Extensions of Stable C^*-Algebras

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Abstract. We show that an extension of two stable C^*-algebras need not be stable. More explicitly we find an extension

$$0 \to C(Z) \otimes \mathcal{K} \to A \to \mathcal{K} \to 0$$

for some (infinite dimensional) compact Hausdorff space Z such that A is not stable. The C^*-algebra A in our example has an approximate unit consisting of projections.

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1 Introduction

It follows from BDF-theory, [1], that for every extension $0 \to \mathcal{K} \to A \to B \to 0$ of separable C^*-algebras one has A is stable if and only if B is stable. This fact prompted the question if every extension of two (separable) stable C^*-algebras is stable, a question we here answer in the negative.
In an earlier paper with J. Hjelmborg, [3], we derived a characterization of stability for C^*-algebras, actually in the hope of providing a positive answer to the extension problem. Later, in [5], the author showed that stability is not a nicely behaved property by providing an example of a (simple, separable) C^*-algebra A such that $M_2(A)$ is stable while A is non-stable. The construction of that example was inspired by ideas of Villadsen from [9]. Again using ideas of Villadsen and of results obtained in [5] and [6] the author found in [8] an example of a simple C^*-algebra that contains both a non-zero finite and an infinite projection. A key ingredient in this construction was a study of projections in the multiplier algebra of $C(Z) \otimes \mathcal{K}$, where Z is the infinite Cartesian product of 2-spheres. In particular, a recipe was derived for deciding when certain projections in this multiplier algebra, arising as infinite sums of Bott projections, are properly infinite. This recipe (restated here in Proposition 2.1) is also a crucial ingredient in the construction of the example given in this note.

2 The Construction

We review some of the notation and some of the results from [8]. Let Z denote the infinite dimensional Cartesian product space $\prod_{n=1}^{\infty} S^2$. Let $p \in M_2(C(S^2)) = C(S^2, M_2)$ be the Bott projection over S^2, so that p is a one-dimensional projection whose Euler class in $H^2(S^2, \mathbb{Z})$ is non-zero. For each non-empty finite subset $I = \{i_1, i_2, \dots, i_k\}$ of $\mathbb{N}$ and for each point $x = (x_1, x_2, x_3, \dots) \in Z$ we define the Bott projections over the copies of S^2 indexed by the set I to be

$$p_I(x_1, x_2, x_3, \dots) = p(x_{i_1}) \otimes p(x_{i_2}) \otimes \cdots \otimes p(x_{i_k}), \tag{2.1}$$

so that p_I belongs to $C(Z, M_2 \otimes \cdots \otimes M_2)$. Identifying $M_2 \otimes \cdots \otimes M_2$ with a sub-C^*-algebra of the algebra $\mathcal{K}$ of compact operators we may view p_I as an element in $C(Z, \mathcal{K}) = C(Z) \otimes \mathcal{K}$. (It is for our purposes only necessary to define p_I up to Murray–von Neumann equivalence.)
Choose a sequence $\{S_j\}_{j=1}^{\infty}$ of isometries in $\mathcal{M}(C(Z) \otimes \mathcal{K})$ with orthogonal range projections such that $\sum_{j=1}^{\infty} S_j S_j^*$ converges strictly to 1. For each sequence $\{q_j\}_{j=1}^{\infty}$ of projections in $C(Z) \otimes \mathcal{K}$ or in $\mathcal{M}(C(Z) \otimes \mathcal{K})$ define

$$\bigoplus_{j=1}^{\infty} q_j \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} S_j q_j S_j^* \in \mathcal{M}(C(Z) \otimes \mathcal{K}).$$

A projection p in a C^*-algebra A is said to be *properly infinite* if there are subprojections p_1 and p_2 of p in A satisfying $p \sim p_1 \sim p_2$ and $p_1 \perp p_2$. Equivalently, p is properly infinite if p is non-zero and

$$\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \precsim \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix},$$

(i.e., $p \oplus p \precsim p$.)
The following proposition was proved in [8, Proposition 4.4 (i)].

Proposition 2.1 *Let $I_1, I_2, \dots$ be a sequence of non-empty, finite subsets of $\mathbb{N}$, and suppose that $\left|\bigcup_{j \in F} I_j\right| \geq |F|$ for all finite subsets F of $\mathbb{N}$. It follows that the projection $\bigoplus_{j=1}^{\infty} p_{I_j}$ in $\mathcal{M}(C(Z) \otimes \mathcal{K})$ is not properly infinite.*

The next lemma is similar to [8, Lemma 3.1].

