $\mathrm{id}_Y/y = Y_{\le y}$ is contractible, we have $H_k(Y, y \mapsto H_0(\mathrm{id}_Y/y,\mathbb{Z})) = H_k(Y,\mathbb{Z})$. The map id_{Y*} is an isomorphism for $0 \le k \le n-1$, from the above long exact sequence. This shows that f_* is an isomorphism for $0 \le k \le n-1$. □

Lemma 3.7. *Let X be a 0-connected poset. Then X is 1-connected if and only if for every local system $\mathcal{F}$ on X and every $x \in X$, the map $\mathcal{F}(x) \to H_0(X,\mathcal{F})$, induced from the inclusion $\{x\} \hookrightarrow X$, is an isomorphism (or equivalently, every local system on X is a isomorphic with a constant local system).*

Proof. If X is 1-connected then by theorem 2.5 and the connectedness of X, one has $\mathcal{F}(x) \xrightarrow{\simeq} H_0(X,\mathcal{F})$ for every $x \in X$. Now let every local system on X be isomorphic with a constant local system. Let $\mathcal{F}: X \to \underline{\mathrm{Set}}$ be in $\mathcal{L}_S(X)$.

Define the functor $\mathcal{G} : X \to \underline{\mathrm{Ab}}$ where $\mathcal{G}(x)$ is the free abelian group generated by $\mathcal{F}(x)$. Clearly $\mathcal{G}$ is a local system and so it is constant system. It follows that $\mathcal{F}$ is isomorphic to a constant functor. So by lemma 2.6, any connected covering space of $|X|$ is isomorphic to $|X|$. This shows that the universal covering of $|X|$, is $|X|$. Note that the universal covering of a connected simplicial simplex exists and is simply connected [18, Chap. 2, Cor. 14 and 15]. Therefore X is 1-connected. □

Theorem 3.8. *Let $f : X \to Y$ be a map of posets and ht_Y a height function on Y. Assume for every $y \in Y$, that $Link_Y^+(y)$ is $(n - \mathrm{ht}_Y(y) - 2)$-connected and f/y is $(\mathrm{ht}_Y(y) - 1)$-connected. Then X is $(n-1)$-connected if and only if Y is $(n-1)$-connected.*

Proof. By 2.1 and 3.6 we may assume $n \geq 2$. So it is enough to prove that X is 1-connected if and only if Y is 1-connected. Let $\mathcal{F} : X \to \underline{\mathrm{Ab}}$ be a local system. Define the functor $\mathcal{G} : Y \to \underline{\mathrm{Ab}}$ with

$$\mathcal{G}(y) = \begin{cases} H_0(f/y, \mathcal{F}) & \text{if } \mathrm{ht}_Y(y) \neq 0 \\ H_0(Link_Y^+(y), y' \mapsto H_0(f/y', \mathcal{F})) & \text{if } \mathrm{ht}_Y(y) = 0 \end{cases}.$$

We prove that $\mathcal{G}$ is a local system. If $\mathrm{ht}_Y(y) \geq 2$ then f/y is 1-connected and by 3.6, $\mathcal{F}|_{f/y}$ is a constant system, so by 3.7, $H_0(f/y, \mathcal{F}) \simeq \mathcal{F}(x)$ for every $x \in f/y$. If $\mathrm{ht}_Y(y) = 1$, then f/y is 0-connected and $Link_Y^+(y)$ is nonempty. Choose $y' \in Y$ such that $y < y'$. Now f/y' is 1-connected and so $\mathcal{F}|_{f/y'}$ is a constant system on f/y'. But $f/y \subset f/y'$, so $\mathcal{F}|_{f/y}$ is a constant system. Since f/y is 0-connected, by 2.5 and the fact that we mentioned before theorem 2.4, $H_0(f/y, \mathcal{F}) \simeq \mathcal{F}(x)$ for every $x \in f/y$. Now let $\mathrm{ht}_Y(y) = 0$. Then $Link_Y^+(y)$ is 0-connected, f/y is nonempty and for every $y' \in Link_Y^+(y)$, $H_0(f/y', \mathcal{F}) \simeq H_0((f/y)^\circ, \mathcal{F})$ where $(f/y)^\circ$ is a component of f/y, which we fix. This shows that the local system $\mathcal{F}' : Link_Y^+(y) \to \underline{\mathrm{Ab}}$ with $y' \mapsto H_0(f/y', \mathcal{F})$ is isomorphic to a constant system, so $H_0(Link_Y^+(y), y' \mapsto H_0(f/y', \mathcal{F})) = H_0(Link_Y^+(y), \mathcal{F}') \simeq \mathcal{F}'(y') \simeq \mathcal{F}(x)$ for every $x \in f/y'$. Therefore $\mathcal{G}$ is a local system.

If Y is 1-connected, by 3.7, $\mathcal{G}$ is a constant system. But it is easy to see that $\mathcal{F} \simeq \mathcal{G} \circ f$. Therefore $\mathcal{F}$ is a constant system. Since X is connected by our homology calculation, by 3.7 we conclude that X is 1-connected. Now let X be 1-connected. If $\mathcal{E}$ is a local system on Y, then $f^*\mathcal{E} := \mathcal{E} \circ f$ is a local system on X. So it is a constant local system. As above we can construct a local system $\mathcal{G}'$ on Y from $\mathcal{F}' := \mathcal{E} \circ f$. This gives a natural transformation from $\mathcal{G}'$ to $\mathcal{E}$ which is an isomorphism. Since $\mathcal{E} \circ f$ is constant, by 2.5 and 3.7 and an argument as above one sees that $\mathcal{G}'$ is constant. Therefore $\mathcal{E}$ is isomorphic to a constant local system and 3.7 shows that Y is 1-connected. □

Remark 3.9. In the proof of the above theorem 3.8 we showed in fact that: Let $f : X \to Y$ be a map of posets and ht_Y a height function on Y and $n \geq 2$. Assume for every $y \in Y$, that $Link_Y^+(y)$ is $(n - \mathrm{ht}_Y(y) - 2)$-connected and f/y

is $(\mathrm{ht}_Y(y)-1)$-connected. Then $f^*: \mathcal{L}_S(Y) \to \mathcal{L}_S(X)$, with $\mathcal{E} \mapsto \mathcal{E} \circ f$ is an equivalence of categories.

Remark 3.10. Theorem 3.8 is a generalization of a theorem of Quillen [16, Thm. 9.1]. We proved that the converse of that theorem is also valid. Our proof is similar in outline to the proof by Quillen. Furthermore, lemma 3.5 is a generalized version of lemma 1.3 from [5]. With more restrictions, Maazen, in [9, Chap. II] gave an easier proof of Quillen's theorem.

4. Homology and homotopy of posets of sequences

Let V be a nonempty set. We denote by $\mathcal{O}(V)$ the poset of finite ordered sequences of distinct elements of V, the length of each sequence being at least one. The partial ordering on $\mathcal{O}(V)$ is defined by refinement: $(v_1, \dots, v_m) \le (w_1, \dots, w_n)$ if and only if there is a strictly increasing map $\phi : \{1, \dots, m\} \to \{1, \dots, n\}$ such that $v_i = w_{\phi(i)}$, in other words, if $(v_1, \dots, v_m)$ is an order preserving subsequence of $(w_1, \dots, w_n)$. If $v = (v_1, \dots, v_m)$ we denote by $|v|$ the length of v, that is $|v| = m$. If $v = (v_1, \dots, v_m)$ and $w = (w_1, \dots, w_n)$, we write $(v_1, \dots, v_m, w_1, \dots, w_n)$ as vw. For $v \in F$, but for such v only, we define F_v to be the set of $w \in F$ such that $wv \in F$. Note that $(F_v)_w = F_{wv}$. A subset F of $\mathcal{O}(V)$ is said to satisfy the *chain condition* if $v \in F$ whenever $w \in F$, $v \in \mathcal{O}(V)$ and $v \le w$. The subposets of $\mathcal{O}(V)$ which satisfy the chain condition are extensively studied in [9], [20] and [4]. In this section we will study them some more.
Let $F \subseteq \mathcal{O}(V)$. For a nonempty set S we define the poset $F\langle S\rangle$ as

$$F\langle S\rangle := \{((v_1, s_1), \dots, (v_r, s_r)) \in \mathcal{O}(V \times S) : (v_1, \dots, v_r) \in F\}.$$

Assume $s_0 \in S$ and consider the injective poset map $l_{s_0} : F \to F\langle S\rangle$ with $(v_1, \dots, v_r) \mapsto ((v_1, s_0), \dots, (v_r, s_0))$. We have clearly a projection $p : F\langle S\rangle \to F$ with $((v_1, s_1), \dots, (v_r, s_r)) \mapsto (v_1, \dots, v_r)$ such that $p \circ l_{s_0} = \mathrm{id}_F$.

