Rolling Factors Deformations and Extensions of Canonical Curves

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Abstract. A tetragonal canonical curve is the complete intersection of two divisors on a scroll. The equations can be written in 'rolling factors' format. For such homogeneous ideals we give methods to compute infinitesimal deformations. Deformations can be obstructed. For the case of quadratic equations on the scroll we derive explicit base equations. They are used to study extensions of tetragonal curves.

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An easy dimension count shows that not all canonical curves are hyperplane sections of $K3$ surfaces. A surface with a given curve as hyperplane section is called an extension of the curve. With this terminology, the general canonical curve has only trivial extensions, obtained by taking a cone over the curve. In this paper we concentrate on extensions of tetragonal curves.
The extension problem is related to deformation theory for cones. This is best seen in terms of equations. Suppose we have coordinates $(x_0 : \cdots : x_n : t)$ on $\mathbb{P}^{n+1}$ with the special hyperplane section given by $t = 0$. We describe an extension W of a variety $V\colon f_j(x_i) = 0$ by a system of equations $F_j(x_i, t) = 0$ with $F_j(x_i, 0) = f_j(x_i)$. We write $F_j(x_i, t) = f_j(x_i) + tf'_j(x_i) + \cdots + a_j t^{d_j}$, where d_j is the degree of F_j. Considering $(x_0, \ldots, x_n, t)$ as affine coordinates on $\mathbb{C}^{n+1} \times \mathbb{C}$ we can read the equations in a different way. The equations $f_j(x_i) = 0$ define the affine cone $C(V)$ over V and $F_j(x_i, t) = 0$ describes a 1-parameter deformation of $C(V)$. The corresponding infinitesimal deformation is $f_j(x_i) \mapsto f'_j(x_i)$, which is a deformation of weight -1. Conversely, given a 1-parameter deformation $F_j(x_i, t) = 0$ of $C(V)$, with F_j homogeneous of degree d_j, we get an extension W of V. For most of the cones considered here the only infinitesimal deformations of negative weight have weight -1 and in that case the versal deformation in negative weight gives a good description of all possible extensions.

As the number of equations typically is much larger than the codimension one needs good ways to describe them. A prime example is a determinantal scheme X: its ideal is generated by the $t \times t$ minors of an $r \times s$ matrix, which gives a compact description of the equations. Following Miles Reid we call this a format. Canonical curves are not themselves determinantal, but they do lie on scrolls: a k-gonal curve lies on a $(k-1)$-dimensional scroll, which is given by the minors of a $2 \times (g-k+1)$ matrix. For $k = 3$ the curve is a divisor on the scroll, given by one bihomogeneous equation, and for $k = 4$ it is a complete intersection, given by two bihomogeneous equations. In these cases there is a simple procedure (*'rolling factors'*) to write out one resp. two sets of equations on $\mathbb{P}^{g-1}$ cutting out the curve on the scroll.

Powerful methods exist to compute infinitesimal deformations without using explicit equations. We used them for the extension problem for hyperelliptic curves of high degree [Stevens 1996] and trigonal canonical curves [Drewes–Stevens 1996]. In these papers also several direct computations with the equations occur. They seem unavoidable for tetragonal curves, the subject of a preprint by James N. Brawner [Brawner 1996]. The results of these computations do not depend on the particular way of choosing the equations cutting out the curve on the scroll. This observation was the starting point of this paper.

We distinguish between different types of deformations and extensions. If only the equations on the scroll are deformed, but not the scroll itself we speak of *pure rolling factors deformations*. A typical extension lies then on the projective cone over the scroll. Such a cone is a special case of a scroll of one dimension higher. If the extension lies on a scroll which is not a cone, the equations of the scroll are also deformed. We have a *rolling factors deformation*. Finally if the extension does not lie on a scroll of one dimension higher we are in the situation of a *non-scrollar* deformation. Non-scrollar extensions of tetragonal curves occur only in connection with Del Pezzo surfaces. Not every infinitesimal deformation of a scroll gives rise to a deformation of complete intersections on it. One needs certain lifting conditions, which are linear equations in the deformation variables of the scroll. Our first main result describes them, depending only on the coefficients of the equations on the scroll.

The next problem is to extend the infinitesimal deformations to a versal deformation. Here we restrict ourselves to the case that all defining equations are quadratic. Our methods thus do not apply to trigonal curves, but we can handle tetragonal curves. Rolling factors obstructions arise. Previously we observed that one can write them down, given explicit equations on $\mathbb{P}^n$ [Stevens 1996, Prop. 2.12]. Here we give formulas depending only on the coefficients of the equations on the scroll. As first application we study base spaces for hyperelliptic cones. The equations have enough structure so that explicit solutions can be given.

Surfaces with canonical hyperplane sections are a classical subject. References to the older literature can be found in Epema's thesis [Epema 1983], which is especially relevant for our purposes. His results say that apart from $K3$ surfaces only rational surfaces or birationally ruled surfaces can occur. Furthermore he describes a construction of such surfaces. Extensions of pure rolling factors type of tetragonal curves fit very well in this description. A general rolling factors extension is a complete intersection on a nonsingular four-dimensional scroll. The classification of such surfaces [Brawner 1997], which we recall below, shows that surfaces with isolated singularities and in particular $K3$s can only occur if the degrees of the equations on the scroll differ at most by 4. A tetragonal curve of high genus with general discrete invariants has no pure rolling factors deformations. Extensions exist if the base equations have a solution. For low genus we have more variables than equations. For the maximal genus where almost all curves have a $K3$ extension we find:

Proposition. *The general tetragonal curve of genus* 15 *is hyperplane section of* 256 *different* $K3$ *surfaces.*

We also look at examples with genus 16 and 17. It is unclear to us which property of a curve makes it have an extension (apart from the property of being a hyperplane section).
The contents of this paper is as follows. First we describe the rolling factors format and explain in detail the equations and relations for the complete intersection of two divisors on a scroll. Next we recall how canonical curves fit into this pattern. In particular we describe the discrete invariants for tetragonal curves. The same is done for $K3$ surfaces. The second section is devoted to the computation of infinitesimal deformations. First non-scrollar deformations are treated, followed by rolling factors deformations. The main result here describes the lifting matrix. As application the dimension of T^1 is determined for tetragonal cones. In the third section the base equations for complete intersections of quadrics on scrolls are derived. As examples base spaces for hyperelliptic cones are studied. The final section describes extensions of tetragonal curves.

1. Rolling factors format.

A subvariety of a determinantal variety can be described by the determinantal equations and additional equations obtained by 'rolling factors' [Reid 1989]. A typical example is the case of divisors on scrolls.
We start with a k-dimensional rational normal scroll $S \subset \mathbb{P}^n$ (for the theory of scrolls we refer to [Reid 1997]). The classical construction is to take k complementary linear subspaces L_i spanning $\mathbb{P}^n$, each containing a parametrized rational normal curve $\phi_i: \mathbb{P}^1 \to C_i \subset L_i$ of degree $d_i = \dim L_i$, and to take for each $p \in \mathbb{P}^1$ the span of the points $\phi_i(p)$. The degree of S is $d = \sum d_i = n-k+1$. If all $d_i > 0$ the scroll S is a $\mathbb{P}^{d-1}$-bundle over $\mathbb{P}^1$. We allow however that $d_i = 0$ for some i. Then S is the image of $\mathbb{P}^{d-1}$-bundle $\widetilde{S}$ over $\mathbb{P}^1$ and $\widetilde{S} \to S$ is a rational resolution of singularities.
To give a coordinate description, we take homogeneous coordinates $(s:t)$ on $\mathbb{P}^1$, and $(z^{(1)} : \cdots : z^{(k)})$ on the fibres. Coordinates on $\mathbb{P}^n$ are $z_j^{(i)} = z^{(i)} s^{d_i - j} t^j$,

with $0 \leq j \leq d_i$, $1 \leq i \leq k$. We give the variable $z^{(i)}$ the weight $-d_i$. The scroll S is given by the minors of the matrix

$$\Phi = \begin{pmatrix} z_0^{(1)} & \dots & z_{d_1-1}^{(1)} & \dots & z_0^{(k)} & \dots & z_{d_k-1}^{(k)} \\ z_1^{(1)} & \dots & z_{d_1}^{(1)} & \dots & z_1^{(k)} & \dots & z_{d_k}^{(k)} \end{pmatrix}.$$

We now consider a divisor on $\widetilde{S}$ in the linear system $|aH - bR|$, where the hyperplane class H and the ruling R generate the Picard group of $\widetilde{S}$. When we speak of degree on $\widetilde{S}$ this will be with respect to H. The divisor can be given by one bihomogeneous equation $P(s,t,z^{(i)})$ of degree a in the $z^{(i)}$, and total degree $-b$. By multiplying $P(s,t,z^{(i)})$ with a polynomial of degree b in $(s:t)$ we obtain an equation of degree 0, which can be expressed as polynomial of degree a in the $z_j^{(i)}$; this expression is not unique, but the difference of two expressions lies in the ideal of the scroll. By the obvious choice, multiplying with $s^{b-m}t^m$, we obtain $b+1$ equations P_m. In the transition from the equation P_m to P_{m+1} we have to increase by one the sum of the lower indices of the factors $z_j^{(i)}$ in each monomial, and we can and will always achieve this by increasing exactly one index. This amounts to replacing a $z_j^{(i)}$, which occurs in the top row of the matrix, by the element $z_{j+1}^{(i)}$ in the bottom row of the same column. This is the procedure of '*rolling factors*'.

Example 1.1. Consider the cone over $2d-b$ points in $\mathbb{P}^d$, lying on a rational normal curve of degree d, with $b < d$. Let the polynomial $P(s,t) = p_0 s^{2d-b} + p_1 s^{2d-b-1}t + \dots + p_{2d-b}t^{2d-b}$ determine the points on the rational curve. We get the determinantal

$$\begin{vmatrix} z_0 & z_1 & \dots & z_{d-1} \\ z_1 & z_2 & \dots & z_d \end{vmatrix}$$

and additional equations P_m. To be specific we assume that $b = 2c$:

$$\begin{aligned} P_0 &= p_0 z_0^2 + p_1 z_0 z_1 + \dots + p_{2d-2c-1} z_{d-c-1} z_{d-c} + p_{2d-2c} z_{d-c}^2 \\ P_1 &= p_0 z_0 z_1 + p_1 z_1^2 + \dots + p_{2d-2c-1} z_{d-c}^2 + p_{2d-2c} z_{d-c} z_{d-c+1} \\ &\vdots \\ P_{2c} &= p_0 z_c^2 + p_1 z_c z_{c+1} + \dots + p_{2d-2c-1} z_{d-1} z_d + p_{2d-2c} z_d^2 \,. \end{aligned}$$

The 'rolling factors' phenomenon can also occur if the entries of the matrix are more general.

Example 1.2. Consider a non-singular hyperelliptic curve of genus 5, with a half-canonical line bundle $L = g_1^2 + P_1 + P_2$ where the P_i are Weierstrass points. According to [Reid 1989], Thm. 3, the ring $R(C,L) = \bigoplus H^0(C,nL)$ is $k[x_1,x_2,y_1,y_2,z_1,z_2]/I$ with I given by the determinantal

$$\begin{vmatrix} x_1 & y_1 & x_2^2 & z_1 \\ x_2 & x_1^2 & y_2 & z_2 \end{vmatrix}$$

and the three rolling factors equations

$$\begin{aligned} z_1^2 &= x_1^2h + y_1^3 + x_2^4y_2 \\ z_1z_2 &= x_1x_2h + y_1^2x_1^2 + x_2^2y_2^2 \\ z_2^2 &= x_2^2h + y_1x_1^4 + y_2^3 \end{aligned}$$

where h is some quartic in x_1, x_2, y_1, y_2.

The description of the syzygies of a subvariety V of the scroll S proceeds in two steps. First one constructs a resolution of $\mathcal{O}_{\widetilde{V}}$ by vector bundles on $\widetilde{S}$ which are repeated extensions of line bundles. Schreyer describes, following Eisenbud, Eagon-Northcott type complexes $\mathcal{C}^b$ such that $\mathcal{C}^b(a)$ is the minimal resolution of $i_*\bigl(\mathcal{O}_{\widetilde{S}}(-aH+bR)\bigr)$ as $\mathcal{O}_{\mathbb{P}^n}$-module, if $b \geq -1$ [Schreyer 1986]. Here $i\colon \widetilde{S} \to \mathbb{P}^n$ is the map defined by H. The resolution of $\mathcal{O}_V$ is then obtained by taking an (iterated) mapping cone.
The matrix Φ defining the scroll can be obtained intrinsically from the multiplication map

$$H^0\mathcal{O}_{\widetilde{S}}(R) \otimes H^0\mathcal{O}_{\widetilde{S}}(H-R) \longrightarrow H^0\mathcal{O}_{\widetilde{S}}(H)\;.$$

In general, given a map $\Phi\colon F \to G$ of locally free sheaves of rank f and g respectively, $f \geq g$, on a variety one defines Eagon-Northcott type complexes $\mathcal{C}^b$, $b \geq -1$, in the following way:

$$\mathcal{C}^b_j = \begin{cases} \bigwedge^j F \otimes S_{b-j}G, & \text{for } 0 \leq j \leq b \\ \bigwedge^{j+g-1} F \otimes D_{j-b-1}G^* \otimes \bigwedge^g G^*, & \text{for } j \geq b+1 \end{cases}$$

with differential defined by multiplication with $\Phi \in F^* \otimes G$ for $j \neq b+1$ and $\bigwedge^g \Phi \in \bigwedge^g F^* \otimes \bigwedge^g G$ for $j = b+1$ in the appropriate term of the exterior, symmetric or divided power algebra.
In our situation $F \cong \mathcal{O}^d_{\mathbb{P}^n}(-1)$ and $G \cong \mathcal{O}^2_{\mathbb{P}^n}$ with Φ given by the matrix of the scroll. Then $\mathcal{C}^b(-a)$ is for $b \geq -1$ the minimal resolution of $\mathcal{O}_{\widetilde{S}}(-aH+bR)$ as $\mathcal{O}_{\mathbb{P}}$-module [Schreyer 1986, Cor. 1.2].
Now let $V \subset S \subset \mathbb{P}^n$ be a 'complete intersection' of divisors $Y_i \sim a_iH - b_iR$, $i = 1, \ldots, l$, on a k-dimensional rational scroll of degree d with $b_i \geq 0$. The resolution of $\mathcal{O}_V$ as $\mathcal{O}_S$-module is a Koszul complex and the iterated mapping cone of complexes $\mathcal{C}^b$ is the minimal resolution [Schreyer 1986, Sect. 3, Example].
To make this resolution more explicit we look at the case $l = 2$, which is relevant for tetragonal curves. The iterated mapping cone is

$$\bigl[\mathcal{C}^{b_1+b_2}(-a_1-a_2) \longrightarrow \mathcal{C}^{b_1}(-a_1) \oplus \mathcal{C}^{b_2}(-a_2)\bigr] \longrightarrow \mathcal{C}^0$$

To describe equations and relations we give the first steps of this complex. We first consider the case that $b_1 \geq b_2 > 0$. We write $\mathcal{O}$ for $\mathcal{O}_{\mathbb{P}^n}$. We get the double complex

$$\begin{array}{ccccc}
\mathcal{O} & \longleftarrow & \bigwedge^2 \mathcal{O}^d(-1) & \longleftarrow & \bigwedge^3 \mathcal{O}^d(-1)\otimes\mathcal{O}^2 \\
\uparrow & & \uparrow & & \\
S_{b_1}\mathcal{O}^2(-a_1)\oplus S_{b_2}\mathcal{O}^2(-a_2) & \longleftarrow & \mathcal{O}^d(-1)\otimes S_{b_1-1}\mathcal{O}^2(-a_1)\oplus\mathcal{O}^d(-1)\otimes S_{b_2-1}\mathcal{O}^2(-a_2) & & \\
\uparrow & & & & \\
S_{b_1+b_2}\mathcal{O}^2(-a_1-a_2) & & & &
\end{array}$$

