Excellent Special Orthogonal Groups

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Abstract. In this paper we give a complete classification of excellent special orthogonal groups SO(q) where q is a regular quadratic form over a field of characteristic 0.

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0. Introduction

Let F be a field of characteristic $\neq 2$ and let φ be a regular quadratic form over F. Then φ is said to be *excellent* if, for any field extension E/F, the anisotropic part of $\varphi_E := \varphi \otimes_F E$ is defined over F. This notion was introduced by M. Knebusch in [Kn1, Kn2]. In [KR], a similar notion for semisimple algebraic groups was introduced and studied for special linear and special orthogonal groups. Let us recall that the main result of [KR] says that the following conditions are equivalent.

(i) *The special orthogonal group* $\mathbb{SO}(\varphi)$ *is excellent.*
(ii) *For every field extension* E/F *there is an element* $a \in E^*$ *and a form* ψ *over* F *such that the anisotropic part of* φ_E *is isomorphic to* $a\psi_E$ *.*

In general, if φ is excellent $\mathbb{SO}(\varphi)$ is also excellent. The converse holds for odd-dimensional forms (see [KR]). For even-dimensional forms there are examples of non-excellent forms φ such that the group $\mathbb{SO}(\varphi)$ is excellent.

We say that the form φ is *quasi-excellent* if the group $\mathbb{SO}(\varphi)$ is excellent. Taking into account the criterion mentioned above, we can rewrite the definition as follows: φ *is quasi-excellent if for any field extension* E/F *there exists a form* ψ *over* F *such that* $(\varphi_E)_{\rm an}$ *is similar to* ψ_E *.* In this case we write $(\varphi_E)_{\rm an} \sim \psi_E$.

To study even-dimensional quasi-excellent forms, it is very convenient to give another definition.

Definition 0.1. We say that a sequence of quadratic forms $\varphi_0, \varphi_1, \dots, \varphi_h$ over F is *quasi-excellent* if the following conditions hold:

- the forms $\varphi_0, \dots, \varphi_{h-1}$ are regular and of dimension > 0 ;
- the form φ_0 is anisotropic and the form φ_h is zero;
- for $i = 1, \dots, h$, we have $((\varphi_0)_{F_i})_{\rm an} \sim (\varphi_i)_{F_i}$ where $F_i = F(\varphi_0, \dots, \varphi_{i-1})$.

Then the number h is called the *height of the sequence.* (It coincides with the height of φ_0 defined by Knebusch in [Kn1], 5.4.)

It is not difficult to show that we have a surjective map (see Lemma 2.2 and Corollary 2.4 below):

$$\{\text{quasi-excellent sequences}\} \to \{\text{even-dim. quasi-excellent anisotropic forms}\}$$
$$(\varphi_0, \varphi_1, \dots, \varphi_h) \mapsto \varphi_0$$

Any regular quadratic form of dimension $n > 0$ over F is isomorphic to a diagonal form $\langle a_1, \dots, a_n \rangle := a_1X_1^2 + \dots + a_nX_n^2$ with $a_1, \dots, a_n \in F^*$ and variables $X_1, \dots, X_n$. A d-fold *Pfister form* is a form of the type

$$\langle\langle a_1, \dots, a_d \rangle\rangle := \langle 1, -a_1 \rangle \otimes \dots \otimes \langle 1, -a_d \rangle \ .$$

Let $(\varphi_0, \varphi_1, \dots, \varphi_h)$ be a quasi-excellent sequence. We prove in Lemma 2.5 that φ_{h-1} is similar to some d-fold Pfister form, and then say that d is the *degree of the sequence.*

Example 0.2. Let $a_1, a_2, \dots, a_d, k_0, k_1, k_2, u, v, c \in F^*$. Set

$$\begin{aligned}
\phi_0 &= k_0 \langle\langle a_1, a_2, \dots, a_{d-1} \rangle\rangle \otimes (\langle\langle u, v \rangle\rangle \perp -c \langle\langle a_d \rangle\rangle) \,, \\
\phi_1 &= k_1 \langle\langle a_1, a_2, \dots, a_{d-1} \rangle\rangle \otimes \langle -u, -v, uv, a_d \rangle \,, \\
\phi_2 &= k_2 \langle\langle a_1, a_2, \dots, a_d \rangle\rangle \,, \\
\phi_3 &= 0 \,.
\end{aligned}$$

Suppose that ϕ_0 , ϕ_1 and ϕ_2 are anisotropic. Then the sequence $(\phi_0, \phi_1, \phi_2, \phi_3)$ is quasi-excellent of degree d (see Lemma 9.1). We notice that $\dim \phi_{h-1} = 2^d$, $\dim \phi_{h-2} = 2^{d+1}$ and $\dim \phi_{h-3} = 3 \cdot 2^d$.

Clearly, the sequences (ϕ_1, ϕ_2, ϕ_3) and (ϕ_2, ϕ_3) are also quasi-excellent. In particular, the forms ϕ_0, ϕ_1, ϕ_2, and ϕ_3 are quasi-excellent.

Definition 0.3. Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence of degree d. We say that the sequence is of the

- "first type" if $\dim \varphi_{h-2} \neq 2^{d+1}$ or $h = 1$
- "second type" if $\dim \varphi_{h-2} = 2^{d+1}$ and, if $h \geq 3$, $\dim \varphi_{h-3} \neq 3 \cdot 2^d$
- "third type" if $\dim \varphi_{h-2} = 2^{d+1}$ and $\dim \varphi_{h-3} = 3 \cdot 2^d$, (here $h \geq 3$).

Example 0.4. Let $(\phi_0, \phi_1, \phi_2, \phi_3)$ be the sequence constructed in Example 0.2. Assume that ϕ_0, ϕ_1 and ϕ_2 are anisotropic.

- The sequence (ϕ_2, ϕ_3) is of the first type,
- The sequence (ϕ_1, ϕ_2, ϕ_3) is of the second type,
- The sequence $(\phi_0, \phi_1, \phi_2, \phi_3)$ is of the third type.

According to Knebusch [Kn2], 7.4, a regular quadratic form ψ is called a *Pfister neighbor*, if there exist a Pfister form π, some $a \in F^*$, and a form η with $\dim \eta < \dim \psi$, such that $\psi \perp \eta \simeq a\pi$. The form η is called the *complementary* form of the Pfister neighbor ψ.

Example 0.5. Let $(\varphi_1, \dots, \varphi_h)$ be a quasi-excellent sequence. Let φ_0 be an anisotropic Pfister neighbor whose complementary form is similar to φ_1. Then the sequence $(\varphi_0, \varphi_1, \dots, \varphi_h)$ is quasi-excellent. Moreover, this sequence is of the same type as the sequence $(\varphi_1, \dots, \varphi_h)$. (Note that $((\varphi_0)_{F_1})_{\mathrm{an}} \sim (\varphi_1)_{F_1}$ by [Kn2], p. 3.)

Clearly, Examples 0.4 and 0.5 give rise to the construction of many examples of quasi-excellent sequences of prescribed type: We start with a quasi-excellent sequence given in Example 0.4. We can then apply the construction presented in Example 0.5 to obtain a new quasi-excellent sequence. Since we can apply the construction in Example 0.5 many times, we get quasi excellent sequences of arbitrary height.

The main goal of this paper is to prove (under certain assumptions) that all quasi-excellent sequences can be constructed by using this recursive procedure. To be more accurate, for sequences of the first type, we prove the following classification result:

Theorem 0.6. *Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence of the first type. Then for any $i < h$ the form φ_i is a Pfister neighbor whose complementary form is similar to φ_{i+1}.*

For sequences of the second and the third type, we state our classification results as conjectures which we will prove for sequences of degree 1. For sequences of arbitrary degree we will deduce our conjectures from some classical conjectures which now seem to be settled for all fields of characteristic 0, cf. [Vo, OVV].

Conjecture 0.7. *Let* $(\varphi_0, \varphi_1, \dots, \varphi_h)$ *be a quasi-excellent sequence of the second type. Then for any* $i < h-2$ *the form* φ_i *is a Pfister neighbor whose complementary form is similar to* φ_{i+1}*. Besides, the forms* φ_{h-2} *and* φ_{h-1} *look as follows:*

$$\varphi_{h-2} \sim \langle\langle a_1, \dots, a_{d-1}\rangle\rangle \otimes \langle -u, -v, uv, a_d\rangle\,,$$
$$\varphi_{h-1} \sim \langle\langle a_1, \dots, a_{d-1}, a_d\rangle\rangle\,.$$

(For $d = 1$ we put $\langle\langle a_1, \dots, a_{d-1}\rangle\rangle = \langle 1\rangle$.)

Conjecture 0.8. *Let* $(\varphi_0, \varphi_1, \dots, \varphi_h)$ *be a quasi-excellent sequence of the third type. Then for any* $i < h-3$ *the form* φ_i *is a Pfister neighbor whose complementary form is similar to* φ_{i+1}*. Besides, the forms* φ_{h-3} *,* φ_{h-2} *and* φ_{h-1} *look as follows:*

$$\varphi_{h-3} \sim \langle\langle a_1, \dots, a_{d-1}\rangle\rangle \otimes (\langle\langle u, v\rangle\rangle \perp -c\,\langle\langle a_d\rangle\rangle),$$
$$\varphi_{h-2} \sim \langle\langle a_1, \dots, a_{d-1}\rangle\rangle \otimes \langle -u, -v, uv, a_d\rangle\,,$$
$$\varphi_{h-1} \sim \langle\langle a_1, \dots, a_{d-1}, a_d\rangle\rangle\,.$$

The main results of this paper are Theorem 0.6 and the following two theorems.

Theorem 0.9. *Conjectures 0.7 and 0.8 are true for quasi-excellent sequences of degree* 1*. The well-known so far unpublished result by Rost, that the Milnor invariant* e^4 *is bijective, implies that 0.7 and 0.8 are also true for sequences of degree* 2 *.*

Theorem 0.10. *Modulo results proved in* [Vo, OVV] *both Conjectures 0.7 and 0.8 are true over any field of characteristic* 0*.*

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1. Notation and background material

We fix a ground field F of characteristic different from 2 and set $F^* = F\backslash\{0\}$. If two quadratic forms φ and ψ are isomorphic we write $\varphi \simeq \psi$. We say that φ and ψ are *similar* if $\varphi \simeq a\psi$ for some $a \in F^*$, and write $\varphi \sim \psi$. A regular quadratic form φ of dimension $\dim\varphi > 0$ is said to be *isotropic* if there is a non-zero vector v in the underlying vector space of φ such that $\varphi(v) = 0$, and *anisotropic* otherwise. The zero form 0 is assumed to be anisotropic. As has been shown by Witt [W], any regular quadratic form φ has a decomposition

$$\varphi \simeq i \times \langle 1, -1\rangle \perp \varphi_{\mathrm{an}}$$

where φ_{an} is anisotropic and $i \geq 0$. Moreover, the number $i =: i(\varphi)$ and, up to isomorphism, the form φ_{an} are uniquely determined by φ . We call $i(\varphi)$ the

Witt index of φ. If $i(\varphi) > 0$ then φ is isotropic. A form $\varphi \neq 0$ is said to be *hyperbolic* if $\varphi_{\mathrm{an}} = 0$. We have a *Witt equivalence* relation $\varphi \sim_{\mathrm{w}} \psi$ defined by

$$\varphi \sim_{\mathrm{w}} \psi \iff \varphi_{\mathrm{an}} \simeq \psi_{\mathrm{an}}$$

The Witt equivalence classes $[\varphi]$ of regular quadratic forms φ over F form a commutative ring $W(F)$ with zero element $[0]$ and unit element $[\langle 1\rangle]$. The operations in this *Witt ring* $W(F)$ are induced by:

$\langle a_1,\dots,a_m\rangle \perp \langle b_1,\dots,b_n\rangle = \langle a_1,\dots,a_m,b_1,\dots,b_n\rangle$

$\langle a_1,\dots,a_m\rangle \otimes \langle b_1,\dots,b_n\rangle = \langle a_1b_1,\dots,a_1b_n,\ \dots,a_mb_1,\dots,a_mb_n\rangle$.

In particular, there is a surjective ring homomorphism

$$e^0 : W(F) \to \mathbb{Z}/2\mathbb{Z},\ [\varphi] \mapsto (\dim\varphi) \bmod 2\mathbb{Z}.$$

Its kernel $I(F) := \ker(e^0)$ is called the *fundamental ideal* of $W(F)$. Since $\langle a,b\rangle \sim_{\mathrm{w}} \langle\langle -a\rangle\rangle \perp -\langle\langle b\rangle\rangle$ the ideal $I(F)$ is generated by the classes of the 1-fold Pfister forms $\langle\langle a\rangle\rangle = \langle 1,-a\rangle$ with $a \in F^*$. Consequently, the nth power ideal $I^n(F)$ of $I(F)$ is generated by the classes of n-fold Pfister forms $\langle\langle a_1,\dots,a_n\rangle\rangle = \langle 1,-a_1\rangle \otimes \cdots \otimes \langle 1,-a_n\rangle$. We will use the *Arason-Pfister Hauptsatz* [AP]:

Theorem 1.1. (Arason-Pfister) *If* $[\varphi] \in I^n(F)$ *and* $\dim\varphi_{\mathrm{an}} < 2^n$ *then* $\varphi \sim_{\mathrm{w}} 0$.

Put $F^{*2} = \{x^2 \in F^* \mid x \in F^*\}$ and $d(\varphi) := (-1)^{\binom{m}{2}}\det(\varphi)$ with $m = \dim\varphi$. Then there is a surjective group homomorphism

$$e^1 : I(F) \to F^*/F^{*2},\ [\varphi] \mapsto d(\varphi)\,F^{*2},$$

satisfying $\ker(e^1) = I^2(F)$, see [Pf1], 2.3.6. For any ideal I in $W(F)$ we write $\varphi \equiv \psi \bmod I$ when $[\varphi \perp -\psi] \in I$. Let $\mu = \langle a_1,\dots,a_m\rangle$ with $a_1,\dots,a_m \in F^*$. If m is odd then $d\big(\langle a_1,\dots,a_m,-d(\mu)\rangle\big) = \prod_{i=1}^m a_i^2 \in F^{*2}$, and we obtain the following remark which will be used for the classification of quasi-excellent sequences of the first type.