Lemma 4.1. *Suppose $F \subseteq \mathcal{O}(V)$ satisfies the chain condition and S is a nonempty set. Assume for every $v \in F$, that F_v is $(n - |v|)$-connected.*
(i) *If $s_0 \in S$ then $(l_{s_0})_* : H_k(F, \mathbb{Z}) \to H_k(F\langle S\rangle, \mathbb{Z})$ is an isomorphism for $0 \le k \le n$.*
(ii) *If F is $\min\{1, n-1\}$-connected, then $(l_{s_0})_* : \pi_k(F, v) \to \pi_k(F\langle S\rangle, l_{s_0}(v))$ is an isomorphism for $0 \le k \le n$.*

Proof. This follows by [4, Prop. 1.6] from the fact that $p \circ l_{s_0} = \mathrm{id}_F$. □

Lemma 4.2. *Let $F \subseteq \mathcal{O}(V)$ satisfies the chain condition. Then $|Link_F^-(v)| \simeq S^{|v|-2}$ for every $v \in F$.*

Proof. Let $v = (v_1, \dots, v_n)$. By definition $Link_F^-(v) = \{w \in F : w < v\} = \{(v_{i_1}, \dots, v_{i_k}) : k < n, i_1 < \dots < i_k\}$. Hence $|Link_F^-(v)|$ is isomorphic to the barycentric subdivision of the boundary of the standard simplex Δ_{n-1}. It is well known that $\partial\Delta_{n-1} \simeq S^{n-2}$, hence $|Link_F^-(v)| \simeq S^{|v|-2}$. □

Theorem 4.3 (Nerve Theorem for Posets). *Let V and T be two nonempty sets, $F \subseteq \mathcal{O}(V)$ and $X \subseteq \mathcal{O}(T)$. Assume $X = \bigcup_{v \in F} X_v$ such that if $v \leq w$ in F, then $X_w \subseteq X_v$. Let F, X and X_v, for every $v \in F$, satisfy the chain condition. Also assume*
(i) *for every $v \in F$, X_v is $(l - |v| + 1)$-acyclic (resp. $(l - |v| + 1)$-connected),*
(ii) *for every $x \in X$, $\mathcal{A}_x := \{v \in F : x \in X_v\}$ is $(l - |x| + 1)$-acyclic (resp. $(l - |x| + 1)$-connected).*
Then $H_k(F, \mathbb{Z}) \simeq H_k(X, \mathbb{Z})$ for $0 \leq k \leq l$ (resp. F is l-connected if and only if X is l-connected).

Proof. Let $F_{\leq l+2} = \{v \in F : |v| \leq l + 2\}$ and let $i : F_{\leq l+2} \to F$ be the inclusion. Clearly $|F_{\leq l+2}|$ is the $(l + 1)$-skeleton of $|F|$, if we consider $|F|$ as a cell complex whose k-cells are the $|F_{\leq v}|$ with $|v| = k + 1$. It is well known that $i_* : H_k(F_{\leq l+2}, \mathbb{Z}) \to H_k(F, \mathbb{Z})$ and $i_* : \pi_k(F_{\leq l+2}, v) \to \pi_k(F, v)$ are isomorphisms for $0 \leq k \leq l$ (see [25], Chap. II, corollary 2.14, and [25], Chap. II, Corollary 3.10 and Chap. IV lemma 7.12.) So it is enough to prove the theorem for $F_{\leq l+2}$ and $X_{\leq l+2}$. Thus assume $F = F_{\leq l+2}$ and $X = X_{\leq l+2}$. We define $Z \subseteq X \times F$ as $Z = \{(x, v) : x \in X_v\}$. Consider the projections

$$f : Z \to F,\ (x, v) \mapsto v \qquad , \qquad g : Z \to X,\ (x, v) \mapsto x.$$

First we prove that $f^{-1}(v) \sim v\backslash f$ and $g^{-1}(x) \sim x\backslash g$, where $\sim$ means homotopy equivalence. By definition $v\backslash f = \{(x, w) : w \geq v,\ x \in X_w\}$. Define $\phi : v\backslash f \to f^{-1}(v)$, $(x, w) \mapsto (x, v)$. Consider the inclusion $j : f^{-1}(v) \to v\backslash f$. Clearly $\phi \circ j(x, v) = \phi(x, v) = (x, v)$ and $j \circ \phi(x, w) = j(x, v) = (x, v) \leq (x, w)$. So by 2.3(ii), $v\backslash f$ and $f^{-1}(v)$ are homotopy equivalent. Similarly $x\backslash g \sim g^{-1}(x)$.
Now we prove that the maps $f^{op} : Z^{op} \to Y^{op}$ and $g^{op} : Z^{op} \to X^{op}$ satisfy the conditions of 3.6. First $f^{op} : Z^{op} \to Y^{op}$; define the height function $\mathrm{ht}_{F^{op}}$ on F^{op} as $\mathrm{ht}_{F^{op}}(v) = l + 2 - |v|$. It is easy to see that $f^{op}/v \simeq v\backslash f \sim f^{-1}(v) \simeq X_v$. Hence f^{op}/v is $(l - |v| + 1)$-acyclic (resp. $(l - |v| + 1)$-connected). But $l - |v| + 1 = (l + 2 - |v|) - 1 = \mathrm{ht}_{F^{op}}(v) - 1$, so f^{op}/v is $(\mathrm{ht}_{F^{op}}(v) - 1)$-acyclic (resp. $(\mathrm{ht}_{F^{op}}(v) - 1)$-connected). Let $n := l + 1$. Clearly $Link^+_{F^{op}}(v) = Link^-_F(v)$. By lemma 4.2, $|Link^-_F(v)|$ is $(|v| - 3)$-connected. But $|v| - 3 = l + 1 - (l + 2 - |v|) - 2 = n - \mathrm{ht}_{F^{op}}(v) - 2$. Thus $Link^+_{F^{op}}(v)$ is $(n - \mathrm{ht}_{F^{op}}(v) - 2)$-acyclic (resp. $(n - \mathrm{ht}_{F^{op}}(v) - 2)$-connected). Therefore by theorem 3.6, $f_* : H_i(Z, \mathbb{Z}) \to H_i(F, \mathbb{Z})$ is an isomorphism for $0 \leq i \leq l$ (resp. by 3.8, F is l-connected if and only if Z is l-connected). Now consider $g^{op} : Z^{op} \to X^{op}$. We saw in the above that $g^{op}/x \simeq x\backslash g \sim g^{-1}(x)$ and $g^{-1}(x) = \{(x, v) : x \in X_v\} \simeq \{v \in F : x \in X_v\}$. It is similar to the case of f^{op} to see that g^{op} satisfies the conditions of theorem 3.6, hence $g_* : H_i(Z, \mathbb{Z}) \to H_i(X, \mathbb{Z})$ is an isomorphism for $0 \leq i \leq l$ (resp. by 3.8, X is l-connected if and only if Z is l-connected). This completes the proof. □

Let K be a simplicial complex and $\{K_i\}_{i \in I}$ a family of subcomplexes such that $K = \bigcup_{i \in I} K_i$. The *nerve* of this family of subcomplexes of K is the simplicial complex $\mathcal{N}(K)$ on the vertex set I so that a finite subset $\sigma \subseteq I$ is in $\mathcal{N}(K)$ if and only if $\bigcap_{i \in \sigma} K_i \neq \varnothing$. The nerve $\mathcal{N}(K)$ of K, with the inclusion relation, is a poset. As we already said we can consider a simplicial complex as a poset of its simplices.

Corollary 4.4 (Nerve Theorem). *Let K be a simplicial complex and $\{K_i\}_{i\in I}$ a family of subcomplexes such that $K = \bigcup_{i\in I} K_i$. Suppose every nonempty finite intersection $\bigcap_{j=1}^{t} K_{i_j}$ is $(l-t+1)$-acyclic (resp. $(l-t+1)$connected). Then $H_k(K,\mathbb{Z}) \simeq H_k(\mathcal{N}(K),\mathbb{Z})$ for $0 \le k \le l$ (resp. K is l-connected if and only if $\mathcal{N}(K)$ is l-connected).*

Proof. Let V be the set of vertices of K. We give a total ordering to V and I. Put $F = \{(i_1,\dots,i_r) : i_1 < \dots < i_r \text{ and } \bigcap_{j=1}^{r} K_{i_j} \neq \varnothing\} \subseteq \mathcal{O}(I)$, $X = \{(x_1,\dots,x_t) : x_1 < \dots < x_t \text{ and } \{x_1,\dots,x_t\} \text{ is a simplex in } K\} \subseteq \mathcal{O}(V)$ and for every $(i_1,\dots,i_r) \in F$, put $X_{(i_1,\dots,i_r)} = \{(x_1,\dots,x_t) \in X : \{x_1,\dots,x_t\} \in \bigcap_{j=1}^{r} K_{i_j}\}$. It is not difficult to see that $F \simeq \mathcal{N}(K)$ and $X \simeq K$. Also one should notice that $\mathcal{A}_x := \{v \in F : x \in X_v\}$ is contractible for $x \in X$. We leave the details to interested readers. □

Remark 4.5. In [7], a special case of the theorem 4.3 is proved. The nerve theorem for a simplicial complex 4.4, in the stated generality, is proved for the first time in [3], see also [2, p. 1850]. For more information about different types of nerve theorem and more references about them see [2, p. 1850].

