A Generalization of Mumford's Geometric Invariant Theory

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Abstract. We generalize Mumford's construction of good quotients for reductive group actions. Replacing a single linearized invertible sheaf with a certain group of sheaves, we obtain a Geometric Invariant Theory producing not only the quasiprojective quotient spaces, but more generally all divisorial ones. As an application, we characterize in terms of the Weyl group of a maximal torus, when a proper reductive group action on a smooth complex variety admits an algebraic variety as orbit space.

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Introduction

Let the reductive group G act regularly on a variety X. In [19], Mumford associates to every G-linearized invertible sheaf $\mathcal{L}$ on X a set $X^{ss}(\mathcal{L})$ of semistable points. He proves that there is a *good quotient* $p\colon X^{ss}(\mathcal{L}) \to X^{ss}(\mathcal{L})/\!\!/G$, that means p is a G-invariant affine regular map and the structure sheaf of the quotient space is the sheaf of invariants.

Mumford's theory is designed for the quasiprojective category: His quotient spaces are always quasiprojective. Conversely, for connected G and smooth X, if a G-invariant open set $U \subset X$ has a good quotient $U \to U/\!\!/G$ with $U/\!\!/G$ quasiprojective, then U is a saturated subset of a set $X^{ss}(\mathcal{L})$ for some G-linearized invertible sheaf $\mathcal{L}$ on X.

However, there frequently occur good quotients with a non quasiprojective quotient space; even if X is quasiaffine and G is a one dimensional torus, see e.g. [2]. For $X = \mathbb{P}_n$ or X a vector space with linear G-action, the situation is reasonably well understood, see [8] and [9]. But for general X, the picture is still far from being complete.

The purpose of this article is to present a general theory for good quotients with so called *divisorial* quotient spaces. Recall from [12] that an irreducible variety Y is divisorial if every $y \in Y$ admits an affine neighbourhood of the form $Y \setminus \operatorname{Supp}(D)$ with an effective Cartier divisor D on Y. This is a considerable generalization of quasiprojectivity. For example, all smooth varieties are divisorial.

Our approach to divisorial good quotient spaces is to replace Mumford's single invertible sheaf $\mathcal{L}$ with a free finitely generated group Λ of Cartier divisors on X. Then a G-linearization of such a group Λ is a certain G-sheaf structure on the graded $\mathcal{O}_X$-algebra $\mathcal{A}$ associated to Λ; for the precise definitions see Section 1.

In Section 2, we associate to every G-linearized group $\Lambda \subset \operatorname{CDiv}(X)$ a set $X^{ss}(\Lambda) \subset X$ of semistable points and a set $X^{s}(\Lambda) \subset X^{ss}(\Lambda)$ of stable points. Theorem 3.1 generalizes Mumford's result on existence of good quotients:

Theorem 1. *For any G-linearized group Λ of Cartier divisors, there is a good quotient $X^{ss}(\Lambda) \to X^{ss}(\Lambda)/\!\!/G$ with a divisorial quotient space $X^{ss}(\Lambda)/\!\!/G$.*

We note here that our quotient spaces are allowed to be non separated; see also the brief discussion at the end of Section 3. As in the classical situation, the restriction of the above quotient map to the set of stable points separates the orbits. In Theorem 4.1, we give a converse of the above result:

Theorem 2. *For $\mathbb{Q}$-factorial, e.g. smooth, X every G-invariant open subset $U \subset X$ with a good quotient such that $U/\!\!/G$ is divisorial occurs as a saturated subset of a set of semistable points $X^{ss}(\Lambda)$.*

As an application, we discuss actions of connected reductive groups G on normal complex varieties X. The starting point is the reduction theorem of A. Białynicki-Birula and J. Święcicka [6, Theorem 5.1]: If some maximal torus $T \subset G$ admits a good quotient $X \to X/\!\!/T$, then there is a "good quotient" for the action of G on X in the category of algebraic spaces.

Examples show that in general, the quotient space really drops out of the category of algebraic varieties, see [7, page 15]. So, there arises a natural question: When there is a good quotient $X \to X/\!\!/G$ in the category of algebraic varieties?

Our answers to this question are formulated in terms of the normalizer $N(T)$ of a maximal torus $T \subset G$. Recall that the connected component of the unit element of $N(T)$ is just T; in other words $N(T)/T$ is finite. The first result is the following, see Theorem 5.1:

Theorem 3. *Let G be a connected reductive group, and let X be a normal complex G-variety. Then the following statements are equivalent:*

i) *There is a good quotient $X \to X/\!\!/G$ with a divisorial prevariety $X/\!\!/G$.*
ii) *There is a good quotient $X \to X/\!\!/N(T)$ with a divisorial prevariety $X/\!\!/N(T)$.*

Moreover, if one of these statements holds with a separated quotient space then so does the other.

We specialize to proper G-actions. It is an easy consequence of the reduction theorem [6, Theorem 5.1] that such an action always admits a "geometric quotient" in the category of algebraic spaces. Fundamental results of Kollár [18], Keel and Mori [15] extend this fact to a more general framework.

In our second result, Theorem 5.2, the words geometric quotient refer to a good quotient (in the category of algebraic varieties) that separates orbits:

Theorem 4. *Suppose that a connected reductive group G acts properly on a $\mathbb{Q}$-factorial complex variety X. Then the following statements are equivalent:*

i) *There exists a geometric quotient $X \to X/G$.*
ii) *There exists a geometric quotient $X \to X/N(T)$.*

Moreover, if one of these statements holds, then the quotient spaces X/G and $X/N(T)$ are separated and $\mathbb{Q}$-factorial.

So, for proper G-actions on $\mathbb{Q}$-factorial varieties, the answer to the above question is encoded in an action of the Weyl group $W := N(T)/T$: A geometric quotient $X \to X/G$ exists in the category of algebraic varieties if and only if the induced action of W on X/T admits an algebraic variety as orbit space.

1. G-linearization and ample groups

Throughout the whole article, we work in the category of algebraic prevarieties over an algebraically closed field $\mathbb{K}$. In particular, the word point refers to a closed point. First we fix the notions concerning group actions and quotients.

In this section, G denotes a linear algebraic group, and X is an irreducible G-prevariety, that means X is an irreducible (possibly non separated) prevariety (over $\mathbb{K}$) together with a regular group action $\sigma\colon G \times X \to X$.

For reductive G, a *good quotient* of the G-prevariety X is a G-invariant affine regular map $p\colon X \to X /\!\!/ G$ of prevarieties such that $p^*\colon \mathcal{O}_{X/\!\!/G} \to p_*(\mathcal{O}_X)^G$ is an isomorphism. By a *geometric quotient* we mean a good quotient that separates orbits. Geometric quotient spaces are denoted by X/G.

Remark 1.1. [22, Theorem 1.1]. Let $p\colon X \to X/\!\!/G$ be a good quotient for an action of a reductive group G. Then we have:

i) For every G-invariant closed set $A \subset X$ the image $p(A) \subset X/\!\!/G$ is closed.
ii) If $A, B \subset X$ are closed G-invariant subsets, then $p(A \cap B)$ equals $p(A) \cap p(B)$.
iii) Each fibre $p^{-1}(y)$ contains exactly one closed G-orbit.
iv) Every G-invariant regular map $X \to X'$ factors uniquely through p.

Now we introduce the basic concepts used in this article, compare also [13] and [14]. When we speak of a subgroup of the group $\mathrm{CDiv}(X)$ of Cartier divisors of X, we always mean a finitely generated free subgroup.

Let $\Lambda \subset \mathrm{CDiv}(X)$ be such a subgroup. Denoting by $\mathcal{A}_D := \mathcal{O}_X(D)$ the sheaf of sections of $D \in \Lambda$, we obtain a Λ-graded $\mathcal{O}_X$-algebra:

$$\mathcal{A} := \bigoplus_{D \in \Lambda} \mathcal{A}_D.$$

The following notion extends Mumford's concept of a G-linearized invertible sheaf to groups of divisors:

Definition 1.2. Fix the canonical G-sheaf structure $(g\cdot f)(x) := f(g^{-1}\cdot x)$ on the structure sheaf $\mathcal{O}_X$.

i) A *G-linearization* of the group Λ is a graded G-sheaf structure on the Λ-graded $\mathcal{O}_X$-algebra $\mathcal{A}$ such that the representation of G on $\mathcal{A}(U)$ is rational for every G-invariant open subset $U \subset X$.
ii) A *strong G-linearization* of the group Λ is a G-linearization of Λ such that on each homogeneous component $\mathcal{A}_D$, $D \in \Lambda$, the G-sheaf structure arises from a G-linearization $\sigma^*(\mathcal{A}_D) \cong \mathrm{pr}_X^*(\mathcal{A}_D)$ in the sense of [19, Definition 1.6].

