On Neeman's Well Generated Triangulated Categories

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Abstract. We characterize Neeman's well generated triangulated categories and discuss some of its basic properties.

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In his recent book [4], Neeman introduces a class of triangulated categories which he calls *well generated*. Although Neeman's definition is not easily stated, it becomes quite clear that for triangulated categories 'well generated' is the appropriate generalization of 'compactly generated' [3]. The well generated categories share many important properties with the compactly generated categories. In addition, the class of well generated categories is closed under various natural constructions, for instance, passing to appropriate localizing subcategories and localizations. Our aim in this note is to provide an equivalent definition for well generated categories which seems to be more natural.

Theorem A. *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. Then $\mathcal{T}$ is well generated in the sense of* [4] *if and only if there exists a set $\mathcal{S}_0$ of objects satisfying:*

(G1) *an object $X \in \mathcal{T}$ is zero provided that $(S, X) = 0$ for all $S \in \mathcal{S}_0$;*
(G2) *for every set of maps $X_i \to Y_i$ in $\mathcal{T}$ the induced map $(S, \coprod_i X_i) \to (S, \coprod_i Y_i)$ is surjective for all $S \in \mathcal{S}_0$ provided that $(S, X_i) \to (S, Y_i)$ is surjective for all i and $S \in \mathcal{S}_0$;*
(G3) *the objects in $\mathcal{S}_0$ are α-small for some cardinal α.*

Here, (X, Y) denotes the maps $X \to Y$. In addition, we recall that an object S is *α-small* if every map $S \to \coprod_{i \in I} X_i$ factors through $\coprod_{i \in J} X_i$ for some $J \subseteq I$ with card $J < \alpha$. Conditions (G1) and (G3) are fairly natural to consider; (G2) is taken from [2] where it is shown that Brown's Representability Theorem holds for a triangulated category $\mathcal{T}$ whenever there is a set $\mathcal{S}_0$ of objects satisfying (G1) – (G2).

Neeman's definition and our characterization of well generated triangulated categories are based on the concept of compactness. Let us explain this. We fix a triangulated category $\mathcal{T}$ with arbitrary coproducts and a cardinal α. Clearly, there exists a unique maximal class $\mathcal{S}$ of α-small objects in $\mathcal{T}$ such that the following holds:

(G4) every map $S \to \coprod_i X_i$ from $S \in \mathcal{S}$ into a coproduct in $\mathcal{T}$ factors through a map $\coprod_i \phi_i \colon \coprod_i S_i \to \coprod_i X_i$ with $S_i \in \mathcal{S}$ for all i.

Simplifying Neeman's terminology, we call the objects in $\mathcal{S}$ *α-compact* and write $\mathcal{T}^\alpha$ for the full subcategory of α-compact objects. We have now the following more explicit description of such compact objects.

Theorem B. *Let $\mathcal{T}$ be a well generated triangulated category and let $\mathcal{S}_0$ be a set of objects satisfying* (G1) – (G2). *Then there exists for every cardinal α a cardinal $\beta \geq \alpha$ such that $X \in \mathcal{T}$ is β-compact if and only if* $\operatorname{card}(S, X) < \beta$ *for all $S \in \mathcal{S}_0$.*

The characterization of well generated triangulated categories uses a result which relates condition (G2) and (G4) to another condition. To state this, let $\mathcal{C}$ be a small additive category and fix a *regular cardinal* α, that is, α is not the sum of fewer than α cardinals, all smaller than α. An *α-product* is a product of less than α factors, and we suppose that α-products exist in $\mathcal{C}$. We denote by $\operatorname{Prod}_\alpha(\mathcal{C}, \mathrm{Ab})$ the category of functors $\mathcal{C} \to \mathrm{Ab}$ into the category of abelian groups which preserve α-products; the morphisms between two functors are the natural transformations.

Theorem C. *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts and α be a regular cardinal. Let $\mathcal{S}_0$ be a set of objects in $\mathcal{T}$ and denote by $\mathcal{S}$ the full subcategory of α-coproducts of objects in $\mathcal{S}_0$. Then the following are equivalent:*

(1) (G2) *holds for $\mathcal{S}_0$ and every object in $\mathcal{S}_0$ is α-small.*
(2) (G4) *holds for $\mathcal{S}$ and every object in $\mathcal{S}$ is α-small.*
(3) *The functor $\mathcal{T} \to \operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$, $X \mapsto (-, X)|_{\mathcal{S}}$, preserves arbitrary coproducts.*