Remark 1.2. *If* $\dim\mu$ *is odd then* $\mu \equiv \langle\, d(\mu)\,\rangle \bmod I^2(F)$.

Of special interest for us is the *function field* $F(\varphi)$ of a regular quadratic form φ. Assuming that $\dim\varphi \geq 2$ and $\varphi \not\simeq \langle 1,-1\rangle$ we let $F(\varphi)$ be the function field of the projective variety defined by φ. Its transcendence degree is $(\dim\varphi) - 2$ and $\varphi_{F(\varphi)}$ is isotropic. Moreover, $F(\varphi)$ is purely transcendental over F iff φ is isotropic (cf. [Kn1], 3.8). We denote by $F(\varphi,\psi)$ the function field of the product of the varieties defined by the forms φ and ψ.

We say that φ is a *subform* of ψ, and write $\varphi \subset \psi$, if φ is isomorphic to an orthogonal summand of ψ. We will use the following two consequences of the *Cassels-Pfister Subform Theorem* [Pf1], 1.3.4:

Theorem 1.3. ([Kn1], 4.4, and [S], 4.5.4 (ii)) *Let* λ *be an anisotropic form and* ρ *be a Pfister form. Then the following conditions are equivalent:*

- *there exists a form* μ *such that* $\lambda \simeq \rho \otimes \mu$,
- *there exists a form* ν *such that* $\lambda \sim_{\mathrm{w}} \rho \otimes \nu$,
- $\lambda_{F(\rho)}$ *is hyperbolic.*

Moreover, in these cases $k\rho \subset \lambda$ *for any* $k \in D(\lambda) := \{a \in F^* \mid \langle a\rangle \subset \varphi\}$.

Theorem 1.4. ([Kn1], 4.5) *Let φ and ψ be forms of dimension ≥ 2 satisfying $\varphi \not\simeq \langle 1, -1\rangle$ and $\psi \not\sim_{\mathrm{w}} 0$. If $\psi_{F(\varphi)} \sim_{\mathrm{w}} 0$ then φ is similar to a subform of ψ, hence* $\dim\varphi \leq \dim\psi$.
Consequently, if $\dim\varphi > \dim\psi$ *then $\psi_{F(\varphi)}$ is not hyperbolic.*

If, in addition, φ and ψ are anisotropic and the dimensions of φ and ψ are separated by a 2-power, then $\psi_{F(\varphi)}$ is *not isotropic* as Hoffmann has shown.

Theorem 1.5. (Hoffmann ([H1], Theorem 1) *Let φ and ψ be anisotropic forms with* $\dim\psi \leq 2^n < \dim\varphi$ *for some $n > 0$. Then $\psi_{F(\varphi)}$ is anisotropic.*

In accordance with the definition given in the introduction, a form φ is a *Pfister neighbor* of a d-fold Pfister form π if $\dim\varphi > 2^{d-1}$ and φ is similar to a subform of π.

Theorem 1.6. (Hoffmann [H1], Corollaries 1, 2) *Let φ be an anisotropic form of dimension $2^n + m$ with $0 < m \leq 2^n$. Then $\dim(\varphi_{F(\varphi)})_{\mathrm{an}} \geq 2^n - m$.*
If, in addition, φ is a Pfister neighbor then $\dim(\varphi_{F(\varphi)})_{\mathrm{an}} = 2^n - m$.

The following theorem is a result by Izhboldin on "virtual Pfister neighbors", cf. [Izh], Theorem 3.5.

Theorem 1.7. (Izhboldin) *Let φ be an anisotropic form of dimension $2^n + m$ with $0 < m \leq 2^n$. Assume that there is a field extension E/F such that φ_E is an anisotropic Pfister neighbor. Then either $\dim(\varphi_{F(\varphi)})_{\mathrm{an}} \geq 2^n$ or* $\dim(\varphi_{F(\varphi)})_{\mathrm{an}} = 2^n - m$.

Theorem 1.8. (Knebusch [Kn2], 7.13) *Let φ and ψ be anisotropic forms such that $(\varphi_{F(\varphi)})_{\mathrm{an}} \simeq \psi_{F(\varphi)}$. Then φ is a Pfister neighbor and $-\psi$ is the complementary form of φ.*

The following theorem is a special case of the *Knebusch-Wadsworth Theorem* [Kn1], 5.8. It will be used in Lemma 2.5 below.

Theorem 1.9. (Knebusch-Wadsworth) *Let φ be an anisotropic form such that $\varphi_{F(\varphi)}$ is hyperbolic. Then φ is similar to a Pfister form.*

Knebusch introduced in [Kn1] a *generic splitting tower* $K_0 \subset K_1 \subset \cdots \subset K_h$ of a form $\psi \not\sim_{\mathrm{w}} 0$ which is easily described as follows. Let $K_0 = F$ and $\psi_0 \simeq \psi_{\mathrm{an}}$ and proceed inductively by letting $K_i = K_{i-1}(\psi_{i-1})$ and $\psi_i \simeq ((\psi_{i-1})_{K_i})_{\mathrm{an}}$. Then h is the *height* of ψ, that is the smallest number such that $\dim\psi_h \leq 1$.

The form ψ is excellent iff all forms ψ_i are defined over F (that is, for each i there exists a form η_i over F such that $\psi_i \simeq (\eta_i)_{K_i}$), cf. [Kn2], 7.14.

Now assume that $\dim\psi$ is even. Then $\psi_{h-1} \simeq a\pi$ for some $a \in K_{h-1}^*$ and some d-fold Pfister form π over K_{h-1} by Theorem 1.9. The form π is called the *leading form* of ψ and the number $d =: \deg\psi$ the *degree* of ψ. We say that ψ is a *good* form if π is defined over F. Then there is, up to isomorphism, a unique d-fold Pfister form τ over F such that $\pi \simeq \tau_{K_{h-1}}$, cf. [Kn2], 9.2, and we will refer to this Pfister form over F as the *leading form* of a good form.

Saying that all odd-dimensional forms have degree 0 and that the zero form has degree ∞ we get a *degree function*, cf. [Kn1], p. 88:

$$\deg : W(F) \to \mathbb{N} \cup \{0\} \cup \{\infty\}\,, \quad [\psi] \mapsto [\deg \psi]\,.$$

For every $n \geq 0$ let $J_n(F) := \{\, [\psi] \in W(F) \mid \deg \psi \geq n \,\}$. Then $J_n(F)$ is an ideal in the Witt ring $W(F)$ and $J_1(F) = I(F)$ is the fundamental ideal. We are now prepared to formulate the next result we will need later.

Theorem 1.10. (Knebusch [Kn2], 9.6, 7.14, and 10.1; Hoffmann [H2]) *Let ψ be an anisotropic good form of degree $d \geq 1$ with leading form τ. Then*

$$\psi \equiv \tau \bmod J_{d+1}(F)\,.$$

If, in addition, ψ is of height 2 then one of the following conditions holds.

- *The form ψ is excellent. In this case, ψ is a Pfister neighbor whose complementary form is similar to τ. In particular, $\dim \psi = 2^N - 2^d$ with $N \geq d+2$, and $\psi_{F(\tau)}$ is hyperbolic.*
- *The form ψ is non-excellent and good. In this case $\dim \psi = 2^{d+1}$ and $\psi_{F(\tau)}$ is similar to an anisotropic $(d+1)$-fold Pfister form.*

We denote by $P_d(F)$ (resp. $GP_d(F)$) the set of all quadratic forms over F which are isomorphic (resp. similar) to d-fold Pfister forms.

Finally, we mention the following well-known facts (e.g., [L], IX.1.1, X.1.6).

Remark 1.11. (i) *Anisotropic forms over F remain anisotropic over purely transcendental extensions of F.*

(ii) *Isotropic Pfister forms are hyperbolic.*

2. Elementary properties of quasi-excellent forms and sequences

Lemma 2.1. *Let $(\varphi_0, \ldots, \varphi_h)$ be a quasi-excellent sequence. Then*

- *all forms φ_i are forms of even dimension,*
- *all forms φ_i are anisotropic,*
- $\dim \varphi_0 > \dim \varphi_1 > \cdots > \dim \varphi_h = 0$,
- *for all $s = 1, \ldots, h$, we have*

$$((\varphi_0)_{F_s})_{\mathrm{an}} \sim ((\varphi_1)_{F_s})_{\mathrm{an}} \sim \cdots \sim ((\varphi_{s-1})_{F_s})_{\mathrm{an}} \sim (\varphi_s)_{F_s}$$

where $F_s = F(\varphi_0, \ldots, \varphi_{s-1})$.

Proof. Obvious from Definition 0.1. □

Lemma 2.2. *Let φ be an anisotropic even-dimensional quasi-excellent form over F. Then there exists a quasi-excellent sequence $(\varphi_0, \varphi_1, \ldots, \varphi_h)$ such that $\varphi_0 = \varphi$.*

Proof. Let us define the forms φ_i recursively. We set $\varphi_0 = \varphi$. Now, we suppose that $i > 0$ and that all forms $\varphi_0, \ldots, \varphi_{i-1}$ are already defined. Also, we can suppose that these forms are of dimension > 0. Put $F_i = F(\varphi_0, \ldots, \varphi_{i-1})$. Since φ is quasi-excellent, there exists a form ψ over F such that $(\varphi_{F_i})_{\mathrm{an}}$ is similar to ψ_{F_i}. We put $\varphi_i = \psi$. If $\varphi_i = 0$ then we are done by setting $h = i$. If $\varphi_i \neq 0$ we repeat the above procedure. $\square$

Lemma 2.3. *Let $(\varphi_0, \varphi_1, \dots, \varphi_h)$ be a quasi-excellent sequence. Let E/F be a field extension such that $(\varphi_0)_E$ is isotropic, and let i be the maximal integer such that all forms $(\varphi_0)_E, \dots, (\varphi_{i-1})_E$ are isotropic. Then $((\varphi_0)_E)_{\mathrm{an}} \sim (\varphi_i)_E$.*

Proof. Since the forms $(\varphi_0)_E, \dots, (\varphi_{i-1})_E$ are isotropic, the field extension $E_i := E(\varphi_0, \dots, \varphi_{i-1})$ is purely transcendental over E. Since $F_i \subset E_i$ and $((\varphi_0)_{F_i})_{\mathrm{an}} \sim (\varphi_i)_{F_i}$, it follows that $((\varphi_0)_{E_i})_{\mathrm{an}} \sim ((\varphi_i)_{E_i})_{\mathrm{an}}$. Since E_i/E is purely transcendental we can use Springer's theorem (e.g., [L], 6.1.7) to obtain $((\varphi_0)_E)_{\mathrm{an}} \sim ((\varphi_i)_E)_{\mathrm{an}}$. By definition of i, the form $(\varphi_i)_E$ is anisotropic. $\square$

Corollary 2.4. *Let $(\varphi_0, \varphi_1, \dots, \varphi_h)$ be a quasi-excellent sequence. Then the form φ_0 is a quasi-excellent even-dimensional form.*

Proof. Obvious from Lemmas 2.1 and 2.3. $\square$

Lemma 2.5. *Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence. Then the form φ_{h-1} is similar to a Pfister form.*

Proof. By Definition 0.1, we have $\varphi_h = 0$ and $((\varphi_{h-1})_{F_h})_{\mathrm{an}} \sim (\varphi_h)_{F_h}$, where $F_h = F(\varphi_0, \dots, \varphi_{h-1})$. Therefore, $(\varphi_{h-1})_{F_h}$ is hyperbolic. Note that $F_h \simeq F(\varphi_{h-1})(\varphi_0, \dots, \varphi_{h-2})$. Since the dimensions of the forms $\varphi_0, \dots, \varphi_{h-2}$ are strictly greater than $\dim \varphi_{h-1}$ and $(\varphi_{h-1})_{F(\varphi_{h-1})(\varphi_0,\dots,\varphi_{h-2})}$ is hyperbolic, it follows from Theorem 1.4 that $(\varphi_{h-1})_{F(\varphi_{h-1})}$ is hyperbolic. By Theorem 1.9, φ_{h-1} is similar to a Pfister form. $\square$

Definition 2.6. Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence.

- By Lemma 2.5, the form φ_{h-1} is similar to some Pfister form $\tau \in P_d(F)$. We say that τ is the *leading form* and d is the *degree* of the sequence. Besides, we say that h is the *height* of the sequence.
- The form φ_{h-2} is called the *pre-leading form* of the sequence. Clearly, here we assume that $h \geq 2$.

Remark 2.7. *Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence. Then its leading form is the leading form of φ_0 as well. In particular, φ_0 is a good form whose height and degree coincide with the height and degree of the sequence.*

We finish this section with a lemma which we will need for the classification of quasi-excellence sequences of the second and third type.