The reason to introduce besides the straightforward generalization 1.2 ii) also the weaker notion 1.2 i), is that in practice the latter is often much easier to handle. However, in many important cases both notions coincide, for example if the component G^0 of the unit element is a torus:

Proposition 1.3. *If X is covered by G^0-invariant affine open subsets, then every G-linearization of Λ is in fact a strong G-linearization of Λ.*

Proof. Assume that $\Lambda \subset \mathrm{CDiv}(X)$ is G-linearized, and let $\mathcal{A}_D$ be a homogeneous component of the associated graded $\mathcal{O}_X$-algebra. Consider a geometric line bundle $p\colon L \to X$ having $\mathcal{A}_D$ as its sheaf of sections. Then the G-sheaf structure of $\mathcal{A}_D$ gives rise to a set theoretical action, namely

$$G \times L \to L, \qquad (g, z) \mapsto g\cdot z := (g\cdot f)(g\cdot p(z)),$$

where for given $z \in L$ we choose any local section f of $\mathcal{A}_D$ satisfying $f(p(z)) = z$. Note that this well defined. In view of [16, Lemma 2.3], we only have to show that this action is regular. Since for fixed $g \in G$ the map $z \mapsto g\cdot z$ is obviously regular, it suffices to show that $G^0 \times L \to L$ is regular.

According to our assumption on X, it suffices to treat the case that X is affine. But then the rational representation of G^0 on the $\mathcal{O}(X)$-algebra

$$A := \bigoplus_{n \in \mathbb{N}} \mathcal{A}_{nD}(X)$$

defines a regular G^0-action on the dual bundle $L' := \mathrm{Spec}(A)$ such that $L' \to X$ becomes equivariant and G^0 acts linearly on the fibres. It is straightforward to check that this G^0-action on L' is dual to the G^0-action on L. Hence also the latter action is regular. $\square$

Concerning existence of linearizations, we have the following generalization of [19, Corollary 1.6], compare [14, Proposition 3.6]:

Proposition 1.4. *Suppose that G is connected and that X is a normal separated variety. Then every group $\Lambda \subset \mathrm{CDiv}(X)$ admits a strongly G-linearized subgroup $\Lambda' \subset \Lambda$ of finite index.*

Proof. Choose a basis $D_1, \ldots, D_r$ of Λ. According to [16, Proposition 2.4], there is a positive integer n such that each sheaf $\mathcal{A}_{nD_i}$ admits a G-linearization in Mumford's sense. Tensoring these linearizations gives the desired strong linearization of the subgroup $\Lambda' \subset \Lambda$ generated by $nD_1, \ldots, nD_r$. □

A more special existence statement for non connected G will be given in 4.2. There is also a uniqueness statement like [19, Proposition 1.4]. Note that in our version, we do not assume G to be connected:

Proposition 1.5. *Suppose that $\Lambda \subset \mathrm{CDiv}(X)$ admits two strong G-linearizations. If $\mathcal{O}^*(X) = \mathbb{K}^*$ holds and G has only finitely many characters, then the two G-linearizations coincide on a subgroup $\Lambda' \subset \Lambda$ of finite index.*

Proof. To distinguish the two G-sheaf structures on the graded $\mathcal{O}_X$-algebra associated to Λ, we denote them by $(g, f) \mapsto g \cdot f$ and $(g, f) \mapsto g * f$. Consider a homogeneous component $\mathcal{A}_D$, and the tensor product

$$\mathcal{A}_D \otimes_{\mathcal{O}_X} \mathcal{A}_{-D}, \qquad g \bullet (f \otimes h) := g \cdot f \otimes g * h.$$

Since as an $\mathcal{O}_X$-module, $\mathcal{A}_D \otimes_{\mathcal{O}_X} \mathcal{A}_{-D}$ is isomorphic to the structure sheaf itself, we obtain a G-sheaf structure on $\mathcal{O}_X$, also denoted by $(g, f) \mapsto g \bullet f$. As it arises from a G-linearization in the sense of [19, Definition 1.6], this G-sheaf structure is of the form

$$(g \bullet f)(x) = \chi(g, x) f(g^{-1} \cdot x)$$

with a function $\chi \in \mathcal{O}^*(G \times X)$. Since we assumed $\mathcal{O}^*(X) \cong \mathbb{K}^*$, the function χ does not depend on the second variable. In fact, χ even turns out to be a character on G.

Now, replacing in this setting D with a multiple nD amounts to replacing χ with χ^n. Thus, taking n to be the order of the character group of G, we see that for any $D \in \Lambda$, the two G-sheaf structures on $\mathcal{A}_{nD}$ coincide. The assertion follows. □

We look a bit closer to the $\mathcal{O}_X$-algebra $\mathcal{A}$ associated to a group $\Lambda \subset \mathrm{CDiv}(X)$. This algebra gives rise to a prevariety $\widehat{X} := \mathrm{Spec}(\mathcal{A})$ and a canonical map $q \colon \widehat{X} \to X$. We list some basic features of this construction:

Remark 1.6. Let $\widehat{X} := \mathrm{Spec}(\mathcal{A})$ and $q \colon \widehat{X} \to X$ be as above. For an open subset $U \subset X$, set $\widehat{U} := q^{-1}(U)$.

i) For a section $f \in \mathcal{A}_D(U)$, let $Z(f) := \mathrm{Supp}(D|_U + \mathrm{div}(f))$ denote the set of its zeroes. Then we have

$$\widehat{U}_f := \{x \in \widehat{U};\ f(x) \neq 0\} = q^{-1}(U \setminus Z(f)).$$

ii) The algebraic torus $H := \mathrm{Spec}(\mathbb{K}[\Lambda])$ acts regularly on $\widehat{X}$ such that every $f \in \mathcal{A}_D(U)$ is homogeneous with respect to the character χ^D, i.e., we have

$$f(t \cdot x) = \chi^D(t) \cdot f(x).$$

iii) The action of H on $\widehat{X}$ is free and the map $q \colon \widehat{X} \to X$ is a geometric quotient for this action.

For the subsequent constructions, it is important to figure out those groups $\Lambda \subset \mathrm{CDiv}(X)$ for which the associated prevariety $\widehat{X}$ over X is in fact a quasiaffine variety. This leads to the following notion:

Definition 1.7. We call the group $\Lambda \subset \mathrm{CDiv}(X)$ *ample on an open subset* $U \subset X$, if there are homogeneous sections $f_1, \ldots, f_r \in \mathcal{A}(U)$ such that the sets $U \setminus Z(f_i)$ are affine and cover U.

If $\Lambda \subset \mathrm{CDiv}(X)$ is ample on X, then we say for short that Λ is ample. So, the prevariety X admits an ample group $\Lambda \subset \mathrm{CDiv}(X)$ if and only if it is *divisorial* in the sense of Borelli [12], i.e., every $x \in X$ has an affine neighbourhood $X \setminus \mathrm{Supp}(D)$ with some effective $D \in \mathrm{CDiv}(X)$.

Remark 1.8. If X is a divisorial prevariety, then the intersection $U \cap U'$ of any two affine open subsets $U, U' \subset X$ is again affine.

In the following statement, we subsume the consequences of the existence of a G-linearized ample group, compare [13, Section 2]. By an affine closure of a quasiaffine variety Y we mean an affine variety $\overline{Y}$ containing Y as an open dense subvariety.

Proposition 1.9. *Let G be a linear algebraic group and let X be a G-prevariety. Suppose that $\Lambda \subset$ $\mathrm{CDiv}(X)$ is G-linearized and ample on some G-invariant open $U \subset X$. Let $\widehat{U} := q^{-1}(U) \subset \widehat{X}$, where $q \colon \widehat{X} \to X$ is as above.*

i) *$\widehat{U}$ is quasiaffine and the representation of G on $\mathcal{O}(\widehat{U})$ induces a regular G-action on $\widehat{U}$ such that the actions of G and $H := \mathrm{Spec}(\mathbb{K}[\Lambda])$ commute and the canonical map $q \colon \widehat{U} \to X$ becomes G-equivariant.*

ii) *For any collection $f_1, \ldots, f_r \in \mathcal{A}(U)$ satisfying the ampleness condition, there exists a $(G \times H)$-equivariant affine closure $\overline{U}$ of $\widehat{U}$ such that the f_i extend to regular functions on $\overline{U}$ and $q^{-1}(U_{f_i}) = \overline{U}_{f_i}$ holds.*

Proof. Use [13, Lemmas 2.4 and 2.5]. ∎

2. Stability notions

Generalizing [19, Definitions 1.7 and 1.8] we shall associate to a linearized group of divisors sets of semistable, stable and properly stable points. Moreover, for ample linearized groups, we give a geometric interpretation of semistability in terms of a generalized nullcone.