Proofs

Let us start with some preparations. Throughout we fix a triangulated category $\mathcal{T}$ with arbitrary coproducts. Let $\mathcal{C}$ be an additive category $\mathcal{C}$. A functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups is *coherent* if there exists an exact sequence

$$(-, X) \longrightarrow (-, Y) \longrightarrow F \longrightarrow 0.$$

The natural transformations between two coherent functors form a set, and the coherent functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ form an additive category with cokernels which we denote by $\widehat{\mathcal{C}}$. A basic tool is the *Yoneda functor*

$$\mathcal{C} \longrightarrow \widehat{\mathcal{C}}, \quad X \mapsto H_X = (-, X).$$

Note that $\widehat{\mathcal{C}}$ is a cocomplete category if $\mathcal{C}$ has arbitrary coproducts; in this case the Yoneda functor preserves all coproducts. Recall from [1] that an object

X in a cocomplete category is *α-presentable* if the functor $(X,-)$ preserves α-directed colimits.

Lemma 1. *Let $\mathcal{C}$ be an additive category with arbitrary coproducts. Then an object $X \in \mathcal{C}$ is α-small if and only if $(-,X)$ is α-presentable in $\widehat{\mathcal{C}}$.*

Proof. Straightforward. $\square$

Suppose that $\mathcal{C}$ has kernels and α-products. Then we denote by $\operatorname{Lex}_\alpha(\mathcal{C}, \mathrm{Ab})$ the category of left exact functors $\mathcal{C} \to \mathrm{Ab}$ which preserve α-products. Given a class $\mathcal{S}$ of objects in $\mathcal{T}$, we denote by $\operatorname{Add}\mathcal{S}$ the closure of $\mathcal{S}$ in $\mathcal{T}$ under all coproducts and direct factors.

Lemma 2. *Let $\mathcal{S}$ be a small additive subcategory of $\mathcal{T}$ and let α be a regular cardinal. Suppose that every $X \in \mathcal{S}$ is α-small and that $\mathcal{S}$ is closed under α-coproducts in $\mathcal{T}$. Then the assignment $F \mapsto F|_\mathcal{S}$ induces an equivalence $f\colon \widehat{\operatorname{Add}\mathcal{S}} \to \operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$.*

Proof. First observe that H_X is α-presentable in $\widehat{\operatorname{Add}\mathcal{S}}$ for every $X \in \mathcal{S}$ by Lemma 1. The inclusion $i\colon \mathcal{S} \to \operatorname{Add}\mathcal{S}$ induces a right exact functor $i^*\colon \widehat{\mathcal{S}} \to \widehat{\operatorname{Add}\mathcal{S}}$ which sends H_X to H_{iX}. This functor identifies $\widehat{\mathcal{S}}$ with the full subcategory of all α-colimits of objects in $\{H_X \mid X \in \mathcal{S}\}$. It follows from Satz 7.8 in [1] that i^* induces a fully faithful functor $j\colon \operatorname{Lex}_\alpha(\widehat{\mathcal{S}}^{\mathrm{op}}, \mathrm{Ab}) \to \widehat{\operatorname{Add}\mathcal{S}}$ which sends a representable functor $(-,X)$ to i^*X and identifies $\operatorname{Lex}_\alpha(\widehat{\mathcal{S}}^{\mathrm{op}}, \mathrm{Ab})$ with the full subcategory of all colimits of objects in $\{H_X \mid X \in \mathcal{S}\}$. Thus j is an equivalence. Now consider the Yoneda functor $h\colon \mathcal{S} \to \widehat{\mathcal{S}}$. It is easily checked that the restriction functor

$$h_*\colon \operatorname{Lex}_\alpha(\widehat{\mathcal{S}}^{\mathrm{op}}, \mathrm{Ab}) \longrightarrow \operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab}), \quad F \mapsto F \circ h,$$

is an equivalence. We have $f \circ j \cong h_*$ and conclude that f is an equivalence. $\square$

Lemma 3. *Let $\mathcal{S}_0$ be a set of objects in $\mathcal{T}$ and let $\mathcal{S} = \operatorname{Add}\mathcal{S}_0$. Then the functor*

$$\mathcal{T} \longrightarrow \widehat{\mathcal{S}}, \quad X \mapsto (-,X)|_\mathcal{S},$$

preserves coproducts if and only if (G2) *holds for $\mathcal{S}_0$.*

Proof. See Lemma 3 in [2]. $\square$

Lemma 4. *Let $\mathcal{S}$ be a set of α-small objects in $\mathcal{T}$ which is closed under α-coproducts. Then* (G2) *and* (G4) *are equivalent for $\mathcal{S}$.*

