Lifting Galois Representations, and a Conjecture of Fontaine and Mazur

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Abstract. Mumford has constructed 4-dimensional abelian varieties with trivial endomorphism ring, but whose Mumford–Tate group is much smaller than the full symplectic group. We consider such an abelian variety, defined over a number field F, and study the associated p-adic Galois representation. For F sufficiently large, this representation can be lifted to $\mathbf{G}_m(\mathbf{Q}_p) \times \mathrm{SL}_2(\mathbf{Q}_p)^3$.

Such liftings can be used to construct Galois representations which are geometric in the sense of a conjecture of Fontaine and Mazur. The conjecture in question predicts that these representations should come from algebraic geometry. We confirm the conjecture for the representations constructed here.

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Introduction

A construction due to Mumford shows that the Mumford–Tate group of a polarized abelian variety $A/\mathbf{C}$ is not, in general, determined by the polarization and the endomorphism ring of A. This construction, which can be found in [Mum69], proves the existence of families of polarized abelian fourfolds where the general fibre has trivial endomorphism algebra but Mumford–Tate group much smaller than the full symplectic group of $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$.

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Previous papers, [Noo95] and [Noo00], brought to light some properties of an abelian variety A arising from this construction. For example, if A can be defined over a number field F, then it can be shown that, after replacing F by a finite extension, A has good reduction at all non-archimedean places of F and ordinary reduction at "most" of those places. Many of these results are derived from the property that the p-adic Galois representations associated to A factor through maps $\rho_p \colon \mathcal{G}_F \to G(\mathbf{Q}_p)$, where G is the Mumford–Tate group of A. They are therefore to a large extent results on the properties of Galois representations of "Mumford's type".

The present work continues the study of the Galois representations in question, in particular their lifting properties. In this respect, this paper is heavily inspired by Wintenberger's wonderful paper [Win95], even though the results proved therein are never actually used. In fact, the conditions of the main theorem (1.1.3) of that paper are not verified in the case considered here and, indeed, our conclusions are weaker as well.

Let G be the Mumford–Tate group of an abelian variety A over a number field F arising from Mumford's construction. It admits a central isogeny $\widetilde{G} \to G$, where the group $\widetilde{G}$ is (geometrically) a product:

$$\widetilde{G}_{\overline{\mathbf{Q}}} \cong \mathbf{G}_m \times \mathrm{SL}_2^3.$$

We fix a prime number p such that $\widetilde{G}$ decomposes over $\mathbf{Q}_p$ and we attempt to lift the p-adic Galois representation ρ associated to A along the isogeny $\widetilde{G}(\mathbf{Q}_p) \to G(\mathbf{Q}_p)$. This is possible after replacing F by a finite extension. The resulting lift $\tilde{\rho}$ naturally decomposes as a direct sum of a character and three 2-dimensional representations. These representations are studied in section 1. They have rather nice properties but for each one of them there exists a finite place of F where it is not potentially semi-stable. To be precise, the 2-dimensional representations are potentially unramified at all finite places of F of residue characteristic different from p, trivial at certain p-adic places of F but not potentially semi-stable (not even of Hodge–Tate type) at the other p-adic places.

In section 1, the properties of the lifting $\tilde{\rho}$ and its direct factors are studied without using the fact that ρ comes from an abelian variety A with Mumford–Tate group G. If we do use that fact, we can say more about the representations in question. This is the subject of section 2, where it is shown in particular how to distinguish the p-adic places where a given 2-dimensional direct factor of $\tilde{\rho}$ is trivial, from the p-adic places where it is not. The results of section 2 are used in section 3 to show that considering two abelian varieties A and A' of Mumford's kind, and taking a tensor product of two representations of the type considered above, one obtains representations which are potentially crystalline at all p-adic places of F and are potentially unramified at all other finite places of F. These representations are unramified outside a finite set of non-archimedean places of F. For a suitable choice of the varieties A and A', one obtains an irreducible representation.

According to a conjecture of Fontaine and Mazur ([FM95, conjecture 1], see section 0), such a representation should "come from algebraic geometry", i. e. be a subquotient of a Tate twist of an étale cohomology group of an algebraic variety over F. Our construction is geometric in some large sense, but not in the sense of the conjecture. We show in the beginning of section 4 that, in general, our representations do not arise (in the sense of Fontaine and Mazur) from the geometric objects having served in their construction.

It is therefore not clear if the conjecture holds in this case. Nevertheless, it is shown in section 4 that our representations do come from geometry, even from the geometry of abelian varieties, in the strong sense of Fontaine and Mazur. The construction of these abelian varieties is based on a construction going back to Shimura, cf. [Del72, §6].

In the final section, some generalizations of the construction of geometric representations are studied. It is shown in particular that any tensor product of the representations constructed in section 2 which is geometric comes from algebraic geometry.

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0 NOTATIONS AND TERMINOLOGY

0.1 NOTATIONS. For any field F, we denote by $\bar{F}$ an algebraic closure of F and we write $\mathcal{G}_F = \mathrm{Gal}(\bar{F}/F) = \mathrm{Aut}_F(\bar{F})$, the group of F-automorphisms of $\bar{F}$. If $\bar{v}$ is a valuation of $\bar{F}$, then $\mathcal{I}_{F,\bar{v}} \subset \mathcal{D}_{F,\bar{v}} \subset \mathcal{G}_F$ are the inertia group and decomposition group of $\bar{v}$. If F is a local field, we write just $\mathcal{I}_F \subset \mathcal{G}_F$ for the inertia subgroup.

In the case of the field $\mathbf{Q}$ of rational numbers, we let $\overline{\mathbf{Q}}$ be its algebraic closure in the field of complex numbers $\mathbf{C}$. As usual, for any prime number p, $\mathbf{Q}_p$ is the p-adic completion of $\mathbf{Q}$ and $\mathbf{C}_p$ is the completion of $\overline{\mathbf{Q}}_p$.

0.2 REPRESENTATIONS OF MUMFORD'S TYPE. Let K be a field of characteristic 0, let G be an algebraic group over K and let V be a faithful K-linear representation of G. As in [Noo00, 1.2], we will say that the pair (G, V) is *of Mumford's type* if

- $\mathrm{Lie}(G)$ has one dimensional centre $\mathfrak{c}$,
- $\mathrm{Lie}(G)_{\bar{K}} \cong \mathfrak{c}_{\bar{K}} \oplus \mathfrak{sl}^3_{2,\bar{K}}$ and
- $\mathrm{Lie}(G)_{\bar{K}}$ acts on $V_{\bar{K}}$ by the tensor product of the standard representations.

We do not require G to be connected.

0.3 A conjecture of Fontaine and Mazur. Let F be a number field and let $\mathcal{G}_F = \mathrm{Aut}_F(\bar{F})$. Fix a prime number p and suppose that ρ is a finite dimensional $\mathbf{Q}_p$-linear representation of $\mathcal{G}_F$. Following Fontaine and Mazur in [FM95, §1], we will say that ρ is *geometric* if

- it is unramified outside a finite set of non-archimedean places of F and
- for each non-archimedean valuation $\bar{v}$ of $\bar{F}$, the restriction of ρ to the decomposition group $\mathcal{D}_{F,\bar{v}}$ is potentially semi-stable.

The meaning of the second condition depends on $\bar{v}$. If $\bar{v}(p) > 0$, then the notion of potential semi-stability is the one defined by Fontaine, see for example [Fon94]. If $\bar{v}(p) = 0$ then the fact that ρ is potentially semi-stable means that the restriction of ρ to the inertia group $\mathcal{I}_{F,\bar{v}}$ is quasi-unipotent.
By Grothendieck's theorem on semi-stability of p-adic Galois representations, the condition of potential semi-stability at $\bar{v}$ is automatically verified in the case where $\bar{v}(p) = 0$. We can therefore restrict our attention to valuations $\bar{v}$ with $\bar{v}|p$. The only fact needed in what follows is that a crystalline representation of $\mathcal{D}_{F,\bar{v}}$ is semi-stable and hence that a potentially crystalline representation is potentially semi-stable.
We say that a finite dimensional, irreducible, $\mathbf{Q}_p$-linear representation of $\mathcal{G}_F$ *comes from algebraic geometry* if it is isomorphic to a subquotient of a Tate twist of an étale cohomology group of an algebraic variety over F. We cite conjecture 1 from [FM95].

0.4. Conjecture (Fontaine–Mazur). *Let F be a number field. An irreducible p-adic representation of $\mathcal{G}_F$ is geometric if and only if it comes from algebraic geometry.*

0.5 The "if" part of the conjecture is true. A representation coming from a proper and smooth F-variety having potentially semi-stable reduction at all places of residue characteristic p is semi-stable by the C_{pst}-conjecture, [Fon94], proven by Tsuji in [Tsu99]. The general case of this part of the conjecture (for irreducible representations) follows by [dJ96]. The "only if" part is known for potentially abelian representations by [FM95, §6]. In [Tay00], Taylor proves the conjecture for representations of $\mathcal{G}_{\mathbf{Q}}$ with values in $\mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ satisfying some supplementary conditions. These results can be generalized to certain two-dimensional $\overline{\mathbf{Q}}_p$-linear representations of $\mathcal{G}_F$, for totally real number fields F.

0.6 Remark. Let F' be a finite extension of F. It follows from [FM95, §4], Remark (b) that the conjecture for representations of $\mathcal{G}_F$ is equivalent to the conjecture for representations of $\mathcal{G}_{F'}$.
For the implication which is still open, this can be seen as follows. Let ρ be a representation such that a Tate twist $\rho_{|\mathcal{G}_{F'}}(n)$ of $\rho_{|\mathcal{G}_{F'}}$ is a subquotient of an

étale cohomology group of an algebraic variety X over F'. Then

$$\mathrm{Ind}_{\mathcal{G}_{F'}}^{\mathcal{G}_F}\left(\rho_{|\mathcal{G}_{F'}}(n)\right)$$

is a subquotient of an étale cohomology group of $\mathrm{Res}_{F'/F}(X)$. The representation $\rho(n)$ is a subquotient of this induced representation.

1 Lifting Galois representations of Mumford's type

1.1 Let p be a prime number, $G_p/\mathbf{Q}_p$ an algebraic group and V_p a faithful representation of G_p such that (G_p, V_p) is of Mumford's type in the sense of 0.2. Throughout this section we will assume that F is a number field or a finite extension of $\mathbf{Q}_\ell$ for some prime number ℓ and that $\rho\colon \mathcal{G}_F \to G_p(\mathbf{Q}_p)$ is a polarizable, continuous representation. The condition that ρ is polarizable means, by definition, that there exists a non-degenerate, alternating, bilinear, $\mathcal{G}_F$-equivariant map $V \times V \to \mathbf{Q}_p(-1)$. Here $\mathbf{Q}_p(-1)$ is the 1-dimensional $\mathbf{Q}_p$-linear representation where $\mathcal{G}_F$ acts through χ^{-1}, the inverse of the cyclotomic character.
We will assume moreover that G_p is connected and that $\mathrm{Lie}(G_p) \cong \mathfrak{c} \oplus \mathfrak{sl}_2^3$. In this case, there exists a central isogeny

$$N\colon \widetilde{G}_p = \mathbf{G}_m \times \mathrm{SL}_2^3 \longrightarrow G_p$$

such that the induced representation of $\widetilde{G}_p$ on V_p is isomorphic to the tensor product $V_0 \otimes V_1 \otimes V_2 \otimes V_3$ of the standard representations of the factors.
We fix a $\mathcal{G}_F$-stable $\mathbf{Z}_p$-lattice $V_{\mathbf{Z}_p} \subset V_p$ such that there exists lattices $V_{i,\mathbf{Z}_p} \subset V_i$ with $V_{\mathbf{Z}_p} = \otimes_{i=1}^4 V_{i,\mathbf{Z}_p}$ and such that, on $V_{\mathbf{Z}_p}$, the polarization form takes values in $\mathbf{Z}_p(-1)$.