Let G be a reductive algebraic group, and let X be an irreducible G-prevariety. Suppose that $\Lambda \subset \mathrm{CDiv}(X)$ is a G-linearized (finitely generated free) subgroup. Denote the associated Λ-graded $\mathcal{O}_X$-algebra by

$$\mathcal{A} = \bigoplus_{D \in \Lambda} \mathcal{A}_D.$$

Definition 2.1. Let G, X, Λ and $\mathcal{A}$ be as above. We say that a point $x \in X$ is

i) *semistable*, if x has an affine neighbourhood $U = X \setminus Z(f)$ with some G-invariant $f \in \mathcal{A}_D(X)$ such that the $D' \in \Lambda$ admitting a G-invariant $f_{D'} \in \mathcal{A}_{D'}(U)$ which is invertible in $\mathcal{A}(U)$ form a subgroup of finite index in Λ,
ii) *stable*, if x is semistable, its orbit $G \cdot x$ is of maximal dimension and $G \cdot x$ is closed in the set of semistable points of X,
iii) *properly stable*, if x is semistable, its isotropy group G_x is finite and $G \cdot x$ is closed in the set of semistable points of X.

Following Mumford's notation, we denote the G-invariant open sets corresponding to the semistable, stable and properly stable points by $X^{ss}(\Lambda)$, $X^s(\Lambda)$ and $X_0^s(\Lambda)$ respectively. If we want to specify the acting group G, we also write $X^{ss}(\Lambda, G)$ etc..

Remark 2.2. Let X be complete, let $D \in \mathrm{CDiv}(X)$ be an effective Cartier divisor and suppose that the invertible sheaf $\mathcal{L} := \mathcal{A}_D$ on X is G-linearized in the sense of [19, Definition 1.6]. Then the induced G-sheaf structure of $\mathcal{A}_D$ extends to a G-linearization of $\Lambda := \mathbb{Z}D$. Moreover,

i) $X^{ss}(\Lambda)$ contains precisely the points of X which are semistable in the sense of [19, Definition 1.7 i)],
ii) $X^s(\Lambda)$ contains precisely the points of X which are stable in the sense of [19, Definition 1.7 ii)],
iii) $X_0^s(\Lambda)$ contains precisely the points of X which are properly stable in the sense of [19, Definition 1.8].

The remainder of this section is devoted to giving a geometric interpretation of semistability. For this, let $U \subset X$ denote any G-invariant open subset such that Λ is ample on U and $X^{ss}(\Lambda)$ is contained in U, for example $U = X^{ss}(\Lambda)$. As usual, let

$$\widehat{X} := \mathrm{Spec}(\mathcal{A}), \qquad q\colon \widehat{X} \to X, \qquad \widehat{U} := q^{-1}(U).$$

Recall from Section 1 that the map $q\colon \widehat{X} \to X$ is a geometric quotient for the action of $H := \operatorname{Spec}(\mathbb{K}[\Lambda])$ on $\widehat{X}$ induced by the Λ-grading of $\mathcal{A}$. Moreover, $\widehat{U}$ is a quasiaffine variety and carries a regular G-action making $q\colon \widehat{U} \to X$ equivariant.

Our description involves two choices. First let $f_1, \ldots, f_r \in \mathcal{A}(X)$ be homogeneous G-invariant sections such that the sets $X \setminus Z(f_i)$ are as in Definition 2.1 i) and cover $X^{ss}(\Lambda)$.

Next choose a $(G \times H)$-equivariant affine closure $\overline{U}$ of $\widehat{U}$ such that the functions $f_i \in \mathcal{O}(\widehat{U})$ extend regularly to $\overline{U}$ and $\overline{U}_{f_i} = \widehat{U}_{f_i}$ holds for each $i = 1, \ldots, r$. Consider the good quotient

$$\overline{p}\colon \overline{U} \to \overline{U}/\!\!/G := \operatorname{Spec}(\mathcal{O}(\overline{U}))^G.$$

Then the quotient variety $\overline{U}/\!\!/G$ inherits a regular action of H such that the map $\overline{p}\colon \overline{U} \to \overline{U}/\!\!/G$ becomes H-equivariant. In this setting, the set $\overline{U} \setminus q^{-1}(X^{ss}(\Lambda))$ takes over the role of the classical nullcone:

Proposition 2.3. *Let $V_0 := \overline{U}/\!\!/G \setminus \overline{p}(\overline{U} \setminus \widehat{U})$, and let $V_1 \subset \overline{U}/\!\!/G$ be the union of all H-orbits with finite isotropy.*

i) *One always has $q^{-1}(X^{ss}(\Lambda)) \subset \overline{p}^{-1}(V_0 \cap V_1)$.*
ii) *If $U = X$, then $q^{-1}(X^{ss}(\Lambda)) = \overline{p}^{-1}(V_0 \cap V_1)$.*

The main point in the proof is to express Condition 2.1 i) in terms of the action of the torus H on the affine variety $\overline{U}/\!\!/G$. Consider more generally an arbitrary algebraic torus T and a quasiaffine T-variety Y.

Lemma 2.4. *The isotropy group T_y of a point $y \in Y$ is finite if and only if there is a homogeneous function $h \in \mathcal{O}(Y)$ such that Y_h is an affine neighbourhood of y and the characters $\chi' \in \operatorname{Char}(T)$ admitting an invertible χ'-homogeneous $h' \in \mathcal{O}(Y_h)$ form a sublattice of finite index in $\operatorname{Char}(T)$.*

Proof. First suppose that T_y is finite. Consider the orbit $B := T \cdot y$. This is a locally closed affine subvariety of Y. The set M consisting of all characters $\chi' \in \operatorname{Char}(T)$ admitting a χ'-homogeneous $h' \in \mathcal{O}(B)$ with $h'(y) = 1$ is a sublattice of $\operatorname{Char}(T)$. We show that M is of full rank:

Otherwise there is a non trivial one parameter subgroup $\lambda\colon \mathbb{K}^* \to T$ such that $\chi \circ \lambda = 1$ holds for every $\chi \in M$. Thus, by the definition of M, all homogeneous functions of $\mathcal{O}(B)$ are constant along $\lambda(\mathbb{K}^*) \cdot y$. As these functions separate the points of B, we conclude $\lambda(\mathbb{K}^*) \subset T_y$. A contradiction.

Now, choose any T-homogeneous function $h \in \mathcal{O}(Y)$ such that Y_h is affine, contains B as a closed subset, and for some base $\chi'_1, \ldots, \chi'_d$ of M the associated functions $h'_i \in \mathcal{O}(B)$ extend to invertible regular homogeneous functions on Y_h. Then this $h \in \mathcal{O}(Y)$ is as desired. The "if" part of the assertion is settled by similar arguments. $\square$

Proof of Proposition 2.3. Let $W := X^{ss}(\Lambda)$ and $\widehat{W} := q^{-1}(W)$. We begin with the inclusion "$\subset$" of assertions i) and ii). First note that $\widehat{W}$ is $\overline{p}$-saturated, because this holds for each $\overline{U}_{f_i}$ and, according to Remark 1.6 i), $\widehat{W}$ is covered by these subsets. In particular, it follows $\overline{p}(\widehat{W}) \subset V_0$.

To verify $\overline{p}(\widehat{W}) \subset V_1$, let $z \in \widehat{W}$. Take one of the f_i with $z \in \overline{U}_{f_i}$. As it is G-invariant, f_i descends to an H-homogeneous function $h \in \mathcal{O}(\overline{U} /\!\!/ G)$. By the properties of f_i, the function h satisfies the condition of Lemma 2.4 for the point $\overline{p}(z)$. Hence $H_{\overline{p}(z)}$ is finite, which means $\overline{p}(z) \in V_1$.

