Motivic Symmetric Spectra

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Abstract. This paper demonstrates the existence of a theory of symmetric spectra for the motivic stable category. The main results together provide a categorical model for the motivic stable category which has an internal symmetric monoidal smash product. The details of the basic construction of the Morel-Voevodsky proper closed simplicial model structure underlying the motivic stable category are required to handle the symmetric case, and are displayed in the first three sections of this paper.

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Introduction

This paper gives a method for importing the stable homotopy theory of symmetric spectra [7] into the motivic stable category of Morel and Voevodsky [14], [16], [17]. This category arises from a closed model structure on a suitably defined category of spectra on a smooth Nisnevich site, and it is fundamental for Voevodsky's proof of the Milnor Conjecture [16]. The motivic stable category acquires an effective theory of smash, or non-abelian tensor products with the results presented here.

Loosely speaking, the motivic stable category is the result of formally inverting the functor $X \mapsto T \wedge X$ within motivic homotopy theory, where T is the quotient of sheaves $\mathbb{A}^1/(\mathbb{A}^1 - 0)$. In this context, a spectrum X, or T-spectrum, consists of pointed simplicial presheaves X^n, $n \geq 0$, together with bonding maps $T \wedge X^n \to X^{n+1}$. The theory is exotic in at least two ways: it lives within the motivic model category, which is a localized theory of simplicial presheaves, and the object T is not a circle in any sense, but is rather motivic equivalent to an honest suspension $S^1 \wedge \mathbb{G}_m$ of the scheme underlying the multiplicative group. Smashing with T is thus a combination of topological and geometric suspensions.

A symmetric spectrum in this category is a T-spectrum Y which is equipped with symmetric group actions $\Sigma_n \times Y^n \to Y^n$ in all levels such that all composite bonding maps $T^{\wedge p} \wedge X^n \to X^{p+n}$ are $(\Sigma_p \times \Sigma_n)$-equivariant. The main theorems of this paper assert that this category of symmetric spectra carries a notion of stable equivalence within the motivic model category which is part of a proper closed simplicial model structure (Theorem 4.15), and such that the forgetful functor to T-spectra induces an equivalence of the stable homotopy category for symmetric spectra with the motivic stable category (Theorem 4.31). This collection of results gives a category which models the motivic stable category, and also has a symmetric monoidal smash product.

The relation between spectra and symmetric spectra in motivic homotopy theory is an exact analogue of that found in ordinary homotopy theory. In this way, every T-spectrum is representable by a symmetric object, but some outstanding examples of T-spectra are intrinsically symmetric. These include the T-spectrum $\mathbf{H}_{\mathbb{Z}}$ which represents motivic cohomology [18].

The principal results of this paper are simple enough to state, but a bit complicated to demonstrate in that their proofs involve some fine detail from the construction of the motivic stable category. It was initially expected, given the experience of [13], that the passage from spectra to symmetric spectra would be essentially axiomatic, along the lines of the original proof of [7]. This remains true in a gross sense, but many of the steps in the proofs of [7] and [13] involve standard results from stable homotopy theory which cannot be taken for granted in the motivic context. In particular, the construction of the motivic stable category is quite special: one proves it by verifying the Bousfield-Friedlander axioms A4 – A6 [2], but the proofs of these axioms involve Nisnevich descent in a non-trivial way, and essentially force the introduction of the concept of flasque simplical presheaf. The class of flasque simplicial presheaves contains all globally fibrant objects, but is also closed under filtered colimit (*unlike* fibrant objects — the assertion to the contrary is a common error) and the "T-loop" functor. It is a key technical point that these constructions also preserve many pointwise weak equivalences, such as those arising from Nisnevich descent.

We must also use a suitable notion of compact object, so that the corresponding loop functors commute with filtered colimits. The class of compact simplicial presheaves is closed under finite smash product and homotopy cofibre, and includes all finite simplicial sets and smooth schemes over a decent base. As a result, the Morel-Voevodsky object belongs to a broader class of compact objects T for which the corresponding categories of T-spectra on the smooth Nisnevich site have closed model structures associated to an adequate notion of stable equivalence. These ideas are the subject of the first two sections of this paper and culminate in Theorem 2.9, which asserts the existence of the model structure.

Theorem 2.9 is proved without reference to stable homotopy groups. This is achieved in part by using an auxilliary closed model structure for T-spectra, for which the cofibrations (respectively weak equivalences) are maps which are cofibrations (respectively motivic weak equivalences) in each level. The fibrant objects for the theory are called injective objects, and one can show (Lemma 2.11) that the functor defined by naive homotopy classes of maps taking values in objects W which are both injective and stably fibrant for the theory detects stable equivalences. This idea was lifted from [7], and appears again for symmetric spectra in Section 4.

It is crucial for the development of the stable homotopy theory of symmetric spectra as presented here (eg. Proposition 4.13, proof of Theorem 4.15) to know that fibre sequences and cofibre sequences of ordinary spectra coincide up to motivic stable equivalence — this is the first major result of Section 3 (Lemma 3.9, Corollary 3.10). The method of proof involves long exact sequences in weighted stable homotopy groups. These groups were introduced in [16], but the present construction is predicated on knowing that a spectrum X is a piece of an asymmetric bispectrum object for which one smashes with the simplicial circle S^1 in one direction and with the scheme $\mathbb{G}_m$ in the other.

The section closes with a proof of the assertion (Theorem 3.11, Corollary 3.16) that the functors $X \mapsto X \wedge T$ and $Y \mapsto \Omega_T Y$ are inverse to each other on the motivic stable category. This proof uses Voevodsky's observation that twisting the 3-fold smash product $T^3 = T^{\wedge 3}$ by a cyclic permutation of order 3 is the identity in the motivic homotopy category — this is Lemma 3.13. This result is also required for showing that the stable homotopy category of symmetric spectra is equivalent to the motivic stable category.

Section 4 contains the main results: the model structure for stable equivalences of symmetric spectra is Theorem 4.15, and the equivalence of stable categories is Theorem 4.31. With all of the material in the previous sections in place, and subject to being careful about the technical difficulties underlying the stability functor for the category of spectra, the derivation of the proper closed simplicial model structure for symmetric spectra follows the method developed in [7] and [13]. The demonstration of the equivalence of stable categories is also by analogy with the methods of those papers, but one has to be a bit more careful again, so that it is necessary to discuss T-bispectra in a limited way.

It would appear that the compactness of T and the triviality of the action of the cyclic permutation on T^3 are minimum requirements for setting up the full machinery of spectra and symmetric spectra, along with the equivalence of stable categories within motivic homotopy theory, at least according to the proofs given here (see also [6]). These features are certainly present for the original categories of presheaves of spectra and symmetric spectra in motivic homotopy theory. This is the case $T = S^1$ for the results of Section 2, and the corresponding thread of results (Theorem 2.9, Remark 3.22) for the motivic stable categories of S^1-spectra and symmetric S^1-spectra concludes in Section 4.5 with an equivalence of motivic stable homotopy categories statement in Theorem 4.40. There is also a rather generic result about the interaction between cofibrations and the smash product in the category of symmetric spectrum objects which obtains in all of the cases at hand — see Proposition 4.41. The motivic stable homotopy theory of S^1-spectra has found recent application in [19].

This paper concludes with two appendices. Appendix A shows that formally inverting a rational point $f : * \to I$ of a simplicial presheaf I on an arbitrary small Grothendieck site gives a closed model structure which is proper (Theorem A.5). This result specializes to a proof that the motivic closed model structure is proper, but does not depend on the object I being an interval in any sense — compare [14, Theorem 2.3.2].

The purpose of Appendix B is to show that the category of presheaves on the smooth Nisnevich site $(Sm|_S)_{Nis}$ inherits a proper closed simplicial model structure from the corresponding category of simplicial presheaves, such that the presheaf category is a model for motivic homotopy theory. The main result is Theorem B.4. The corresponding sheaf theoretic result appears as Theorem B.6, and this is the foundation of the Morel-Voevodsky category of spaces model for motivic homotopy theory. I have included this on the grounds that it so far appears explicitly nowhere else, though the alert reader can cobble a proof together from the ideas in [14]. The only particular claim to originality of the results presented in Appendix B is the observation that the Morel-Voevodsky techniques also make sense on the presheaf level.

This paper has gone through a rather long debugging phase that began with its appearance under the original title "$\mathbb{A}^1$-local symmetric spectra" on the K-theory preprint server in September, 1998. I would like to thank a group of referees for their remarks and suggestions. One such remark was that the proof of Lemma 3.14 in the original version was incorrect, and should involve Voevodsky's Lemma 3.13. The corrected form of this result now appears as Theorem 3.11. Another suggestion was to enlarge the class of base schemes from fields to Noetherian schemes S of finite dimension, and this has been done here — the only technical consequence was the necessity to strengthen Lemma 3.13 to a statement that holds over the integers.

There has been a rather substantial shift in language with the present version of the paper. In particular, the use of the term "motivic homotopy theory" has become standard recently, and is incorporated here in place of either the old homotopy theoretic convention "f-local theory" [4] for the localized theory associated to a rational point $f : * \to \mathbb{A}^1$, or the "$\mathbb{A}^1$-homotopy theory" of [14]. Motivic homotopy theory is the fundamental object of discussion; at the risk of confusing readers who like to start in the middle, "weak equivalence" means "motivic weak equivalence" and similarly fibrations and cofibrations are in the motivic closed model structure, unless explicit mention is made to the contrary.

This work owes an enormous debt to that of Fabien Morel, Jeff Smith and Vladimir Voevodsky, and to conversations with all three; I would like to take this opportunity to thank them. Several of the main results of the first two sections of this paper were announced in some form in [16], while the unstable Nisnevich descent technique that is so important here was brought to my

attention by Morel, and appears in [14].

The conversations that I refer to took place at a particularly stimulating meeting on the homotopy theory of algebraic varieties at the Mathematical Sciences Research Institute in Berkeley in May, 1998. The idea for this project was essentially conceived there, while Appendix A was mostly written a few weeks prior during a visit to Université Paris VII. I thank both institutions for their hospitality and support.

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1 Preliminaries

1.1 Motivic homotopy theory

One starts with a rational point $f : * \to \mathbb{A}^1$ of the affine line $\mathbb{A}^1$ in the category of smooth schemes $(Sm|_S)_{Nis}$ of finite type over a scheme S of finite dimension, equipped with the Nisnevich topology. The empty scheme $\emptyset$ is a member of this category.

The localization theory arising from "formally inverting" the map f in the standard, or local homotopy theory of simplicial presheaves on $(Sm|_S)_{Nis}$ is the motivic homotopy theory for the scheme S — it has been formerly called both the f-local theory [4] and the $\mathbb{A}^1$-homotopy theory [14].

The standard homotopy theory of simplicial presheaves arises from a proper closed model structure that exists quite generally [9], [12] for simplicial presheaves on arbitrary small Grothendieck sites. In cases, like the Nisnevich site, where stalks are available, a *local weak equivalence* (or *stalkwise weak equivalence*) is a map of simplicial presheaves which induces a weak equivalence of simplicial sets in all stalks. A *cofibration* is a monomorphism of simplicial presheaves, and a *global fibration* is a map which has the right lifting property with respect to all maps which are cofibrations and local weak equivalences. A proper closed simplicial model structure for simplicial sheaves on an arbitrary Grothendieck site arises from similar definitions (cofibrations are monomorphisms, local weak equivalences are defined stalkwise, and global fibrations are defined by a lifting property), and the resulting homotopy category for simplicial sheaves is equivalent to the homotopy category associated to the closed model structure on simplicial presheaves. In particular, the associated sheaf map $\eta : X \to \tilde{X}$ from a simplicial presheaf to its associated simplicial sheaf is a local weak equivalence, since it induces an isomorphism on stalks. In the local theory, a *globally fibrant model* of a simplicial presheaf or sheaf X is a local weak equivalence $X \to W$ such that W is globally fibrant.