1.2 Definition. We say that $\rho(\mathcal{G}_F)$ is *sufficiently small* if it stabilizes $V_{\mathbf{Z}_p}$ and if all elements of $\rho(\mathcal{G}_F)$ are congruent to 1 mod p if $p > 2$ and congruent to 1 mod 4 if $p = 2$.

1.3 Remarks.

1.3.1 For any polarizable, continuous representation $\rho\colon \mathcal{G}_F \to G_p(\mathbf{Q}_p)$ as in the beginning of 1.1, there exist a lattice $V_{\mathbf{Z}_p}$ as in 1.1 and a finite extension $F' \supset F$ such that $\rho(\mathcal{G}_{F'})$ is sufficiently small.

1.3.2 If $\rho(\mathcal{G}_F)$ is sufficiently small, then it does not contain any non-trivial elements of finite order. This implies in particular that ρ is unramified at a finite place v of F of residue characteristic different from p if and only if it is potentially unramified at v.

1.4. Lemma. *If $\rho(\mathcal{G}_F)$ is sufficiently small, then ρ lifts uniquely to a continuous group homomorphism $\tilde{\rho}\colon \mathcal{G}_F \to \widetilde{G}_p(\mathbf{Q}_p)$ with the property that all elements of $\tilde{\rho}(\mathcal{G}_F)$ are congruent to* 1 mod p *if* $p > 2$ *and to* 1 mod 4 *if* $p = 2$.

Proof. The map $\mathrm{GL}_2^3 \to G_p$ obtained by taking the tensor product of the standard representations is surjective on $\mathbf{Q}_p$-valued points and even on $\mathbf{Q}_p$-valued points of GL_2^3 and G_p satisfying the above congruence condition. By Hensel's lemma, the determinant of an element of $\mathrm{GL}_2(\mathbf{Q}_p)$ which satisfies this congruence condition is the square of an element of $\mathbf{G}_m(\mathbf{Q}_p)$ satisfying the same condition, which proves that if $\rho(\mathcal{G}_F)$ is sufficiently small, then any element lifts to an element of $\widetilde{G}_p(\mathbf{Q}_p)$, congruent to 1 mod $2p$.
One has $\ker(N) = \{(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3) \mid \epsilon_i = \pm 1, \epsilon_0\epsilon_1\epsilon_2\epsilon_3 = 1\}$. As the congruence condition implies that $-1 \not\equiv 1$, the lifting with this property is unique so we obtain the unique lift $\tilde{\rho}\colon \mathcal{G}_F \to \widetilde{G}_p(\mathbf{Q}_p)$ from the lemma, for the moment just a continuous map. The uniqueness of the lifting from $G_p(\mathbf{Q}_p)$ to $\widetilde{G}_p(\mathbf{Q}_p)$ implies that $\tilde{\rho}$ is a group homomorphism. $\square$

1.5 DEFINITION. If $\rho(\mathcal{G}_F)$ is sufficiently small, then we call the lifting $\tilde{\rho}\colon \mathcal{G}_F \to \widetilde{G}_p(\mathbf{Q}_p)$ of the lemma the *canonical lifting* of ρ.

1.6 THE REPRESENTATIONS ρ_i. For any $\rho\colon \mathcal{G}_F \to G_p(\mathbf{Q}_p)$ and any lifting $\tilde{\rho}\colon \mathcal{G}_F \to \widetilde{G}_p(\mathbf{Q}_p)$, there exists a finite extension F' of F such that $\rho(\mathcal{G}_{F'})$ is sufficiently small and such that the restriction of $\tilde{\rho}$ to $\mathcal{G}_{F'}$ is its canonical lifting. Replacing F by such a finite extension F', we will assume in the sequel that this is already the case over F.
By composing $\tilde{\rho}$ with the projections on the factors of $\widetilde{G}_p(\mathbf{Q}_p)$, we get a character $\rho_0\colon \mathcal{G}_F \to \mathbf{G}_m(\mathbf{Q}_p)$ and representations $\rho_i\colon \mathcal{G}_F \to \mathrm{SL}_2(\mathbf{Q}_p)$ for $i = 1, 2, 3$. The facts that the image of ρ is sufficiently small and that $\tilde{\rho}$ is its canonical lift imply that the $\rho_i(\mathcal{G}_F)$ do not contain any elements of finite order other than the identity.
The following lemma is obvious.

1.7. LEMMA. *Let notations and hypotheses be as in 1.6. In particular, $\rho(\mathcal{G}_F)$ is sufficiently small and $\tilde{\rho}$ is its canonical lifting. If ρ is unramified at some finite place of F of residue characteristic different from p, then so are $\tilde{\rho}$ and the ρ_i.*

1.8. LEMMA. *Let ρ, $\tilde{\rho}$ and the ρ_i (for $i = 0, 1, 2, 3$) be as in 1.6 and lemma 1.7. Then the character ρ_0 satisfies $\rho_0^2 = \chi^{-1}$, where $\chi\colon \mathcal{G}_F \to \mathbf{Z}_p^*$ is the cyclotomic character.*

Proof. The condition that ρ is polarizable means that it respects a symplectic form up to a scalar and that the multiplier is the inverse of the cyclotomic character. This means that the determinant of ρ is χ^{-4}. The representation $\rho_1 \otimes \rho_2 \otimes \rho_3$ has trivial determinant, so the determinant of its product by ρ_0 is ρ_0^8 and hence $\rho_0^8 = \chi^{-4}$. The lemma follows because $\rho_0(\mathcal{G}_F)$ and $\rho(\mathcal{G}_F)$ and hence $\chi(\mathcal{G}_F)$ are congruent to 1 mod $2p$. $\square$

1.9. Proposition. *Under the assumptions of the lemmas, let v be a p-adic valuation of F and let $\mathcal{I} = \mathcal{I}_{F,\bar{v}}$ be the inertia group of a valuation $\bar{v}$ of $\bar{F}$ lying over v. Assume that the restriction of ρ to $\mathcal{I}$ is Hodge–Tate (resp. crystalline). For $i \in \{1, 2, 3\}$, either*

- *the restriction of ρ_i to $\mathcal{I}$ is trivial, or*
- *the restriction of $\rho_0 \cdot \rho_i$ to $\mathcal{I}$ is Hodge–Tate (resp. crystalline).*

Proof. One has an isomorphism $G_p^{\mathrm{ad}}(\mathbf{Q}_p) \cong \mathrm{PSL}_2(\mathbf{Q}_p)^3$ and it can be shown as in [Noo00, 3.5] that the projection of $\rho(\mathcal{I})$ on exactly two of the factors is trivial. We sketch the argument.
Let $\mu_{\mathrm{HT}} \colon \mathbf{G}_{m,\mathbf{C}_p} \to G_{p,\mathbf{C}_p}$ be the cocharacter associated to the Hodge–Tate decomposition of $\rho_{|\mathcal{I}}$. As the kernel of the isogeny $N \colon \widetilde{G}_p \to G_p$ is annihilated by 2, the square

$$\mu_{\mathrm{HT}}^2 \colon \mathbf{G}_{m,\mathbf{C}_p} \to G_{p,\mathbf{C}_p}$$

lifts to a cocharacter $\tilde{\mu} \colon \mathbf{G}_{m,\mathbf{C}_p} \to \widetilde{G}_{p,\mathbf{C}_p}$. One has $\tilde{\mu} = (\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3)$, where $\tilde{\mu}_0 \colon \mathbf{G}_{m,\mathbf{C}_p} \to \mathbf{G}_{m,\mathbf{C}_p}$ and $\tilde{\mu}_i \colon \mathbf{G}_{m,\mathbf{C}_p} \to \mathrm{SL}_{2,\mathbf{C}_p}$ for $i = 1, 2, 3$. Since $V_{p,\mathbf{C}_p}$ is the direct sum of two eigenspaces for μ_{HT} and hence for $\tilde{\mu}$, exactly one of the maps $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3$ is non-trivial. The Zariski closure of the image of μ_{HT} therefore projects non-trivially on exactly one of the factors of $G^{\mathrm{ad}}_{p,\mathbf{Q}_p} \cong (\mathrm{PSL}_{2,\mathbf{Q}_p})^3$. The theorem of Sen, [Ser78, Théorème 2], implies the statement concerning the projection of $\rho(\mathcal{I})$.
To end the proof, we can assume that the projection of $\rho(\mathcal{I})$ on the first factor of $G_p^{\mathrm{ad}}(\mathbf{Q}_p)$ is non-trivial. The hypotheses that $\rho(\mathcal{G}_F)$ is sufficiently small and that $\tilde{\rho}$ is the canonical lifting of ρ imply that $(\rho_2)_{|\mathcal{I}}$ and $(\rho_3)_{|\mathcal{I}}$ are trivial. Since the tensor product $\rho_{|\mathcal{I}} = (\rho_0\rho_1)_{|\mathcal{I}} \otimes (\rho_2 \otimes \rho_3)_{|\mathcal{I}}$ is Hodge–Tate (resp. crystalline), this implies that $(\rho_0\rho_1)_{|\mathcal{I}}$ is Hodge–Tate (resp. crystalline) as well. □

1.10 Definition. Notations being as above, let $1 \le i \le 3$ and let $\bar{v}$ be a p-adic valuation of $\bar{F}$. We say that $(\rho_i)_{|\mathcal{I}_{F,\bar{v}}}$ is *of the first kind* if $(\rho_0\rho_i)_{|\mathcal{I}_{F,\bar{v}}}$ is potentially crystalline. We say that $(\rho_i)_{|\mathcal{I}_{F,\bar{v}}}$ is *of the second kind* if its image is trivial.

1.11 Remark. Let (G_p, V_p) be of Mumford's type as in 1.1, let F be a number field and assume that A/F is a polarizable abelian fourfold such that, for some identification $V_p = \mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$, the Galois representation of $\mathcal{G}_F$ on V_p factors through a map $\rho \colon \mathcal{G}_F \to G_p(\mathbf{Q}_p)$. Then ρ is a representation of the type considered in 1.1. This is the setting in which the results of this section will be applied.
Abelian varieties with this property exist by [Mum69, §4] and [Noo95, 1.7]. By [Noo00, 2.2], such varieties have potentially good reduction at all non-archimedean places of F. This implies that for each p-adic place $\bar{v}$ of $\bar{F}$, the restriction of ρ to the decomposition group $\mathcal{D}_{F,\bar{v}} \subset \mathcal{G}_F$ is potentially crystalline. At all other finite places, ρ is potentially unramified in any case and unramified if $\rho(\mathcal{G}_F)$ is sufficiently small.