Lemma 4.6. *Let $F \subseteq \mathcal{O}(V)$ satisfy the chain condition and let $\mathcal{G} : F^{op} \to \underline{\mathrm{Ab}}$ be a functor. Then the natural map $\psi : \bigoplus_{v\in F,\ |v|=1} \mathcal{G}(v) \to H_0(F^{op},\mathcal{G})$ is surjective.*

Proof. By definition $C_0(F^{op},\mathcal{G}) = \bigoplus_{v\in F^{op}} \mathcal{G}(v)$, $C_1(F^{op},\mathcal{G}) = \bigoplus_{v<v'\in F^{op}} \mathcal{G}(v)$ and we have the chain complex

$$\dots \to C_1(F^{op},\mathcal{G}) \xrightarrow{\partial_1} C_0(F^{op},\mathcal{G}) \to 0,$$

where $\partial_1 = d_0^1 - d_1^1$. Again by definition $H_0(F^{op},\mathcal{G}) = C_0(F^{op},\mathcal{G})/\partial_1$. Now let $w \in F$ and $|w| \ge 2$. Then there is a $v \in F$, $v \le w$, with $|v| = 1$. So $w < v$ in F^{op}, and we have the component $\partial_1|_{\mathcal{G}(w)} : \mathcal{G}(w) \to \mathcal{G}(w) \oplus \mathcal{G}(v)$, $x \mapsto d_0^1(x) - d_1^1(x) = d_0^1(x) - x$. This shows that $\mathcal{G}(w) \subseteq \mathrm{im}\partial_1 + \mathrm{im}\psi$. Therefore $H_0(F^{op},\mathcal{G})$ is generated by the groups $\mathcal{G}(v)$ with $|v| = 1$. □

Theorem 4.7. *Let V and T be two nonempty sets, $F \subseteq \mathcal{O}(V)$ and $X \subseteq \mathcal{O}(T)$. Assume $X = \bigcup_{v\in F} X_v$ such that if $v \le w$ in F, then $X_w \subseteq X_v$ and let F, X and X_v, for every $v \in F$, satisfy the chain condition. Also assume*
(i) *for every $v \in F$, X_v is $\min\{l-1, l-|v|+1\}$-connected,*
(ii) *for every $x \in X$, $\mathcal{A}_x := \{v \in F : x \in X_v\}$ is $(l-|x|+1)$-connected,*
(iii) *F is l-connected.*
Then X is $(l-1)$-connected and the natural map

$$\bigoplus_{v\in F,\ |v|=1} (i_v)_* : \bigoplus_{v\in F,\ |v|=1} H_l(X_v,\mathbb{Z}) \to H_l(X,\mathbb{Z})$$

is surjective, where $i_v : X_v \to X$ is the inclusion. Moreover, if for every v with $|v| = 1$, there is an l-connected Y_v with $X_v \subseteq Y_v \subseteq X$, then X is also l-connected.

Proof. If $l = -1$, then everything is easy. If $l = 0$, then for v of length one, X_v is nonempty, so X is nonempty. This shows that X is (-1)-connected. Also, every connected component of X intersects at least one X_w and therefore also contains a connected component of an X_v with $|v| = 1$. This gives the surjectivity of the homomorphism

$$\bigoplus_{v\in F,\ |v|=1} (i_v)_* : \bigoplus_{v\in F,\ |v|=1} H_0(X_v,\mathbb{Z}) \to H_0(X,\mathbb{Z}).$$

Now assume that, for every v of length one, $X_v \subseteq Y_v$ where Y_v is connected. We prove, in a combinatorial way, that X is connected. Let $x, y \in X$, $x \in X_{(v_1)}$ and $y \in X_{(v_2)}$ where $(v_1), (v_2) \in F$. Since F is connected, there is a sequence $(w_1), \dots, (w_r) \in F$ such that they give a path, in F, from (v_1) to (v_2), that is

$$\begin{array}{ccccccccc} (v_1) & & & & (w_1) & & (w_r) & & & & (v_2) \\ & \searrow & & \swarrow & & \cdots & & \searrow & & \swarrow & \\ & & (v_1,w_1) & & & & & & (w_r,v_2) & & \end{array} .$$

Since $Y_{(v_1)}$ is connected, $x \in X_{(v_1)} \subseteq Y_{(v_1)}$ and $X_{(v_1,w_1)} \neq \varnothing$, there is an element $x_1 \in X_{(v_1,w_1)}$ such that there is a path, in $Y_{(v_1)}$, from x to x_1. Now $x_1 \in Y_{(w_1)}$. Similarly we can find $x_2 \in X_{(w_1,w_2)}$ such that there is a path, in $Y_{(w_1)}$, from x_1 to x_2. Now $x_2 \in Y_{(w_2)}$. Repeating this process finitely many times, we find a path from x to y. So X is connected.
Hence we assume that $l \geq 1$. As we said in the proof of theorem 4.3, we can assume that $F = F_{\leq l+2}$ and $X = X_{\leq l+2}$ and we define Z, f and g as we defined them there. Define the height function $\mathrm{ht}_{F^{op}}$ on F^{op} as $\mathrm{ht}_{F^{op}}(v) = l + 2 - |v|$. As we proved in the proof of theorem 4.3, $f^{op}/v \simeq v\backslash f \sim f^{-1}(v) \simeq X_v$. Thus f^{op}/v is $(\mathrm{ht}_{F^{op}}(v) - 1)$-connected if $|v| > 1$ and it is $(\mathrm{ht}_{F^{op}}(v) - 2)$-connected if $|v| = 1$ and also $|Link^+_{F^{op}}(v)|$ is $(l+1-\mathrm{ht}_{F^{op}}(v)-2)$-connected. By theorem 3.1, we have the first quadrant spectral sequence

$$E^2_{p,q} = H_p(F^{op}, v \mapsto H_q(f^{op}/v,\mathbb{Z})) \Rightarrow H_{p+q}(Z^{op},\mathbb{Z}).$$

For $0 < q \leq \mathrm{ht}_{F^{op}}(v) - 2$, $H_q(f^{op}/v,\mathbb{Z}) = 0$. Define $\mathcal{G}_q : F^{op} \to \underline{\mathrm{Ab}}$, $\mathcal{G}_q(v) = H_q(f^{op}/v,\mathbb{Z})$. Then $\mathcal{G}_q(v) = 0$ for $\mathrm{ht}_{F^{op}}(v) \geq q + 2$, $q > 0$. By lemma 3.5, $H_p(F^{op}, \mathcal{G}_q) = 0$ for $p \leq l + 1 - (q + 2)$. Therefore $E^2_{p,q} = 0$ for $p + q \leq l - 1$, $q > 0$. If $q = 0$, arguing similarly to the proof of theorem 3.6, we get $E^2_{p,0} = 0$ if $0 < p \leq l - 1$ and $E^2_{0,0} = \mathbb{Z}$. Also by the fact that F^{op} is l-connected we get the surjective homomorphism $H_l(F^{op}, v \mapsto \tilde{H}_0(f^{op}/v,\mathbb{Z})) \twoheadrightarrow E^2_{l,0}$. Since $l \geq 1$, $\tilde{H}_0(f^{op}/v,\mathbb{Z}) = 0$ for all $v \in F^{op}$ with $\mathrm{ht}_{F^{op}}(v) \geq 1$ and so $H_l(F^{op}, v \mapsto \tilde{H}_0(f^{op}/v,\mathbb{Z})) = 0$ by lemma 3.5. Therefore $E^2_{l,0} = 0$. Let

$$\mathcal{G}'_q : F^{op} \to \underline{\mathrm{Ab}}, \quad \mathcal{G}'_q(v) = \begin{cases} 0 & \text{if } \mathrm{ht}_{F^{op}}(v) < l + 1 \\ H_q(f^{op}/v,\mathbb{Z}) & \text{if } \mathrm{ht}_{F^{op}}(v) = l + 1 \end{cases}$$

and

$$\mathcal{G}''_q : F^{op} \to \underline{\mathrm{Ab}}, \quad \mathcal{G}''_q(v) = \begin{cases} H_q(f^{op}/v, \mathbb{Z}) & \text{if } \mathrm{ht}_{F^{op}}(v) < l+1 \\ 0 & \text{if } \mathrm{ht}_{F^{op}}(v) = l+1. \end{cases}$$

Then we have the short exact sequence $0 \to \mathcal{G}'_q \to \mathcal{G}_q \to \mathcal{G}''_q \to 0$ and the associated long exact sequence

$$\begin{aligned} \cdots \to H_{l-q}(F^{op}, \mathcal{G}'_q) \to H_{l-q}(F^{op}, \mathcal{G}_q) \to \\ H_{l-q}(F^{op}, \mathcal{G}''_q) \to H_{l-q-1}(F^{op}, \mathcal{G}'_q) \to \cdots . \end{aligned}$$

If $q > 0$, then $\mathcal{G}''_q(v) = 0$ for $0 < q \le \mathrm{ht}_{F^{op}}(v) - 1$ and so by lemma 3.5, $H_p(F^{op}, \mathcal{G}''_q) = 0$ for $p + q \le l$, $q > 0$. Also if $|v| = 1$ then $H_0(f^{op}/v, \mathbb{Z}) = 0$ for $0 < q \le \mathrm{ht}_{F^{op}}(v) - 2 = l - 1$. This shows $\mathcal{G}'_q = 0$ for $0 < q \le l - 1$. From the long exact sequence and the above calculation we get

$$E^2_{p,q} = \begin{cases} \mathbb{Z} & \text{if } p = q = 0 \\ 0 & \text{if } 0 < p + q \le l, \ q \ne l. \end{cases}$$

$$\begin{array}{r|cccccc} l+1 & * & & & & & E^2_{p,q} \\ l & * & * & & & & \\ & 0 & 0 & * & & & \\ & \vdots & \vdots & \ddots & \ddots & & \\ & 0 & \cdot & \cdot & \ddots & * & \\ 0 & \mathbb{Z} & 0 & \cdots & 0 & 0 & * \\ \hline & 0 & 1 & & & l & l+1 \end{array}$$

Thus for $0 \le p+q \le l$, $q \ne l$, $E^2_{p,q} \simeq \cdots \simeq E^\infty_{p,q}$ and there exist an integer r such that $E^2_{0,l} \twoheadrightarrow \cdots \twoheadrightarrow E^r_{0,l} \simeq E^{r+1}_{0,l} \simeq \cdots \simeq E^\infty_{0,l}$. Hence we get a surjective map $H_0(F^{op}, v \mapsto H_l(f^{op}/v, \mathbb{Z})) \twoheadrightarrow H_l(Z^{op}, \mathbb{Z})$. By lemma 4.6, we have a surjective map $\bigoplus_{v \in F, \, |v|=1} H_l(f^{op}/v, \mathbb{Z}) \twoheadrightarrow H_l(Z^{op}, \mathbb{Z})$.