The equations for V consist of the determinantal ones plus two sets of additional equations obtained by rolling factors: the two equations $P^{(1)}$, $P^{(2)}$ defining V on the scroll give rise to b_1+1 equations $P^{(1)}_m$ and b_2+1 equations $P^{(2)}_m$.
To describe the relations we introduce the following notation. A column in the matrix Φ has the form $(z^{(i)}_j, z^{(i)}_{j+1})$. We write symbolically $(z_\alpha, z_{\alpha+1})$, where the index α stands for the pair ${}^{(i)}_{j}$ and $\alpha+1$ means adding 1 to the lower index. More generally, if $\alpha = {}^{(i)}_{j}$ and $\alpha' = {}^{(i')}_{j'}$ then the sum $\alpha+\alpha' := j+j'$ only involves the lower indices. To access the upper index we say that α is of type i. The rolling factors assumption is that two consecutive additional equations are of the form

$$P_m = \sum_\alpha p_{\alpha,m} z_\alpha,$$
$$P_{m+1} = \sum_\alpha p_{\alpha,m} z_{\alpha+1}.$$

where the polynomials $p_{\alpha,m}$ depend on the z-variables and the sum runs over all possible pairs $\alpha = {}^{(i)}_{j}$. To roll from P_{m+1} to P_{m+2} we collect the 'coefficients' in the equation P_{m+1} in a different way: we also have $P_{m+1} = \sum_\alpha p_{\alpha,m+1} z_\alpha$. We write the scrollar equations as $f_{\alpha\beta} = z_\alpha z_{\beta+1} - z_{\alpha+1} z_\beta$. The relations between them are

$$R_{\alpha,\beta,\gamma} = f_{\alpha,\beta} z_\alpha - f_{\alpha,\gamma} z_\beta + f_{\beta,\gamma} z_\alpha,$$
$$S_{\alpha,\beta,\gamma} = f_{\alpha,\beta} z_{\alpha+1} - f_{\alpha,\gamma} z_{\beta+1} + f_{\beta,\gamma} z_{\alpha+1},$$

which corresponds to the term $\bigwedge^3 \mathcal{O}^d(-1)\otimes\mathcal{O}^2$ in Schreyer's resolution. The second line yields relations involving the two sets of $P^{(n)}_m$:

$$R^n_{\beta,m} = P^{(n)}_{m+1} z_\beta - P^{(n)}_m z_{\beta+1} - \sum_\alpha f_{\beta,\alpha} p^{(n)}_{\alpha,m},$$

where $n = 1,2$ and $0 \le m < b_i$. We note the following relation:

$$R^n_{\beta,m} z_\gamma - R^n_{\beta,m} z_\beta - \sum R^n_{\beta,\gamma,\alpha} p^{(n)}_{\alpha,m} = P^{(n)}_m f_{\beta,\gamma} - f_{\beta,\gamma} P^{(n)}_m.$$

The right hand side is a Koszul relation; the second factor in each product is considered as coefficient. There are similar expressions involving $z_{\gamma+1}$, $z_{\beta+1}$

and $S_{\beta,\gamma,\alpha}$. Finally by multiplication with suitable powers of s and t the Koszul relation $P^{(1)}P^{(2)} - P^{(2)}P^{(1)}$ gives rise to $b_1 + b_2 + 1$ relations — this is the term $S_{b_1+b_2}\mathcal{O}^2(-a_1 - a_2)$.
In case $b_1 > b_2 = 0$ the resolution is

$$\begin{array}{ccccc}
\mathcal{O} & \leftarrow & \bigwedge^2 \mathcal{O}^d(-1) & \leftarrow & \bigwedge^3 \mathcal{O}^d(-1) \otimes \mathcal{O}^2 \\
\uparrow & & \uparrow & & \\
S_{b_1}\mathcal{O}^2(-a_1) \oplus \mathcal{O}^2(-a_2) & \leftarrow & \mathcal{O}^d(-1) \otimes S_{b_1-1}\mathcal{O}^2(-a_1) \oplus \bigwedge^2 \mathcal{O}^d(-1-a_2) & & \\
\uparrow & & & & \\
S_{b_1}\mathcal{O}^2(-a_1 - a_2) & & & &
\end{array}$$

The new term expresses the Koszul relations between the one equation $P^{(2)}$ and the determinantal equations (which had previously been expressible in terms of rolling factors relations). For the computation of deformations these relations may be ignored.
Finally, if $b_2 = -1$, the equations change drastically.

(1.3) *Canonical curves* [Schreyer 1986].
A k-gonal canonical curve lies on a $(k-1)$-dimensional scroll of degree $d = g - k + 1$. We write D for the divisor of the g^1_k. To describe the type $S(e_1, ..., e_{k-1})$ of the scroll we introduce the numbers

$$f_i = h^0(C, K - iD) - h^0(C, K - (i+i)D) = k + h^0(iD) - h^0((i+1)D)$$

for $i \geq 0$ and set

$$e_i = \#\{j \mid f_j \geq i\} - 1 \ .$$

In particular, e_1 is the minimal number i such that $h^0((i+1)D) - h^0(iD) = k$ and it satisfies therefore $e_1 \leq \frac{2g-2}{k}$.
A trigonal curve lies on a scroll of type $S(e_1, e_2)$ and degree $d = e_1 + e_2 = g - 2$ with

$$\frac{2g-2}{3} \geq e_1 \geq e_2 \geq \frac{g-4}{3}$$

as a divisor of type $3H - (g-4)R$. The minimal resolution of $\mathcal{O}_C$ is given by the mapping cone

$$\mathcal{C}^{d-2}(-3) \longrightarrow \mathcal{C}^0 \ .$$

Introducing bihomogeneous coordinates $(x : y; s : t)$ and coordinates $x_i = xs^{e_1-i}t^i$, $y_i = ys^{e_2-i}t^i$ we obtain the scroll

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{e_1-1} & y_0 & y_1 & \dots & y_{e_2-1} \\ x_1 & x_2 & \dots & x_{e_1} & y_1 & y_2 & \dots & y_{e_2} \end{pmatrix}$$

and a bihomogeneous equation for C

$$P = A_{2e_1-e_2+2}x^3 + B_{e_1+2}x^2y + C_{e_2+2}xy^2 + D_{2e_2-e_1+2}y^3$$

where $A_{2e_1-e_2+2}$ is a polynomial in $(s:t)$ of degree $2e_1-e_2+2$ and similarly for the other coefficients. By rolling factors P gives rise to $g-3$ extra equations. The inequality $e_1 \leq \frac{2g-2}{3}$ can also be explained from the condition that the curve C is nonsingular, which implies that the polynomial P is irreducible, and therefore the degree $2e_2-e_1+2 = 2g-2-3e_1$ of the polynomial $D_{2e_2-e_1+2}$ is nonnegative. The other inequality follows from this one because $e_1 = g-e_2-2$, but also by considering the degree of $A_{2e_1-e_2+2}$.

A tetragonal curve of genus $g \geq 5$ is a complete intersection of divisors $Y \sim 2H - b_1R$ and $Z \sim 2H - b_2R$ on a scroll of type $S(e_1, e_2, e_3)$ of degree $d = e_1 + e_2 + e_3 = g - 3$, with $b_1 + b_2 = d - 2$, and

$$\frac{g-1}{2} \geq e_1 \geq e_2 \geq e_3 \geq 0$$

We introduce bihomogeneous coordinates $(x:y:z;s:t)$. Then Y is given by an equation

$$P = P_{1,1}x^2 + P_{1,2}xy + \cdots + P_{3,3}z^2$$

with P_{ij} (if nonzero) a polynomial in $(s:t)$ of degree $e_i + e_j - b_1$ and likewise Z has equation

$$Q = Q_{1,1}x^2 + Q_{1,2}xy + \cdots + Q_{3,3}z^2$$

with $\deg Q_{ij} = e_i + e_j - b_2$.

The minimal resolution is of type discussed above, because the condition $-1 \leq b_2 \leq b_1 \leq d-1$ is satisfied: the only possibility to have a divisor of type $2H-bR$ with $b \geq d$ is to have $e_1 = e_2 = d/2$, $e_3 = 0$ and $b = d$, but then the equation P is of the form $\alpha x^2 + \beta xy + \gamma y^2$ with constant coefficients, so reducible. If $b_2 = -1$ also cubics are needed to generate the ideal, so the curve admits also a g^1_3 or g^2_5; this happens only up to $g = 6$. We exclude these cases and assume that $b_2 \geq 0$.

Lemma 1.4. *We have* $b_1 \leq 2e_2$ *and* $b_2 \leq 2e_3$.

Proof. If $b_1 > 2e_2$ the polynomials P_{22}, P_{23} and P_{33} vanish so P is reducible and therefore C. If $b_2 > 2e_3$ then P_{33} and Q_{33} vanish. This means that the section $x = y = 0$ is a component of $Y \cap Z$ on the $\mathbb{P}^2$-bundle whose image in $\mathbb{P}^{g-1}$ is the scroll (if $e_3 > 0$ the scroll is nonsingular, but for $e_3 = 0$ it is a cone). As the arithmetic genus of $Y \cap Z$ is g and its image has to be the nonsingular curve C of genus g, the line cannot be a component. $\square$

This Lemma is parts 2 – 4 in [Brawner 1997, Prop. 3.1]. Its last part is incorrect. It states that $b_1 \leq e_1 + e_3$ if $e_3 > 0$, and builds upon the fact that Y has only isolated singularities. However the discussion in [Schreyer 1986] makes clear that this need not be the case.

The surface Y fibres over $\mathbb{P}^1$. There are now two cases, first that the general fibre is a non-singular conic. In this case one of the coefficients P_{13}, P_{23} or P_{33} is nonzero, giving indeed $b_1 \leq e_1 + e_3$.

The other possibility is that each fibre is a singular conic. Then Y is a birationally ruled surface over a (hyper)elliptic curve E with a rational curve $\overline{E}$ of double points, the canonical image of E, and C does not intersect $\overline{E}$. This means that the section $\overline{E}$ of the scroll does not intersect the surface Z, so if one inserts the parametrisation of $\overline{E}$ in the equation of Z one obtains a non-zero constant. Let the section be given by polynomials in $(s:t)$, which if nonzero have degree $d_s - e_1$, $d_s - e_2$, $d_s - e_3$. Inserting them in the polynomial Q gives a polynomial of degree $2d_s - b_2$. So b_2 is even and $2d_s = b_2 \leq 2e_3$. On the other hand $d_s - e_3 \geq 0$ so $d_s = e_3$ and $b_2 = 2e_3$. The genus of E satisfies $p_a(E) = b_2/2 + 1$. If $b_1 > e_1 + e_3$, then Y is singular along the section $x = y = 0$. An hyperelliptic involution can also occur if $b_1 \leq e_1 + e_3$.
We have shown:

Lemma 1.5. *If Y is singular, in particular if $b_1 > e_1 + e_3$, then $b_2 = 2e_3$.*

Finally we analyze the case $b_2 = 0$ (cf. [Brawner 1996]).

Lemma 1.6. *A nonsingular tetragonal curve is bi-elliptic or lies on a Del Pezzo surface if and only if $b_2 = 0$. The first case occurs for $e_3 = 0$, and the second for the values $(2,0,0)$, $(1,1,0)$, $(2,1,0)$, $(1,1,1)$, $(3,1,0)$, $(2,2,0)$, $(2,1,1)$, $(3,2,0)$, $(2,2,1)$, $(4,2,0)$, $(3,2,1)$ or $(2,2,2)$ of the triple (e_1, e_2, e_3).*

Proof. If the curve is bielliptic or lies on a Del Pezzo, the g^1_4 is not unique, which implies that the scroll is not unique. This is only possible if $b_2 = 0$ by [Schreyer 1986], p. 127. Then C is the complete intersection of a quadric and a surface Y of degree $g-1$, which is uniquely determined by C.
The inequality $e_1 + e_2 + e_3 - 2 = b_1 \leq 2e_2$ shows that $e_3 \leq e_2 - e - 1 + 2 \leq 2$. If the general fibre of Y over $\mathbb{P}^1$ is non-singular we have $b_1 \leq e_1 + e_3$. This gives $e_2 \leq 2$ and $b_1 \leq 4$. The possible values are now easily determined. If the general fibre of Y is singular then $e_3 = b_2/2 = 0$ and Y is an elliptic cone.
□

(1.7) *$K3$ surfaces.*
Let X be a $K3$ surface (with at most rational double point singularities) on a scroll. If the scroll is nonsingular the projection onto $\mathbb{P}^1$ gives an elliptic fibration on X, whose general fibre is smooth. This is even true if the scroll is singular: the strict transform $\widetilde{X}$ on $\widetilde{S}$ has only isolated singularities.
We start with the case of divisors. A treatment of such scrollar surfaces with an elliptic fibration can be found in [Reid 1997, 2.11]. One finds:

Lemma 1.8. *For the general $F \in |3H - kR|$ on a scroll $S(e_1, e_2, e_3)$ the general fibre of the elliptic fibration is a nonsingular cubic curve if and only if $k \leq 3e_2$ and $k \leq e_1 + 2e_3$.*

If one fixes k and $e_1 + e_2 + e_3$ these conditions limit the possible distribution of the integers (e_1, e_2, e_3). By the adjunction formula one has $k = e_1 + e_2 + e_3 - 2$ for a $K3$ surface. In this case we obtain 12 solutions, which fall into 3 deformation types of scrolls, according to $\sum e_i \pmod 3$:

$$(e+2,e,e-2) \to (e+1,e,e-1) \to (e,e,e)$$
$$(e+3,e,e-2) \to (e+2,e,e-1) \to (e+1,e+1,e-1) \to (e+1,e,e)$$
$$(e{+}4,e,e{-}2) \to (e{+}3,e,e{-}1) \to (e{+}2,e{+}1,e{-}1) \to (e{+}2,e,e) \to (e{+}1,e{+}1,e)$$

The general element of the linear system can only have singularities at the base locus. The base locus is the section $(0{:}0{:}1)$ if and only if $k > 3e_3$ and there is a singularity at the points $(s{:}t)$ where both $A_{e_1+2e_3-k}$ and $A_{e_2+2e_3-k}$ vanish. The assumption that the coefficients are general implies now that $\deg A_{e_2+2e_3-k} < 0$ and $\deg A_{e_1+2e_3-k} > 0$.
In the 12 cases above this occurs only for $(e+3,e,e-1)$ and $(e+2,e,e-1)$. In the first case the term y^2z is also missing, yielding that there is an A_2-singularity at the only zero of $A_{e_2+2e_3-k}$, whereas the second case gives an A_1. The scroll $S_{e+4,e,e-2}$ deforms into $S_{e+3,e,e-1}$, but the general $K3$-surface on it does not deform to a $K3$ on $S_{e+3,e,e-1}$, but only those with an A_2-singularity. These results hold if all $e_i > 0$; we leave the modifications in case $e_3 = 0$ to the reader.
The tetragonal case is given as exercise in [Reid 1997] and the complete solution (modulo some minor mistakes) can be found in [Brawner 1997]. We give the results:

LEMMA 1.9. *For the general complete intersection of divisors of type $2H-b_1R$ and $2H-b_2R$ on a scroll S_{e_1,e_2,e_3,e_4} the general fibre of the elliptic fibration is a nonsingular quartic curve if and only if either*
α: $b_1 \le e_1+e_3$, $b_1 \le 2e_2$ and $b_2 \le 2e_4$, or
β: $b_1 \le e_1+e_4$, $b_1 \le 2e_2$, $2e_4 < b_2 \le 2e_3$ and $b_2 \le e_2+e_4$.

PROPOSITION 1.10. *The general element is singular at a point of the section $(0:0:0:1)$ if the invariants satisfy in addition one of the following conditions:*
1α: $b_2 < 2e_4$, $b_1 > e_1+e_4$.
$1\beta i$: $b_2 \le e_3+e_4$, $e_2+e_4 < b_1 < e_1+e_4$.
$1\beta ii$: $b_2 > e_3+e_4$, and $e_1+e_2+2e_4 > b_1+b_2$.
There is a singularity with $z \ne 0$ if
2α: $b_1 > e_2+e_3$, $e_1+e_3 > b_1 > e_1+e_4$.
$2\alpha\beta$: $e_1+e_4 \ge b_1 > e_2+e_3$ and
i: if $b_2 \le 2e_4$ then $2(e_1+e_3+e_4) > 2b_1+b_2$
ii: if $2e_4 < b_2 \le e_3+e_4$ then $e_1+2e_3+e_4 > b_1+b_2$
iii: $e_3+e_4 < b_2 < 2e_3$

For $K3$ surfaces we need $b_1+b_2 = e_1+e_2+e_3+e_4-2$. We give a table listing the possibilities under this assumption, cf. [Brawner 1997, Table A.1–A.4].
The table lists the possible values for (b_1,b_2) and gives for each pair the invariants (e_1,e_2,e_3,e_4) of the scrolls on which the curve can lie. These form one deformation type with adjacencies going vertically, except $S_{e+2,e,e,e}$ and