One says that a simplicial presheaf X on the Nisnevich site is *motivic fibrant* if it is globally fibrant for the Nisnevich topology, and has the right lifting property with respect to all simplicial presheaf inclusions

$$(f, j) : (\mathbb{A}^1 \times A) \cup_A B \to \mathbb{A}^1 \times B$$

arising from $f : * \to \mathbb{A}^1$ and all cofibrations $j : A \to B$. A simplicial presheaf map $g : X \to Y$ is said to be a *motivic weak equivalence* if it induces a weak equivalence of simplicial sets

$$g^* : \mathbf{hom}(Y, Z) \to \mathbf{hom}(X, Z)$$

in function complexes for every motivic fibrant object Z. A *cofibration* is a monomorphism of simplicial presheaves, just as in the local theory. A map $p : Z \to W$ is a *motivic fibration* if it has the right lifting property with respect to all maps which are simultaneously motivic weak equivalences and cofibrations. The homotopy theory arising from the following theorem is effectively the motivic homotopy theory of Morel and Voevodsky:

Theorem 1.1. *The category $\mathbf{S}\mathrm{Pre}(Sm|_S)_{Nis}$ of simplicial presheaves on the smooth Nisnevich site of the scheme S, together with the classes of cofibrations, motivic weak equivalences and motivic fibrations, satisfies the axioms for a proper, closed simplicial model category.*

The simplicial structure is the usual one for simplicial presheaves: the *function complex* $\mathbf{hom}(X, Y)$ for simplicial presheaves X and Y has n-simplices consisting of all simplicial presheaf maps $X \times \Delta^n \to Y$. Most of Theorem 1.1 is derived in [4], meaning that all except the properness assertion is proved there. Morel and Voevodsky demonstrate properness in [14] — an alternative proof appears in Appendix A (Theorem A.5) of this paper. Recall that a closed model category is said to be *proper* if the class of weak equivalences is closed under pullback along fibrations and pushout along cofibrations.

Recall [4] a map $g : X \to Y$ of simplicial presheaves is a *pointwise weak equivalence* if each map $g : X(U) \to Y(U)$, U smooth over S, in sections is a weak equivalence of simplicial sets. Similarly, g is said to be a *pointwise fibration* if all maps $g : X(U) \to Y(U)$ are Kan fibrations.

The standard equivalence of the local homotopy theories for simplicial presheaves and simplicial sheaves is inherited by all localized theories, and induces an equivalence of the homotopy category arising from Theorem 1.1 with the homotopy category for a corresponding closed model structure for simplicial sheaves. This holds quite generally [4, Theorem 1.2], but in the case at hand, more explicit definitions and proofs are quite easy to see: say that a map $p : X \to Y$ of simplicial sheaves on $(Sm|_S)_{Nis}$ is a *motivic fibration* if it is a global fibration of simplicial sheaves and has the right lifting property with respect to all simplicial sheaf inclusions $(f, j) : (\mathbb{A}^1 \times A) \cup_A B \to \mathbb{A}^1 \times B$. Then a map is a motivic fibration of simplicial sheaves if and only if it is a motivic fibration in the simplicial presheaf category.

In particular (see the discussion preceding Lemma 1.6) a simplicial sheaf or presheaf Z is motivic fibrant if and only if it is globally fibrant and the projection $U \times \mathbb{A}^1 \to U$ induces a weak equivalence of simplicial sets $Z(U) \simeq Z(U \times \mathbb{A}^1)$ for all smooth S-schemes U. Thus, if Y is a motivic fibrant simplicial presheaf and the simplicial sheaf $G\tilde{Y}$ is a globally fibrant model of its associated simplicial sheaf $\tilde{Y}$, then the map $Y \to G\tilde{Y}$ is a pointwise weak equivalence, so that $G\tilde{Y}$ is motivic fibrant. The two following statements are therefore equivalent for a simplicial sheaf map $g : X \to Y$:

1) the map g induces a weak equivalence $g^* : \mathbf{hom}(Y, Z) \to \mathbf{hom}(X, Z)$ for all motivic fibrant simplicial sheaves Z,

2) the map g is a motivic weak equivalence in the simplicial presheaf category.

Say that a map g which satisfies either of these properties is a *motivic weak equivalence* of simplicial sheaves. A *cofibration* of simplicial sheaves is a levelwise monomorphism, or a cofibration in the simplicial presheaf category.

Theorem 1.2. *1) The category* $\mathbf{S}\,\mathrm{Shv}(Sm|_S)_{Nis}$ *of simplicial sheaves on the smooth Nisnevich site of the scheme* S*, together with the classes of cofibrations, motivic weak equivalences and motivic fibrations, satisfies the axioms for a proper, closed simplicial model category.*

2) The forgetful functor and the associated sheaf functor together determine an adjoint equivalence of motivic homotopy categories

$$\mathrm{Ho}(\mathbf{S}\mathrm{Pre}(Sm|_S)_{Nis}) \simeq \mathrm{Ho}(\mathbf{S}\,\mathrm{Shv}(Sm|_S)_{Nis}).$$

The first part of Theorem 1.2 is proved in [14], and is the basis for their discussion of motivic homotopy theory. The second part says that the simplicial presheaf category gives a second model for motivic homotopy theory. Other models arising from ordinary (not simplicial) sheaves and presheaves are discussed in Appendix B.

Proof of Theorem 1.2. The equivalence of the homotopy categories is trivial, once the first statement is proved. For the closed model structure of part 1), there is really just a factorization axiom to prove. Any map $f : X \to Y$ of simplicial sheaves has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & Y \end{tikzcd}$$

in the simplicial presheaf category, where j is a motivic weak equivalence and a cofibration and p is a motivic fibration. Then the composite map

$$X \xrightarrow{i} Z \xrightarrow{\eta} \tilde{Z}$$

is a motivic weak equivalence and a cofibration of simplicial sheaves, where η is the associated sheaf map. Form the diagram

$$\begin{tikzcd} Z \arrow[r, "\eta"] \arrow[dr, "p"'] & \tilde{Z} \arrow[r, "i"] & W \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

where i is a trivial cofibration and π is a global fibration of simplicial sheaves. This same diagram is a local weak equivalence of cofibrant and globally fibrant objects over Y, and so the map $Z \to W$ is a homotopy equivalence and therefore a pointwise weak equivalence. Finally (see Lemma 1.5), a motivic fibration of simplicial presheaves can be characterized as a global fibration $X \to Y$ such that the induced map

$$X(U \times \mathbb{A}^1) \to X(U) \times_{Y(U)} Y(U \times \mathbb{A}^1)$$

is a weak equivalence of simplicial sets for all smooth S-schemes U. It follows that π is a motivic fibration of simplicial sheaves. $\square$

1.2 Controlled fibrant models

This section is technical, and should perhaps be read in conjunction with some motivation, such as one finds in the proofs of Proposition 2.15 and Corollary 2.16. This material is used to produce generating sets of trivial cofibrations in a variety of contexts. In particular, essential use is made of these ideas for symmetric spectrum objects in the proofs of Theorem 4.2 and Proposition 4.4.

The proofs in [4] and Appendix A hold for arbitrary choices of rational point $* \to I$ of any simplicial presheaf on any small Grothendieck site $\mathcal{C}$. At that level of generality, and in the language of [4], suppose α is an infinite cardinal which is an upper bound for the cardinality of the set $\mathrm{Mor}(\mathcal{C})$ of morphisms of $\mathcal{C}$. Pick a rational point $f : * \to I$, and suppose that I is α-bounded in the sense that all sets of simplices of all sections $I(U)$ have cardinality bounded above by α. This map f is a cofibration, and we are entitled to a corresponding f-localization homotopy theory for the category $\mathbf{S}Pre(\mathcal{C})$, according to the results of [4].

In particular, one says that a simplicial presheaf Z is *f-local* if Z is globally fibrant, and the map $Z \to *$ has the right lifting property with respect to all inclusions

$$(* \times L_U\Delta^n) \cup_{(*\times Y)} (I \times Y) \subset I \times L_U\Delta^n \tag{1.1}$$

arising from all subobjects $Y \subset L_U\Delta^n$. It follows that $Z \to *$ has the right lifting property with respect to all inclusions

$$(* \times B) \cup_{(*\times A)} (I \times A) \subset I \times B$$

arising from cofibrations $A \to B$. The map

$$f^* : \mathbf{hom}(I \times Y, Z) \to \mathbf{hom}(* \times Y, Z)$$

is therefore a weak equivalence for all simplicial presheaves Y if Z is f-local, and so all induced maps

$$\mathbf{hom}(I \times L_U\Delta^n, Z) \to \mathbf{hom}((I \times Y) \cup_{(*\times Y)} (* \times L_U\Delta^n), Z)$$

are trivial fibrations of simplicial sets.

A simplicial presheaf map $g : X \to Y$ is an *f-equivalence* if the induced map

$$g^* : \mathbf{hom}(Y, Z) \to \mathbf{hom}(X, Z)$$

is a weak equivalence of simplicial sets for all f-local objects Z. The original map $f : * \to I$ is an f-equivalence, and the maps

$$f \times 1_Y : * \times Y \to I \times Y$$

and the inclusions

$$(* \times B) \cup_{(*\times A)} (I \times A) \subset I \times B$$

are f-equivalences. A map $p : X \to Y$ is an *f-fibration* if it has the right lifting property with respect to all cofibrations of simplicial presheaves which are f-equivalences.

It is a consequence of Theorem 4.6 of [4] that the category $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ with the cofibrations, f-equivalences and f-fibrations, together satisfy the axioms for a closed simplicial model category. This result specializes to the closed model structure of Theorem 1.1 in the case of simplicial presheaves on the smooth Nisnevich site of S. Note as well that, very generally, the f-local objects coincide with the f-fibrant objects.

Pick cardinals λ and κ such that

$$\lambda = 2^{\kappa} > \kappa > 2^{\alpha}.$$

As part of the proof of [4, Theorem 4.6], it is shown that there is a functor $X \mapsto \mathcal{L}X$ defined on simplicial presheaves X together with a natural transformation $\eta_X : X \to \mathcal{L}X$ which is an f-fibrant model for X, such that the following properties hold:

L1: $\mathcal{L}$ preserves local weak equivalences.

L2: $\mathcal{L}$ preserves cofibrations.

L3: Let β be any cardinal with $\beta \geq \alpha$. Let $\{X_j\}$ be the filtered system of sub-objects of X which are β-bounded. Then the map

$$\varinjlim \mathcal{L}(X_j) \to \mathcal{L}X$$

is an isomorphism.

L4: Let γ be an ordinal number of cardinality strictly greater than 2^{α}. Let $X : \gamma \to \mathbf{S}$ be a diagram of cofibrations so that for all limit ordinals $s < \gamma$ the induced map

$$\varinjlim{}_{t<s} X(t) \to X(s)$$

is an isomorphism. Then $\varinjlim{}_{t<\gamma} \mathcal{L}(X(t)) \cong \mathcal{L}(\varinjlim{}_{t<\gamma} X(t))$.

L5: If X is λ-bounded, then $\mathcal{L}X$ is λ-bounded.

L6: Let Y, Z be two subobjects of X. Then

$$\mathcal{L}(Y) \cap \mathcal{L}(Z) = \mathcal{L}(Y \cap Z)$$

in $\mathcal{L}X$.

L7: The functor $\mathcal{L}$ is continuous; that is, it extends to a natural morphism of simplicial sets

$$\mathcal{L} : \mathrm{HOM}(X, Y) \to \mathrm{HOM}(\mathcal{L}X, \mathcal{L}Y)$$

compatible with composition.

In fact, the map $\eta_X : X \to \mathcal{L}X$ is a cofibration and an f-weak equivalence, which is constructed by a transfinite small object argument. The size of the construction, or rather the ordinal number that defines $\mathcal{L}X$ as a filtered colimit, is the cardinal κ (see [4, p.42]).

The demonstration of the statement L7 further involves the construction of a functorial pairing

$$\phi : \mathcal{L}X \times L \to \mathcal{L}(X \times K)$$

for simplicial presheaves X and simplicial sets L, and which satisfies a short list of compatibility conditions. This pairing induces a natural pointed map

$$\phi : \mathcal{L}X \wedge K \to \mathcal{L}(X \wedge K)$$

for pointed simplicial presheaves X and pointed simplicial sets K such that the following properties hold:

L8: the map

$$\phi : (\mathcal{L}X) \wedge \Delta^0_+ \to \mathcal{L}(X \wedge \Delta^0_+)$$

is the canonical isomorphism,

L9: the triangle

$$\begin{tikzcd} X \wedge K \arrow[r, "\eta_X \wedge K"] \arrow[dr, "\eta_{X \wedge K}"'] & (\mathcal{L}X) \wedge K \arrow[d, "\phi"] \\ & \mathcal{L}(X \wedge K) \end{tikzcd}$$

commutes, and

L10: the diagram

$$\begin{tikzcd} (\mathcal{L}X) \wedge K \wedge L \arrow[r, "\phi"] \arrow[d, "\phi \wedge L"'] & \mathcal{L}(X \wedge K \wedge L) \\ (\mathcal{L}(X \wedge K)) \wedge L \arrow[ur, "\phi"'] & \end{tikzcd}$$

commutes.

These statements are analogues of the standard properties for the unpointed pairing, and are consequences of same. In fact, nothing in the argument prevents L and K from being arbitrary simplicial presheaves, and we shall work with the more general pairing.