2 Galois representations of abelian varieties of Mumford's type

2.1 Fix an abelian fourfold A over a number field F and an embedding $F \subset \mathbf{C}$. Let $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ and let $G \subset \mathrm{GL}(V)$ be an algebraic subgroup such that (G, V) is of Mumford's type, with G connected. We assume that the morphism $h\colon S \to \mathrm{GL}(V \otimes \mathbf{R})$ determining the Hodge structure of $A_{\mathbf{C}}$ factors through $G_{\mathbf{R}}$. This condition is equivalent to the condition that the Mumford–Tate group of $A_{\mathbf{C}}$ is contained in G.
Such abelian varieties exist by [Mum69, §4], and [Noo95] implies that there also exist abelian varieties of this type with the additional property that the Mumford–Tate group is equal to G. It is explained in [Noo01, 1.5] that any abelian variety satisfying the above condition whose Mumford–Tate group is equal to G can be obtained as a fibre of one of the families constructed in [Mum69, §4]. As in [Noo01] one can draw the following conclusions.
There is a central isogeny $\widetilde{G} \to G$, where $\widetilde{G} = \mathbf{G}_m \times \widetilde{G}^{\mathrm{der}}$ and $\widetilde{G}^{\mathrm{der}}$ is an algebraic group over $\mathbf{Q}$ such that $\widetilde{G}^{\mathrm{der}}_{\overline{\mathbf{Q}}} \cong \mathrm{SL}^3_{2,\overline{\mathbf{Q}}}$. The group $\widetilde{G}^{\mathrm{der}}$ can be described in the following way. There exist a totally real cubic number field K and a quaternion division algebra D with centre K, with $\mathrm{Cor}_{K/\mathbf{Q}}(D) \cong \mathrm{M}_8(\mathbf{Q})$ and with $D \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathrm{M}_2(\mathbf{R}) \times \mathbf{H} \times \mathbf{H}$ such that

$$\widetilde{G}^{\mathrm{der}} = \{x \in D \mid \mathrm{Nrd}(x) = 1\},$$

considered as an algebraic group over $\mathbf{Q}$. Here $\mathbf{H}$ is the algebra of real quaternions and $\mathrm{Nrd}\colon D \to K$ is the reduced norm. The group G can be identified with the image of $D^\times$ in $\mathrm{Cor}_{K/\mathbf{Q}}(D) \cong \mathrm{M}_8(\mathbf{Q})$ under the norm map $N\colon D \to \mathrm{Cor}_{K/\mathbf{Q}}(D)$ and therefore has a natural 8-dimensional representation. This representation is isomorphic to V. Note that for any field L containing a normal closure of K, one has

$$\widetilde{G}^{\mathrm{der}}_L \cong \prod_{\varphi\colon K \hookrightarrow L} G_\varphi,$$

where each G_φ is an L-form of SL_2.
In what follows, we will refer to K and D as the number field and the division algebra *associated to G*.

2.2 The Hodge decomposition on $V_{\mathbf{C}} = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{C}) \cong \mathrm{H}^1_{\mathrm{dR}}(A_{\mathbf{C}}/\mathbf{C})$ is determined by a cocharacter

$$\mu_{\mathrm{HdR}}\colon \mathbf{G}_{m,\mathbf{C}} \longrightarrow G_{\mathbf{C}}$$

such that $\mathbf{G}_m$ acts by the character $\cdot^i$ on the subspace $H^{i,1-i} \subset \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{C})$. Let $C_{\mathrm{HdR}} \subset G_{\mathbf{C}}$ be the conjugacy class of μ_{HdR}. It follows from [Noo01, 1.2] (and is easy to check) that the field of definition in $\mathbf{C}$ of C_{HdR} is isomorphic to K. In what follows we will identify K with this subfield of $\mathbf{C}$ and thus assume that $K \subset \mathbf{C}$. One deduces that C_{HdR} can be defined over $F \subset \mathbf{C}$ if and only if $K \subset F$, and hence that, up to replacing F by a finite extension, one can

assume that C_{HdR} can be defined over F. From now on, we will assume that this is the case.
The arguments used in the proof of proposition 1.9 also apply to μ_{HdR}. Thus μ_{HdR}^2 lifts to a map $\tilde{\mu}\colon \mathbf{G}_{m,\mathbf{C}} \to \widetilde{G}_{\mathbf{C}}$. As in *loc. cit.*, the fact that there are two eigenspaces in $V_{\mathbf{C}}$ for the action of $\mathbf{G}_m$ implies that the projection of $\tilde{\mu}$ on one and only one factor $\mathrm{SL}_{2,\mathbf{C}}$ of $\widetilde{G}_{\mathbf{C}}$ is non trivial. This implies that C_{HdR} projects non-trivially on exactly one factor $\mathrm{PSL}_{2,\mathbf{C}}$ of $G_{\mathbf{C}}^{\mathrm{ad}}$, namely the factor corresponding to the embedding $K \subset \mathbf{C}$ fixed above.

2.3 Let p be a prime number such that $\widetilde{G}_{\mathbf{Q}_p} \cong \mathbf{G}_m \times \mathrm{SL}_2^3$. If K and D are the number field and the division algebra associated to G, this condition is equivalent to p being completely split in K and D being split at all places of K above p. The factors SL_2 of the above product correspond to the p-adic valuations of K. As $K \subset F$ by assumption, all Hodge classes on $A_{\mathbf{C}}$ that are invariant for the $G(\mathbf{Q})$-action on the rational cohomology are defined over F, so the Galois representation associated to A factors through a map $\rho\colon \mathcal{G}_F \to G(\mathbf{Q}_p)$. This implies that we find ourselves in the situation of 1.1, with F a number field and with $G_{\mathbf{Q}_p}$ playing the role of the group G_p.
After further enlarging F, the image of ρ is sufficiently small. The constructions of 1.6 provide a character $\rho_0\colon \mathcal{G}_F \to \mathbf{G}_m(\mathbf{Q}_p)$ and representations $\rho_1, \rho_2, \rho_3\colon \mathcal{G}_F \to \mathrm{SL}_2(\mathbf{Q}_p)$. The above remarks give rise to a bijective correspondence between the ρ_i ($i = 1, 2, 3$) and the p-adic valuations v_1, v_2, v_3 of K, with ρ_i corresponding to v_i.

2.4 We summarize the notations and hypotheses in effect at this point.

- (G, V) is a pair of Mumford's type (as defined in 0.2) with G connected.
- K is the number field associated to G.
- p is a prime number which splits completely in K and such that the group G is split at p.
- $F \subset \mathbf{C}$ is a number field A/F an abelian variety.
- $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ is an identification such that the Mumford–Tate group of A is contained in G.
- Inclusions $K \subset F \subset \mathbf{C}$ are fixed such that the field of definition in $\mathbf{C}$ of the conjugacy class $C_{\mathrm{HdR}} \subset G_{\mathbf{C}}$ of μ_{HdR} is equal to K.
- The image of the Galois representation $\rho\colon \mathcal{G}_F \to G(\mathbf{Q}_p)$ is sufficiently small.

2.5. Proposition. *Under the above hypotheses, let v be a p-adic valuation of F and $\mathcal{I} = \mathcal{I}_{F,\bar{v}}$ be the inertia group of a valuation $\bar{v}$ of $\bar{F}$ lying over v. Suppose that $i \in \{1, 2, 3\}$. Then $(\rho_i)_{|\mathcal{I}}$ is of the first kind if and only if $v_{|K} = v_i$.*

Proof. We already saw (cf. proposition 1.9 and its proof) that the projection of $\rho(\mathfrak{I})$ on one and only one factor of $G^{\mathrm{ad}}(\mathbf{Q}_p)$ is non-trivial. This is the factor corresponding to v_i if and only if $(\rho_i)_{|\mathfrak{I}}$ is of the first kind. After renumbering the v_i and the ρ_i, we can assume that $v_{|K} = v_1$ and it suffices to show that the projection of $\rho(\mathfrak{I})$ on the factor of $G^{\mathrm{ad}}(\mathbf{Q}_p)$ corresponding to v_1 is non-trivial. By 2.2, the conjugacy class $C_{\mathrm{HdR}} \subset G_{\mathbf{C}}$ projects non-trivially to exactly one factor $\mathrm{PSL}_{2,\mathbf{C}}$ of

$$G^{\mathrm{ad}}_{\mathbf{C}} = \prod_{K \hookrightarrow \mathbf{C}} \mathrm{PSL}_{2,\mathbf{C}}.$$

Since the field of definition of C_{HdR} is equal to K, this must be the factor corresponding to the embedding $K \subset F \subset \mathbf{C}$ fixed in 2.2 (cf. 2.4).
Let $\iota\colon \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ be an embedding such that the composite

$$F \subset \overline{\mathbf{Q}} \xrightarrow{\ \iota\ } \mathbf{C}_p$$

induces the valuation v on F. For each K-variety Y, we denote by $Y \otimes_K \mathbf{C}_p$ the base extension of Y to $\mathbf{C}_p$ via the embedding induced by ι and the inclusion $K \subset \overline{\mathbf{Q}}$.
The Hodge–Tate decomposition associated to $\rho_{|\mathfrak{I}}$ is determined by a cocharacter $\mu_{\mathrm{HT}}\colon \mathbf{G}_{m,\mathbf{C}_p} \to G_{\mathbf{C}_p}$ and we let C_{HT} be its conjugacy class in $G_{\mathbf{C}_p}$. Since [Bla94, theorem 0.3] implies conjecture 1 of [Win88], it follows from [Win88, proposition 7] that $C_{\mathrm{HdR}} \otimes_K \mathbf{C}_p = C_{\mathrm{HT}}$, as subvarieties of $(G_K) \otimes_K \mathbf{C}_p = G_{\mathbf{C}_p}$. The conjugacy class C_{HT} thus projects non-trivially onto the factor of $G^{\mathrm{ad}}_{\mathbf{C}_p}$ corresponding to the inclusion ι. As $\iota\colon \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ restricts to an inclusion $K \hookrightarrow \mathbf{Q}_p$ inducing the valuation v_1 on K, it follows from proposition 1.9 and its proof that $\rho(\mathfrak{I})$ projects non-trivially to the factor of $G^{\mathrm{ad}}_{\mathbf{Q}_p}$ corresponding to v_1. □

2.6 THE SPECIAL CASE. We keep the above notations. In this section, it was not assumed that the Mumford–Tate group of A is equal to G. We investigate the case where these two groups are not equal, so let $T \subset G$ the Mumford–Tate group of A and assume that $T \neq G$. As T is a reductive $\mathbf{Q}$-group containing the scalars, it has to be a torus of G. This means that A corresponds to a special point of an appropriate Shimura variety associated to (G, X), where X denotes the $G(\mathbf{R})$-conjugacy class of the morphism $h\colon S \to G_{\mathbf{R}}$ determining the Hodge structure on $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{R})$.
It follows from [Noo01, §3] that there exists a totally imaginary extension L of K which splits D (with D the division algebra associated to G as in 2.1) and such that T is the image in G of $L^\times \subset D^\times$. If $\widetilde{T} \subset \widetilde{G}$ denotes the connected component of the inverse image of T, one has $\widetilde{T} = \mathbf{G}_m \times (\widetilde{G}^{\mathrm{der}} \cap L^\times)$. There is an identification

$$\widetilde{G}^{\mathrm{der}} \cap L^\times = \{x \in L^\times \mid x\bar{x} = 1\},$$

where the right hand side is considered as an algebraic group over $\mathbf{Q}$. Again by [Noo01, §3], the field L is the reflex field of the CM-type of A.

Assume that p is completely split in L. Then

$$\widetilde{T}_{\mathbf{Q}_p} \cong \mathbf{G}_m \times (\mathbf{G}_m)^3,$$

where the first factor lies in the centre of $\widetilde{G}$ and the other factors identify with maximal tori of the factors $\mathrm{SL}_{2,\mathbf{Q}_p}$ of $\widetilde{G}_{\mathbf{Q}_p}$. In this case, the representations ρ_1, ρ_2 and ρ_3 each decompose as a direct sum of two 1-dimensional representations given by opposite characters, so $\tilde{\rho}$ decomposes as a sum of 1-dimensional representations.