(b_1,b_2)	(e_1,e_2,e_3,e_4)	$\mathcal{M}$	base	sings
$(2e,2e-2)$	$(e+3,e+1,e-1,e-3)$	17	B_{e-1}	$--$
	$(e+3,e,e-1,e-2)$	15	B_{e-1}	A_3
	$(e+2,e+1,e-1,e-2)$	16	B_{e-1}	A_1
	$(e+2,e,e,e-2)$	16	B_{e-2}	$--$
	$(e+2,e,e-1,e-1)$	15	B_{e-1}	$2A_1$
	$(e+1,e+1,e-1,e-1)$	16	B_{e-1}	$--$
	$(e+1,e,e,e-1)$	17	B_{e-1}	$--$
	(e,e,e,e)	17	$\emptyset$	$--$
$(2e-1,2e-1)$	$(e+1,e+1,e,e-2)$	17	B_{e-2}	$--$
	$(e+1,e,e,e-1)$	17	B_{e-1}	$--$
	(e,e,e,e)	18	$\emptyset$	$--$
$(2e+1,2e-2)$	$(e+4,e+1,e-1,e-3)$	17	B_{e-1}	$--$
	$(e+3,e+1,e-1,e-2)$	16	B_{e-1}	A_1
	$(e+2,e+1,e-1,e-1)$	16	B_{e-1}	$--$
	$(e+1,e+1,e,e-1)$	17	B_e	$--$
$(2e,2e-1)$	$(e+2,e+1,e,e-2)$	17	B_{e-2}	$--$
	$(e+2,e,e,e-1)$	15	B_{e-1}	A_1
	$(e+1,e+1,e,e-1)$	17	B_{e-1}	$--$
	$(e+1,e,e,e)$	18	$\emptyset$	$--$
$(2e+2,2e-2)$	$(e+5,e+1,e-1,e-3)$	18	B_{e-1}	$--$
	$(e+4,e+1,e-1,e-2)$	17	B_{e-1}	A_1
	$(e+3,e+1,e-1,e-1)$	17	B_{e-1}	$--$
	$(e+2,e+1,e,e-1)$	18	B_e	$--$
	$(e+1,e+1,e+1,e-1)$	18	B_{e-1}	$--$
$(2e+1,2e-1)$	$(e+3,e+1,e,e-2)$	16	B_e	$--$
	$(e+2,e+1,e,e-1)$	16	B_e	$--$
	$(e+1,e+1,e,e)$	17	B_e	$--$
$(2e,2e)$	$(e+2,e+2,e,e-2)$	17	B_{e-2}	$--$
	$(e+2,e+1,e,e-1)$	16	B_{e-1}	A_1
	$(e+1,e+1,e+1,e-1)$	17	B_{e-1}	$--$
	$(e+2,e,e,e)$	15	$\emptyset$	$--$
	$(e+1,e+1,e,e)$	17	$\emptyset$	$--$
$(2e+2,2e-1)$	$(e+4,e+1,e,e-2)$	16	B_e	A_1
	$(e+3,e+1,e,e-1)$	16	B_e	A_1
	$(e+2,e+1,e,e)$	17	B_e	A_1
	$(e+1,e+1,e+1,e)$	17	B_e	A_1
$(2e+1,2e)$	$(e+3,e+2,e,e-2)$	17	B_e	$--$
	$(e+3,e+1,e,e-1)$	15	B_e	A_2
	$(e+2,e+2,e,e-1)$	16	B_e	A_1
	$(e+2,e+1,e+1,e-1)$	17	B_{e-1}	$--$
	$(e+2,e+1,e,e)$	16	B_e	$--$
	$(e+1,e+1,e+1,e)$	18	B_e	$--$

$S_{e+1,e+1,e+1,e-1}$ which do not deform into each other but are both deformations of $S_{e+2,e+1,e,e-1}$ and both deform to $S_{e+1,e+1,e,e}$. Furthermore we give the number of moduli for each family, in the column $\mathcal{M}$.
In the table we also list the base locus of $|2H-b_1R|$ (which contains that of $|2H-b_2R|$). The base locus is a subscroll, for which we use the following notation [Reid 1997, 2.8]: we denote by B_a the subscroll corresponding to the subset of all e_i with $e_i \leq a$, defined by the equations $z^{(j)}=0$ for $e_j > a$. We give the number and type of the singularities of the general element; the number given in the second half of [Brawner 1997, Table A.2] is not correct.
As example of the computations we look at $(e+3,e,e-1,e-2)$ with $(b_1,b_2)=(2e,2e-2)$. The two equations have the form

$$p_1xw+p_2xz+p_0y^2+p_3yx+p_6x^2$$
$$q_0z^2+q_0yw+q_3xw+q_1yz+q_4xz+q_2y^2+q_5yx+q_8x^2\ ,$$

where the index denotes the degree in $(s:t)$. We first use coordinate transformations to simplify these equations. By replacing y, z and w by suitable multiples we may assume that the three constant polynomials are 1. Now replacing z by $z-\frac{1}{2}q_1y-\frac{1}{2}q_4x$ removes the yz and xz terms. We then replace y by $y-q_3x$ to get rid of the xw term. By changing w we finally achieve the form $z^2+yw+q_8x^2$. By a change in $(s:t)$ we may assume that $p_1=s$. We now look at the affine chart $(w=1,t=1)$ and find $y=-z^2-q_8x^2$, which we insert in the other equation to get an equation of the form $x(s+p_2z+\cdots)+z^4$, which is an A_3.
We leave it again to the reader to analyze which further singularities can occur if $e_4=0$.

2. Infinitesimal deformations.

Deformations of cones over complete intersections on scrolls need not preserve the rolling factors format. We shall study in detail those who do. Many deformations of negative weight are of this type.

Definition 2.1. A *pure rolling factors deformation* is a deformation in which the scroll is undeformed and only the equations on the scroll are perturbed.

This means that the deformation of the additional equations can be written with the rolling factors. Such deformations are always unobstructed. However this is not the only type of deformation for which the scroll is not changed. In weight zero one can have deformations inside the scroll, where the type $(b_1,\dots,b_l)$ changes.

Definition 2.2. A (general) *rolling factors deformation* is a deformation in which the scroll is deformed and the additional equations are written in rolling factors with respect to the deformed scroll.

The equations for the total space of a 1-parameter rolling factors deformation describe a scroll of one dimension higher, containing a subvariety of the same codimension, again in rolling factors format. Deformations over higher dimensional base spaces may be obstructed. Again in weight zero one can have deformations of the scroll, where also the type $(b_1,\dots,b_l)$ changes.

Finally there are *non-scrollar deformations*, where the perturbation of the scrollar equations does not define a deformation of the scroll. Examples of this phenomenon are easy to find (but difficult to describe explicitly). A trigonal canonical curve is a divisor in a scroll, whereas the general canonical curve of the same genus g is not of this type: the codimension of the trigonal locus in moduli space is $g-4$.

Example 2.3. To give an example of a deformation inside a scroll, we let C be a tetragonal curve in $\mathbb{P}^9$ with invariants $(2,2,2;3,1)$. Then there is a weight 0 deformation to a curve of type $(2,2,2;2,2)$. To be specific, let C be given by $P=sx^2+ty^2+(s+t)z^2$, $Q=t^3x^2+s^3y^2+(s^3-t^3)z^2$. We do not deform the scroll, but only the additional equations:

$$
\begin{array}{ll}
x_0^2+y_0y_1+z_0^2 & +\varepsilon(z_1^2-x_1^2)\\
x_0x_1+y_1^2+z_0z_1 & +\varepsilon(z_1z_2-x_1x_2)\\
x_1^2+y_1y_2+z_1^2 & +\varepsilon(y_0y_1+z_0z_1)\\
x_1x_2+y_2^2+z_1z_2 & +\varepsilon(y_1^2+z_1^2)\\[4pt]
x_1x_2+y_0^2+z_0^2-z_1z_2 & \\
x_2^2+y_0y_1+z_0z_1-z_2^2 &
\end{array}
$$

For $\varepsilon\neq 0$ we can write the ideal as

$$
\begin{array}{ll}
x_0^2+y_0y_1+z_0^2 & +\varepsilon(z_1^2-x_1^2)\\
x_0x_1+y_1^2+z_0z_1 & +\varepsilon(z_1z_2-x_1x_2)\\
x_1^2+y_1y_2+z_1^2 & +\varepsilon(z_2^2-x_2^2)\\[4pt]
x_0x_1+y_1^2+z_0z_1 & +\varepsilon(y_0^2+z_0^2)\\
x_1^2+y_1y_2+z_1^2 & +\varepsilon(y_0y_1+z_0z_1)\\
x_1x_2+y_2^2+z_1z_2 & +\varepsilon(y_1^2+z_1^2)
\end{array}
$$

We can describe this deformation in the following way. Write $Q=sQ_s+tQ_t$. The two times three equations above are obtained by rolling factors from $sP-\varepsilon Q_t$ and $tP+\varepsilon Q_s$. We may generalize this example.

Lemma 2.4. *Let V be a complete intersection of divisors of type $aH-b_1R$, $aH-b_2R$, given by equations P, Q. If it is possible to write $Q=sQ_s+t^{b_1-b_2-1}Q_t$ then the equations $sP-\varepsilon Q_t$, $t^{b_1-b_2-1}P+\varepsilon Q_s$ give a deformation to a complete intersection of type $aH-(b_1-1)R$, $aH-(b_2+1)R$.*

In general one has to combine such a deformation with a deformation of the scroll.

(2.5) *Non-scrollar deformations.*

Example 2.6. As mentioned before such deformations must exist in weight zero for trigonal cones. We proceed with the explicit computation of embedded deformations. We start from the normal bundle exact sequence

$$0 \longrightarrow N_{S/C} \longrightarrow N_C \longrightarrow N_S \otimes \mathcal{O}_C \longrightarrow 0\ .$$

As C is a curve of type $3H-(g-4)R$ on S we have that $C\cdot C = 3g+6$ and $H^1(C, N_{S/C}) = 0$. So we are interested in $H^0(C, N_S \otimes \mathcal{O}_C)$, and more particularly in the cokernel of the map $H^0(S, N_S) \longrightarrow H^0(C, N_S \otimes \mathcal{O}_C)$, as $H^0(S, N_S)$ gives deformations of the scroll.

Proposition 2.7. *The cokernel of the map $H^0(S, N_S) \longrightarrow H^0(C, N_S \otimes \mathcal{O}_C)$ has dimension $g-4$.*

Proof. An element of $H^0(C, N_S \otimes \mathcal{O}_C)$ is a function φ on the equations of the scroll such that the generators of the module of relations map to zero in $\mathcal{O}_C$ and it lies in the image of $H^0(S, N_S)$ if the function values can be lifted to $\mathcal{O}_S$ such that the relations map to $0 \in \mathcal{O}_S$. Therefore we perform our computations in $\mathcal{O}_S$.
We have to introduce some more notation. Using the equations described in (1.3) we have three types of scrollar equations, $f_{i,j} = x_i x_{j+1} - x_{i+1} x_j$, $g_{i,j} = y_i y_{j+1} - y_{i+1} y_j$ and mixed equations $h_{i,j} = x_i y_{j+1} - x_{i+1} y_j$. The scrollar relations come from doubling a row in the matrix and there are two ways to do this. The equations resulting from doubling the top row can be divided by s, and the other ones by t, so the result is the same.
A relation involving only equations of type $f_{i,j}$ gives the condition

$$xs^{e_1-i-1}t^i\varphi(f_{j,k}) - xs^{e_1-j-1}t^j\varphi(f_{i,k}) + xs^{e_1-k-1}t^k\varphi(f_{i,j}) = 0 \in \mathcal{O}_C$$

which may be divided by x. As the image $\varphi(f_{i,j})$ is quadratic in x and y the resulting left hand side cannot be a multiple of the equation of C, so we have

$$s^{e_1-i-1}t^i\varphi(f_{j,k}) - s^{e-1-j-1}t^j\varphi(f_{i,k}) + s^{e_1-k-1}t^k\varphi(f_{i,j}) = 0 \in \mathcal{O}_S$$

and the analogous equation involving only the $g_{i,j}$ equations.
For the mixed equations we get

$$xs^{e_1-i-1}t^i\varphi(h_{j,k}) - xs^{e_1-j-1}t^j\varphi(h_{i,k}) + ys^{e_2-k-1}t^k\varphi(f_{i,j}) = \psi_{i,j;k}P \in \mathcal{O}_S$$

with $\psi_{i,j;k}$ of degree $e_1+e_2-3 = g-5$ and analogous ones involving $g_{i,j}$ with coefficients $\psi_{i;j,k}$. These coefficients are not independent, but satisfy a systems of equations coming from the syzygies between the relations. They can also be verified directly. We obtain

$$s^{e_1-i-1}t^i\psi_{j,k;l} - s^{e_1-j-1}t^j\psi_{i,k;l} + s^{e_1-k-1}t^k\psi_{i,j;l} = 0 \in \mathcal{O}_S$$

and

$$xs^{e_1-i-1}t^i\psi_{j;k,l} - xs^{e_1-j-1}t^j\psi_{i;k,l} + ys^{e_2-k-1}t^k\psi_{i,j;l} - ys^{e_2-l-1}t^l\psi_{i,j;k} = 0$$

The last set of equations shows that $s^{e_2-k-1}t^k\psi_{i,j;l} = s^{e_2-l-1}t^l\psi_{i,j;k}$ (rolling factors!) and therefore $\psi_{i,j;k} = s^{e_2-k-1}t^k\psi_{i,j;}$ with $\psi_{i,j;}$ of degree e_1-2. This yields the equations

$$s^{e_1-i-1}t^i\psi_{j,k;} - s^{e_1-j-1}t^j\psi_{i,k;} + s^{e_1-k-1}t^k\psi_{i,j;} = 0$$

Our next goal is to express all $\psi_{i,j;}$ in terms of the $\psi_{i,i+1;}$ (where $0 \le i \le e_1-2$). First we observe by using the last equation for the triples $(0,i,i+1)$ and $(i,i+1,e_1-1)$ that $\psi_{i,i+1;}$ is divisible by t^i and by s^{e_1-i-2} so $\psi_{i,i+1;} = s^{e_1-i-2}t^ic_i$ for some constant c_i. By induction it then follows that $\psi_{i,j;} = s^{e_1-i-2}t^ic_{j-1} + s^{e_1-i-3}t^{i+1}c_{j-2} + \cdots + s^{e_1-j-1}t^{j-1}c_i$, so the solution of the equations depends on e_1-1 constants. Similarly one finds e_2-1 constants d_i for the $\psi_{i;j,k}$ so altogether $e_1+e_2-2 = g-4$ constants.
Finally we can solve for the perturbations of the equations. We give the formulas in the case that all d_i and all c_i but one are zero, say $c_\gamma = 1$. This implies that $\psi_{i,j;} = 0$ if $\gamma \notin [i,j)$ and $\psi_{i,j;} = s^{e_1-i-j+\gamma-1}t^{i+j-\gamma-1}$ if $\gamma \in [i,j)$; under the last assumption $\psi_{i,j;k} = s^{g-4-i-j-k+\gamma}t^{i+j+k-\gamma-1}$. We take $\varphi(f_{i,j}) = 0$ if $\gamma \notin [i,j)$. It follows that for a fixed k the $\varphi(h_{i,k})$ with $i \le \gamma$ are related by rolling factors, as are the $\varphi(h_{i,k})$ with $i > \gamma$. This reduces the mixed equations with fixed k to one, which can be solved for in a uniform way for all k. To this end we write the equation P as

$$(s^{2e_1-e_2-\gamma+2}A^+_\gamma + t^{\gamma+1}A^-_{2e_1-e_2-\gamma+1})x^3 + y(B_{e_1+2}x^2 + C_{e_2+2}xy + D_{2e_2-e_1+2}y^2)$$

which we will abbreviate as $(s^{2e_1-e_2-\gamma+2}A^+ + t^{\gamma+1}A^-)x^3 + yE$. We set

$$\begin{array}{rcll} \varphi(f_{i,j}) & = & 0, & \text{if } \gamma \notin [i,j) \\ \varphi(f_{i,j}) & = & s^{e_1-i-j-1+\gamma}t^{i+j-\gamma-1}E, & \text{if } \gamma \in [i,j) \\ \varphi(g_{i,j}) & = & 0, & \\ \varphi(h_{i,k}) & = & -s^{e_2-1-k-i+\gamma}t^{i+k}A^-x^2, & \text{if } i \le \gamma \\ \varphi(h_{i,k}) & = & s^{2e_1+1-i-k}t^{i+k-\gamma-1}A^+x^2, & \text{if } i > \gamma \end{array}$$

This is well defined, because all exponents of s and t are positive. □

A similar computation can be used to show that all elements of $T^1(\nu)$ with $\nu > 0$ can be written rolling factors type. However, even more is true, they can be represented as pure rolling factors deformations, see [Drewes–Stevens 1996], where a direct argument is given.

We generalize the above discussion to the case of a complete intersection of divisors of type $aH - b_iR$ (with the same $a \ge 2$) on a scroll

$$\begin{pmatrix} z_0^{(1)} & \dots & z_{d_1-1}^{(1)} & \dots & z_0^{(k)} & \dots & z_{d_k-1}^{(k)} \\ z_1^{(1)} & \dots & z_{d_1}^{(1)} & \dots & z_1^{(k)} & \dots & z_{d_k}^{(k)} \end{pmatrix}.$$

We have equations $f_{ij}^{(\alpha\beta)} = z_i^{(\alpha)} z_{j+1}^{(\beta)} - z_{i+1}^{(\alpha)} z_j^{(\beta)}$. The lowest degree in which non rolling factors deformations can occur is $a-3$. We get the conditions

$$z^{(\alpha)} s^{d_\alpha - i-1} t^i \varphi\bigl(f_{jk}^{(\beta\gamma)}\bigr) - z^{(\beta)} s^{d_\beta - j-1} t^j \varphi\bigl(f_{ik}^{(\alpha\gamma)}\bigr) + z^{(\gamma)} s^{d_\gamma - k-1} t^k \varphi\bigl(f_{ij}^{(\alpha\beta)}\bigr) \\ = \sum_l \psi_{ijk;l}^{(\alpha\beta\gamma)} P^{(l)}$$

with the $\psi_{ijk;l}^{(\alpha\beta\gamma)}$ homogeneous polynomials in $(s{:}t)$ of degree $b_l - 1$. The relations between these polynomials come from the syzygies of the scroll: we add four of these relations, multiplied with a term linear in the $z^{(\alpha)}$; then the left hand side becomes zero, leading to a relation (in $\mathcal{O}_S$) between the $P^{(l)}$. As we are dealing with a complete intersection, the relations are generated by Koszul relations. Because the coefficients of the relation obtained are linear in the $z^{(\alpha)}$, they cannot lie in the ideal generated by the $P^{(l)}$ (as $a \geq 2$), so they vanish and we obtain for each l equations

$$z^{(\alpha)} s^{d_\alpha - i-1} t^i \psi_{jkm;l}^{(\beta\gamma\delta)} - z^{(\beta)} s^{d_\beta - j-1} t^j \psi_{ikm;l}^{(\alpha\gamma\delta)} \\ + z^{(\gamma)} s^{d_\gamma - k-1} t^k \psi_{ijm;l}^{(\alpha\beta\delta)} - z^{(\delta)} s^{d_\delta - m-1} t^m \psi_{ijk;l}^{(\alpha\beta\gamma)} = 0\ .$$

Here some of the $\alpha, \ldots, \delta$ may coincide. If e.g. δ is different from α, β and γ, then $\psi_{ijk;l}^{(\alpha\beta\gamma)} = 0$. If there are at least four different indices (e.g. if the scroll is nonsingular of dimension at least four) then δ can always be chosen in this way, so all coefficients vanish and every deformation of degree $a-3$ is of rolling factors type.