Specializing this construction to the case of pointed simplicial presheaves on $(Sm|_S)_{Nis}$ gives *controlled fibrant model* construction $\eta_X : X \to \mathcal{L}X$ for a simplicial presheaf X. The construction is controlled in the sense that the cardinality of $\mathcal{L}X$ has a specific bound if the cardinality of the original object X is well behaved, by L5. Also, the functor $X \mapsto \mathcal{L}X$ is compatible with smash product pairings in the sense that every pointed simplicial presheaf map $\sigma : X \wedge T \to Y$ induces a commutative diagram

$$\begin{tikzcd} X \wedge T \arrow[rr, "\sigma"] \arrow[d, "\eta_X \wedge 1_T"'] \arrow[dr, "\eta_{X\wedge T}"] & & Y \arrow[d, "\eta_Y"] \\ \mathcal{L}X \wedge T \arrow[r, "\phi"'] & \mathcal{L}(X \wedge T) \arrow[r, "\mathcal{L}\sigma"'] & \mathcal{L}Y \end{tikzcd} \tag{1.2}$$

1.3 Nisnevich descent

We shall need an unstable variant of the Nisnevich descent theorem [15]. The version of this result given in [11, p.296] says if a presheaf of spectra F on the Nisnevich site satisfies the *cd*-excision property, then any stably fibrant model $j : F \to GF$ for the Nisnevich topology is a stable equivalence in all sections.

A simplicial presheaf Z is said to have the *cd-excision property* (aka. B.G. property in [14]) if any *elementary Cartesian square*

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"'] & X \end{tikzcd} \tag{1.3}$$

of smooth schemes over k with p étale, i an open immersion and $p^{-1}(X-U) \cong X - U$ induces a homotopy Cartesian diagram of simplicial sets

$$\begin{tikzcd} Z(X) \arrow[r] \arrow[d] & Z(U) \arrow[d] \\ Z(V) \arrow[r] & Z(U \times_X V) \end{tikzcd}$$

The *cd*-excision property for presheaves of spectra is the stable analog of this requirement.

The unstable Nisnevich descent theorem is the following:

Theorem 1.3. *A simplicial presheaf Z on the site $(Sm|_S)_{Nis}$ has the cd-excision property if and only if any globally fibrant model $j : Z \to GZ$ for Z induces weak equivalences of simplicial sets $Z(U) \to GZ(U)$ in all sections.*

This is the simplicial presheaf analogue of a result for simplicial sheaves [14, 3.1.16].

Proof. Morel and Voevodsky point out that any globally fibrant simplicial sheaf has the *cd*-excision property [14, 3.1.15] and they show [14, 3.1.18] that if a map

$f : X \to Y$ is a local weak equivalence of simplicial presheaves and both have the *cd*-excision property, then f consists of weak equivalences $f : X(U) \to Y(U)$ in all sections.

Any simplicial sheaf which is globally fibrant within the simplicial sheaf category is also globally fibrant as a simplicial presheaf. It follows that the canonical map $\eta : Z \to \tilde{Z}$ taking values in the associated sheaf $\tilde{Z}$ gives rise to a diagram

$$\begin{tikzcd} Z \arrow[r, "\eta"] \arrow[d, "j_Z"'] & \tilde{Z} \arrow[d, "i_{\tilde{Z}}"] \\ GZ \arrow[r, "\eta_*"'] & G\tilde{Z} \end{tikzcd}$$

where all maps are local weak equivalences and $G\tilde{Z}$ is globally fibrant in the simplicial sheaf category. In particular, η_* is a local weak equivalence of globally fibrant simplicial presheaves, and hence consists weak equivalences $GZ(U) \to G\tilde{Z}(U)$ in all sections, since weakly equivalent globally fibrant models are homotopy equivalent. It follows in particular that any globally fibrant simplicial presheaf has the *cd*-excision property. Thus, if Z has the *cd*-excision property, any globally fibrant model consists of weak equivalences $Z(U) \to GZ(U)$ in sections, by the Morel-Voevodsky result, and the converse is obvious. $\square$

All of the hard work in the proof of Theorem 1.3 was done by Morel and Voevodsky. The original stable form of the Nisnevich descent theorem for the smooth site $(Sm|_S)_{Nis}$ is a corollary:

Corollary 1.4. *Suppose that Z is a presheaf of spectra on the smooth Nisnevich site $(Sm|_S)_{Nis}$. Then a stably fibrant model $j : Z \to GZ$ consists of stable equivalences $Z(U) \to GZ(U)$ in all sections if and only if the presheaf of spectra Z satisfies the (stable) cd-excision property.*

Proof. The presheaf of spectra Z satisfies the stable *cd*-excision property if and only if any elementary Cartesian diagram (1.3) induces a homotopy Cartesian diagram

$$\begin{tikzcd} Z(X) \arrow[r] \arrow[d] & Z(U) \arrow[d] \\ Z(V) \arrow[r] & Z(U \times_X V) \end{tikzcd}$$

of spectra with respect to stable equivalence. It follows that a presheaf of spectra Z has the stable *cd*-excision property if and only if each of the simplicial presheaves $Q\,\mathrm{Ex}^\infty Z^n$ has the *cd*-excision property. The maps $Q\,\mathrm{Ex}^\infty Z \to GZ$ are level weak equivalences of presheaves of Ω-spectra and all simplicial presheaves GZ^n are globally fibrant. It follows that Z has the stable *cd*-excision property if and only if all of the maps in sections $Q\operatorname{Ex}^\infty Z^n(U) \to GZ^n(U)$ are weak equivalences of pointed simplicial sets, and this holds if and only if all maps $Z(U) \to GZ(U)$ are stable equivalences of spectra. □

The *cd*-excision property is preserved by taking filtered colimits. Thus, if

$$Z_1 \to Z_2 \to Z_3 \to \cdots$$

is an inductive system of maps between simplicial presheaves which are globally fibrant for the Nisnevich topology, then any choice of globally fibrant model

$$j : \varinjlim Z_i \to G(\varinjlim Z_i)$$

for the Nisnevich topology is a pointwise weak equivalence.

Let's return briefly to a gross level of generality. Suppose that X and Y are simplicial presheaves on a site $\mathcal{C}$. For $U \in \mathcal{C}$, write $\mathcal{C} \downarrow U$ for the category whose objects are morphism $V \to U$ and whose morphisms are commutative triangles. There is a standard functor $Q_U : \mathcal{C} \downarrow U \to \mathcal{C}$ which is defined by taking the morphism

$$\begin{tikzcd} V_1 \arrow[rr, "\alpha"] \arrow[dr] & & V_2 \arrow[dl] \\ & U & \end{tikzcd}$$

to the morphism $\alpha : V_1 \to V_2$ of $\mathcal{C}$. Write $X|_U$ for the composite of the simplicial presheaf X with the functor Q_U. Any map $\phi : V \to U$ of $\mathcal{C}$ defines a functor $\phi_* : \mathcal{C} \downarrow V \to \mathcal{C} \downarrow U$ on objects $V_1 \to V$ by composition with ϕ, and obviously $Q_U \cdot \phi_* = Q_V$.

The *internal hom complex* $\mathbf{Hom}(X,Y)$ is a simplicial presheaf on $\mathcal{C}$ which is defined by

$$\mathbf{Hom}(X,Y)(U) = \mathbf{hom}(X|_U, Y|_U).$$

Evaluation in U-sections defines natural maps

$$ev_U : \mathbf{hom}(X|_U, Y|_U) \times X(U) \to Y(U)$$

which together give a natural *evaluation map*

$$ev : \mathbf{Hom}(X,Y) \times X \to Y.$$

This evaluation map defines a natural bijection

$$\hom(Z \times X, Y) \cong \hom(Z, \mathbf{Hom}(X,Y)),$$

or exponential law, for simplicial presheaves X, Y and Z on an arbitrary Grothendieck site $\mathcal{C}$.

The main homotopical fact about internal hom complexes is the following expanded version of Quillen's axiom SM7:

Lemma 1.5. *Suppose that $i : A \to B$ is a cofibration and that $p : X \to Y$ is a global fibration of simplicial presheaves. Then the induced map*

$$(i^*, p_*) : \mathbf{Hom}(B, X) \to \mathbf{Hom}(A, X) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(B, Y)$$

is a global fibration, which is trivial if either i or p is a local weak equivalence.

Proof. By adjointness, the claim follows from the assertion that the cofibration $i : A \to B$ and another cofibration $j : C \to D$ together determine a cofibration

$$(A \times D) \cup_{(A\times C)} (B \times C) \hookrightarrow B \times D$$

which is a local weak equivalence if either i or j is a local weak equivalence. This is checked stalkwise, or with a Boolean localization argument [12]. □

Recall that a motivic fibrant simplicial presheaf Z on $(Sm|_S)_{Nis}$ is an object which is globally fibrant for the Nisnevich topology and has the right lifting property with respect to all simplicial presheaf inclusions

$$(\mathbb{A}^1 \times A) \cup_A B \xrightarrow{(f,j)} \mathbb{A}^1 \times B$$

arising from $f : * \to \mathbb{A}^1$ and all cofibrations $j : A \to B$. The lifting property is equivalent to the assertion that the induced global fibration

$$f^* : \mathbf{Hom}(\mathbb{A}^1, Z) \to \mathbf{Hom}(*, Z) \cong Z$$

is a trivial global fibration. It follows that a simplicial presheaf Z is motivic fibrant if and only if Z is globally fibrant and all projections $U \times \mathbb{A}^1 \to U$ induce weak equivalences of simplicial sets $Z(U) \to Z(U \times \mathbb{A}^1)$. This observation is essentially well known, and was proved by Morel and Voevodsky in [14].

We can now prove the following:

Lemma 1.6. *Suppose given an inductive system*

$$Z_1 \to Z_2 \to Z_2 \to \cdots$$

of motivic fibrant simplicial presheaves on $(Sm|_S)$, and let

$$j : \varinjlim Z_i \to G(\varinjlim Z_i)$$

be a choice of globally fibrant model for the Nisnevich topology. Then the simplicial presheaf $G(\varinjlim Z_i)$ is motivic fibrant.

Proof. The map j is a pointwise weak equivalence by Nisnevich descent, and the the simplicial presheaf maps

$$pr^* : Z_i(U) \to Z_i(U \times \mathbb{A}^1)$$

induce a weak equivalence on the filtered colimit, and so $G(\varinjlim Z_i)$ is motivic fibrant. □

We shall make constant use of the following variant of Lemma 1.6:

Corollary 1.7. *Suppose that $X_1 \to X_2 \to \dots$ is an inductive system of motivic fibrant simplicial presheaves on $(Sm|_S)_{Nis}$. Then any motivic fibrant model*

$$j : \varinjlim X_i \to Z$$

is a pointwise weak equivalence.

1.4 Flasque simplicial presheaves

Say that a simplicial presheaf X on $(Sm|_S)_{Nis}$ is *flasque* if X is a presheaf of Kan complexes and every finite collection $U_i \hookrightarrow U$, $i = 1, \dots, n$ of subschemes of a scheme U induces a Kan fibration

$$X(U) \cong \mathbf{hom}(U, X) \xrightarrow{i^*} \mathbf{hom}(\cup_{i=1}^n U_i, X).$$

Here, the union is taken in the presheaf category, so that the simplicial set

$$\mathbf{hom}(\cup_{i=1}^n U_i, X)$$

is an iterated fibre product of the simplicial sets $X(U_i)$.

Every globally fibrant simplicial presheaf is flasque, and the class of flasque simplicial presheaves is closed under filtered colimits. Note that the condition for X to be flasque says that the map $X(U) \to X(V)$ associated to the singleton set consisting of a subscheme $V \hookrightarrow U$ is a Kan fibration.

Lifting problems

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathbf{hom}(U, X) \arrow[d, "i^*"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathbf{hom}(\cup_{i=1}^n U_i, X) \end{tikzcd}$$

and their solutions are equivalent to diagrams of simplicial presheaf maps

$$\begin{tikzcd} (\cup_{i=1}^n U_i \times \Delta^n) \cup_{(\cup_{i=1}^n U_i \times \Lambda^n_k)} U \times \Lambda^n_k \arrow[r] \arrow[d] & X \\ U \times \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

One says more generally that a map $p : X \to Y$ of simplicial presheaves is flasque if it is a pointwise fibration and has the right lifting property with respect to all maps

$$(\cup_{i=1}^n U_i \times \Delta^n) \cup_{(\cup_{i=1}^n U_i \times \Lambda^n_k)} U \times \Lambda^n_k \hookrightarrow U \times \Delta^n \tag{1.4}$$

arising from all finite collections U_i, $i = 1, \ldots, n$ of subschemes of schemes U. Equivalently, the map p is flasque if and only if the simplicial set map

$$\mathbf{hom}(U, X) \xrightarrow{(i^*, p_*)} \mathbf{hom}(\cup_{i=1}^n U_i, X) \times_{\mathbf{hom}(\cup_{i=1}^n U_i, Y)} \mathbf{hom}(U, Y)$$

is a Kan fibration.