Now consider the map $g^{op} : Z^{op} \to X^{op}$ and define the height function $\mathrm{ht}_{X^{op}}(x) = l + 2 - |x|$ on X^{op}. Arguing similarly to the proof of theorem 4.3 one sees that $g_* : H_k(Z, \mathbb{Z}) \to H_k(X, \mathbb{Z})$ is an isomorphism for $0 \le k \le l$. Therefore we get a surjective map $\bigoplus_{v \in F, \, |v|=1} H_l(X_v, \mathbb{Z}) \twoheadrightarrow H_l(X, \mathbb{Z})$. We call it ψ. We prove that this map is the same map that we claimed. For v of length one consider the commutative diagram of posets

$$\begin{tikzcd} & \{v\}^{op} \arrow[rr] & & F^{op} \\ f^{-1}(\{v\})^{op} \arrow[ur] \arrow[rr] \arrow[dr] & & Z^{op} \arrow[ur] \arrow[dr] & \\ & X_v^{op} \arrow[rr] & & X^{op} \end{tikzcd}$$

By functoriality of the spectral sequence for the above diagram and lemma 4.6 we get the commutative diagram

$$\begin{array}{ccc}
H_l(f_v^{op}/v,\mathbb{Z}) & \xrightarrow{(j_v)_*} & \bigoplus_{v\in F,|v|=1} H_l(f^{op}/v,\mathbb{Z}) \\
\downarrow & & \downarrow \\
H_0(\{v\}^{op}, v\mapsto H_l(f_v^{op}/v,\mathbb{Z})) & \longrightarrow & H_0(F^{op}, v\mapsto H_l(f^{op}/v,\mathbb{Z})) \\
\downarrow & & \downarrow \\
H_l(f^{-1}(v)^{op},\mathbb{Z}) & \longrightarrow & H_l(Z^{op},\mathbb{Z}) \\
\downarrow & & \downarrow \\
H_l(X_v^{op},\mathbb{Z}) & \xrightarrow{(i_v^{op})_*} & H_l(X^{op},\mathbb{Z})
\end{array}$$

where $j_v : f_v^{op}/v \to f^{op}/v$ is the inclusion which is a homotopy equivalence as we already mentioned. It is not difficult to see that the composition of homomorphisms in the left column of the above diagram induces the identity map from $H_l(X_v,\mathbb{Z})$, the composition of homomorphisms in the right column of above diagram induces the surjective map ψ and the last row induces the homomorphism $(i_v)_*$. This show that $(i_v)_* = \psi|_{H_l(X_v,\mathbb{Z})}$. This completes the proof of surjectiveness.

Now let for v of length one $X_v \subseteq Y_v$ where Y_v is l-connected. Then we have the commutative diagram

$$\begin{array}{ccc}
H_l(X_v,\mathbb{Z}) & \xrightarrow{(i_v)_*} & H_l(X,\mathbb{Z}) \\
& \searrow \quad \nearrow & \\
& H_l(Y_v,\mathbb{Z}) &
\end{array}.$$

By the assumption $H_l(Y_v,\mathbb{Z})$ is trivial and this shows that $(i_v)_*$ is the zero map. Hence by the surjectivity, $H_l(X,\mathbb{Z})$ is trivial. If $l\ge 2$, the nerve theorem 4.3 says that X is simply connected and by the Hurewicz theorem 2.1, X is l-connected. So the only case that is left is when $l=1$. By theorem 3.8, X is 1-connected if and only if Z is 1-connected. So it is enough to prove that Z^{op} is 1-connected. Note that as we said, we can assume that $F=F_{\le 3}$ and $X=X_{\le 3}$. Suppose $\mathcal{F}$ is a local system on Z^{op}. Define the functor $\mathcal{G} : F^{op}\to \underline{\mathrm{Ab}}$, as

$$\mathcal{G}(y) = \begin{cases} H_0(f^{op}/v,\mathcal{F}) & \text{if } |v|=1,2 \\ H_0(Link^+_{F^{op}}(v), v'\mapsto H_0(f^{op}/v',\mathcal{F})) & \text{if } |v|=3 \end{cases}.$$

We prove that $\mathcal{G}$ is a local system on F^{op}. Put $Z_w := g^{-1}(Y_w)$ for $|w|=1$. If $|v|=1,2$, then f^{op}/v is 0-connected and $f^{op}/v\subseteq Z_w^{op}$, where $w\le v$, $|w|=1$. By remark 3.9 we can assume that $\mathcal{F}=\mathcal{E}\circ g^{op}$ where $\mathcal{E}$ is a local system on X^{op}. Then $\mathcal{F}|_{Z_w^{op}} = \mathcal{E}|_{Y_w^{op}}\circ g^{op}|_{Z_w^{op}}$. Since Y_w^{op} is 1-connected, $\mathcal{E}|_{Y_w^{op}}$ is a constant local system. This shows that $\mathcal{F}|_{Z_w^{op}}$ is a constant local system. So $\mathcal{F}|_{f^{op}/v}$ is a constant local system and since f^{op}/v is 0-connected we have $H_0(f^{op}/v,\mathbb{Z})\simeq \mathcal{F}(x)$, for every $x\in f^{op}/v$. If $|v|=3$, with an argument similar to the proof of the theorem 3.8 and the above discussion one can get $\mathcal{G}(v)\simeq\mathcal{F}(x)$ for every $x\in f^{op}/v$. This shows that $\mathcal{G}$ is a local system on F^{op}. Hence it is a constant local system, because F^{op} is 1-connected. It is easy to see that $\mathcal{F}\simeq\mathcal{G}\circ f$. Therefore $\mathcal{F}$ is a constant system. Since X is connected by our homology calculation, by 3.7 we conclude that X is 1-connected. This completes the proof. $\square$

5. Posets of unimodular sequences

Let R be an associative ring with unit. A vector $(r_1, \dots, r_n) \in R^n$ is called unimodular if there exist $s_1, \dots, s_n \in R$ such that $\Sigma_{i=1}^n r_i s_i = 1$, or equivalently if the submodule generated by this vector is a free summand of the left R-module R^n. We denote the standard basis of R^n by $e_1, \dots, e_n$. If $n \leq m$, we assume that R^n is the submodule of R^m generated by $e_1, \dots, e_n \in R^m$.
We say that a ring R satisfies the *stable range condition* (S_m), if $m \geq 1$ is an integer so that for every unimodular vector $(r_0, r_1, \dots, r_m) \in R^{m+1}$, there exist $t_1, \dots, t_m$ in R such that $(r_1 + r_0 t_1, \dots, r_m + r_0 t_m) \in R^m$ is unimodular. We say that R has *stable rank* m, we denote it with $\mathrm{sr}(R) = m$, if m is the least number such that (S_m) holds. If such a number does not exist we say that $\mathrm{sr}(R) = \infty$.
Let $\mathcal{U}(R^n)$ denote the subposet of $\mathcal{O}(R^n)$ consisting of unimodular sequences. Recall that a sequence of vectors $v_1, \dots, v_k$ in R^n is called unimodular when $v_1, \dots, v_k$ is basis of a free direct summand of R^n. Note that if $(v_1, \dots, v_k) \in \mathcal{O}(R^n)$ and if $n \leq m$, it is the same to say that $(v_1, \dots, v_k)$ is unimodular as a sequence of vectors in R^n or as a sequence of vectors in R^m. We call an element $(v_1, \dots, v_k)$ of $\mathcal{U}(R^n)$ a *k-frame.*

Theorem 5.1 (Van der Kallen). *Let R be a ring with $\mathrm{sr}(R) < \infty$ and $n \leq m+1$. Let δ be 0 or 1. Then*
(i) $\mathcal{O}(R^n + \delta e_{n+1}) \cap \mathcal{U}(R^m)$ *is $(n - \mathrm{sr}(R) - 1)$-connected.*
(ii) $\mathcal{O}(R^n + \delta e_{n+1}) \cap \mathcal{U}(R^m)_v$ *is $(n - \mathrm{sr}(R) - |v| - 1)$-connected for all $v \in \mathcal{U}(R^m)$.*

Proof. See [20, Thm. 2.6]. $\square$

Example 5.2. Let R be a ring with $\mathrm{sr}(R) < \infty$. Let $n \geq \mathrm{sr}(R) + k + 1$ and assume $(v_1, \dots, v_k) \in \mathcal{U}(R^{2n})$. Set $W = e_2 + \Sigma_{i=2}^n Re_{2i}$. Renumbering the basis one gets by theorem 5.1 that the poset $F := \mathcal{O}(W) \cap \mathcal{U}(R^{2n})_{(v_1, \dots, v_k)}$ is $((n-1) - \mathrm{sr}(R) - k - 1)$-connected. Since $n \geq \mathrm{sr}(R) + k + 1$, it follows that F is not empty. This shows that there is $v \in W$ such that $(v, v_1, \dots, v_k) \in \mathcal{U}(R^{2n})$. We will need such result in the next section but with a different method we can prove a sharper result. Compare this with lemma 5.4.