3 Construction of geometric Galois representations

3.1 Fix two representations (G, V) and (G', V') of Mumford's type and assume that the number field associated (as in 2.1) to both groups G and G' is the same totally real cubic field K. We will assume that both G and G' are connected. Also fix a number field $F \subset \mathbf{C}$, two 4-dimensional abelian varieties A/F and A'/F and identifications $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ and $V' = \mathrm{H}^1_{\mathrm{B}}(A'(\mathbf{C}), \mathbf{Q})$ such that the Mumford–Tate groups of A and A' are contained in G and G' respectively. In analogy with 2.2, let $\mu_{\mathrm{HdR}} \colon \mathbf{G}_{m,\mathbf{C}} \to G_{\mathbf{C}}$ (resp. $\mu'_{\mathrm{HdR}} \colon \mathbf{G}_{m,\mathbf{C}} \to G'_{\mathbf{C}}$) be the morphism determining the Hodge decomposition on $V \otimes \mathbf{C}$ (resp. $V' \otimes \mathbf{C}$) and let C_{HdR} (resp. C'_{HdR}) be the conjugacy class of μ_{HdR} (resp. μ'_{HdR}). We assume that the field of definition in $\mathbf{C}$ of C_{HdR} is equal to that of C'_{HdR} and we identify K with this field, just as in 2.2. Replacing F by a finite extension if necessary, we will assume that $K \subset F$.
Let D and D' be the division algebras associated to G and G' respectively and assume that p is a prime number which splits completely in K and such that

$$D \otimes_{\mathbf{Q}} \mathbf{Q}_p \cong D' \otimes_{\mathbf{Q}} \mathbf{Q}_p \cong \mathrm{M}_2(\mathbf{Q}_p)^3.$$

As in 2.3, the Galois representations associated to A and A' factor through morphisms $\rho \colon \mathcal{G}_F \longrightarrow G(\mathbf{Q}_p)$ and $\rho' \colon \mathcal{G}_F \longrightarrow G'(\mathbf{Q}_p)$. After replacing F by a finite extension, we may assume that ρ and ρ' have sufficiently small image in the sense of 1.2.
Let v_1, v_2 and v_3 be the p-adic valuations of K and order the factors of

$$\widetilde{G}_{\mathbf{Q}_p} \cong \mathbf{G}_{m,\mathbf{Q}_p} \times \prod_{K \hookrightarrow \mathbf{Q}_p} \mathrm{SL}_{2,\mathbf{Q}_p} \cong \widetilde{G}'_{\mathbf{Q}_p}$$

in such a way that the ith factor SL_2 in each product corresponds to the embedding $K \hookrightarrow \mathbf{Q}_p$ inducing v_i. Applying the construction of 1.6 to ρ and ρ', we obtain a character $\rho_0 = \rho'_0$ and Galois representations

$$\rho_i, \rho'_i \colon \mathcal{G}_F \longrightarrow \mathrm{SL}_2(\mathbf{Q}_p)$$

for $i = 1, 2, 3$. We will consider the representation

$$\tau = \rho_1 \otimes \rho'_1 \colon \mathcal{G}_F \longrightarrow \mathrm{SL}_2(\mathbf{Q}_p)^2 \longrightarrow \mathrm{GL}_4(\mathbf{Q}_p), \tag{3.1.$*$}$$

where the last arrow is defined by the action of $\mathrm{SL}_2(\mathbf{Q}_p)^2$ on $\mathbf{Q}_p^2 \otimes \mathbf{Q}_p^2 \cong \mathbf{Q}_p^4$.

3.2 Let $\bar{v}$ be a p-adic place of $\bar{F}$ and let $\mathcal{I} = \mathcal{I}_{\bar{F},\bar{v}} \subset \mathcal{G}_F$ be the corresponding inertia group. If the restriction of $\bar{v}$ to K is equal to v_1, then the representations $(\rho_1)_{|\mathcal{I}}$ and $(\rho_1')_{|\mathcal{I}}$ are both of the first kind by proposition 2.5. It follows that the tensor product $(\rho_1 \otimes \rho_1')_{|\mathcal{I}}$ is potentially crystalline, as it is the twist by χ of the tensor product of the two potentially crystalline representations $\rho_0\rho_1$ and $\rho_0\rho_1'$.
If the restriction of $\bar{v}$ to K is different from v_1, then the representations $(\rho_1)_{|\mathcal{I}}$ and $(\rho_1')_{|\mathcal{I}}$ are both of the second kind by proposition 2.5. In this case, the condition that $\rho(\mathcal{G}_F)$ and $\rho'(\mathcal{G}_F)$ are sufficiently small implies that $(\rho_1 \otimes \rho_1')_{|\mathcal{I}}$ is trivial. It follows that $(\rho_1 \otimes \rho_1')_{|\mathcal{I}}$ is potentially crystalline in this case as well. In both cases, it follows from [Fon94, 5.1.5] that $(\rho_1 \otimes \rho_1')_{|\mathcal{D}_{F,\bar{v}}}$ is potentially crystalline. This proves the following theorem.

3.3. Theorem. *Assume that the hypotheses of 2.4 are verified for $K \subset F \subset \mathbf{C}$, A/F, G, p and $\rho\colon \mathcal{G}_F \to G(\mathbf{Q}_p)$ and for $K \subset F \subset \mathbf{C}$, A'/F, G', p and $\rho'\colon \mathcal{G}_F \to G'(\mathbf{Q}_p)$. Let*

$$\tau = \rho_1 \otimes \rho_1' \colon \mathcal{G}_F \longrightarrow \mathrm{SL}_2(\mathbf{Q}_p)^2 \longrightarrow \mathrm{GL}_4(\mathbf{Q}_p)$$

be the representation of (3.1.∗). Then τ is a geometric Galois representation in the sense of 0.3. If F is sufficiently large then τ has good reduction everywhere.

3.4. Proposition. *Let p, K, G and G' be as in 3.1, subject to the condition that the associated division algebras D and D' both satisfy*

$$D \otimes \mathbf{R} \cong D' \otimes \mathbf{R} \cong \mathrm{M}_2(\mathbf{R}) \times \mathbf{H} \times \mathbf{H}$$

and are both split at the same real place of K. Then one can choose the abelian varieties A and A' used in the construction of τ in 3.1 such that $\mathrm{Lie}(\tau(\mathcal{G}_F)) = \mathfrak{sl}_2^2$. In that case τ is an irreducible representation of $\mathcal{G}_F$.

Proof. From the surjections $\mathbf{G}_{m,\mathbf{Q}} \times G^{\mathrm{der}} \to G$ and $\mathbf{G}_{m,\mathbf{Q}} \times (G')^{\mathrm{der}} \to G'$ one deduces a surjection $\mathbf{G}_m^2 \times G^{\mathrm{der}} \times (G')^{\mathrm{der}} \to G \times G'$. Let $\Delta \subset \mathbf{G}_m^2$ be the diagonal and let $G'' \subset G \times G'$ be the image of $\Delta \times G^{\mathrm{der}} \times (G')^{\mathrm{der}}$. The representations V and V' and the projections $G'' \to G$ and $G'' \to G'$ induce a representation of G'' on $V'' = V \oplus V'$. The representations V and V' both carry a bilinear form which is G- resp. G'-invariant up to a scalar. These forms are also G''-invariant up to a scalar and their multipliers are equal. It follows that V'' can be endowed with a bilinear form, G''-invariant up to scalars.
Fix isomorphisms $D \otimes \mathbf{R} \cong \mathrm{M}_2(\mathbf{R}) \times \mathbf{H} \times \mathbf{H} \cong D' \otimes \mathbf{R}$. Any pair of morphisms $h_D\colon S \to D_{\mathbf{R}}^\times$ and $h_{D'}\colon S \to (D')_{\mathbf{R}}^\times$ both conjugate to

$$\begin{aligned} S &\longrightarrow \mathrm{GL}_{2,\mathbf{R}} \times \mathbf{H}^\times \times \mathbf{H}^\times \\ z = a + bi &\mapsto \left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, 1, 1 \right) \end{aligned}$$

for the above identifications defines a morphism $h'' \colon S \to G''_{\mathbf{R}}$. As in [Mum69, §4], one shows that these data define the Hodge structure of an abelian variety. One can choose h_D and $h_{D'}$ in such a way that the image of h'' is Zariski dense in G'', i. e. is not contained in a proper subgroup of G'' defined over $\mathbf{Q}$. This implies that there exists an abelian variety $A''_{\mathbf{C}}$ over $\mathbf{C}$ with Mumford–Tate group equal to G'' and such that the representation of its Mumford–Tate group on $\mathrm{H}^1_{\mathrm{B}}(A''_{\mathbf{C}}(\mathbf{C}), \mathbf{Q})$ is isomorphic to the representation of G'' on V''.
It follows from [Noo95, 1.7] and its proof that there exists an abelian variety A'' over a number field $F'' \subset \mathbf{C}$ with Mumford–Tate group G'', such that the representation of its Mumford–Tate group on $\mathrm{H}^1_{\mathrm{B}}(A''(\mathbf{C}), \mathbf{Q})$ is isomorphic to (G'', V'') and with the property that that the image of the Galois representation $\rho'' \colon \mathcal{G}_{F''} \to G''(\mathbf{Q}_p)$ is open. By construction of the representation (G'', V'') as a direct sum, A'' is isogenous to a product $A \times A'$, where the Mumford–Tate groups of A and A' are G and G' respectively. The image of

$$(\rho, \rho') \colon \mathcal{G}_F \longrightarrow G(\mathbf{Q}_p) \times G'(\mathbf{Q}_p)$$

is open in $G''(\mathbf{Q}_p)$. This implies that the image of $\tau \colon \mathcal{G}_F \to \mathrm{GL}_4(\mathbf{Q}_p)$ is open in $H(\mathbf{Q}_p)$, where $H \subset \mathrm{GL}_{4,\mathbf{Q}_p}$ is the image of $(\mathrm{SL}_{2,\mathbf{Q}_p})^2$ acting on $\mathbf{Q}_p^2 \otimes \mathbf{Q}_p^2 \cong \mathbf{Q}_p^4$ by the tensor product of the standard representations. Since the representation of H on $\mathbf{Q}_p^4$ is irreducible, the same thing is true for τ. □

3.5 Remark. As noted in 2.1, the condition that

$$D \otimes \mathbf{R} \cong D' \otimes \mathbf{R} \cong \mathrm{M}_2(\mathbf{R}) \times \mathbf{H} \times \mathbf{H}$$

is equivalent to the conditions that there are abelian varieties A and A' such that D and D' are the algebras associated to the Mumford–Tate groups of A and A' respectively.

3.6 A special case. Let us return to the notations of theorem 3.3, i. e. we assume that the hypotheses of 2.4 are verified for $K \subset F \subset \mathbf{C}$, A/F, G, p and $\rho \colon \mathcal{G}_F \to G(\mathbf{Q}_p)$ and for $K \subset F \subset \mathbf{C}$, A'/F, G', p and $\rho' \colon \mathcal{G}_F \to G'(\mathbf{Q}_p)$.
From now on we moreover assume that the Mumford–Tate group of A is equal to G whereas that of A' is strictly contained in G'. The condition on A implies that $\mathrm{End}(A_{\bar{F}}) = \mathbf{Z}$. Since the representation ρ is semisimple, as $\rho(\mathcal{G}_F) \subset G(\mathbf{Q}_p)$ and because

$$\mathrm{End}_{\mathcal{G}_F}\left(\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)\right) = \mathbf{Q}_p,$$

one easily deduces that $\rho(\mathcal{G}_F)$ is open in $G(\mathbf{Q}_p)$.
As was seen in 2.6, the condition on A' implies that, after replacing F by a finite extension again if necessary, A' is of CM-type and its Mumford–Tate group is a maximal torus $T' \subset G'$. Let L' be the reflex field of the CM-type of A'. The assumption that F is sufficiently large for the Mumford–Tate group of A' to be a torus is equivalent to the condition that $L' \subset F$. By 2.6, the map

$\tilde{\rho}'\colon \mathcal{G}_F \to \widetilde{G}'(\mathbf{Q}_p)$ factors through $\widetilde{T}'(\mathbf{Q}_p)$, where $\widetilde{T}' \subset \widetilde{G}'$ is a maximal torus which identifies with a subgroup of $\mathbf{G}_m \times (L')^\times$.
Suppose that p splits completely in L'. Then the representation ρ_1' decomposes as a direct sum of two opposite characters ψ_1 and ψ_1^{-1} and it follows from 3.3 that the product $\psi_1\rho_1$ is geometric with potentially good reduction everywhere. By construction, the image of $\psi_1\rho_1$ is Zariski dense in GL_2, so $\psi_1\rho_1$ is an irreducible representation. Note that, as F contains the reflex field L' of the CM-type of A' (cf. 2.6), it is a totally imaginary field. We have thus constructed a 2-dimensional geometric representation of the absolute Galois group of a totally imaginary field F.
Of course, the same statement holds for $\psi_1^{-1}\rho_1$ and similar constructions can be carried out using the decompositions of ρ_2' as $\psi_2 \oplus \psi_2^{-1}$ and of ρ_3' as $\psi_3 \oplus \psi_3^{-1}$. All the above statements are true for the 6 representations $\psi_i^{\pm 1}\rho_i$ (for $i = 1, 2, 3$). The representation that will be of interest in section 4 is the direct sum

$$\sigma = \psi_1\rho_1 \oplus \psi_1^{-1}\rho_1 \oplus \psi_2\rho_2 \oplus \psi_2^{-1}\rho_2 \oplus \psi_3\rho_3 \oplus \psi_3^{-1}\rho_3. \tag{3.6.$*$}$$