Lemma 2.8. *Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence with $h \geq 2$. Let E/F be an extension such that $(\varphi_0)_E$ is an anisotropic Pfister neighbor whose complementary form is similar to $(\varphi_2)_E$. Then $\dim \varphi_1$ is a power of 2 and $\dim \varphi_0 = 2 \dim \varphi_1 - \dim \varphi_2$.*

Proof. Let us write $\dim\varphi_0$ in the form $\dim\varphi_0 = 2^n + m$ with $0 < m \leq 2^n$. Since $(\varphi_2)_E$ is similar to the complementary form of $(\varphi_0)_E$, we have $\dim\varphi_2 = 2^{n+1} - \dim\varphi_0 = 2^n - m$. Since $\dim\varphi_1 = \dim((\varphi_0)_{F(\varphi_0)})_{\mathrm{an}}$, Theorem 1.7 shows that either $\dim\varphi_1 \geq 2^n$ or $\dim\varphi_1 = 2^n - m$. The equality $\dim\varphi_1 = 2^n - m$ is obviously false because $\dim\varphi_2 = 2^n - m$. Therefore, $\dim\varphi_1 \geq 2^n$. If $\dim\varphi_1 = 2^n$ then $\dim\varphi_0 = 2^n + m = 2\cdot 2^n - (2^n - m) = 2\dim\varphi_1 - \dim\varphi_2$ and the proof is complete. Hence, we can assume that $\dim\varphi_1 > 2^n$. Then $2^n < \dim\varphi_1 < \dim\varphi_0 = 2^n + m$. Therefore $\dim\varphi_1$ can be written in the form $2^n + m_1$ with $0 < m_1 < m \leq 2^n$. Let $K = F(\varphi_0)$. Then Lemma 2.1 shows that $((\varphi_1)_{K(\varphi_1)})_{\mathrm{an}}$ is similar to $(\varphi_2)_{K(\varphi_1)}$. Hence, $\dim((\varphi_1)_{K(\varphi_1)})_{\mathrm{an}} = \dim\varphi_2$. Since $\dim\varphi_1 = 2^n + m_1$, Theorem 1.6 shows that $\dim\varphi_2 = \dim((\varphi_1)_{K(\varphi_1)})_{\mathrm{an}} \geq 2^n - m_1$. Since $\dim\varphi_2 = 2^n - m$, we get $m_1 \geq m$. This contradicts to the inequality $m_1 < m$ proved earlier. □

3. Inductive properties of quasi-excellent sequences

In this section we study further properties of a quasi-excellent sequence $(\varphi_0, \ldots, \varphi_h)$ of degree d with leading form τ. Then we derive some results on its pre-leading form $\gamma := \varphi_{h-2}$. In particular, we show that $\dim\gamma$ is either 2^{d+1} or $2^N - 2^d$ with $N \geq d+2$ and that $(\varphi_i)_{F(\gamma,\tau)}$ is hyperbolic for all $i = 0, \ldots, h-1$.

Lemma 3.1. *Let $(\varphi_0, \ldots, \varphi_h)$ be a quasi-excellent sequence and let $E = F(\varphi_1)$.*

- *If $(\varphi_0)_E$ is isotropic then $(\varphi_1, \varphi_2, \ldots, \varphi_h)$ is a quasi-excellent sequence and $((\varphi_0)_E)_{\mathrm{an}} \sim (\varphi_2)_E$.*
- *If $(\varphi_0)_E$ is anisotropic then $\big((\varphi_0)_E, (\varphi_2)_E, (\varphi_3)_E, \ldots, (\varphi_h)_E\big)$ is a quasi-excellent sequence.*

Proof. Let $F_i = F(\varphi_0, \ldots, \varphi_{i-1})$ and $F_{0,i} = F(\varphi_1, \ldots, \varphi_{i-1})$. Assume that φ_0 is isotropic over $F(\varphi_1)$. Then the extension $F_i/F_{0,i}$ is purely transcendental for all $i \geq 2$. By Lemma 2.1, we have $((\varphi_1)_{F_i})_{\mathrm{an}} \sim (\varphi_i)_{F_i}$ for all $i \geq 1$. Since $F_i/F_{0,i}$ is purely transcendental for $i \geq 2$, we have $((\varphi_1)_{F_{0,i}})_{\mathrm{an}} \sim (\varphi_i)_{F_{0,i}}$ for all $i \geq 2$. This means that the sequence $(\varphi_1, \varphi_2, \ldots, \varphi_h)$ is quasi-excellent. Now Lemma 2.3 implies that $((\varphi_0)_E)_{\mathrm{an}} \sim (\varphi_2)_E$. The last statement is obvious from Definition 0.1. □

Lemma 3.2. *Let $(\varphi_0, \ldots, \varphi_h)$ be a quasi-excellent sequence. Suppose that φ_0 is a Pfister neighbor whose complementary form is similar to φ_1. Then the sequence $(\varphi_1, \varphi_2, \ldots, \varphi_h)$ is quasi-excellent.*

Proof. By Lemma 3.1, it suffices to show that $(\varphi_0)_{F(\varphi_1)}$ is isotropic. By assumption, there is a form $\eta \sim \varphi_1$ and a Pfister form π such that $\varphi_0 \perp \eta \sim \pi$. Since $\eta_{F(\varphi_1)}$ is isotropic, the form $\pi_{F(\varphi_1)}$ must be hyperbolic. Since $\dim\varphi_0 > \dim\eta$ it follows that $(\varphi_0)_{F(\varphi_1)}$ is isotropic. □

Lemma 3.3. *Let* $(\varphi_0, \ldots, \varphi_h)$ *be a quasi-excellent sequence of height* $h \geq 2$ *with leading form* $\tau \in P_d(F)$ *and let* $F_i = F(\varphi_0, \ldots, \varphi_{i-1})$ *. Then the sequence*

$$\big((\varphi_i)_{F_i}, (\varphi_{i+1})_{F_i}, \ldots, (\varphi_h)_{F_i}\big)$$

is quasi-excellent of height $h-i$ *with leading form* $\tau_{F_i} \in P_d(F_i)$ *for* $1 \leq i < h$ *.*

Proof. By Lemma 2.1, the forms $(\varphi_s)_{F_s}$ are anisotropic for $s = 1, \ldots, h$. Thus $(\varphi_s)_{F_i}$ is anisotropic for fixed $i < h$ and $s = i, \ldots, h$. In particular, τ_{F_i} is anisotropic since $\tau_{F_i} \sim (\varphi_{h-1})_{F_i}$ by Definition 2.6. Now the result is obvious. □

Lemma 3.4. *Let* $(\varphi_0, \ldots, \varphi_h)$ *be a quasi-excellent sequence of degree* d *with leading form* τ *. Then for all* $i = 0, \ldots, h-1$*, we have* $\varphi_i \equiv \tau \bmod J_{d+1}(F)$ *. In particular,* $\deg(\varphi_i) = \deg(\tau) = d$ *for all* $i = 0, \ldots, h-1$ *.*

Proof. By Remark 2.7 and Theorem 1.10, we have $\varphi_0 \equiv \tau \bmod J_{d+1}(F)$. Using Lemma 3.3 we obtain from Remark 2.7 and Theorem 1.10 that

$$(\varphi_i)_{F_i} \equiv \tau_{F_i} \bmod J_{d+1}(F_i)$$

for all $i = 1, \ldots, h-1$. Since $\dim \varphi_0 > \ldots > \dim \varphi_{h-2} > 2^d = \dim \varphi_{h-1}$, the canonical map $J_d(F_{i-1})/J_{d+1}(F_{i-1}) \to J_d(F_i)/J_{d+1}(F_i)$ is injective for $i = 1, \ldots, h-1$ and $F_0 = F$ as Knebusch [Kn1], 6.11, has shown. Thus the composed map $J_d(F)/J_{d+1}(F) \to J_d(F_i)/J_{d+1}(F_i)$ is also injective. Hence $\varphi_i \equiv \tau \bmod J_{d+1}(F)$ for $i = 1, \ldots, h-1$. The second statement follows from the first since $J_d(F)$ is an ideal in the Witt ring $W(F)$. □

Lemma 3.5. *Let* γ *and* τ *be anisotropic forms. Suppose that* $\dim \gamma = 2^{d+1}$ *and* $\tau \in P_d(F)$ *for suitable* d*. Suppose also that* $\gamma_{F(\gamma)}$ *is not hyperbolic and* $\gamma_{F(\gamma,\tau)}$ *is hyperbolic. Then the form* $(\gamma_{F(\gamma)})_{\mathrm{an}}$ *is similar to* $\tau_{F(\gamma)}$ *.*

Proof. Let $K = F(\gamma)$. By assumption, the form $\gamma_{K(\tau)}$ is hyperbolic. Thus Theorem 1.3 implies that there exists a K-form μ such that $(\gamma_K)_{\mathrm{an}} \simeq \tau_K \otimes \mu$. Since $\dim \tau = 2^d$ and $\dim(\gamma_K)_{\mathrm{an}} = \dim(\gamma_{F(\gamma)})_{\mathrm{an}} < \dim \gamma = 2^{d+1}$, it follows that $\dim \mu < 2^{d+1}/2^d = 2$. Hence, $\dim \mu = 0$ or 1.

If $\dim \mu = 0$ then $(\gamma_K)_{\mathrm{an}} = 0$. Then $\gamma_{F(\gamma)} = \gamma_K$ is hyperbolic. We get contradiction to the hypothesis of the lemma.

If $\dim \mu = 1$, then the isomorphism $(\gamma_K)_{\mathrm{an}} \simeq \tau_K \otimes \mu$ shows that $(\gamma_K)_{\mathrm{an}}$ is similar to τ_K. The lemma is proved. □

Proposition 3.6. *Let* $(\varphi_0, \ldots, \varphi_h)$ *be a quasi-excellent sequence with leading form* $\tau \in P_d(F)$ *and pre-leading form* $\gamma = \varphi_{h-2}$ *. Then*

(1) $\dim \gamma = 2^{d+1}$ *or* $\dim \gamma = 2^N - 2^d$ *with* $N \geq d+2$*.*

(2) *If* $\dim \gamma = 2^{d+1}$ *then* γ *is a good non-excellent form of height* 2 *and degree* d *with leading form* τ*.*

(3) *If* $\dim \gamma \neq 2^{d+1}$ *then* γ *is excellent and* $\gamma_{F(\tau)}$ *is hyperbolic.*

Proof. (1). Let $E = F_{h-2} = F(\varphi_0, \dots, \varphi_{h-3})$. By Lemma 3.3 with $i = h-2$, the sequence $(\gamma_E, (\varphi_{h-1})_E, (\varphi_h)_E)$ is quasi-excellent of height 2 with leading form $\tau_E \in P_d(E)$. Thus Remark 2.7 implies that γ_E is a good form of height 2, degree d, and leading form τ_E. By Theorem 1.10, there are two types of good forms of height 2, *non-excellent* and *excellent*.

If γ_E is good non-excellent of height 2 and degree d, then $\dim \gamma = 2^{d+1}$.

If γ_E is excellent form of height 2 and degree d, then $\dim \gamma = 2^N - 2^d$ with $N \geq d+2$.

(2). Assume that $\dim \gamma = 2^{d+1}$. We have to prove that $(\gamma_{F(\gamma)})_{\mathrm{an}}$ is similar to $\tau_{F(\gamma)}$. By Lemma 3.5, it suffices to verify that $\gamma_{F(\gamma)}$ is not hyperbolic and $\gamma_{F(\gamma,\tau)}$ is hyperbolic.

Since $E(\gamma) = F_{h-1}$, we have $(\gamma_{E(\gamma)})_{\mathrm{an}} \sim (\varphi_{h-1})_{E(\gamma)} \sim \tau_{E(\gamma)}$ by Lemma 2.1 and Definition 2.6. This shows that $\gamma_{F(\gamma)}$ is not hyperbolic and that $\gamma_{E(\gamma,\tau)}$ is hyperbolic. Since $E = F(\varphi_0, \dots, \varphi_{h-3})$ is the function field of forms of dimension $> \dim \varphi_{h-2} = \dim \gamma$ and $\gamma_{E(\gamma,\tau)}$ is hyperbolic, it follows from Theorem 1.4 that $\gamma_{F(\gamma,\tau)}$ is also hyperbolic. By Lemma 3.5, we are done.

(3). If $\dim \gamma \neq 2^{d+1}$ then γ_E is an excellent form of height 2 and degree d with the leading form τ_E. In this case $\gamma_{E(\tau)}$ is hyperbolic by Theorem 1.10. Hence $\gamma_{F(\tau)}$ is also hyperbolic by Theorem 1.4. □

Proposition 3.7. *Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence with leading form $\tau \in P_d(F)$ and pre-leading form $\gamma = \varphi_{h-2}$. Then $(\varphi_i)_{F(\gamma,\tau)}$ is hyperbolic for all $i = 0, \dots, h-1$.*

Proof. If $h = 1$ then $\varphi_0 \sim \tau$ by Definition 2.6, hence $(\varphi_0)_{F(\tau)}$ is hyperbolic. If $h = 2$ then the statement is obvious as well. Thus, we can assume that $h \geq 3$. We use induction on h.

Let $E = F(\varphi_0)$. By Lemma 3.3, $((\varphi_1)_E, (\varphi_2)_E, \dots, (\varphi_h)_E)$ is quasi-excellent. By induction assumption, $(\varphi_i)_{E(\gamma,\tau)}$ is hyperbolic for all $i = 1, \dots, h-1$. Since $E(\gamma,\tau) = F(\gamma,\tau,\varphi_0)$ and $\dim \varphi_0$ is strictly greater than the dimensions of all forms $\varphi_1, \dots, \varphi_{h-1}$, Theorem 1.4 shows that the forms $(\varphi_i)_{F(\gamma,\tau)}$ are hyperbolic for all $i = 1, \dots, h-1$.

Now, it suffices to prove that $(\varphi_0)_{F(\gamma,\tau)}$ is hyperbolic. We consider three cases and use the following *observation*. Since $(\varphi_1)_{F(\gamma,\tau)}$ is hyperbolic, hence isotropic, it follows that $F(\gamma,\tau,\varphi_1)$ is purely transcendental over $F(\gamma,\tau)$.

Case 1. The form $(\varphi_0)_{F(\varphi_1)}$ is isotropic.