Suppose now the scroll is a cone over a nonsingular 3-dimensional scroll, i.e. we have three different indices at our disposal. Then every $\psi_{ijk;l}^{(\alpha\beta\gamma)}$ with at most two different upper indices vanishes, and the ones with three different indices satisfy rolling factors equations. We conclude that for pairwise different α, β, γ

$$\psi_{ijk;l}^{(\alpha\beta\gamma)} = s^{d-i-j-k-3} t^{i+j+k} \psi'_l$$

with $d = d_\alpha + d_\beta + d_\gamma$ the degree of the scroll.

Finally, for the cone over a 2-dimensional scroll we get similar computations as in the trigonal example above.

PROPOSITION 2.8. *A tetragonal cone (with $g > 5$) has non-scrollar deformations of degree -1 if and only if $b_2 = 0$. If the canonical curve lies on a Del Pezzo surface then the dimension is 1. If the curve is bielliptic then the dimension is $b_1 = g - 5$.*

Proof. First suppose $e_3 > 0$. Then the only possibly non zero coefficients are the ψ'_l, which have degree $b_l + 2 - \sum e_i$. As $b_1 + b_2 = \sum e_i - 2$ they do not vanish iff $b_2 = 0$. In this case the computation yields one non rolling factors deformation of the Del Pezzo surface on which the curve lies.

If $e_3 = 0$, then $b_2 = 0$. For a bielliptic curve the methods above yield $(e_1 - 1) + (e_2 - 1) = b_1 = g - 5$ non-scrollar deformations (a detailed computation is given in [Brawner 1996]). Suppose now that the curve lies on a (singular) Del Pezzo surface. If $b_1 = e_1 > e_2 = 2$ then the equation P contains the monomial xz with nonzero coefficient, which we take to be 1, while there is no monomial yz. After a coordinate transformation we may assume that the same holds in case $e_1 = e_2 = 2$. Let $\varphi(h_{i,k}) \equiv \zeta_{i,k} z \bmod (x, y)$. In the equation

$$xs^{e_1-i-1}t^i\varphi(g_{j,k}) - ys^{e_2-j-1}t^j\varphi(h_{i,k}) + ys^{e_2-k-1}t^k\varphi(h_{i,j}) = \psi_{i;j,k}P$$

holding in $\mathcal{O}_S$ the monomial yz occurs only on the left hand side, which shows that the $\zeta_{i,k}$ are of rolling factors type in the first index. Being constants, they vanish. This means that in the equation

$$xs^{e_1-i-1}t^i\varphi(h_{j,k}) - xs^{e_1-j-1}t^j\varphi(h_{i,k}) + ys^{e_2-k-1}t^k\varphi(f_{i,j}) = \psi_{i,j;k}P$$

the monomial xz does not occur on the left hand side and therefore $\psi_{i,j;k} = 0$. We find only $e_2 - 1 = 1$ non rolling factors deformation. If $e_1 = 2$, $e_2 = 1$ we find one deformation. Finally, if $e_1 = 3$, $e_2 = 1$ then there is only one type of mixed equation. We have two constants c_0 and c_1. Let the coefficient of xz in P be $p_0s + p_1t$. We obtain the equations

$$\begin{aligned} s\zeta_{1,0} - t\zeta_{0,0} &= c_0(p_0s + p_1t) \\ s\zeta_{2,0} - t\zeta_{1,0} &= c_1(p_0s + p_1t) \end{aligned}$$

from which we conclude that $p_0c_0 + p_1c_1 = 0$, giving again only one non rolling factors deformation. □

(2.9) *Rolling factors deformations of degree* -1.
We look at the miniversal deformation of the scroll:

$$\begin{pmatrix} z_0^{(1)} & \dots & z_{d_1-2}^{(1)} & z_{d_1-1}^{(1)} & z_0^{(2)} & \dots & z_{d_k-2}^{(k)} & z_{d_k-1}^{(k)} \\ z_1^{(1)} + \zeta_1^{(1)} & \dots & z_{d_1-1}^{(1)} + \zeta_{d_1-1}^{(1)} & z_{d_1}^{(1)} & z_1^{(2)} & \dots & z_{d_k-1}^{(k)} + \zeta_{d_k-1}^{(k)} & z_{d_k}^{(k)} \end{pmatrix}$$

To compute which of those deformations can be lifted to deformations of a complete intersection on the scroll we have to compute perturbations of the additional equations.
We assume that we have a complete intersection of divisors of type $aH - b_iR$ (with the same $a \geq 2$).
Extending the notation introduced before we write the columns in the matrix symbolically as $(z_\alpha, z_{\alpha+1}+\zeta_{\alpha+1})$. In order that this makes sense for all columns we introduce dummy variables $\zeta_0^{(i)}$ and $\zeta_{d_i}^{(i)}$ with the value 0.
The Koszul type relations give no new conditions, but the relation

$$P_{m+1}z_\beta - P_m z_{\beta+1} - \sum_\alpha p_{\alpha,m} f_{\beta\alpha} = 0$$

gives as equation in the local ring for the perturbations P'_m of P_m:

$$P'_{m+1}z_\beta - P'_m z_{\beta+1} - \sum_\alpha p_{\alpha,m}(\zeta_{\alpha+1}z_\beta - z_\alpha\zeta_{\beta+1}) = 0\ .$$

In particular we see that we can look at one equation on the scroll at a time. As $\sum p_{\alpha,m}z_\alpha = P_m$ the coefficient of $\zeta_{\beta+1}$ vanishes. Because $tz_\beta - sz_{\beta+1} = 0$ we get a condition which is independent of β:

$$sP'_{m+1} - tP'_m - s\sum_\alpha p_{\alpha,m}\zeta_{\alpha+1} = 0$$

This has to hold in the local ring, but as the degree of the $p_{\alpha,m}$ is lower than that of the equations defining the complete intersection on the scroll (here we use the assumption that all degrees a are equal), it holds on the scroll. From it we derive the equation

$$s^bP'_b - t^bP'_0 = \sum_{m=0}^{b-1}\sum_\alpha s^{m+1}t^{b-m-1}p_{\alpha,m}\zeta_{\alpha+1} \tag{S}$$

which has to be solved with P'_b and P'_0 polynomials in the z_α of degree $a-1$. We determine the monomials on the right hand side.
The result depends on the chosen equations, but only on P_0 and P_b and not on the intermediate ones, provided they are obtained by rolling factors.

Example 2.10. Let $b = 4$. We take variables $y_i = s^{3-i}t^iy$, $z_i = s^{3-i}t^iz$ with deformations η_i, ζ_i, and roll from y_0z_0 to y_2z_2 in two different ways:

$$y_0z_0 \to y_1z_0 \to y_1z_1 \to y_2z_1 \to y_2z_2$$
$$y_0z_0 \to y_0z_1 \to y_0z_2 \to y_1z_2 \to y_2z_2$$

This gives as right-hand side of the equation (S) in the two cases

$$s^4t^3z\eta_1 + s^4t^3y\zeta_1 + s^5t^2z\eta_2 + s^5t^2y\zeta_2$$
$$s^4t^3y\zeta_1 + s^5t^2y\zeta_2 + s^4t^3z\eta_1 + s^5t^2z\eta_2$$

which is the same expression. Similarly, if we roll from z_0^2 to z_2^2 we get

$$2s^4t^3z\zeta_1 + 2s^5t^2z\zeta_2$$

However, if we roll in the last step from y_1z_2 to y_1z_3 we get

$$s^4t^3y\zeta_1 + s^5t^2y\zeta_2 + s^4t^3z\eta_1$$

(remember that we have no deformation parameter ζ_3).

To analyze the general situation it is convenient to use multi-index notation. The equation P of a divisor in $|aH - bR|$ may then be written as

$$P = \sum_{|I|=a} \sum_{j=0}^{\langle e,I\rangle - b} p_{I,j} s^{\langle e,I\rangle - b - j} t^j z^I \, .$$

Here $e = (e_1, \dots, e_k)$ is the vector of degrees and z^I stands for $(z^{(1)})^{i_1} \cdot \ldots \cdot (z^{(k)})^{i_k}$.

Proposition 2.11. *The lifting condition for the equations P_m is that for each I with $|I| = a-1$ and $\langle e, I\rangle < b-1$ the following $b - \langle e, I\rangle - 1$ linear equations hold:*

$$\sum_{l=1}^{k} \sum_{j=0}^{\langle e,I+\delta_l\rangle - b} (i_l+1) p_{I+\delta_l,j} \zeta^{(l)}_{j+n} = 0 \, ,$$

where $0 < n < b - \langle e, I\rangle$

Proof. We look at a monomial $s^{\langle e,I'\rangle - b - j} t^j z^{I'}$. In rolling from P_0 to P_m we go from z_A to z_{A+B}. Here we write a monomial as product of a factors: $z_{\alpha_1} \cdots z_{\alpha_a}$ with i'_l factors of type l. Let $I' = I + \delta_l$ with δ_l the lth unit vector. The monomial leads to an expression in which the coefficient of z^I is

$$\sum_{\{q|\alpha_q \text{ of type } l\}} \sum_{r=1}^{\beta_q} s^{\langle e,I\rangle + r - j + \alpha_q} t^{b - r + j - \alpha_q} \zeta_{\alpha_q + r}$$

We stress that the choice of α_q can be very different for different j.
We collect all contributions and look at the coefficient of $s^{\langle e,I\rangle + n} t^{b-n} z^I$ with $0 < n < b - \langle e, I\rangle$. This cannot be realized as left-hand side of equation (S). Because $b - n = b - r + j - \alpha_q$ this coefficient is

$$\sum_{l=1}^{k} \sum_{j=0}^{\langle e,I+\delta_l\rangle - b} (i_l+1) p_{I+\delta_l,j} \zeta^{(l)}_{j+n} = 0 \, ,$$

We note that all terms really occur: in rolling from z_A to z_{A+B} we have to increase the qth factor sufficiently many times, because $\langle e, I\rangle < b - 1$. $\square$

Example 2.12: trigonal cones. Let the curve be given by the bihomogeneous equation

$$F = A_{2a-m+2} z^3 + B_{a+2} z^2 w + C_{m+2} z w^2 + D_{2m-a+2} w^3$$

then there are only conditions for $I = (0, 2)$, i.e. for w^2, as $a+m = g-2 > b-1$. So if $2m < b - 1 = g - 5$ we get $b - 2m - 1 = a - m - 3$ equations on the deformation variables $\zeta_1, \dots, \zeta_{a-1}, \omega_1, \dots, \omega_{m-1}$

$$\sum_{j=0}^{m+2} c_j \zeta_{j+n} + 3 \sum_{j=0}^{2m-a+2} d_j \omega_{j+n} = 0 \, ,$$

as stated in [Drewes–Stevens 1996, 3.11]. We have a system of linear equations so we can write the coefficient matrix. It consists of two blocks $(\,\mathcal{C}\mid\mathcal{D}\,)$ with $\mathcal{C}$ of the form

$$\begin{pmatrix}
c_0 & c_1 & c_2 & \dots & c_{m+2} & 0 & 0 & \dots & 0 & 0 & 0\\
0 & c_0 & c_1 & \dots & c_{m+1} & c_{m+2} & 0 & \dots & 0 & 0 & 0\\
0 & 0 & c_0 & \dots & c_m & c_{m+1} & c_{m+2} & \dots & 0 & 0 & 0\\
 & & & \ddots & & & & \ddots & & & \\
0 & 0 & 0 & \dots & c_0 & c_1 & c_2 & \dots & c_{m+2} & 0 & 0\\
0 & 0 & 0 & \dots & 0 & c_0 & c_1 & \dots & c_{m+1} & c_{m+2} & 0\\
0 & 0 & 0 & \dots & 0 & 0 & c_0 & \dots & c_m & c_{m+1} & c_{m+2}
\end{pmatrix}$$

and $\mathcal{D}$ similarly. Obviously this system has maximal rank.

The Proposition gives a system of linear equations and we call the coefficient matrix *lifting matrix*. It was introduced for tetragonal cones in [Brawner 1996]. In general the lifting matrix will have maximal rank, but it is a difficult question to decide when this happens.

Example 2.13: trigonal $K3$s. We take the invariants (e,e,e), $b=3e-2$ with $e\geq 3$. The $K3$ lies on $\mathbb{P}^2\times\mathbb{P}^1$ and is given by an equation of bidegree $(3,2)$. Now there are six Is with $|I|=2$ each giving rise to $e-3$ equations in $3(e-1)$ deformation variables. In general the matrix has maximal rank, but for special surfaces the rank can drop. Consider an equation of type $p_1x^3+p_2y^3+p_3z^3$ with the p_i quadratic polynomials in $(s{:}t)$ without common or multiple zeroes. Then the surface is smooth. The lifting equations corresponding to the quadratic monomials xy, xz and yz vanish identically and the lifting matrix reduces to a block-diagonal matrix of rank $3(e-3)$. The kernel has dimension 6, but the corresponding deformations are obstructed: an extension of the $K3$ would be a Fano 3-fold with isolated singularities lying as divisor of type $3H-bR$ on a scroll $S(e_1,e_2,e_3,e_4)$, and a computation reveals that such a Fano can only exist for $\sum e_i\leq 8$.

We can say something more for the lifting conditions coming from one quadratic equation.

PROPOSITION 2.14. *The lifting matrix for one quadratic equation has dependent rows if and only if the generic fibre has a singular point on the subscroll B_{b-1}.*

Proof. The equation P on the scroll can be written in the form ${}^tz\Pi z$ with Π a symmetric $k\times k$ matrix with polynomials in $(s:t)$ as entries. The condition that there is a singular section of the form $z=(0,\dots,0,z^{(l+1)}(s,t),\dots,z^{(k)}(s,t))$ with $e_l>b-1$ is that ${}^tz\Pi=0$ or ${}^tz_{>l}\Pi_{>l}=0$ where $z_{>l}=(z^{(l+1)}(s,t),\dots,z^{(k)}(s,t))$ and $\Pi_{>l}$ is the matrix consisting of the last $k-l$ rows of Π. The resulting system of equations for the coefficients of the polynomials $z^{(i)}(s,t)$ gives exactly the lifting matrix. □

(2.15) *Tetragonal curves.* Most of the following results are contained in the preprint [Brawner 1996]. We have two equations on the scroll and the lifting matrix M can have rows coming from both equations. We first suppose that $b_2 > 0$. Then also $e_3 > 0$ and the number of columns of M is always $\sum(e_i - 1) = g - 6$, but the number of rows depends on the values of $(e_1, e_2, e_3; b_1, b_2)$: it is $\sum_{i,j} \max(0, b_i - e_j - 1)$.

Theorem 2.16. *Let X be the cone over a tetragonal canonical curve and suppose that $b_2 > 0$. Then $\dim T^1_X(-2) = 0$. Suppose that the g^1_4 is not composed with an involution of genus $\frac{b_2}{2} + 1$.*

1) If $b_1 < e_1 + 1$ or $b_2 < e_3 + 1$ or $g \leq 15$ then $\dim T^1_X(-1) = 9 + \dim \operatorname{Cork} M$.

2) If $b_1 \geq e_1 + 1$, $b_2 \geq e_3 + 1$ and $g > 15$ then $9 + \dim \operatorname{Cork} M \leq \dim T^1_X(-1) \leq \frac{g+3}{6} + 6 + \dim \operatorname{Cork} M$ and the maximum is obtained for g of the form $6n - 3$ and $(e_1, e_2, e_3; b_1, b_2) = (3n-2, 2n-2, n-2; 4n-4, 2n-4)$.

3) For generic values of the moduli $\dim \operatorname{Cork} M = 0$.