Let $T_{L'}$ be the kernel of the map $(L')^\times \to K^\times$ induced by the field norm and define $T_K = K^\times \cap T_{L'}$, seen as group schemes over $\mathbf{Q}$. This implies that $T_{L'} = \widetilde{T}' \cap (\widetilde{G}')^{\mathrm{der}}$ and $T_K = \{x \in K^\times \mid x^2 = 1\}$. Let H' be the algebraic $\mathbf{Q}$-group defined by the short exact sequence

$$1 \longrightarrow T_K \xrightarrow{x \mapsto (x, x^{-1})} \widetilde{G}^{\mathrm{der}} \times T_{L'} \longrightarrow H' \longrightarrow 1. \tag{3.6.$\dagger$}$$

The Galois representation on $\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes \mathrm{H}^1_{\text{ét}}(A'_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)$ factors through $(G \times T'/\mathbf{G}_m)(\mathbf{Q}_p)$, where

$$\begin{aligned} \mathbf{G}_{m,\mathbf{Q}} &\longrightarrow G \times T' \\ z &\mapsto (z \cdot \mathrm{id}, z^{-1} \cdot \mathrm{id}). \end{aligned}$$

The maps $\widetilde{G}^{\mathrm{der}} \to G$ and $T_{L'} \to T'$ induce a map $H' \to (G \times T'/\mathbf{G}_m)$ and the representation σ defined in (3.6.$*$) is a lifting to $H'(\mathbf{Q}_p)$ of the representation of $\mathcal{G}_F$ on the Tate twist

$$\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes \mathrm{H}^1_{\text{ét}}(A'_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(1).$$

The facts that $\rho(\mathcal{G}_F)$ is open in $G(\mathbf{Q}_p)$ and that $\rho'(\mathcal{G}_F)$ is open in $T'(\mathbf{Q}_p)$ imply that $\sigma(\mathcal{G}_F)$ is open in $H'(\mathbf{Q}_p)$.

4 The geometric origin

4.1. Proposition. *Let $\tau\colon \mathcal{G}_F \to \mathrm{GL}_4(\mathbf{Q}_p)$ be a representation as in theorem 3.3 and assume that $\mathrm{Lie}(\mathrm{im}(\tau)) \cong \mathfrak{sl}_2^2$ (resp. let $\psi_i^{\pm 1}\rho_i$ be as in 3.6). Let A/F and A'/F be the abelian varieties serving in the construction of τ (resp. $\psi_i^{\pm 1}\rho_i$). Then there do not exist $n, m \in \mathbf{Z}$ such that τ (resp. $\psi_i^{\pm 1}\rho_i$) is isomorphic to a subquotient of a Tate twist of*

$$\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)^{\otimes n} \otimes \mathrm{H}^1_{\text{ét}}(A'_{\bar{F}}, \mathbf{Q}_p)^{\otimes m}.$$

Proof. We first give the proof for $\psi_1\rho_1$, the other $\psi_i^{\pm}\rho_i$ are handled by identical arguments.
Assume that $n, m \in \mathbf{Z}$ such that $\psi_1\rho_1$ is isomorphic to a subquotient of a Tate twist of $\mathrm{H}^1_{\text{ét}}(A_{\bar F}, \mathbf{Q}_p)^{\otimes n} \otimes \mathrm{H}^1_{\text{ét}}(A'_{\bar F}, \mathbf{Q}_p)^{\otimes m}$. Then ρ_1 is a twist by a character of a subquotient of $\mathrm{H}^1_{\text{ét}}(A_{\bar F}, \mathbf{Q}_p)^{\otimes n}$ and hence a twist by a character of a subquotient of

$$\rho_1^{\otimes n} \otimes \rho_2^{\otimes n} \otimes \rho_3^{\otimes n}.$$

It follows from 3.6 that $\tilde\rho(\mathcal{G}_F)$ is open in $\mathbf{G}_m(\mathbf{Q}_p) \times \mathrm{SL}_2(\mathbf{Q}_p)^3$. The representation theory of SL_2^3 therefore implies that $\rho_1^{\otimes n}$ contains an irreducible factor isomorphic to ρ_1 and that $\rho_2^{\otimes n}$ and $\rho_3^{\otimes n}$ both contain an invariant line. The first condition implies that n is even and the second one that n is odd, a contradiction which proves the proposition.
To prove the result for τ, we use the notations of theorem 3.3. The condition on $\mathrm{im}(\tau)$ implies that $\rho(\mathcal{G}_F)$ and $\rho'(\mathcal{G}_F)$ are not commutative, so G and G' are the Mumford–Tate groups of A and A' respectively. As noted in 3.6, this implies that $\rho(\mathcal{G}_F)$ (resp. $\rho'(\mathcal{G}_F)$) is open in $G(\mathbf{Q}_p)$ (resp. $G'(\mathbf{Q}_p)$).
The Mumford–Tate group of $A \times A'$ is the group $G'' \subset G \times G'$ introduced in the proof of proposition 3.4, so $(\rho, \rho')(\mathcal{G}_F) \subset G''(\mathbf{Q}_p)$. The Zariski closure H_p of the image of (ρ, ρ') is a reductive algebraic subgroup of $G''_{\mathbf{Q}_p}$ containing the centre.
We will show that $H_p = G''$ by proving that its rank is equal to 6. Considering the restrictions to the different inertia subgroups, it is clear that ρ'_1 is isomorphic to neither ρ_2 nor ρ_3. The condition on $\mathrm{im}(\tau)$ implies that ρ'_1 is not isomorphic to ρ_1 either. Together with the fact that $\rho(\mathcal{G}_F)$ is open in $G(\mathbf{Q}_p)$, this implies that the rank of H_p is at least 4.
Let ℓ be a prime number which is inert in the cubic number field K. The Zariski closure H_ℓ of the ℓ-adic Galois representation associated to $A \times A'$ is a reductive algebraic subgroup of $G''_{\mathbf{Q}_\ell}$. It follows from [Ser81, §3] (cf. [LP92, 6.12, 6.13]) that the ranks of H_p and H_ℓ are equal, so H_ℓ is of rank at least 4. As the adjoint group $G''_{\mathbf{Q}_\ell}$ is a product of two $\mathbf{Q}_\ell$-simple groups of rank 3, this gives $H_\ell = G''_{\mathbf{Q}_\ell}$. We conclude that H_ℓ and hence H_p are of rank 6.
The fact that $H_p = G''_{\mathbf{Q}_p}$ implies that $(\rho, \rho')(\mathcal{G}_F)$ is open in $G''(\mathbf{Q}_p)$. The statement about τ can now be proved using the representation theory of SL_2^6 in an argument similar to the one used for $\psi_1\rho_1$. □

4.2 Remark. In the proposition, the condition that $\mathrm{Lie}(\mathrm{im}(\tau)) \cong \mathfrak{sl}_2^2$ implies that τ is irreducible. This is essential for the conclusion of the proposition to hold. To see this, consider the representation $\rho_1 \otimes \rho_1$, which is reducible and decomposes as the sum of the trivial representation $\wedge^2\rho_1$ and $\mathrm{Sym}^2\rho_1$. The only interesting representation of the two is $\mathrm{Sym}^2\rho_1$. It is a quotient of $\mathrm{H}^1_{\text{ét}}(A_{\bar F}, \mathbf{Q}_p)^{\otimes 2}$.

4.3 The special case considered in 3.6. From now on and up to and including theorem 4.12, we place ourselves in the situation of 3.6. In particular, the abelian varieties A and A' have Mumford–Tate groups G and $T' \subset G'$ respectively. Moreover, we let D be the division algebra associated to G, as in 2.1, and let L' be the reflex field of the CM-type of A'. We saw in in 2.6 that the Mumford–Tate group of A' is the image of $(L')^\times$ in G'.

4.4 The construction of the abelian variety B. By [Noo01, proposition 1.5], the morphism $h\colon S \to G_{\mathbf{R}}$ determining the Hodge structure on $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ lifts uniquely to a map

$$h_D\colon S \longrightarrow D^\times_{\mathbf{R}} \cong \mathrm{GL}_{2,\mathbf{R}} \times \mathbf{H}^\times \times \mathbf{H}^\times$$

conjugate to

$$z = a + bi \mapsto \left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, 1, 1 \right).$$

As $D \otimes_{\mathbf{Q}} \mathbf{R} = \prod_{K \hookrightarrow \mathbf{R}} D \otimes_K \mathbf{R}$, the factors in the above product correspond to the real embeddings φ_1, φ_2 and φ_3 of K (in that order) and the composite of φ_1 with the inclusion $\mathbf{R} \subset \mathbf{C}$ is the embedding $K \hookrightarrow \mathbf{C}$ fixed in 2.2. On the $'$-side, there is an isomorphism

$$(L')^\times_{\mathbf{R}} \cong \prod_{K \hookrightarrow \mathbf{R}} S, \tag{4.4.$*$}$$

the factors still being indexed by the φ_i. Let

$$h'_{L'} = (1, \mathrm{id}, \mathrm{id})\colon S \longrightarrow \prod_{K \hookrightarrow \mathbf{R}} S \cong (L')^\times_{\mathbf{R}}, \tag{4.4.$\dagger$}$$

where the trivial component is the one corresponding to φ_1.
Let H denote the algebraic group defined by the short exact sequence

$$1 \longrightarrow K^\times \xrightarrow{x \mapsto (x, x^{-1})} D^\times \times (L')^\times \longrightarrow H \longrightarrow 1. \tag{4.4.$\ddagger$}$$

The group H' defined by (3.6.†) identifies with a subgroup of this group and H and H' naturally act on $W_B = D \otimes_K L'$. For any pair of maps $h_1\colon S \to D^\times_{\mathbf{R}}$ and $h_2\colon S \to (L')^\times_{\mathbf{R}}$, we will write $h_1 \cdot h_2$ for the composite of the product (h_1, h_2) with the projection onto $H_{\mathbf{R}}$.
The morphisms h_D and $h'_{L'}$ give rise to $h_H = h_D \cdot h'_{L'}\colon S \to H_{\mathbf{R}}$ and via the action of H on W_B, this gives a Hodge structure on W_B, cf. [Del72, §6]. By [Del72, pp. 161–162], this is the Hodge structure on the Betti cohomology of some polarizable complex abelian variety $B_{\mathbf{C}}$. By construction, $W_B = \mathrm{H}^1_{\mathrm{B}}(B_{\mathbf{C}}(\mathbf{C}), \mathbf{Q})$ and $\dim(B_{\mathbf{C}}) = 12$. The totally real field K is contained in the centre of D, so L' is contained in the centre of $D \otimes_K L'$. It follows that L' acts on $B_{\mathbf{C}}$ by isogenies.