An $n \times k$-matrix B with $n < k$ is called unimodular if B has a right inverse. If B is an $n \times k$-matrix and $C \in GL_k(R)$, then B is unimodular if and only if CB is unimodular. A matrix of the form $\begin{pmatrix} 1 & u \\ 0 & B \end{pmatrix}$, where u is a row vector with coordinates in R, is unimodular if and only if the matrix B is unimodular.
We say that the ring R satisfies the *stable range condition* (S_k^n) if for every $n \times (n+k)$-matrix B, there exists a vector $r = (r_1, \dots, r_{n+k-1})$ such that

$B\begin{pmatrix} 1 & r \\ 0 & I_{n+k-1} \end{pmatrix} = \begin{pmatrix} u & B' \end{pmatrix}$, where the $n \times (n+k-1)$-matrix B' is unimodular and u is the first column of the matrix B. Note that (S_k^1) is the same as (S_k).

Theorem 5.3 (Vaserstein). *For every $k \geq 1$ and $n \geq 1$, a ring R satisfies* (S_k) *if and only if it satisfies* (S_k^n).

Proof. The definition of (S_k^n) and the proof of this theorem is similar to the theorem [22, Thm. 3′] of Vaserstein. □

Lemma 5.4. *Let R be ring with $\mathrm{sr}(R) < \infty$ and let $n \geq \mathrm{sr}(R)+k$. Then for every $(v_1, \dots, v_k) \in \mathcal{U}(R^{2n})$ there is a $v \in e_2 + \Sigma_{i=2}^n Re_{2i}$ such that $(v, v_1, \dots, v_k) \in \mathcal{U}(R^{2n})$.*

Proof. There is a permutation matrix $A \in GL_{2n}(R)$ such that $(e_2 + \Sigma_{i=2}^n Re_{2i})A = e_1 + \Sigma_{j=n+2}^{2n} Re_j$. Let $w_i = v_iA$ for $i = 1, \dots, k$. So $(w_1, \dots, w_k) \in \mathcal{U}(R^{2n})$. Consider the $k \times 2n$-matrix B whose i-th row is the vector w_i. By theorem 5.3 there exists a vector $r = (r_2, \dots, r_{2n})$ such that $B\begin{pmatrix} 1 & r \\ 0 & I_{2n-1} \end{pmatrix} = \begin{pmatrix} u_1 & B_1 \end{pmatrix}$, where the $k \times (2n-1)$-matrix B_1 is unimodular and u_1 is the first column of the matrix B. Now let $s = (s_3, \dots, s_{2n})$ such that $B_1 \begin{pmatrix} 1 & s \\ 0 & I_{2n_2} \end{pmatrix} = \begin{pmatrix} u_2 & B_2 \end{pmatrix}$, where the $k \times (2n-2)$-matrix B_2 is unimodular and u_2 is the first column of the matrix B_1. Now clearly

$$B\begin{pmatrix} 1 & r \\ 0 & I_{2n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & I_{2n-2} \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & B_2 \end{pmatrix}.$$

By continuing this process, n times, we find a $2n \times 2n$ matrix C of the form

$$\begin{pmatrix} 1 & * & * & * & \\ & \ddots & * & * & \mathrm{N} \\ & & 1 & * & * \\ & 0 & & 1 & * \\ & & & & I_{n-1} \end{pmatrix}$$

where N is an $(n-1)\times(n-1)$ matrix and $BC = (L \mid M)$ where L is a $k \times (n+1)$ matrix and M is a unimodular $k \times (n-1)$ matrix. Now let
$t = (t_{n+2}, \dots, t_{2n}) = -(\text{first row of } N)$. Then

$$\begin{pmatrix} 1 & 0 & \dots & 0 & t_{n+2} & \dots & t_{2n} \\ & & & B & & & \end{pmatrix} C = \begin{pmatrix} 1 & * & * & 0 & \dots & 0 \\ * & * & * & & M & \end{pmatrix}.$$

Since M is unimodular the right hand side of the above equality is unimodular. This shows that the matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & t_{n+2} & \dots & t_{2n} \\ & & & B & & & \end{pmatrix}$$

is unimodular. Put $w = (1, 0, \dots, 0, t_{n+2}, \dots, t_{2n})$. Then $(w, w_1, \dots, w_k) \in \mathcal{U}(R^{2n})$. Now $v = wA^{-1}$ is the one that we are looking for. $\square$

6. HYPERBOLIC SPACES AND SOME POSETS

Let there be an involution on R, that is an automorphism of the additive group of R, $R \to R$ with $r \mapsto \overline{r}$, such that $\overline{\overline{r}} = r$ and $\overline{rs} = \overline{s}\,\overline{r}$. Let ϵ be an element in the center of R such that $\epsilon\overline{\epsilon} = 1$. Set $R_\epsilon := \{r - \epsilon\overline{r} : r \in R\}$ and $R^\epsilon := \{r \in R : \epsilon\overline{r} = -r\}$ and observe that $R_\epsilon \subseteq R^\epsilon$. A *form parameter* relative to the involution and ϵ is a subgroup Λ of $(R, +)$ such that $R_\epsilon \subseteq \Lambda \subseteq R^\epsilon$ and $\overline{r}\Lambda r \subseteq \Lambda$, for all $r \in R$. Notice that R_ϵ and R^ϵ are form parameters. We denote them by $\Lambda_{\min}$ and $\Lambda_{\max}$, respectively. If there is an s in the center of R such that $s + \overline{s} \in R^*$, in particular if $2 \in R^*$, then $\Lambda_{\min} = \Lambda_{\max}$.
Let $e_{i,j}(r)$ be the $2n \times 2n$-matrix with $r \in R$ in the (i, j) place and zero elsewhere. Consider $Q_n = \Sigma_{i=1}^n e_{2i-1,2i}(1) \in M_{2n}(R)$ and $F_n = Q_n + \epsilon\ {}^tQ_n = \Sigma_{i=1}^n (e_{2i-1,2i}(1) + e_{2i,2i-1}(\epsilon)) \in GL_{2n}(R)$. Define the bilinear map $h : R^{2n} \times R^{2n} \to R$ by $h(x, y) = \Sigma_{i=1}^n (\overline{x_{2i-1}} y_{2i} + \epsilon\overline{x_{2i}} y_{2i-1})$ and $q : R^{2n} \to R/\Lambda$ by $q(x) = \Sigma_{i=1}^n \overline{x_{2i-1}} x_{2i} \bmod \Lambda$, where $x = (x_1, \dots, x_{2n})$, $y = (y_1, \dots, y_{2n})$ and $\overline{x} = (\overline{x_1}, \dots, \overline{x_{2n}})$. The triple (R^{2n}, h, q) is called a *hyperbolic space*. By definition the *unitary group* relative Λ is the group

$$U_{2n}^\epsilon(R, \Lambda) := \{A \in GL_{2n}(R) : h(xA, yA) = h(x, y), q(xA) = q(x), x, y \in R\}.$$

For more general definitions and the properties of these spaces and groups see [8].

Example 6.1. (i) Let $\Lambda = \Lambda_{\max} = R$. Then $U_{2n}^\epsilon(R, \Lambda) = \{A \in GL_{2n}(R) : h(xA, yA) = h(x, y) \text{ for all } x, y \in R^{2n}\} = \{A \in GL_{2n}(R) : {}^t\overline{A}F_nA = F_n\}$. In particular if $\epsilon = -1$ and if the involution is the identity map id_R, then $\Lambda_{\max} = R$. In This case $U_{2n}^\epsilon(R, \Lambda_{\max}) := Sp_{2n}(R)$ is the usual symplectic group. Note that R is commutative in this case.
(ii) Let $\Lambda = \Lambda_{\min} = 0$. Then $U_{2n}^\epsilon(R, \Lambda) = \{A \in GL_{2n}(R) : q(xA) = q(x) \text{ for all } x \in R^{2n}\}$. In particular if $\epsilon = 1$ and if the involution is the identity map id_R, then $\Lambda_{\min} = 0$. In this case $U_{2n}^\epsilon(R, \Lambda_{\min}) := O_{2n}(R)$ is the usual orthogonal group. As in the symplectic case, R is necessarily commutative.
(iii) Let $\epsilon = -1$ and the involution is not the identity map id_R. If $\Lambda = \Lambda_{\max}$ then $U_{2n}^\epsilon(R, \Lambda) := U_{2n}(R)$ is the classical unitary group corresponding to the involution.