Lemma 2.2 *Let I be a non-empty finite subset of $\mathbb{N}$ and let e be a constant one-dimensional projection in $C(Z) \otimes \mathcal{K}$ (so that e corresponds to the trivial complex line bundle over Z). Then $e \precsim \bigoplus_{j=1}^{n} p_I$ whenever $n > |I|$.*

Proof: Write $I = \{i_1, i_2, \dots, i_k\}$, define $\rho \colon Z \to (S^2)^k$ by

$$\rho(x_1, x_2, x_3, \dots) = (x_{i_1}, x_{i_2}, \dots, x_{i_k}), \qquad (x_1, x_2, x_3, \dots) \in Z,$$

and let $\widehat{\rho}\colon C((S^2)^k) \to C(Z)$ be its induced map. Use (2.1) to see that p_I belongs to the image of $\widehat{\rho} \otimes \mathrm{id}_{\mathcal{K}}$, and hence that $\bigoplus_{j=1}^n p_I = (\widehat{\rho} \otimes \mathrm{id}_{\mathcal{K}})(q)$ for some n-dimensional projection q in $C((S^2)^k) \otimes \mathcal{K}$. The projection q corresponds to an n-dimensional complex vector bundle ξ over $(S^2)^k$. Since

$$\dim(\xi) \;=\; n \;>\; (n-1) \;\geq\; k \;\geq\; \tfrac{1}{2}\big(\dim((S^2)^k) - 1\big),$$

it follows from Husemoller, [4, 9.1.2], that ξ dominates a trivial complex line bundle. Translated into a statement about projections, this means that $f \precsim q$, where f is a constant one-dimensional projection in $C((S^2)^k) \otimes \mathcal{K}$. But then

$$e \;\sim\; (\widehat{\rho} \otimes \mathrm{id}_{\mathcal{K}})(f) \;\precsim\; (\widehat{\rho} \otimes \mathrm{id}_{\mathcal{K}})(q) \;=\; \bigoplus_{j=1}^{n} p_I.$$

□

We also need the following lemma to decide that our extension is not stable. The lemma is contained in [7, Proposition 6.8] and it is a consequence of [3, Corollary 4.3]. Recall that the multiplier algebra of a stable C^*-algebra contains $B(H)$, the bounded operators on a separable Hilbert space H, as a unital sub-C^*-algebra, so the unit of the multiplier algebra of a stable C^*-algebra is a properly infinite projection.

Lemma 2.3 *Let A be a separable C^*-algebra and let I be an essential ideal in A (so that A is a sub-C^*-algebra of $\mathcal{M}(I)$). If A contains a projection Q such that $1 - Q$ is not a properly infinite projection in $\mathcal{M}(I)$, then A is not stable.*

Proof: Assume to reach a contradiction that A is stable and let Q be a projection in A. It then follows from [3, Corollary 4.3] that $(1-Q)A(1-Q)$ is stable. The C^*-algebra $(1-Q)I(1-Q)$ must also be stable, being an ideal in the stable C^*-algebra $(1-Q)A(1-Q)$, and so its multiplier algebra is properly infinite. The multiplier algebra of $(1-Q)I(1-Q)$ is isomorphic to $(1-Q)\mathcal{M}(I)(1-Q)$. Therefore $1-Q$ is a properly infinite projection in $\mathcal{M}(I)$, in contradiction with the assumption in the lemma. □

Our main result below shows that not all extensions of two stable C^*-algebras are stable. We keep the notation Z for the space $\prod_{j=1}^{\infty} S^2$, and $\mathcal{K}$ is the algebra of compact operators.

Theorem 2.4 *There is an extension of C^*-algebras*

$$0 \longrightarrow C(Z) \otimes \mathcal{K} \longrightarrow A \longrightarrow \mathcal{K} \longrightarrow 0 \tag{2.2}$$

such that A is non-stable and such that A contains an approximate unit consisting of projections.

Proof: Let J denote the C^*-algebra $C(Z) \otimes \mathcal{K}$. Choose a one-dimensional constant projection e in J. Choose mutually disjoint subset $I_2, I_3, \dots$ of $\mathbb{N}$ such that I_n has $n-1$ elements. Choose mutually orthogonal projections $q_{n,j}$ in $\mathcal{M}(J)$, for $n \in \mathbb{N}$ and $1 \le j \le n$, such that the sum

$$Q = \sum_{n=1}^{\infty} \sum_{j=1}^{n} q_{n,j}$$

converges strictly in $\mathcal{M}(J)$ and such that

$$q_{1,1} \sim e, \qquad q_{n,1} \sim q_{n,2} \sim \cdots \sim q_{n,n} \sim p_{I_n}, \quad n \ge 2.$$

We claim that $Q \sim 1$ in $\mathcal{M}(J)$. Indeed, observe that

$$1 \;\sim\; \bigoplus_{n=1}^{\infty} e \;\precsim\; \sum_{n=1}^{\infty} \sum_{j=1}^{n} q_{n,j} \;=\; Q \;\le\; 1,$$

where the second relation follows from Lemma 2.2. This shows that $Q \oplus Q \le 1 \oplus 1 \precsim 1 \precsim Q$ because the unit 1 is properly infinite, and hence Q is a properly infinite projection. The two projections 1 and Q define the same element of $K_0(\mathcal{M}(J))$ because this group is trivial. It therefore follows from Cuntz [2, Section 1] that $Q \sim 1$.