Note in particular that a simplicial presheaf X is flasque if and only if the map $X \to *$ is flasque. The class of flasque maps is clearly stable under pullback.

One also has the following:

Lemma 1.8. *Suppose that $p : X \to Y$ is a flasque map of simplicial presheaves, and suppose that $j : A \hookrightarrow B$ is an inclusion of schemes. Then the induced map*

$$\mathbf{Hom}(B, X) \xrightarrow{(j^*, p_*)} \mathbf{Hom}(A, X) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(B, Y)$$

is flasque.

Proof. The map in U-sections induced by (j^*, p_*) is isomorphic to the map

$$X(B \times U) \to X(A \times U) \times_{Y(A \times U)} Y(B \times U)$$

which is induced by restriction along the subscheme $A \times U$ of $B \times U$. This map is a Kan fibration since p is flasque, so that (j^*, p_*) is a pointwise Kan fibration.

Any lifting problem for the cofibration (1.4) and the map (j^*, p_*) is equivalent to the extension problem for the map $p : X \to Y$ corresponding to the collection of subschemes consisting of $U_i \times B$, $i = 1, \ldots, n$, as well as $U \times A$ of the scheme $U \times B$. □

Corollary 1.9. *Suppose that X is a flasque simplicial presheaf and that B is a scheme. Then $\mathbf{Hom}(B, X)$ is flasque.*

Proof. If X is flasque, then $\mathbf{Hom}(\emptyset, X)$ is the constant simplicial presheaf on the Kan complex $X(\emptyset)$, and is therefore flasque. The inclusion $\emptyset \subset B$ induces a flasque map $\mathbf{Hom}(B, X) \to \mathbf{Hom}(\emptyset, X)$, by Lemma 1.8, so that $\mathbf{Hom}(B, X)$ is flasque. □

Corollary 1.10. *Suppose that X is a pointed flasque simplicial presheaf and that $j : A \hookrightarrow B$ is an inclusion of schemes. Then $\mathbf{Hom}_*(B/A, X)$ is flasque.*

Proof. $\mathbf{Hom}_*(B/A, X)$ is the fibre of the flasque map $j^* : \mathbf{Hom}(B, X) \to \mathbf{Hom}(A, X)$. □

Lemma 1.11. *Suppose that the simplicial presheaf X is flasque, and that $j : K \hookrightarrow L$ is an inclusion of simplicial sets. Then the simplicial presheaf map*

$$j^* : \mathbf{hom}(L, X) \to \mathbf{hom}(K, X)$$

is flasque.

Proof. Write $X^L = \mathbf{hom}(L,X)$. We must solve the lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathbf{hom}(U,X^L) \arrow[d, "(i^*,j^*)"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathbf{hom}(\cup_i U_i, X^L) \times_{\mathbf{hom}(\cup_i U_i, X^K)} \mathbf{hom}(U, X^K) \end{tikzcd}$$

An adjointness argument says that this problem is isomorphic to the lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathbf{hom}(U,X)^L \arrow[d, "(i^*,j^*)"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathbf{hom}(\cup_i U_i, X)^L \times_{\mathbf{hom}(\cup_i U_i, X)^K} \mathbf{hom}(U, X)^K \end{tikzcd}$$

But i^* is a fibration, so the lifting problem is solved by SM7 for simplicial sets. □

Lemma 1.12. *Suppose that $g : A \to B$ is a map of schemes, and that X is a pointed flasque simplicial presheaf. Let M_g denote the mapping cylinder for g in the simplicial presheaf category, and let $C_g = M_g/A$ be the homotopy cofibre. Then the standard cofibration $j : A \hookrightarrow M_g$ associated to g induces a flasque map*

$$j^* : \mathbf{Hom}(M_g, X) \to \mathbf{Hom}(A, X).$$

The simplicial presheaves $\mathbf{Hom}(M_g, X)$ and $\mathbf{Hom}_*(C_g, X)$ are flasque.

Proof. The second claim follows from the first. The mapping cylinder M_g is defined by a pushout diagram

$$\begin{tikzcd} A \sqcup A \arrow[r, "g \sqcup 1_A"] \arrow[d, "(d^0,d^1)"'] & B \sqcup A \arrow[d, "d_*"] \\ A \times \Delta^1 \arrow[r] & M_g \end{tikzcd}$$

and the map j is the composite

$$A \xrightarrow{in_R} B \sqcup A \xrightarrow{d_*} M_g.$$

The map $d = (d^0, d^1)$ induces a flasque map

$$\mathbf{Hom}(A \times \Delta^1, X) \xrightarrow{d^*} \mathbf{Hom}(A \times \partial\Delta^1, X),$$

by Lemma 1.11 since $\mathbf{Hom}(A, X)$ is flasque by Corollary 1.9. Flasque maps are closed under pullback, so the map

$$d^* : \mathbf{Hom}(M_g, X) \to \mathbf{Hom}(B \sqcup A, X)$$

is flasque. The inclusion $in_R : A \to B \sqcup A$ induces the projection map

$$\mathbf{Hom}(B, X) \times \mathbf{Hom}(A, X) \to \mathbf{Hom}(A, X)$$

which is flasque since the simplicial presheaf $\mathbf{Hom}(B, X)$ is flasque. Flasque maps are closed under composition, so we're done. □

Example 1.13. Suppose that T is the quotient $\mathbb{A}^1/(\mathbb{A}^1 - 0)$, and suppose that X is a flasque simplicial presheaf. Then the object $\mathbf{Hom}_*(T, X)$ is the fibre of the flasque map

$$\mathbf{Hom}(\mathbb{A}^1, X) \xrightarrow{i^*} \mathbf{Hom}(\mathbb{A}^1 - 0, X),$$

which is induced by the inclusion $i : \mathbb{A}^1 - 0 \subset \mathbb{A}^1$, so that $\mathbf{Hom}_*(T, X)$ is flasque by Corollary 1.10.

There is an isomorphism

$$\mathbf{Hom}(U, X)(V) \cong X(U \times V),$$

which is natural for all objects U and V of the underlying site. It follows that there is a fibre sequence

$$\mathbf{Hom}_*(T, X)(U) \to X(\mathbb{A}^1 \times U) \to X((\mathbb{A}^1 - 0) \times U)$$

if X is flasque, so that the functor $\mathbf{Hom}_*(T,\ )$ preserves pointwise weak equivalences of flasque simplicial presheaves. It follows as well that the functor $\mathbf{Hom}_*(T,\ )$ preserves filtered colimits of simplicial presheaves.

Example 1.14. Suppose that K is a finite pointed simplicial set, identified with a constant simplicial presheaf. Then there is an isomorphism

$$\mathbf{Hom}_*(K, X) \cong \mathbf{hom}_*(K, X),$$

and the functor $\mathbf{hom}_*(K,\ )$ is flasque by Lemma 1.11. The functor $\mathbf{hom}_*(K,\ )$ preserves pointwise weak equivalences of pointed simplicial presheaves consisting of Kan complexes, so that it preserves pointwise weak equivalences of flasque simplicial presheaves. The functor $\mathbf{hom}(K,\ )$ commutes with all filtered colimits since K is finite.

2 Motivic stable categories

In this section, we work exclusively with spectrum objects defined by T on the smooth Nisnevich site $(Sm|_S)_{Nis}$, where T is a pointed simplicial presheaf which is compact in the sense described below; examples of such T include the quotient $\mathbb{A}^1/(\mathbb{A}^1 - 0)$ and all constant simplicial presheaves associated to pointed finite simplicial sets. The object of the section is to develop a stable homotopy theory of spectrum objects defined by T, or T-spectra, in the motivic context. The motivic stable category of Morel and Voevodsky arises as a special case, as does a motivic stable homotopy theory for ordinary S^1-spectra.

Warning: We shall work almost entirely within the motivic closed model structure henceforth. In particular, all fibrations will be motivic fibrations and all weak equivalences will be motivic weak equivalences, unless explicit mention is made to the contrary.

Formally, if T is a pointed simplicial presheaf, then a *T-spectrum* X consists of pointed simplicial presheaves X^n, $n \geq 0$, and pointed maps $\sigma : T \wedge X^n \to X^{n+1}$. The maps σ are called *bonding maps*; it is a fact of life (see Section 3.4) that it matters whether one writes $T \wedge X^n$ or $X^n \wedge T$ in the description of these maps — I shall always display them by smashing with T on the left.

There is an obvious category $\mathbf{Spt}_T(Sm|_S)_{Nis}$ of T-spectra. If T is the Morel-Voevodsky object $\mathbb{A}^1/(\mathbb{A}^1 - 0)$ then the corresponding category of T-spectra is the basis for the motivic stable category.

2.1 The level structures

For arbitrary pointed simplicial presheaves T, there are two preliminary closed model structures on T-spectra which are analogous to the level fibration and level cofibration structures for ordinary presheaves of spectra (aka. S^1-spectra in this language), but where the level equivalences are motivic weak equivalences.

Say that a map $f : X \to Y$ of T-spectra is a

1) *level cofibration* if all component maps $f : X^n \to Y^n$ are cofibrations of simplicial presheaves,
2) *level fibration* if all component maps $f : X^n \to Y^n$ are fibrations (ie. motivic fibrations),
3) *level equivalence* if all component maps $f : X^n \to Y^n$ are motivic weak equivalences

A *cofibration* is a map which has the left lifting property with respect to all maps which are level fibrations and level weak equivalences. An *injective fibration* is a map which has the right lifting property with respect to all maps which are level cofibrations and level equivalences.

Lemma 2.1. *1) The category $\mathbf{Spt}_T((Sm|_S)_{Nis})$ of T-spectra, together with the classes of cofibrations, level equivalences and level fibrations, satisfies the axioms for a proper closed simplicial model category.*

2) The category $\mathbf{Spt}_T((Sm|_S)_{Nis})$, together with the classes of level cofibrations, level equivalences and injective fibrations, satisfies the axioms for a proper closed simplicial model category.

Proof. For the first part (following [2]), suppose that a map $i : A \to B$ satisfies

a) $i^0 : A^0 \to B^0$ is a cofibration of simplicial presheaves, and

b) each map $i_* : T \wedge B^n \cup_{T \wedge A^n} A^{n+1} \to B^{n+1}$ is a cofibration.

Then i is a cofibration. Further, if i^0 and all maps i_* as above are cofibrations and equivalences, then i is a level equivalence as well as a cofibration. These two observations are the basis of proof for the factorization axiom CM5. Further, it's a consequence of the factorization axiom that every cofibration satisfies the two properties above. The axiom CM4 follows, and the rest of the axioms are trivial.

For the second statement, suppose that α is an infinite cardinal which is an upper bound for the cardinality of the set of morphisms $\mathrm{Mor}((Sm|_S)_{Nis})$. As in [4], choose a cardinal $\kappa > 2^\alpha$ and set $\lambda = 2^\kappa$. The axioms sE1 − sE7 of [4] and their consequences apply to categories of T-spectra. We verify the bounded cofibration axiom sE7; the remaining axioms are easily verified, giving statement 2) according to the methods of [4].

Recall that the classes of cofibrations and equivalences of simplicial presheaves on $(Sm|_S)_{Nis}$ together satisfy the bounded cofibration condition for the cardinal λ in the sense that, given a diagram

$$\begin{tikzcd} & X \arrow[d, "i"] \\ A \arrow[r, "j"'] & Y \end{tikzcd} \tag{2.1}$$

such that the cofibration i is an equivalence and the subobject A of Y is λ-bounded, there is a λ-bounded suboject B of Y with $A \subset B$, with $B \cap X \hookrightarrow B$ an equivalence.

