Pseudodifferential Analysis on Continuous Family Groupoids

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Abstract. We study properties and representations of the convolution algebra and the algebra of pseudodifferential operators associated to a continuous family groupoid. We show that the study of representations of the algebras of pseudodifferential operators of order zero completely reduces to the study of the representations of the ideal of regularizing operators. This recovers the usual boundedness theorems for pseudodifferential operators of order zero. We prove a structure theorem for the norm completions of these algebras associated to groupoids with invariant filtrations. As a consequence, we obtain criteria for an operator to be compact or Fredholm. We end with a discussion of the significance of these results to the index theory of operators on certain singular spaces. For example, we give a new approach to the question of the existence of spectral sections for operators on coverings of manifolds with boundary. We expect that our results will also play a role in the analysis on more general singular spaces.

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Introduction

For the proof of his measured index theorem for $\mathcal{C}^{\infty,0}$-foliations [3], Connes introduced pseudodifferential operators on the holonomy groupoid. Two closely related constructions of algebras of pseudodifferential operators for general differentiable groupoids were proposed in [28, 29]. Coming from microlocal analysis of pseudodifferential operators on manifolds with corners, a similar construction was suggested and used by Melrose [21], without mentioning groupoids explicitly. While the initial motivation for these constructions was completely different, eventually it was realized that they all can be used to formalize various constructions with pseudodifferential operators related to adiabatic limits, scattering or spectral problems, and index theoretical computations.
Recall that a groupoid is a small category in which every morphism is invertible. The domain map $d : \mathcal{G}^{(1)} \longrightarrow M$ associates to a morphism $g : d(g) \to r(g)$ its domain $d(g)$, which is an object in M. This yields a decomposition of the set of morphisms

$$\mathcal{G}^{(1)} = \bigcup_{x \in M} d^{-1}(x)\,.$$

The basic idea for a pseudodifferential calculus on groupoids is to consider families $(P_x)_{x\in M}$ of pseudodifferential operators on the "fibers" $\mathcal{G}_x := d^{-1}(x)$ that are equivariant with respect to the action of the groupoid induced by composing compatible morphisms. All that is needed for this construction is in fact a smooth structure on the sets $\mathcal{G}_x$, $x \in M$. Of course, there is a maze of possible ways to glue these fibers together. Let us only mention *differentiable groupoids*, where the fibers, roughly speaking, depend smoothly on the parameter $x \in M$, and *continuous family groupoids* introduced by Paterson [32] generalizing the holonomy groupoid of a $\mathcal{C}^{\infty,0}$-foliation as considered in [3]. In that case, the dependence on x is merely continuous in an appropriate sense – see Section 1 for precise definitions.
In [28, 29] pseudodifferential operators were introduced on differentiable groupoids; one of the main results is that (under appropriate restrictions on the distributional supports) pseudodifferential operators can be composed. Though the definition is rather simple, these algebras of pseudodifferential operators recover many previously known classes of operators, including families, adiabatic limits, and longitudinal operators on foliations. Moreover, for manifolds with corners, the pseudodifferential calculus identifies with a proper subalgebra of the "b"–calculus [18, 19]. It is possible, however, to recover the "b"–calculus in the framework of groupoids, as shown by the second autor in [27].
In this paper, we first extend the construction of [28, 29] to continuous family groupoids. This more general setting enables us to freely restrict pseudodifferential operators to invariant subsets of the groupoid; these restrictions are necessary to fully understand the Fredholm properties of pseudodifferential operators on groupoids. To see what are the technical problems when dealing with the more restrictive setting of differentiable groupoids, simply note that for instance, the boundary of a manifold with corners is not a manifold with

corners anymore, so that the class of differentiable groupoids is not stable under restriction to invariant subsets.

In the main part of the paper, we set up some analytical foundations for a general pseudodifferential analysis on groupoids. This covers among others the existence of bounded representations on appropriate Hilbert spaces, and criteria for Fredholmness or compactness. This is inspired in part by the central role played by groupoids in the work of Connes on index theory and by some questions in spectral theory [13]. In both cases it is natural to consider norm closures of the algebras of pseudodifferential operators that are of interest, so the study of these complete algebras plays a prominent role in our paper. We show that (and how) the geometry of the space of objects of a given groupoid is reflected by the structure of the C^*-algebra generated by the operators of order zero. The morphism of restricting to invariant subsets is an important tool in this picture; these homomorphisms are in fact necessary ingredients to understand Fredholm and representation theory of pseudodifferential operators.

Let us now briefly describe the contents of each section. The first section introduces continuous family groupoids and explains how the results of [28] and [29] can be extended to this setting. However, we avoid repeating the same proofs.

In the second section, we discuss restriction maps, which are a generalization of the indicial maps of Melrose. The third section contains the basic results on the boundedness and representations of algebras of pseudodifferential operators on a continuous family groupoid. We prove that all boundedness results for pseudodifferential operators of order zero reduce to the corresponding results for regularizing operators. This is obtained using a variant of the approach of Hörmander [6]. In the fourth section, we study the structure of the norm closure of the groupoid algebras and obtain a canonical composition series of a groupoid algebra, if there is given an invariant stratification of the space of units. This generalizes a result from [22]. As a consequence of this structure theorem, we obtain characterizations of Fredholm and of compact operators in these algebras in Theorem 4:

- *An order zero operator between suitable L^2-spaces is* Fredholm *if, and only if, it is elliptic* (*i.e.,* its principal symbol is invertible) *and its restrictions to all strata of lower dimension are invertible as operators between certain natural Hilbert spaces.*
- *An order zero operator is* compact *if, and only if, its principal symbol and all its restrictions to strata of lower dimension vanish.*

See Theorem 4 for the precise statements. These characterizations of compactness and Fredholmness are classical results for compact manifolds without boundary (which correspond in our framework to the product groupoid $\mathcal{G} = M \times M$, M compact without boundary). For other classes of operators, characterizations of this kind were obtained previously for instance in [17, 19, 20, 22, 23, 33, 41].

The last section contains two applications. The first one is a discussion of the relation between the adiabatic groupoid $\mathcal{G}_{ad}$ canonically associated to a groupoid $\mathcal{G}$ and index theory of pseudodifferential operators. The second application is to operators on a covering $\widetilde{M}$ of a manifold with boundary M, with group of deck transformations denoted by Γ. We prove that every invariant, b-pseudodifferential, elliptic operator on $\widetilde{M}$ has a perturbation by regularizing operators of the same kind that is $C^*(\Gamma)$-Fredholm in the sense of Mishenko and Fomenko. This was first proved in [14] using "spectral sections."

A differentiable groupoid is a particular case of a continuous family groupoid. In particular, all results of this section remain valid for differentiable groupoids, when they make sense. This also allows us to recover most of the results of [12].

The dimension of the fibers $\mathcal{G}_x$ is constant in x on each component of M. For simplicity, we agree throughout the paper to assume that M is connected and denote by n the common dimension of the fibers $\mathcal{G}_x$.

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1 Basic Definitions

We begin this section by recalling some definitions involving groupoids. Then we review and extend some results from [28, 29] on pseudodifferential operators, from the case of differentiable groupoids to that of continuous family groupoids. In the following, we shall use the framework of [28, 29], and generalize it to the context of continuous family groupoids.

A *small category* is a category whose class of morphisms is a set. The class of objects of a small category is then a set as well. By definition, a *groupoid* is a small category $\mathcal{G}$ in which every morphism is invertible. See [37] for general references on groupoids.

We now fix some notation and make the definition of a groupoid more explicit. The set of objects (or *units*) of $\mathcal{G}$ is denoted by M or $\mathcal{G}^{(0)}$. The set of morphisms (or *arrows*) of a groupoid $\mathcal{G}$ is denoted by $\mathcal{G}^{(1)}$. We shall sometimes write $\mathcal{G}$ instead of $\mathcal{G}^{(1)}$, by abuse of notation. For example, when we consider a space of functions on $\mathcal{G}$, we actually mean a space of functions on $\mathcal{G}^{(1)}$. We will denote by $d(g)$ [respectively $r(g)$] the *domain* [respectively, the *range*] of the morphism $g : d(g) \to r(g)$. We thus obtain functions

$$d, r : \mathcal{G}^{(1)} \longrightarrow M = \mathcal{G}^{(0)} \tag{1}$$

that will play an important role in what follows. The multiplication gh of $g, h \in \mathcal{G}^0$ is defined if, and only if, $d(g) = r(h)$. A groupoid $\mathcal{G}$ is completely

determined by the spaces M and $\mathcal{G}$ and by the structural morphisms: d, r, multiplication, inversion, and the inclusion $M \to \mathcal{G}$.
In [3], A. Connes defined the notion of a $\mathcal{C}^{\infty,0}$-foliation. This leads to the definition of a *continuous family groupoid* by Paterson [32]. Let us summarize this notion.
By definition, a *continuous family groupoid* is a locally compact topological groupoid such that $\mathcal{G}$ is covered by some open subsets Ω and:

- each chart Ω is homeomorphic to two open subsets of $\mathbb{R}^k \times \mathcal{G}^{(0)}$, $T \times U$ and $T' \times U'$ such that the following diagram is commutative:

$$
\begin{tikzcd}
 & T' \times U' \arrow[dl] & & \Omega \arrow[ll, "\simeq"'] \arrow[rr, "\simeq"] \arrow[dl, "r"] \arrow[dr, "d"'] & & T \times U \arrow[dr] & \\
U' & & r(\Omega) \arrow[ll, "="'] & & d(\Omega) \arrow[rr, "="] & & U
\end{tikzcd}
$$

- each coordinate change is given by $(t, u) \mapsto (\phi(t, u), u)$ where ϕ is of class $\mathcal{C}^{\infty,0}$, *i.e.* $u \mapsto \phi(., u)$ is a continuous map from U to $\mathcal{C}^\infty(T, T')$.

In addition, one requires that the composition and the inversion be $\mathcal{C}^{\infty,0}$ morphisms.
Generally, we will transform several concepts from the smooth to the $\mathcal{C}^{\infty,0}$ setting. The definitions are in the same spirit as the definition of a continuous family groupoid, and the reader can fill in the necessary details without any difficulties. For instance, the restriction $A(\mathcal{G})$ of the d-vertical tangent bundle $T_d\mathcal{G} = \bigcup_{g \in \mathcal{G}} T_g \mathcal{G}_{d(g)}$ of $\mathcal{G}$ to the space of units is called the *Lie algebroid of* $\mathcal{G}$; it is a $\mathcal{C}^{\infty,0}$-vector bundle.
We now review pseudodifferential operators, the main focus being on the definition and properties of the algebra $\Psi^\infty(\mathcal{G})$ of pseudodifferential operators on a continuous family groupoid $\mathcal{G}$, and its variant, $\Psi^\infty(\mathcal{G}; E)$, the algebra of pseudodifferential operators on $\mathcal{G}$ acting on sections of a vector bundle.
Consider a complex vector bundle E on the space of units M of a continuous family groupoid $\mathcal{G}$, and let $r^*(E)$ be its pull-back to $\mathcal{G}$. Right translations on $\mathcal{G}$ define linear isomorphisms

$$U_g : \mathcal{C}^\infty(\mathcal{G}_{d(g)}, r^*(E)) \to \mathcal{C}^\infty(\mathcal{G}_{r(g)}, r^*(E)) : (U_g f)(g') = f(g'g) \in (r^*E)_{g'} \quad (2)$$

which are defined because $(r^*E)_{g'} = (r^*E)_{g'g} = E_{r(g')}$.
Let $B \subset \mathbb{R}^n$ be an open subset. Define the space $\mathcal{S}^m(B \times \mathbb{R}^n)$ of symbols on the bundle $B \times \mathbb{R}^n \to B$ as in [8] to be the set of smooth functions $a : B \times \mathbb{R}^n \to \mathbb{C}$ such that

$$|\partial_y^\alpha \partial_\xi^\beta a(y, \xi)| \leq C_{K,\alpha,\beta}(1 + |\xi|)^{m-|\beta|} \quad (3)$$

for any compact set $K \subset B$, for any multi-indices α and β, for any $x \in K$, and for any $\xi \in \mathbb{R}^n$. An element of one of our spaces $\mathcal{S}^m$ should more properly be said to have "order less than or equal to m"; however, by abuse of language we will say that it has "order m."