We come to the inclusion "$\supset$" of assertion ii). Let $y \in V_0 \cap V_1$. Lemma 2.4 provides an $h \in \mathcal{O}(\overline{X} /\!\!/ G)$, homogeneous with respect to some $\chi^D \in \mathrm{Char}(H)$, such that $y \in V := (\overline{X} /\!\!/ G)_h$ holds and the $D' \in \Lambda$ admitting an invertible $\chi^{D'}$-homogeneous function on V form a subgroup of finite index in Λ. Suitably modifying h, we achieve additionally $V \subset V_0 \cap V_1$.

Now, consider a point $z \in \overline{p}^{-1}(y)$. Since $y \in V_0$, we have $z \in \widehat{X}$. We have to show that $q(z)$ is semistable. For this, consider the G-invariant homogeneous section $f := \overline{p}^*(h)|_{\widehat{X}}$ of $\mathcal{A}_D(X)$. By the choice of h, this f fulfills the conditions of Definition 2.1 i) and thus the point $q(z)$ is in fact semistable. □

Corollary 2.5. *Let $\Lambda \subset \mathrm{CDiv}(X)$ be an ample G-linearized group.*

i) *A point $x \in X^{ss}(\Lambda)$ with an orbit $G \cdot x$ of maximal dimension is stable if and only if for any $z \in q^{-1}(x)$ the orbit $G \cdot z$ is closed in $\widehat{X}$.*
ii) *A point $x \in X^{ss}(\Lambda)$ with finite isotropy group G_x is properly stable if and only if for any $z \in q^{-1}(x)$ the orbit $G \cdot z$ is closed in $\widehat{X}$.* □

3. The quotient of the set of semistable points

Let G be a reductive algebraic group, and let X be a G-prevariety. In this section we show that any set of semistable points admits a good quotient. The result generalizes [19, Theorem 1.10].

Theorem 3.1. *Let $\Lambda \subset \mathrm{CDiv}(X)$ be a G-linearized subgroup. Then there exists a good quotient $p \colon X^{ss}(\Lambda) \to X^{ss}(\Lambda) /\!\!/ G$ and the quotient space $X^{ss}(\Lambda) /\!\!/ G$ is a divisorial prevariety.*

An immediate consequence of this result is that the set of stable points admits a geometric quotient. More precisely, by the properties of good quotients we have:

Remark 3.2. In the notation of 3.1, the set $X^s(\Lambda)$ is p-saturated and the restriction $p \colon X^s(\Lambda) \to p(X^s(\Lambda))$ is a geometric quotient.

In the proof of Theorem 3.1, we make use of the following observation on geometric quotients for torus actions, compare [1, Proposition 1.5]:

Lemma 3.3. *Let T be an algebraic torus and suppose that Y is an irreducible quasiaffine T-variety with geometric quotient $p\colon Y \to Y/T$. Then Y/T is a divisorial prevariety.*

Proof. We may assume that T acts effectively. Set for short $Z := Y/T$. Given a point $z \in Z$, choose a T-homogeneous $f \in \mathcal{O}(Y)$ such that $U := Y_f$ is an affine neighbourhood of $p^{-1}(z)$. Consider the affine neighbourhood $V := p(U)$ of z. We show that $B := Z \setminus V$ is the support of an effective Cartier divisor on Y.

Let $\chi \in \mathrm{Char}(T)$ be the weight of the above $f \in \mathcal{O}(Y)$. Since T acts effectively with geometric quotient, all isotropy groups T_y are finite. So we can use Lemma 2.4 to cover Y by T-invariant affine open sets U_i admitting invertible functions $g_i \in \mathcal{O}(U_i)$ that are homogeneous with respect to some common multiple $m\chi$.

Each $h_i := f^m/g_i \in \mathcal{O}(U_i)$ is T-invariant and hence we have $h_i = p^*(h_i')$ with a regular function h_i' defined on $V_i := p(U_i)$. By construction, the zero set of h_i' is just $B \cap V_i$. Since every h_i'/h_j' is regular and invertible on $V_i \cap V_j$, the functions h_i' yield local equations for an effective Cartier divisor E on Z having support B. □

Proof of Theorem 3.1. As usual, let $\mathcal{A}$ be the graded $\mathcal{O}_X$-algebra associated to Λ. We consider the corresponding prevariety $\widehat{X} := \mathrm{Spec}(\mathcal{A})$ and the map $q\colon \widehat{X} \to X$. Recall that the latter is a geometric quotient for the action of $H := \mathrm{Spec}(\mathbb{K}[\Lambda])$ on $\widehat{X}$. Set for short $W := X^{ss}(\Lambda)$. Surely, Λ is ample on W. Proposition 1.9 yields that $\widehat{W} := q^{-1}(W)$ is a quasiaffine variety. Moreover, $\widehat{W}$ carries a G-action that commutes with the action of H and makes $q\colon \widehat{W} \to W$ equivariant. Choose $f_1, \ldots, f_r \in \mathcal{A}(X)$ satisfying the conditions of Definition 2.1 such that W is covered by the affine sets $X \setminus Z(f_i)$, and set $h_i := f_i|_W$.

Choose a $(G \times H)$-equivariant affine closure $\overline{W}$ of $\widehat{W}$ such that the above $h_i \in \mathcal{O}(\widehat{W})$ extend regularly to $\overline{W}$ and satisfy $\overline{W}_{h_i} = \widehat{W}_{h_i}$. The set $\widehat{W}$ is saturated with respect to the good quotient $\overline{p}\colon \overline{W} \to \overline{W}/\!\!/G$ because this holds for the sets $\overline{W}_{h_i}$. Consequently, restricting $\overline{p}$ to $\widehat{W}$ yields a good quotient $\widehat{p}\colon \widehat{W} \to \widehat{W}/\!\!/G$.

Moreover, Proposition 2.3 i) tells us that H acts with at most finite isotropy groups on $\widehat{W}/\!\!/G$. Thus, there is a geometric quotient $\widehat{W}/\!\!/G \to (\widehat{W}/\!\!/G)/H$. By Lemma 3.3, the quotient space is a divisorial prevariety. Since good quotients are categorical, we obtain a commutative diagram

$$\begin{array}{ccc} \widehat{W} & \xrightarrow{\widehat{p}} & \widehat{W}/\!\!/G \\ {\scriptstyle /H}\downarrow & & \downarrow{\scriptstyle /H} \\ W & \longrightarrow & (\widehat{W}/\!\!/G)/H \end{array}$$

Now it is straightforward to check that the induced map $W \to (\widehat{W/\!\!/G})/H$ is the desired good quotient for the action of G on W. $\square$

We conclude this section with a short discussion of the question, when the quotient space $X^{ss}(\Lambda)/\!\!/G$ is separated. Translating the usual criterion for separateness in terms on functions on the quotient space to the setting of invariant sections of the $\mathcal{O}_X$-algebra $\mathcal{A}$ of a G-linearized group Λ, we obtain:

Remark 3.4. Let $\Lambda \subset \mathrm{CDiv}(X)$ be a G-linearized group on a G-variety X, and let $X^{ss}(\Lambda)$ be covered by $X \setminus Z(f_i)$ with G-invariant sections $f_1, \ldots, f_r \in \mathcal{A}(X)$ as in 2.1 i). The quotient space $X^{ss}/\!\!/G$ is separated if and only if for any two indices i, j the multiplication map defines a surjection in degree zero:

$$\mathcal{A}(X)^G_{(f_i)} \otimes \mathcal{A}(X)^G_{(f_j)} \to \mathcal{A}(X)^G_{(f_i f_j)}.$$

In the classical setting [19, Definition 1.7], the group Λ is of rank one, and the above sections f_i are of positive degree. In particular, for suitable positive powers n_i, all sections $f_i^{n_i}$ are of the same degree, and Remark 3.4 implies that the resulting quotient space is always separated.