4.5. Lemma. *Let $B_{\mathbf{C}}$ be as above. There exist a number field $F' \subset \mathbf{C}$ and an abelian variety B over F' such that $B_{\mathbf{C}} = B \otimes_{F'} \mathbf{C}$ and $L' \subset \mathrm{End}^0(B)$.*

Proof. It suffices to show that $B_{\mathbf{C}}$ admits a model over $\overline{\mathbf{Q}}$. Let X be the conjugacy class of the morphism $h\colon S \to G_{\mathbf{R}}$ defining the Hodge structure on $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$, let X_D be the conjugacy class of $h_D\colon S \to D^\times$ and Y that of h_H. By [Del72, §6], this gives rise to Shimura data (G, X), $(D^\times, X_D)$, $((L')^\times, \{h'_{L'}\})$ and (H, Y) and morphisms $(D^\times, X_D) \to (G, X)$ and

$$(D^\times \times (L')^\times, X_D \times \{h'_{L'}\}) \to (H, Y).$$

For appropriate compact open subgroups $C \subset G(\mathbf{A}^f)$, $C_D \subset D^\times(\mathbf{A}^f)$, $C_H \subset H(\mathbf{A}^f)$ and $C_{D,L} \subset (D^\times \times (L')^\times)(\mathbf{A}^f)$ we obtain a diagram

$$\begin{array}{ccc} {}_{C_{D,L}}M_{\overline{\mathbf{Q}}}(D^\times \times (L')^\times, X_D \times \{h'_{L'}\}) & \longrightarrow & {}_{C_H}M_{\overline{\mathbf{Q}}}(H, Y) \\ \downarrow & & \\ {}_{C_D}M_{\overline{\mathbf{Q}}}(D^\times, X_D) & & \\ \downarrow & & \\ {}_{C}M_{\overline{\mathbf{Q}}}(G, X) & & \end{array}$$

of morphisms between the weakly canonical models of the associated Shimura varieties.

We dispose of faithful representations of G and of H, and via these representations, the $\mathbf{C}$-valued points of the Shimura varieties ${}_CM_{\overline{\mathbf{Q}}}(G, X)$ and ${}_{C_H}M_{\overline{\mathbf{Q}}}(H, Y)$ correspond to isogeny classes of polarized abelian varieties endowed with extra structure. If the above compact subgroups are sufficiently small, all the Shimura varieties in question are smooth and both ${}_CM_{\overline{\mathbf{Q}}}(G, X)$ and ${}_{C_H}M_{\overline{\mathbf{Q}}}(H, Y)$ carry families of abelian varieties such that the fibre over a $\mathbf{C}$-valued point of the Shimura variety lies in the isogeny class corresponding to that point. One can moreover choose the compact subgroups above such that the vertical maps are finite.

After fixing a level structure on $A_{\overline{\mathbf{Q}}}$, it corresponds to a point

$$a \in {}_CM_{\overline{\mathbf{Q}}}(G, X)(\overline{\mathbf{Q}}).$$

By construction, there is a $\mathbf{C}$-valued point

$$\tilde{a} \in {}_{C_{D,L}}M_{\overline{\mathbf{Q}}}(D^\times \times (L')^\times, X_D \times \{h'_{L'}\})(\mathbf{C})$$

in the fibre over a of the vertical map such that $\tilde{a}$ maps to a point $b \in {}_{C_H}M_{\overline{\mathbf{Q}}}(H, Y)(\mathbf{C})$ corresponding to the isogeny class of $B_{\mathbf{C}}$ endowed with an appropriate level structure. Because the vertical arrows are finite, it follows that $\tilde{a}$ and hence b are defined over $\overline{\mathbf{Q}}$ and this implies that $B_{\mathbf{C}}$ admits a model over $\overline{\mathbf{Q}}$. $\square$

4.6 The construction of the abelian variety C. We keep the notations of 4.3 and 4.4. The morphism $h\colon S \to T'_{\mathbf{R}}$ determining the Hodge structure of $A'_{\mathbf{C}}$ on $V' = \mathrm{H}^1_{\mathrm{B}}(A'(\mathbf{C}), \mathbf{Q})$ lifts to a map $h_{L'}\colon S \to (L')^{\times}_{\mathbf{R}}$ inducing an isomorphism of S with the factor of $(L')^{\times}_{\mathbf{R}} \cong \prod_{K \hookrightarrow \mathbf{R}} S$ corresponding to $\varphi_1\colon K \hookrightarrow \mathbf{R}$ and inducing the trivial map on the other factors.
Let $h'_{L'}$ be as in (4.4.†) and denote by $\overline{h'_{L'}}$ its composite with the involution of S induced by complex conjugation. We define yet another map $h''_{L'}\colon S \to (L')^{\times}_{\mathbf{R}}$ as the product $\overline{h'_{L'}} h_{L'}$, where the product is taken for the commutative group structure on $(L')^{\times}_{\mathbf{R}}$. Together with the natural action of L' on $W_C = L'$, this defines a Hodge structure on W_C. This Hodge structure is the Hodge structure of $C_{\mathbf{C}}$ for an abelian variety C of CM-type over a number field $F'' \subset \mathbf{C}$. One has $\dim(C) = 3$ and $L' \subset \mathrm{End}^0(C)$. By construction, $W_C = \mathrm{H}^1_{\mathrm{B}}(C(\mathbf{C}), \mathbf{Q})$ and $\dim_{L'}(W_C) = 1$.

4.7 Remark. Possibly after replacing the identification of equation (4.4.∗) on the first factor by its complex conjugate, one can assume that $h_{L'}$ is given by

$$h_{L'} = (\mathrm{id}, 1, 1)\colon S \longrightarrow \prod_{K \hookrightarrow \mathbf{R}} S \cong (L')^{\times}_{\mathbf{R}}.$$

Doing so, the map $h''_{L'}$ is given by $z \mapsto (z, \bar{z}, \bar{z})$.

4.8 The construction of the motives m' and m. In what follows, we will work in the category $\mathcal{C}$ of motives for absolute Hodge cycles as described in [DM82], especially section 6 of that paper. Recall that this category is constructed as Grothendieck's category of motives except where it concerns the morphisms. These are defined to be given by absolute Hodge classes, not by cycle classes as usual.
We keep the assumptions of 4.3. Replacing F by a finite extension and extending the base field of the varieties in question, we will assume from now on that the abelian varieties A and A' (of 3.6), B (of 4.5) and C (of 4.6) are F-varieties and that L' acts on B and C (over F). The motives $h^1(B)$ and $h^1(C)$ belong to the category $\mathcal{C}_{(L')}$ of objects of $\mathcal{C}$ endowed with L'-action. One can thus form the tensor product

$$m' = h^1(B) \otimes_{L'} h^1(C),$$

still belonging to $\mathcal{C}_{(L')}$, as in [DM82, pp. 155–156].
Let $\widetilde{K}$ be the normal closure of K in $\mathbf{C}$. As K is totally real, one has $\widetilde{K} \subset \mathbf{R}$. The decomposition of the algebra $K \otimes_{\mathbf{Q}} \widetilde{K}$ as a product of fields gives rise to a system (u_1, u_2, u_3) of orthogonal idempotents in $K \otimes_{\mathbf{Q}} \widetilde{K}$, indexed be the real embeddings $\varphi_1, \varphi_2, \varphi_3$ of K. For $1 \leq i < j \leq 3$, let $u_{i,j} = u_i + u_j$.
Let $m' \otimes_{\mathbf{Q}} \widetilde{K}$ be the external tensor product as in [DM82, pp. 155–156]. As $K \subset L'$, it acts on m' and one deduces a $\widetilde{K}$-linear action of $K \otimes_{\mathbf{Q}} \widetilde{K}$ on $m' \otimes_{\mathbf{Q}} \widetilde{K}$,

given by a $\mathrm{Gal}(\widetilde{K}/K)$-equivariant $\widetilde{K}$-algebra morphism

$$K \otimes_{\mathbf{Q}} \widetilde{K} \longrightarrow \mathrm{End}_{\widetilde{K}}(m' \otimes_{\mathbf{Q}} \widetilde{K}) = \mathrm{End}(m') \otimes_{\mathbf{Q}} \widetilde{K}.$$

This action induces an action of $K \otimes_{\mathbf{Q}} \widetilde{K}$ on $\wedge^3_{\widetilde{K}}(m' \otimes_{\mathbf{Q}} \widetilde{K})$ given by a $\mathrm{Gal}(\widetilde{K}/K)$-equivariant multiplicative map (not a morphism of algebras)

$$K \otimes_{\mathbf{Q}} \widetilde{K} \longrightarrow \mathrm{End}_{\widetilde{K}}(\wedge^3_{\widetilde{K}}(m' \otimes_{\mathbf{Q}} \widetilde{K})) = \mathrm{End}(\wedge^3 m') \otimes_{\mathbf{Q}} \widetilde{K}.$$

For $1 \leq i < j \leq 3$, let $u'_{i,j}$ be the image of $u_{i,j}$ in $\mathrm{End}(\wedge^3 m') \otimes_{\mathbf{Q}} \widetilde{K}$. As

$$\{u_{2,3}, u_{1,3}, u_{1,2}\} \subset K \otimes_{\mathbf{Q}} \widetilde{K}$$

is a $\mathrm{Gal}(\widetilde{K}/K)$-invariant subset, the sum $u'_{2,3} + u'_{1,3} + u'_{1,2} \in \mathrm{End}(\wedge^3 m') \otimes_{\mathbf{Q}} \widetilde{K}$ is also $\mathrm{Gal}(\widetilde{K}/K)$-invariant. This element therefore determines an element $u' \in \mathrm{End}(\wedge^3 m')$. The AH-motive m is defined to be the kernel of u' on $\wedge^3 m'$.

4.9 Remark. Intuitively, the aim of this construction is to pass from an object m' which can be expressed, after tensoring with $\widetilde{K}$, as a direct sum, to the tensor product of the direct factors. This tensor product descends to an object m over $\mathbf{Q}$.

4.10. Proposition. *Let notations and assumptions be as in 4.3–4.8. In particular, the Mumford–Tate group of A is equal to G, that of A' is T', the varieties A, A', B and C are defined over F and L' acts on B and on C. Then there is an isomorphism*

$$m \cong (h^1(A) \otimes h^1(A')(-2))^8$$

of absolute Hodge motives.

Proof. The main theorem (2.11) of Deligne's paper [Del82] states that the spaces of Hodge cycles and of absolute Hodge cycles on an abelian variety coincide. As noted in [DM82, 6.25], this implies that to prove the proposition, it suffices to show that the Hodge structures on the Betti realizations of the motives m and $(h^1(A) \otimes h^1(A')(-2))^8$ are isomorphic.
First consider $h^1(A)$, the Betti realization of which is $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$. The Mumford–Tate group G of A admits a morphism $D^\times \to G$, which makes V into a representation of $D^\times$. The Hodge structure on V is determined by the action of S on $V_{\mathbf{R}}$ given by the morphism $h_D \colon S \to D^\times_{\mathbf{R}}$ of 4.4. Similarly, $V' = \mathrm{H}^1_{\mathrm{B}}(A'(\mathbf{C}), \mathbf{Q})$, the Betti realization of $h^1(A')$, is made into a representation of $(L')^\times$ by the morphism $(L')^\times \to T'$, and the map $h_{L'} \colon S \to (L')^\times_{\mathbf{R}}$ of 4.6 determines the Hodge structure on V'.
In these representations, the subgroup $K^\times \subset D^\times$, resp. $K^\times \subset (L')^\times$ acts through the map $K^\times \to \mathbf{G}_{m,\mathbf{Q}}$ induced by the field norm $\mathrm{N}_{K/\mathbf{Q}}$. It follows that the action of $D^\times \times (L')^\times$ on $V \otimes_{\mathbf{Q}} V'$, the Betti realization of $h^1(A) \otimes h^1(A')$, factors through the group H defined by (4.4.‡). Passing to $\overline{\mathbf{Q}}$, one has

$$H_{\overline{\mathbf{Q}}} \cong \prod_{K \hookrightarrow \overline{\mathbf{Q}}} \left(\mathrm{GL}_2 \times \mathbf{G}_m^2 / \mathbf{G}_m \right),$$

where the map $\mathbf{G}_m \to \mathrm{GL}_2 \times \mathbf{G}_m^2$ on each factor is given by $z \mapsto (z \cdot \mathrm{id}, z^{-1}, z^{-1})$. The representation of $H_{\overline{\mathbf{Q}}}$ on $V \otimes V' \otimes \overline{\mathbf{Q}}$ induces a representation of

$$\left(\mathrm{GL}_{2,\overline{\mathbf{Q}}} \times \mathbf{G}^2_{m,\overline{\mathbf{Q}}}\right)^3$$

on this space which is isomorphic to the tensor product of the standard representations of its factors GL_2 and $\mathbf{G}_m^2$ on $\overline{\mathbf{Q}}^2$.