A symbol $a \in \mathcal{S}^m(B \times \mathbb{R}^n)$ is called *classical* (or polyhomogeneous) if it has an asymptotic expansion as an infinite sum of homogeneous symbols $a \sim \sum_{k=0}^{\infty} a_{m-k}$, a_l homogeneous of degree l:

$$a_l(y, t\xi) = t^l a_l(y, \xi) \quad \text{if} \quad \|\xi\| \geq 1$$

and $t \geq 1$. ("Asymptotic expansion" is used here in the sense that for each $N \in \mathbb{N}$, the difference $a - \sum_{k=0}^{N-1} a_{m-k}$ belongs to $\mathcal{S}^{m-N}(B \times \mathbb{R}^n)$.) The space of classical symbols will be denoted by $\mathcal{S}^m_{\mathrm{cl}}(B \times \mathbb{R}^n)$; its topology is given by the semi-norms induced by the inequalities (3). We shall be working exclusively with classical symbols in this paper.

This definition immediately extends to give spaces $\mathcal{S}^m_{\mathrm{cl}}(E; F)$ of symbols on E with values in F, where $\pi : E \to B$ and $F \to B$ are smooth Euclidean vector bundles. These spaces, which are independent of the metrics used in their definition, are sometimes denoted $\mathcal{S}^m_{\mathrm{cl}}(E; \pi^*(F))$. Taking $E = B \times \mathbb{R}^n$ and $F = \mathbb{C}$ one recovers $\mathcal{S}^m_{\mathrm{cl}}(B \times \mathbb{R}^n) = \mathcal{S}^m_{\mathrm{cl}}(B \times \mathbb{R}^n; \mathbb{C})$.

Recall that an operator $T : \mathcal{C}^\infty_c(U) \to \mathcal{C}^\infty(V)$ is called *regularizing* if, and only if, it has a smooth distribution (or Schwartz) kernel. For any open subset W of $\mathbb{R}^n$ and any complex valued symbol a on $T^*W = W \times \mathbb{R}^n$, let

$$a(y, D_y) : \mathcal{C}^\infty_c(W) \to \mathcal{C}^\infty(W)$$

be given by

$$a(y, D_y)u(y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iy\cdot\xi} a(y, \xi) \hat{u}(\xi) d\xi \,. \tag{4}$$

Then, by definition, a pseudodifferential operator P on B is a continuous, linear map $P : \mathcal{C}^\infty_c(B) \to \mathcal{C}^\infty(B)$ that is locally of the form $P = a(y, D_y) + R$, where R is a regularizing operator.

We shall sometimes refer to pseudodifferential operators acting on a smooth manifold as *ordinary* pseudodifferential operators, in order to distinguish them from pseudodifferential operators on groupoids, a class of operators we now define (and which are really *families* of ordinary pseudodifferential operators). Throughout this paper, we shall denote by $(P_x, x \in M)$ a family of order m pseudodifferential operators P_x, acting on the spaces $\mathcal{C}^\infty_c(\mathcal{G}_x, r^*(E))$ for some vector bundle E over M. Operators between sections of two *different* vector bundles E_1 and E_2 are obtained by considering $E = E_1 \oplus E_2$. (See also below.)

Definition 1 *A family $(P_x, x \in M)$ as above is called* continuous *if, and only if, for any open chart $V \subset \mathcal{G}$, homeomorphic to $W \times d(V)$, and for any $\phi \in \mathcal{C}^{\infty,0}_c(V)$, we can find a continuous family of symbols $(a_x, x \in d(V))$ with*

$$a_x \in \mathcal{S}^m_{\mathrm{cl}}(T^*W; \mathrm{End}(E))$$

such that $\phi P_x \phi$ corresponds to $a_x(y, D_y)$ under the diffeomorphism $\mathcal{G}_x \cap V \simeq W$, for each $x \in d(V)$.

Thus, we require that the operators P_x be given in local coordinates by symbols a_x that depend smoothly on longitudinal variables (with respect to d) and continuously on transverse variables.
Let us denote by $\mathcal{D}$ the density bundle of the Lie algebroid $A(\mathcal{G})$.

Definition 2 *An order m, invariant pseudodifferential operator P on a continuous family groupoid $\mathcal{G}$, acting on sections of the vector bundle E, is a continuous family $(P_x, x \in M)$ of order m, classical pseudodifferential operators P_x acting on $\mathcal{C}_c^\infty(\mathcal{G}_x, r^*(E \otimes \mathcal{D}^{1/2}))$ that satisfies*

$$P_{r(g)}U_g = U_g P_{d(g)}, \tag{5}$$

for any $g \in \mathcal{G}$, where U_g is as in (2).

This definition is a generalization of the one in [28, 29]; moreover, we have replaced the bundle E by $E \otimes \mathcal{D}^{1/2}$.
Let us denote by $\mathcal{C}^{-\infty}(Y; E) := \mathcal{C}_c^\infty(Y, E' \otimes \Omega)'$ the space of distributions on a smooth manifold Y with coefficients in the bundle E; here E' is the dual bundle of E, and $\Omega = \Omega(Y)$ is the bundle of 1-densities on Y.
We fix from now on a Hermitian metric on E, and we use it to identify E', the dual of E, with $\overline{E}$, the *complex conjugate* of E. Of course, $\overline{E} \simeq E$.
For a family of pseudodifferential operators $P = (P_x, x \in \mathcal{G}^{(0)})$ acting on $\mathcal{G}_x$, let us denote by K_x the distributional kernel of P_x

$$\begin{aligned} & K_x \in \mathcal{C}^{-\infty}(\mathcal{G}_x \times \mathcal{G}_x; r_1^*(E \otimes \mathcal{D}^{1/2}) \otimes r_2^*(E \otimes \mathcal{D}^{1/2})' \otimes \Omega_2) \\ \simeq\ & \mathcal{C}^{-\infty}(\mathcal{G}_x \times \mathcal{G}_x; r_1^*(E \otimes \mathcal{D}^{1/2}) \otimes r_2^*(E \otimes \mathcal{D}^{1/2})). \end{aligned} \tag{6}$$

Here Ω_2 is the pull-back of the bundle of vertical densities $r^*(\mathcal{D})$ on $\mathcal{G}_x$ to $\mathcal{G}_x \times \mathcal{G}_x$ via the second projection. These distributional kernels are obtained using Schwartz' kernel theorem. Let us denote

$$\mathrm{END}(E) := r^*(E \otimes \mathcal{D}^{1/2}) \otimes d^*(E^* \otimes \mathcal{D}^{1/2}).$$

The space of kernels of pseudodifferential operators on $\mathcal{G}_x$ is denoted, as usual, by $I^m(\mathcal{G}_x \times \mathcal{G}_x, \mathcal{G}_x; \mathrm{END}(E))$ where $\mathcal{G}_x \hookrightarrow \mathcal{G}_x \times \mathcal{G}_x$ is embedded as the diagonal [8].
Let $\mu_1(g', g) = g'g^{-1}$. We define the support of the operator P to be

$$\text{supp }(P) = \overline{\cup_x \mu_1(\text{ supp }(K_x))} \subset \mathcal{G}. \tag{7}$$

The family $P = (P_x, x \in \mathcal{G}^{(0)})$ is called *uniformly supported* if, and only if, supp (P) is a compact subset of $\mathcal{G}^{(1)}$. The composition PQ of two uniformly supported families of operators $P = (P_x, x \in M)$ and $Q = (Q_x, x \in M)$ on $\mathcal{G}^{(1)}$ is defined by pointwise multiplication:

$$PQ = (P_x Q_x, x \in M).$$

Since

$$\text{supp }(PQ) \subset \text{ supp }(P) \text{ supp }(Q),$$

the product is also uniformly supported. The action of a family $P = (P_x)$ on sections of $r^*(E)$ is also defined pointwise, as follows. For any smooth section $f \in \mathcal{C}^{\infty,0}(\mathcal{G}, r^*(E))$, let f_x be the restriction $f|_{\mathcal{G}_x}$. If each f_x has compact support and $P = (P_x, x \in \mathcal{G}^{(0)})$ is a family of ordinary pseudodifferential operators, then we define Pf such that its restrictions to the fibers $\mathcal{G}_x$ are given by

$$(Pf)_x = P_x(f_x).$$

Let $\mathcal{G}$ be a continuous family groupoid. The space of order m, invariant, *uniformly* supported pseudodifferential operators on $\mathcal{G}$, acting on sections of the vector bundle E will be denoted by $\Psi^{m,0}(\mathcal{G}; E)$. For the trivial bundle $E = M \times \mathbb{C}$, we write $\Psi^{m,0}(\mathcal{G}; E) = \Psi^{m,0}(\mathcal{G})$. Furthermore, let $\Psi^{\infty,0}(\mathcal{G}; E) = \cup_{m\in\mathbb{Z}}\Psi^{m,0}(\mathcal{G}; E)$ and $\Psi^{-\infty,0}(\mathcal{G}; E) = \cap_{m\in\mathbb{Z}}\Psi^{m,0}(\mathcal{G}; E)$.
Thus, $P \in \Psi^{m,0}(\mathcal{G}; E)$ is actually a continuous family $P = (P_x, x \in \mathcal{G}^{(0)})$ of ordinary pseudodifferential operators. It is sometimes more convenient to consider the convolution kernels of these operators. Let $K_x(g, g')$ be the Schwartz kernel of P_x, a distribution on $\mathcal{G}_x \times \mathcal{G}_x$, as above; thus $(K_x)_{x\in M}$ is a continuous family, equivariant with respect to the action of $\mathcal{G}$:

$$\forall g_0 \in \mathcal{G}, \forall g \in \mathcal{G}_{r(g_0)}, \forall g' \in \mathcal{G}_{d(g_0)}, K_{r(g_0)}(g, g'g_0^{-1}) = K_{d(g_0)}(gg_0, g').$$

We can therefore define

$$k_P(g) = K_{d(g)}(g, d(g)) \tag{8}$$

which is a distribution on $\mathcal{G}$, *i.e.* a continuous linear form on $\mathcal{C}^{\infty,0}(\mathcal{G})$.
We denote by $I_c^{m,0}(\mathcal{G}, M; \mathrm{END}(E))$ the space of distributions k on $\mathcal{G}$ such that for any $x \in \mathcal{G}$ the distribution defined by

$$K_x(g, g') := k(g{g'}^{-1})$$

is a pseudodifferential kernel on $\mathcal{G}_x \times \mathcal{G}_x$, *i.e.* $K_x \in I^m(\mathcal{G}_x \times \mathcal{G}_x, \mathcal{G}_x; \mathrm{END}(E))$, and the family $(K_x)_{x\in M}$ is continuous. Let us denote by $\mathcal{S}_{\mathrm{cl}}^{m,0}(A^*(\mathcal{G}); \mathrm{End}(E))$ the space of continuous families $(a_x)_{x\in M}$ with $a_x \in \mathcal{S}_{\mathrm{cl}}^{m}(T_x\mathcal{G}_x; \mathrm{End}(E))$. For $P \in \Psi^{m,0}(\mathcal{G}; E)$, let

$$\sigma_m(P) \in \mathcal{S}_{\mathrm{cl}}^{m,0}(A^*(\mathcal{G}); \mathrm{End}(E))/\mathcal{S}_{\mathrm{cl}}^{m-1,0}(A^*(\mathcal{G}); \mathrm{End}(E)) \tag{9}$$

be defined by

$$\sigma_m(P)(\xi) = \sigma_m(P_x)(\xi),$$

if $\xi \in A(\mathcal{G})_x$. Note that the principal symbol of P determines the principal symbols of the individual operators P_x by the invariance with respect to right translations. More precisely, we have $\sigma_m(P_x) = r^*(\sigma(P))|_{T^*\mathcal{G}_x}$. As in the classical situation, it is convenient to identify the space on the right hand side in (9) with sections of a certain bundle $\mathcal{P}_m$. Let $S^*(\mathcal{G})$ be the cosphere bundle of $\mathcal{G}$, that is, $S^*(\mathcal{G}) = (A^*(\mathcal{G}) \setminus 0)/\mathbb{R}_+^*$ is the quotient of the vector bundle $A^*(\mathcal{G})$ with the zero section removed by the action of positive real numbers. Let $\mathcal{P}_m$ be the bundle on $S^*(\mathcal{G})$ whose sections are $\mathcal{C}^{\infty,0}$-functions f on $A^*(\mathcal{G}) \setminus 0$ that are homogeneous of degree m. Then the quotient space in Equation (9) can certainly be identified with the space $\mathcal{C}_c^{\infty,0}(S^*(\mathcal{G}), \mathrm{End}(E) \otimes \mathcal{P}_m)$, thus, we have $\sigma_m(P) \in \mathcal{C}_c^{\infty,0}(S^*(\mathcal{G}), \mathrm{End}(E) \otimes \mathcal{P}_m)$. The following theorem is a generalization of results from [26, 28, 29] and extends some well-known properties of the calculus of pseudodifferential operators on smooth manifolds.