As soon as we leave the classical setting, the above reasoning may fail, and we can obtain nonseparated quotient spaces, as the following two simple examples show. Both examples arise from the hyperbolic $\mathbb{K}^*$-action on the affine plane. In the first one we present a group Λ of rank one defining a nonseparated quotient space:

Example 3.5. Let the onedimensional torus $T := \mathbb{K}^*$ act diagonally on the punctured affine plane $X := \mathbb{K}^2 \setminus \{(0,0)\}$ via

$$t \cdot (z_1, z_2) := (tz_1, t^{-1}z_2).$$

Consider the group $\Lambda \subset \mathrm{CDiv}(X)$ generated by the principal divisor $D := \mathrm{div}(z_1)$. Since D is T-invariant, the group Λ is canonically T-linearized. We claim that the corresponding set of semistable points is

$$X^{ss}(\Lambda) = X.$$

To verify this claim, let $\mathcal{A}$ denote the graded $\mathcal{O}_X$-algebra associated to Λ, and consider the T-invariant sections

$$f_1 := 1 \in \mathcal{A}_D(X), \qquad f_2 := z_1 z_2 \in \mathcal{A}_{-D}(X).$$

Then the sets $X \setminus Z(f_1)$ and $X \setminus Z(f_2)$ form an affine cover of X. Moreover, we have T-invariant invertible sections:

$$1 \in \mathcal{A}_D(X \setminus Z(f_1)), \qquad \frac{1}{z_1 z_2} \in A_D(X \setminus Z(f_2)).$$

So, $f_1, f_2 \in \mathcal{A}(X)$ satisfy the conditions of Definition 2.1 i), and the claim is verified. The quotient space $Y := X^{ss}(\Lambda)/\!\!/T$ is the affine line with doubled zero. In particular, Y is a nonseparated prevariety.

In view of Remark 3.4, we obtain always separated quotient spaces when starting with a group $\Lambda = \mathbb{Z}D$, where D is a divisor on a complete G-variety X. In this setting, the lack of enough invariant sections of degree zero on the sets $X \setminus Z(f_i)$ occurs for groups Λ of higher rank:

Example 3.6. Let the onedimensional torus $T := \mathbb{K}^*$ act diagonally on the projective plane $X := \mathbb{P}_2$ via

$$t\cdot[z_0, z_1, z_2] := [z_0, tz_1, t^{-1}z_2].$$

Consider the group $\Lambda \subset \mathrm{CDiv}(X)$ generated by the divisors $D_1 := E_0 + E_1$ and $D_2 := E_0 + E_2$, where E_i denotes the prime divisor $V(X; z_i)$. Since the divisors D_i are T-invariant, the group Λ is canonically T-linearized. We claim that the corresponding set of semistable points is

$$X^{ss}(\Lambda) = X \setminus \{[1,0,0],[0,1,0],[0,0,1]\}.$$

To check this claim, denote the right hand side by U. Let $\mathcal{A}$ again denote the graded $\mathcal{O}_X$-algebra associated to Λ, and consider the T-invariant sections

$$f_1 := 1 \in \mathcal{A}_{D_1}(X), \qquad f_2 := 1 \in \mathcal{A}_{D_2}(X), \qquad f_3 := \frac{z_1 z_2}{z_0^2} \in \mathcal{A}_{D_1+D_2}(X).$$

For the respective zero sets of these sections we have

$$Z(f_1) = V(X; z_0 z_1), \qquad Z(f_2) = V(X; z_0 z_2), \qquad Z(f_3) = V(X; z_1 z_2).$$

So, the set U is indeed the union of the affine sets $X \setminus Z(f_i)$. Moreover, we have invertible sections

$$\begin{aligned} 1 &\in \mathcal{A}_{D_1}(X \setminus Z(f_1)), & \frac{z_0^2}{z_1 z_2} &\in \mathcal{A}_{D_2}(X \setminus Z(f_1)),\\ 1 &\in \mathcal{A}_{D_2}(X \setminus Z(f_2)), & \frac{z_0^2}{z_1 z_2} &\in \mathcal{A}_{D_1}(X \setminus Z(f_2)),\\ \frac{z_1 z_2}{z_0^2} &\in \mathcal{A}_{2D_1}(X \setminus Z(f_3)), & \frac{z_1 z_2}{z_0^2} &\in \mathcal{A}_{2D_2}(X \setminus Z(f_3)). \end{aligned}$$

Thus $f_1, f_2, f_3 \in \mathcal{A}(X)$ satisfy the conditions of Definition 2.1 i). Since the fixed points $[1,0,0]$, $[0,1,0]$ and $[0,0,1]$ occur as limit points of suitable T-orbits through U, they cannot be semistable. The claim is verified.

Note that $X^{ss}(\Lambda)$ equals in fact the set of (properly) stable points. The quotient space $Y := X^{ss}(\Lambda)/\!\!/T$ is a projective line with doubled zero. In particular, Y is a nonseparated prevariety.

4. Good quotients for $\mathbb{Q}$-factorial G-varieties

Let G be a not necessarily connected reductive group, and let X be an irreducible G-prevariety. In [19, Converse 1.13], Mumford shows that, provided X is a smooth variety and G is connected, every open subset U with a geometric quotient $U \to U/G$ such that U/G is quasiprojective arises in fact from a set of stable points.

Here we generalize this statement to non connected G and open subsets with a divisorial good quotient space. Assume that X is $\mathbb{Q}$-factorial, i.e., X is normal and for each Weil divisor D on X, some multiple of D is Cartier. Moreover, suppose that X is of affine intersection, i.e., for any two open affine subsets of X their intersection is again affine.

To formulate our result, let $U \subset X$ be an open G-invariant set of the G-prevariety X such that there exists a good quotient $U \to U /\!\!/ G$. Then we have:

Theorem 4.1. *If $U /\!\!/ G$ is divisorial, then there exists a G-linearized group $\Lambda \subset \mathrm{CDiv}(X)$ such that U is contained in $X^{ss}(\Lambda)$ and is saturated with respect to the quotient map $X^{ss}(\Lambda) \to X^{ss}(\Lambda) /\!\!/ G$.*

For the proof of this statement, we need two lemmas. The first one is an existence statement on *canonical linearizations*:

Let H be any linear algebraic group. We say that a Weil divisor E on a normal H-prevariety Y is *H-tame*, if $\mathrm{Supp}(E)$ is H-invariant and for any two prime cycles E_1, E_2 of E with $E_2 = h \cdot E_1$ for some $h \in H$ their multiplicities in E coincide.

Lemma 4.2. *Let $\Lambda \subset \mathrm{CDiv}(Y)$ be a group consisting of H-tame divisors. Then Λ admits a canonical H-linearization, namely*

$$\mathcal{A}_E(U) \to \mathcal{A}_E(h \cdot U), \qquad (h \cdot f)(x) := f(h^{-1} \cdot x).$$

Proof. First we note that the canonical action of H on $\mathbb{K}(Y)$ induces indeed a H-sheaf structure on the sheaf $\mathcal{A}_E$ of an H-tame Cartier divisor E on Y. This follows from the fact that for $f \in \mathbb{K}(Y)$, the order of a translate $h \cdot f$ along a prime divisor E_0 of E is given by

$$\mathrm{ord}_{E_0}(h \cdot f) = \mathrm{ord}_{h^{-1} \cdot E_0}(f).$$

We still have to show that for every H-invariant open set $V \subset Y$, the representation of H on $\mathcal{A}_E(V)$ is regular. Consider the maximal separated subsets $V_1, \ldots, V_r$ of $V \setminus \mathrm{Supp}(E)$, see [3, Theorem I]. Their intersection V' is H-invariant, and $\mathcal{A}_E(V)$ injects H-equivariantly into $\mathcal{O}(V')$. Hence [16, Section 2.5] gives the claim. □

Now, consider a normal prevariety Y with effective $E_1, \ldots, E_r \in \mathrm{CDiv}(Y)$ such that the sets $V_i := Y \setminus \mathrm{Supp}(E_i)$ are affine and cover Y. Let $\Gamma \subset \mathrm{CDiv}(Y)$ be the subgroup generated by $E_1, \ldots, E_r$. Denote the associated Γ-graded $\mathcal{O}_Y$-algebra by

$$\mathcal{B} := \bigoplus_{E \in \Gamma} \mathcal{B}_E := \bigoplus_{E \in \Gamma} \mathcal{O}_Y(E).$$

Lemma 4.3. *In the above setting, every open set $V_i = Y \setminus \mathrm{Supp}(E_i)$ is covered by finitely many open affine subsets $V_{ij} \subset V_i$ with the following properties:*

i) *$V_{ij} = Y \setminus Z(h_{ij})$ with some $h_{ij} \in \mathcal{B}_{n_i E_i}(Y)$, where $n_i \in \mathbb{N}$,*

ii) *for each $k = 1, \ldots, r$ there exists an $h_{ijk} \in \mathcal{B}_{E_k}(V_{ij})$ without zeroes in V_{ij}.*