The Hodge structure on $V \otimes V'$, considered as the Betti realization of $h^1(A) \otimes h^1(A')$, is given by $h_D \cdot h_{L'} \colon S \to H_{\mathbf{R}}$. Multiplying the composite map

$$S \xrightarrow{h_D \cdot h_{L'}} H_{\mathbf{R}} \longrightarrow \mathrm{GL}(V \otimes V' \otimes \mathbf{R})$$

by the square of the norm map $S \to \mathbf{G}_{m,\mathbf{R}}$, one gets the Hodge structure of the Betti realization of the Tate twist $h^1(A) \otimes h^1(A')(-2)$.

The Betti realization of m' is $\mathrm{H}^1_{\mathrm{B}}(B(\mathbf{C}), \mathbf{Q}) \otimes_{L'} \mathrm{H}^1_{\mathrm{B}}(C(\mathbf{C}), \mathbf{Q}) = W_B \otimes_{L'} W_C$ and the group H of (4.4.‡) naturally acts on this space. Over $\overline{\mathbf{Q}}$ we have a decomposition

$$(W_B \otimes_{L'} W_C) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} = (W_1 \oplus W_2 \oplus W_3)^2$$

of the induced representation of $H_{\overline{\mathbf{Q}}}$. As before, the factors correspond to the embeddings $K \hookrightarrow \overline{\mathbf{Q}}$. The representation of $H_{\overline{\mathbf{Q}}}$ on W_i is isomorphic to the one induced by the representation of the ith factor $\mathrm{GL}_2 \times \mathbf{G}_m^2$ on $\overline{\mathbf{Q}}^2 \otimes_{\overline{\mathbf{Q}}} \overline{\mathbf{Q}}^2$ by tensor product.

The Hodge structure on the tensor product $W_B \otimes_{L'} W_C$ is obtained by multiplying the map h_H defining the Hodge structure of $B_{\mathbf{C}}$ by $h''_{L'}$, the map determining the Hodge structure of C. Since h_H is defined in 4.4 as the product $h_D \cdot h'_{L'}$ and since $h''_{L'} = h_{L'} \overline{h'_{L'}}$, it follows that $h_H h''_{L'} = h_D \cdot h_{L'} h'_{L'} \overline{h'_{L'}}$. Writing $N' = h'_{L'} \overline{h'_{L'}}$, this implies that the Hodge structure on $W_B \otimes_{L'} W_C$ is determined by

$$h_H h''_{L'} = h_D \cdot h_{L'} N' \colon S \longrightarrow H_{\mathbf{R}}.$$

The image of $N' \colon S \to (L')^\times_{\mathbf{R}}$ lies in $K^\times_{\mathbf{R}}$ and N' is given by

$$\begin{aligned} N' \colon S &\longrightarrow \prod_{K \hookrightarrow \mathbf{R}} \mathbf{G}_{m,\mathbf{R}} \cong K^\times_{\mathbf{R}} \\ z &\mapsto (1, z\bar{z}, z\bar{z}). \end{aligned}$$

With this information, the Betti realization W_m of m can be computed. It is the kernel of the endomorphism of $\wedge^3(W_B \otimes_{L'} W_C)$ induced by the map u' of 4.8. For $1 \le i < j \le 3$, the endomorphism $u_{i,j} \in \mathrm{End}_{\overline{\mathbf{Q}}}(W_B \otimes_{L'} W_C \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})$ induces the identity on W_i and W_j and zero on the remaining factor. Since $W_m \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ is the kernel of $u'_{\overline{\mathbf{Q}}}$ on $\wedge^3(W_B \otimes_{L'} W_C) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$, it follows that there is an isomorphism

$$W_m \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} \cong W_1^2 \otimes W_2^2 \otimes W_3^2 = (W_1 \otimes W_2 \otimes W_3)^8$$

of representations of $H_{\overline{\mathbf{Q}}}$. It follows that $V \otimes V' \cong W_m$ as representations of H.
We still have to show that the Hodge structures are the same. In view of what we already know about these Hodge structures, it suffices to show that the action of S on $W_m \otimes \mathbf{R}$ defined by $N' \colon S \to K_{\mathbf{R}}^\times \subset H_{\mathbf{R}}$ and the representation of H on W_m is equal to scalar multiplication by the square of the norm $S \to \mathbf{G}_{m,\mathbf{R}}$. This in turn follows immediately from the above description of N' and the fact that $K^\times \subset H$ acts on W_m through the map $K^\times \to \mathbf{G}_{m,\mathbf{Q}}$ induced by the field norm $\mathrm{N}_{K/\mathbf{Q}}$. □

4.11. Corollary. *Keep the hypotheses of the proposition. For any prime number p, there is an isomorphism of $\mathcal{G}_F$-modules between*

$$\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathrm{H}^1_{\text{ét}}(A'_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(-2)\right)^8$$

and the étale p-adic realization of m. This representation is a subquotient of $\mathrm{H}^6_{\text{ét}}(B_{\overline{\mathbf{Q}}} \times C_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)$.

4.12. Theorem. *Let notations and assumptions be as in 4.3. In particular, the Mumford–Tate group of A is equal to G and that of A' is T'. Assume moreover that A, A', B and C are defined over F and that $\rho(\mathcal{G}_F)$ and $\rho'(\mathcal{G}_F)$ are sufficiently small. Let the representation*

$$\sigma \colon \mathcal{G}_F \to H'(\mathbf{Q}_p) \subset H(\mathbf{Q}_p)$$

be as in (3.6.∗). Then there exists a finite extension F' of F such that the restriction of $\sigma \oplus \sigma$ to $\mathcal{G}_{F'}$ is isomorphic to the representation of $\mathcal{G}_{F'}$ on

$$\mathrm{H}^1_{\text{ét}}(B_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes_{L' \otimes \mathbf{Q}_p} \mathrm{H}^1_{\text{ét}}(C_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(1).$$

In particular, the representations $\psi_i^{\pm}\rho_i$ (for $i = 1, 2, 3$) defined in 3.6 come from algebraic geometry in the sense of 0.3.

Proof. The first statement of the theorem implies that $\sigma \colon \mathcal{G}_F \to H'(\mathbf{Q}_p)$ differs from a representation of $\mathcal{G}_F$ on a subquotient of an étale cohomology group of an algebraic variety by a finite character. As σ is the direct sum of the irreducible representations $\psi_i^{\pm}\rho_i$, the second statement of the theorem follows. This leaves the first statement to be proven. From the constructions of B and C and from the proof of proposition 4.10, it is clear that the Mumford–Tate group of the Hodge structure on $\mathrm{H}^1_{\mathrm{B}}(B(\mathbf{C}), \mathbf{Q}) \otimes_{L'} \mathrm{H}^1_{\mathrm{B}}(C(\mathbf{C}), \mathbf{Q})$ is contained in H and that the Mumford–Tate group of $\mathrm{H}^1_{\mathrm{B}}(B(\mathbf{C}), \mathbf{Q}) \otimes_{L'} \mathrm{H}^1_{\mathrm{B}}(C(\mathbf{C}), \mathbf{Q})(1)$ is contained in H'. By [Del82, 2.9, 2.11], this implies that there is a finite extension F' of F such that L' acts on $B_{F'}$ and on $C_{F'}$ and such that the representation of $\mathcal{G}_{F'}$ on

$$W^{(p)} = \mathrm{H}^1_{\text{ét}}(B_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes_{L' \otimes \mathbf{Q}_p} \mathrm{H}^1_{\text{ét}}(C_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(1)$$

factors through a morphism $\sigma' \colon \mathcal{G}_{F'} \to H'(\mathbf{Q}_p)$. The hypotheses that $D \otimes_{\mathbf{Q}} \mathbf{Q}_p \cong \mathrm{M}_2(\mathbf{Q}_p)^3$ and that p splits completely in L' imply that $W^{(p)}$ decomposes as the direct sum of two isomorphic representations of the group $H'_{\mathbf{Q}_p}$. We contend that, possibly after replacing F' by a finite extension, $W^{(p)}$ is isomorphic to $V^{(p)}$, the $\mathcal{G}_{F'}$ module underlying $\sigma \oplus \sigma$.
The decomposition

$$\sigma = \bigoplus_{i=1,2,3} \left(\psi_i \rho_i \oplus \psi_i^{-1} \rho_i\right)$$

gives a decomposition $V^{(p)} = (V_1 \oplus V_2 \oplus V_3)^2$. As σ lifts the Galois representation on $\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes \mathrm{H}^1_{\text{ét}}(A'_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(1)$ from $(G \times T'/\mathbf{G}_m)(\mathbf{Q}_p)$ to $H'(\mathbf{Q}_p)$, one has

$$\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes \mathrm{H}^1_{\text{ét}}(A'_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(1) = V_1 \otimes V_2 \otimes V_3.$$

If $u' \in \mathrm{End}(\wedge^3 V^{(p)})$ is constructed as in 4.8, then $\ker(u') = (V_1 \otimes V_2 \otimes V_3)^8$. In analogy with the proof of proposition 4.10, one has $W^{(p)} = (W_1 \oplus W_2 \oplus W_3)^2$ as representations of $H_{\mathbf{Q}_p}$ and $(W_1 \otimes W_2 \otimes W_3)^8$ is the étale p-adic realization of the motive m. It follows from the corollary 4.11 that

$$(V_1 \otimes V_2 \otimes V_3)^8 \cong (W_1 \otimes W_2 \otimes W_3)^8$$

as representations of $\mathcal{G}_{F'}$. Hence σ and σ' have the same projection to $H'(\mathbf{Q}_p)/\ker(N)(\mathbf{Q}_p)$. Here $N \colon T_K \to \{\pm 1\}$ is the map induced by the norm $\mathrm{N}_{K/\mathbf{Q}}$, where T_K is, as in 3.6, the group scheme $\{x \in K^\times \mid x^2 = 1\}$. As $T_K(\mathbf{Q}_p)$ is finite, this implies that $\sigma_{|\mathcal{G}_{F'}}$ and $\sigma'_{|\mathcal{G}_{F'}}$ differ by a finite character and thus proves the theorem. □

4.13. COROLLARY. *We return to the hypotheses of theorem 3.3, i. e. we only assume that the Mumford–Tate groups of A and A' are contained in G and G' respectively. Then the representation $\tau \colon \mathcal{G}_F \to \mathrm{GL}_4(\mathbf{Q}_p)$ of (3.1.∗), which is geometric by theorem 3.3, comes from algebraic geometry.*

Proof. By remark 0.6, it is sufficient to prove this after replacing F by a finite extension. Apart from the varieties A and A' serving in the construction of τ in 3.1, we choose, after enlarging F if necessary, an auxiliary abelian variety A'' with Mumford–Tate group contained in a group of Mumford's type, which is of CM-type, and such that p splits completely in the reflex field L'' of its CM-type. As in 3.6, let ψ_1 be one of the characters in which the representation ρ_1'' decomposes. For F sufficiently large, the above theorem implies that $\psi_1\rho_1$ and $\psi_1^{-1}\rho_1'$ come from algebraic geometry. It follows that the same is true for $\tau = (\psi_1\rho_1) \otimes (\psi_1^{-1}\rho_1')$. □

5 Other geometric representations

5.1 Let p be a prime number, F a number field and let $\mathrm{Rep}(\mathcal{G}_F)$ be the tannakian category of finite dimensional, continuous $\mathbf{Q}_p$-linear representations of the absolute Galois group $\mathcal{G}_F$. Consider all abelian fourfolds A/F such that the conditions of 2.4 are verified for some identification $V_A = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ and some algebraic group $G_A \subset \mathrm{GL}(V_A)$. We denote the p-adic Galois representation associated to A by $\rho_A \colon \mathcal{G}_F \to G_A(\mathbf{Q}_p)$ and define $\mathcal{M}$ to be the full tannakian subcategory of $\mathrm{Rep}(\mathcal{G}_F)$ generated by the ρ_A for A running through the above set of abelian varieties and the Tate object $\mathbf{Q}_p(1)$.
For any A as above, we denote by $\tilde{\rho}_A$ the canonical lifting of ρ_A constructed in lemma 1.4. Let $\mathcal{L}$ be the full tannakian subcategory of $\mathrm{Rep}(\mathcal{G}_F)$ generated by all liftings $\tilde{\rho}_A$ and $\mathbf{Q}_p(1)$. Obviously, $\mathcal{M}$ is a full subcategory of $\mathcal{L}$.
Let $\tilde{\rho}$ be an object of $\mathcal{L}$, it is a representation of $\mathcal{G}_F$ on a finite dimensional $\mathbf{Q}_p$-vector space V. We let $\widetilde{G}$ be the Zariski closure of the image of $\tilde{\rho}$ in $\mathrm{GL}(V)$. The category generated by $\tilde{\rho}$ contains an object ρ of $\mathcal{M}$ such that there is a central isogeny $\widetilde{G} \to G$, where G denotes the Zariski closure of the image of ρ.