Suppose now that the objects and maps of diagram (2.1) are in the category of T-spectra, where i is a level equivalence and a level cofibration and A is λ-bounded. There is a simplicial presheaf B^0 with $A^0 \subset B^0 \subset Y^0$ such that B^0 is λ-bounded and the cofibration $B^0 \cap X^0 \hookrightarrow B^0$ is an equivalence. Write j' for the inclusion $B^0 \hookrightarrow Y^0$ and use the diagram

$$\begin{tikzcd} T \wedge A^0 \arrow[r] \arrow[d, "\sigma"'] & T \wedge B^0 \arrow[d, "\sigma\cdot(T\wedge j')"] \\ A^1 \arrow[r, "j"'] & Y^1 \end{tikzcd}$$

to show that there is a λ-bounded subobject $\overline{A}^1 \subset Y^1$ such that the map

$$A^1 \cup_{T \wedge A^0} T \wedge B^0 \to Y^1$$

factors through $\overline{A}^1$. There is a λ-bounded subobject $B^1 \subset Y^1$ with $\overline{A}^1 \subset B^1$ such that the cofibration $B^1 \cap X^1 \hookrightarrow B^1$ is an equivalence. This is the beginning of an inductive construction which produces a λ-bounded subobject B of the T-spectrum Y with $A \subset B$ such that the level cofibration $B \cap X \hookrightarrow B$ is a level equivalence. □

Insofar as the factorization axiom CM5 in part (2) of Lemma 2.1 is covertly proved by using a small object argument, there is a natural injective model construction: there is a natural map of T-spectra $i_X : X \to IX$, such that i_X is a level cofibration and a level equivalence, and IX is injective. More generally, any level equivalence $X \to Y$ with Y injective is said to be an *injective model* for X.

There is a natural *level fibrant model* $j_X : X \to JX$, meaning that j_X is a cofibration and a level equivalence and JX is level fibrant. This can be constructed directly from the small object arguments, or by using the controlled fibrant object construction $X \mapsto \mathcal{L}X$ of [4] (see also Section 1.2). Note as well that every injective object is level fibrant.

2.2 Compact objects

Say that a simplicial presheaf X on $(Sm|_S)_{Nis}$ is *motivic flasque* if

1) X is flasque, and

2) every map $X(U) \to X(\mathbb{A}^1 \times U)$ induced by the projection $\mathbb{A}^1 \times U \to U$ is a weak equivalence of simplicial sets.

Every motivic fibrant simplicial presheaf on $(Sm|_S)_{Nis}$ is motivic flasque, and the class of motivic flasque simplicial presheaves is closed under filtered colimits.

A pointed simplicial presheaf T on the smooth Nisnevich site is said to be *compact* if the following conditions hold:

C1: All inductive systems $Y_1 \to Y_2 \to \dots$ of pointed simplicial presheaves induce isomorphisms

$$\mathbf{Hom}_*(T, \varinjlim Y_i) \cong \varinjlim \mathbf{Hom}_*(T, Y_i).$$

C2: If X is motivic flasque, then so is $\mathbf{Hom}_*(T, X)$.

C3: The functor $\mathbf{Hom}_*(T, \ )$ takes pointwise weak equivalences of motivic flasque simplicial presheaves to pointwise weak equivalences.

The following result generates examples of compact simplicial presheaves:

Lemma 2.2. *1) If $A \hookrightarrow B$ is an inclusion of schemes, then the quotient B/A is compact.*

2) All finite pointed simplicial sets K are compact.

3) All pointed schemes U in the underlying site $(Sm|_S)_{Nis}$ are compact.

4) If T_1 and T_2 are compact, then $T_1 \vee T_2$ and $T_1 \wedge T_2$ and are compact.

5) If $g : T_1 \to T_2$ is a map of compact simplicial presheaves, then the pointed mapping cylinder M_g and the homotopy cofibre C_g are compact.

Proof. If X is motivic flasque, then $\mathbf{Hom}_*(B/A, X)$ is flasque by Corollary 1.10. We also know that there is an isomorphism

$$\mathbf{Hom}(B, X)(V) \cong X(B \times V)$$

and a pointwise fibre sequence

$$\mathbf{Hom}_*(B/A, X) \to \mathbf{Hom}(B, X) \to \mathbf{Hom}(A, X) \tag{2.2}$$

All maps

$$\mathbf{Hom}(B, X)(V) \to \mathbf{Hom}(B, X)(V \times \mathbb{A}^1)$$

induced by projection are weak equivalences of simplicial sets. It follows that $\mathbf{Hom}_*(B/A, X)$ is motivic flasque. The functor $X \mapsto \mathbf{Hom}_*(B/A, X)$ preserves filtered colimits of simplicial presheaves. The fibre sequences (2.2) imply that the functor $\mathbf{Hom}_*(B/A, \ )$ preserves pointwise weak equivalences of motivic flasque simplicial presheaves, giving 1).

Statement 2) is proved by first observing that there is a natural isomorphism

$$\mathbf{Hom}_*(K, X) \cong \mathbf{hom}_*(K, X).$$

The functor $X \mapsto \mathbf{hom}_*(K, X)$ preserves filtered colimits since K is a finite simplicial set. The statement C3 is trivial, and C2 follows from Lemma 1.11, and the functor $X \mapsto \mathbf{hom}_*(K, X)$ preserves pointwise weak equivalences of pointed presheaves of Kan complexes.

Statement 3) is a consequence of statement 1), and the smash product part of statement 4) is an adjointness argument.

Suppose that X is motivic flasque. The diagram

$$\begin{tikzcd} T_1 \vee T_1 \arrow[r] \arrow[d] & T_1 \vee T_2 \arrow[d] \\ T_1 \wedge \Delta^1_+ \arrow[r] & M_g \end{tikzcd}$$

that defines the pointed mapping cylinder M_g induces a pullback diagram

$$\begin{tikzcd} \mathbf{Hom}_*(M_g, X) \arrow[r] \arrow[d] & \mathbf{Hom}_*(T_1 \wedge \Delta^1_+, X) \arrow[d] \\ \mathbf{Hom}_*(T_1 \vee T_2, X) \arrow[r] & \mathbf{Hom}_*(T_1 \vee T_1, X) \end{tikzcd} \tag{2.3}$$

and the map

$$\mathbf{Hom}_*(T_1 \wedge \Delta^1_+, X) \to \mathbf{Hom}_*(T_1 \vee T_1, X)$$

is flasque, by the pointed version of Lemma 1.11. $\mathbf{Hom}_*(M_g, X)$ is therefore flasque. The composite

$$\mathbf{Hom}_*(M_g, X) \to \mathbf{Hom}_*(T_1 \vee T_2, X) \to \mathbf{Hom}(T_2, X)$$

is also flasque, and so the pointwise homotopy fibre $\mathbf{Hom}_*(C_g, X)$ is flasque. The objects other than $\mathbf{Hom}_*(M_g, X)$ in the pointwise fibre square (2.3) take the projections $U \times \mathbb{A}^1 \to U$ to weak equivalences. Properness for simplicial sets therefore implies that the simplicial presheaves $\mathbf{Hom}_*(M_g, X)$ and $\mathbf{Hom}_*(C_g, X)$ are motivic flasque. Similarly, the functors $\mathbf{Hom}_*(M_g, \ )$ and $\mathbf{Hom}_*(C_g, \ )$ preserve pointwise weak equivalences of motivic flasque objects. Both functors preserve filtered colimits, since they are built in finitely many steps from functors that do the same. We have proved statement 5). □

Remark 2.3. One can show that statement 1) of Lemma 2.2 follows from statement 5), but the presented proof is easier. Statement 1) implies that the Morel-Voevodsky object $T = \mathbb{A}^1/(\mathbb{A}^1 - 0)$ is compact.

2.3 The stable closed model structure

Suppose that T is a compact pointed simplicial presheaf on the smooth Nisnevich site $(Sm|_S)_{Nis}$.

The *T-loops functor* $\Omega_T Y$ is defined for pointed simplicial presheaves Y in terms of internal hom by

$$\Omega_T Y = \mathbf{Hom}_*(T, Y).$$

The T-loops functor is right adjoint to smashing with T, and so the bonding maps $\sigma : T \wedge X^n \to X^{n+1}$ of a presheaf of T-spectra X can equally well be specified by their adjoints $\sigma_* : X^n \to \Omega_T X^{n+1}$, up to a twist: σ_* is the adjoint of the composite

$$X^n \wedge T \xrightarrow[\cong]{t} T \wedge X^n \xrightarrow{\sigma} X^{n+1},$$

where t is the isomorphism which flips smash factors.

The T-loops functor $\Omega_T X$ is defined on T-spectra X by setting $(\Omega_T X)^n = \Omega_T(X^n)$, and by specifying that the bonding map $\sigma : T \wedge \Omega_T X^n \to \Omega_T X^{n+1}$ should be adjoint to the composite

$$T \wedge \Omega_T X^n \wedge T \xrightarrow{T \wedge ev} T \wedge X^n \xrightarrow{\sigma} X^{n+1}.$$

The T-loops functor $X \mapsto \Omega_T X$ is right adjoint to the functor $Y \mapsto Y \wedge T$ which is defined by smashing with T on the right. More generally, there is a function complex functor $X \mapsto \mathbf{Hom}_*(A, X)$ for all T-spectra X and pointed simplicial presheaves A, and this functor is right adjoint to the functor $X \mapsto X \wedge A$ defined by smashing on the right with A in the obvious way.

Just as in ordinary stable homotopy theory (see [11, Chapter 1]), there is a *fake T-loops spectrum* $\Omega_T^\ell X$, with

$$(\Omega_T^\ell X)^n = \Omega_T(X^n),$$

and with bonding maps adjoint to the morphisms

$$\Omega_T(\sigma_*) : \Omega_T(X^n) \to \Omega_T^2(X^{n+1}).$$

The fake T-loop suspension functor is right adjoint to the *fake suspension functor* $Y \mapsto \Sigma_T^\ell Y$, where $\Sigma_T^\ell Y^n = T \wedge Y^n$ and the bonding maps $T \wedge \Sigma_T^\ell Y^n \to \Sigma_T^\ell Y^{n+1}$ are defined to be the morphisms $T \wedge \sigma : T^2 \wedge Y^n \to T \wedge Y^{n+1}$. Generally, the superscript ℓ for "left": the functor $X \mapsto \Omega_T^\ell X$ is the right adjoint of $Y \mapsto \Sigma_T^\ell Y$, which is defined by smashing with T on the left.

Remark 2.4. The fake T-loop spectrum $\Omega_T^\ell X$ is *not* isomorphic to the T-loop spectrum $\Omega_T X$, since the adjoint $\sigma_* : \Omega_T X^n \to \Omega_T^2 X^{n+1}$ of the bonding map $\sigma : T \wedge \Omega_T X^n \to \Omega_T X^{n+1}$ differs from the map $\Omega_T \sigma_*$ by a twist of loop factors. This phenomenon is the source of much of the technical fun in stable homotopy theory, and the present discussion is no exception — see the proof of Theorem 3.11.

The maps σ_* determine a natural morphism of T-spectra

$$\sigma_* : X \to \Omega_T^\ell X[1],$$

where the shifted T-spectrum $X[1]$ is defined by $X[1] = X^{n+1}$. The T-spectrum $Q_T X$ is defined to be the inductive colimit of the system

$$X \xrightarrow{\sigma_*} \Omega_T^\ell X[1] \xrightarrow{\Omega_T^\ell \sigma_*[1]} (\Omega_T^\ell)^2 X[2] \xrightarrow{(\Omega_T^\ell)^2 \sigma_*[2]} \cdots$$

Write $\eta_X : X \to Q_T X$ for the associated canonical map. We shall be particularly interested in the composite map

$$X \xrightarrow{j_X} JX \xrightarrow{\eta_{JX}} Q_T JX,$$

which will be denoted by $\tilde{\eta}_X$. The functor Q_T is sometimes called the *stabilization functor*, for the object T.

A map $g : X \to Y$ of T-spectra is said to be a *stable equivalence* if it induces a level equivalence

$$Q_T J(g) : Q_T JX \to Q_T JY.$$

Observe that g is a stable equivalence if and only if it induces a level equivalence

$$IQ_T J(g) : IQ_T JX \to IQ_T JY.$$

More usefully, perhaps, it is a consequence of Corollary 1.7 that g is a stable equivalence if and only if the induced map $Q_T J(g)$ is a pointwise equivalence of motivic flasque simplicial presheaves in all levels.

A *stable fibration* is a map which has the right lifting property with respect to all maps which are cofibrations and stable equivalences. A T-spectrum X is said to be *stably fibrant* if the map $T \to *$ is a stable fibration.

We shall prove the following statements:

A4 Every level equivalence is a stable equivalence

A5 The maps

$$\tilde{\eta}_{Q_T JX}, Q_T J(\tilde{\eta}_X) : Q_T JX \to (Q_T J)^2 X$$

are stable equivalences.

A6 Stable equivalences are closed under pullback along stable fibrations, and stable equivalences are closed under pushout along cofibrations.

Lemma 2.5. *The statements* A4 *and* A5 *hold for T-spectra.*

Proof. If $g : X \to Y$ is a level equivalence between T-spectra such that X and Y are level fibrant, then g is a pointwise weak equivalence of motivic flasque objects in all levels, and so all $\Omega_T^n g$ and $Q_T g$ are level pointwise equivalences by C2 and C3. This proves A4.