Then $(\varphi_0)_{F(\gamma,\tau,\varphi_1)}$ is isotropic. Thus $(\varphi_0)_{F(\gamma,\tau)}$ is isotropic too by the above observation. Since the forms $(\varphi_i)_{F(\gamma,\tau)}$ are hyperbolic for all $i = 1, \dots, h-1$, Lemma 2.3 applies with $i = h$ so that $((\varphi_0)_{F(\gamma,\tau)})_{\mathrm{an}} \sim (\varphi_h)_{F(\gamma,\tau)} = 0$.

Case 2. The form $(\varphi_0)_{F(\varphi_1)}$ is anisotropic and $h = 3$.

In this case $\gamma = \varphi_1$. Let $E = F(\varphi_1) = F(\gamma)$. By Lemma 3.1, the sequence $((\varphi_0)_E, (\varphi_2)_E, 0)$ is quasi-excellent of height 2.

Clearly, $\dim \varphi_2 = \dim \tau = 2^d$, and $\dim \varphi_0 > \dim \varphi_1 = \dim \gamma \geq 2^{d+1}$ by Proposition 3.6. Since $\dim(\varphi_0)_E \neq 2^{d+1}$ it follows that $(\varphi_0)_E$ is excellent of height 2 with leading form τ_E. Hence $(\varphi_0)_{E(\tau)}$ is hyperbolic (see Theorem 1.10). Since $E(\tau) = F(\gamma, \tau)$, we are done.

Case 3. The form $(\varphi_0)_{F(\varphi_1)}$ is anisotropic and $h \geq 4$.

Let $E = F(\varphi_1)$. By Lemma 3.1, the sequence

$$((\varphi_0)_E, (\varphi_2)_E, \ldots, (\varphi_{h-2})_E, (\varphi_{h-1})_E, 0)$$

is a quasi-excellent of height $h-1$. Clearly, τ_E is the leading form and $\gamma_E = (\varphi_{h-2})_E$ is the pre-leading form of this sequence (we note, that here we use the condition $h \geq 4$). Applying the induction hypothesis, we see that the form $(\varphi_0)_{E(\gamma,\tau)}$ is hyperbolic. Therefore, $(\varphi_0)_{F(\gamma,\tau)}$ is hyperbolic by the above observation. □

4. Classification theorem for sequences of the first type

Recall that a quasi-excellent sequence $(\varphi_0, \ldots, \varphi_h)$ of degree d is of the *first type* if $\dim \varphi_{h-2} \neq 2^{d+1}$ or if $h = 1$.

LEMMA 4.1. *Let $(\varphi_0, \ldots, \varphi_h)$ be a quasi-excellent sequence of the first type with leading form $\tau \in P_d(F)$. Then*

(i) *the form $(\varphi_i)_{F(\tau)}$ is hyperbolic for all $i = 0, \ldots, h-1$,*

(ii) *for every $i = 0, \ldots, h-1$ there exists an odd-dimensional form μ_i such that $\varphi_i \simeq \mu_i \otimes \tau$,*

(iii) *φ_0 is a Pfister neighbor, whose complementary form is similar to φ_1.*

Proof. (i). For $h = 1$ the statement follows from Remark 1.11. Now assume that $h \geq 2$, and put $\gamma = \varphi_{h-2}$. By Proposition 3.6 (3), the form $\gamma_{F(\tau)}$ is isotropic. Hence, the extension $F(\gamma, \tau)/F(\tau)$ is purely transcendental. This implies, since $(\varphi_i)_{F(\gamma,\tau)}$ is hyperbolic by 3.7, that $(\varphi_i)_{F(\tau)}$ is hyperbolic.

(ii). By Theorem 1.3 and (i), there exists a form μ_i such that $\varphi_i \simeq \mu_i \otimes \tau$. Thus it suffices to prove that $\dim \mu_i$ is odd. If we assume that μ_i is an even-dimensional form, then we get $[\varphi_i] \in I(F) \cdot I^d(F) = I^{d+1}(F)$. This contradicts to Lemma 3.4, where we have proved that $\deg(\varphi_i) = d$ for all $i = 0, \ldots, h-1$, since $I^{d+1}(F) \subset J_{d+1}(F)$ by [Kn1], 6.6.

(iii). Let $K = F(\varphi_0)$. By Definition 0.1, there exists $x \in K^*$ such that $((\varphi_0)_K)_{\mathrm{an}} \simeq x\,(\varphi_1)_K$. By (ii), this implies

$$(*) \qquad (\mu_0 \otimes \tau)_K \sim_{\mathrm{w}} x\,(\mu_1 \otimes \tau)_K\,.$$

Let $s_0 = d(\mu_0)$ and $s_1 = d(\mu_1)$. Since μ_0 and μ_1 are both of odd dimension, we have $\mu_0 \equiv \langle s_0 \rangle \mod I^2(F)$ and $\mu_1 \equiv \langle s_1 \rangle \mod I^2(F)$ by Remark 1.2. Thus $s_0\tau_K \equiv (\mu_0 \otimes \tau)_K \mod I^{d+2}(K)$ and $x(\mu_1 \otimes \tau)_K \equiv xs_1\tau_K \mod I^{d+2}(K)$, since $\tau \in I^d(F)$. This yields $s_0\tau_K \equiv xs_1\tau_K \mod I^{d+2}(K)$ by $(*)$. Setting $s = s_0 s_1$ we obtain $s\tau_K \equiv x\tau_K \mod I^{d+2}(K)$. Theorem 1.1 now shows that $s\tau_K \simeq x\tau_K$.

Therefore, (ii) and the above yield $((\varphi_0)_K)_{\mathrm{an}} \simeq x(\varphi_1)_K \simeq x(\mu_1 \otimes \tau)_K \simeq (\mu_1)_K \otimes x\tau_K \simeq s(\mu_1 \otimes \tau)_K \simeq (s\varphi_1)_K$. Theorem 1.8 now shows that φ_0 is a Pfister neighbor whose complementary form is isomorphic to $-s\varphi_1$. □

The following theorem proves Theorem 0.6.

Theorem 4.2. *Let $(\varphi_0,\dots,\varphi_h)$ be a sequence of anisotropic forms. Then this sequence is a quasi-excellent sequence of the first type if and only if the following two conditions hold.*

- *For any $i=0,\dots,h-1$, the form φ_i is a Pfister neighbor whose complementary form is similar to φ_{i+1}.*
- *The form φ_h is zero.*

Proof. Obvious induction by using Lemma 3.2 and Lemma 4.1 (iii). The converse direction follows from Example 0.5 starting with the sequence $\{\varphi_{h-1},\,\varphi_h=0\}$. □

5. Quasi-excellent sequences modulo some ideals

Let $I(F)$ be the ideal of classes of even-dimensional forms in the Witt ring $W(F)$, and let $I^n(F)$ denote the nth power of $I(F)$. We need the following proposition for the classification of sequences of the second and third type.

Proposition 5.1. *Let $(\varphi_0,\varphi_1,\dots,\varphi_h)$ be a quasi-excellent sequence. Suppose that there exists an integer k such that $1\le k<h$ with the following property: for all $i=0,\dots,k-1$, there exists $f_i\in F^*$ such that $\varphi_i\equiv f_i\,\varphi_k \bmod I^{m+1}(F)$ where m is a minimal integer such that $\dim\varphi_k<2^m$.*

Then φ_0 is a Pfister neighbor whose complementary form is similar to φ_1.

Proof. For convenience, we will include in our consideration also the case where $k=0$ and prove, by induction, the following two properties:

(a) If $k\ge 1$ then φ_0 is a Pfister neighbor whose complementary form is similar to φ_1.
(b) If $k\ge 0$ then for any $x\in F^*$ the conditions $\varphi_0\equiv x\varphi_0 \bmod I^{m+1}(F)$ and $\dim\varphi_k<2^m$ imply that $\varphi_0\simeq x\varphi_0$.

For a given k we denote properties (a) and (b) by $(a)_k$ and $(b)_k$ correspondingly. The plan of the proof of Proposition 5.1 will be the following.

- We start with the proof of property $(b)_0$.
- For $k\ge 1$, we prove that $(b)_{k-1}\Rightarrow(a)_k$.
- For $k\ge 1$, we prove that $(b)_{k-1}\Rightarrow(b)_k$.

Lemma 5.2. *Condition (b) holds in the case $k=0$.*

Proof. If $k=0$ we have in (b) the conditions $[\varphi_0\perp -x\varphi_0]\in I^{m+1}(F)$ and $\dim\varphi_0<2^m$, in particular $\dim(\varphi_0\perp -x\varphi_0)<2^{m+1}$. By Theorem 1.1, the form $\varphi_0\perp -x\varphi_0$ is hyperbolic. Therefore $\varphi_0\simeq x\varphi_0$. □

Lemma 5.3. *Let $k\ge 1$. Then property $(b)_{k-1}$ (stated for all quasi-excellent sequences over all fields of characteristic $\ne 2$) implies property $(a)_k$.*

Proof. Consider a sequence $(\varphi_0, \dots, \varphi_h)$ as in Proposition 5.1. Then, by assumption, there exist $f_0, f_1 \in F^*$ such that $\varphi_0 \equiv f_0\varphi_k \bmod I^{m+1}(F)$ and $\varphi_1 \equiv f_1\varphi_k \bmod I^{m+1}(F)$. Hence $\varphi_0 \equiv f_0 f_1 \varphi_1 \bmod I^{m+1}(F)$.

Let $E = F(\varphi_0)$. By Definition 0.1, there exists $x \in E^*$ such that $((\varphi_0)_E)_{\mathrm{an}} \simeq x(\varphi_1)_E$. Hence $x(\varphi_1)_E \equiv (\varphi_0)_E \equiv f_0 f_1 (\varphi_1)_E \bmod I^{m+1}(E)$. Property $(b)_{k-1}$ stated for the quasi-excellent sequence $((\varphi_1)_E, (\varphi_2)_E, \dots, (\varphi_k)_E, \dots, (\varphi_h)_E)$ shows that $x(\varphi_1)_E \simeq f_0 f_1 (\varphi_1)_E$. Hence $((\varphi_0)_E)_{\mathrm{an}} \simeq x(\varphi_1)_E \simeq (f_0 f_1 \varphi_1)_E$. By Theorem 1.8, φ_0 is a Pfister neighbor whose complementary form is similar to φ_1. □

Lemma 5.4. *Let $k \geq 1$. Then property $(b)_{k-1}$ (stated for all quasi-excellent sequences over all fields of characteristic $\neq 2$) implies property $(b)_k$.*

Proof. Consider a sequence $(\varphi_0, \dots, \varphi_h)$ as in Proposition 5.1. We assume that property $(b)_{k-1}$ holds. Then property $(a)_k$ also holds (see previous lemma). This means that there exist $a, s \in F^*$, an integer $n > 0$, and an n-fold Pfister form π such that $a\pi \simeq \varphi_0 \perp -s\varphi_1$ and $\dim \varphi_1 < 2^{n-1}$.

Since $2^{n-1} > \dim \varphi_1 \geq \dim \varphi_k$, the definition of m yields $n - 1 \geq m$. Therefore $[a\pi] \in I^n(F) \subset I^{m+1}(F)$. Hence, $\varphi_0 \equiv s\varphi_1 \bmod I^{m+1}(F)$. Now, let $x \in F^*$ be as in $(b)_k$. In other words, $\varphi_0 \equiv x\varphi_0 \bmod I^{m+1}(F)$. Then $s\varphi_1 \equiv \varphi_0 \equiv x\varphi_0 \equiv sx\varphi_1 \bmod I^{m+1}(F)$. Hence, $\varphi_1 \equiv x\varphi_1 \bmod I^{m+1}(F)$. Property $(b)_{k-1}$, applied to the quasi-excellent sequence

$$((\varphi_1)_{F(\varphi_0)}, (\varphi_2)_{F(\varphi_0)}, \dots, (\varphi_k)_{F(\varphi_0)}, \dots, (\varphi_h)_{F(\varphi_0)})$$

shows that $(\varphi_1)_{F(\varphi_0)} \simeq x(\varphi_1)_{F(\varphi_0)}$. Hence the form $\varphi_1 \otimes \langle\langle x \rangle\rangle$ is hyperbolic over $F(\varphi_0)$. Since φ_0 is a Pfister neighbor of π, it follows that $F(\varphi_0, \pi)/F(\pi)$ is purely transcendental. Thus $\varphi_1 \otimes \langle\langle x \rangle\rangle$ is also hyperbolic over $F(\pi)$. Since $\dim(\varphi_1 \otimes \langle\langle x \rangle\rangle) < 2^{n-1} \cdot 2 = \dim \pi$, Theorem 1.4 shows that $\varphi_1 \otimes \langle\langle x \rangle\rangle$ is hyperbolic, hence $\varphi_1 \simeq x\varphi_1$. Moreover, $a\pi \otimes \langle\langle x \rangle\rangle \simeq (\varphi_0 \perp -s\varphi_1) \otimes \langle\langle x \rangle\rangle$ is isotropic and therefore hyperbolic (since π is a Pfister form). Hence $a\pi \simeq xa\pi$. Since $a\pi \simeq \varphi_0 \perp -s\varphi_1$ and $\varphi_1 \simeq x\varphi_1$, it follows that $\varphi_0 \simeq x\varphi_0$. □

Clearly, the three lemmas complete the proof of Proposition 5.1. □

6. Five classical conjectures

Let $H^n(F) := H^n(F, \mathbb{Z}/2\mathbb{Z})$ be the nth Galois cohomology group. Let $I^0(F) := W(F)$ be the Witt ring and $I^1(F) := I(F)$ be the fundamental ideal in $W(F)$ of classes of even-dimensional forms. In §1, we have considered the homomorphisms

$$e^0 : I^0(F) \to \mathbb{Z}/2\mathbb{Z} \simeq H^0(F) \quad \text{and} \quad e^1 : I^1(F) \to F^*/F^{*2} \simeq H^1(F)$$

defined by the *dimension* and the *discriminant* respectively. Denoting by ${}_2\mathrm{Br}(F)$ the 2-torsion part of the Brauer group of F we obtain a homomorphism $e^2 : I^2(F) \to {}_2\mathrm{Br}(F) \simeq H^2(F)$ defined by the *Clifford algebra.* We have $\ker(e^n) = I^{n+1}(F)$ for $n = 0, 1, 2$.