Proof. If $b_2 > 0$ there are only rolling factors deformations in negative degrees. In particular $\dim T^1_X(-2) = 0$. The number of pure rolling factors deformations is $\rho = \sum_{i,j} \max(e_j - b_i + 1, 0)$. The number of rows in the lifting matrix is $\sum_{i,j} \max(0, b_i - e_j - 1) = 3(b_1 + b_2) - 2(e_1 + e_2 + e_3 + 3) + \rho = g - 15 + \rho$. If $\rho > 9$ the number of rows exceeds the number of columns and $\dim T^1_X(-1) = \rho + \dim \operatorname{Cork} M$; otherwise it is $9 + \dim \operatorname{Cork} M$. So we have to estimate ρ.
As the g^1_4 is not composed we have $b_1 \leq e_1 + e_3$. Together with $b_1 \leq 2e_2$ we get $3b_1 \leq 2g - 6$ and $3b_2 \geq g - 9$; from $e_1 \leq \frac{g-1}{2}$ we now derive $e_1 - b_2 + 1 \leq \frac{g-1}{2} + 1 - \frac{g-9}{3}$. Also $b_2 = e_1 + e_2 + e_3 - 2 - b_1 \geq e_2 - 2$, so $e_2 - b_2 + 1 \leq 3$.
2) Suppose first that $b_1 \geq e_1 + 1$ and $b_2 \geq e_3 + 1$. Then $\rho = \max(0, e_1 - b_2 + 1) + \max(0, e_2 - b_2 + 1) \leq \frac{g+3}{6} + 6$. Equality is achieved iff $e_1 = (g-1)/2$, $b_2 = (g-9)/3$ and $e_2 = b_2 + 2$, so g has the form $6n - 3$ and $(e_1, e_2, e_3; b_1, b_2) = (3n-2, 2n-2, n-2; 4n-4, 2n-4)$.
1) In all other cases $\rho \leq 9$: if $b_1 \geq e_1 + 1$, but $b_2 < e_3 + 1$ then $\rho = (e_1 - b_2 + 1) + (e_2 - b_2 + 1) + (e_3 - b_2 + 1) = g - 3b_2 \leq 9$. If $e_2 + 1 \leq b_1 < e_1 + 1$ then $\rho = (e_1 - b_1 + 1) + (e_1 - b_2 + 1) + \max(0, e_2 - b_2 + 1) + \max(0, e_3 - b_2 + 1) = 2e_1 + 7 - g + \max(0, e_2 - b_2 + 1) + \max(0, e_3 - b_2 + 1)$. As $b_2 > 0$ we have that $\max(0, e_2 - b_2 + 1) + \max(0, e_3 - b_2 + 1) > \max(0, e_2 + e_3 - b_2 + 1)$. But from $b_1 \leq e_1$ it follows that $b_2 \geq e_2 + e_3 - 2$. So $\rho \leq (g-1) + 7 - g + 3 = 9$. If $b_1 < e_2 + 1$ then $\rho \leq 2e_1 + 2e_2 + 4 - 2b_1 - 2b_2 + 2e_3 = 8$.
3) It is easy to construct lifting matrices of maximal rank for all possible numbers of blocks occurring. □

Now we consider the case that the g^1_4 is composed with an involution of genus $g' = \frac{b_2}{2} + 1$. So if $b_2 > 0$, then $g' > 1$. After a coordinate transformation we may assume that the surface Y is singular along the section $x = y = 0$, so its equation depends only on x and y: $P = P(x, y; s, t)$. We may assume that Q has the form $Q = z^2 + Q'(x, y; s, t)$. Let M_{xy} be the submatrix of the lifting matrix consisting of the blocks coming from P and Q' and the ξ and η deformations.

Theorem 2.17. *Let X be a tetragonal canonical cone such that the g^1_4 is composed with an involution of genus $g' > 1$. Then* $\dim T^1_X(-1) = e_1 + e_2 - 2e_3 + 6 + \operatorname{Cork} M_{xy}$.

Proof. The rows in the lifting matrix M coming from the first equation and the variable z vanish identically. The second equation gives a z-block which is an identity matrix of size $b_2 - e_3 - 1 = e_3 - 1$, so all ζ variables have to vanish. What remains is the matrix M_{xy} which has $e_1 + e_2 - 2$ columns. The number of rows is $\max(0, e_2 - e_3 - 3) + \max(0, e_1 - e_3 - 3) + \max(0, 2e_3 - e_1 - 1) + \max(0, 2e_3 - e_2 - 1)$. We estimate the last two terms with $e_3 - 1$ and the first two by $e_2 - e_3$, resp. $e_1 - e_3$. Therefore the number of rows is at most $e_1 + e_2 - 2$. For each term which contributes 0 to the sum we have pure rolling factors deformations, so if the matrix has maximal rank the dimension of $T^1_X(-1)$ is $e_1 + e_2 - 2 - (2e_3 - 8)$. □

Example 2.18. It is possible that the lifting matrix M does not have full rank even if the g^1_4 is not composed. An example with invariants $(6, 5, 5; 7, 7)$ is the curve given by the equations $(s^5 + t^5)x^2 + s^3y^2 + t^3z^2$, $s^5x^2 + (s^3 - t^3)(y - z)^2 + 2t^3z^2$. The matrix is

$$\left(\begin{array}{ccc|cccc|cccc} 0 & \dots & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ \hline 0 & \dots & 0 & 2 & 0 & 0 & -2 & -2 & 0 & 0 & 2 \\ 0 & \dots & 0 & -2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 \end{array}\right).$$

Finally we mention the case $b_2 = 0$. Bielliptic curves ($e_3 = 0$) are treated in [Ciliberto–Miranda 1992], curves on a Del Pezzo in [Brawner 1996] (but he overlooks those with $e_3 = 0$). Now there is only one equation coming from Q, which can be perturbed arbitrarily. As the z variable does not enter the scroll, we have one coordinate transformation left. The lifting matrix involves only rows coming from the equation P. One checks that the matrix M resp. M_{xy} has maximal rank and the number of rows does not exceed the number of columns. Together with the number of non-scrollar deformations (Prop. 2.8) this yields the following result, where we have excluded the complete intersection case $g = 5$.

Proposition 2.19. *Let X be the cone over a tetragonal canonical curve C with $b_2 = 0$ and $g > 5$. Then* $\dim T^1_X(-2) = 1$.
1) If C lies on a Del Pezzo surface then $\dim T^1_X(-1) = 10$.
2) If C is bielliptic ($e_3 = 0$), then $\dim T^1_X(-1) = 2g - 2$.

Remark 2.20. For all non-hyperelliptic canonical cones the dimension of $T^1_X(\nu)$ with $\nu \geq 0$ is the same. The Wahl map easily gives $\dim T^1_X(0) = 3g - 3$, $\dim T^1_X(1) = g$, $\dim T^1_X(2) = 1$ and $\dim T^1_X(\nu) = 0$ for $\nu \geq 3$ (see e.g. [Drewes–Stevens 1996], 3.3).

3. Rolling factors obstructions.

Rolling factors deformations can be obstructed. We first give a general result on the dimension of T^2. For the case of quadratic equations on the scroll one can actually write down the base equations.

Proposition 3.1. *Let X be the cone over a complete intersection of divisors of type $aH - b_iR$ with $b_i > 0$ (and the same $a \geq 2$) on a scroll. If $a > 2$, then $\dim T^2_X(-a) = \sum(b_i - 1)$, and $\dim T^2_X(-a) \geq \sum(b_i - 1)$ in case $a = 2$.*

Proof. Let $\psi \in \operatorname{Hom}(R/R_0, \mathcal{O}_X)$ be an homogeneous element of degree $-a$. The degree of $\psi(R_{\alpha,\beta,\gamma})$ is $3 - a$, so ψ vanishes on the scrollar relations, if $a > 2$. If $a = 2$ we can assert that the functions vanishing on the scrollar relations span a subspace of $T^2_X(-2)$.
As the degree of the relation $R^n_{\alpha,m}$ is $a + 1$, the image $\psi(R^n_{\alpha,m})$ is a linear function of the coordinates. The relations

$$R^n_{\alpha,m} z_\beta - R^n_{\beta,m} z_\alpha - \sum R^n_{j,k,\gamma} p^n_{\gamma,m} = P^{(n)}_m f_{\alpha,\beta} - f_{\alpha,\beta} P^{(n)}_m .$$

imply that the $\psi(R^n_{\alpha,m})$ are also in rolling factors form. A basis (of the relevant subspace) of $\operatorname{Hom}(R/R_0, \mathcal{O}_X)(-a)$ consists of the $2\sum b_i$ elements $\psi^i_{l,s}(R^j_{\alpha,m}) = \delta_{ij}\delta_{lm} z_\alpha$, $\psi^i_{l,t}(R^j_{\alpha,m}) = \delta_{ij}\delta_{lm} z_{\alpha+1}$, where $0 \leq m < b_j$. The image of $P^{(i)}_m$ in $\operatorname{Hom}(R/R_0, \mathcal{O}_X)(-a)$ is $\psi^i_{m-1,s} - \psi^i_{m,t}$, if $0 < m < b_i$, $-\psi^i_{0,t}$ for $m = 0$, and $\psi^i_{b_i-1,s}$ for $m = b_i$. The quotient has dimension $\sum(b_i - 1)$. □

For $a = 2$ only the rolling factors obstructions will contribute to the base equations. A more detailed study could reveal if there are other obstructions. Typically this can happen, if there exist non-scrollar deformations. As example we mention Wahl's result for tetragonal cones that $\dim T^2_X(-2) = g - 7 = b_1 + b_2 - 2$, if $b_2 > 0$, whereas for a curve on a Del Pezzo the dimension is $2(g - 6)$ [Wahl 1997, Thm. 5.6].
In the quadratic case we can easily write the base equations, given a first order lift of the scrollar deformations. We can consider each equation on the scroll separately, so we will suppress the upper index of the additional equations in our notation. We may assume that we have pure rolling factors deformations ρ_α and that the lifting conditions are satisfied. We can write the perturbation of the equation P_m as

$$P_m(z) + P'_m(z, \zeta, \rho) .$$

Note that P'_m is linear in z. Now we have the following result [Stevens 1996].

Proposition 3.2. *The maximal extension of the infinitesimal deformation defined by the P'_m is given by the $b - 1$ base equations*

$$P'_m(\zeta, \zeta, \rho) - P_m(\zeta) = 0 ,$$

with $1 \leq m \leq b - 1$.

Proof. We also suppress ρ from the notation. We have to lift the relations $R_{\beta,m}$. As the lifting equations are satisfied we can write

$$P'_{m+1}(z, \zeta) z_\beta - P'_m(z, \zeta) z_{\beta+1} - \sum p_{\alpha,m}(z) \zeta_{\alpha+1} z_\beta = \sum f_{\beta\gamma} d_\gamma(\zeta) ,$$

because the left hand side lies in the ideal of the scroll. This identity involving quadratic monomials in the z-variables can be lifted to the deformation of the scroll. We write $\tilde{f}_{\beta\alpha}$ for the deformed equation $z_\beta(z_{\alpha+1}+\zeta_{\alpha+1})-(z_{\beta+1}+\zeta_{\beta+1})z_\alpha$. We get

$$P'_{m+1}(z+\zeta,\zeta)z_\beta - P'_m(z,\zeta)(z_{\beta+1}+\zeta_{\beta+1}) - \sum p_{\alpha,m}(z+\zeta)z_{\alpha+1}z_\beta = \sum \tilde{f}_{\beta\gamma}d_\gamma(\zeta).$$

We now lift the relation $R_{\beta,m}$:

$$\begin{aligned}\big(P_{m+1}(z)+P'_{m+1}(z+\zeta,\zeta)-P_{m+1}(\zeta)\big)\,z_\beta-\left(P_m(z)+P'_m(z,\zeta)\right)(z_{\beta+1}+\zeta_{\beta+1})\\ -\sum \tilde{f}_{\beta\alpha}p_{\alpha,m}(z)-\sum \tilde{f}_{\beta\gamma}d_\gamma(\zeta)=0.\end{aligned}$$

If $1\le m\le b-1$, then P_m occurs in a relation as first and as second term. Therefore $P'_m(z,\zeta)$ and $P'_m(z+\zeta,\zeta)-P_m(\zeta)$ have to be equal. These equations correspond to the $b-1$ elements of $T^2_X(-2)$, constructed above. □

Example 3.3. We continue with our rolling factors example 2.10. We look at two ways of rolling:

$$\begin{gathered} y_0z_0\to y_1z_0\to y_1z_1\to y_2z_1\to y_2z_2\\ y_0z_0\to y_0z_1\to y_0z_2\to y_1z_2\to y_1z_3\end{gathered}$$

The equation for P'_0 and P'_4 has a unique solution with $P'_0=0$. We get

$$\begin{array}{rcrr} P'_0 &=& 0, & 0\\ P'_1 &=& \eta_1z_0, & y_0\zeta_1\\ P'_2 &=& \eta_1z_1+y_1\zeta_1, & y_1\zeta_1+y_0\zeta_2\\ P'_3 &=& \eta_1z_2+y_2\zeta_1+\eta_2z_1, & y_2\zeta_1+y_1\zeta_2+\eta_1z_2\\ P'_4 &=& \eta_1z_3+y_3\zeta_1+\eta_2z_2+y_2\zeta_2, & y_3\zeta_1+y_2\zeta_2+\eta_1z_3 \end{array}$$

The resulting base equations are in both cases

$$0,\eta_1\zeta_1,\eta_1\zeta_2+\eta_2\zeta_1$$

In general the quadratic base equations are not uniquely determined. They can be modified by multiples of the linear lifting equations, if such are present. The other source of non-uniqueness is the possibility of coordinate transformations using the pure rolling factors variables.

THEOREM 3.4. *Let $P=\sum p_{I,k}s^{\langle e,I\rangle-b-k}t^kz^I$ define a divisor of type $2H-bR$. It leads to quadratic base equations $\pi_1,\ \dots,\ \pi_{b-1}$. The coefficient $p_{I,k}$ gives the following term in π_m. We write $z^I=xy$ and assume that $e_x\ge e_y$.*

I. If $e_x<b$ then for $m\le k$ the term is $-\sum_{l=m}^{k}\eta_{k-l+m}\xi_l$, while for $m>k$ it is

$$\sum_{l=\max(k+m-e_y+1,k+1)}^{\min(e_x-1,m-1)}\eta_{k-l+m}\xi_l\ .$$

II. *If $e_x \geq b$ then for $m \leq k+b-e_x$ the term is $-\sum_{l=m+e_x-b}^{k} \eta_{k-l+m}\xi_l$, while for $m > k+b-e_x$ it is*

$$\sum_{l=\max(k+m-e_y+1,k+1)}^{\min(e_x-b+m-1,k+m-1)} \eta_{k-l+m}\xi_l \ .$$

Furthermore, if $e_x \geq b$ the $e_x - b + 1$ pure rolling factors deformations involving x contribute $\rho_0\xi_m + \cdots + \rho_{e_x-b}\xi_{m+e_x-b}$ to π_m.

Proof. We need explicit equations P_m. The monomial $s^{e-x+e_y-b-k}t^k xy$ gives a rolling monomial $x_{i(m)}y_{j(m)}$, where $i(m)+j(m) = k+m$. Let $i(0) = i$, $i(b) = i'$, $j(0) = j$ and $j(b) = j'$. We have to compute P'_m. Equation (S) gives

$$s^b P'_b - t^b P'_0 = \sum_{l=1}^{j'-j} s^{e_x-k+j+l}t^{k+b-j-l}x\eta_{j+l} + \sum_{n=1}^{i'-i} s^{e_y-k+i+n}t^{k+b-i-n}y\xi_{i+n} \ ,$$

which we rewrite as

$$s^b P'_b - t^b P'_0 = \sum_{l=j+1}^{j'} s^{e_x-k+l}t^{k+b-l}x\eta_l + \sum_{l=i+1}^{i'} s^{e_y-k+l}t^{k+b-l}y\xi_l \ .$$

Case I: $e_x < b$. The condition $k+b \leq e_x+e_y$ implies $k < e_y$. We solve for P'_0:

$$P'_0 = -\sum_{l=j+1}^{k} x_{k-l}\eta_l - \sum_{l=i+1}^{k} y_{k-l}\xi_l \ .$$

For the P'_m we formally write the formula

$$P'_m = -\sum_{l=j+1}^{k} x_{k-l+m}\eta_l - \sum_{l=i+1}^{k} y_{k-l+m}\xi_l + \sum_{l=j+1}^{j(m)} x_{k-l+m}\eta_l + \sum_{l=i+1}^{i(m)} y_{k-l+m}\xi_l \ .$$

This expression can involve non-existing x or y variables: for y this happens if $k-l+m > e_y$, or $l < m+k-e_y$. The terms in the two sums involving y cancel. If $i(m) < k$, then the smallest non-canceling term has $l = i(m)+1$ and $i(m)+1 \geq i(m)+j(m)-e_y = k+m-e_y$. If $i(m) > k$ we a sum of positive terms starting with $k+1$. If $k < l < m+k-e_y$ then our monomial contributes to the lifting conditions, and we can leave out this term. The sum therefore now starts at $\max(k+1, m+k-e_y)$. Keeping this in mind we determine the term in the base equation π_m from the formal formula. To this end we change the summation variable in the sums containing x-variables and arrive, using $i(m)+j(m) = k+m$, at

$$\begin{aligned} -\sum_{l=m}^{m+i-1} \xi_l\eta_{k-l+m} - \sum_{l=i+1}^{k} \eta_{k-l+m}\xi_l + \sum_{l=i(m)}^{m+i-1} \xi_l\eta_{k-l+m} + \sum_{l=i+1}^{i(m)} \eta_{k-l+m}\xi_l - \xi_{i(m)}\eta_{j(m)} \\ = -\sum_{l=m}^{m+i-1} \xi_l\eta_{k-l+m} - \sum_{l=i+1}^{k} \eta_{k-l+m}\xi_l + \sum_{l=i+1}^{m+i-1} \eta_{k-l+m}\xi_l \ . \end{aligned}$$

If $k \geq m$ the terms from $l = m$ to $l = k$ occur twice with a minus sign and once with a plus. Otherwise all negative terms cancel, but we have to take the lifting conditions into account.