Theorem 1 *Let $\mathcal{G}$ be a continuous family groupoid. Then*

$$\Psi^{m,0}(\mathcal{G}; E)\Psi^{m',0}(\mathcal{G}; E) \subset \Psi^{m+m',0}(\mathcal{G}; E),$$

$\sigma_{m+m'}(PQ) = \sigma_m(P)\sigma_{m'}(Q)$, and the map $P \mapsto k_P$ establishes an isomorphism $\Psi^{m,0}(\mathcal{G}; E) \ni P \longmapsto k_P \in I_c^{m,0}(\mathcal{G}, M; \mathrm{END}(E))$.
Moreover, the principal symbol σ_m gives rise to a short exact sequence

$$0 \to \Psi^{m-1,0}(\mathcal{G}; E) \to \Psi^{m,0}(\mathcal{G}; E) \xrightarrow{\sigma_m} \mathcal{C}_c^{\infty,0}(S^*(\mathcal{G}), \mathrm{End}(E) \otimes \mathcal{P}_m) \to 0. \quad (10)$$

It follows that $\Psi^{-\infty,0}(\mathcal{G}; E)$ is a two-sided ideal of $\Psi^{\infty,0}(\mathcal{G}; E)$. Another consequence of the above theorem is that we obtain the asymptotic completeness of the spaces $\Psi^{m,0}(\mathcal{G})$: If $P_k \in \Psi^{m,0}(\mathcal{G})$ is a sequence of operators such that the order of $P_k - P_{k+1}$ converges to $-\infty$ and the kernels k_{P_k} have support contained in a fixed compact set, then there exists $P \in \Psi^{m,0}(\mathcal{G})$, such that the order of $P - P_k$ converges to $-\infty$.
Using an observation from [29], we can assume that E is the trivial one dimensional bundle $M \times \mathbb{C}$. Indeed, we can realize E as a sub-bundle of a trivial bundle $M \times \mathbb{C}^n$, with the induced metric. Let e be the orthogonal projection onto E, which is therefore an $n \times n$ matrix, and hence it is a multiplier of $\Psi^{\infty,0}(\mathcal{G})$. Then $\Psi^{m,0}(\mathcal{G}; E) \simeq eM_n(\Psi^{m,0}(\mathcal{G}))e$, for each m, and for $m = \infty$, it is an isomorphism of algebras. In the last section we shall consider operators between *different* vector bundles. They can be treated similarly, as follows. Suppose E_0 and E_1 are two vector bundles on M and (P_x), $x \in M$, is a family of pseudodifferential operators $P_x \in \Psi^{m,0}(\mathcal{G}_x; r^*(E_0), r^*(E_1))$ satisfying the usual conditions: (P_x) is a continuous family of invariant, uniformly supported operators. The set of such operators will be denoted $\Psi^{m,0}(\mathcal{G}; E_0, E_1)$. It can be defined using the spaces $\Psi^{m,0}(\mathcal{G})$ by the following procedure. Choose embeddings of E_0 and E_1 into the trivial bundle $\mathbb{C}^N$ such that E_i can be identified with the range of a projection $e_i \in M_N(\mathcal{C}^\infty(M)$. Then $\Psi^{m,0}(\mathcal{G}; E_0, E_1) \simeq e_1 M_N(\Psi^{m,0}(\mathcal{G}))e_0$, as filtered vector spaces.
Sometimes it is convenient to get rid of the density bundles in the definition of various algebras associated to a continuous family groupoid. This can easily be achieved as follows. The bundle $\mathcal{D}$ is trivial, but not canonically. Choose a positive, nowhere vanishing section ω of $\mathcal{D}$. Its pull-back, denoted $r^*(\omega)$, restricts to a nowhere vanishing density on each fiber $\mathcal{G}_x$, and hence defines

a smooth measure μ_x, with support $\mathcal{G}_x$. From the definition, we see that the measures μ_x are invariant with respect to right translations.
The choice of ω as above gives rise to an isomorphism

$$\Phi_\omega : \Psi^{-\infty,0}(\mathcal{G}) \to \mathcal{C}_c^{\infty,0}(\mathcal{G}),$$

such that the convolution product becomes

$$f_0 * f_1(g) = \int_{\mathcal{G}_x} f_0(gh^{-1}) f_1(h) d\mu_x(h),$$

where $g, h \in \mathcal{G}$ and $x = d(g)$. If we change ω to $\phi^{-2}\omega$, then we get $\Phi_{\phi^{-2}\omega}(f)(g) = \phi(r(g))\Phi_\omega(f)(g)\phi(d(g))$, and μ_x changes to $(\phi \circ r)^{-1}\mu_x$. See Ramazan's thesis [36] for the question of the existence of Haar systems on groupoids.

2 RESTRICTION MAPS

Let $A \subset M$ and let $\mathcal{G}_A := d^{-1}(A) \cap r^{-1}(A)$. Then $\mathcal{G}_A$ is a groupoid with units A, called the *reduction of* $\mathcal{G}$ *to* A. An *invariant* subset $A \subset M$ is a subset such that $d(g) \in A$ implies $r(g) \in A$. Then $\mathcal{G}_A = d^{-1}(A) = r^{-1}(A)$.
In this section we establish some elementary properties of the *restriction map*

$$\mathcal{R}_Y : \Psi^{\infty,0}(\mathcal{G}) \to \Psi^{\infty,0}(\mathcal{G}_Y)\,,$$

associated to a closed, invariant subset $Y \subset M$. Then we study the properties of these indicial maps. For algebras acting on sections of a smooth, hermitian bundle E on M, this morphism becomes

$$\mathcal{R}_Y : \Psi^{\infty}(\mathcal{G}; E) \to \Psi^{\infty,0}(\mathcal{G}_Y; E|_Y).$$

Then Y is the space of units of the reduction $\mathcal{G}_Y$ and $d^{-1}(Y) = r^{-1}(Y)$ is the space of arrows of $\mathcal{G}_Y$, hence

$$\mathcal{G}_Y = (Y, d^{-1}(Y))$$

is a continuous family groupoid for the structural maps obtained by restricting the structural maps of $\mathcal{G}$ to $\mathcal{G}_Y$. As before, we identify the groupoid $\mathcal{G}_Y$ with its set of arrows $d^{-1}(Y)$.
Clearly, $\mathcal{G}_Y = d^{-1}(Y)$ is a disjoint union of d-fibers $\mathcal{G}_x$, so if $P = (P_x, x \in \mathcal{G}^{(0)})$ is a pseudodifferential operator on $\mathcal{G}$, we can restrict P to $d^{-1}(Y)$ and obtain

$$\mathcal{R}_Y(P) := (P_x, x \in Y),$$

which is a family of operators acting on the fibers of $d : \mathcal{G}_Y = d^{-1}(Y) \to Y$ and satisfies all the conditions necessary to define an element of $\Psi^{\infty,0}(\mathcal{G}_Y)$. This leads to a map

$$\mathcal{R}_Y = \mathcal{R}_{Y,M} : \Psi^{\infty,0}(\mathcal{G}) \to \Psi^{\infty,0}(\mathcal{G}_Y), \tag{11}$$

which is easily seen to be an algebra morphism.
If $Z \subset Y$ are two closed invariant subsets of M, we also obtain a map

$$\mathcal{R}_{Z,Y} : \Psi^{\infty,0}(\mathcal{G}_Y) \to \Psi^{\infty,0}(\mathcal{G}_Z), \tag{12}$$

defined analogously. The following proposition summarizes the properties of the maps $\mathcal{R}_Y$.

Proposition 1 *Let $Y \subset M$ be a closed, invariant subset. Using the notation above, we have:*
(i) *The convolution kernel $k_{\mathcal{R}_Y(P)}$ of $\mathcal{R}_Y(P)$ is the restriction of k_P to $d^{-1}(Y)$.*
(ii) *The map $\mathcal{R}_Y$ is an algebra morphism with $\mathcal{R}_Y(\Psi^{m,0}(\mathcal{G})) = \Psi^{m,0}(\mathcal{G}_Y)$ and*

$$\mathcal{C}_0(M \setminus Y) I_c^{m,0}(\mathcal{G}, M; \mathrm{END}(E)) \subseteq \ker(\mathcal{R}_Y).$$

(iii) *If $Z \subset Y$ is a closed invariant submanifold, then $\mathcal{R}_Z = \mathcal{R}_{Z,Y} \circ \mathcal{R}_Y$.*
(iv) *If $P \in \Psi^{m,0}(\mathcal{G})$, then $\sigma_m(\mathcal{R}_Y(P)) = \sigma_m(P)$ on $S^*(\mathcal{G}_Y) = S^*(\mathcal{G})|_Y$.*

Proof: The definition of k_P, equation (8), is compatible with restrictions, and hence (i) follows from the definitions.
The surjectivity of $\mathcal{R}_Y$ follows from the fact that the restriction

$$I_c^{m,0}(\mathcal{G}, M) \to I_c^{m,0}(\mathcal{G}_Y, Y)$$

is surjective. Finally, (iii) and (iv) follow directly from the definitions. □

Consider now an *open* invariant subset $\mathcal{O} \subset M$, instead of a *closed* invariant subset $Y \subset M$. Then we still can consider the reduction $\mathcal{G}_{\mathcal{O}} = (\mathcal{O}, d^{-1}(\mathcal{O}))$, which is also a continuous family groupoid, and hence we can define $\Psi^{\infty,0}(\mathcal{G}_{\mathcal{O}})$. If moreover $\mathcal{O}$ is the complement of a closed invariant subset $Y \subset M$, then we can extend a family $(P_x) \in \Psi^{\infty,0}(\mathcal{G}_{\mathcal{O}})$ to be zero outside $\mathcal{O}$, which gives an inclusion $\Psi^{\infty,0}(\mathcal{G}_{\mathcal{O}}) \subset \Psi^{\infty,0}(\mathcal{G})$. Clearly, $\Psi^{\infty,0}(\mathcal{G}_{\mathcal{O}}) \subset \ker(\mathcal{R}_Y)$, but they are not equal in general, although we shall see later on that the norm closures of these algebras are the same.

3 Continuous representations

As in the classical case of pseudodifferential operators on a compact manifold (without corners) M, the algebra $\Psi^{0,0}(\mathcal{G})$ of operators of order 0 acts by bounded operators on various Hilbert spaces. It is convenient, in what follows, to regard these actions from the point of view of representation theory. Unlike the classical case, however, there are many (non-equivalent, irreducible, bounded, and infinite dimensional) representations of the algebra $\Psi^{0,0}(\mathcal{G})$, in general. The purpose of this section is to introduce the class of representations in which we are interested and to study some of their properties. A consequence of our results is that in order to construct and classify bounded representations of $\Psi^{0,0}(\mathcal{G})$, it is essentially enough to do this for $\Psi^{-\infty,0}(\mathcal{G})$.

Let $\mathcal{D}^{1/2}$ be the square root of the density bundle

$$\mathcal{D} = | \wedge^n A(\mathcal{G})|,$$

as before. If $P \in \Psi^{m,0}(\mathcal{G})$ consists of the family $(P_x, x \in M)$, then each P_x acts on

$$V_x = C_c^\infty(\mathcal{G}_x; r^*(\mathcal{D}^{1/2})).$$

Since $r^*(\mathcal{D}^{1/2}) = \Omega^{1/2}_{\mathcal{G}_x}$ is the bundle of half densities on $\mathcal{G}_x$, we can define a hermitian inner product on V_x, and hence also the formal adjoint P_x^* of P_x. The following lemma establishes that $\Psi^{\infty,0}(\mathcal{G})$ is stable with respect to taking (formal) adjoints. (The formal adjoint P^* of a pseudodifferential operator P is the pseudodifferential operator that satisfies $(P^*\phi, \psi) = (\phi, P\psi)$, for all *compactly supported,* smooth 1/2-densities ϕ and ψ.) Let

$$\mathrm{END}(\mathcal{D}^{1/2}) := r^*(\mathcal{D}^{1/2}) \otimes d^*(\mathcal{D}^{1/2}).$$

Lemma 1 *If $P = (P_x, x \in M) \in \Psi^{m,0}(\mathcal{G})$, then $(P_x^*, x \in M) \in \Psi^{m,0}(\mathcal{G})$. Moreover,*

$$k_{P^*}(g) = \overline{k_P}(g^{-1}) \in I_c^{\infty,0}(\mathcal{G}, M; \mathrm{END}(\mathcal{D}^{1/2})), \tag{13}$$

and hence $\sigma_m(P^*) = \overline{\sigma_m(P)}$.