The map $Q_T J(j_X) : Q_T JX \to Q_T J^2 X$ is a level equivalence by A4. There is a commutative diagram

$$\begin{tikzcd}
Q_T J^2 X \arrow[r, "Q_T J(\eta_{JX})"] & Q_T J Q_T JX \\
Q_T JX \arrow[u, "Q_T(j_{JX})"] \arrow[r, "Q_T(\eta_{JX})"'] & Q_T Q_T JX \arrow[u, "Q_T(j_{Q_T JX})"']
\end{tikzcd}$$

The vertical map $Q_T(j_{JX})$ is a level equivalence because j_{JX} is a pointwise weak equivalence of motivic flasque simplicial presheaves in each level, and Q_T preserves such by C2 and C3. All maps $Q_T(\eta_Z)$ are isomorphisms by C1 and a cofinality argument. The map $j_{Q_T JX}$ is a pointwise weak equivalence of motivic flasque simplicial presheaves in each level by Corollary 1.7, and so the map $Q_T(j_{Q_TJX})$ has the same property by C2 and C3. It follows that $Q_TJ(\eta_{JX})$ and $Q_TJ(\tilde{\eta}_X)$ are level equivalences.

There is a commutative diagram

$$\begin{tikzcd} JQ_TJX^n \arrow[r, "\sigma_*"] & \Omega_T JQ_TJX^{n+1} \\ Q_TJX^n \arrow[u, "j_{Q_TJX}", "\simeq"'] \arrow[r, "\sigma_*"'] & \Omega_T Q_TJX^{n+1} \arrow[u, "\Omega_T(j_{Q_TJX})"'] \end{tikzcd}$$

The map j_{Q_TJX} is a level pointwise equivalence by Corollary 1.7, the lower map σ_* is an isomorphism by a cofinality argument and C1, and the map $\Omega_T(j_{Q_TJX})$ is a pointwise weak equivalence of motivic flasque simplicial presheaves by C2 and C3. It follows that all maps $\sigma_* : JQ_TJX^n \to \Omega_T JQ_TJX^{n+1}$ are pointwise weak equivalences, and so the map

$$\eta_{JQ_TJX} : JQ_TJX \to Q_TJQ_TJX$$

is a level equivalence. In particular, the composite

$$Q_TJX \xrightarrow{j_{Q_TJX}} JQ_TJX \xrightarrow{\eta_{JQ_TJX}} Q_TJQ_TJX$$

is a level equivalence. □

Lemma 2.6. *The class of stable equivalences is closed under pullback along level fibrations.*

Proof. Suppose given a pullback diagram

$$\begin{tikzcd} A \times_Y X \arrow[r, "g_*"] \arrow[d] & X \arrow[d, "p"] \\ A \arrow[r, "g"'] & Y \end{tikzcd}$$

in which g is a stable equivalence and p is a level fibration. We want to show that g_* is a stable equivalence.

By properness of the level structure and A4, we can assume that all objects are level fibrant. Every level equivalence $C \to D$ of level fibrant objects consists of pointwise weak equivalences $C^n \to D^n$ of motivic flasque simplicial presheaves, so Q_T takes each level equivalence of level fibrant objects to a map of T-spectra which consists of pointwise weak equivalences in all levels. All induced maps $Q_TA^n \to Q_TY^n$ are pointwise weak equivalences. The maps $p_* : Q_TX^n \to Q_TY^n$ are filtered colimits of pointwise Kan fibrations, and are therefore pointwise Kan fibrations. Finally, Q_T preserves pullbacks and the ordinary simplicial set category is proper, so the maps

$$Q_T(g_*) : Q_T(A \times_Y X)^n \to Q_TX^n$$

are pointwise weak equivalences of simplicial presheaves. □

Every stable fibration is a level fibration, because every level equivalence is a stable equivalence. Lemma 2.6 therefore implies the first statement of A6.

The statements A4 and A5 together imply a Bousfield-Friedlander recognition principle for stable fibrations (see Lemma A.9 of [2]):

Lemma 2.7. *A map $p : X \to Y$ is a stable fibration if p is a level fibration and the diagram*

$$\begin{tikzcd} X \arrow[r, "\tilde{\eta}_X"] \arrow[d, "p"'] & Q_T JX \arrow[d, "p_*"] \\ Y \arrow[r, "\tilde{\eta}_Y"'] & Q_T JY \end{tikzcd}$$

is level homotopy Cartesian.

In particular, a T-spectrum X is stably fibrant if X is level fibrant and the maps $\sigma_* : X^n \to \Omega_T X^{n+1}$ are equivalences (or pointwise weak equivalences). We shall need the converse assertion:

Lemma 2.8. *Suppose that X is stably fibrant. Then X is level fibrant, and all maps $\sigma_* : X^n \to \Omega_T X^{n+1}$ are pointwise weak equivalences.*

Proof. The composite

$$X \xrightarrow{j_X} JX \xrightarrow{\eta_{IX}} Q_T JX \xrightarrow{i_{Q_T JX}} IQ_T JX$$

is a stable equivalence by Lemma 2.5, and the object $IQ_T JX$ is stably fibrant since all maps

$$\sigma_* : IQ_T JX^n \to \Omega_T IQ_T JX^{n+1}$$

are pointwise weak equivalences. Write $\mu_X : X \to IQ_T JX$ for this composite.

Factorize μ_X as

$$\begin{tikzcd} X \arrow[rr, "\mu_X"] \arrow[dr, "\alpha"'] & & IQ_T JX \\ & Z \arrow[ur, "\pi"'] & \end{tikzcd}$$

where π is a level fibration and a level equivalence, and α is a cofibration. Then π is a stable fibration since it has the right lifting property with respect to all cofibrations. It follows that Z is stably fibrant and all maps $\sigma_* : Z^n \to \Omega_T Z^{n+1}$ are pointwise weak equivalences. Also, the map $\alpha : X \to Z$ is a cofibration and a stable equivalence. The object X is therefore a retract of Z, and so the maps $\sigma^* : X^n \to \Omega_T X^{n+1}$ are pointwise weak equivalences. ∎

Theorem 2.9. *Suppose that T is a compact object on the smooth Nisnevich site $(Sm|_S)_{Nis}$. Then the category of T-spectra on that site, together with the classes of cofibrations, stable equivalences and stable fibrations, satisfies the axioms for a proper closed simplicial model category.*

The homotopy category $\mathrm{Ho}(\mathbf{Spt}_T(Sm|_S)_{Nis})$ associated to the stable model structure of Theorem 2.9 is the *motivic stable category* of T-spectra on the smooth Nisnevich site. In the particular case where $T = \mathbb{A}^1/(\mathbb{A}^1 - 0)$, the category $\mathrm{Ho}(\mathbf{Spt}_T(Sm|_S)_{Nis})$ is the motivic stable category of Morel and Voevodsky — it is often denoted by $\mathcal{SH}(S)$.

Proof. The axioms CM1 – CM3 are trivial to verify. We also know (Lemma A.8 of [2], but this is also a direct consequence of Lemma 2.7) that a map p is a stable fibration and a stable equivalence if and only if it is a level fibration and a level equivalence. The existence of the cofibration-trivial fibration factorization of CM5 follows, as does CM4.

It is a consequence of Lemma 2.7 and Lemma 2.8 that a level fibration between stably fibrant objects must be a stable fibration.

To prove the remaining part of CM5, suppose given a map $g : X \to Y$ of T-spectra. Form the diagram

$$\begin{tikzcd}
X \arrow[rr, "\mu_X"] \arrow[dr, "\alpha_*"] \arrow[dd, "g"'] & & IQ_TJX \arrow[dr, "\alpha"] \arrow[dd, "g_*" near end] & \\
& Y \times_{IQ_TJY} Z \arrow[rr, "\mu_*" near start, crossing over] \arrow[dl, "p_*"] & & Z \arrow[dl, "p"] \\
Y \arrow[rr, "\mu_Y"'] & & IQ_TJY &
\end{tikzcd}$$

where p is a level fibration and α is a cofibration and a level equivalence. Then Z is level fibrant, and the maps $\alpha : IQ_TJX^n \to Z^n$ are pointwise equivalences of motivic flasque simplicial presheaves, so Z is stably fibrant. Thus, p is a stable fibration.

The map μ_* is a stable equivalence by Lemma 2.6, so that α_* is a stable equivalence. Factorize α_* as

$$\begin{tikzcd}
X \arrow[r, "\alpha'"] \arrow[dr, "\alpha_*"'] & W \arrow[d, "\pi"] \\
& Y \times_{IQ_TJY} Z
\end{tikzcd}$$

where α' is a cofibration and π is a level fibration and a level equivalence. Then α' is also a stable equivalence, and π is a stable fibration, so $f = (p_*\pi) \cdot \alpha'$ is a factorization of f as a stable fibration following a cofibration which is a stable equivalence, giving CM5.

Part of the properness assertion was proved in Lemma 2.6. For the cofibration statement, form a pushout diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "j"'] & C \arrow[d] \\ B \arrow[r, "g_*"'] & B \cup_A C \end{tikzcd}$$

where j is a cofibration and g is a stable equivalence. We must show that g_* is a stable equivalence. By properness of the level structure and by taking a suitable factorization in the level structure, we can assume that g is a cofibration. But then it's a standard fact about closed model categories that trivial cofibrations are closed under pushout.

We must finally verify Quillen's axiom SM7. Suppose that $i : K \to L$ is a cofibration of pointed simplicial sets and that $\alpha : A \to B$ is a cofibration of T-spectra. We must show that the cofibration

$$(A \wedge L) \cup_{(A \wedge K)} (B \wedge K) \to B \wedge L$$

is a stable equivalence if either j is a stable equivalence or i is a weak equivalence of simplicial sets. The case where i is a weak equivalence is a consequence of the corresponding result for the level structure. The remaining case is verified by showing that the cofibration $\alpha \wedge L : A \wedge L \to B \wedge L$ is a stable equivalence if α is a stable equivalence.

From Lemma 2.8, one sees that if W is both stably fibrant and injective, then so is $\mathbf{hom}_*(L, W)$. Also one can identify the set $[X, W]$ of stable homotopy classes of maps with $\pi_0 \mathbf{hom}(X, W)$ in the sense that the natural map

$$\pi_0 \mathbf{hom}(X, W) \to [X, W]$$

is a bijection. In effect, there is a trivial level fibration $\pi : X' \to X$ with X' cofibrant which induces an isomorphism

$$\pi_0 \mathbf{hom}(X, W) \cong \pi_0 \mathbf{hom}(X', W)$$

since W is injective and all T-spectra are cofibrant in the injective model structure (see Remark 2.10 following this proof), while $\pi_0 \mathbf{hom}(X', W) \cong [X', W] \cong [X, W]$ since X' is cofibrant and W is stably fibrant. There is an isomorphism

$$\mathbf{hom}(X, \mathbf{hom}_*(L, W)) \cong \mathbf{hom}(X \wedge L, W),$$

and so there is a natural bijection

$$[X, \mathbf{hom}_*(L, W)] \cong [X \wedge L, W]$$

of morphisms in the stable homotopy category. From Lemma 2.11 below, one sees that a map $g : X \to Y$ is a stable equivalence if and only if it induces a bijection $g^* : [Y, W] \to [X, W]$ of morphisms in the homotopy category for all injective stably fibrant objects W. It follows that $\alpha \wedge L$ is a stable equivalence if α is a stable equivalence. $\square$

Remark 2.10. In general, every map $f : A \to B$ between cofibrant objects in a closed model category has a factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "j"'] & & B \\ & C \arrow[ur, "\pi"'] & \end{tikzcd}$$

where j is a cofibration and π is left inverse to a trivial cofibration — this is really just the standard mapping cylinder construction. It follows that, in a simplicial model category, if W is fibrant and $g : A \to B$ is a weak equivalence of cofibrant objects, then the induced map

$$g^* : \mathbf{hom}(B, W) \to \mathbf{hom}(A, W)$$

is a weak equivalence of Kan complexes. This is certainly so if g is a trivial cofibration, and then one uses the above factorization to see the more general case.

Lemma 2.11. *A map $g : X \to Y$ is a stable equivalence if and only if it induces bijections*

$$g^* : [Y, W] \xrightarrow{\cong} [X, W]$$

of morphisms in the stable (equivalently, level) homotopy category for all stably fibrant injective objects W.

Proof. Every stable equivalence clearly induces a bijection

$$g^* : [Y, W] \xrightarrow{\cong} [X, W]$$

for all stably fibrant injective objects W.