For each integer $n > 0$ let $(a_1) \cdot \ldots \cdot (a_n)$ denote the cup-product where (a_i) is the class of $a_i \in F^*$ in $H^1(F)$ for $i = 1, \ldots, n$. The following five conjectures are believed to be true for all fields F of characteristic $\neq 2$.

Conjecture 6.1. (Milnor conjecture). *Let $n \geq 0$ be an integer. Then there exists a homomorphism*

$$e^n : I^n(F) \to H^n(F)$$

such that $\langle\langle a_1, \ldots, a_n \rangle\rangle \mapsto (a_1) \cdot \ldots \cdot (a_n)$. *Moreover, the homomorphism e^n induces an isomorphism*

$$e^n : I^n(F)/I^{n+1}(F) \simeq H^n(F).$$

Conjecture 6.2. *For any $\pi \in P_m(F)$ and all integers $n \geq m \geq 0$, we have*

$$\ker(H^n(F) \to H^n(F(\pi))) = e^m(\pi)\, H^{n-m}(F).$$

Conjecture 6.3. *We have $J_n(F) = I^n(F)$ for all integers $n \geq 0$.*

Conjecture 6.4. *Let φ be an anisotropic form over F. If $[\varphi] \in I^n(F)$ and $2^n \leq \dim \varphi < 2^n + 2^{n-1}$ then $\dim \varphi = 2^n$.*

Conjecture 6.5. *Let γ be an even-dimensional anisotropic form. Assume that γ is a good non-excellent form of height 2 with leading form $\tau \in P_n(F)$. Then there exists $\tau_0 \in P_{n-1}(F)$, ($\tau_0 = \langle 1 \rangle$ if $n = 1$), and $a, b, c \in F^*$ such that*

- *γ is similar to $\tau_0 \otimes \langle -a, -b, ab, c \rangle$,*
- *$\tau \simeq \tau_0 \otimes \langle\langle c \rangle\rangle$.*

Remark 6.6. *For proving Conjecture 6.5 it suffices to show that there exists $\tau_0 \in P_{n-1}(F)$ such that $\gamma \simeq \tau_0 \otimes \gamma'$ where $\dim \gamma' = 4$.*

Proof. Write $\gamma' = \langle r, s, t, u \rangle$ with $r, s, t, u \in F^*$. Then setting $d = rst$, we obtain $d\gamma' \simeq \langle -a, -b, ab, c \rangle$ with $a = -st$, $b = -rt$, and $c = du$. This shows $\gamma \sim \tau_0 \otimes \psi$ where $\psi = \langle -a, -b, ab, c \rangle$. Since $\langle d(\psi) \rangle \simeq \langle c \rangle$ by definition of $d(\psi) = (-1)^{\binom{4}{2}} \det(\psi)$ and since τ is the leading form of γ, it follows from [Kn1], 6.12, that $\tau \simeq \tau_0 \otimes \langle\langle c \rangle\rangle$. $\square$

Remark 6.7. Recent results of Voevodsky [Vo] and Orlov-Vishik-Voevodsky [OVV] show that Conjectures 6.1, 6.2, and 6.3 hold for all fields of characteristic 0. These three conjectures were proved earlier in the cases $n \leq 4$ and characteristic $\neq 2$, (cf. [Pf2], [KRS], and [Kahn]).

Conjecture 6.4 is proved for all fields of characteristic 0 by Vishik [Vi]. In the case $n \leq 4$, it is proved for all fields of characteristic $\neq 2$ (see [H4]).

Conjecture 6.5 is proved in the case $n \leq 3$ for all fields of characteristic $\neq 2$, (see Remark 6.6 and [Kahn]). Moreover, it follows from Proposition 3.6 and from [Kahn], Proposition 4.3 (b), that Conjectures 6.3 and 6.4 for degree $n+1$ imply Conjecture 6.5 for degree n.

Definition 6.8. Let $d \geq 0$ be an integer, and let F be a field. We say that *condition A_d holds for F* if F is of characteristic $\neq 2$ and if for all field extensions F'/F the following conjectures hold:

- Conjecture 6.1 for all $n \leq d+2$,
- Conjecture 6.2 for $n \leq d+2$,
- Conjecture 6.3 for $n = d+1$,
- Conjecture 6.4 for $n = d+2$,
- Conjecture 6.5 for $n = d > 0$.

Theorem 6.9. *Let F be a field of characteristic $\neq 2$. If $d=0$ or $d=1$ then condition A_d holds for F. If $d=2$ then condition A_d holds for F, possibly with the exception of the bijectivity of the homomorphism $e^4 : I^4(F)/I^5(F) \to H^4(F)$.*

Proof. Conjecture 6.1 holds for $n=0$ by definition of the ideal $I(F)$. It has been proved by Pfister for $n=1$, cf. [Pf1], 2.3.6, and by Merkurjev for $n=2$, cf. [M1]. The existence of e^3 has been proved by Arason [Ara], Satz 5.7, and the bijectivity of e^3 by Rost [R] and independently by Merkurjev-Suslin [MS]. The existence of e^4 has been proved by Jacob-Rost [JR] and independently by Szyjewsi [Sz]. The bijectivity of e^4 was claimed by Rost (unpublished).

Conjecture 6.2 holds $n \leq 4$, cf. [KRS].

Conjecture 6.3. For $n = 1, 2$ see [Kn1], 6.2; for $n=3$ (and $n=4$), see [Kahn], Théorème 2.8.

Conjecture 6.4 is trivial for $n=2$. For $n=3$ it is due to Pfister and for $n=4$ to Hoffmann, see [H4], Main Theorem for $n=4$, and 2.9 for $n=3$.

Conjecture 6.5. For $n=1$ see [Kn1], 5.10. For for $n=2$ see Remark 6.6 and [Kahn], Corollaire 2.1. □

Remark 6.7 gives rise to the following theorem.

Theorem 6.10. *Let $d \geq 0$ be an integer. Modulo results proved in* [Vo, OVV] *condition A_d holds for all fields of characteristic* 0.

We are going to prove some consequences of the above conjectures.

Let $GP_n(F)$ denote the set of quadratics forms over F which are similar to n-fold Pfister forms, and let $H^n(K/F) = \ker(H^n(F) \to H^n(K))$.

Lemma 6.11. *Let F be a field of characteristic $\neq 2$ and let $\pi \in GP_r(F)$ for some integer $r \geq 0$. Suppose that Conjecture 6.2 holds for $n = m = r$ and for all field extensions of F. Then*

(1) *for any extension K/F, we have* $H^r(K(\pi)/F) = H^r(K/F) + H^r(F(\pi)/F)$;

(2) *for any form φ over F, we have* $H^r(F(\varphi,\pi)/F) = H^r(F(\varphi)/F) + H^r(F(\pi)/F)$.

Proof. (1) Let $u \in H^r(K(\pi)/F)$. Then $u_K \in H^r(K(\pi)/K)$. Conjecture 6.2 applied with $n = m = r$ shows that $u_K = \ell \cdot e^r(\pi_K)$ with $\ell \in H^0(K) \simeq \mathbb{Z}/2\mathbb{Z}$.

Hence $(u - \ell \cdot e^r(\pi)) \in H^r(K/F)$. Therefore, $u \in H^r(K/F) + \ell \cdot e^r(\pi) \subset H^r(K/F) + H^r(F(\pi)/F)$.

(2) It suffices to set $K = F(\varphi)$ in (1). □

Lemma 6.12. *Let $d > 0$ be an integer and F be a field such that condition A_d holds for F. Let γ be an even-dimensional anisotropic form which is good non-excellent of degree d and height 2. Then*

$$H^{d+2}(F(\gamma)/F) = \{e^{d+2}(\gamma \otimes \langle\langle f\rangle\rangle) \mid f \in F^* \textit{ with } \gamma \otimes \langle\langle f\rangle\rangle \in GP_{d+2}(F)\}\,.$$

Proof. Let τ, τ_0 and $a, b, c \in F^*$ be as in Conjecture 6.5. We can assume that $\gamma = \tau_0 \otimes \langle -a, -b, ab, c\rangle$. Let $\pi = \tau_0 \otimes \langle\langle a, b\rangle\rangle$.

Clearly, $\gamma \sim_{\mathrm{w}} \pi \perp -\tau$. Hence $(\gamma_{F(\pi)})_{\mathrm{an}} \simeq (-\tau_{F(\pi)})_{\mathrm{an}}$. Since $\dim\gamma > \dim\tau$, it follows that $\gamma_{F(\pi)}$ is isotropic. Hence $F(\gamma,\pi)/F(\pi)$ is purely transcendental, forcing $H^{d+2}(F(\gamma)/F) \subset H^{d+2}(F(\pi)/F)$. Conjecture 6.2 shows that $H^{d+2}(F(\pi)/F) = e^{d+1}(\pi)H^1(F)$. Hence, an arbitrary element of the group $H^{d+2}(F(\gamma)/F)$ is of the form $e^{d+1}(\pi)\cdot(s) = e^{d+2}(\pi \otimes \langle\langle s\rangle\rangle)$ with $s \in F^*$. Let $\rho = \pi \otimes \langle\langle s\rangle\rangle$. Then $\rho \in P_{d+2}(F)$ and $e^{d+2}(\rho_{F(\gamma)}) = 0$. By Conjecture 6.1, the form $\rho_{F(\gamma)}$ is hyperbolic. Hence γ is similar to a subform of ρ of by Theorem 1.4. Let γ^* and $t \in F^*$ be such that $t\gamma \perp -\gamma^* \simeq \rho$. The forms γ and γ^* are *half-neighbors*. By [H3, Prop. 2.8], there exists $k \in F^*$ such that $\gamma^* \simeq k\gamma$. Then $\gamma \otimes \langle\langle tk\rangle\rangle \simeq t\rho \in GP_{d+2}(F)$. To complete the proof, it suffices to notice that $e^{d+2}(t\rho) = e^{d+2}(\rho)$. □

Lemma 6.13. *Let $d > 0$ be an integer and F be a field such that condition A_d holds for F . Assume that γ is an even-dimensional anisotropic form which is good non-excellent of height 2 with leading form $\tau \in P_d(F)$. Now, let φ be a form such that $\varphi \equiv \tau \bmod I^{d+1}(F)$ and $\varphi_{F(\gamma,\tau)}$ is hyperbolic. Also assume that there exists an extension E/F such that $\dim(\varphi_E)_{\mathrm{an}} = 2^{d+1}$. Then the following is true.*

(1) *There exists $f \in F^*$ such that*

$$\varphi \equiv f\gamma \mod I^{d+2}(F)\,.$$

(2) *If we suppose additionally that $\dim(\varphi_{F(\gamma)})_{\mathrm{an}} < \dim\gamma$, then there exists $f \in F^*$ such that*

$$\varphi \equiv f\gamma \mod I^{d+3}(F)\,.$$

Proof. (1). Let $\psi = \varphi \perp -\tau$. By assumption, we have $[\psi] \in I^{d+1}(F)$. Hence, we can consider the element $e^{d+1}(\psi) \in H^{d+1}(F)$. Since $\varphi_{F(\gamma,\tau)}$ and $\tau_{F(\tau)}$ are hyperbolic, it follows that $\psi_{F(\gamma,\tau)}$ is also hyperbolic. Hence, $e^{d+1}(\psi) \in H^{d+1}(F(\gamma,\tau)/F)$. By Conjecture 6.5, we can assume that $\gamma = \tau_0 \otimes \langle -a, -b, ab, c\rangle$ and $\tau = \tau_0 \otimes \langle\langle c\rangle\rangle$ where $\tau_0 \in P_{d-1}(F)$.

Let $\pi = \tau_0 \otimes \langle\langle a,b\rangle\rangle \in P_{d+1}(F)$. Then $\gamma \sim_{\mathrm{w}} \pi \perp -\tau$. Hence $\gamma_{F(\tau)} \sim_{\mathrm{w}} \pi_{F(\tau)}$. Therefore, by Lemma 6.11(2) and Conjecture 6.2 we have

$$\begin{aligned} H^{d+1}(F(\gamma,\tau)/F) &= H^{d+1}(F(\pi,\tau)/F) \\ &= H^{d+1}(F(\pi)/F) + H^{d+1}(F(\tau)/F) \\ &= e^{d+1}(\pi)H^0(F) + e^d(\tau)H^1(F)\,. \end{aligned}$$

Since $e^d(\tau)H^1(F) = \{e^{d+1}(\tau \otimes \langle\langle s\rangle\rangle) \,|\, s \in F^*\}$, it follows that any element of the group $H^{d+1}(F(\gamma,\tau)/F)$ has one of the following forms:

- either $e^{d+1}\big[\pi \perp (\tau \otimes \langle\langle s\rangle\rangle)\big]$
- or $e^{d+1}(\tau \otimes \langle\langle s\rangle\rangle)$.

Since $e^{d+1}(\psi) \in H^{d+1}(F(\gamma,\tau)/F)$ and $\psi = \varphi \perp -\tau$, Conjecture 6.1 shows that

- either $\varphi \perp -\tau \equiv \pi \perp (\tau \otimes \langle\langle s\rangle\rangle) \bmod I^{d+2}(F)$
- or $\varphi \perp -\tau \equiv \tau \otimes \langle\langle s\rangle\rangle \bmod I^{d+2}(F)$.