Proof: It follows directly from the invariance of the family P_x that the family P_x^* is invariant. The support $\operatorname{supp}(P^*) = \operatorname{supp}(k_{P^*}) \subset \mathcal{G}$ of the reduced kernel k_{P^*} is $\iota(\operatorname{supp}(P)) = \{g^{-1}, g \in \operatorname{supp}(P)\}$, also a compact set. Since the adjoint of a continuous family is a continuous family, we obtain that $(P_x^*)_{x\in M}$ defines an element of $\Psi^{m,0}(\mathcal{G})$.
We now obtain the explicit formula (13) for the kernel of k_{P^*} stated above. Suppose first that $P \in \Psi^{-n-1,0}(\mathcal{G})$. Then the convolution kernel k_P of P is a compactly supported continuous section of $\mathrm{END}(\mathcal{D}^{1/2})$, and the desired formula follows by direct computation. In general, we can choose $P_m \in \Psi^{-n-1,0}(\mathcal{G})$ such that $k_{P_m} \to k_P$ as distributions. Then $k_{P_m^*} \to k_{P^*}$ as distributions also, which gives (13) in general. $\square$

Having defined the involution $*$ on $\Psi^{\infty,0}(\mathcal{G})$, we can now introduce the representations we are interested in. Fix $m \in \{0\} \cup \{\pm\infty\}$ and let $\mathcal{H}_0$ be a dense subspace of a Hilbert space.

Definition 3 *A bounded $*$-representation of $\Psi^{m,0}(\mathcal{G})$ on the inner product space $\mathcal{H}_0$ is a morphism $\rho : \Psi^{m,0}(\mathcal{G}) \to \mathrm{End}(\mathcal{H}_0)$ satisfying*

$$(\rho(P^*)\xi, \eta) = (\xi, \rho(P)\eta) \tag{14}$$

and, if $P \in \Psi^{0,0}(\mathcal{G})$,

$$\|\rho(P)\xi\| \leq C_P\|\xi\|, \tag{15}$$

for all $\xi, \eta \in \mathcal{H}_0$, where $C_P > 0$ is independent of ξ. One defines similarly bounded $*$-representations of the algebras $\Psi^{m,0}(\mathcal{G}; E)$.

Note that for $m > 0$ and $P \in \Psi^{m,0}(\mathcal{G})$, $\rho(P)$ does not have to be bounded, even if ρ is bounded. However, $\rho(P)$ will be a densely defined operator with $\rho(P^*) \subset \rho(P)^*$.

Theorem 2 *Let $\mathcal{H}$ be a Hilbert space and let $\rho : \Psi^{-\infty,0}(\mathcal{G}; E) \to \mathrm{End}(\mathcal{H})$ be a bounded $*$-representation. Then ρ extends to a bounded $*$-representation of $\Psi^{0,0}(\mathcal{G}; E)$ on $\mathcal{H}$ and to a bounded $*$-representation of $\Psi^{\infty,0}(\mathcal{G}; E)$ on the subspace $\mathcal{H}_0 := \rho(\Psi^{-\infty,0}(\mathcal{G}; E))\mathcal{H}$ of $\mathcal{H}$. Moreover, any extension of ρ to a $*$-representation of $\Psi^{0,0}(\mathcal{G}; E)$ is bounded and is uniquely determined provided that $\mathcal{H}_0$ is dense in $\mathcal{H}$.*

Proof: We assume that $E = \mathbb{C}$ is a trivial line bundle, for simplicity. The general case can be treated in exactly the same way. We first address the question of the existence of the extension ρ with the desired properties. Let $P \in \Psi^{m,0}(\mathcal{G})$. If $\mathcal{H}_0$ is not dense in $\mathcal{H}$, we let $\rho(P) = 0$ on the orthogonal complement of $\mathcal{H}_0$. Thus, in order to define $\rho(P)$, we may assume that $\mathcal{H}_0$ is dense in $\mathcal{H}$.
On $\mathcal{H}_0$ we let

$$\rho(P)\xi = \rho(PQ)\eta,$$

if $\xi = \rho(Q)\eta$, for some $Q \in \Psi^{-\infty,0}(\mathcal{G})$ and $\eta \in \mathcal{H}$; however, we need to show that this is well-defined and that it gives rise to a bounded operator for each $P \in \Psi^{0,0}(\mathcal{G})$. Thus, we need to prove that $\sum_{k=1}^N \rho(PQ_k)\xi_k = 0$, if $P \in \Psi^{0,0}(\mathcal{G})$ and $\sum_{k=1}^N \rho(Q_k)\xi_k = 0$, for some $Q_k \in \Psi^{-\infty,0}(\mathcal{G})$ and $\xi_k \in \mathcal{H}$.
We will show that, for each $P \in \Psi^{0,0}(\mathcal{G})$, there exists a constant $C_P > 0$ such that

$$\|\sum_{k=1}^N \rho(PQ_k)\xi_k\| \leq C_P \|\sum_{k=1}^N \rho(Q_k)\xi_k\|. \tag{16}$$

This will prove that $\rho(P)$ is well defined and bounded at the same time. To this end, we use an argument of [8]. Let $M \geq |\sigma_0(P)| + 1$, $M \in \mathbb{R}$, and let

$$b = (M^2 - |\sigma_0(P)|^2)^{1/2}. \tag{17}$$

Then $b - M$ is in $\mathcal{C}_c^{\infty,0}(S^*(\mathcal{G}))$, and it follows from Theorem 1 that we can find $Q_0 \in \Psi^{0,0}(\mathcal{G})$ such that $\sigma_0(Q_0) = b - M$. Let $Q = Q_0 + M$. Using again Theorem 1, we obtain, for

$$R = M^2 - P^*P - Q^*Q \in \Psi^{0,0}(\mathcal{G}),$$

that

$$\sigma_0(R) = \sigma_0(M^2 - P^*P - Q^*Q) = 0,$$

and hence $R \in \Psi^{-1,0}(\mathcal{G})$. We can also assume that Q is self-adjoint. We claim that we can choose Q so that R is of order $-\infty$. Indeed, by the asymptotic completeness of the space of pseudodifferential operators, it is enough to find Q such that R is of arbitrary low order and has principal symbol in a fixed compact set. So, assume that we have found Q such that R has order $-m$. Then, if we let $Q_1 = Q + (RS + SR)/4$, where S is a self-adjoint parametrix of Q (*i.e.,* $SQ - 1$ and $QS - 1$ have negative order), then $R_1 = M^2 - P^*P - Q_1^2$ has lower order.

So, assume that R has order $-\infty$, and let

$$\xi = \sum_{k=1}^{N} \rho(Q_k)\xi_k, \quad \eta = \sum_{k=1}^{N} \rho(PQ_k)\xi_k, \quad \text{and} \quad \zeta = \sum_{k=1}^{N} \rho(QQ_k)\xi_k. \tag{18}$$

Then we have

$$\begin{aligned}
&(\eta, \eta) \qquad (19)\\
= &\sum_{j,k=1}^{N} (\rho(Q_k^* P^* P Q_j)\xi_j, \xi_k)\\
= &\sum_{j,k=1}^{N} \big(M^2(\rho(Q_k Q_j)\xi_j, \xi_k) - (\rho(Q_k^* Q^* Q Q_j)\xi_j, \xi_k) - (\rho(Q_k^* R Q_j)\xi_j, \xi_k)\big)\\
= &M^2\|\xi\|^2 - \|\zeta\|^2 - (\rho(R)\xi, \xi) \leq (M^2 + \|\rho(R)\|)\|\xi\|^2.
\end{aligned}$$

The desired representation of $\Psi^{0,0}(\mathcal{G})$ on $\mathcal{H}$ is obtained by extending $\rho(P)$ by continuity to $\mathcal{H}$.

To extend ρ further to $\Psi^{\infty,0}(\mathcal{G})$, we proceed similarly: we want

$$\rho(P)\rho(Q)\xi = \rho(PQ)\xi,$$

for $P \in \Psi^{\infty,0}(\mathcal{G})$ and $Q \in \Psi^{-\infty,0}(\mathcal{G})$. Let ξ and η be as in Equation (18). We need to prove that $\eta = 0$ if $\xi = 0$. Now, because $\mathcal{H}_0$ is dense in $\mathcal{H}$, we can find T_j in A_ρ the norm closure of $\rho(\Psi^{-\infty,0}(\mathcal{G}))$ and $\eta_j \in \mathcal{H}$ such that $\eta = \sum_{j=1}^{N} T_j \eta_j$. Choose an approximate unit u_α of the C^*-algebra A_ρ, then $u_\alpha T_j \to T_j$ (in the sense of generalized sequences). We can replace then the generalized sequence (net) u_α by a subsequence, call it u_m such that $u_m T_j \to T_j$, as $m \to \infty$. By density, we may assume $u_m = \rho(R_m)$, for some $R_m \in \Psi^{-\infty,0}(\mathcal{G})$. Consequently, $\rho(R_m)\eta \to \eta$, as $m \to \infty$. Then

$$\eta = \lim \sum_{k=1}^{N} \rho(R_m)\rho(PQ_k)\xi_k = \lim \sum_{k=1}^{N} \rho(R_m P)\rho(Q_k)\xi_k = 0,$$

because $R_m P \in \Psi^{-\infty,0}(\mathcal{G})$.

We now consider the uniqueness of the extension of ρ to $\Psi^{0,0}(\mathcal{G})$. First, the uniqueness of this extension acting on the closure of $\mathcal{H}_0 = \rho(\Psi^{-\infty,0}(\mathcal{G}))\mathcal{H}$ is immediate. This implies the boundedness of any extension of ρ to $\Psi^{0,0}(\mathcal{G})$ if $\mathcal{H}_0$ is dense.

In general, a completely similar argument applies to give that on the orthogonal complement of $\mathcal{H}_0$ any extension of ρ to $\Psi^{0,0}(\mathcal{G})$ factors through a representation of $\Psi^{0,0}(\mathcal{G})/\Psi^{-1,0}(\mathcal{G})$, and hence it is again bounded. $\square$

Let $x \in M$, then the *regular representation* π_x associated to x is the natural representation of $\Psi^{\infty,0}(\mathcal{G})$ on $C_c^\infty(\mathcal{G}_x; r^*(\mathcal{D}^{1/2}))$, that is $\pi_x(P) = P_x$. (Because $C_c^\infty(\mathcal{G}_x; r^*(\mathcal{D}^{1/2}))$ consists of half-densities, it has a natural inner product and a natural Hilbert space completion.) As for locally compact groups, the regular representation(s) will play an important role in our study and are one of the main sources of examples of bounded $*$-representations.

Assume that M is connected, so that all the manifolds $\mathcal{G}_x$ have the same dimension n. We now proceed to define a Banach norm on $\Psi^{-n-1,0}(\mathcal{G})$. This norm depends on the choice of a trivialization of the bundle of densities $\mathcal{D}$, which then gives rise to a right invariant system of measures μ_x on $\mathcal{G}_x$. Indeed, for $P \in \Psi^{-n-1,0}(\mathcal{G})$, we use the chosen trivialization of $\mathcal{D}$ to identify k_P, which is a priori a continuous family of distributions, with a compactly supported, $\mathcal{C}^{\infty,0}$-function on $\mathcal{G}$, still denoted k_P. We then define

$$\|P\|_1 = \sup_{x \in M} \left\{ \int_{\mathcal{G}_x} |k_P(g^{-1})| d\mu_x(g), \int_{\mathcal{G}_x} |k_P(g)| d\mu_x(g) \right\}. \tag{20}$$

If we change the trivialization of $\mathcal{D}$, then we obtain a new norm $\|P\|_1'$, which is however related to the original norm by $\|P\|_1' = \|\phi P \phi^{-1}\|_1$, for some continuous function $\phi > 0$ on M. This shows that the completions of $\Psi^{-n-1,0}(\mathcal{G})$ with respect to $\| \ \|_1$ and $\| \ \|_1'$ are isomorphic.

Corollary 1 *Let $x \in M$. Then the regular representation π_x is a bounded $*$-representation of $\Psi^{0,0}(\mathcal{G})$ such that $\|\pi_x(P)\| \leq \|P\|_1$, if $P \in \Psi^{-n-1,0}(\mathcal{G})$.*

Proof: Suppose first that $P \in \Psi^{-n-1,0}(\mathcal{G})$. Then the convolution kernel k_P of P, which is a priori a distribution, turns out in this case to be a compactly supported continuous section of

$$\mathrm{END}(\mathcal{D}^{1/2}) = r^*(\mathcal{D}^{1/2}) \otimes d^*(\mathcal{D}^{1/2}).$$

Choose a trivialization of $\mathcal{D}$, which then gives trivializations of $d^*(\mathcal{D})$ and $r^*(\mathcal{D})$. Also denote by μ_x the smooth measure on $\mathcal{G}_x$ obtained from the trivialization of $\Omega_{\mathcal{G}_x} = r^*(\mathcal{D})$, so that $L^2(\mathcal{G}_x, r^*(\mathcal{D}^{1/2}))$ identifies with $L^2(\mathcal{G}_x, \mu_x)$. Using the same trivialization, we identify k_P with a continuous, compactly supported function.