For the converse, assume that all such maps g^* are bijections. The injective stably fibrant model $X \to IQ_TJX$ is a stable equivalence, so it suffices to assume that X and Y are both stably fibrant and injective. But then g must be a homotopy equivalence: the homotopy inverse of g is a pre-image under g^* of the class of 1_X for the case $W = X$. □

With the proof of Theorem 2.9 now completely in hand, Lemma 2.11 can be bootstrapped to the following:

Corollary 2.12. *A map $g : X \to Y$ of T-spectra is a stable equivalence if and only if it induces a weak equivalence*

$$g^* : \mathbf{hom}(Y, W) \to \mathbf{hom}(X, W)$$

of Kan complexes for all stably fibrant injective objects W.

Proof. If $g : X \to Y$ is a level equivalence, then the induced map

$$g^* : \mathbf{hom}(Y, W) \to \mathbf{hom}(X, W)$$

is a weak equivalence for all stably fibrant injective objects W, since all objects in the injective simplicial model structure are cofibrant and we can use Remark 2.10.

Suppose that $g : X \to Y$ is a stable equivalence. Then there is a diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{g}"] \arrow[d, "\pi_X"'] & \tilde{Y} \arrow[d, "\pi_Y"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

such that $\tilde{X}$ and $\tilde{Y}$ are cofibrant and the maps π_X and π_Y are trivial level fibrations. Then, for example, π_X induces a weak equivalence $\pi_X^* : \mathbf{hom}(X, W) \to \mathbf{hom}(\tilde{X}, W)$ for all stably fibrant injective objects W by the previous paragraph. It suffices, therefore, to assume that X and Y are cofibrant, but then Remark 2.10 can be used in the stable simplicial model structure to show that g^* is a weak equivalence of simplicial sets.

For the reverse direction, suppose that $g^* : \mathbf{hom}(Y, W) \to \mathbf{hom}(X, W)$ is a weak equivalence for all stably fibrant injective W. Then by computing in π_0, the induced map

$$g^* : [Y, W] \to [X, W]$$

of morphisms in the homotopy category is a bijection for all stably fibrant injective W, and Lemma 2.11 can be applied. □

2.4 Change of suspension

Any map $\theta : T_1 \to T_2$ of pointed simplicial presheaves on the site $(Sm|_S)_{Nis}$ induces a functor

$$\theta^* : \mathbf{Spt}_{T_2}(Sm|_S)_{Nis} \to \mathbf{Spt}_{T_1}(Sm|_S)_{Nis},$$

by precomposing the bonding maps with θ. More precisely, for any T_2-spectrum X, $\theta^* X$ is the T_1-spectrum with $(\theta^* X)^n = X^n$, with bonding maps given by the composites

$$T_1 \wedge X^n \xrightarrow{\theta \wedge 1} T_2 \wedge X^n \xrightarrow{\sigma} X^{n+1}.$$

There is homotopical content to this construction when T_1 and T_2 are compact and θ is an equivalence:

Proposition 2.13. *Suppose that $\theta : T_1 \to T_2$ is a weak equivalence of compact objects on the site $(Sm|_S)_{Nis}$. Then the functor θ^* induces an equivalence of motivic stable homotopy categories*

$$\theta^* : \mathrm{Ho}(\mathbf{Spt}_{T_2}(Sm|_S)_{Nis}) \to \mathrm{Ho}(\mathbf{Spt}_{T_1}(Sm|_S)_{Nis}).$$

Proof. Write σ_θ for the bonding maps of $\theta^* X$. The functor θ^* clearly preserves level equivalences, level fibrations and level cofibrations. If X is level fibrant, there is a diagram

$$\begin{tikzcd}
X^n \arrow[r, "\sigma"] \arrow[dr, "\sigma_\theta"'] & \Omega_{T_2} X^{n+1} \arrow[r, "\Omega_{T_2}\sigma"] \arrow[d, "\theta^*"] & \Omega^2_{T_2} X^{n+2} \arrow[d, "\theta^*"] & \cdots \\
& \Omega_{T_1} X^{n+1} \arrow[r, "\Omega_{T_1}\sigma"] \arrow[dr, "\Omega_{T_1}\sigma_\theta"'] & \Omega_{T_1}\Omega_{T_2} X^{n+2} \arrow[d, "\Omega_{T_1}\theta^*"] & \cdots \\
& & \Omega^2_{T_1} X^{n+2} & \cdots
\end{tikzcd}$$

All vertical maps are pointwise weak equivalences, so there are induced natural pointwise weak equivalences $\theta^* : Q_{T_2} X^n \to Q_{T_1}\theta^* X^n$ for level fibrant objects X. It follows that $g : X \to Y$ is a stable equivalence of T_2-spectra if and only if $\theta^* g : \theta^* X \to \theta^* Y$ is a stable equivalence of presheaves of T_1-spectra. In particular, θ^* induces a functor

$$\theta^* : \mathrm{Ho}(\mathbf{Spt}_{T_2}(Sm|_S)_{Nis}) \to \mathrm{Ho}(\mathbf{Spt}_{T_1}(Sm|_S)_{Nis}).$$

on stable homotopy categories. It also follows, using Lemma 2.7, that θ^* preserves stable fibrations.

To go further, we must presume that θ is a cofibration as well as an equivalence. This suffices, since the factorization trick of Remark 2.10 involves the mapping cylinder, and we have Lemma 2.2.

Given this new assumption, one can further show that θ^* preserves cofibrations: given a cofibration $i : A \to B$ of T_2-spectra, there is a pushout diagram

$$\begin{tikzcd}
(T_1 \wedge B^n) \cup_{(T_1 \wedge A^n)} (T_2 \wedge B^n) \arrow[r] \arrow[d, "(\theta,i)_*"'] & (T_1 \wedge B^n) \cup_{(T_1 \wedge A^n)} A^{n+1} \arrow[d, "\theta_*"] \\
T_2 \wedge B^n \arrow[r] & (T_2 \wedge B^n) \cup_{(T_2 \wedge A^n)} A^{n+1}
\end{tikzcd}$$

in which $(\theta, i)_*$ is a cofibration. The canonical map

$$(T_1 \wedge B^n) \cup_{(T_1 \wedge A^n)} A^{n+1} \to B^{n+1}$$

for $\theta^* i$ is the composite

$$(T_1 \wedge B^n) \cup_{(T_1 \wedge A^n)} A^{n+1} \xrightarrow{\theta_*} (T_2 \wedge B^n) \cup_{(T_2 \wedge A^n)} A^{n+1} \to B^{n+1},$$

so $\theta^* i$ is a cofibration of T_1-spectra if i is a cofibration of T_2-spectra.

Every stably fibrant T_1-spectrum X is of the form $X = \theta^* \overline{X}$ for some stably fibrant T_2-spectrum $\overline{X}$. To see this, let $\overline{X}^n = X^n$, and choose bonding maps $\overline{\sigma} : T_2 \wedge X^n \to X^{n+1}$ making the following diagram commute:

$$\begin{tikzcd} T_1 \wedge X^n \arrow[r, "\sigma"] \arrow[d, "\theta\wedge 1"'] & X^{n+1} \\ T_2 \wedge X^n \arrow[ur, "\overline{\sigma}"'] & \end{tikzcd}$$

One gets away with this because $\theta \wedge 1$ is a trivial cofibration. It follows that every stably fibrant T_1-spectrum X is stably equivalent to a T_1-spectrum $\theta^* Y$, where Y is a stably fibrant and cofibrant T_2-spectrum.

To finish off the proof, the idea is to show that $\theta : T_1 \to T_2$ induces a weak equivalence of Kan complexes

$$\mathbf{hom}(A, X) \xrightarrow{\theta_*} \mathbf{hom}(\theta^* A, \theta^* X)$$

for all cofibrant A and stably fibrant X. Computing in π_0 implies that θ induces bijections

$$\theta^* : [Y, X] \xrightarrow{\cong} [\theta^* Y, \theta^* X]$$

for all stably fibrant, cofibrant objects X and Y. The desired result then follows from basic category theory.

We show that θ^* is a weak equivalence of Kan complexes by showing that, given any solid arrow diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathbf{hom}(A, X) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathbf{hom}(\theta^* A, \theta^* X) \end{tikzcd}$$

a dotted arrow exists such that

1) the upper triangle commutes, and

2) the lower triangle commutes up to homotopy which is constant on $\partial\Delta^n$.

This homotopy lifting property is implied by the following: given any solid arrow commutative diagrams

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "j"'] & X \\ B \arrow[ur, dotted, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} \theta^* A \arrow[r, "\theta^*\alpha"] \arrow[d, "\theta^* j"'] & \theta^* X \\ \theta^* B \arrow[ur, "f"'] & \end{tikzcd}$$

with A is cofibrant, j is a cofibration and X is stably fibrant, then the dotted arrow g exists making the diagram of T_2-spectra commute, and there is a homotopy $\theta^* g \simeq f$ which is constant at $\theta^*\alpha$ on $\theta^* A$.

This last property is proved by a homotopy extension argument which depends on the assumption that θ is a trivial cofibration. The method is to inductively find the dotted arrows h and g making the following diagrams simultaneously commute

$$\begin{tikzcd}
B^{n+1} \arrow[d, "d^1"'] \arrow[dr, "f"] & \\
B^{n+1} \ltimes \Delta^1 \arrow[r, dotted, "h"] & X^{n+1} \\
B^{n+1} \arrow[u, "d^0"] \arrow[ur, dotted, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
T_1 \wedge B^n \ltimes \Delta^1 \arrow[r, "T_1 \wedge h"] \arrow[d, "\theta\wedge 1"'] & T_1 \wedge X^n \arrow[d, "\theta\wedge 1"] \\
T_2 \wedge B^n \ltimes \Delta^1 \arrow[d, "\sigma \ltimes \Delta^1"'] & T_2 \wedge X^n \arrow[d, "\sigma"] \\
B^{n+1} \ltimes \Delta^1 \arrow[r, dotted, "h"'] & X^{n+1}
\end{tikzcd}$$

$$\begin{tikzcd}
A^{n+1} \ltimes \Delta^1 \arrow[r, "pr"] \arrow[d, "j \ltimes \Delta^1"'] & A^{n+1} \arrow[d, "\alpha"] \\
B^{n+1} \ltimes \Delta^1 \arrow[r, dotted, "h"'] & X^{n+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
T_2 \wedge B^n \arrow[r, "T_2 \wedge g"] \arrow[d, "\sigma"'] & T_2 \wedge X^n \arrow[d, "\sigma"] \\
B^{n+1} \arrow[r, dotted, "g"'] & X^{n+1}
\end{tikzcd}$$

The inclusion of

$$(A^{n+1} \cup (T_1 \wedge B^n)) \ltimes \Delta^1 \cup (A^{n+1} \cup (T_2 \wedge B^n)) \ltimes \partial\Delta^1$$

in $(A^{n+1} \cup (T_2 \wedge B^n)) \ltimes \Delta^1$ is a trivial cofibration since θ is trivial, so that the composite homotopy

$$T_1 \wedge B^n \ltimes \Delta^1 \xrightarrow{T_1 \wedge h} T_1 \wedge X^n \xrightarrow{\theta\wedge 1} T_2 \wedge X^n \xrightarrow{\sigma} X^{n+1}$$

extends to a homotopy $\tilde{h} : T_2 \wedge B^n \ltimes \Delta^1 \to X^{n+1}$ from $f \cdot \sigma$ to $\sigma \cdot (T_2 \wedge g)$ which is constant on A^{n+1}. The homotopy $\tilde{h}$ extends to the desired map h in the usual way, since the map

$$(A^{n+1} \cup (T_2 \wedge B^n)) \ltimes \Delta^1 \cup B^{n+1} \times \{0\} \to B^{n+1} \ltimes \Delta^1$$

is a trivial cofibration. $\square$

2.5 Bounded cofibrations

The commutativity of the diagram (1.2) for the controlled fibrant model construction $X \mapsto \mathcal{L}X$ of Section 1 implies that this construction can be promoted to the category of T-spectra. More explicitly, there is a natural level fibrant model $\eta_X : X \to \mathcal{L}X$ defined for T-spectra such that the map η_X is a level cofibration and a level equivalence. The standard properties of the functor $\mathcal{L}$ (see Section 1.1) pass to the spectrum level, and so the functor $\mathcal{L}$ is an example of a functor $F : \mathbf{Spt}_T(Sm|_S)_{Nis} \to \mathbf{Spt}_T(Sm|_S)_{Nis}$ which satisfies the following:

L1: F preserves level weak equivalences.

L2: F preserves level cofibrations.