Proof. Let $y \in V_i$ and consider an affine open neighbourhood $V \subset V_i$ of y such that on V we have $E_k = \operatorname{div}(h'_k)$ with some $h'_k \in \mathcal{O}(V)$ for all k. Then each $h_k := 1/h'_k$ is a section of $\mathcal{B}_{E_k}(V)$ without zeroes in V. By suitably shrinking V, we achieve $V = X \setminus Z(h)$ with some $h \in \mathcal{B}_{n_i E_i}(X)$ and some $n_i \in \mathbb{N}$. Since finitely many of such V cover V_i, the assertion follows. $\square$

Proof of Theorem 4.1. Since the quotient space $Y := U /\!\!/ G$ is divisorial, we find effective $E_1, \ldots, E_r \in \operatorname{CDiv}(Y)$ such that the sets $V_i := Y \setminus \operatorname{Supp}(E_i)$ are affine and cover Y. Let V_{ij}, h_{ij} and h_{ijk} as in Lemma 4.3. Consider the quotient map $p\colon U \to Y$ and the pullback divisors

$$D'_i := p^*(E_i) \in \operatorname{CDiv}(U).$$

Then every $U_i := p^{-1}(V_i)$ is affine and equals $U \setminus \operatorname{Supp}(D'_i)$. Moreover, since they are locally defined by invariant functions, we see that the divisors D'_i are G-tame. Since X is $\mathbb{Q}$-factorial and of affine intersection, we can construct G-tame effective divisors $D_i \in \operatorname{CDiv}(X)$ with the following properties:

i) $D_i|_U = m_i D'_i$ holds with some $m_i \in \mathbb{N}$ and we have $X \setminus \operatorname{Supp}(D_i) = U_i$,
ii) for some $l_i \in \mathbb{N}$, every $f_{ij} := p^*(h_{ij}^{l_i})$ extends to a global section of $\mathcal{O}_X(D_i)$ and satisfies $X \setminus Z(f_{ij}) = p^{-1}(V_{ij})$.

Let $\Lambda \subset \operatorname{CDiv}(X)$ denote the group generated by the divisors $D_1, \ldots, D_r$, and let $\mathcal{A}$ be the associated graded $\mathcal{O}_X$-algebra. Lemma 4.2 tells us that the group Λ is canonically G-linearized by setting $g{\cdot}f(x) := f(g^{-1}{\cdot}x)$ on the homogeneous components of $\mathcal{A}$.

Note that the set $U \subset X$ is covered by the affine open subsets $U_{ij} := p^{-1}(V_{ij})$. Thus, using the pullback data f_{ij} and

$$f_{ijk} := p^*(h_{ijk}^{m_i}) \in \mathcal{A}_{D_i}(U_{ij}),$$

it is straightforward to check $U \subset X^{ss}(\Lambda)$. Moreover, since the U_{ij} are defined by the G-invariant sections f_{ij}, we see that they are saturated with respect to the quotient map $p'\colon X^{ss}(\Lambda) \to X^{ss}(\Lambda)/\!\!/G$. Hence U is p'-saturated in $X^{ss}(\Lambda)$. $\square$

Corollary 4.4. *Let the algebraic torus T act effectively and regularly on a $\mathbb{Q}$-factorial variety X, and let $U \subset X$ be the union of all T-orbits with finite isotropy group. If $\dim(X \setminus U) < \dim(T)$, then U is the set of semistable points of a T-linearized group $\Lambda \subset \operatorname{CDiv}(X)$.*

Proof. By [23, Corollary 3], there is a geometric quotient $U \to U/T$. Using Proposition 1.9 and Lemma 3.3, we see that U/T is a divisorial prevariety. Theorem 4.1 provides a T-linearized group $\Lambda \subset \operatorname{CDiv}(X)$ such that $X^{ss}(\Lambda)$ contains U as a saturated subset with respect to $p\colon X^{ss}(\Lambda) \to X^{ss}(\Lambda)/T$.

Semicontinuity of the fibre dimension of p and $\dim(X \setminus U) < \dim(T)$ imply $U = X^{ss}(\Lambda)$. □

The classical example of a generic $\mathbb{C}^*$-action on the Grassmannian of two dimensional planes in $\mathbb{C}^4$, compare also [5] and [25], fits into the setting of the above observation:

EXAMPLE 4.5. Realize the complex Grassmannian $X := G(2;4)$ via Plücker relations as a quadric hypersurface in the complex projective space $\mathbb{P}_5$:

$$X = V(\mathbb{P}_5; z_0z_5 - z_1z_4 + z_2z_3).$$

This allows us to define a regular action of the one dimensional torus $T = \mathbb{C}^*$ on X in terms of coordinates:

$$t \cdot [z_0, z_1, z_2, z_3, z_4, z_5] := [tz_0, t^2z_1, t^3z_2, t^3z_3, t^4z_4, t^5z_5].$$

This T-action has six fixed points. Let $U \subset X$ be the complement of the fixed point set. It is well known that the quotient space $Y := U/T$ is a nonseparated prevariety which is covered by four projective open subsets. Moreover, Y contains two nonprojective complete open subsets, see [5, Remark 1.6] and [25, Example 6.4].

According to Corollary 4.4, the set U can be realized as the set of semistable points of a T-linearized group of divisors. Let us do this explicitly. Consider for example the prime divisors $D_1 := V(X; z_1)$ and $D_2 := V(X; z_4)$ and the group

$$\Lambda := \mathbb{Z}D_1 \oplus \mathbb{Z}D_2 \subset \mathrm{CDiv}(X).$$

Then the group Λ is canonically T-linearized. We show that $X^{ss}(\Lambda) = U$ holds. Let $\mathcal{A}$ denote the graded $\mathcal{O}_X$-algebra associated to Λ, and consider the following T-invariant sections $f_{ij} \in \mathcal{A}(X)$:

$$f_{01} := \frac{z_0^2z_4}{z_1^3} \in \mathcal{A}_{4D_1-D_2}(X), \qquad h_{01} := 1 \in \mathcal{A}_{D_1}(X \setminus Z(f_{01})),$$

$$f_{02} := \frac{z_0z_2}{z_1^2} \in \mathcal{A}_{2D_1}(X), \qquad h_{02} := \frac{z_2^3}{z_0z_4^2} \in \mathcal{A}_{2D_2}(X \setminus Z(f_{02})),$$

$$f_{03} := \frac{z_0z_3}{z_1^2} \in \mathcal{A}_{2D_1}(X), \qquad h_{03} := \frac{z_3^3}{z_0z_4^2} \in \mathcal{A}_{2D_2}(X \setminus Z(f_{03})),$$

$$f_{04} := \frac{z_0^2z_4}{z_1^3} \in \mathcal{A}_{3D_1}(X), \qquad h_{04} := 1 \in \mathcal{A}_{D_2}(X \setminus Z(f_{04})),$$

$$f_{05} := \frac{z_0^3z_5}{z_1^4} \in \mathcal{A}_{4D_1}(X), \qquad h_{05} := \frac{z_0z_5^3}{z_4^4} \in \mathcal{A}_{4D_2}(X \setminus Z(f_{05})),$$

$$f_{12} := \frac{z_2^2}{z_1z_4} \in \mathcal{A}_{2D_1+D_2}(X), \qquad h_{12} := 1 \in \mathcal{A}_{D_1}(X \setminus Z(f_{12})),$$

$$f_{13} := \frac{z_3^2}{z_1z_4} \in \mathcal{A}_{2D_1+D_2}(X), \qquad h_{13} := 1 \in \mathcal{A}_{D_1}(X \setminus Z(f_{13})),$$

$$f_{14} := 1 \in \mathcal{A}_{D_1+D_2}(X), \qquad h_{14} := 1 \in \mathcal{A}_{D_1}(X \setminus Z(f_{14})),$$

$$
\begin{aligned}
f_{15} &:= \frac{z_1 z_5^2}{z_4^3} \in \mathcal{A}_{3D_2}(X), & h_{15} &:= 1 \in \mathcal{A}_{D_1}(X \setminus Z(f_{15})),\\
f_{24} &:= \frac{z_2^2}{z_1 z_4} \in \mathcal{A}_{D_1+2D_2}(X), & h_{24} &:= 1 \in \mathcal{A}_{D_2}(X \setminus Z(f_{24})),\\
f_{25} &:= \frac{z_2 z_5}{z_4^2} \in \mathcal{A}_{2D_2}(X), & h_{25} &:= \frac{z_2^3}{z_1^2 z_5} \in \mathcal{A}_{2D_1}(X \setminus Z(f_{25})),\\
f_{34} &:= \frac{z_3^2}{z_1 z_4} \in \mathcal{A}_{D_1+2D_2}(X), & h_{34} &:= 1 \in \mathcal{A}_{D_2}(X \setminus Z(f_{34})),\\
f_{35} &:= \frac{z_3 z_5}{z_4^2} \in \mathcal{A}_{2D_2}(X), & h_{35} &:= \frac{z_3^3}{z_1^2 z_5} \in \mathcal{A}_{2D_1}(X \setminus Z(f_{35})),\\
f_{45} &:= \frac{z_1 z_5^2}{z_4^3} \in \mathcal{A}_{4D_2-D_1}(X), & h_{45} &:= 1 \in \mathcal{A}_{D_2}(X \setminus Z(f_{45})).
\end{aligned}
$$

By definition, we have $Z(f_{ij}) = V(X; z_i z_j)$ for the set of zeroes of f_{ij}. Consequently, U is the union of the affine open subsets $X_{ij} := X \setminus Z(f_{ij})$. Moreover, every h_{ij} is invertible over X_{ij}, and the claim follows.