The action of P_x on $C_c^\infty(\mathcal{G}_x)$ is given then by

$$P_x u(g) = \int_{\mathcal{G}_x} k_P(gh^{-1}) u(h) d\mu_x(h).$$

Let $y = r(g)$, $x = d(g) = d(h)$, and $z = r(h)$. Then the integrals

$$\int_{\mathcal{G}_x} |k_P(gh^{-1})| d\mu_x(h) = \int_{\mathcal{G}_y} |k_P(h^{-1})| d\mu_y(h),$$

and

$$\int_{\mathcal{G}_x} |k_P(gh^{-1})| d\mu_x(g) = \int_{\mathcal{G}_z} |k_P(g)| d\mu_z(g)\,,$$

are uniformly bounded by a constant M that depends only on k_P and the trivialization of $\mathcal{D}$, but not on $g \in \mathcal{G}_x$ or $h \in \mathcal{G}_x$. A well-known estimate implies then that P_x is bounded on $L^2(\mathcal{G}_x, \mu_x)$ with

$$\|\pi_x(P)\| = \|P_x\| \leq M.$$

Then Theorem 2 gives the result. □

Note that the boundedness of order zero operators depends essentially on the fact that we use *uniformly supported* operators. For properly supported operators this is not true, as seen by considering the multiplication operator with an unbounded function $f \in \mathcal{C}(M)$.
Define now the *reduced norm* of P by

$$\|P\|_r = \sup_x \|\pi_x(P)\| = \sup_x \|P_x\|, \quad x \in M.$$

Then $\|P\|_r$ is the norm of the operator $\pi(P) := \prod \pi_x(P)$ acting on the Hilbert space direct sum

$$l^2 - \bigoplus_{x \in M} L^2(\mathcal{G}_x).$$

The Hilbert space l^2-direct sum space $l^2 - \oplus_{x \in M} L^2(\mathcal{G}_x)$ is called the space of the *total regular* representation. Also, let

$$\|P\| = \sup_\rho \|\rho(P)\|,$$

where ρ ranges through all bounded $*$-representations ρ of $\Psi^{0,0}(\mathcal{G})$ such that

$$\|\rho(P)\| \leq \|P\|_1,$$

for all $P \in \Psi^{-\infty,0}(\mathcal{G})$ and for some fixed choice of the measures μ_x corresponding to a trivialization of $\mathcal{D}$.
The following result shows, in particular, that we have $\|P\|_r \leq \|P\| < \infty$, for all $P \in \Psi^{0,0}(\mathcal{G})$, which is not clear a priori from the definition.

COROLLARY 2 *Let $P \in \Psi^{0,0}(\mathcal{G})$, then $\|P\|$ and $\|P\|_r$ are finite and we have the inequalities $\|\mathcal{R}_Y(P)\|_r \leq \|P\|_r$ and $\|\mathcal{R}_Y(P)\| \leq \|P\|$, for any closed invariant submanifold Y of M.*

PROOF: Consider the product representation $\pi = \prod_{x \in M} \pi_x$ of $\Psi^{-\infty,0}(\mathcal{G})$ acting on

$$\mathcal{H} := \prod L^2(\mathcal{G}_x; \mathcal{D}^{1/2}).$$

It follows from Corollary 1 that π is bounded. By Theorem 2, π is bounded on $\Psi^{0,0}(\mathcal{G})$. This shows that $\|P\|_r := \|\pi(P)\|$ is finite for all $P \in \Psi^{0,0}(\mathcal{G})$.
Moreover, we have

$$\|\mathcal{R}_Y(P)\|_r = \sup_{y \in Y} \|\pi_y(P)\| \leq \sup_{x \in M} \|\pi_x(P)\| = \|P\|_r.$$

The rest is proved similarly. □

Denote by $\mathfrak{A}(\mathcal{G})$ [respectively, by $\mathfrak{A}_r(\mathcal{G})$] the closure of $\Psi^{0,0}(\mathcal{G})$ in the norm $\| \ \|$ [respectively, in the norm $\| \ \|_r$]. Also, denote by $C^*(\mathcal{G})$ [respectively, by $C^*_r(\mathcal{G})$] the closure of $\Psi^{-\infty,0}(\mathcal{G})$ in the norm $\| \ \|$ [respectively, in the norm $\| \ \|_r$].
We also obtain an extension of the classical results on the boundedness of the principal symbol and of results on the distance of an operator to the regularizing ideal. In what follows, $S^*(\mathcal{G})$ denotes the space of rays in $A^*(\mathcal{G})$, as in Section 1. By choosing a metric on $A(\mathcal{G})$, we may identify $S^*(\mathcal{G})$ with the subset of vectors of length one in $A^*(\mathcal{G})$.

Corollary 3 *Let $P \in \Psi^{0,0}(\mathcal{G})$. Then the distance from P to $C^*_r(\mathcal{G})$ in $\mathfrak{A}(\mathcal{G})$ is $\|\sigma_0(P)\|_\infty$. Similarly,* $\operatorname{dist}(P, C^*(\mathcal{G})) = \|\sigma_0(P)\|_\infty$*, for all $P \in \Psi^{0,0}(\mathcal{G})$. Consequently, the principal symbol extends to continuous algebra morphisms $\mathfrak{A}_r(\mathcal{G}) \to \mathcal{C}_0(S^*(\mathcal{G}))$ and $\mathfrak{A}(\mathcal{G}) \to \mathcal{C}_0(S^*(\mathcal{G}))$ with kernels $C^*_r(\mathcal{G})$ and $C^*(\mathcal{G})$, respectively.*

Proof: Let $P = (P_x)$. Then, by classical results,

$$\|\sigma_0(P)\|_\infty = \sup_{x \in M} \|\sigma_0(P_x)\|_\infty \leq \sup_{x\in M} \|P_x\| = \|P\|_r \leq \|P\|.$$

This proves the first part of this corollary.
Consider now the morphism $\rho : \Psi^{0,0}(\mathcal{G}) \to \mathfrak{A}_r(\mathcal{G})/C^*_r(\mathcal{G})$. Then we proceed as in the proof of Theorem 2, but we take $M = \|\sigma_0(P)\|_\infty + \epsilon$ in the definition of b of Equation (17), where $\epsilon > 0$ is small but fixed. Since we may assume that $\mathfrak{A}_r(\mathcal{G})/C^*_r(\mathcal{G})$ is embedded in the algebra of bounded operators on a Hilbert space, we may apply the same argument as in the proof of Theorem 2, and construct $Q \in \Psi^{0,0}(\mathcal{G}) + \mathbb{C}1$ and $R \in \Psi^{-\infty,0}(\mathcal{G})$ such that $P^*P = M^2 - Q^*Q - R$. Then Equation (19) gives $\|\rho(P)\| \leq M$, because ρ vanishes on $\Psi^{-\infty,0}(\mathcal{G})$. □

We shall continue to denote by σ_0 the extensions by continuity of the principal symbol map $\sigma_0 : \Psi^{0,0}(\mathcal{G}) \to \mathcal{C}^{\infty,0}_c(S^*(\mathcal{G}))$. The above corollary extends to operators acting on sections of a vector bundle E, in an obvious way.
See [28] for a result related to Corollary 3. Another useful consequence is the following.

Corollary 4 *Using the above notation, we have that $\Psi^{-\infty,0}(\mathcal{G})$ is dense in $\Psi^{-1,0}(\mathcal{G})$ in the $\| \cdot \|$-norm, and hence $\Psi^{-1,0}(\mathcal{G}) \subset C^*(\mathcal{G})$ and $\Psi^{-1,0}(\mathcal{G}) \subset C^*_r(\mathcal{G})$.*

4 Invariant filtrations

Let $\mathcal{G}$ be a continuous family groupoid with space of units denoted by M. In order to obtain more insight into the structure of the algebras $\mathfrak{A}(\mathcal{G})$ and $\mathfrak{A}_r(\mathcal{G})$, we shall make certain assumptions on $\mathcal{G}$.

Definition 4 An invariant filtration $Y_0 \subset Y_1 \subset \ldots \subset Y_n = M$ *is an increasing sequence of closed invariant subsets of M.*

Fix now an invariant filtration $Y_0 \subset Y_1 \subset \ldots \subset Y_n = M$. For each k, we have restriction maps $\mathcal{R}_{Y_k}$ and we define ideals $\mathfrak{I}_k$ as follows:

$$\mathfrak{I}_k = \ker \mathcal{R}_{Y_{k-1}} \cap C^*(\mathcal{G}). \tag{21}$$

(by convention, we define $\mathfrak{I}_0 = C^*(\mathcal{G})$).
We shall also consider the "L^1-algebra" $L^1(\mathcal{G})$ associated to a groupoid $\mathcal{G}$; it is obtained as the completion of $\Psi^{-\infty,0}(\mathcal{G}) \simeq \mathcal{C}_c^{\infty,0}(\mathcal{G})$ (using a trivialization of the density bundle $\mathcal{D}$) in the $\| \ \|_1$-norm, defined in Equation (20). More precisely, $L^1(\mathcal{G})$ is the completion of $\mathcal{C}_c^{\infty,0}(\mathcal{G})$ in the algebra of bounded operators on

$$\ell^\infty - \bigoplus \big(L^1(\mathcal{G}_x) \oplus L^\infty(\mathcal{G}_x)\big),$$

so it is indeed an algebra. If $Y \subset M$ is invariant, then we obtain sequences

$$0 \to L^1(\mathcal{G}_{M \setminus Y}) \to L^1(\mathcal{G}) \to L^1(\mathcal{G}_Y) \to 0\,, \tag{22}$$

and

$$0 \to C^*(\mathcal{G}_{M \setminus Y}) \to C^*(\mathcal{G}) \to C^*(\mathcal{G}_Y) \to 0\,. \tag{23}$$

LEMMA 2 *The sequences* (22) *and* (23) *are exact.*

PROOF: The exactness of (22) follows from the fact that the two functions,

$$x \longmapsto \int_{\mathcal{G}_x} |f(g)| d\mu_x(g) \quad \text{and} \quad x \longmapsto \int_{\mathcal{G}_x} |f(g^{-1})| d\mu_x(g),$$

are continuous in x for $f \in L^1(\mathcal{G})$.
Indeed, to prove exactness in (22), let $f \in L^1(\mathcal{G})$ be a function that vanishes in $L^1(\mathcal{G}_Y)$. Then we can find $\phi_n \in \mathcal{C}_c^\infty(M \smallsetminus Y)$ such that $\|f - \phi_n f\|_1 < 1/n$, by the continuity of the above two function. Choose $f_n \in \mathcal{C}_c^{\infty,0}(\mathcal{G})$ such that $\|f_n - f\|_1 \to 0$, as $n \to \infty$. Then $\phi_n f_n \in \mathcal{C}_c^{\infty,0}(\mathcal{G}_{M \setminus Y})$ and $\|\phi_n f_n - f\|_1 \to 0$, as $n \to \infty$.
Let π be an irreducible representation of $C^*(\mathcal{G})$ that vanishes on $C^*(\mathcal{G}_{M \setminus Y})$. To prove the exactness of (23), it is enough to prove that π comes from a representation of $C^*(\mathcal{G}_Y)$. Now π vanishes on $L^1(\mathcal{G}_{M \setminus Y})$, and hence it induces a bounded $*$-representation of $L^1(\mathcal{G}_Y)$, by the exactness of (22). This proves the exactness of (23). □

The exactness of the second exact sequence was proved in [37], and is true in general for locally compact groupoids. It is worthwhile mentioning that the corresponding results for reduced C^*-algebras is not true (at least in the general setting of locally compact groupoids).
We then have the following generalization of some results from [9, 22, 26]:

THEOREM 3 *Let $\mathcal{G}$ be a continuous family groupoid with space of units M and $Y_0 \subset Y_1 \subset \ldots \subset Y_n = M$ be an invariant filtration. Then Equation* (21) *defines a composition series*

$$(0) \subset \mathfrak{I}_n \subset \mathfrak{I}_{n-1} \subset \ldots \subset \mathfrak{I}_0 \subset \mathfrak{A}(\mathcal{G})\,,$$

with not necessarily distinct ideals, such that $\mathfrak{I}_0$ *is the norm closure of* $\Psi^{-\infty,0}(\mathcal{G})$ *and* $\mathfrak{I}_k$ *is the norm closure of* $\Psi^{-\infty,0}(\mathcal{G}_{M\setminus Y_{k-1}})$*. The subquotients are determined by* $\sigma_0 : \mathfrak{A}_M/\mathfrak{I}_0 \xrightarrow{\sim} \mathcal{C}_0(S^*(\mathcal{G}))$*, and by*

$$\mathfrak{I}_k/\mathfrak{I}_{k+1} \simeq C^*(\mathcal{G}_{Y_k \setminus Y_{k-1}}) \,, \quad 0 \leq k \leq n \,.$$

The above theorem extends right away to operators acting on sections of a vector bundle E, the proof being exactly the same.
Proof: We have that $\mathfrak{I}_0$ is the closure of $\Psi^{-\infty,0}(\mathcal{G})$, by definition. By the Corollaries 3 and 4, $\mathfrak{I}_0$ is also the kernel of σ_0.
The rest of the theorem follows by applying Lemma 2 to the groupoids $\mathcal{G}_{M\setminus Y_{k-1}}$ and the closed subsets $Y_k \setminus Y_{k-1}$ of $M \setminus Y_{k-1}$, for all k. $\square$

The above theorem leads to a characterization of compactness and Fredholmness for operators in $\Psi^{0,0}(\mathcal{G})$, a question that was discussed also in [12, 26]. This generalizes the characterization of Fredholm operators in the "b-calculus" or one of its variants on manifolds with corners, see [19, 23]. Characterizations of compact and of Fredholm operators on manifolds with more complicated boundaries were obtained in [11, 17], see also [41].
Recall that the *product groupoid* with units X is the groupoid with set of arrows $X \times X$, so there exists exactly one arrow between any two points of X, and we have $(x, y)(y, z) = (x, z)$.