5.2. Theorem. *Let τ be an irreducible representation of $\mathcal{G}_F$ contained in $\mathcal{L}$ which is geometric. Then τ comes from algebraic geometry.*

Proof. Each $\tilde{\rho}_A$ is a direct sum of a (fixed) character ρ_0 and three 2-dimensional representations $\rho_{A,1}$, $\rho_{A,2}$ and $\rho_{A,3}$. It follows that $\mathcal{L}$ is generated by ρ_0 and the $\rho_{A,i}$, for $i = 1, 2, 3$ and A running through the abelian varieties considered in 5.1. This implies that each object of $\mathcal{L}$ is a subquotient of a representation of the form

$$\tau = \rho_0^k \cdot \bigotimes_{j=1}^{r} \rho_j, \tag{5.2.$*$}$$

with $k \in \mathbf{Z}$, and where $\rho_j = \rho_{A_j, i_j}$ for some an abelian variety A_j of the type considered above and some integer $i_j \in \{1, 2, 3\}$. It therefore suffices to consider representations of this form.
Let τ be as in (5.2.$*$). It follows from 3.6 and 4.12 that, for each j, there exist a finite extension F_j of F and a character ψ_j of $\mathcal{G}_{F_j}$ such that $\psi_j \tau_{|\mathcal{G}_{F_j}}$ comes from algebraic geometry. Let F' be a finite extension of F containing all F_j, define the character ψ of $\mathcal{G}_{F'}$ by $\psi = (\rho_0^{-k}) \prod \psi_j$ and let

$$\tau' = \bigotimes_{j=1}^{r} (\psi_i \rho_j)_{|\mathcal{G}_{F'}} = \psi \tau_{|\mathcal{G}_{F'}}.$$

This defines a geometric representation $\mathcal{G}_{F'}$. After replacing F' by a finite extension, τ' is a Tate twist of a subquotient of an étale cohomology group of an algebraic (even abelian) variety over F'.
Now assume that τ and hence $\tau_{|\mathcal{G}_{F'}}$ are geometric. This implies that ψ is a geometric representation of $\mathcal{G}_{F'}$ as well. If follows from [FM95, §6], that ψ

comes from algebraic geometry and this in turn implies that $\tau_{|\mathcal{G}_{F'}} = \psi^{-1}\tau'$ is a Tate twist of a subquotient of an étale cohomology group of an algebraic variety. By remark 0.6, the same thing is true for τ. $\square$

5.3 Using the proof of theorem 5.2, one can describe the geometric representations contained in $\mathcal{L}$. We keep the notations used above. Let A/F be an abelian variety as in 5.1, G_A its Mumford–Tate group and v a p-adic valuation of F. Let

$$\mu_{\mathrm{HT},A,v}\colon \mathbf{G}_{m,\mathbf{C}_p} \to G_{A,\mathbf{C}_p}$$

be the cocharacter defined by the Hodge–Tate decomposition of $\rho_{|\mathcal{I}_{F,\bar v}}$ for some valuation $\bar v$ of $\bar F$ with $v = \bar v_{|F}$. We saw in the proof of proposition 1.9 that $\mu^2_{\mathrm{HT},A,v}$ lifts to $\tilde\mu_{A,v}\colon \mathbf{G}_{m,\mathbf{C}_p} \to \widetilde{G}_{A,\mathbf{C}_p}$. This lifting is conjugate to the map

$$\begin{aligned} \mathbf{G}_{m,\mathbf{C}_p} &\longrightarrow \mathbf{G}_{m,\mathbf{C}_p} \times \mathrm{SL}_{2,\mathbf{C}_p} \times \mathrm{SL}_{2,\mathbf{C}_p} \times \mathrm{SL}_{2,\mathbf{C}_p} \cong \widetilde{G}_{A,\mathbf{C}_p} \\ z &\mapsto \left(z, \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \mathrm{id}, \mathrm{id} \right). \end{aligned}$$

The factor SL_2 on which $\tilde\mu_{A,v}$ is non trivial is the factor to which the image of $\mathcal{I}_{F,\bar v}$ projects non-trivially.
The representations

$$\psi_j\rho_j\colon \mathcal{G}_F \to \mathrm{GL}_2(\mathbf{Q}_p)$$

occurring in the expression for τ' are geometric and the maps $\tilde\mu_{A,v}$ define cocharacters $\tilde\mu_{j,v}\colon \mathbf{G}_{m,\mathbf{C}_p} \to \mathrm{GL}_{2,\mathbf{C}_p}$. The above description of the $\tilde\mu_{A,v}$ and the choice of the ψ_j imply that each $\tilde\mu_{j,v}$ is a square of a cocharacter of $\mathrm{GL}_{2,\mathbf{C}_p}$. Similarly, the liftings $\tilde\mu_{A,v}$ and the expression of the character

$$\psi\colon \mathcal{G}_F \to \mathbf{G}_m(\mathbf{Q}_p)$$

as a product of ρ_0^{-k} and the ψ_j define a map $\tilde\mu_v\colon \mathbf{G}_{m,\mathbf{C}_p} \to \mathbf{G}_{m,\mathbf{C}_p}$. It follows from proposition 1.9, its proof, and the description of the $\tilde\mu_{A,v}$, that $\psi_{|\mathcal{I}_{F,\bar v}}$ is the nth power of $(\rho_0)_{|\mathcal{I}_{F,\bar v}}$ for some integer n and that $\tilde\mu_v(z) = z^n$. This means that ψ is potentially semi-stable at v if and only if n is even, which is the case if and only if $\tilde\mu_v$ is a square.
The condition that ψ is potentially semi-stable at v is equivalent to τ being potentially semi-stable at v. This proves the following proposition, characterizing the geometric representations in $\mathcal{L}$.

5.4. Proposition. *For each object $\tilde\rho\colon \mathcal{G}_F \to \mathrm{GL}(V)$ of $\mathcal{L}$, there exists a central isogeny $N\colon \widetilde{G} \to G$ of algebraic groups over $\mathbf{Q}_p$ such that $\widetilde{G}$ is the Zariski closure of the image of $\tilde\rho$ and such that $N \circ \tilde\rho$ belongs to $\mathcal{M}$. The representation ρ is semi-stable at v if and only if the cocharacter*

$$\mu_{\mathrm{HT},v}\colon \mathbf{G}_{m,\mathbf{C}_p} \to G_{\mathbf{C}_p},$$

defined by the Hodge–Tate decomposition of $\rho_{|\mathcal{I}_{F,\bar{v}}}$ *(for a place* $\bar{v}$ *of* $\bar{F}$ *lying over* v*), lifts to* $\widetilde{G}_{\mathbf{C}_p}$*. The representation* ρ *is geometric if and only if* $\mu_{\mathrm{HT},v}$ *lifts for each valuation* v *of* F*.*

5.5. Corollary. *Let* F *be a number field,* p *a prime number and* m *be a motive belonging to the category of absolute Hodge motives generated by the motives of the abelian varieties considered in 5.1. Assume that* G *is the Mumford–Tate group of* m *and let* $\rho\colon \mathcal{G}_F \to G(\mathbf{Q}_p)$ *the associated* p*-adic Galois representation. Suppose that* $\widetilde{G} \to G$ *is a central isogeny such that* ρ *lifts to a representation* $\tilde{\rho}\colon \mathcal{G}_F \to \widetilde{G}(\mathbf{Q}_p)$ *belonging to* $\mathcal{L}$*. Then the following statements are equivalent.*

1. $\tilde{\rho}$ *is geometric.*
2. *For some* p*-adic valuation* $\bar{v}$ *of* $\bar{F}$*, the restriction* $\tilde{\rho}_{|\mathcal{I}_{F,\bar{v}}}$ *is potentially semi-stable.*
3. *The cocharacter* $\mu_{\mathrm{HdR}}\colon \mathbf{G}_{m,\mathbf{C}} \to G_{\mathbf{C}}$ *defining the Hodge filtration on the Betti realization of* m *lifts to* $\widetilde{G}_{\mathbf{C}}$*.*
4. $\tilde{\rho}$ *comes from algebraic geometry.*

Proof. By hypothesis, the conjugacy class C_{HdR} of μ_{HdR} can be defined over F. It was noted in the proof of proposition 2.5 that, for any embedding $\bar{F} \hookrightarrow \mathbf{C}_p$ inducing the valuation $\bar{v}$, the conjugacy class of $\mu_{\mathrm{HT},v}$ is equal to $C_{\mathrm{HdR}} \otimes_F \mathbf{C}_p$. Using proposition 5.4, this proves that 2 implies 3 and that 3 implies 1.
The theorem 5.2 shows that 1 implies 4. The remaining implications are left to the reader. □

5.6 Remarks.

5.6.1 If condition 3 of the corollary is verified, then it follows from [Win95, Théorème 2.1.7] that there exists a finite extension F' of F such that the representations $\mathcal{G}_{F'} \to G(\mathbf{Q}_\ell)$, for ℓ ranging through the prime numbers, simultaneously lift to $\widetilde{G}(\mathbf{Q}_\ell)$.

5.6.2 The geometric representations considered in this paper are obtained by lifting representations coming from algebraic geometry along isogenies of algebraic groups $\widetilde{H} \to H$ over $\mathbf{Q}$. Conjecturally, a representation $\mathcal{G}_F \to H(\mathbf{Q}_p)$ which comes from algebraic geometry corresponds to a morphism $f\colon G_M^0 \to H$, where G_M^0 is the connected component of the motivic Galois group, see for example [Ser94]. The geometric representations considered in this paper are obtained by lifting along an isogeny $\widetilde{H} \to H$ with the property that the morphism $\mathbf{G}_m \times \mathbf{G}_m \to H_{\mathbf{C}}$ determining the (mixed) Hodge structure on the corresponding Betti cohomology group lifts to $\widetilde{H}$.
In [Ser94, 8.1], Serre asks whether the derived group of G_M^0 is simply connected. He notes that an argument due to Deligne shows that if this is the case, then

the fact that the morphism determining the Hodge structure lifts to $\widetilde{H}$ implies that $f\colon G_M^0 \to H$ lifts to a morphism $G_M^0 \to \widetilde{H}$. This in turn should imply that the lifted Galois representation comes from algebraic geometry. An affirmative answer to Serre's question, in conjunction with the standard conjectures involved in the theory of motives, therefore proves the Fontaine–Mazur conjecture for the representations considered in this paper. The above discussion also indicates that, conversely, the conjecture of Fontaine and Mazur is unlikely to be true if the answer to Serre's question is negative.

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