Case II: $e_x \geq b$. Now there are $e_x - b + 1$ pure rolling factors deformations present: we can perturb P_m with $\rho_0 x_m + \cdots + \rho_{e_x-b} x_{m+e_x-b}$. These contribute $\rho_0\xi_m + \cdots + \rho_{e_x-b}\xi_{m+e_x-b}$ to the equation π_m.
We can roll using only the x variable: $x_{i+m}y_j$, with $i+j = k$ and $i+b \leq e_x$. We take $i = k$ if $k+b \leq e_x$ and $i = e_x - b$ otherwise. We get

$$s^b P'_b - t^b P'_0 = \sum_{l=i+1}^{i+b} s^{e_y-k+l} t^{k+b-l} y\xi_l \;.$$

We solve:

$$P'_0 = -\sum_{l=i+1}^{k} p y_{k-l}\xi_l$$

and

$$P'_m = -\sum_{l=i+1}^{k} y_{k-l+m}\xi_l + \sum_{l=i+1}^{i+m} y_{k-l+m}\xi_l \;.$$

Again if $k < l < m + k - e_y$ our monomial contributes to the lifting conditions, and the sum starts at $\max(k+1, m+k-e_y)$. We get as contribution to π_m

$$-\sum_{l=i+1}^{k} \eta_{k-l+m}\xi_l + \sum_{l=i+1}^{i+m-1} \eta_{k-l+m}\xi_l \;.$$

Taking the lifting conditions and our choice of i into account we get the statement of the theorem. $\square$

Example 3.5: Case I. Let $b = 7$, $e_x = 5$ and $e_y = 4$. Consider the equation $P = (p_0s^2 + p_1st + p_2t^2)xy$. This leads to the following six equations:

$$\begin{array}{llll}
\pi_1 = & & & -p_2(\xi_1\eta_2 + \xi_2\eta_1) \\
\pi_2 = & p_0\xi_1\eta_1 & & \qquad -p_2\xi_2\eta_2 \\
\pi_3 = & p_0(\xi_1\eta_2 + \xi_2\eta_1) & +p_1\xi_2\eta_2 & \\
\pi_4 = & p_0(\xi_1\eta_3 + \xi_2\eta_2) + \xi_3\eta_1 & +p_1(\xi_2\eta_3 + \xi_3\eta_2) & +p_2\xi_3\eta_3 \\
\pi_5 = & p_0(\xi_2\eta_3 + \xi_3\eta_2) + \xi_4\eta_1 & +p_1(\xi_3\eta_3 + \xi_4\eta_2) & +p_2\xi_4\eta_3 \\
\pi_6 = & p_0(\xi_3\eta_3 + \xi_4\eta_2) & +p_1\xi_4\eta_3 &
\end{array}$$

If we write a matrix with the coefficients of the p_i in the columns with rows coming from the equations π_m we find that the first $k+1$ rows form a skew symmetric matrix. This is due to the specific choices made in the above proof. One can also get any other block to be skew symmetric by using the lifting conditions. In this example they are $p_0\eta_1 + p_1\eta_2 + p_2\eta_3 = 0$, $p_0\xi_1 + p_1\xi_2 + p_2\xi_3 = 0$ and $p_0\xi_2 + p_1\xi_3 + p_2\xi_4 = 0$. From the skew symmetry we can conclude:

Proposition 3.6. *If $e_y \le e_x < b$ then the $b-1$ equations π_m coming from the equation $P = (\sum_{j=0}^{k} p_j s^{k-j} t^k)xy$, where $b+k = e_x + e_y$, satisfy $b-k-1$ linear relations $\sum_{j=0}^{k} p_j \pi_{i+j} = 0$, for $0 < i < b-k$.*

Example 3.7: case II. Let $b=4$, $e_x = 5$ and $e_y = 3$. Consider the equation $P = (p_0 s^4 + p_1 s^3 t + p_2 s^2 t^2 + p_3 s t^3 + p_4 t^4)xy$. This leads to the following three equations:

$$
\begin{aligned}
\pi_1 &= \rho_0\xi_1 + \rho_1\xi_2 \quad - p_2\xi_2\eta_1 - p_3(\xi_2\eta_2 + \xi_3\eta_1)p_4(\xi_3\eta_2 + \xi_4\eta_1)\\
\pi_2 &= \rho_0\xi_2 + \rho_1\xi_3 \quad + p_0\xi_1\eta_1 + p_1\xi_2\eta_1 - p_3\xi_3\eta_2 - p_4\xi_4\eta_2\\
\pi_3 &= \rho_0\xi_3 + \rho_1\xi_4 \quad + p_0(\xi_1\eta_2 + \xi_2\eta_1) + p_1(\xi_2\eta_2 + \xi_3\eta_1) + p_2\xi_3\eta_2
\end{aligned}
$$

(3.8) *Hyperelliptic cones* (cf. [Stevens 1996]). Let X be the cone over a hyperelliptic curve C embedded with a line bundle L of degree $d \ge 2g+3$. Then $\dim T^1_X(-1) = 2g+2$. The curve lies on a scroll of degree $d-g-1$ as curve of type $2H - (d-2g-2)R$. The number of rolling factors equations is $d-2g-3$, so we have at least as many equations as variables if $d > 4g+4$. In that case only conical deformations exist, so all deformations in negative degree are obstructed.
The easiest case to describe is $L = ng^1_2$. The curve C has an affine equation $y^2 = \sum_{k=0}^{2g+2} p_k t^k$, which gives the bihomogeneous equation $(\sum_{k=0}^{2g+2} p_k s^{2g+2-k} t^k)x^2 - y^2 = 0$. The line bundle L embeds C in a scroll $S(n, n-g-1)$, and there are $2n-2g-1$ rolling factors equations P_m, coming from $p(s,t)x^2 - y^2$. The lifting matrix is a block diagonal matrix with the y-block equal to $-2I_{n-g-2}$, and the x-block a $(n-2g-3)\times(n-1)$ matrix, so the dimension of the space of lifting deformations of the scroll is $2g+2$ if $n \ge 2g+3$. If $n \le 2g+3$, the x-block is not present, and all $n-1$ ξ-deformations lift. Furthermore there are $2g+3-n$ pure rolling factors deformations. This shows again that $\dim T^1_X(-1) = 2g+2$.

Proposition 3.9. *If $n \ge 2g+3$ the base space in negative degrees is a zero-dimensional complete intersection of $2g+2$ quadratic equations.*

Proof. We may assume that the highest coefficient p_{2g+2} in $p(s,t)$ equals 1. The lifting equations allow now to eliminate the variables $\xi_{2g+3}, \dots, \xi_{n-1}$. The base equations π_m involve only the ξ_i and are therefore not linearly independent. Because $p_{2g+2} = 1$ we can discard all π_m with $m > 2g+2$. The first $2g+2$ equations involve only the first $2g+2$ variables. This shows that we have the same system of equations for all $n \ge 2g+3$. As we know that there are no deformations over a positive dimensional base, we conclude that the base space is a complete intersection of $2g+2$ equations. □

Remark 3.10. The fact that the system of equations above defines a complete intersection can also be seen directly. In fact we have the following result:

Lemma 3.11. *The system of $e = b-1$ equations π_m in $e-1$ variables ξ_i coming from one polynomial $P_{b-2}(s,t)x^2$ is a zero-dimensional complete intersection if and only if $P_{b-2}(s,t)$ has no multiple roots.*

Proof. First we note that there are only $b-2$ linearly independent equations. We put $\xi_i = s^{b-i-1}t^i$. Then

$$\begin{aligned}\pi_m &= \sum_{k=0}^{b-2}(m-k-1)p_k s^{2b-k-m-2}t^{k+m-2}\\ &= s^{b-m}t^{m-2}\big(\sum(b-2-k)p_k s^{b-2-k}t^k + (m+1-b)\sum p_k s^{b-2-k}t^k\big) \qquad .\end{aligned}$$

The form $P(s,t)$ has multiple roots if and only if $P(s,t)$ and $s\frac{\partial}{\partial s}P(s,t)$ have a common zero $(s_0 : t_0)$. Then $\xi_i = s_0^{b-i-1}t_0^i$ is a nontrivial solution to the system of equations.
We show the converse by induction. One first checks that a linear transformation in $(s:t)$ does not change the isomorphism type of the ideal. We apply a transformation such that $s=0$ is a single root of P, so $p_0 = 0$ but $p_1 \neq 0$. The equations $\pi_2, \dots, \pi_{b-1}$ now do not involve the variable ξ_1 and are by the induction hypotheses a complete intersection in $e-2$ variables, so their zero set is the ξ_1-axis with multiple structure. The equation π_1 has the form $-p_1\xi_1^2 + \dots$, so the whole system has a zero-dimensional solution set. □

Remark 3.12. For $\deg L = 4g+4$ the base space is a cone over 2^{2g+1} points in a very special position: there exist $2g+2$ hyperplanes $\{l_i = 0\}$ such that the base is given by $l_i^2 = l_j^2$ [Stevens 1996]. We can make this more explicit in the case $L = (2g+2)g_2^1$. Again the y-block of the lifting matrix is a multiple of the identity, but now there is also one rolling factors deformation parameter ρ. More generally, we look the equations coming from $p(s,t)x^2$ with $\deg p = b = e$. We get base equations $\Pi_m = \rho\xi_m + \pi_m$, where π_m is a quadratic equation in the ξ-variables only. One solution is clearly $\xi_i = 0$ for all i. To find the others we eliminate ρ:

$$\operatorname{Rank}\begin{pmatrix}\pi_1 & \pi_2 & \dots & \pi_{e-1}\\ \xi_1 & \xi_2 & \dots & \xi_{e-1}\end{pmatrix} \le 1\ . \qquad (**)$$

The equations Π_m can be changed by changing ρ, but this system is independent of such changes. Write inhomogeneously $p(t) = p_0 + p_1 t + \dots + p_{e-1}t^{e-1} + t^e = \prod(t-\alpha_i)$, where the α_i are the roots of $p(t)$.

Lemma 3.13. *The e points $P_i = (1 : \alpha_i : \alpha_i^2 : \dots : \alpha_i^{e-2})$ are solutions to the system $(**)$.*

Proof. Let α be a root of p and insert $\xi_i = \alpha^{i-1}$ in the system $(**)$. We simplify the matrix by column operations: subtract α times the jth column from the $(j+1)$st column, starting at the end. The matrix has clearly rank 1, if $\pi_{j+1}(\alpha) - \alpha\pi_j(\alpha) = 0$, where $\pi_j(\alpha)$ is the result of substituting $\xi_i = \alpha^{i-1}$ in the equation π_j. The coefficient p_k occurs in $\pi_j(\alpha)$ in the term $lp_k\alpha^{j+k-2}$ for some integer l, and in the term $(l+1)p_k\alpha^{j+k-1}$ in $\pi_{j+1}(\alpha)$. Therefore $\pi_{j+1}(\alpha) - \alpha\pi_j(\alpha) = -\sum p_k\alpha^{j+k-1} = -\alpha^{j-1}p(\alpha) = 0$. □

The remaining solutions are found in the following way. Divide the set of roots into two subsets I and J. The points P_i lie on a rational normal curve. Therefore the points P_i with $i \in I$ span a linear subspace L_I of dimension $|I|-1$.

Claim. *The intersection point $P_I := L_I \cap L_J$ is a solution to* $(**)$.

The proof is a similar but more complicated computation. We determine here only the point P_I. The condition that the point $\sum_{i\in I}\lambda_i P_i$ lies in L_J is that

$$\operatorname{Rank}\begin{pmatrix} \sum\lambda_i & \dots & \sum\lambda_i\alpha_i^{e-2} \\ 1 & \dots & \alpha_{j_1}^{e-2} \\ \vdots & & \vdots \\ 1 & \dots & \alpha_{j_{|J|}}^{e-2} \end{pmatrix} = |J|\,.$$

We find the resulting linear equations on the λ_i by extending the matrix to a square matrix by adding $|I|-2$ rows of points on the rational normal curve, for which we take roots. Then only two λ_i survive, and they come with a Vandermonde determinant as coefficient. Upon dividing by common factors we get $(\prod_{i\neq i_1,i_2}(\alpha_{i_1}-\alpha_i))\lambda_{i_1} + (\prod_{i\neq i_1,i_2}(\alpha_{i_2}-\alpha_i))\lambda_{i_2} = 0$. We multiply with $\alpha_{i_1}-\alpha_{i_2}$. Noting that $\prod_{i\neq i_1}(\alpha_{i_1}-\alpha_i) = p'(\alpha_{i_1})$ (with $p'(t)$ the derivative of $p(t)$) we get $p'(\alpha_{i_1})\lambda_{i_1} = p'(\alpha_{i_2})\lambda_{i_2}$.
We write out the equations for $e=5$:

$$\begin{gathered} \rho\xi_1 - p_1\xi_1^2 - 2p_2\xi_1\xi_2 - p_3\xi_2^2 - 2p_4\xi_2\xi_3 - p_5(2\xi_2\xi_4+\xi_3^2) \\ \rho\xi_2 + p_0\xi_1^2 - p_2\xi_2^2 - p_4\xi_3^2 - 2p_5\xi_3\xi_4 \\ \rho\xi_3 + 2p_0\xi_1\xi_2 + p_1\xi_2^2 - p_3\xi_3^2 - p_5\xi_4^2 \\ \rho\xi_4 + p_0(2\xi_1\xi_3+\xi_2^2) + 2p_1\xi_2\xi_3 + p_2\xi_3^2 + 2p_3\xi_3\xi_4 + p_4\xi_4^2 \end{gathered}$$

Let α be a root of $p_0+p_1t+p_2t^2+p_3t^3+p_4t^4+t^5$, and $\beta,\dots,\varepsilon$ the remaining roots. Write σ_i' for the ith symmetric function of these four roots. Then a solution is $\xi_i=\alpha^{i-1}$, $\rho=\alpha^4-\alpha^3\sigma_1'-\alpha^2\sigma_2'-\alpha\sigma_3'+\sigma_4'$. Given two roots α and β we get a solution $\xi_i=(\gamma-\beta)(\delta-\beta)(\varepsilon-\beta)\alpha^i+(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon)\beta^i$. To write ρ we set $\mu=(\gamma-\beta)(\delta-\beta)(\varepsilon-\beta)$, $\lambda=(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon)$ and σ_i'' the ith symmetric function in γ, δ and ε. Then and $\rho=\mu(\alpha^4-\alpha^2(\alpha+2\beta)\sigma_1''-\alpha^2\sigma_2''-\alpha\sigma_3'')+\lambda(\beta^4-\beta^2(\alpha+2\alpha)\sigma_1''-\beta^2\sigma_2''-\beta\sigma_3'')$. The hyperplane through $(1:0:0:0:0)$, P_γ, P_δ and P_ε is $l^-_{\alpha\beta}=\sigma_3''\xi_1-\sigma_2''\xi_2+\sigma_1''\xi_3-\xi_4$. In it lie also $P_{\gamma\delta}$, $P_{\gamma\varepsilon}$, $P_{\delta\varepsilon}$ and $P_{\alpha\beta}$. The hyperplane containing the remaining points is $l^+_{\alpha\beta}=\rho-(\alpha+\beta)l^-_{\alpha\beta}+2\sigma_3''\xi_2+2\alpha\beta\xi_3$. We put $l_a=\rho-2\sigma_4'\xi_1+2\sigma_3'\xi_2+2\alpha\sigma_1'\xi_3-2\alpha\xi_4$. Then $l_\alpha^2-l_\beta^2=4(\alpha-\beta)l^-_{\alpha\beta}l^+_{\alpha\beta}$.

4. Tetragonal curves.

An extension of a canonical curve yields a surface with the given canonical curve as hyperplane section. Surfaces with canonical hyperplane sections were studied in Dick Epema's thesis [Epema 1983]. Only a limited list of surfaces can occur.

THEOREM 4.1([Epema 1983], Cor. I.5.5 and Cor. II.3.3). *Let W be a surface with canonical hyperplane sections. Then one of the following holds:*
(a) *W is a $K3$ surface with at most rational double points as singularities,*
(b) *W is a rational surface with one minimally elliptic singularity and possibly rational double points,*
(c) *W is a birationally ruled surface over an elliptic curve Γ with as non-rational singularities either*
  i) *two simple elliptic singularities with exceptional divisor isomorphic to Γ, or*
  ii) *one Gorenstein singularity with $p_g = 2$,*
(d) *W is a birationally ruled surface over a curve Γ of genus $q \geq 2$ with one non-rational singularity with $p_g = q+1$, whose exceptional divisor contains exactly one non-rational curve isomorphic to Γ.*

Case (c) occurs for bi-elliptic curves (see below). If we exclude them and curves of low genus on Del Pezzo surfaces, then all extensions of tetragonal curves are of rolling factors type. The surface W has therefore to occur in our classification of complete intersection surfaces on scrolls. In particular, $K3$ surfaces can only occur if $b_1 \leq b_2 + 4$. This has consequences for deformations of tetragonal cones.

PROPOSITION 4.2. *Pure rolling factors deformations are always unobstructed. If $e_3 > 0$ and $b_1 > b_2 + 4$ the remaining deformations are obstructed.*

Proof. The first statement follows directly from the form of the equations. For the second we note that the total space of a nontrivial one-parameter deformation of a scroll with $e_3 > 0$ is a scroll with $e_4 > 0$. □

By taking hyperplane sections of a general element in each of the families of the classification we obtain for all g tetragonal curves with $b_1 \leq b_2 + 4$ lying on $K3$ surfaces (with at most rational double points). To realize the other types of surfaces we give a construction, which goes back to [Du Val 1933]. His construction was generalized to the non-rational case in [Epema 1983]. In our situation we want a given curve to be a hyperplane section. A general construction for given hyperplane sections of regular surfaces is given in [Wahl 1998].