Theorem 4 *Suppose that, using the notation of the above theorem, the restriction of* $\mathcal{G}$ *to* $M\setminus Y_{n-1}$ *is the product groupoid, and that the regular representation* π_x *is injective on* $\mathfrak{A}(\mathcal{G})$ *(for one, and hence for all* $x \in M \setminus Y_{n-1}$*).*
(i) The algebra $\mathfrak{A}(\mathcal{G})$ *contains (an ideal isomorphic to) the algebra of compact operators acting on* $L^2(M\setminus Y_{n-1})$*, where on* $M\setminus Y_{n-1}$ *we consider the (complete) metric induced by a metric on* $A(\mathcal{G})$*.*
(ii) An operator $P \in \Psi^{0,0}(\mathcal{G})$ *is compact on* $L^2(M \setminus Y_{n-1})$ *if, and only if, the principal symbol* $\sigma_0(P)$ *vanishes, and* $\mathcal{R}_{Y_{n-1}}(P) = 0 \in \Psi^{0,0}(\mathcal{G}_{Y_{n-1}})$*.*
(iii) An operator $P \in \mathfrak{A}(\mathcal{G})$ *is Fredholm on* $L^2(M \setminus Y_{n-1})$ *if, and only if,* $\sigma_0(P)(\xi)$ *is invertible for all* $\xi \in S^*(\mathcal{G})$ *(which can happen only when* M *is compact) and* $\mathcal{R}_{Y_{n-1}}(P)$ *is invertible in* $\mathfrak{A}(\mathcal{G}_{Y_{n-1}})$*.*
In (ii) and (iii), we may assume, more generally, that we have $P \in \mathfrak{A}(\mathcal{G})$ *or* $P \in M_N(\mathfrak{A}(\mathcal{G}))$*.*

The above theorem extends right away to operators acting on sections of a vector bundle E, the proof being exactly the same. If the representation(s) π_x, $x \in M \smallsetminus Y_{n-1}$, are not injective, then the above theorem gives only sufficient conditions for an operator P as above to be compact or Fredholm.
Proof: First we need to prove the following lemma:

Lemma 3 *Let* $Y \subset M$ *be an invariant subset and let* $S^*\mathcal{G}_Y$ *be the restriction of the cosphere bundle of* $A(\mathcal{G})$*,* $S^*\mathcal{G} \to M$*, to* Y*. Then the following sequence is exact:*

$$0 \longrightarrow C^*(\mathcal{G}_{M\setminus Y}) \to \mathfrak{A}(\mathcal{G}) \xrightarrow{\mathcal{R}_Y \oplus \sigma_0} \mathfrak{A}(\mathcal{G}_Y) \times_{C_0(S^*\mathcal{G}_Y)} C_0(S^*\mathcal{G}) \longrightarrow 0 \,.$$

Above we have denoted by $\mathfrak{A}(\mathcal{G}_Y) \times_{C_0(S^*\mathcal{G}_Y)} C_0(S^*\mathcal{G})$ the fibered product algebra obtained as the pair of elements (P, f), $P \in \mathfrak{A}(\mathcal{G}_Y)$ and $f \in C_0(S^*\mathcal{G})$ that map to the same element in $C_0(S^*\mathcal{G}_Y)$.

Proof: This exact sequence comes out directly from the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & C^*(\mathcal{G}_{M\setminus Y}) & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \to & C^*(\mathcal{G}) & \to & \mathfrak{A}(\mathcal{G}) & \xrightarrow{\sigma_0} & C_0(S^*\mathcal{G}) & \to & 0 \\
 & & \downarrow & & \downarrow \mathcal{R}_Y & & \downarrow & & \\
0 & \to & C^*(\mathcal{G}_Y) & \to & \mathfrak{A}(\mathcal{G}_Y) & \to & C_0(S^*\mathcal{G}_Y) & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

□

We can now prove the theorem itself.

(i) As $\mathcal{G}_{M\setminus Y_{n-1}} = (M \setminus Y_{n-1}) \times (M \setminus Y_{n-1})$, its C^*-algebra is isomorphic to that of compact operators on $L^2(M \setminus Y_{n-1})$.

(ii) A bounded operator $P \in \mathcal{L}(\mathcal{H})$, acting on the Hilbert space $\mathcal{H}$, is compact if, and only if, its image in the Calkin algebra $Q(\mathcal{H}) := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is zero. The assumption that π_x is injective guarantees that the induced map $\mathfrak{A}(\mathcal{G})/C^*(\mathcal{G}_{M\setminus Y_{n-1}}) \to Q(\mathcal{H})$ is also injective. Then the lemma above, applied to Y_{n-1}, implies that P is compact if and only if

$$P \in \ker(\mathcal{R}_{Y_{n-1}} \oplus \sigma_0) = \ker \mathcal{R}_{Y_{n-1}} \cap \ker \sigma_0 \,.$$

(iii) By Atkinson's theorem, a bounded operator $P \in \mathcal{L}(\mathcal{H})$ is Fredholm if, and only if, its image in the Calkin algebra $Q(\mathcal{H}) := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is invertible. Also, recall that an injective morphism ρ of C^*-algebras has the property that $\rho(T)$ is invertible if, and only if, T is invertible. Thus we can use again the fact that the morphism $\mathfrak{A}(\mathcal{G})/C^*(\mathcal{G}_{M\setminus Y_{n-1}}) \to Q(\mathcal{H})$ induced by π_x is injective to conclude that P is Fredholm if, and only if, $(\mathcal{R}_{Y_{n-1}} \oplus \sigma_0)(P)$ is invertible, *i.e.* if, and only if, $\mathcal{R}_{Y_{n-1}}(P)$ and $\sigma_0(P)$ are invertible. □

The significance of Theorem 3 is that often in practice we can find nice invariant filtrations of M, possibly given by a stratification of M, such that the subquotients $C^*(\mathcal{G}_{Y_k\setminus Y_{k-1}})$ have a relatively simpler structure than that of $C^*(\mathcal{G})$. In that case, the ideal structure reflects the geometric structure of M [12]. In this context, let us mention only the edge-calculus on manifolds with boundary [13, 16], and the b- resp. cusp-calculus, or, slightly more general, the c_n-calculus on manifolds with corners [10, 13, 22, 23, 26, 27].

Let us now assume, until the end of this section, that $\mathcal{G}$ is a differentiable groupoid. In many cases, the subquotients $C^*(\mathcal{G}_{Y_k\setminus Y_{k-1}})$ are then related to foliation algebras, to which the results of [38] can be applied. Actually, we

can always find an ideal in $\mathfrak{A}(\mathcal{G})$, whose structure resembles that of foliation algebras. The construction of this ideal goes as follows. Recall that for a differentiable groupoid $\mathcal{G}$ there is a canonical map $q : A(\mathcal{G}) \to TM$ of vector bndles, called the *anchor map.*
Consider, for each $k \leq n = \dim \mathcal{G}_x$, the set

$$X_k \subset M$$

of points x such that the dimension of $q(A(\mathcal{G})_x) \subset T_xM$ is k. Then

$$Y_k := X_0 \cup X_1 \cup \cdots \cup X_k$$

is a closed subset of M. It is known [15], that $\dim(q(A(\mathcal{G})_y))$ is constant for $y \in r(\mathcal{G}_x)$, and hence each X_k is an *invariant* subset of M. If p is the largest integer for which $X_p \neq \emptyset$, then $\mathcal{O} = X_p$ is an open invariant subset of M, foliated by the sets $r(\mathcal{G}_x)$, $x \in \mathcal{O}$. Denote by $\mathcal{F}$ this foliation of $\mathcal{O}$ and by $T\mathcal{F} = q(A(\mathcal{G})|_{\mathcal{O}})$ its tangent space. The set $\mathcal{O}$ will be called the *maximal regular open subset* of M.
Finally, still in the setting of differentiable groupoids, let us define a representation π of $\Psi^{\infty,0}(\mathcal{G})$ on $\mathcal{C}_c^\infty(M)$by

$$(\pi(P)u) \circ r = P(u \circ r)\,;$$

this representation is called the *vector representation.*

Lemma 4 *Let $\mathcal{O}$ be an invariant open subset of M. Then the operator $\pi(P)$ map $\mathcal{C}_c^\infty(\mathcal{O})$ to itself.*

Proof: The support of $\pi(P)u$ is contained in the product

$$\text{supp }(P)\text{ supp }(u),$$

a compact subset of M, which, we claim, does not intersect $Y := M \smallsetminus \mathcal{O}$. Indeed, if we assume by contradiction that

$$y \in Y \cap \text{ supp }(P)\text{ supp }(u),$$

then the intersection of supp $(P)^{-1}Y$ and supp (u) is not empty. However, this is not possible since we have supp $(P)^{-1}Y \subset Y$, by the invariance of Y, and supp $(u) \subset \mathcal{O}$. □

The representation of $\Psi^{\infty,0}(\mathcal{G})$ on $\mathcal{C}_c^\infty(\mathcal{O})$ obtained in the above lemma will be denoted by $\pi_{\mathcal{O}}$. In particular, $\pi_M = \pi$.
Let $\mathcal{F}$ be the foliation of the maximal regular open subset $\mathcal{O}$ of M. Also, let $\Omega_{\mathcal{F}}$ be the bundle of densities along the fibers of $\mathcal{F}$. The bundle $\Omega_{\mathcal{F}}$ is trivial and the notion of *positive* section of $\Omega_{\mathcal{F}}$ is defined invariantly. Recall that a *transverse measure* μ on $\mathcal{F}$ is a linear map $\mu : \mathcal{C}_c(\mathcal{O}, \Omega_{\mathcal{F}}) \to \mathbb{C}$ such that

$\mu(f) \geq 0$ if f is positive. A transverse measure μ on $\mathcal{O}$ gives rise to an inner product $(\, , \,)_\mu$ on

$$C_c^{\infty,0}(\mathcal{O};\Omega_{\mathcal{F}}^{1/2})$$

by the formula $(f,g)_\mu := \mu\big(f\overline{g})\big)$. Let $L^2(\mathcal{O},d\mu)$ be the completion of $\mathcal{C}_c^{\infty,0}(\mathcal{O},\Omega_{\mathcal{F}})$ with respect to the Hilbert space norm $\|f\|_\mu = (f,f)_\mu^{1/2}$.

THEOREM 5 *For any transverse measure μ on $\mathcal{O}$ the representation $\pi_{\mathcal{O}}$ extends to a bounded $*$–representation of $\mathfrak{A}(\mathcal{G})$ on $L^2(\mathcal{O},d\mu)$.*

PROOF: The results from [3] and, more specifically, [38] show that $\pi_{\mathcal{O}}$ extends to a bounded $*$-representation of $C^*(\mathcal{G}_{M\smallsetminus\mathcal{O}})$. Since $C^*(\mathcal{G}_{M\smallsetminus\mathcal{O}})$ is an ideal in $\mathfrak{A}(\mathcal{G})$, we can extend further $\pi_{\mathcal{O}}$ to a bounded $*$-representation of $\mathfrak{A}(\mathcal{G})$ acting on the same Hilbert space. □

The above construction generalizes to give a large class of representations of the algebras $\mathfrak{A}(\mathcal{G})$ for groupoids $\mathcal{G}$ whose spaces of units are endowed with some specific filtrations. Let us assume that $Y_0 \subset Y_1 \subset \ldots \subset Y_n = M$ is an invariant filtration, such that for each stratum $S_k = Y_k \setminus Y_{k-1}$, the map $r : \mathcal{G}_x \to S_k$ has the same rank for all $x \in S_k$. When this is the case, we shall call $M = \cup S_k$ a *regular invariant* stratification. Then each S_k is invariant and foliated by the orbits of $\mathcal{G}$ (whose leaves are the sets $r(\mathcal{G}_x)$, $x \in S$). In particular, each S_k is an invariant open subset of Y_k, and hence plays the role of $\mathcal{O}$ above for the groupoid $\mathcal{G}_{Y_k}$.