Consider the first case where $\varphi \perp -\tau \equiv \pi \perp (\tau \otimes \langle\langle s\rangle\rangle) \bmod I^{d+2}(F)$. Clearly, we can compute $[\varphi]$ modulo $I^{d+2}(F)$. In our computation, we note that $[\pi]$ and $[\tau \otimes \langle\langle s\rangle\rangle]$ belong to $I^{d+1}(F)$. Hence for any $x \in F^*$, we have $x\pi \equiv \pi$ and $x\tau \otimes \langle\langle s\rangle\rangle \equiv \tau \otimes \langle\langle s\rangle\rangle \bmod I^{d+2}(F)$. Besides, we recall that $\gamma \sim_{\mathrm{w}} \pi \perp -\tau$. Now, we have the following calculation $\varphi \equiv \tau \perp \pi \perp (\tau \otimes \langle\langle s\rangle\rangle) \equiv \tau \perp -\pi \perp -(\tau \otimes \langle\langle s\rangle\rangle) \equiv -\gamma \perp ((\gamma \perp -\pi) \otimes \langle\langle s\rangle\rangle) \equiv s\pi \perp -\pi \perp -s\gamma \equiv -s\gamma \bmod I^{d+2}(F)$. Hence $\varphi \equiv f\gamma \bmod I^{d+2}(F)$ with $f = -s$.

Now we consider the second case where $\varphi \perp -\tau \equiv \tau \otimes \langle\langle s\rangle\rangle \bmod I^{d+1}(F)$. Here, we get $\varphi \equiv \tau \perp (\tau \otimes \langle\langle s\rangle\rangle) \equiv \tau \perp -(\tau \otimes \langle\langle s\rangle\rangle) \equiv s\tau \bmod I^{d+2}(F)$. By the assumption of the lemma there exists a field extension E/F such that $\dim(\varphi_E)_{\mathrm{an}} = 2^{d+1}$. Since $\varphi \equiv s\tau \bmod I^{d+2}(F)$, we have $(\varphi_E)_{\mathrm{an}} \equiv s\tau_E \bmod I^{d+2}(E)$. Since $\dim(\varphi_E)_{\mathrm{an}} + \dim \tau_E = 2^{d+1} + 2^d < 2^{d+2}$, Theorem 1.1 shows that $(\varphi_E)_{\mathrm{an}} \simeq s(\tau_E)_{\mathrm{an}}$. This contradicts to the inequality $\dim(\varphi_E)_{\mathrm{an}} = 2^{d+1} > 2^d \geq \dim(\tau_E)_{\mathrm{an}}$.

(2). Let f be as in (1). Set $\psi = \varphi \perp -f\gamma$. We have $\psi \in I^{d+2}(F)$. Since $\dim(\varphi_{F(\gamma)})_{\mathrm{an}} < \dim\gamma$, we have $\dim(\psi_{F(\gamma)})_{\mathrm{an}} < 2\dim\gamma = 2\cdot 2^{d+1} = 2^{d+2}$. By Theorem 1.1, the form $\psi_{F(\gamma)}$ is hyperbolic. Hence $e^{d+2}(\psi) \in H^{d+2}(F(\gamma)/F)$. By Lemma 6.12, there exists $s \in F^*$ such that $\gamma \otimes \langle\langle s\rangle\rangle \in GP_{d+2}(F)$ and $e^{d+2}(\psi) = e^{d+2}(\gamma \otimes \langle\langle s\rangle\rangle)$. Thus $\psi \equiv \gamma \otimes \langle\langle s\rangle\rangle \bmod I^{d+3}(F)$ by Conjecture 6.1. Since $\gamma \otimes \langle\langle s\rangle\rangle \in GP_{d+2}(F)$, we have $\gamma \otimes \langle\langle s\rangle\rangle \equiv -f\gamma \otimes \langle\langle s\rangle\rangle \bmod I^{d+3}(F)$. Therefore, $\varphi \equiv \psi \perp f\gamma \equiv (\gamma \otimes \langle\langle s\rangle\rangle) \perp f\gamma \equiv -f(\gamma \otimes \langle\langle s\rangle\rangle) \perp f\gamma \equiv fs\gamma \bmod I^{d+3}(F)$. □

Proposition 6.14. *Let $d > 0$ be an integer and F be a field such that condition A_d holds for F. Let $(\varphi_0, \varphi_1, \ldots, \varphi_h)$ be a quasi-excellent sequence with leading form $\tau \in P_d(F)$ and pre-leading form $\gamma = \varphi_{h-2}$. Suppose that this sequence is of the second or third type (in particular $h \geq 2$). Let $\varphi = \varphi_i$ with $i \leq h-2$.*

(1) *We have $\varphi \equiv \tau \bmod I^{d+1}(F)$.*
(2) *There exists $f \in F^*$ such that $\varphi \equiv f\gamma \bmod I^{d+2}(F)$.*
(3) *If $\dim(\varphi_{F(\gamma)})_{\mathrm{an}} < \dim\gamma$ then there exists $f \in F^*$ such that*
$\varphi \equiv f\gamma \bmod I^{d+3}(F)$.

Proof. (1). Follows from Lemma 3.4 and Conjecture 6.3.

(2) and (3). Definition 0.3 shows that $\dim\gamma = 2^{d+1}$. By Proposition 3.6(2), γ is a good non-excellent form of height 2 and degree d. By Proposition 3.7, $\varphi_{F(\gamma,\tau)}$ is hyperbolic. Lemma 2.1 yields $\dim(\varphi_E)_{\mathrm{an}} = \dim\varphi_{h-2} = 2^{d+1}$ where $E = F_{h-2} = F(\varphi_0,\dots,\varphi_{h-3})$. Now, Lemma 6.13 completes the proof. □

Corollary 6.15. *Let $(\varphi,\gamma,\tau,0)$ be a quasi-excellent sequence with $\tau \in P_d(F)$ and $\dim\gamma = 2^{d+1}$. Then $\dim\varphi \geq 3\cdot 2^d$. Moreover, if $\dim\varphi = 3\cdot 2^d$ then $\varphi_{F(\gamma)}$ is anisotropic.*

Proof. Let $E = F(\gamma)$. If φ_E is anisotropic, then the sequence $(\varphi_E,\tau_E,0)$ is quasi-excellent by Lemma 3.1. Since $\dim\varphi_E = \dim\varphi > \dim\gamma = 2^{d+1}$, it follows from Proposition 3.6 that $\dim\varphi = 2^N - 2^d$ for some $N \geq d+2$. Therefore, $\dim\varphi \geq 2^{d+2} - 2^d = 3\cdot 2^d$.

Now, we assume that φ_E is isotropic. Then $(\varphi_E)_{\mathrm{an}} \sim \tau_E$ by Lemma 3.1 and hence $\dim(\varphi_E)_{\mathrm{an}} = 2^d < \dim\gamma$. By Proposition 6.14, there exists $f \in F^*$ such that $\varphi \equiv f\gamma \mod I^{d+3}(F)$. Suppose that $\dim\varphi \leq 3\cdot 2^d$. Then $\dim\varphi + \dim\gamma \leq 3\cdot 2^d + 2^{d+1} < 2^{d+3}$. By Theorem 1.1, we get $\varphi \simeq f\gamma$. This is a contradiction because $\dim\varphi > \dim\gamma$. □

7. Classification theorem for sequences of the second type

In Definition 6.8 we formulated the condition A_d for a field F. We showed that A_d is true for $d = 0, 1$ and $\mathrm{char}(F) \neq 2$ and that (based on results in [Vo, OVV]) A_d is true for all $d \geq 0$ and all fields of characteristic 0, cf. Theorems 6.9 and 6.10.

The main purpose of this section is to prove the following

Theorem 7.1. *Let $d > 0$ be an integer and F be a field such that condition A_d holds for F. Let $(\varphi_0,\varphi_1,\dots,\varphi_h)$ be a quasi-excellent sequence of the second type and of degree d. Then*

- *for all $i < h-2$ the form φ_i is a Pfister neighbor whose complementary form is similar to φ_{i+1},*
- *the sequence $(\varphi_{h-2},\varphi_{h-1},\varphi_h)$ is quasi-excellent of the second type.*

First of all, we state the following corollary (which proves Theorems 0.9 and 0.10 for the quasi-excellent sequences of second type).

Corollary 7.2. *Let $d > 0$ be an integer and F be a field such that condition A_d holds for F. Then Conjecture 0.7 holds for all quasi-excellent sequences of degree d over F.*

Proof. By Theorem 7.1, it suffices to consider the case $h = 2$. In this case, the required result follows immediately from Proposition 3.6(2) and Conjecture 6.5. □

Now, we return to Theorem 7.1. We will prove this theorem by using induction on h. In the case where $h=2$ the statement is obvious. Thus we can assume that $h \geq 3$. In what follows we will suppose that $h \geq 3$ and Theorem 7.1 holds for all quasi-excellent sequences of height $< h$.

We start with the following lemma.

Lemma 7.3. *If $(\varphi_0)_{F(\varphi_1)}$ is anisotropic then $(\varphi_0)_{F(\varphi_1)}$ is a Pfister neighbor whose complementary form is similar to $(\varphi_2)_{F(\varphi_1)}$.*

Proof. Let $E = F(\varphi_1)$. By Lemma 3.1, the sequence $((\varphi_0)_E, (\varphi_2)_E, \ldots, 0)$ is quasi-excellent of height $h-1$. Let us consider two cases, $h \geq 4$ and $h = 3$.

If $h \geq 4$ then the sequence $((\varphi_0)_E, (\varphi_2)_E, \ldots, (\varphi_h)_E)$ is of the second type. Then Theorem 7.1 (stated for sequences of height $< h$) completes the proof.

If $h = 3$ then the quasi-excellent sequence $((\varphi_0)_E, (\varphi_2)_E, 0)$ is of the first type because $\dim \varphi_0 > \dim \varphi_1 = 2^{d+1}$. In this case, Theorem 4.2 completes the proof. □

The following lemma shows that the situation described in Lemma 7.3 is actually impossible.

Lemma 7.4. *The form $(\varphi_0)_{F(\varphi_1)}$ is isotropic.*

Proof. Assume the contrary, $(\varphi_0)_{F(\varphi_1)}$ is anisotropic. Then Lemmas 7.3 and 2.8 show that $\dim \varphi_1$ is a power of 2 and $\dim \varphi_0 = 2 \dim \varphi_1 - \dim \varphi_2$. Let $K = F(\varphi_0)$. The sequence $((\varphi_1)_K, \ldots, (\varphi_h)_K)$ is quasi-excellent of height $h-1$ by Lemma 3.3. Clearly, this sequence is of the second type. We consider the two cases $h \geq 4$ and $h = 3$. If $h \geq 4$ then Theorem 7.1 (stated for sequences of height $< h$) shows that $(\varphi_1)_K$ is a Pfister neighbor whose complementary form is similar to the non-zero form $(\varphi_2)_K$. This in particular shows that $\dim \varphi_1$ is not a power of 2. We get a contradiction. Now, we assume that $h = 3$. In other words, we have the sequence $(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ of the second type. By Definitions 0.3 and 2.6, we have $\dim \varphi_1 = 2^{d+1}$, $\dim \varphi_2 = 2^d$ and $\dim \varphi_0 \neq 3 \cdot 2^d$. On the other hand, $\dim \varphi_0 = 2 \dim \varphi_1 - \dim \varphi_2 = 2 \cdot 2^{d+1} - 2^d = 3 \cdot 2^d$. We get a contradiction. □

Corollary 7.5. *The sequence $(\varphi_1, \varphi_2, \ldots, \varphi_h)$ is a quasi-excellent sequence of the second type and of degree d.*

Proof. Follows from Lemmas 7.4, and 3.1. □

Lemma 7.6. *Let $\gamma = \varphi_{h-2}$. Then $\dim((\varphi_i)_{F(\gamma)})_{\mathrm{an}} \leq 2^d$ for all $i = 0, \ldots, h$.*

Proof. Using induction and Corollary 7.5 we see that $\dim((\varphi_i)_{F(\gamma)})_{\mathrm{an}} \leq 2^d$ for all $i \geq 1$. Now, it suffices to prove that $\dim((\varphi_0)_{F(\gamma)})_{\mathrm{an}} \leq 2^d$. Since $h \geq 3$, we have $\dim((\varphi_1)_{F(\gamma)})_{\mathrm{an}} \leq 2^d = \dim \varphi_{h-1} < \dim \varphi_1$. Hence $(\varphi_1)_{F(\gamma)}$ is isotropic forcing that $F(\gamma, \varphi_1)/F(\gamma)$ is purely transcendental. Thus 7.4 yields that $(\varphi_0)_{F(\gamma)}$ is isotropic. By Lemma 2.3, there exists $i > 0$ such that $((\varphi_0)_{F(\gamma)})_{\mathrm{an}} \sim (\varphi_i)_{F(\gamma)} = ((\varphi_i)_{F(\gamma)})_{\mathrm{an}}$. Hence $\dim((\varphi_0)_{F(\gamma)})_{\mathrm{an}} = \dim((\varphi_i)_{F(\gamma)})_{\mathrm{an}} \leq 2^d$. □

Corollary 7.7. *There exists $f_i \in F^*$ such that $\varphi_i \equiv f_i\gamma \bmod I^{d+3}(F)$ for all $i = 0, \ldots, h-2$.*

Proof. Obvious consequence of Proposition 6.14 and Lemma 7.6. □

Corollary 7.8. *If $h \geq 3$ then φ_0 is a Pfister neighbor whose complementary form is similar to φ_1.*

Proof. Corollary 7.7 shows that the condition of Proposition 5.1 holds in the case $k = h-2$ and $m = d+2$. □

Proof of Theorem 7.1. If $h \geq 3$, Corollaries 7.8 and 7.5 show that

- φ_0 is a Pfister neighbor whose complementary form is similar to φ_1,
- the sequence $(\varphi_1, \varphi_2, \ldots, \varphi_h)$ is quasi-excellent of the second type.

After that, an evident induction completes the proof. □

8. Classification theorem for sequences of the third type

We proceed similarly as in the previous section. The main purpose is to prove the following theorem.