Let σ be the permutation of the set of natural numbers given by $\sigma(2i) = 2i-1$ and $\sigma(2i-1) = 2i$. For $1 \le i, j \le 2n$, $i \ne j$, and every $r \in R$ define

$$E_{i,j}(r) = \begin{cases} I_{2n} + e_{i,j}(r) & \text{if } i = 2k-1, j = \sigma(i), r \in \Lambda \\ I_{2n} + e_{i,j}(r) & \text{if } i = 2k, j = \sigma(i), \overline{r} \in \Lambda \\ I_{2n} + e_{i,j}(r) + e_{\sigma(j),\sigma(i)}(-\overline{r}) & \text{if } i + j = 2k,\ i \ne j \\ I_{2n} + e_{i,j}(r) + e_{\sigma(j),\sigma(i)}(-\epsilon^{-1}\overline{r}) & \text{if } i \ne \sigma(j),\ i = 2k-1,\ j = 2l \\ I_{2n} + e_{i,j}(r) + e_{\sigma(j),\sigma(i)}(\epsilon\overline{r}) & \text{if } i \ne \sigma(j),\ i = 2k,\ j = 2l-1 \end{cases}$$

where I_{2n} is the identity element of $GL_{2n}(R)$. It is easy to see that $E_{i,j}(r) \in U_{2n}^{\epsilon}(R,\Lambda)$. Let $EU_{2n}^{\epsilon}(R,\Lambda)$ be the group generated by the $E_{i,j}(r)$, $r \in R$. We call it *elementary unitary group.*
A nonzero vector $x \in R^{2n}$ is called isotropic if $q(x) = 0$. This shows automatically that if x is isotropic then $h(x,x) = 0$. We say that a subset S of R^{2n} is isotropic if for every $x \in S$, $q(x) = 0$ and for every $x, y \in S$, $h(x,y) = 0$. If $h(x,y) = 0$, then we say that x is perpendicular to y. We denote by $\langle S \rangle$ the submodule of R^{2n} generated by S, and by $\langle S \rangle^{\perp}$ the submodule consisting of all the elements of R^{2n} which are perpendicular to all the elements of S.
From now, we fix an involution, an ϵ, a form parameter Λ and we consider the triple (R^{2n}, h, q) as defined above.

Definition 6.2 (Transitivity condition). Let $r \in R$ and define $C_r^{\epsilon}(R^{2n},\Lambda) = \{x \in \mathrm{Um}(R^{2n}) : q(x) = r \bmod \Lambda\}$, where $\mathrm{Um}(R^{2n})$ is the set of all unimodular vectors of R^{2n}. We say that R satisfies the transitivity condition (T_n), if $EU_{2n}^{\epsilon}(R,\Lambda)$ acts transitively on $C_r^{\epsilon}(R^{2n},\Lambda)$, for every $r \in R$. It is easy to see that $e_1 + re_2 \in C_r^{\epsilon}(R^{2n},\Lambda)$.

Definition 6.3 (Unitary stable range). We say that a ring R satisfies the unitary stable range condition (US_m) if R satisfies the conditions (S_m) and (T_{m+1}). We say that R has unitary stable rank m, we denote it with $\mathrm{usr}(R)$, if m is the least number such that (US_m) is satisfied. If such a number does not exist we say that $\mathrm{usr}(R) = \infty$. Clearly $\mathrm{sr}(R) \leq \mathrm{usr}(R)$.

Remark 6.4. Our definition of unitary stable range is a little different than the one in [8]. In fact if (USR_{m+1}) satisfied then, by [8, Chap. VI, Thm. 4.7.1], (US_m) is satisfied where (USR_{m+1}) is the unitary stable range as defined in [8, Chap. VI, 4.6]. In comparison with the absolute stable rank $\mathrm{asr}(R)$ from [10], we have that if $m \geq \mathrm{asr}(R) + 1$ or if the involution is the identity map (so R is commutative) and $m \geq \mathrm{asr}(R)$ then (US_m) is satisfied [10, 8.1].

Example 6.5. Let R be a commutative Noetherian ring where the dimension d of the maximal spectrum $\mathrm{Mspec}(R)$ is finite. If A is a finite R-algebra then $\mathrm{usr}(A) \leq d + 1$ (see [21, Thm. 2.8], [8, Thm. 6.1.4]). In particular if R is local ring or more generally a semilocal ring then $\mathrm{usr}(R) = 1$ [8, 6.1.3].

Lemma 6.6. *Let R be a ring with $\mathrm{usr}(R) < \infty$. Assume $n \geq \mathrm{usr}(R) + k$ and $(v_1, \dots, v_k) \in \mathcal{U}(R^{2n})$. Then there is a hyperbolic basis $\{x_1, y_1, \dots, x_n, y_n\}$ of R^{2n} such that $v_1, \dots, v_k \in \langle x_1, y_1, \dots, x_k, y_k \rangle$.*

Proof. The proof is by induction on k. If $k = 1$, by definition of unitary stable range there is an $E \in EU_{2n}^{\epsilon}(R,\Lambda)$ such that $v_1 E = e_1 + re_2$. So the base of the induction is true. Let $k \geq 2$ and assume the induction hypothesis. Arguing as in the base of the induction we can assume that $v_1 = (1, r, 0, \dots, 0)$, $r \in R$. Let $W = e_2 + \Sigma_{i=2}^{n} Re_{2i}$. By lemma 5.4, choose $w \in W$ so that $(w, v_1, \dots, v_k) \in \mathcal{U}(R^{2n})$. Then $(w, v_1 - rw, v_2, \dots, v_k) \in \mathcal{U}(R^{2n})$. But $(w, v_1 - rw)$ is a hyperbolic pair, so there is an $E \in EU_{2n}^{\epsilon}(R,\Lambda)$ such that $wE = e_{2n-1}, (v_1 - rw)E = e_{2n}$ by [8, Chap. VI, Thm. 4.7.1]. Let

$(wE, (v_1 - rw)E, v_2E, \dots, v_kE) =: (w_0, w_1, \dots, w_k)$ where $w_i = (r_{i,1}, \dots, r_{i,2n})$. Put $u_i = w_i - r_{i,2n-1}e_{2n-1} - r_{i,2n}e_{2n}$ for $2 \le i \le k$. Then $(u_2, \dots, u_k) \in \mathcal{U}(R^{2n-2})$. Now by induction there is a hyperbolic basis $\{a_2, b_2, \dots, a_n, b_n\}$ of R^{2n-2} such that $u_i \in \langle a_2, b_2, \dots, a_k, b_k\rangle$. Let $a_1 = e_{2n-1}$ and $b_1 = e_{2n}$. Then $w_i \in \langle a_1, b_1, \dots, a_k, b_k\rangle$. But $v_1E = w_1 + rwE = e_{2n} + re_{2n-1}$, $v_iE = w_i$ for $2 \le i \le k$ and considering $x_i = a_iE^{-1}$, $y_i = b_iE^{-1}$, one sees that $v_1, \dots, v_k \in \langle x_1, y_1, \dots, x_k, y_k\rangle$. □

Definition 6.7. Let $Z_n = \{x \in R^{2n} : q(x) = 0\}$. We define the poset $\mathcal{U}'(R^{2n})$ as $\mathcal{U}'(R^{2n}) := \mathcal{O}(Z_n) \cap \mathcal{U}(R^{2n})$.

Lemma 6.8. *Let R be a ring with* $\mathrm{sr}(R) < \infty$ *and $n \le m$. Then*
(i) $\mathcal{O}(R^{2n}) \cap \mathcal{U}'(R^{2m})$ *is $(n - \mathrm{sr}(R) - 1)$-connected,*
(ii) $\mathcal{O}(R^{2n}) \cap \mathcal{U}'(R^{2m})_v$ *is $(n - \mathrm{sr}(R) - |v| - 1)$-connected for every $v \in \mathcal{U}'(R^{2m})$,*
(iii) $\mathcal{O}(R^{2n}) \cap \mathcal{U}'(R^{2m}) \cap \mathcal{U}(R^{2m})_v$ *is $(n - \mathrm{sr}(R) - |v| - 1)$-connected for every $v \in \mathcal{U}(R^{2m})$.*

Proof. Let $W = \langle e_2, e_4, \dots, e_{2n}\rangle$ and $F := \mathcal{O}(R^{2n}) \cap \mathcal{U}'(R^{2m})$. It is easy to see that $\mathcal{O}(W) \cap F = \mathcal{O}(W) \cap \mathcal{U}(R^{2m})$ and $\mathcal{O}(W) \cap F_u = \mathcal{O}(W) \cap \mathcal{U}(R^{2m})_u$ for every $u \in \mathcal{U}'(R^{2m})$. By theorem 5.1, the poset $\mathcal{O}(W) \cap F$ is $(n - \mathrm{sr}(R) - 1)$-connected and the poset $\mathcal{O}(W) \cap F_u$ is $(n - \mathrm{sr}(R) - |u| - 1)$-connected for every $u \in F$. It follows from lemma [20, 2.13 (i)] that F is $(n - \mathrm{sr}(R) - 1)$-connected. The proof of (ii) and (iii) is similar to the proof of (i). □

Lemma 6.9. *Let R be a ring with* $\mathrm{usr}(R) < \infty$ *and let $(v_1, \dots, v_k) \in \mathcal{U}'(R^{2n})$. If $n \ge \mathrm{usr}(R) + k$ then $\mathcal{O}(\langle v_1, \dots, v_k\rangle^{\perp}) \cap \mathcal{U}'(R^{2n})_{(v_1, \dots, v_k)}$ is $(n - \mathrm{usr}(R) - k - 1)$-connected.*