Proof. It is clear that (G4) implies (G2). To prove the converse, let $S \to \coprod_i X_i$ be a map in $\mathcal{T}$ with $S \in \mathcal{S}$. Choose for every i a map $\psi_i\colon \coprod_j S_{ij} \to X_i$ with $S_{ij} \in \mathcal{S}$ for all j such that every map $X \to X_i$ with $X \in \mathcal{S}$ factors through ψ_i. Then (G2) implies that the map $S \to \coprod_i X_i$ factors through $\amalg_i \psi_i\colon \coprod_i \coprod_j S_{ij} \to \coprod_i X_i$. Using the fact that S is α-small and that $\mathcal{S}$ has α-coproducts, we can replace for each i the coproduct $\coprod_j S_{ij}$ by some $S_i \in \mathcal{S}$. $\square$

Proof of Theorem C. The equivalence of (1) and (2) is Lemma 4 and it remains to show that (1) and (3) are equivalent. We fix a set of objects $\mathcal{S}_0$ in $\mathcal{T}$ and a regular cardinal α. The full subcategory of α-coproducts of objects in $\mathcal{S}_0$ is denoted by $\mathcal{S}$. We can write

$$f\colon \mathcal{T} \longrightarrow \operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto (-,X)|_{\mathcal{S}},$$

as composite

$$f\colon \mathcal{T} \xrightarrow{g} \widehat{\operatorname{Add}\mathcal{S}_0} = \widehat{\operatorname{Add}\mathcal{S}} \xrightarrow{h} \operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$$

where $gX = (-,X)|_{\operatorname{Add}\mathcal{S}_0}$ and $hF = F|_{\mathcal{S}}$. Now suppose that every object in $\mathcal{S}_0$ is α-small and that (G2) holds. Then it follows from Lemma 3 that g preserves coproducts and Lemma 2 implies that h is an equivalence. We conclude that f preserves coproducts.
Conversely, suppose that f preserves coproducts. It follows that the right exact functor $f^*\colon \widehat{\mathcal{T}} \to \operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$ which sends H_X to fX preserves colomits. Now identify $\operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$ with $\operatorname{Lex}_\alpha(\widehat{\mathcal{S}}^{\mathrm{op}}, \mathrm{Ab})$ as in the proof of Lemma 2. Using Satz 5.5 in [1], it is not hard to see that f^* has a left adjoint which sends $(-,X)$ in $\operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$ to H_X. A left adjoint of a functor which preserves α-directed colimits, sends α-presentable objects to α-presentable objects. But the representable functors are α-presentable in $\operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$. We conclude from Lemma 1 that every $X \in \mathcal{S}$ is α-small. The first part of the proof shows that (G2) holds as well. □

Remark. The proof of Theorem C does not use the triangulated structure of $\mathcal{T}$. One needs that $\mathcal{T}$ is an additive category with arbitrary coproducts and that weak kernels exist in $\mathcal{T}$.

Recall that a full triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ is *localizing* if $\mathcal{S}$ is closed under arbitrary coproducts. Given a regular cardinal α, we call $\mathcal{S}$ *α-localizing* if $\mathcal{S}$ is closed under α-coproducts and direct factors. For example, the full subcategory $\mathcal{T}_\alpha$ of α-small objects in $\mathcal{T}$ is α-localizing. The full subcategory $\mathcal{T}^\alpha$ of α-compact objects is α-localizing as well.

LEMMA 5. *Let $\mathcal{S}_0$ be a set of α-small objects satisfying* (G1) – (G2) *and denote by $\mathcal{S}$ the smallest α-localizing subcategory containing $\mathcal{S}_0$. Then $\mathcal{S} = \mathcal{T}^\alpha$. Moreover:*

1. *The objects in $\mathcal{S}$ form, up to isomorphism, a set of α-small objects satisfying* (G2).
2. *Every set of α-small objects satisfying* (G2) *is contained in $\mathcal{S}$.*

Proof. First we prove (1) and (2); the proof for $\mathcal{S} = \mathcal{T}^\alpha$ is given at the end.
(1) Using the equivalent condition (G4) from Lemma 4, it is straightforward to check that (G2) is preserved if we pass to the closure with respect to forming triangles and α-coproducts. The closure $\mathcal{S}$ can be constructed explicitly, and this shows that the isomorphism classes of objects form a set.
(2) Let $\mathcal{S}_1$ be a set of α-small objects satisfying (G2). We denote by $\mathcal{S}'$ the smallest α-localizing subcategory containing $\mathcal{S}_0 \cup \mathcal{S}_1$ and claim that $\mathcal{S}' = \mathcal{S}$.