In fact, using $\mathrm{Pic}_T(X) \cong \mathbb{Z}^2$, it is not hard to show that besides the T-invariant open subsets $W \subset X$ admitting a projective quotient variety $W/\!\!/T$, the subset U is the only open subset of the form $X^{ss}(\Lambda)$ with a T-linearized group $\Lambda \subset \mathrm{CDiv}(X)$.

5. Reduction theorems for good quotients

In this section, G is a connected reductive group and the field $\mathbb{K}$ is of characteristic zero. Fix a maximal torus $T \subset G$ and denote by $N(T)$ its normalizer in G. The first result of this section relates existence of a good quotient by G to existence of a good quotient by $N(T)$:

Theorem 5.1. *For a normal G-prevariety X, the following statements are equivalent:*

i) *There is a good quotient $X \to X/\!\!/G$ with a divisorial prevariety $X/\!\!/G$.*
ii) *There is a good quotient $X \to X/\!\!/N(T)$ with a divisorial prevariety $X/\!\!/N(T)$.*

Moreover, if one of these statements holds with a separated quotient space, then so does the other.

Note that if X admits a divisorial good quotient space, then X itself is divisorial. In the second result, we specialize to geometric quotients. Recall that an action of G on X is said to be proper, if the map $G \times X \to X \times X$ sending (g, x) to $(g \cdot x, x)$ is proper.

Theorem 5.2. *Suppose that G acts properly on a $\mathbb{Q}$-factorial variety X. Then the following statements are equivalent:*

i) *There exists a geometric quotient $X \to X/G$.*

ii) *There exists a geometric quotient* $X \to X/N(T)$.

Moreover, if one of these statements holds, then the quotient spaces X/G *and* $X/N(T)$ *are separated* $\mathbb{Q}$*-factorial varieties.*

As an immediate consequence, we obtain the following statement on orbit spaces by the special linear group $\mathrm{SL}_2(\mathbb{K})$, which applies for example to the problem of moduli for n ordered points on the projective line, compare [20] and [4, Section 5]:

Corollary 5.3. *Let* $\mathrm{SL}_2(\mathbb{K})$ *act properly on an open subset* $U \subset X$ *of a* $\mathbb{Q}$*-factorial toric variety* X *such that some maximal torus* $T \subset \mathrm{SL}_2(\mathbb{K})$ *acts by means of a homomorphism* $T \to T_X$ *to the big torus* $T_X \subset X$*. Then there is a geometric quotient* $U \to U/\mathrm{SL}_2(\mathbb{K})$.

Proof. Since $\mathrm{SL}_2(\mathbb{K})$ acts properly, there is a geometric quotient $U \to U/T$. Let $U' \subset X$ be a maximal open subset such that $U \subset U'$ and there is a geometric quotient $U' \to U'/T$. Then the set U' is invariant under the big torus T_X, see e.g. [24, Corollary 2.4]. Thus the geometric quotient space $Y' := U'/T$ is again a toric variety.

In particular, any two points $y, y' \in Y'$ admit a common affine neighbourhood in Y'. But this property is inherited by $Y := U/T$. Thus, since $W := N(T)/T$ is of order two, we obtain a geometric quotient $Y \to Y/W$. The composition of $U \to Y$ and $Y \to Y/W$ is a geometric quotient for the action of $N(T)$ on U. So Theorem 5.2 gives the claim. □

We come to the proof of Theorems 5.1 and 5.2. We make use of the following well known fact on semisimple groups:

Lemma 5.4. *If* G *is semisimple then the character group of* $N(T)$ *is finite.*

Proof. It suffices to show that for each $\widetilde{\chi} \in \mathrm{Char}(N(T))$, the restriction $\chi := \widetilde{\chi}|_T$ is trivial. Clearly χ is fixed under the action of the Weyl group $W = N(T)/T$ on $\mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Char}(T)$ induced by the $N(T)$-action

$$(n \cdot \alpha)(t) := \alpha(n^{-1}tn)$$

on $\mathrm{Char}(T)$. On the other hand, W acts transitively on the set of Weyl chambers associated to the root system determined by $T \subset G$. Consequently, χ lies in the closure of every Weyl chamber and hence is trivial. □

Proof of Theorem 5.1. The implication "i)⇒ii)" is easy, use [21, Lemma 4.1]. To prove the converse, we first reduce to the case that G is semisimple: Let $R \subset G$ be the radical of G. Then R is a torus, and we have $R \subset T$. In particular, there is a good quotient $X \to X'$ for the action of R on X. Thus we obtain a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "/\!\!/ N(T)"] \arrow[dr, "/\!\!/ R"'] & & X/\!\!/ N(T) \\ & X' \arrow[ur] & \end{tikzcd}$$

Consider the induced action of the connected semisimple group $G' := G/R$ on X'. The image T' of T under the projection $G \to G'$ is a maximal torus of G'. Moreover, $N(T')$ is the image of $N(T)$ under $G \to G'$. Thus, the upwards arrow of the above diagram is a good quotient for the action of $N(T')$ on X'.

To proceed, we only have to derive from the existence of a good quotient $X' \to X'/\!\!/ N(T')$ that there is a good quotient $X' \to X'/\!\!/ G'$ with a divisorial prevariety $X'/\!\!/ G'$. In other words, we may assume from the beginning that the group G is semisimple.

Let $p\colon X \to X/\!\!/ N(T)$ denote the good quotient. Using Lemmas 4.2 and 4.3 we can construct a canonically $N(T)$-linearized ample group $\Lambda \subset X$ consisting of $N(T)$-tame divisors such that we have

$$X^{ss}(\Lambda, N(T)) = X.$$

Note that this equality also holds for any subgroup $\Lambda' \subset \Lambda$ of finite index in Λ. We construct now such a subgroup $\Lambda' \subset \Lambda$ for which the canonical $N(T)$-linearization of Λ' extends to a strong G-linearization. The first step is to realize X as an open G-invariant subset of a certain G-prevariety Y with $\mathcal{O}(Y) = \mathbb{K}$.

Consider the maximal separated open subsets $X_1, \ldots , X_m \subset X$, see [3, Theorem I]. Since G is connected, it leaves these sets invariant. By Sumihiro's Equivariant Completion Theorem [23, Theorem 3], we find G-equivariant open embeddings $X_i \to Z_i$ into complete G-varieties Z_i. Applying equivariant normalization, we achieve that each Z_i is normal.

Let Y_i denote the union of X_i with the set of regular points of Z_i. Note that $\mathcal{O}(Y_i) = \mathbb{K}$. Define Y to be the G-equivariant gluing of the varieties Y_i along the invariant open subsets $X_i \subset Y_i$. Then we have $\mathcal{O}(Y) = \mathbb{K}$. Moreover, all points of $Y \setminus X$ are regular points of Y.

By closing components, every Cartier divisor $D \in \Lambda$ extends to a Cartier divisor on Y. Let $\Gamma \subset \mathrm{CDiv}(Y)$ denote the (free) group of Cartier divisors generated by these extensions. Lemma 4.2 ensures that the canonical $N(T)$-linearization of Λ extends to a canonical $N(T)$-linearization of the group Γ. By [23, Corollary 2] and Proposition 1.3, this linearization is even a strong one.