Proof. By lemma 6.6 there is a hyperbolic basis $\{x_1, y_1, \dots, x_n, y_n\}$ of R^{2n} such that $v_1, \dots, v_k \in \langle x_1, y_1, \dots, x_k, y_k\rangle$. Let $W = \langle x_{k+1}, y_{k+1}, \dots, x_n, y_n\rangle \simeq R^{2(n-k)}$ and $F := \mathcal{O}(\langle v_1, \dots, v_k\rangle^{\perp}) \cap \mathcal{U}'(R^{2n})_{(v_1, \dots, v_k)}$. It is easy to see that $\mathcal{O}(W) \cap F = \mathcal{O}(W) \cap \mathcal{U}'(R^{2n})$. Let $V = \langle v_1, \dots, v_k\rangle$, then $\langle x_1, y_1, \dots, x_k, y_k\rangle = V \oplus P$ where P is a (finitely generated) projective module. Consider $(u_1, \dots, u_l) \in F \backslash \mathcal{O}(W)$ and let $u_i = x_i + y_i$ where $x_i \in V$ and $y_i \in P \oplus W$. One should notice that $(u_1 - x_1, \dots, u_l - x_l) \in \mathcal{U}(R^{2n})$ and not necessarily in $\mathcal{U}'(R^{2n})$. It is not difficult to see that $\mathcal{O}(W) \cap F_{(u_1, \dots, u_l)} = \mathcal{O}(W) \cap \mathcal{U}'(R^{2n}) \cap \mathcal{U}(R^{2n})_{(u_1 - x_1, \dots, u_l - x_l)}$. By lemma 6.8, $\mathcal{O}(W) \cap F$ is $(n - k - \mathrm{usr}(R) - 1)$-connected and $\mathcal{O}(W) \cap F_u$ is $(n - k - \mathrm{usr}(R) - |u| - 1)$-connected for every $u \in F \backslash \mathcal{O}(W)$. It follows from lemma [20, 2.13 (i)] that F is $(n - \mathrm{usr}(R) - k - 1)$-connected. □

7. Posets of isotropic and hyperbolic unimodular sequences

Let $\mathcal{IU}(R^{2n})$ be the set of sequences $(x_1, \dots, x_k)$, $x_i \in R^{2n}$, such that $x_1, \dots, x_k$ form a basis for an isotropic direct summand of R^{2n}. Let $\mathcal{HU}(R^{2n})$ be the set of sequences $((x_1, y_1), \dots, (x_k, y_k))$ such that $(x_1, \dots, x_k)$, $(y_1, \dots, y_k) \in \mathcal{IU}(R^{2n})$, $h(x_i, y_j) = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta. We call $\mathcal{IU}(R^{2n})$ and $\mathcal{HU}(R^{2n})$ the poset of isotropic unimodular sequences and the poset of hyperbolic unimodular sequences, respectively. For $1 \le k \le n$, let $\mathcal{IU}(R^{2n}, k)$ and $\mathcal{HU}(R^{2n}, k)$ be the set of all elements of length k of $\mathcal{IU}(R^{2n})$ and $\mathcal{HU}(R^{2n})$ respectively. We call the elements of $\mathcal{IU}(R^{2n}, k)$ and $\mathcal{HU}(R^{2n}, k)$ the isotropic k-frames and the hyperbolic k-frames, respectively. Define the poset $\mathcal{MU}(R^{2n})$ as the set of $((x_1, y_1), \dots, (x_k, y_k)) \in \mathcal{O}(R^{2n} \times R^{2n})$ such that, (i) $(x_1, \dots, x_k) \in \mathcal{IU}(R^{2n})$, (ii) for each i, either $y_i = 0$ or $(x_j, y_i) = \delta_{ji}$, (iii) $\langle y_1, \dots, y_k\rangle$ is isotropic. We identify $\mathcal{IU}(R^{2n})$ with $\mathcal{MU}(R^{2n}) \cap \mathcal{O}(R^{2n} \times \{0\})$ and $\mathcal{HU}(R^{2n})$ with $\mathcal{MU}(R^{2n}) \cap \mathcal{O}(R^{2n} \times (R^{2n}\backslash\{0\}))$.

Lemma 7.1. *Let R be a ring with $\mathrm{usr}(R) < \infty$. If $n \ge \mathrm{usr}(R) + k$ then $EU_{2n}^{\epsilon}(R, \Lambda)$ acts transitively on $\mathcal{IU}(R^{2n}, k)$ and $\mathcal{HU}(R^{2n}, k)$.*

Proof. The proof is by induction on k. If $k = 1$, by definition $EU_{2n}^{\epsilon}(R, \Lambda)$ acts transitively on $\mathcal{IU}(R^{2n}, 1)$ and by [8, Chap. VI, Thm. 4.7.1] the group $EU_{2n}^{\epsilon}(R, \Lambda)$ acts transitively on $\mathcal{HU}(R^{2n}, 1)$. The rest is an easy induction and the fact that for every isotropic k-frame $(x_1, \dots, x_k)$ there is an isotropic k-frame $(y_1, \dots, y_k)$ such that $((x_1, y_1), \dots, (x_k, y_k))$ is a hyperbolic k-frame [8, Chap. I, Cor. 3.7.4]. □

Lemma 7.2. *Let R be a ring with $\mathrm{usr}(R) < \infty$, and let $n \ge \mathrm{usr}(R) + k$. Let $((x_1, y_1), \dots, (x_k, y_k)) \in \mathcal{HU}(R^{2n})$, $(x_1, \dots, x_k) \in \mathcal{IU}(R^{2n})$ and $V = \langle x_1, \dots, x_k\rangle$. Then*
(i) $\mathcal{IU}(R^{2n})_{(x_1,\dots,x_k)} \simeq \mathcal{IU}(R^{2(n-k)})\langle V\rangle$,
(ii) $\mathcal{HU}(R^{2n}) \cap \mathcal{MU}(R^{2n})_{((x_1,0),\dots,(x_k,0))} \simeq \mathcal{HU}(R^{2n})_{((x_1,y_1),\dots,(x_k,y_k))}\langle V \times V\rangle$,
(iii) $\mathcal{HU}(R^{2n})_{((x_1,y_1),\dots,(x_k,y_k))} \simeq \mathcal{HU}(R^{2(n-k)})$.

Proof. See [5], the proof of lemma 3.4 and the proof of Thm. 3.2. □

For a real number l, by $\lfloor l \rfloor$ we mean the largest integer n with $n \le l$.

Theorem 7.3. *The poset $\mathcal{IU}(R^{2n})$ is $\lfloor \frac{n-\mathrm{usr}(R)-2}{2} \rfloor$-connected and $\mathcal{IU}(R^{2n})_x$ is $\lfloor \frac{n-\mathrm{usr}(R)-|x|-2}{2} \rfloor$-connected for every $x \in \mathcal{IU}(R^{2n})$.*

Proof. If $n \le \mathrm{usr}(R)$, the result is clear, so let $n > \mathrm{usr}(R)$. Let $X_v = \mathcal{IU}(R^{2n}) \cap \mathcal{U}'(R^{2n})_v \cap \mathcal{O}(\langle v\rangle^{\perp})$, for every $v \in \mathcal{U}'(R^{2n})$, and put $X := \bigcup_{v \in F} X_v$ where $F = \mathcal{U}'(R^{2n})$. It follows from lemma 7.1 that $\mathcal{IU}(R^{2n})_{\le n-\mathrm{usr}(R)} \subseteq X$. So to treat $\mathcal{IU}(R^{2n})$, it is enough to prove that X is $\lfloor \frac{n-\mathrm{usr}(R)-2}{2} \rfloor$-connected. First we prove that X_v is $\lfloor \frac{n-\mathrm{usr}(R)-|v|-2}{2} \rfloor$-connected for every $v \in F$. The proof is by descending induction on $|v|$. If $|v| > n - \mathrm{usr}(R)$, then $\lfloor \frac{n-\mathrm{usr}(R)-|v|-2}{2} \rfloor < -1$. In this case there is nothing to prove. If $n - \mathrm{usr}(R) - 1 \le |v| \le n - \mathrm{usr}(R)$, then $\lfloor \frac{n-\mathrm{usr}(R)-|v|-2}{2} \rfloor = -1$, so we must prove that X_v is nonempty. This follows from lemma 6.6. Now assume $|v| \le n - \mathrm{usr}(R) - 2$ and assume by induction that X_w is $\lfloor \frac{n-\mathrm{usr}(R)-|w|-2}{2} \rfloor$-connected for every w, with $|w| > |v|$. Let $l = \lfloor \frac{n-\mathrm{usr}(R)-|v|-2}{2} \rfloor$, and observe that $n - |v| - \mathrm{usr}(R) \ge l + 2$. Put $T_w = \mathcal{IU}(R^{2n}) \cap \mathcal{U}'(R^{2n})_{wv} \cap \mathcal{O}(\langle wv\rangle^{\perp})$ where $w \in G_v = \mathcal{U}'(R^{2n})_v \cap \mathcal{O}(\langle v\rangle^{\perp})$ and put $T := \bigcup_{w \in G_v} T_w$. It follows by lemma 6.6 that $(X_v)_{\le n-|v|-\mathrm{usr}(R)} \subseteq T$. So it is enough to prove that T is l-connected. The poset G_v is l-connected by lemma