COROLLARY 5 *Let $\mathcal{G}$ be a differentiable groupoid with space of units M. Assume that $M = \cup S$ is a regular, invariant stratification. Then any non-zero transverse measure on a stratum S gives rise to a $*$-representation of $\mathfrak{A}(\mathcal{G})$.*

PROOF: Any transverse measure on S_k gives rise to a representation of the C^*-algebra $\mathfrak{A}(\mathcal{G}_{Y_k})$, by the above theorem. Then use the restriction morphism $\mathcal{R}_{Y_k} : \mathfrak{A}(\mathcal{G}) \to \mathfrak{A}(\mathcal{G}_{Y_k})$ to obtain the desired representations. □

In the following, we shall denote by $\otimes_{min}$ the *minimal* tensor product of C^*–algebras, defined using the tensor product of Hilbert spaces, see [40]. We shall use the following well-known result several times in the last section.

PROPOSITION 2 *If $\mathcal{G}_i$, $i = 0, 1$, are two differential groupoids, then*

$$C_r^*(\mathcal{G}_0 \times \mathcal{G}_1) \simeq C_r^*(\mathcal{G}_0) \otimes_{min} C_r^*(\mathcal{G}_1).$$

PROOF: Let M_0 and M_1 be the space of units of $\mathcal{G}_0$ and $\mathcal{G}_1$, and define

$$\mathcal{H}_i = \bigoplus_{x\in M_i} L^2((\mathcal{G}_i)_x)$$

[respectively, $\mathcal{H} = \oplus_{x\in M_0\times M_1} L^2((\mathcal{G}_0 \times \mathcal{G}_1)_x)$] to be the space of the total regular representation of $\Psi^{-\infty}(\mathcal{G}_i)$ [respectively, of $\Psi^{-\infty}(\mathcal{G}_0 \times \mathcal{G}_1)$]. Then the reduced C^*-algebras $C^*_r(\mathcal{G}_i)$ [respectively, $C^*_r(\mathcal{G}_0 \times \mathcal{G}_1)$] are the completions of $\Psi^{-\infty}(\mathcal{G}_i)$ [respectively, of $\Psi^{-\infty}(\mathcal{G}_0 \times \mathcal{G}_1)$] acting on $\mathcal{H}_i$ [respectively, on $\mathcal{H}$]. Let $\overline{\otimes}$ be the completed tensor product of Hilbert spaces. Since $\mathcal{H} \simeq \mathcal{H}_0 \overline{\otimes} \mathcal{H}_1$, the isomorphism

$$C^*_r(\mathcal{G}_0 \times \mathcal{G}_1) \simeq C^*_r(\mathcal{G}_0) \otimes_{min} C^*_r(\mathcal{G}_1)$$

follows directly from the definition of the minimal C^*-algebra tensor product of $C^*_r(\mathcal{G}_0)$ and $C^*_r(\mathcal{G}_1)$ as the completion of $C^*_r(\mathcal{G}_0) \otimes C^*_r(\mathcal{G}_1)$ acting on $\mathcal{H}_0 \otimes \mathcal{H}_1$. $\square$

5 Applications to index theory on singular spaces

We now discuss in greater detail two examples, the adiabatic limit groupoid and the "b-Γ–groupoid." The first example is relevant for the index theory on singular manifolds, or open manifolds with a uniform structure at infinity; it generalizes the construction of the tangent groupoid, that plays a key role in index theory as showed in [4]. The second example is related to the theory of elliptic (or Fredholm) boundary value problems.

Let X be a locally compact space and B be a Banach algebra. We shall denote, as usual, by $\mathcal{C}_0(X; B)$ the space of continuous functions $X \to B$ that vanish in norm at infinity. Also, recall that $K_i(\mathcal{C}_0(\mathbb{R}, B)) \simeq K_{i-1}(B)$ and $K_0(\mathcal{C}_0(X)) \simeq K^0(X)$.

If $\mathcal{G}$ is a continuous family groupoid with space of units M, then we construct its *adiabatic groupoid*, denoted $\mathcal{G}_{ad}$, as follows. First, the space of units of $\mathcal{G}_{ad}$ is $M \times [0, \infty)$.

The underlying set of the groupoid $\mathcal{G}_{ad}$ is the disjoint union:

$$\mathcal{G}_{ad} = A(\mathcal{G}) \times \{0\} \cup \mathcal{G} \times (0, \infty).$$

We endow $A(\mathcal{G}) \times \{0\}$ with the structure of a commutative bundle of Lie groups and $\mathcal{G} \times (0, \infty)$ with the product (or pointwise) groupoid structure. Then the groupoid operations of $\mathcal{G}_{ad}$ are such that $A(\mathcal{G}) \times \{0\}$ and $\mathcal{G} \times (0, \infty)$ are subgroupoids with the induced structure.

Now let us endow this groupoid with a continuous family groupoid structure. Let us consider an atlas (Ω).

Let Ω be a chart of $\mathcal{G}$, such that $\Omega \cap \mathcal{G}^{(0)} \neq \emptyset$; one can assume without loss of generality that $\Omega \simeq T \times U$ with respect to d, and $\Omega \simeq T' \times U$ with respect to r; let us denote by ϕ and ψ these homeomorphisms. Thus, if $x \in U$, $\mathcal{G}_x \simeq T$, and $A(\mathcal{G})_U \simeq \mathbb{R}^k \times U$. Let $(\Theta_x)_{x \in U}$ (resp. $(\Theta'_x)_{x \in U}$) be a continuous family of diffeomorphisms from $\mathbb{R}^k$ to T (resp. T') such that $\iota(x) = \phi(\Theta_x(0), x)$ (resp. $\iota(x) = \psi(\Theta'_x(0), x)$, where ι denotes the inclusion of $\mathcal{G}^{(0)}$ into $\mathcal{G}$).

Then $\overline{\Omega} = A(\mathcal{G})_U \times \{0\} \cup \Omega \times (0, \infty)$ is an open subset of $\mathcal{G}_{ad}$, homeomorphic to $\mathbb{R}^k \times U \times \mathbb{R}_+$ with respect to d and to r as follows:

$$\begin{aligned}
\overline{\phi}(\xi,u,\alpha) &= \begin{cases} (\phi(\Theta_u(\alpha\xi),u),\alpha) & \text{if } \alpha \neq 0 \\ (\xi,u,0) & \text{if } \alpha = 0 \end{cases} \\
\overline{\psi}(\xi,u,\alpha) &= \begin{cases} \big((\phi(\Theta_u(\alpha\xi),u))^{-1},\alpha\big) & \text{if } \alpha \neq 0 \\ (\xi,u,0) & \text{if } \alpha = 0 \end{cases}
\end{aligned}$$

This defines an atlas of $\mathcal{G}_{ad}$, endowing it with a continuous family groupoid structure.

The tangent groupoid of $\mathcal{G}$ is defined to be the restriction of $\mathcal{G}_{ad}$ to $M\times[0,1]$. We are interested in the adiabatic groupoid (or in the tangent groupoid) because it may be used to formalize certain constructions in index theory, as we shall show below.

First, note that

$$M \times [0,\infty) = M \times \{0\} \cup M \times (0,\infty)$$

is an invariant stratification of the space of units. Consequently, Theorem 3 gives rise to the short exact sequence

$$0 \to SC^*(\mathcal{G}) := \mathcal{C}_0((0,\infty), C^*(\mathcal{G})) \to C^*(\mathcal{G}_{ad}) \to \mathcal{C}_0(A^*(\mathcal{G})) \to 0.$$

The boundary map ∂ of the K-theory six term exact sequence associated to the above exact sequence of C^*-algebras then provides us with a map

$$\mathrm{ind}_a : K^i(A^*(\mathcal{G})) = K_i(\mathcal{C}_0(A^*(\mathcal{G}))) \xrightarrow{\partial} K_{i+1}(SC^*(\mathcal{G})) \simeq K_i(C^*(\mathcal{G})), \tag{24}$$

the *analytic index morphism*, which we shall discuss below in relation with the Fredholm index. Remark that this morphism does not necessarily take its values in $\mathbb{Z}$; however, in the case of the groupoid $M\times M$ of a smooth manifold M one has $K_i(C^*(M\times M) = \mathbb{Z}$.

We assume from now on, for simplicity, that M, the space of units of $\mathcal{G}$, is compact. Let $P = (P_x) \in \Psi^{m,0}(\mathcal{G}; E_0, E_1)$ be a family of *elliptic* operators acting on sections of $r^*(E_0)$, with values sections of $r^*(E_1)$, for some bundles E_0 and E_1 on M. (Here "elliptic" means, as before, that the principal symbol is invertible.) We shall denote the pull-backs of E_0 and E_1 to $A^*(\mathcal{G})$ by the same letters. Then the triple $(E_0, E_1, \sigma_m(P))$ defines an element $[\sigma_m(P)]$ in $K^0(A^*(\mathcal{G}))$, the K-theory groups with compact supports of $A^*(\mathcal{G})$. Furthermore, the morphism ind_a provides us with an element $\mathrm{ind}_a([\sigma_m(P)])$, which we shall also write as $\mathrm{ind}_a(P)$, and call the *analytic index* of the family P. As we shall see below, this construction generalizes the usual analytic (or Fredholm) index of elliptic operators.

Suppose now that $M = \cup S$ is an invariant stratification of the space of units of the continuous family groupoid $\mathcal{G}$. Then we obtain a natural, invariant stratification of the space of units of $\mathcal{G}_{ad}$ as

$$M \times [0,\infty) = \bigcup_S \big(S \times (0,\infty)\big) \cup M \times \{0\}.$$

To recover the Fredholm index, we shall assume that there exists a unique stratum of maximal dimension in the above stratification, let us call it $S_{\max}$, and let us assume that the restriction of $\mathcal{G}$ to $S_{\max}$ is the product groupoid:

$$\mathcal{G}_{S_{\max}} := r^{-1}(S_{\max}) = d^{-1}(S_{\max}) \simeq S_{\max} \times S_{\max}. \tag{25}$$

Let $\mathcal{K}$ denote the algebra of compact operators on $L^2(S_{\max}) \simeq L^2(M)$. Then $S_{\max} \times (0,\infty)$ is the unique stratum of maximal dimension of $M \times [0,\infty)$, and the ideal associated to it is $S\mathcal{K} := \mathcal{C}_0((0,\infty),\mathcal{K})$. This leads to an exact sequence of C^*-algebras

$$0 \to S\mathcal{K} \to C^*(\mathcal{G}_{ad}) \to Q(\mathcal{G}_{ad}) \to 0, \tag{26}$$

where $Q(\mathcal{G}_{ad}) := C^*(\mathcal{G}_{ad})/\mathcal{C}_0((0,\infty),\mathcal{K})$.
As above, this exact sequence of C^*-algebras leads to a six term exact sequence in K-theory, and hence to a map

$$\mathrm{ind}_f : K_0(Q(\mathcal{G}_{ad})) \xrightarrow{\partial} K_1(S\mathcal{K}) \simeq \mathbb{Z}. \tag{27}$$

(The second isomorphism is obtained from the boundary map associated to the exact sequence

$$0 \to S\mathcal{K} \to \mathcal{C}_0((0,\infty],\mathcal{K}) \to \mathcal{K} \to 0.)$$

Below we shall use the "graph projection" of a densely defined, unbounded operator P, which we now define. Let τ be a smooth, even function on $\mathbb{R}$ satisfying $\tau(x^2)^2x^2 = e^{-x^2}(1-e^{-x^2})$. Then *the graph projection* of P is

$$\mathbb{B}(P) = \begin{bmatrix} 1-e^{-P^*P} & \tau(P^*P)P^* \\ \tau(PP^*)P & e^{-PP^*} \end{bmatrix}. \tag{28}$$

This projection is also called the Bott or the Wasserman projection by some authors. Also, let

$$e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Denote by $\rho : Q(\mathcal{G}_{ad}) \to \mathcal{C}_0(A^*(\mathcal{G}))$ the canonical projection, and let

$$\rho_* : K_i(Q(\mathcal{G}_{ad})) \to K_i(\mathcal{C}_0(A^*(\mathcal{G})))$$

be the morphism induced on K-theory. For operators of positive order, we shall consider Sobolev spaces on $S_{\max}$, defined using the bounded geometry metric on $S_{\max}$ obtained from a metric on $A(\mathcal{G})$ (recall that we assume M to be compact, and that $S_{\max}$ is smooth as it is a fiber of $\mathcal{G}$). We then have the following result.