CONSTRUCTION 4.3. *Let Y be a surface containing the curve C and let $D \in |-K_Y|$ be an anticanonical divisor. Let $\widetilde{Y}$ be the blow up of Y in the scheme $Z = C \cap D$. If the linear subsystem $\mathcal{C}'$ of $|C|$ with base scheme Z has dimension g, it associated map contracts D and blows down $\widetilde{Y}$ to a surface $\overline{Y}$ with C as canonical hyperplane section.*

Let $\mathcal{I}_Z$ be the ideal sheaf of Z. Then we have the exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{I}_Z\mathcal{O}_Y(C) \longrightarrow \mathcal{O}_C(C-Z) \longrightarrow 0$$

and by the adjunction formula $\mathcal{O}_C(C-Z) = K_C$. If $h^0(\mathcal{I}_Z\mathcal{O}_Y(C)) = g+1$ then the map $H^0(\mathcal{I}_Z\mathcal{O}_Y(C)) \longrightarrow H^0(K_C)$ is surjective, a condition which is

automatically satisfied if Y is a regular surface. This yields that the special hyperplane section is the curve C in its canonical embedding.
Suppose that Y is not regular. By Epema's classification Y is then a birationally ruled surface, over a curve Γ of genus q. Let $\widetilde{C}$ be the strict transform of C on $\widetilde{Y}$ and $\overline{C}$ its image on $\overline{Y}$. Then $H^0(\mathcal{I}_Z\mathcal{O}_Y(C)) = H^0(\mathcal{O}_{\widetilde{Y}}(\widetilde{C})) = H^0(\mathcal{O}_{\overline{Y}}(\overline{C}))$. We look at the exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{Y}} \longrightarrow \mathcal{O}_{\overline{Y}}(\overline{C}) \longrightarrow \mathcal{O}_{\overline{C}}(\overline{C}) = K_C \longrightarrow 0\ .$$

We compute $H^1(\mathcal{O}_{\overline{Y}})$ with the spectral sequence for the map $\pi\colon \widetilde{Y} \longrightarrow \overline{Y}$. This gives us the long exact sequence

$$0 \longrightarrow H^1(\mathcal{O}_{\overline{Y}}) \longrightarrow H^1(\mathcal{O}_{\widetilde{Y}}) \longrightarrow H^0(R^1\pi_*\mathcal{O}_{\widetilde{Y}}) \longrightarrow H^2(\mathcal{O}_{\overline{Y}}) \longrightarrow 0$$

in which $\dim H^1(\mathcal{O}_{\widetilde{Y}}) = q$. We choose D in such a way that the composed map $H^1(\mathcal{O}_\Gamma) \longrightarrow R^1\pi_*\mathcal{O}_{\widetilde{Y}} \longrightarrow H^1(\mathcal{O}_{\widetilde{D}})$, where $\widetilde{D}$ is the exceptional divisor of the map π, is injective. Then the map $H^0(\mathcal{I}_Z\mathcal{O}_Y(C)) \longrightarrow H^0(K_C)$ is surjective.
To apply the construction we need a surface on which the curve C lies. In the tetragonal case a natural candidate is the surface Y of type $2H - b_1R$ on the scroll.
We first assume that $e_1 < b_1$, so there are no pure rolling factors deformations coming from the first equation on the scroll. The canonical divisor of the scroll S is $-3H + (b_1 + b_2)R$ [Schreyer 1986, 1.7]. So an anticanonical divisor on Y is of type $H - b_2R$. Let $T = \tau_{e_1-b_2}(s,t)x + \tau_{e_2-b_2}(s,t)y + \tau_{e_3-b_2}(s,t)z$ be the equation of such a divisor. Sections of $\mathcal{I}_Z\mathcal{O}_Y(C)$ are Q (which defines C), and $x_iT = s^{e_1-i}t^ixT$, y_iT and z_iT. With coordinates $(t : x_i : y_i : z_i)$ on $\mathbb{P}^g$ we get by rolling factors $b_2 + 1$ equations $\widetilde{Q}_m$ from the relation $Q(\tau_{e_1-b_2}(s,t)x + \tau_{e_2-b_2}(s,t)y + \tau_{e_3-b_2}(s,t)z) = (Q_{1,1}x^2 + \cdots + Q_{3,3}z^2)T$. As t is also a coordinate on the four-dimensional scroll, which is the cone over S, we can write the equation on the scroll as

$$Q_{1,1}x^2 + \cdots + Q_{3,3}z^2 - (\tau_{e_1-b_2}x + \cdots + \tau_{e_3-b_2}z)t\ .$$

We analyze the resulting singularities. If Y is a rational surface, we have a anticanonical divisor D which has arithmetic genus 1, giving a minimally elliptic singularity on the total space of the deformation.
If Y is a ruled surface over a hyperelliptic curve Γ, then D passes through the double locus. This gives an exceptional divisor with Γ as only non-rational curve.

Example 4.4. Let $(e_1, e_2, e_3; b_1, b_2) = (3n-2, 2n-2, n-2; 4n-4, 2n-4)$. If the coefficient of xz does not vanish, we may bring the equation P onto the form $xz - y^2$. The second equation has the form $z^2 + q_nzy + q_{2n}zx + q_{3n}xy + q_{4n}x^2$ from which z may be eliminated to obtain a quartic equation for y. The case of a cyclic curve $y^4 + q_{4n}x^4$ is a special instance. The equation P gives a

square lifting matrix in which the antidiagonal blocks are square unit matrices. Therefore the only deformations are pure rolling factors deformations, coming from the second equation, in number $(n+3)+3 = \frac{g+3}{6}+6$. We have $T = \tau_{n+2}x + \tau_2 y$. The section $(0:0:1)$ is always a component of D. If $t_2 \not\equiv 0$ we have a cusp singularity, but if $\tau_2 \equiv 0$ the section occurs with multiplicity 2 in D.
If however the coefficient of xz vanishes, the surface Y is singular. After a coordinate transformation its equation is $y^2 + p_{2n}x^2$, the other equation being $z^2 + q_{3n}xy + q_{4n}x^2$. In this case the lifting matrix has (up to a factor $\frac{1}{2}$) the following block structure

$$\begin{pmatrix} \Pi & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}$$

so there are $2n$ ξ-deformations, on which we have $4n-5$ base equations coming from the equation P. Of these are only $2n$ linearly independent, defining a zero-dimensional complete intersection (see Lemma 3.11). These deformations are therefore obstructed, leaving us again with only the pure rolling factors deformations. The curve D consists of the double locus and in general $2n+4$ lines.

The same computation as above works for bielliptic cones. In that case one has a deformation of weight -2. The total space is a surface in weighted projective space $\mathbb{P}(1,\dots,1,2)$. Replacing the deformation parameter t by t^2 we get a surface in ordinary $\mathbb{P}^g$. This is a surface with two simple elliptic singularities. The most general surface of this type is the intersection of our elliptic cone with one dimensional vertex with the hypersurface given by

$$\widetilde{Q} = z^2 + Q(x_i, y_i) + tl(x_i, y_i) + at^2 \ ,$$

where $l(x_i, y_i)$ is a linear form in the coordinates x_i, y_i. If the coefficient a vanishes, we get a surface with one singularity with $p_g = 2$. The construction above gives an equation of the form $\widetilde{Q} = z^2 + \cdots + azt$, which after a coordinate transformation becomes $z^2 + \cdots - \frac{1}{4}a^2t^2$.

PROPOSITION 4.5. *For bielliptic cones of genus $g > 10$ the only deformations of negative weight are pure rolling factors deformations.*

Proof. Each infinitesimal deformation of the bielliptic cone induces an infinitesimal deformation of the cone over the projective cone over the elliptic curve. The same holds therefore for complete deformations of negative weight. It is well-known that the cone over an elliptic curve of degree at least 10 has only obstructed deformations of negative weight. Therefore the deformation of the elliptic cone is trivial and the only possibility is to deform the last quadratic equation. □

On the other hand, non-scrollar extension do occur for bielliptic curves with $g \leq 10$ and for tetragonal curves on Del Pezzo surfaces.

Example 4.6. A bielliptic curve of genus 10 lies on the projective cone over an elliptic curve of degree 9. Such a cone is can be smoothed to the triple Veronese embedding of $\mathbb{P}^2$. Let W be a $K3$ surface of degree 2, a double cover of $\mathbb{P}^2$ branched along a sextic curve. We re-embed W with $|3L|$, where L is the pull-back of a line on $\mathbb{P}^2$. The image lies on the cone over the Verones embedding. A hyperplane section through the vertex of the cone is a bielliptic curve, whereas the general hyperplane section has a g^2_6. This example, due to [Donagi–Morrison 1989], is the only case where the gonality of smooth curves in a base-point-free ample linear system on a $K3$ surface is not constant [Ciliberto–Pareschi 1995].

Now we look at the case that also the first set of equations admit pure rolling factors deformations.

Lemma 4.7. *If $e_1 \geq b_1$ then $e_1 \leq b_1 + 2$ and $b_1 \leq b_2 + 4$.*

Proof. Under the assumption $e_1 \geq b_1$ we have $e_2 + e_3 - 2 \leq b_2 \leq 2e_3$ so $e_2 \leq e_3 + 2$ and $b_1 \leq 2e_2 \leq e_2 + e_3 + 2 \leq b_2 + 4$. Furthermore $e - 1 = b_1 + b_2 + 2 - e_2 - e_3 \leq b_1 + 2$. □

It is now easy to list all 18 possibilities, ranging from $(2e + 2, e, e; 2e, 2e)$ to $(2e + 4, e + 2, e; 2e + 4, 2e)$. A look at the table of tetragonal $K3$ surfaces reveals that all possibilities are realizable as special sections of $K3$-surfaces; e.g., the hyperplane section $x_{e+2} = y_0$ of a $K3$ with invariants $(e+2, e+2, e+2, e; 2e+4, 2e)$ yields the last case.
On the other hand, every family of $K3$ surfaces contains degenerate elements with singularities of higher genus. Those can be constructed with Epema's construction and in fact he gives rather complete results for quartic hypersurfaces [Epema 1983]. The classification of such surfaces is due to [Rohn 1884] and is quite involved. In those cases the rational or ruled surfaces on which the canonical curve lies are not evident. For pure rolling factors extensions the situation is better; in fact, we can make the following simple observation.

Proposition 4.8. *Let W be a pure rolling factors extension of tetragonal curve, which is not bi-elliptic. It lies on the cone over the 3-dimensional scroll S with vertex in $p = (0 : \ldots : 0 : 1)$ and the projection from the point p yields a surface $Y \subset \mathbb{P}^{g-1}$ on which C lies.*

Example 4.9. If $b_1 > e_1$ then X lies on the cone over the surface Y on the scroll and the projection is just this surface Y, so we get the construction described above.

Example 4.10. Consider the curve with invariants $(8, 4, 2; 8, 4)$. In general a pure rolling factors extension leads to a $K3$-surface (with an ordinary double point). It is the case $e = 3$ of $(e + 5, e + 1, e - 1, e - 3; 2e + 2, 2e - 1)$ from the table; the singularity appears because the section $(0 : 0 : 0 : 1)$ is contracted. To find the equation of Y on the scroll we have to eliminate the last coordinate w. The deformed equation $\widetilde{P}$ is $P + axw$, while $\widetilde{Q} = Q + byw + c_2(s,t)xw$ with a

and b nonzero constants. The equation of Y is therefore $(by+c_2(s,t)x)P-axQ$, which defines a divisor of type $3H-(b_1+b_2)R$ on the scroll.

Example 4.11: the case $(2e+2,e,e;2e,2e)$. We first derive a normal form for the equations P and Q. We start with the restriction to $x=0$. We have a pencil of quadrics so we may choose the first equation as y^2 and the second as z^2. We get:

$$\begin{aligned} &P\colon y^2+p_{e+2}xz+p_{2e+4}x^2\\ &Q\colon z^2+q_{e+2}xy+q_{2e+4}x^2\ . \end{aligned}$$

There are $3+3$ pure rolling factors deformations:

$$\begin{aligned} &\widetilde{P}\colon P+(\rho_0s^2+\rho_1st+\rho_2t^2)x\\ &\widetilde{Q}\colon Q+(\tau_0s^2+\tau_1st+\tau_2t^2)x\ . \end{aligned}$$

If the polynomials $\rho:=\rho_0s^2+\rho_1st+\rho_2t^2$ and $\tau:=\tau_0s^2+\tau_1st+\tau_2t^2$ are proportional, so $\lambda\rho+\mu\tau=0$, then the surface Y is the surface $\lambda P+\mu Q=0$ from the pencil. In general the anticanonical divisor D contains the two sections given by $x=0$, $\lambda y^2+\mu z^2=0$ and the singularity on the deformation is a cusp singularity. If ρ and τ have $0\le\gamma<2$ roots in common, the projected surface is a divisor of type $2H-(2e-2+\gamma)R$. In general we get a simple elliptic singularity.

To describe the remaining deformations we look at the lifting matrix, which is a block matrix

$$\begin{pmatrix} 0 & 2I & 0\\ \Pi_{e+2} & 0 & 0\\ \Xi_{e+2} & 0 & 0\\ 0 & 0 & 2I \end{pmatrix}$$

of size $(4e-4)\times(4e-1)$. Its rank is $2e-2$ if p_{e+2} and q_{e+2} both vanish identically, and lies between $3e-3$ and $4e-4$ otherwise. The solution space has dimension $\gamma\ge3$ with strict inequality iff the polynomials p_{e+2} and q_{e+2} have γ roots in common. The η and ζ deformations vanish. Therefore the base equations depend only on p_{2e+4} and q_{2e+4}. They are $2(2e-1)$ quadratic equations on $2e+1+6$ variables, which may or may not have solutions.

We now turn to the other deformations in general. A dimension count shows that the general tetragonal curve of genus $g>15$ cannot lie on a $K3$ surface, so the deformations are obstructed. For a general tetragonal cone we have that $\dim T^1_X=9$. There are $(b_1-1)+(b_2-1)=g-7$ quadratic base equations. Compare this with the dimension of T^2:

Theorem 4.12([Wahl, Thm. 5.9]). *Let X be a tetragonal cone with $e_3>0$. Then $\dim T^2_X(-k)=0$ for $k>2$ and $\dim T^2_X(-k)=g-7$ if $b_2>0$. If $b_2=0$, then $\dim T^2_X(-k)=2(g-6)$.*

In particular, if $g>15$ we have more equations than variables and in general there are no solutions. For special moduli solutions do exist and one expects in general exactly one solution.

(4.13) *The case* $g = 15$. Consider the most general situation, of equal invariants: $e_1 = e_2 = e_3 = 4$, $b_1 = b_2 = 5$. In this case there are no pure rolling factor deformations and no lifting conditions.

Proposition 4.14. *The general tetragonal curve with* $e_1 = e_2 = e_3 = 4$, $b_1 = b_2 = 5$ *is hyperplane section of* 256 *different* $K3$ *surfaces.*

Proof. We have 8 homogeneous quadratic equations in 9 variables, which define a complete intersection of degree 2^8. We give an explicit example. Take the curve, given by the equations

$$\begin{gathered}(s^3+t^3)x^2+(s^3+2t^3)y^2+(s^3-2t^3)z^2\\(s^2+t^2)(s-t)x^2+s^2(s+t)y^2+t^3z^2\end{gathered}$$

on the scroll. The base equations are formed according to Thm. 3.4. One computes that indeed we have a complete intersection, which is non-singular. □

It is very difficult to find solutions to such equations, and I have not succeeded to do so in the specific example. Note that the absence of mixed terms in x, y and z on the scroll means that the automorphism group of the curve has order at least eight and it operates on the base space: given one solution one finds three other ones by multiplying all ξ_i or all ζ_i by -1.

Remark 4.15. Alternatively one can start with a $K3$ surface and take a general hyperplane section. Therefore we look at complete intersections of two surfaces of type $2H-5R$ on a scroll of type $(3,3,3,3)$. Such a $K3$ surface can have infinitesimal deformations of negative weight (which are always obstructed). The lifting matrix for the $K3$ has size 8×8. The equations P and Q on the scroll are pencils of quadrics. In general such a pencil has 4 singular fibres and by taking a suitable linear combination we may suppose that P has the form

$$sX^2+tY^2+(s+t)Z^2+(s-t)W^2\,.$$

The polynomial Q is then a general pencil with 20 coefficients, of which one can be made to vanish by subtracting a multiple of P. This shows that these $K3$ surfaces depend on 18 moduli. Let $Q=(a_{11}s+b_{11}t)X^2+2(a_{12}s+b_{12}t)XY+\ldots+(a_{44}s+b_{44}t)W^2$. Then the lifting matrix is

$$2\begin{pmatrix}1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&1&0&0\\0&0&0&0&0&0&1&-1\\a_{11}&b_{11}&a_{12}&b_{12}&a_{13}&b_{13}&a_{14}&b_{14}\\a_{12}&b_{12}&a_{22}&b_{22}&a_{23}&b_{23}&a_{24}&b_{24}\\a_{13}&b_{13}&a_{23}&b_{23}&a_{33}&b_{33}&a_{34}&b_{34}\\a_{14}&b_{14}&a_{24}&b_{24}&a_{34}&b_{34}&a_{44}&b_{44}\end{pmatrix}.$$

For nonsingular $K3$ surfaces this matrix has at least rank 5, and it is possible to write down examples with exactly rank 5. Rank 4 can be realized with surfaces with isolated singularities. An explicit example (with a slightly different basis for the pencil) is

$$P = sX^2 + tY^2 + (s+t)Z^2$$
$$Q = sX^2 - tY^2 + (s-t)W^2$$

with ordinary double points at $sX = tY = (s+t)Z = (s-t)W = 0$. The hyperplane section $X_3 + Z_2 + W_1 + Y_0 = t^3X + s^2tZ + st^2W + s^3Y$ does not pass through the singular points and defines a smooth tetragonal curve with $e_i = 4$. The base space for this curve is still a complete intersection, but the line corresponding to the singular $K3$ surface is a multiple solution.