Consider the full subcategory $\mathcal{T}'$ of objects $Y \in \mathcal{T}$ such that every map $X \to Y$ with $X \in \mathcal{S}'$ factors through some object in $\mathcal{S}$. Using the fact that (G4) holds for $\mathcal{S}'$ by Lemma 4, it is straightforward to check that $\mathcal{T}'$ is a localizing subcategory containing $\mathcal{S}_0$. The Corollary of Theorem A in [2] shows that $\mathcal{T}$ has no proper localizing subcategory containing $\mathcal{S}_0$. Thus $\mathcal{T}' = \mathcal{T}$ and id_X factors through some object in $\mathcal{S}$ for every $X \in \mathcal{S}'$. We conclude that $\mathcal{S}' = \mathcal{S}$. The proof for the equality $\mathcal{S} = \mathcal{T}^\alpha$ is the same as in (2) if we replace $\mathcal{S}'$ by $\mathcal{T}^\alpha$. □

Lemma 6. *Suppose that the isomorphism classes of objects in $\mathcal{T}^\alpha$ form a set. Then an object $X \in \mathcal{T}$ belongs to $\mathcal{T}^\alpha$ if and only if X is α-compact in the sense of* [4].

Proof. It is automatic from the Definition 1.9 of an α-compact object in [4] that $X \in \mathcal{T}$ belongs to $\mathcal{T}^\alpha$ if X is α-compact in the sense of [4]. Theorem 1.8 in [4] shows that the objects in $\mathcal{T}$ which are α-compact in the sense of [4], form the unique maximal subcategory $\mathcal{S} \subseteq \mathcal{T}_\alpha$ such that the canonical functor $\mathcal{T} \to \mathrm{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$ preserves coproducts. On the other hand, Theorem C shows that $\mathcal{T}^\alpha$ has precisely this property. □

We are now in a position to prove our main result. To this end recall from Definition 1.15 in [4] that $\mathcal{T}$ is *well generated* if there is a regular cardinal α such that condition (2) in the following theorem holds.

Theorem A. *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. Then the following are equivalent for a regular cardinal α:*

(1) *There exists a set of α-small objects satisfying* (G1) – (G2).
(2) *The isomorphism classes of objects in $\mathcal{T}^\alpha$ form a set, and $\mathcal{T}$ is the smallest localizing subcategory containing $\mathcal{T}^\alpha$.*

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{S}_0$ be a set of α-small objects satisfying (G1) – (G2). It follows from Lemma 5 that the isomorphism classes in $\mathcal{T}^\alpha$ form a set. The Corollary of Theorem A in [2] shows that $\mathcal{T}$ has no proper localizing subcategory containing $\mathcal{S}_0$.
(2) $\Rightarrow$ (1) Choose a representative set $\mathcal{S}_0$ of objects in $\mathcal{T}^\alpha$. It follows from Lemma 4 that (G2) holds for $\mathcal{S}_0$. To check (G1) let $\mathcal{Y}$ be the class of objects $Y \in \mathcal{T}$ satisfying $(S, Y) = 0$ for all $S \in \mathcal{S}_0$. Then the objects $X \in \mathcal{T}$ satisfying $(X, Y) = 0$ for all $Y \in \mathcal{Y}$ form a localizing subcategory $\mathcal{X}$ containing $\mathcal{S}_0$. Thus $\mathcal{X} = \mathcal{T}$ and $\mathcal{Y} = \{0\}$. We conclude that $\mathcal{S}_0$ is a set of α-small objects satisfying (G1) – (G2). □

The following immediate consequence of Theorem A is due to Neeman [4].

Corollary. *Let $\mathcal{T}$ be a well generated triangulated category. Then $\mathcal{T} = \bigcup_\alpha \mathcal{T}^\alpha$ where α runs through all cardinals.*

We end this note with a proof of Theorem B.

Proof of Theorem B. Let $\mathcal{S}_0$ be a set of objects satisfying (G1) – (G2) and fix a cardinal α. We suppose that $\mathcal{T}$ is well generated. Therefore the objects in $\mathcal{S}_0$ are α'-small for some cardinal α'. It follows from Theorem C in [2] that there is a cardinal $\beta \geq \alpha + \alpha'$ such that an object $X \in \mathcal{T}$ belongs to the smallest β-localizing subcategory containing $\mathcal{S}_0$ if and only if $\mathrm{card}(S, X) < \beta$ for all $S \in \mathcal{S}_0$. The assertion now follows from Lemma 5. $\square$

Remark. There is an explicit description for the cardinal β in Theorem B; see Theorem C in [2].

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References

[1] P. Gabriel and F. Ulmer, Lokal präsentierbare Kategorien, Springer Lecture Notes in Math. 221 (1971).
[2] H. Krause, A Brown Representability Theorem via coherent functors, Topology, to appear.
[3] A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability, J. Amer. Math. Soc. 9 (1996), 205–236.
[4] A. Neeman, Triangulated categories, Annals of Mathematics Studies 148, Princeton University Press (2001).

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