Proposition 3 *Let $S_{\max}$ and $\mathcal{G}$ be as above, $\mathcal{G}_{S_{\max}} \simeq S_{\max} \times S_{\max}$. Assume that the regular representation $\pi_x : \mathfrak{A}(\mathcal{G}) \to \mathcal{L}(L^2(\mathcal{G}_x)) \simeq \mathcal{L}(L^2(S_{\max}))$, associated to some $x \in S_{\max}$, is injective. If $P \in \Psi^{m,0}(\mathcal{G}; E_0, E_1)$ is a Fredholm* differential *operator $H^m(S_{\max}) \to L^2(S_{\max})$, then it defines a canonical class $[P] \in K_0(Q(\mathcal{G}_{ad}))$ such that $\rho_*([P]) = [\sigma_m(P)]$ and $\mathrm{ind}_f([P])$ coincides with the Fredholm index of P.*

Proof: We may assume that the family P consists of operators of order $m > 0$. The Fredholmness of P implies that $\sigma_m(P)$ is invertible outside the zero section, by Theorem 4.
Because P is a family of differential operator of order m, we can define a new family $Q \in \Psi^{m,0}(\mathcal{G}_{ad})$ by

$$Q_{(x,t)} = t^m P_x, \quad \text{if} \quad (x,t) \in M \times (0,\infty),$$

and

$$Q_{(x,0)} = \sigma_m(P),$$

a polynomial function on $A(\mathcal{G})^*_x \times \{0\}$ (the complete symbol of a homogeneous differential operator on $A(\mathcal{G})_x$).
Moreover, let $\mathcal{C}_0((0,\infty],\mathcal{K})$ be the space of all continuous functions $(0,\infty) \to \mathcal{K}$ vanishing for $t \to 0$ and having limits for $t \to \infty$. Also, let us denote by $\mathcal{B}$ the algebra $\mathcal{B} := C^*(\mathcal{G}_{ad}) + \mathcal{C}_0((0,\infty],\mathcal{K})$.
Consider now the graph projection $\mathbb{B}(Q)$. The algebra $\mathcal{B}$ can be identified with a subalgebra of $\mathcal{C}_0((0,\infty],\mathcal{L}(L^2(S_{\max}))$, naturally. Then $\mathbb{B}(Q)$ identifies with the function whose value at $t > 0$ is $\mathbb{B}(t^m P)$. Because of

$$\lim_{t\to\infty} \mathbb{B}(t^m P) = \begin{pmatrix} 1 - \pi_{N(P)} & 0 \\ 0 & \pi_{N(P^*)} \end{pmatrix},$$

where $\pi_{N(P)}$ [respectively $\pi_{N(P^*)}$] stands for the orthogonal projection onto the kernel $N(P)$ [respectively the cokernel $N(P^*)$], we have $\mathbb{B}(t^m P) - e_0 \in M_N(\mathcal{B})$, thus, we obtain a class $[\mathbb{B}(t^m P)] - [e_0] \in K_0(\mathcal{B})$. (The scalar matrix e_0 was defined shortly before the statement of this theorem.) Let us observe now that the C^*-algebra $\mathcal{B}$ fits into the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{C}_0((0,\infty),\mathcal{K}) & \longrightarrow & C^*(\mathcal{G}_{ad}) & \longrightarrow & Q(\mathcal{G}_{ad}) & \longrightarrow & 0 \\
 & & \big\downarrow id & & \big\downarrow & & \big\downarrow id\oplus 0 & & \\
0 & \longrightarrow & \mathcal{C}_0((0,\infty),\mathcal{K}) & \longrightarrow & \mathcal{B} & \xrightarrow{q} & Q(\mathcal{G}_{ad})\oplus\mathcal{K} & \longrightarrow & 0 \\
 & & \big\uparrow id & & \big\uparrow & & \big\uparrow 0\oplus id & & \\
0 & \longrightarrow & \mathcal{C}_0((0,\infty),\mathcal{K}) & \longrightarrow & \mathcal{C}_0((0,\infty],\mathcal{K}) & \longrightarrow & \mathcal{K} & \longrightarrow & 0
\end{array}$$

We shall use this information in the following way. The two right vertical morphisms identify $K_*(q(\mathcal{B})) \cong K_*(Q(\mathcal{G}_{ad})) \oplus K_*(\mathcal{K})$. We shall decompose accordingly the elements in $K_*(q(\mathcal{B}))$. Thus, there exists a uniquely defined class $[P] \in K_0(Q(\mathcal{G}_{ad}))$ satisfying

$$\begin{aligned} q_*([\mathbb{B}(t^m P)] - [e_0]) &= \left([P], \left[\begin{pmatrix} 1 - \pi_{N(P)} & 0 \\ 0 & \pi_{N(P^*)} \end{pmatrix}\right] - [e_0]\right) \\ &= ([P], -[\pi_{N(P)}] + [\pi_{N(P^*)}]). \end{aligned}$$

The property $\rho_*([P]) = [\sigma_m(P)]$ is now an immediate consequence of the definitions.
Let now $\partial : K_0(\cdot) \longrightarrow K_1(\mathcal{C}_0((0,\infty),\mathcal{K})) \cong K_0(\mathcal{K}) \cong \mathbb{Z}$ be the boundary maps of the corresponding three cyclic 6-term exact sequences. Note that all the previous isomorphisms are canonical, as explained above. Then we get

$$0 = \partial q_*([\mathbb{B}(t^m P)] - [e_0]) = \partial[P] + \partial(-[\pi_{N(P)}] + [\pi_{N(P^*)}]) = \mathrm{ind}_f([P]) - \mathrm{ind}(P),$$

where $\mathrm{ind}(P)$ is the Fredholm index. This completes the proof. □

We now briefly discuss a different example, that of elliptic operators on coverings of manifolds with boundary and interpret in our framework the existence of spectral sections considered in [14] and [24]. We need to remind first two constructions, the first one is that of the "b-groupoid" associated to a manifold with boundary and the second one is that of the groupoid associated to a covering of a manifold without boundary.
Let M be a manifold with boundary with fundamental group Γ acting on $\widetilde{M}$. We need to first recall the definition of the *b-groupoid of M*. For the simplicity of the presentation, we shall assume that the boundary of M is *connected*. Not all results extend to the case when ∂M is not connected. (We are indebted to Severino Melo for this remark.) The b-groupoid $\mathcal{G}_{M,b}$ is a submanifold of the b-stretched product defined by Melrose [18, 19]. It consists of the disjoint union of $\partial M \times \partial M \times \mathbb{R}$ and $(M \smallsetminus \partial M) \times (M \smallsetminus \partial M)$, with groupoid operations induced from the product groupoid structures on each component. Choose a defining function f of the boundary of M. Then the topology on $\mathcal{G}_{M,b}$ is such that

$$(y_n, z_n) \to (y, z, t) \in \partial M \times \partial M \times \mathbb{R}$$

if, and only if,

$$(y_n, z_n) \to (y, z)$$

in $M \times M$ and

$$\log f(y_n) - \log f(z_n) \to t.$$

Second, recall that if π is a discrete group that acts freely on the space X, then $(X \times X)/\pi$ is naturally a groupoid with units X/π, such that the domain and the range maps are the projections onto the first and, respectively, onto the second variable. The composition is such that $(x, y)\pi \circ (y, z)\pi = (x, z)\pi$.
We are now ready to define the *b-Γ-groupoid* associated to a covering

$$\Gamma \to \widetilde{M} \to M$$

(with Γ the group of deck transformations) of a manifold with boundary M. This groupoid will be denoted $\mathcal{G}_{\widetilde{M},b}$. To form the groupoid $\mathcal{G}_{\widetilde{M},b}$, we proceed in a similar way, by combining the above two constructions. Consider first the actions of Γ on $\widetilde{M} \setminus \partial\widetilde{M}$ and on $\partial\widetilde{M}$ to form the induced groupoids

$$\mathcal{G}_1 = \big((\widetilde{M} \setminus \partial\widetilde{M}) \times (\widetilde{M} \setminus \partial\widetilde{M})\big)/\Gamma$$

and

$$\mathcal{G}_2 = \big(\partial\widetilde{M} \times \partial\widetilde{M}\big)/\Gamma.$$

Then we form the disjoint union

$$\mathcal{G}_{\widetilde{M},b} := (\mathcal{G}_2 \times \mathbb{R}) \cup \mathcal{G}_1,$$

to which we give the structure of a continuous family groupoid by requiring that the projection $\mathcal{G}_{\widetilde{M},b} \to \mathcal{G}_{M,b}$ is a local diffeomorphism. (Here $\mathcal{G}_{M,b}$ is Melrose's b-groupoid, as above.)

Let $\mathcal{T}_0$ denote $C^*(\mathcal{G}_{I,b})$, if I is the manifold with boundary $[0,\infty)$. Then $\mathcal{T}_0$ is the closure of the algebra of Wiener-Hopf operators acting on $[0,\infty)$, and hence its closure is isomorphic to the (non-unital) Toeplitz algebra, defined as the kernel of the evaluation at 1 of the symbol map of a Toeplitz operator. In other words, if $\mathcal{T}$ denotes the C^*-algebra of Toeplitz operators on the unit circle, then we have an exact sequence

$$0 \to \mathcal{T}_0 \to \mathcal{T} \to \mathbb{C} \to 0.$$

The structure of the C^*-algebra $C^*(\mathcal{G}_{\widetilde{M},b})$ is given by the isomorphism

$$C^*(\mathcal{G}_{\widetilde{M},b}) \cong \mathcal{T}_0 \otimes C^*(\Gamma) \otimes \mathcal{K}$$

and similarly for the reduced algebras. Since the reduced Toeplitz algebra $\mathcal{T}_0$ is contractible, the K-groups of $C^*(\mathcal{G}_{\widetilde{M},b})$ vanish, and hence the map

$$p_* : K_*(\mathfrak{A}(\mathcal{G}_{\widetilde{M},b})) \longrightarrow K_*(\mathfrak{A}(\mathcal{G}_{\widetilde{M},b})/C^*(\mathcal{G}_{\widetilde{M},b})) \cong K_*(\mathcal{C}(S^*M)),$$

induced by the canonical projection $p : \mathfrak{A}(\mathcal{G}_{\widetilde{M},b}) \longrightarrow \mathfrak{A}(\mathcal{G}_{\widetilde{M},b})/C^*(\mathcal{G}_{\widetilde{M},b})$, is an isomorphism. On the other hand, let $p_1 : \mathfrak{A}(\mathcal{G}_{\widetilde{M},b}) \longrightarrow \mathfrak{A}(\mathcal{G}_{\widetilde{M},b})/C^*(\mathcal{G}_1)$ and $q : \mathfrak{A}(\mathcal{G}_{\widetilde{M},b})/C^*(\mathcal{G}_1) \longrightarrow \mathfrak{A}(\mathcal{G}_{\widetilde{M},b})/C^*(\mathcal{G}_{\widetilde{M},b})$ be the obvious projection maps. Then we have $p = q \circ p_1$, which gives the surjectivity of the induced map

$$q_* : K_*(\mathfrak{A}(\mathcal{G}_{\widetilde{M},b})/C^*(\mathcal{G}_1)) \longrightarrow K_*(\mathfrak{A}(\mathcal{G}_{\widetilde{M},b})/C^*(\mathcal{G}_{\widetilde{M},b})) \cong K_*(\mathcal{C}(S^*M))\,.$$

(Here it is essential to assume that ∂M is connected.)

Consequently, any elliptic operator $P \in M_N(\Psi^{m,0}(\mathcal{G}_{\widetilde{M},b}))$ (which identifies with a Γ-invariant b-pseudodifferential operator acting on a trivial vector bundle on $\widetilde{M}$) has a perturbation by an element in $M_N(\Psi^{-\infty,0}(\mathcal{G}_{\widetilde{M},b}))$ to an operator in $M_N(\Psi^{m,0}(\mathcal{G}_{\widetilde{M},b}))$ that has an invertible boundary indicial map. (This is proved as the corresponding statement in [31].) Consequently, this perturbation is $C^*(\Gamma)$-Fredholm, in the sense of Mishenko and Fomenko, [25]. The existence of this kind of perturbations was obtained before in [14] using the concept of spectral section. A completely similar argument can be used in the study of elliptic boundary value problems for families of elliptic operators to prove that every family of elliptic b-pseudodifferential operators on a manifold with boundary has a perturbation by a family of regularizing operators that makes this family a family of Fredholm operators. This result was obtained in [24] also using spectral sections.

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