(4.16) *The case* $g = 16$. The curves lying on a $K3$ form a codimension one subspace in the moduli space of tetragonal curves of genus $g = 16$. In terms of the coefficients of the equations of the scroll one gets an equation of high degree. It makes no sense to write it. We will not study the most general case $(5,4,4;6,5)$ but $(5,5,3;6,5)$. These curves form a codimension two subspace in moduli. The computations will show that the condition of being a hyperplane section has again codimension one. The lifting matrix need not have full rank. We have $b_1 = 2e_3$, and the g^1_4 can be composed.
Suppose that the coefficient of z^2 in the first equation on the scroll does not vanish. With a coordinate transformation we may assume that the equation has the form $z^2 + P_4(s,t;x,y)$ with P_4 of degree 4 in $(s:t)$ and quadratic in $(x:y)$. Then we can take Q to be without z^2 term. Let $q_{0;1}s^3 + \ldots + q_{3;1}t^3$ be the coefficient of xz and $q_{0;2}s^3 + \ldots + q_{3;2}t^3$ that of yz. The rows of the 3×8 lifting matrix come only from the monomial z:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 1 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 1 \\ \hline q_{0;1} & q_{1;1} & q_{2;1} & q_{3;1} & | & q_{0;2} & q_{1;2} & q_{2;2} & q_{3;2} & | & 0 & 0 \end{pmatrix}$$

The matrix has rank 3 if some $q_{i,j}$ does not vanish, but rank 2 if they all vanish; then the surface $\{Q = 0\}$ has a singular line.
The deformation variables ζ_1, ζ_2 vanish. We have two pure rolling factors deformations ρ_1 and ρ_2 in the second set of additional equations, and there are scrollar deformations $\xi_1, \ldots, \eta_4$. Between those exist a linear relation given by the third line of the matrix. The equations for the base can be written down independently of this linear relation, because the ζ_i vanish.
We give a specific example: $z^2 + t^4y^2 + (s^4 + t^4)x^2$ and $(s^5 + t^5)x^2 + (s^5 - t^5)y^2 + q_1(s,t)xz + q_2(s,t)yz$. We get the following nine equations:

$$-2\xi_2\xi_3 - 2\xi_1\xi_4 - 2\eta_2\eta_3 - 2\eta_1\eta_4$$
$$\xi_1^2 - \xi_3^2 - 2\xi_2\xi_4 - \eta_3^2 - 2\eta_2\eta_4$$
$$2\xi_1\xi_2 - 2\xi_3\xi_4 - 2\eta_3\eta_4$$

$$
\begin{gathered}
2\xi_1\xi_3+\xi_2^2-\xi_4^2-\eta_4^2\\
2\xi_1\xi_4+2\xi_2\xi_3\\
\rho_1\xi_1+\rho_2\eta_1-\xi_3^2-2\xi_2\xi_4+\eta_3^2+2\eta_2\eta_4\\
\rho_1\xi_2+\rho_2\eta_2+\xi_1^2+\eta_1^2-2\xi_3\xi_4+2\eta_3\eta_4\\
\rho_1\xi_3+\rho_2\eta_3+2\xi_1\xi_2+2\eta_1\eta_2-\xi_4^2+\eta_4^2\\
\rho_1\xi_4+\rho_2\eta_4+2\xi_1\xi_3+\xi_2^2+2\eta_1\eta_3+\eta_2^2
\end{gathered}
$$

Also in general we have 5 equations π_m and 4 equations $\rho_1\xi_m+\rho_2\eta_m+\chi_m$. The pure rolling factors equations are never obstructed. We have as solution to the equations therefore the (ρ_1,ρ_2)-plane with a non reduced structure. Given a general value of (ρ_1,ρ_2) we can eliminate say the η_i variables. We are then left with 5 equations π_i depending only on the x_i. Their quadratic parts satisfy a relation with constant coefficients, but even more is true: this relation can be lifted to the equations themselves. So the component has multiplicity 16. The general fibre over the reduced component has a simple elliptic singularity of degree 10.
To find the other solutions we eliminate ρ_1 and ρ_2. This gives the condition

$$
\operatorname{Rank}\begin{pmatrix}\chi_1&\chi_2&\chi_3&\chi_4\\ \xi_1&\xi_2&\xi_3&\xi_4\\ \eta_1&\eta_2&\eta_3&\eta_4\end{pmatrix}\le 2
$$

which defines a codimension 2 variety of degree 11. In general the 5 equations π_m cut out a subset of codimension 7 and degree 352. But if

$$
\operatorname{Rank}\begin{pmatrix}\xi_1&\xi_2&\xi_3&\xi_4\\ \eta_1&\eta_2&\eta_3&\eta_4\end{pmatrix}\le 1 \tag{R}
$$

the full equations have only solutions in the (ρ_1,ρ_2)-plane. Even if this rank condition defines a codimension 3 subspace, there are always solutions. To see this we set $\xi_i=s^{4-i}t^{i-1}\xi$ and $\eta_i=s^{4-i}t^{i-1}\eta$. The equations π_m are satisfied if $\frac{\partial}{\partial s}P_4(s,t;\xi,\eta)=0$ and $\frac{\partial}{\partial t}P_4(s,t;\xi,\eta)=0$. This is the intersection of two curves of type $(2,3)$ on the scroll $S_{3,3}\cong\mathbb{P}^1\times\mathbb{P}^1$ and there are 12 such intersection points. Those points give multiple solutions. One can compute that the rank of the Jacobi matrix of the system of equations (R) together with the π_m is five. By taking a suitable general example one finds that the multiplicity is in fact 4, and 48 is the degree of the solution of the system.

Proposition 4.17. *The general tetragonal cone with invariants* $(5,5,3;6,5)$, *which is composed with an involution of genus* 4, *has* 302 *smoothing components. The base space of a non-composed cone can be identified with a hyperplane section of the base of the corresponding composed one and only the smoothing components of lying in this hyperplane give smoothing components of the non-composed cone.*

This means that for fixed polynomials P, Q the existence of smoothings depends on an equation of degree 302 in the eight variables $q_{i;j}$. For special values the number of smoothing components may go down. This happens in the specific example given, where the condition (R) gives two-dimensional 'false' solutions. Here there are only 238 smoothing components. Besides the hyperelliptic involution the curve has another automorphism which acts on the base space. The only solutions I have found are easy to see:

$$\eta_1 = \eta_2 = \xi_3 = \xi_4 = \xi_1 + \xi_2 = \eta_3 + \eta_4 = \rho_1 + \xi_1 = \rho_2 + \eta_4 = \xi_1^2 - \eta_4^2 = 0$$

We take $\xi_1 = \eta_3 = \rho_2 = \delta$ and $\xi_2 = \eta_4 = \rho_1 = -\delta$. The total space is a surface on a scroll of type $(4,3,3,3)$ with bihomogeneous coordinates $(W,X,Y,Z;s,t)$. We set $Y_i = y_i$, $X_i = x_{i+2}$ and $Z_i = z_i$ for $i = 0,\dots,3$. The hyperplane section is $\delta = W_2 + X_0 + X_1 + Y_2 + Y_3$, so if $\delta = 0$ we have $X = t^2x$, $Y = s^2y$, $Z = z$ and $W = -(s+t)(x+y)$. The lifting equation is now $q_{0;1} - q_{1;1} + q_{2;2} - q_{3;3} = 0$. One computes that the surface is given by

$$\begin{gathered} 2X^2 + Y^2 - 2(X-Y)W(s-t) + W^2(s-t)^2 + Z^2 \\ 2Y^2s - XW(s^2 - 2st + 2t^2) + YW(2s^2 - 2st + t^2) + W^2(s-t)(s^2 - st + t^2) \\ -XZ(sq_{2;2} + tq_{3;2}) - YZ(sq_{0;1} + tq_{1;1}) - ZW(s^2q_{0;1} + st(q_{2;2} - q_{3;2}) + t^2q_{3;2}) \end{gathered}$$

This is a $K3$ surface with an A_1-singularity.
For even more special values of the coefficients there may be higher dimensional smoothing components. This happens e.g. for $P = z^2 + t^4y^2 + s^4x^2$ and the same Q as above, where the equations π_m have the solution $\xi_1 = \xi_2 = \eta_3 = \eta_4 = 0$, giving rise to an extra component of degree 15, which is the cone over three rational normal curves of degree 5. Then all tetragonal on Y have smoothings, but depending on the position of the hyperplane the number may increase.

(4.18) *The case* $(b_1, b_2) = (8,4)$. In this case there exist five families of $K3$-surfaces, three of which have the maximal dimension 18. The general hyperplane section of the scroll $S_{8,4,2,0}$ is a scroll $S_{8,4,2}$ while for both $S_{5,4,3,2}$ and $S_{4,4,4,2}$ it is $S_{5,5,4}$. One computes that the tetragonal curves of type $(2H - 8R, 2H - 4R)$ on $S_{8,4,2}$ depend on 29 moduli and those on $S_{5,5,4}$ depend on 34 moduli.

PROPOSITION 4.19. *The general tetragonal curve of type* $(8,4,2;8,4)$ *has only pure rolling factors extensions. If the* g^1_4 *is composed with an involution of genus* 3, *then there are in general* 91 *smoothing components not of this type.*

Remark 4.20. The tetragonal curve can be a special hyperplane section of a $K3$ surface on $S_{7,4,2,1}$, $S_{6,4,2,2}$, $S_{5,4,3,2}$ or $S_{4,4,4,2}$. Therefore the genericity assumption cannot be dropped.

Proof. After a coordinate we may assume that P has the form $p_8x^2 + y^2 + p_2xz$. The g^1_4 is composed with an involution of genus 3 if and only if $p_2 \equiv 0$. In that case Q may be taken in the form $q_{12}x^2 + q_8xy + z^2$. That the curve is nonsingular implies that p_8 has no multiple roots. If the g^1_4 is not composed, the term z^2

may be absent in Q, and p_8 may have multiple roots. For the general curve this does not occur. We look therefore at curves given by

$$P\colon p_8x^2+y^2+p_2xz$$
$$Q\colon q_{12}x^2+q_8xy+z^2\ .$$

The lifting matrix is a block matrix

$$\begin{pmatrix} 0 & 2I & 0 \\ \Pi & 0 & 0 \\ 0 & 0 & 2I \end{pmatrix}$$

with Π giving the equations $p_{2,0}\xi_i+p_{2,1}\xi_{i+1}+p_{2,2}\xi_{i+2}=0$. There is one pure rolling factors deformation for the first equation, and $5+1$ for the second. The equation P leads to 7 base equations π_m in the 8 variables ρ, ξ_1, …, ξ_7. The 128 solutions are described above. The equations coming from Q are

$$\begin{aligned}\rho_1\xi_1+\rho_2\xi_2+\rho_3\xi_3+\rho_4\xi_4+\rho_5\xi_5+\chi_1&=0\\ \rho_1\xi_2+\rho_2\xi_3+\rho_3\xi_4+\rho_4\xi_5+\rho_5\xi_6+\chi_2&=0\\ \rho_1\xi_3+\rho_2\xi_4+\rho_3\xi_5+\rho_4\xi_6+\rho_5\xi_7+\chi_3&=0\end{aligned}$$

We view this as inhomogeneous linear equations for the ρ_i. The coefficient matrix

$$M=\begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 \\ \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 \\ \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 \end{pmatrix}$$

is the transpose of the coefficient matrix of the equations $p_{2,0}\xi_i+p_{2,1}\xi_{i+1}+p_{2,2}\xi_{i+2}=0$, viewed as equations for the coefficients of p_2. If for a given solution of the equations π_m the matrix M has not full rank, then there exists a non-composed pencil admitting the same solution. But then also $p_{2,0}\chi_1+p_{2,1}\chi_2+p_{2,2}\chi_3=0$, an equation which in general is not satisfied. We have 8 solutions which lie on a rational normal curve and 28 solutions on the secant variety of this curve. The equations of the secant variety are the maximal minors of M. Only for 91 solutions the matrix M has full rank. □

In the general case we get components of dimension $3+1$ (the y-rolling factors deformation does not enter the equations), for solutions not on the rational curve, but on its secant variety the component has dimension 5, while we get a 6-dimensional component if p_8 and q_{12} have a common root. This does not contradict the fact that all smoothing components of Gorenstein surface singularities have the same dimension, because we here only look at the restriction to negative degree.

Proposition 4.21. *The general hyperplane section of a $K3$ surface of type $(5,4,3,2;8,4)$ or $(4,4,4,2;8,4)$ is a tetragonal curve of type $(5,5,4;8,4)$, which lies on a rational surface with two double points.*

Proof. We use coordinate transformations on the scroll to bring the hyperplane section into a normal form, while we suppose the coefficients of

the equations to be general. Let as usual $(X, Y, Z, W; s, t)$ be coordinates on the scroll. Let the hyperplane section be $a_0W_0 + a_1W_1 + a_2W_2 + \cdots = (a_0s^2 + a_1st + a_2t^2)W + \cdots$. By a transformation in (s, t) we achieve that $a_0 = a_2 = 0$, so the equation is $W_1 + \cdots$. First consider the case $(4, 4, 4, 2)$. By a suitable transformation $w \mapsto W + a_2(s, t)X + b_2(s, t)Y + c_2(s, t)Z$ we remove all terms with index 1, 2 or 3, leaving $W_1 + a_0X_0 + b_0Y_0 + c_0Z_0 + a_4X_4 + b_4Y_4 + c_4Z_4$. Taking $a_0X + b_0Y + c_0Z$ as new X and $a_4X + b_4Y + c_4Z$ as new Y brings us finally to $X_4 + W_1 + Y_0$. With coordinates $(x, y, z; s, t)$ for the scroll $S_{5,5,4}$ we get the hyperplane section by setting $Z = z$, $X = sx$, $Y = ty$ and $W = -t^3x - s^3y$. The equation P does not involve the variable W so we have quadratic singularities if $sx = ty = z = 0$, which gives the points $s = y = z = 0$ and $t = x = z = 0$.
In the case $(5, 4, 3, 2)$ we can achieve $W_1 + Z_0 + Z_3$ and we get the curve by $X = x$, $Y = z$, $Z = sty$ and $W = -(s^3 + t^3)y$. The equation $P\colon p_2X^2 + p_1XY + p_0Y^2 + XZ$ now gives $p_2x^2 + p_1xz + p_0z^2 + stxy$, which for general p_i has singular points at $x = z = st = 0$. □

To investigate the sufficiency of these conditions we look at the general cone of type $(5, 5, 4; 8, 4)$. We may suppose that P has the form $z^2 + P_2(x, y)$. The equation $P_2(x, y)$ describes a curve of type $(2, 2)$ on $S_{5,5} \cong \mathbb{P}^1 \times \mathbb{P}^1$. If this curve has a singular point, we may assume that it lies in the point $x = s = 0$. Under the assumption that the coefficient of $stxy$ does not vanish we can transform the equation into the form $(as^2 + bt^2)x^2 + 2stxy + cs^2y^2$ and unfolding the singularity we get the equation

$$P_2 = (as^2 + bt^2)x^2 + 2stxy + (cs^2 + dt^2)y^2 \ .$$

One can then write out the lifting conditions and base equations coming from the equation P. The result is that they have only trivial solutions if and only if $abcd((ad + bc - 1)^2 - 4abcd) \neq 0$, if and only if the curve P_2 is nonsingular. If a singularity is present we assume it to be in $x = s = 0$, so $d = 0$. The equation Q gives three base equations, in which 2+2 pure rolling factors variables can enter. We analyze what happens if there is a second singularity. For $b = d = 0$ the equation P_2 is divisible by s, and we do not find extensions. In case $a = d = 0$ the curve P_2 splits into two curves of type $(1, 1)$; we get two components with deformed scroll $S_{4,4,4,2}$. For $c = d = 0$ we have intersection of a line with a curve of type $(2, 1)$ and we find two components with deformed scroll $S_{5,4,3,2}$.

Remark 4.22. For the general tetragonal cone with large g we found $\dim T^1(-1) = 9$, but all deformations are obstructed. For special curves extensions may exist; also the dimension can be higher. Both conditions seem to be independent. As the number of base equations we find is always $g - 7$, having more variables increases the chances of finding solutions. In the borderline case studied above this may suffice to force the existence, but in general it does not. On the other, taking a general hyperplane section of a general tetragonal $K3$ surface will give a cone with $\dim T^1(-1) = 9$. It would be interesting to find a property of a canonical curve which gives a sufficient condition for the existence of an extension.

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