Theorem 8.1. *Let $d > 0$ be an integer and F be a field such that condition A_d holds for F. Let $(\varphi_0, \varphi_1, \ldots, \varphi_h)$ be a quasi-excellent sequence of the third type and of degree d. Then*

- *for all $i < h-3$ the form φ_i is a Pfister neighbor whose complementary form is similar to φ_{i+1},*
- *the sequence $(\varphi_{h-3}, \varphi_{h-2}, \varphi_{h-1}, 0)$ is quasi-excellent of the third type.*

We will prove this theorem by using induction on h. In the case where $h = 3$ the statement is obvious. Thus we can assume that $h \geq 4$. In what follows we will suppose that $h \geq 4$ and Theorem 8.1 holds for all quasi-excellent sequences of height $< h$.

Lemma 8.2. *If $(\varphi_0)_{F(\varphi_1)}$ is anisotropic then $(\varphi_0)_{F(\varphi_1)}$ is a Pfister neighbor whose complementary form is similar to $(\varphi_2)_{F(\varphi_1)}$.*

Proof. Let $E = F(\varphi_1)$. By Lemma 3.1, the sequence $((\varphi_0)_E, (\varphi_2)_E, \ldots, 0)$ is quasi-excellent of height $h-1$. Let us consider two cases, $h \geq 5$ and $h = 4$.

If $h \geq 5$ then the sequence $((\varphi_0)_E, (\varphi_2)_E, \ldots, (\varphi_h)_E)$ is of the third type. Then Theorem 8.1 (stated for sequences of height $< h$) completes the proof.

If $h = 4$ then the quasi-excellent sequence $((\varphi_0)_E, (\varphi_2)_E, (\varphi_3)_E, 0)$ has the second type because $\dim \varphi_0 > \dim \varphi_1 = 3 \cdot 2^d$. In this case, Theorem 7.1 completes the proof. □

Lemma 8.3. *The form $(\varphi_0)_{F(\varphi_1)}$ is isotropic.*

Proof. Assume the contrary, $(\varphi_0)_{F(\varphi_1)}$ is anisotropic. Then Lemmas 8.2 and 2.8 show that $\dim\varphi_1$ is a power of 2. Let $K = F(\varphi_0)$. The sequence $((\varphi_1)_K, \dots, (\varphi_h)_K)$ is quasi-excellent of height $h-1$ by Lemma 3.3. Clearly, this sequence is of the third type. We consider two cases, $h \geq 5$ and $h = 4$. If $h \geq 5$ then Theorem 8.1 (stated for sequences of height $< h$) shows that $(\varphi_1)_K$ is a Pfister neighbor whose complementary form is similar to the non-zero form $(\varphi_2)_K$. This in particular shows that $\dim\varphi_1$ is not a power of 2. We get a contradiction. Now, we assume that $h = 4$. Then $\dim\varphi_1 = \dim\varphi_{h-3} = 3\cdot 2^d$ is not a power of 2, a contradiction. □

Corollary 8.4. *The sequence $(\varphi_1, \varphi_2, \dots, \varphi_h)$ is a quasi-excellent sequence of the third type and of degree d.*

Proof. Obvious in view of Lemmas 8.3, and 3.1. □

In what follows we use the following notation:
τ is the leading form. Clearly, we can assume that $\varphi_{h-1} = \tau$;
$\gamma = \varphi_{h-2}$ is the pre-leading form;
$\lambda = \varphi_{h-3}$ is "pre-pre-leading" form.
Thus, our quasi-excellent sequence looks as follows: $(\varphi_0, \dots, \varphi_{h-4}, \lambda, \gamma, \tau, 0)$

Lemma 8.5. *For all $i = 0, \dots, h-1$, we have* $\dim((\varphi_i)_{F(\lambda,\gamma)})_{\mathrm{an}} = 2^d$.

Proof. It follows from Lemma 2.1 that $(\tau)_{F(\lambda,\gamma)}$ is anisotropic.

Using induction and Corollary 8.4, we see that $\dim((\varphi_i)_{F(\lambda,\gamma)})_{\mathrm{an}} = 2^d$ for all $i = 1, \dots, h-1$. In particular, $(\varphi_1)_{F(\lambda,\gamma)}$ is isotropic. Hence $F(\lambda,\gamma,\varphi_1)/F(\lambda,\gamma)$ is purely transcendental. Since $(\varphi_0)_{F(\varphi_1)}$ is isotropic by Lemma 8.3, it follows that $(\varphi_0)_{F(\lambda,\gamma)}$ is also isotropic. By Lemma 2.3, there exists $i > 0$ such that $((\varphi_0)_{F(\lambda,\gamma)})_{\mathrm{an}} \sim (\varphi_i)_{F(\lambda,\gamma)} = ((\varphi_i)_{F(\lambda,\gamma)})_{\mathrm{an}}$.

Hence $\dim((\varphi_0)_{F(\lambda,\gamma)})_{\mathrm{an}} = \dim((\varphi_i)_{F(\lambda,\gamma)})_{\mathrm{an}} = 2^d$. □

Proposition 8.6. *For any $i = 0, \dots, h-3$ there exists $f_i \in F^*$ such that $\varphi_i \equiv f_i\lambda \bmod I^{d+3}(F)$.*

Proof. There is $s_i \in F^*$ such that $\varphi_i \equiv s_i\gamma \bmod I^{d+2}(F)$ for $i = 0, \dots, h-3$ by Proposition 6.14(2).

Changing notation $\varphi_i := s_i\varphi_i$, we can assume that $\varphi_i \equiv \gamma \bmod I^{d+2}(F)$ for all $i = 0, \dots, h-3$. In particular, $\lambda = \varphi_{h-3} \equiv \gamma \bmod I^{d+2}(F)$. Hence, we get the element $e^{d+2}(\lambda\perp -\gamma) \in H^{d+2}(F)$.

Now, we fix an integer $i \leq h-3$ and set $\varphi = \varphi_i$. We have $\varphi \equiv \gamma \equiv \lambda \bmod I^{d+2}(F)$. Hence, we get the elements $e^{d+2}(\varphi\perp -\gamma)$ and $e^{d+2}(\varphi\perp -\lambda)$ in $H^{d+2}(F)$. Recall that $H^n(F'/F) := \ker(H^n(F) \to H^n(F'))$.

Lemma 8.7. (1) $e^{d+2}(\varphi\perp -\gamma) \in H^{d+2}(F(\lambda,\gamma)/F)$,
(2) $e^{d+2}(\varphi\perp -\gamma) \notin H^{d+2}(F(\gamma)/F)$,
(3) $e^{d+2}(\varphi\perp -\lambda) \in H^{d+2}(F(\gamma)/F)$.

Proof. To prove item (1), it suffices to verify that $(\varphi \perp -\gamma)_{F(\lambda,\gamma)}$ is hyperbolic. By Lemma 8.5, we have $\dim(\varphi_{F(\lambda,\gamma)})_{\mathrm{an}} \le 2^d$ and $\dim(\gamma_{F(\lambda,\gamma)})_{\mathrm{an}} \le 2^d$. Hence, $\dim((\varphi \perp -\gamma)_{F(\lambda,\gamma)})_{\mathrm{an}} \le 2^{d+1} < 2^{d+2}$. Since $[\varphi\perp -\gamma] \in I^{d+2}(F)$, Theorem 1.1 shows that $(\varphi \perp -\gamma)_{F(\lambda,\gamma)}$ is hyperbolic.

(2) Assume that $e^{d+2}(\varphi\perp-\gamma) \in H^{d+2}(F(\gamma)/F)$. Let $E = F(\varphi_0,\dots,\varphi_{h-4})$. Then $(\varphi_E)_{\mathrm{an}} \sim (\varphi_{h-3})_E = \lambda_E$ by Lemma 2.1. Hence, there exists $s \in E^*$ such that $(\varphi_E)_{\mathrm{an}} \simeq s\lambda_E$. Therefore, $e^{d+2}(s\lambda_E\perp -\gamma_E) \in H^{d+2}(E(\gamma)/E)$. Hence $e^{d+2}(s\lambda_{E(\gamma)}\perp -\gamma_{E(\gamma)}) = 0$.

Then Conjecture 6.1 implies that $[s\lambda_{E(\gamma)} \perp -\gamma_{E(\gamma)}] \in I^{d+3}(E(\gamma))$. Since $\dim\lambda + \dim\gamma = 3\cdot 2^d + 2^{d+1} < 2^{d+3}$, Theorem 1.1 shows that $s\lambda_{E(\gamma)} \perp -\gamma_{E(\gamma)}$ is hyperbolic. Thus $\dim(\lambda_{E(\gamma)})_{\mathrm{an}} \le \dim\gamma < \dim\lambda$. Hence, $\lambda_{E(\gamma)}$ is isotropic. By Lemma 3.3, the sequence $(\lambda_E,\gamma_E,\tau_E,0)$ is quasi-excellent. By Corollary 6.15, the form $\lambda_{E(\gamma)}$ is anisotropic. We get a contradiction.

(3). Set $K = F(\gamma)$. Then we have a non-zero element $e^{d+2}(\varphi_K \perp -\gamma_K)$ in the group $H^{d+2}(K(\lambda)/K)$ by (1) and (2).

Since $(\gamma_K)_{\mathrm{an}}$ is similar to τ_K by Lemma 3.5 and Proposition 3.7, there exists $s \in K^*$ such that $(\gamma_K)_{\mathrm{an}} \simeq s\tau_K$. Since $[\lambda \perp -\gamma] \in I^{d+2}(F)$ we obtain that $[\lambda_K \perp -s\tau_K] \in I^{d+2}(K)$. Computing $\dim\lambda + \dim\tau = 3\cdot 2^d + 2^d = 2^{d+2}$, we conclude from the *Arason-Pfister Hauptsatz* that there is a form $\pi \in GP_{d+2}(K)$ such that $\pi \simeq \lambda_K \perp -s\tau_K$, (see Theorem 1.1, and [AP], p. 174, Korollar 3). It follows that $\lambda_{K(\pi)}$ is isotropic, since $\dim\lambda > \dim\tau$, and $\pi_{K(\pi)}$ is hyperbolic. Hence, $K(\pi,\lambda)/K(\pi)$ is purely transcendental. Since by (1) and (2) we have $0 \ne e^{d+2}(\varphi_K \perp -\gamma_K) \in H^{d+2}(K(\pi,\lambda)/K)$ we see that

$$0 \ne e^{d+2}(\varphi_K \perp -\gamma_K) \in H^{d+2}(K(\pi)/K).$$

Thus Conjecture 6.2 shows that $e^{d+2}(\varphi_K \perp -\gamma_K) = e^{d+2}(\pi)$. Clearly, this yields $e^{d+2}(\varphi_K \perp -\gamma_K \perp -\pi) = 0$. Since $\pi \simeq \lambda_K \perp -s\tau_K \sim_{\mathrm{w}} \lambda_K \perp -\gamma_K$ we have $\varphi_K \perp -\gamma_K \perp -\pi \sim_{\mathrm{w}} (\varphi \perp -\lambda)_K$. Hence, $e^{d+2}(\varphi \perp -\lambda)_K = 0$. Therefore, $e^{d+2}(\varphi \perp -\lambda) \in H^{d+2}(K/F)$. □

By Proposition 3.6(2), γ is a good non-excellent form of height 2 and degree d. Now, Lemma 6.12 and item (3) of Lemma 8.7 show that there exists $f \in F^*$ such that $e^{d+2}(\lambda \perp -\varphi) = e^{d+2}(\gamma \otimes \langle\langle f\rangle\rangle)$. By Conjecture 6.1, we have

$$\lambda \perp -\varphi \equiv \gamma \otimes \langle\langle f\rangle\rangle \mod I^{d+3}(F).$$

Since $\gamma \equiv \lambda \bmod I^{d+2}(F)$, it follows that $\lambda \perp -\varphi \equiv \lambda \otimes \langle\langle f\rangle\rangle \simeq \lambda \perp -f\lambda$ $(\mathrm{mod}\ I^{d+3}(F))$. Therefore, $\varphi \equiv f\lambda \bmod I^{d+3}(F)$. This completes the proof of Proposition 8.6. □

Corollary 8.8. *If $h \ge 3$ then φ_0 is a Pfister neighbor whose complementary form is similar to φ_1.*

Proof. Proposition 8.6 shows that the condition of Proposition 5.1 holds in the case $k = h-3$ and $m = d+2$. □

Proof of Theorem 8.1. In case $h \ge 4$, Corollaries 8.8 and 8.4 show that

- φ_0 is a Pfister neighbor whose complementary form is similar to φ_1 ,
- the sequence $(\varphi_1, \varphi_2, \ldots, \varphi_h)$ is quasi-excellent of the third type.

After that, an evident induction completes the proof. $\square$

9. Quasi-excellent sequences of type 3 and height 3

The main purpose of this section is to complete the classification of quasi-excellent sequences of the third type. The results of the previous section show that it suffices to consider only quasi-excellent sequences of height 3.