Choose an isometry S in $\mathcal{M}(J)$ such that $SS^* = Q$. Upon replacing Q and $q_{n,j}$ by S^*QS and $S^*q_{n,j}S$ we can assume that $Q = 1$ and hence that $\sum_{n=1}^{\infty} \sum_{j=1}^{n} q_{n,j} = 1$. Put

$$Q_j = \sum_{n=j}^{\infty} q_{n,j}, \qquad j \in \mathbb{N},$$

so that $\{Q_j\}_{j=1}^{\infty}$ is a sequence of mutually orthogonal projections in $\mathcal{M}(J)$ with $\sum_{j=1}^{\infty} Q_j = 1$. Notice that $Q_j \sim Q_{j+1} + q_{j,1}$ for all j. With $\pi \colon \mathcal{M}(J) \to \mathcal{M}(J)/J$ the quotient mapping we get $\pi(Q_1) \sim \pi(Q_2) \sim \cdots$. It follows that there is a *-homomorphism $\varphi \colon \mathcal{K} \to \mathcal{M}(J)/J$ such that $\varphi(e_{jj}) = \pi(Q_j)$ where $\{e_{ij}\}_{i,j=1}^{\infty}$ is a system of matrix units for $\mathcal{K}$. Put $A = \pi^{-1}(\varphi(\mathcal{K}))$ so that we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, hook] \arrow[d, equal] & A \arrow[r, "\rho"] \arrow[d, hook] & \mathcal{K} \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & J \arrow[r, hook] & \mathcal{M}(J) \arrow[r, "\pi"'] & \mathcal{M}(J)/J \arrow[r] & 0
\end{tikzcd}$$

The projection Q_1 belongs to A and

$$1 - Q_1 \;=\; \sum_{j=2}^{\infty} Q_j \;=\; \sum_{j=2}^{\infty} \sum_{n=j}^{\infty} q_{n,j} \;=\; \sum_{n=2}^{\infty} \sum_{j=2}^{n} q_{n,j} \;\sim\; \bigoplus_{n=2}^{\infty} \bigoplus_{j=2}^{n} p_{I_n}.$$

It follows from Proposition 2.1 and the choice of the sets I_n that $1 - Q_1$ is not properly infinite, and Lemma 2.3 now yields that A is not stable.
Put $P_n = Q_1 + \cdots + Q_n$. We show that $\{P_n\}_{n=1}^{\infty}$ is an approximate unit for A. Notice that $\{\rho(P_n)\}_{n=1}^{\infty}$ is an approximate unit for $\mathcal{K}$ and that $P_n \to 1$ strictly. Let a in A and $\varepsilon > 0$ be given. Then $\|\rho(a - P_m a)\| \leq \varepsilon/2$ for some m. Find x in J such that $\|\rho(a - P_m a)\| = \|a - P_m a - x\|$. Find next n such that $\|x - P_n x\| \leq \varepsilon/2$. Then $\|a - P_m a - P_n x\| \leq \varepsilon$, and therefore

$$\|(1 - P_k)a\| \leq \varepsilon + \|(1 - P_k)(P_m a + P_n x)\| = \varepsilon$$

for all $k \geq \max\{n, m\}$. □

Our example leaves open several questions regarding extensions of stable C^*-algebras (see also [7]).

Question 2.5 Let

$$0 \longrightarrow J \longrightarrow A \underset{\lambda}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

be a split exact extension with J and B stable (and separable). Does it follow that A is stable?

Question 2.6 Suppose that I and J are stable (separable) ideals of a C^*-algebra A. Does it follow that $I + J$ is stable?

Question 2.6 can equivalently be phrased as follows: Does every (separable) C^*-algebra A have a *greatest* stable ideal, i.e., a stable ideal that contains all stable ideals of A? (See [7].) It can be shown that the canonical ideal $C(Z) \otimes \mathcal{K}$ of the C^*-algebra A appearing in Theorem 2.4 is a greatest stable ideal in A. Hence even when a C^*-algebra A has a greatest stable ideal I it may be that the quotient A/I has a non-zero stable ideal.
The two questions below were suggested by Eberhard Kirchberg.

Question 2.7 Suppose that

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

is an extension of (separable) C^*-algebras, and suppose that J and B are stable and that A is of real rank zero. Does it follow that A is stable?

Question 2.8 Suppose that

$$0 \longrightarrow J \longrightarrow A \longrightarrow \mathcal{O}_2 \otimes \mathcal{K} \longrightarrow 0$$

is an extension where J is stable and separable. Does it follow that A is stable? What if we replace $\mathcal{O}_2 \otimes \mathcal{K}$ by its cone $C_0((0,1]) \otimes \mathcal{O}_2 \otimes \mathcal{K}$?

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