We claim that some subgroup $\Gamma' \subset \Gamma$ of finite index admits a strong G-linearization. Let $\mathcal{B}$ be the graded $\mathcal{O}_Y$-algebra associated to Γ. For each homogeneous component $\mathcal{B}_{E,i} := \mathcal{B}_E|_{Y_i}$, some power $\mathcal{B}_{nE,i}$ admits a G-linearization as in [16, Proposition 2.4]. Since G is semisimple, these linearizations are unique, see [19, Proposition 1.4]. Thus they define G-sheaf structures on the $\mathcal{O}_{Y_i}$-algebras

$$\mathcal{B}_i := \bigoplus_{E \in \Gamma_i} \mathcal{B}_{E,i},$$

for suitable subgroups $\Gamma_i \subset \Gamma$ of finite index. Again by uniqueness of strong G-linearizations, we can patch the above G-sheaf structures together to the desired strong G-linearization on the intersection $\Gamma' \subset \Gamma$ of the subgroups $\Gamma_i \subset \Gamma$, and our claim is proved.

Now, since the character group of $N(T)$ is finite, Proposition 1.5 tells us that on some subgroup $\Gamma'' \subset \Gamma'$ of finite index, the canonical $N(T)$-linearization and the one induced by the G-linearization coincide. Thus restricting Γ'' to X provides the desired subgroup $\Lambda' \subset \Lambda$ of finite index. We replace Λ with Λ'.

In order to obtain a quotient of X by G, we want to apply Theorem 3.1. So we have to show that $X^{ss}(\Lambda, G)$ equals X. For this, let $\mathcal{A}$ be the graded $\mathcal{O}_X$-algebra associated to Λ, and set $\widehat{X} := \operatorname{Spec}(\mathcal{A})$. Moreover, let $q\colon \widehat{X} \to X$ be the canonical map and $H := \operatorname{Spec}(\mathbb{K}[\Lambda])$ the torus acting on $\widehat{X}$. Note that

$$X^{ss}(\Lambda, T) = X^{ss}(\Lambda, N(T)) = X.$$

Choose G-invariant homogeneous $f_1, \ldots, f_r \in \mathcal{A}(X)$ and T-invariant homogeneous $h_1, \ldots, h_s \in \mathcal{A}(X)$ such that the complements $X \setminus Z(f_i)$ and $X \setminus Z(h_i)$ satisfy the condition of Definition 2.1 i) and

$$\begin{aligned} X^{ss}(\Lambda, G) &= (X \setminus Z(f_1)) \cup \ldots \cup (X \setminus Z(f_r)), \\ X^{ss}(\Lambda, T) &= (X \setminus Z(h_1)) \cup \ldots \cup (X \setminus Z(h_r)). \end{aligned}$$

Since Λ is ample, Proposition 1.9 yields a $(G \times H)$-equivariant affine closure $\overline{X}$ of $\widehat{X}$ such that the f_i and the h_j extend to regular functions on $\overline{X}$ satisfying $\overline{X}_{f_i} = \widehat{X}_{f_i}$ and $\overline{X}_{h_j} = \widehat{X}_{h_j}$. Moreover, we obtain a commutative diagram of H-equivariant maps:

$$\begin{tikzcd} \overline{X} \arrow[rr, "\overline{p}_G", "/\!\!/ G"'] \arrow[dr, "/\!\!/ T", "\overline{p}_T"'] & & \overline{X} /\!\!/ G \\ & \overline{X} /\!\!/ T \arrow[ur] & \end{tikzcd}$$

Now, let $x \in X$, and assume that x is not semistable with respect to G. Choose $z \in q^{-1}(x)$, and let $y := \overline{p}_G(z)$. By Proposition 2.3 ii), the assumption $x \notin X^{ss}(\Lambda, G)$ amounts to $y \in \overline{p}_G(\overline{X} \setminus \widehat{X})$ or to an isotropy group H_y of positive dimension.

First suppose that we have $y \in \overline{p}_G(\overline{X} \setminus \widehat{X})$. Let $G \cdot z'$ be the closed orbit in $\overline{p}_G^{-1}(y)$. Then $G \cdot z'$ is contained in $\overline{X} \setminus \widehat{X}$. Moreover, the Hilbert-Mumford-Birkes Lemma [10], provides a maximal torus $T' \subset G$ such that the closure of $T' \cdot z$ intersects $G \cdot z'$.

Let $g \in G$ with $gT'g^{-1} = T$. Then the closure of $T{\cdot}g{\cdot}z$ contains a point $z'' \in G{\cdot}z'$. Surely, $\overline{p}_T(g{\cdot}z)$ equals $\overline{p}_T(z'')$. Thus, since $z'' \in \overline{X} \setminus \widehat{X}$, Proposition 2.3 ii) tells us that $g{\cdot}x = q(g{\cdot}z)$ is not semistable with respect to T. A contradiction.

As the situation $y \in \overline{p}_G(\overline{X} \setminus \widehat{X})$ is excluded, the isotropy group H_y is of positive dimension, and the whole fibre $\overline{p}_G^{-1}(y)$ is contained in $\widehat{X}$. Let $H_0 \subset H_y$ be the connected component of the neutral element. Then H_0 acts freely on the fibre $\overline{p}_G^{-1}(y)$, and the closed orbit $G{\cdot}z' \subset \overline{p}_G^{-1}(y)$ is invariant by H_0.

Let $\mu\colon g \mapsto g{\cdot}z'$ denote the orbit map. Since the actions of G and H_0 commute, $G' := \mu^{-1}(H_0{\cdot}z')$ is a subgroup of G. Since $H_0{\cdot}z' \cong H_0$, there is a torus $S' \subset G'$ with $\mu(S') = H_0{\cdot}z'$, use for example [11, Proposition IV.11.20].

Let $T' \subset G$ be a maximal torus with $S' \subset T'$ and choose $g \in G$ with $T = gT'g^{-1}$. Then $H_0{\cdot}g{\cdot}z'$ equals $(gS'g^{-1}){\cdot}g{\cdot}z'$. According to Proposition 2.3 ii), the point $q(g{\cdot}z')$ is not semistable with respect to T. A contradiction. So, every $x \in X$ is semistable with respect to G, and the implication "ii)⇒i)" is proved.

We come to the supplement concerning separateness. Clearly, existence of a good quotient $X \to X/\!\!/G$ with $X/\!\!/G$ separated implies that also the quotient space $X/\!\!/N(T)$ is separated.

For the converse, suppose that $X \to X/\!\!/N(T)$ exists with a separated divisorial $X/\!\!/N(T)$. Then there is a good quotient $X \to X/\!\!/T$ with a separated quotient space $X/\!\!/T$, and [6, Theorem 5.4] implies that also the quotient space $X/\!\!/G$ is separated. □

In the proof of Theorem 5.2, we shall use that geometric quotient spaces of proper actions inherit $\mathbb{Q}$-factoriality. By the lack of a reference for this presumably well-known fact, we give here a proof:

Lemma 5.5. *Suppose that a reductive group H acts regularly with finite isotropy groups on a variety Y and that there is a geometric quotient $p\colon Y \to Y/H$. If Y is $\mathbb{Q}$-factorial, then so is Y/H.*

Proof. Assume that Y is $\mathbb{Q}$-factorial, and let $E \subset Y/H$ be a prime divisor. Then $p^{-1}(E)$ is a union of prime divisors $D_1, \ldots, D_r$. Some multiple mD of the divisor $D := D_1 + \ldots + D_r$ is Cartier. Using Lemma 4.2 and Proposition 1.3, we see that the group of Cartier divisors generated by mD is canonically strongly H-linearized.

Enlarging m, we achieve that the sheaf $\mathcal{A}_{mD}$ is equivariantly isomorphic to the pullback $p^*(\mathcal{L})$ of some invertible sheaf $\mathcal{L}$ on Y/H, use e.g. [17, Proposition 4.2]. The canonical section $1 \in \mathcal{A}_{mD}(Y)$ is H-invariant and hence induces a section $f \in \mathcal{L}(Y/H)$ having precisely E as its set of zeroes. □

Proof of Theorem 5.2. If one of the quotients exists, then by [19, Section 0.4] and Lemma 5.5, the quotient space is separated and $\mathbb{Q}$-factorial. Now, existence of a geometric quotient $X \to X/G$ surely implies existence of a geometric quotient $X/N(T)$. Conversely, if $X/N(T)$ exists, then it is $\mathbb{Q}$-factorial. Hence Theorem 5.1 yields a geometric quotient $X \to X/G$. □

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