Lemma 9.1. *The sequence $(\phi_0, \phi_1, \phi_2, \phi_3)$ in Example 0.2 is quasi-excellent.*

Proof. Set $\varrho = \langle\langle a_1, \ldots, a_{d-1}\rangle\rangle$ and $K = F(\phi_0)$. Then $(\phi_1)_K$ is anisotropic by Theorem 1.5. We have $i((\phi_0)_K) \geq \dim\rho = 2^{d-1}$ by [HR], Lemma 2.5 ii, hence $\dim((\phi_0)_K)_{\rm an} \leq 2^{d+1} = \dim(\phi_1)_K$. Set $\eta = k_0\rho\otimes\langle\langle u, v, c\rangle\rangle$. Then

$$\begin{aligned}\phi_0 \perp -c\,k_0k_1\phi_1 &\simeq k_0\rho\otimes(\langle\langle u,v\rangle\rangle \perp -c\,\langle\langle a_d\rangle\rangle \perp -c\,\langle -u,-v,uv,a_d\rangle)\\ &\simeq k_0\rho\otimes(\langle\langle u,v,c\rangle\rangle \perp \langle c\,a_d, -c\,a_d\rangle)\\ &\simeq \eta \perp \rho\otimes\langle 1,-1\rangle\ .\end{aligned}$$

The form $\eta_{F(\eta)}$ is hyperbolic. But $(\phi_1)_{F(\eta)}$ is anisotropic by Theorem 1.5, hence $((\phi_0)_{F(\eta)})_{\rm an} \sim (\phi_1)_{F(\eta)}$. Since there is an F-place $K \to F(\eta)\cup\{\infty\}$ it follows that $\dim((\phi_0)_K)_{\rm an} \geq \dim(\phi_1)_{F(\eta)} = 2^{d+1}$, cf [Kn1], Proposition 3.1. Thus $\dim((\phi_0)_K)_{\rm an} = 2^{d+1} = \dim(\phi_1)_K$.

If η_K is hyperbolic then it follows that $((\phi_0)_K)_{\rm an} \sim (\phi_1)_K$. Otherwise, η_K is anisotropic, and $((\phi_0)_K)_{\rm an} \perp -c\,k_0k_1(\phi_1)_K \simeq \eta_K$. This shows that for every $x\in K^*$, the forms $(x(\phi_0)_K)_{\rm an}$ and $(\phi_1)_K$ are *half-neighbors* in the sense of [H3], p. 258. Since $(\phi_0)_K$ is isotropic there is an $x\in K^*$ such that $x(\phi_0)_K \simeq (\rho\otimes\langle\langle u,v\rangle\rangle)_K \perp (-\rho\otimes\langle\langle a_d\rangle\rangle)_K$, e.g., [HR], Lemma 2.5 i. Thus $(x(\phi_0)_K)_{\rm an}$ is a $(2^{d+1}, 2^d)$-Pfister form in the sense of [H3], p. 262. Now, [H3], Proposition 2.8, shows that $((\phi_0)_K)_{\rm an} \sim (\phi_1)_K$.

By Theorem 1.5, the forms ϕ_1, ϕ_2 remain anisotropic over $K = F(\phi_0)$. Thus we consider ϕ_1, ϕ_2 as forms over K and show that $((\phi_1)_{K(\phi_1)})_{\rm an}) \sim (\phi_2)_{K(\phi_1)}$. We have

$$\begin{aligned}\phi_1 \perp k_1\,k_2\,\phi_2 &\simeq k_1\rho\otimes(\langle -u,-v,uv,a_d\rangle \perp \langle\langle a_d\rangle\rangle)\\ &\sim_{\rm w} k_1\rho\otimes\langle\langle u,v\rangle\rangle\ .\end{aligned}$$

Set $\psi = k_1\rho\otimes\langle\langle u,v\rangle\rangle$. Then $\dim\psi = 2^{d+1} = \dim\phi_1$. Since $(\phi_2)_{K(\psi)}$ is anisotropic by Theorem 1.5 and since $\psi_{K(\psi)}$ is hyperbolic, the form $(\phi_1)_{K(\psi)}$ is not hyperbolic. There is a K-place $K(\phi_1) \to K(\psi)\cup\{\infty\}$ forcing that $(\phi_1)_{K(\phi_1)}$ is not hyperbolic. But the form $(\phi_1)_{K(\phi_1,\phi_2)}$ is hyperbolic, for otherwise, since $(\phi_2)_{K(\phi_1,\phi_2)}$ is hyperbolic, we would have the contradiction $\dim((\phi_1)_{(K(\phi_1,\phi_2)})_{\rm an} = \dim\psi = 2^{d+1} = \dim(\phi_1)_{K(\phi_1,\phi_2)}$. Now Lemma 3.5 yields that $((\phi_1)_{K(\phi_1)})_{\rm an}) \sim (\phi_2)_{K(\phi_1)}$. $\square$

The main result of this section is the following

Proposition 9.2. *Let $d > 0$ be an integer and F be a field such that condition A_d holds for F. Let $(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ be a quasi-excellent sequence of degree d and of the third type. Then this sequence looks as in Example 0.2.*

Clearly, this proposition together with the results of the previous sections completes the proof of Theorems 0.9 and 0.10.

We say that a form φ is *divisible* by a form ρ if there is a form χ such that $\varphi \simeq \rho \otimes \chi$.

Lemma 9.3. *Let φ and ψ be anisotropic forms. Suppose that φ and ψ are divisible by a Pfister form ρ (including the case $\rho = \langle 1 \rangle$). Then there exist forms φ_0, ψ_0, μ such that*

- *φ_0, ψ_0 and μ are divisible by ρ,*
- *$\varphi \simeq \varphi_0 \perp \mu$ and $\psi \simeq \psi_0 \perp \mu$,*
- *$(\varphi \perp -\psi)_{\mathrm{an}} \simeq \varphi_0 \perp -\psi_0$.*

Proof. Let μ be a form of maximal dimension satisfying the following conditions:

(a) μ is divisible by ρ,
(b) $\mu \subset \varphi$ and $\mu \subset \psi$.

Then there exist forms φ_0 and ψ_0 such that $\varphi \simeq \varphi_0 \perp \mu$ and $\psi \simeq \psi_0 \perp \mu$. Since φ, ψ and μ are divisible by ρ, it follows from Theorem 1.3 that φ_0 and ψ_0 are also divisible by ρ. Now, it suffices to prove that $(\varphi \perp -\psi)_{\mathrm{an}} \simeq \varphi_0 \perp -\psi_0$.

Since $\varphi \perp -\psi \simeq (\varphi_0 \perp \mu) \perp -(\psi_0 \perp \mu) \sim_{\mathrm{w}} \varphi_0 \perp -\psi_0$, it suffices to prove that the form $\varphi_0 \perp -\psi_0$ is anisotropic. Suppose the contrary. Then the forms φ_0 and ψ_0 have a common value, say $\ell \in F^*$. By Theorem 1.3, we have $\ell\rho \subset \varphi_0$ and $\ell\rho \subset \psi_0$. Setting $\tilde{\mu} = \mu \perp \ell\rho$, we see that $\tilde{\mu}$ satisfies conditions (a) and (b). Since $\dim \tilde{\mu} > \dim \mu$, we get a contradiction to the definition of μ. □

Lemma 9.4. *Let φ and ψ be anisotropic forms being divisible by a Pfister form $\rho \in P_{d-1}(F)$ (where $\rho = \langle 1 \rangle$ if $d = 1$). Suppose that $\dim \varphi = 3 \cdot 2^d$, $\dim \psi = 2^{d+1}$, and $(\varphi \perp -\psi)_{\mathrm{an}} \in GP_{d+2}(F)$. Then there exist $u, v, a_d, c \in F^*$ such that*

$$\varphi \sim \rho \otimes (\langle\langle u, v \rangle\rangle \perp -c \langle\langle a_d \rangle\rangle) \quad \textit{and} \quad \psi \sim \rho \otimes \langle -u, -v, uv, a_d \rangle .$$

Proof. Let φ_0, ψ_0 and μ be as in Lemma 9.3. We have

$$\begin{aligned} 2 \dim \mu &= (\dim \varphi + \dim \psi) - (\dim \varphi_0 + \dim \psi_0) \\ &= \dim \varphi + \dim \psi - \dim(\varphi \perp -\psi)_{\mathrm{an}} \\ &= 3 \cdot 2^d + 2^{d+1} - 2^{d+2} = 2^d. \end{aligned}$$

Hence, $\dim \mu = 2^{d-1}$. Since μ is divisible by ρ, there exists $s \in F^*$ such that $\mu \simeq s\rho$. Clearly, $\dim \psi_0 = \dim \psi - \dim \mu = 2^{d+1} - 2^{d-1} = 3 \cdot 2^{d-1}$. Since ψ_0 is divisible by $\rho \in P_{d-1}(F)$ there exist $k, u, v \in F^*$ such that

$$\psi_0 \simeq k\rho \otimes \langle 1, -u, -v \rangle .$$

Then $\psi \simeq (k\rho \otimes \langle 1, -u, -v \rangle) \perp s\rho \simeq kuv\rho \otimes \langle uv, -v, -u, a_d \rangle$ with $a_d = skuv$. Thus $\psi \sim \rho \otimes \langle -u, -v, uv, a_d \rangle$.

Put $\pi := \rho \otimes \langle\langle u,v\rangle\rangle \in P_{d+1}(F)$. Since $\psi_0 \simeq k\rho \otimes \langle 1,-u,-v\rangle$, it is easily checked that $k\pi \simeq \psi_0 \perp kuv\rho$. Hence, $(\psi_0)_{F(\pi)}$ is isotropic.

Let $\eta := -k(\varphi \perp -\psi)_{\mathrm{an}} \simeq -k(\varphi_0 \perp -\psi_0)$. By the hypotheses of the lemma, $\eta \in GP_{d+2}(F)$. Since $k\psi_0 \simeq \rho \otimes \langle 1,-u,-v\rangle$ represents 1, it follows that η represents 1. Hence, $\eta \in P_{d+2}(F)$. Since $k\psi_0 \subset \eta$ and $(\psi_0)_{F(\pi)}$ is isotropic, it follows that $\eta_{F(\pi)}$ is isotropic. Thus $\eta \simeq \pi \otimes \eta_0$ for some form η_0 by Theorem 1.3. Since $\eta \in P_{d+2}(F)$ and $\pi \in P_{d+1}(F)$, it follows that $\eta \simeq \pi \otimes \langle\langle c\rangle\rangle$ for suitable $c \in F^*$. Hence, $\eta \simeq \rho \otimes \langle\langle u,v,c\rangle\rangle$. Clearly, $uv \in D(\eta)$. Since η is a Pfister form, we obtain $\eta \simeq uv\eta$, (see [S], 2.10.4). By definition of η we have $k\eta \sim_{\mathrm{w}} \psi \perp -\varphi$. Hence,

$$\begin{aligned}\varphi &\sim_{\mathrm{w}} \psi \perp -k\eta \simeq \psi \perp -kuv\eta \simeq kuv\rho \otimes \langle uv,-v,-u,a_d\rangle \perp -kuv\rho \otimes \langle\langle u,v,c\rangle\rangle\\ &\sim_{\mathrm{w}} kuv\rho \otimes (\langle\langle u,v\rangle\rangle \perp -\langle\langle a_d\rangle\rangle) \perp -kuv\rho \otimes (\langle\langle u,v\rangle\rangle \perp -c\,\langle\langle u,v\rangle\rangle)\\ &\sim_{\mathrm{w}} kuv\rho \otimes (c\,\langle\langle u,v\rangle\rangle \perp -\langle\langle a_d\rangle\rangle) \simeq kuvc\rho \otimes (\langle\langle u,v\rangle\rangle \perp -c\,\langle\langle a_d\rangle\rangle)\,.\end{aligned}$$

Since $\dim\varphi = 3\cdot 2^d = \dim \rho \otimes (\langle\langle u,v\rangle\rangle \perp -c\,\langle\langle a_d\rangle\rangle)$, we get
$\varphi \simeq kuvc\rho \otimes (\langle\langle u,v\rangle\rangle \perp -c\,\langle\langle a_d\rangle\rangle)$. □

Proof of Proposition 9.2. Let $\tau \in P_d(F)$ be the leading form. Clearly, we can assume that $\varphi_2 = \tau$. In other words, we have a quasi-excellent sequence of the form $(\lambda,\gamma,\tau,0)$ with $\dim\lambda = 3\cdot 2^d$, $\dim\gamma = 2^{d+1}$ and $\dim\tau = 2^d$. By Proposition 6.14, there exists $s \in F^*$ such that $\lambda \equiv s\gamma \bmod I^{d+2}(F)$. Replacing λ by $s\lambda$, we can assume that $\lambda \equiv \gamma \bmod I^{d+2}(F)$. Let $\xi := (\lambda \perp -\gamma)_{\mathrm{an}}$. Clearly, $\xi \not\sim_{\mathrm{w}} 0$ and $[\xi] \in I^{d+2}(F)$. By Theorem 1.1, we have $\dim\xi \geq 2^{d+2}$. Since $\dim\xi \leq \dim\lambda + \dim\gamma = 3\cdot 2^d + 2^{d+1} = 5\cdot 2^d < 3\cdot 2^{d+1}$, Conjecture 6.4 yields $\dim\xi = 2^{d+2}$. Hence $(\lambda \perp -\gamma)_{\mathrm{an}} = \xi \in GP_{d+2}(F)$, cf. [AP], Kor. 3.

By Proposition 3.6, γ is a good non-excellent form of height 2 with leading form τ. By Conjecture 6.5, there exists $\rho \in P_{d-1}(F)$ such that γ and τ are divisible by ρ. Then $\gamma_{F(\rho)}$ and $\tau_{F(\rho)}$ are isotropic. Hence $F(\rho,\gamma,\tau)/F(\rho)$ is purely transcendental. Since the form $\lambda_{F(\gamma,\tau)}$ is hyperbolic by Proposition 3.7, it follows that $\lambda_{F(\rho)}$ is hyperbolic. Therefore λ is divisible by ρ (see Theorem 1.3). Applying Lemma 9.4 to the forms ρ, $\varphi = \lambda$, and $\psi = \gamma$, we see that there exists $u, v, a_d \in F^*$ such that

$$\lambda \sim \rho \otimes (\langle\langle u,v\rangle\rangle \perp -c\,\langle\langle a_d\rangle\rangle) \quad \text{and} \quad \gamma \sim \rho \otimes \langle -u,-v,uv,a_d\rangle\ .$$

It follows that $\tau \sim \rho \otimes \langle\langle a_d\rangle\rangle$ by [Kn1], 6.12. The proof is complete. □

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