6.9. By induction, T_w is $\lfloor\frac{n-\mathrm{usr}(R)-|v|-|w|-2}{2}\rfloor$-connected. But $\min\{l-1, l-|w|+1\} \le \lfloor\frac{n-\mathrm{usr}(R)-|v|-|w|-2}{2}\rfloor$, so T_w is $\min\{l-1, l-|w|+1\}$-connected. For every $y \in T$, $\mathcal{A}_y = \{w \in G_v : y \in T_w\}$ is isomorphic to $\mathcal{U}'(R^{2n})_{vy} \cap \mathcal{O}(\langle vy\rangle^{\perp})$ so by lemma 6.9, it is $(l-|y|+1)$-connected. Let $w \in G_v$ with $|w|=1$. For every $z \in T_w$ we have $wz \in X_v$, so T_w is contained in a cone, call it C_w, inside X_v. Put $C(T_w) = T_w \cup (C_w)_{\le n-|v|-\mathrm{usr}(R)}$. Thus $C(T_w) \subseteq T$. The poset $C(T_w)$ is l-connected because $C(T_w)_{\le n-|v|-\mathrm{usr}(R)} = (C_w)_{\le n-|v|-\mathrm{usr}(R)}$. Now by theorems 5.1 and 4.7 , T is l-connected. In other words, we have now shown that X_v is $\lfloor\frac{n-\mathrm{usr}(R)-|v|-2}{2}\rfloor$-connected. By knowing this one can prove, in a similar way, that X is $\lfloor\frac{n-\mathrm{usr}(R)-2}{2}\rfloor$-connected. (Just pretend that $|v|=0$.)
Now consider the poset $\mathcal{IU}(R^{2n})_x$ for an $x=(x_1,\dots,x_k) \in \mathcal{IU}(R^{2n})$. The proof is by induction on n. If $n=1$, everything is easy. Similarly, we may assume $n-\mathrm{usr}(R)-|x| \ge 0$. Let $l = \lfloor\frac{n-\mathrm{usr}(R)-|x|-2}{2}\rfloor$. By lemma 7.2, $\mathcal{IU}(R^{2n})_x \simeq \mathcal{IU}(R^{2(n-|x|)})\langle V\rangle$, where $V = \langle x_1,\dots,x_k\rangle$. In the above we proved that $\mathcal{IU}(R^{2(n-|x|)})$ is l-connected and by induction, the poset $\mathcal{IU}(R^{2(n-|x|)})_y$ is $\lfloor\frac{n-|x|-\mathrm{usr}(R)-|y|-2}{2}\rfloor$-connected for every $y \in \mathcal{IU}(R^{2(n-|x|)})$. But $l-|y| \le \lfloor\frac{n-|x|-\mathrm{usr}(R)-|y|-2}{2}\rfloor$. So $\mathcal{IU}(R^{2(n-|x|)})\langle V\rangle$ is l-connected by lemma 4.1. Therefore $\mathcal{IU}(R^{2n})_x$ is l-connected. $\square$

Theorem 7.4. *The poset $\mathcal{HU}(R^{2n})$ is $\lfloor\frac{n-\mathrm{usr}(R)-3}{2}\rfloor$-connected and $\mathcal{HU}(R^{2n})_x$ is $\lfloor\frac{n-\mathrm{usr}(R)-|x|-3}{2}\rfloor$-connected for every $x \in \mathcal{HU}(R^{2n})$.*

Proof. The proof is by induction on n. If $n=1$, then everything is trivial. Let $F = \mathcal{IU}(R^{2n})$ and $X_v = \mathcal{HU}(R^{2n}) \cap \mathcal{MU}(R^{2n})_v$, for every $v \in F$. Put $X := \bigcup_{v\in F} X_v$. It follows from lemma 7.1 that $\mathcal{HU}(R^{2n})_{\le n-\mathrm{usr}(R)} \subseteq X$. Thus to treat $\mathcal{HU}(R^{2n})$, it is enough to prove that X is $\lfloor\frac{n-\mathrm{usr}(R)-3}{2}\rfloor$-connected, and we may assume $n \ge \mathrm{usr}(R)+1$. Take $l = \lfloor\frac{n-\mathrm{usr}(R)-3}{2}\rfloor$ and $V = \langle v_1,\dots,v_k\rangle$, where $v=(v_1,\dots,v_k)$. By lemma 7.2, there is an isomorphism $X_v \simeq \mathcal{HU}(R^{2(n-|v|)})\langle V \times V\rangle$, if $n \ge \mathrm{usr}(R)+|v|$. By induction $\mathcal{HU}(R^{2(n-|v|)})$ is $\lfloor\frac{n-|v|-\mathrm{usr}(R)-3}{2}\rfloor$-connected and again by induction $\mathcal{HU}(R^{2(n-|v|)})_y$ is $\lfloor\frac{n-|v|-\mathrm{usr}(R)-|y|-3}{2}\rfloor$-connected for every $y \in \mathcal{HU}(R^{2(n-|v|)})$. So by lemma 4.1 , X_v is $\lfloor\frac{n-|v|-\mathrm{usr}(R)-3}{2}\rfloor$-connected. Thus the poset X_v is $\min\{l-1, l-|v|+1\}$-connected. Let $x = ((x_1,y_1),\dots,(x_k,y_k))$. It is easy to see that $\mathcal{A}_x = \{v \in F : x \in X_v\} \simeq \mathcal{IU}(R^{2n})_{(x_1,\dots,x_k)}$. By the above theorem 7.3, $\mathcal{A}_x$ is $\lfloor\frac{n-\mathrm{usr}(R)-k-2}{2}\rfloor$-connected. But $l-|x|+1 \le \lfloor\frac{n-\mathrm{usr}(R)-k-2}{2}\rfloor$, so $\mathcal{A}_x$ is $(l-|x|+1)$-connected. Let $v=(v_1) \in F$, $|v|=1$, and let $D_v := \mathcal{HU}(R^{2n})_{(v_1,w_1)} \simeq \mathcal{HU}(R^{2(n-1)})$ where $w_1 \in R^{2n}$ is a hyperbolic dual of $v_1 \in R^{2n}$. Then $D_v \subseteq X_v$ and D_v is contained in a cone, call it C_v, inside $\mathcal{HU}(R^{2n})$. Take $C(D_v) := D_v \cup (C_v)_{\le n-\mathrm{usr}(R)}$. By induction D_v is $\lfloor\frac{n-1-\mathrm{usr}(R)-3}{2}\rfloor$-connected and so $(l-1)$-connected. Let $Y_v = X_v \cup C(D_v)$. By the Mayer-Vietoris theorem and the fact that $C(D_v)$ is l-connected, we get the exact sequence

$$\tilde{H}_l(D_v,\mathbb{Z}) \stackrel{(i_v)_*}{\to} \tilde{H}_l(X_v,\mathbb{Z}) \to \tilde{H}_l(Y_v,\mathbb{Z}) \to 0.$$

where $i_v : D_v \to X_v$ is the inclusion. By induction $(D_v)_w$ is $\lfloor\frac{n-1-\mathrm{usr}(R)-|w|-3}{2}\rfloor$-connected and so $(l-|w|)$-connected, for $w \in D_v$. By lemma 4.1(i) and lemma 7.2, $(i_v)_*$ is an isomorphism, and by exactness of the above sequence we get $\tilde{H}_l(Y_v,\mathbb{Z}) = 0$. If $l \geq 1$ by the Van Kampen theorem $\pi_1(Y_v,x) \simeq \pi_1(X_v,x)/N$ where $x \in D_v$ and N is the normal subgroup generated by the image of the map $(i_v)_* : \pi_1(D_v,x) \to \pi_1(X_v,x)$. Now by lemma 4.1(ii), $\pi_1(Y_v,x)$ is trivial. Thus by the Hurewicz theorem 2.1, Y_v is l-connected. By having all this we can apply theorem 4.7 and so X is l-connected. The fact that $\mathcal{HU}(R^{2n})_x$ is $\lfloor\frac{n-\mathrm{usr}(R)-|x|-3}{2}\rfloor$-connected follows from the above and lemma 7.2. □

Remark 7.5. One can define a more generalized version of hyperbolic space $H(P) = P \oplus P^*$ where P is a finitely generated projective module. Charney in [5, 2.10] introduced the posets $\mathcal{IU}(P)$, $\mathcal{HU}(P)$ and conjectured that if P contains a free summand of rank on rank n then $\mathcal{IU}(P)$ and $\mathcal{HU}(P)$ are in fact highly connected. We leave it as exercise to the interested reader to prove this conjecture using the theorems 7.3 and 7.4 as in the proof of lemma 6.8. In fact one can prove that if P contains a free summand of rank n then $\mathcal{IU}(P)$ is $\lfloor\frac{n-\mathrm{usr}(R)-2}{2}\rfloor$-connected and $\mathcal{HU}(P)$ is $\lfloor\frac{n-\mathrm{usr}(R)-3}{2}\rfloor$-connected. Also, by assuming the high connectivity of the $\mathcal{IU}(R^{2n})$, Charney proved that $\mathcal{HU}(R^{2n})$ is highly connected. Our proof is different and relies on our theory, but we use ideas from her paper, such as the lemma 7.2 and her lemma 4.1, which is a modified version of work of Maazen [9].

8. Homology stability

From theorem 7.4 one can get the homology stability of unitary groups. The approach is well known.

Remark 8.1. To prove homology stability of this type one only needs high acyclicity of the corresponding poset, not high connectivity. But usually this type of posets are also highly connected. Here we also proved the high connectivity. In particular we wished to confirm the conjecture of Charney [5, 2.10], albeit with different bounds (see 7.5).

Theorem 8.2. *Let R be a ring with $\mathrm{usr}(R) < \infty$ and let the action of the unitary group on the Abelian group A is trivial. Then the homomorphism Inc_* : $H_k(U^\epsilon_{2n}(R,\Lambda),A) \to H_k(U^\epsilon_{2n+2}(R,\Lambda),A)$ is surjective for $n \geq 2k+\mathrm{usr}(R)+2$ and injective for $n \geq 2k+\mathrm{usr}(R)+3$.*

Proof. See [5, Section 4] and theorem 7.4. □

Remark 8.3. With the result of the previous section one also can prove homology stability of the unitary groups with twisted coefficients. For more information in this direction see [20, §5] and [5, 4.2].

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