L3: Let β be any cardinal with $\beta \geq \alpha$. Let $\{X_j\}$ be the filtered system of sub-objects of X which are β-bounded. Then the map

$$\varinjlim F(X_j) \to FX$$

is an isomorphism.

L4: Let γ be an ordinal number of cardinality strictly greater than 2^α. Let $X : \gamma \to \mathbf{Spt}_T(Sm|_S)_{Nis}$ be a diagram of level cofibrations so that for all limit ordinals $s < \gamma$ the induced map

$$\varinjlim_{t<s} X(t) \to X(s)$$

is an isomorphism. Then $\varinjlim_{t<\gamma} F(X(t)) \cong F(\varinjlim_{t<\gamma} X(t))$.

L5: If X is λ-bounded, then FX is λ-bounded.

L6: Let Y, Z be two subobjects of X. Then

$$FY \cap FZ = F(Y \cap Z)$$

in FX.

L7: The functor F is continuous; that is, it extends to a natural morphism of simplicial sets

$$F : \mathbf{hom}(X, Y) \to \mathbf{hom}(FX, FY)$$

compatible with composition.

Recall that the cardinals λ and κ are chosen such that

$$\lambda = 2^\kappa > \kappa > 2^\alpha,$$

where α is an upper bound on the cardinality of the set of morphisms of (the chosen approximation for) the smooth Nisnevich site.

Remark 2.14. If the spectrum X has extra structure, such as a symmetric structure, then that structure is preserved by the functor $X \mapsto \mathcal{L}X$: the pairings

$$\mathcal{L}X^n \wedge L \xrightarrow{\phi} \mathcal{L}(X^n \wedge L)$$

satisfy properties L9 and L10 in Section 1.1, and are natural in L and X^n so that they respect all symmetric group actions.

Say that a map $g : X \to Y$ of T-spectra is an F-equivalence if it induces a level weak equivalence $Fg : FX \to FY$.

Proposition 2.15. *Suppose that the functor*

$$F : \mathbf{Spt}_T(Sm|_S)_{Nis} \to \mathbf{Spt}_T(Sm|_S)_{Nis}$$

satisfies the conditions L1 – L7*. Then the class of cofibrations of T-spectra which are F-equivalences satisfies the bounded cofibration condition for the cardinal λ.*

Proof. The class of maps of T-spectra which are level cofibrations and level equivalences satisfies the bounded cofibration condition for the cardinal λ. To see this, recall that the category of simplicial presheaves satisfies the bounded cofibration condition with respect to the cardinal λ, since λ is an upper bound for the cardinality of the set of morphisms of the underlying site [4, Lemma 2.3]. Then use the argument for the second part of Lemma 2.1.

Suppose that $i : X \hookrightarrow Y$ is a cofibration in the category of T-spectra, and that $j : A \hookrightarrow Y$ is a subobject of Y. Then the restriction $X \cap A \to A$ is a cofibration of T-spectra (so that the statement of the Proposition makes sense). The claim for S^1-spectra was proved in Lemma 3.1 of [4]. There is nothing special about the simplicial circle S^1 in that argument, so the same argument obtains here.

Alternatively, the key is to show that the map

$$j_* : (T \wedge A^n) \cup_{(T\wedge(A^n \cap X^n))} (A^{n+1} \cap X^{n+1}) \to (T \wedge Y^n) \cup_{(T\wedge X^n)} X^{n+1}$$

is an inclusion in all presheaves of simplices for all n. But

$$\begin{aligned}(T \wedge A^n) &\cup_{(T\wedge(A^n \cap X^n))} (A^{n+1} \cap X^{n+1}) \\ &= ((T - *) \times (A^n - X^n)) \sqcup (A^{n+1} \cap X^{n+1}),\end{aligned}$$

at the simplex level, while

$$(T \wedge Y^n) \cup_{(T\wedge X^n)} X^{n+1} = ((T - *) \times (Y^n - X^n)) \sqcup X^{n+1},$$

and the map between the two is obvious.

Let $X \to Y$ be an F-equivalence and a cofibration of T-spectra, and let $A \subseteq Y$ be a λ-bounded sub-object. Inductively define a chain of λ-bounded sub-objects $A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq Y$ over λ, and a chain of sub-objects

$$F(A) = F(A_0) \subseteq X_1 \subseteq F(A_1) \subseteq X_2 \subseteq F(A_2) \subseteq \cdots F(Y),$$

also over λ, with the property that the cofibration

$$F(X) \cap X_s \to X_s$$

is a level weak equivalence. Set $B = \varinjlim_{s<\kappa} A_s$. Then, by L6,

$$F(X \cap B) = F(X) \cap F(B) = \varinjlim_{s<\kappa} F(X) \cap X_s \\ \to \varinjlim_{s<\kappa} X_s \cong F(B)$$

is a level weak equivalence, and so $X \cap B \hookrightarrow B$ is an F-equivalence.

The A_s and X_s are defined recursively. Suppose $s+1$ is a successor ordinal and A_s has been defined. Then, since A_s is λ-bounded, $F(A_s)$ is λ-bounded by L5. The map $F(X) \to F(Y)$ is a cofibration and a level equivalence, so there is a λ-bounded sub-object $X_{s+1} \subseteq F(Y)$ so that $F(A_s) \subseteq X_{s+1}$ and $F(X) \cap X_{s+1} \to X_{s+1}$ is a level weak equivalence. Since there is a filtered colimit $F(Y) = \varinjlim F(Y_j)$ indexed over the λ-bounded subobjects Y_j by L3, there is a λ-bounded subobject A'_{s+1} of Y so that $X_{s+1} \subset F(A'_{s+1})$. Set $A_{s+1} = A_s \cup A_{s+1}$. Finally suppose that s is a limit ordinal, and set

$$X_s = \varinjlim_{t<s} F(A_t) \cong \varinjlim_{t<s} X_t.$$

Then X_s is λ-bounded and $F(X) \cap X_s \to X_s$ is a level weak equivalence. Choose $A'_s \subset Y$ such that A'_s is λ-bounded and $X_s \subset F(A'_s)$. Set $A_s = \varinjlim_{t<s} A_t \cup A'_s$. □

Corollary 2.16. *The class of cofibrations which are stable equivalences satisfies the bounded cofibration condition with respect to the cardinal λ.*

Proof. The functor $X \mapsto Q_T \mathcal{L} X$ is an example of a functor F satisfying the conditions for Proposition 2.15. □

3 Fibre and cofibre sequences

The purpose of this section is to show that the standard calculus of fibre and cofibre sequences can be promoted to the motivic stable category, with the help of a suitable theory of stable homotopy groups with weights. The outcomes include detection of motivic stable equivalences by presheaves of weighted stable homotopy groups, and a collection of results which together assert that fibre and cofibre sequences are indistinguishable in the motivic stable category.

The last part of this section is devoted to showing that the various standard flavors of suspension functors (ie. left, right, and shift) are equivalent. These results turn out to be special, and depend on knowing Voevodsky's observation that the cyclic permutation of order 3 acts trivially on $T^3 = T^{\wedge 3}$ in the motivic homotopy category. The Voevodsky result appears here as Lemma 3.13.

3.1 Exact sequences for S^1-spectra

Recall that Lemma 2.2 asserts, in part, that finite pointed simplicial sets are compact. The simplicial circle S^1 is finite, so that Theorem 2.9 implies that there is a proper closed simplicial model structure on the category

$$\mathbf{Spt}(Sm|_S)_{Nis} = \mathbf{Spt}_{S^1}(Sm|_S)_{Nis}$$

for S^1-spectra on the smooth Nisnevich site, for which the weak equivalences are the motivic stable equivalences. Our first job is to show that the traditional facts about fibre and cofibre sequences of ordinary spectra have analogues in this setting.

Lemma 3.1. *Suppose that a map $g : X \to Y$ of S^1-spectra is an ordinary local stable equivalence. Then g is a motivic stable equivalence.*

Recall [10], [11] that a map $g : X \to Y$ of presheaves of spectra is a *local stable equivalence* if it induces an isomorphism on all sheaves of ordinary stable homotopy groups.

Proof. If an S^1-spectrum W is motivic injective and motivic stably fibrant, it must be injective and stably fibrant for the local theory. It follows that ordinary stable homotopy classes $[X, W]$ coincide with naive homotopy classes $\pi(X, W)$ and hence with level homotopy classes $[X, W]$ in the motivic theory for all such W and all S^1-spectra X. Thus, every stable equivalence $g : X \to Y$ induces a bijection

$$g^* : [Y, W] \to [X, W]$$

in level homotopy classes for the motivic theory if W is motivic injective and motivic stably fibrant. Lemma 2.11 implies that g is a motivic stable equivalence. □

Corollary 3.2. *Suppose that*

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a level motivic fibre sequence of S^1-spectra. Then the induced map $p_* : X/F \to Y$ is a motivic stable equivalence.

Proof. This is a consequence of the corresponding result for ordinary spectra, and Lemma 3.1. □

All weak equivalences, stable equivalences, fibrations and so on will be tacitly assumed to be motivic henceforth. We shall drop the use of the term "motivic", except when it is necessary to include it for clarity.

Lemma 3.3. *Suppose given a commutative diagram of S^1-spectra*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"'] & B_1 \arrow[r] \arrow[d, "f_2"] & C_1 \arrow[d, "f_3"] \\ A_2 \arrow[r] & B_2 \arrow[r] & C_2 \end{tikzcd}$$

in which the horizontal sequences are level cofibre sequences. Then if any two of f_1, f_2 or f_3 are stable equivalences, then so is the third.

Proof. We will show that f_1 is a stable equivalence if f_2 and f_3 are stable equivalences. The other two cases are similar.

The idea is to show that precomposition with f_1 induces a weak equivalence

$$f_1^* : \mathbf{hom}(A_2, W) \to \mathbf{hom}(A_1, W)$$

of function complexes for any stably fibrant injective object W. The map of cofibre sequences induces a comparison diagram of fibre sequences

$$\begin{tikzcd} \mathbf{hom}(C_2, W) \arrow[r] \arrow[d, "f_3^*"'] & \mathbf{hom}(B_2, W) \arrow[r] \arrow[d, "f_2^*"] & \mathbf{hom}(A_2, W) \arrow[d, "f_1^*"] \\ \mathbf{hom}(C_1, W) \arrow[r] & \mathbf{hom}(B_1, W) \arrow[r] & \mathbf{hom}(A_1, W) \end{tikzcd}$$

The level equivalences $W \to \Omega W[1]$ of stably fibrant injective objects give all spaces in this diagram the structure of infinite loop spaces, and f_2^* and f_3^* are the maps at level 0 for stable equivalences of spectra. The map f_1^* is therefore the level 0 part of a stable equivalence of stably fibrant spectra, and so f_1^* is a weak equivalence of simplicial sets. □

We now have the following consequence of Corollary 3.2 and Lemma 3.3:

Corollary 3.4. *Suppose given a commutative diagram of S^1-spectra*

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d, "f_1"'] & X_1 \arrow[r] \arrow[d, "f_2"] & Y_1 \arrow[d, "f_3"] \\ F_2 \arrow[r] & X_2 \arrow[r] & Y_2 \end{tikzcd}$$

in which the horizontal sequences are level fibre sequences. Then if any two of f_1, f_2 or f_3 are stable equivalences, then so is the third.

Recall that a map $g : X \to Y$ is a stable equivalence of S^1-spectra if and only if it induces a pointwise level equivalence $g_* : QJX \to QJY$. The functor $QJ = Q_{S^1}J$ produces presheaves of infinite loop spaces, so that g_* is a pointwise level equivalence if and only if it induces pointwise isomorphisms

$$\pi_n QJX(U) \cong \pi_n QJY(U)$$

in all homotopy groups. The group $\pi_n QJX(U)$ can be identified up to isomorphism with the filtered colimit of the system

$$[S^{n+r}, X^r|_U] \to [S^{n+r+1}, X^{r+1}|_U] \to \cdots,$$

where S^t denotes the t-fold smash product of the constant simplicial presheaf associated to the simplicial circle S^1, and the morphisms in the motivic homotopy category are computed over the scheme U. This filtered colimit may be computed without reference to a level fibrant model for X; we define a presheaf $\pi_n X$ of *stable homotopy groups* for X in U-sections to be the filtered colimit of this system. A map $g : X \to Y$ is a motivic stable equivalence if and only if it induces presheaf isomorphisms $\pi_n X \cong \pi_n Y$ for all $n \in \mathbb{Z}$.

Warning: The presheaves of groups $\pi_n X$ are defined by morphisms in the motivic homotopy category. Despite the notation, they do not coincide with the stable homotopy group presheaves of X, but rather with the stable homotopy group presheaves of a motivic stably fibrant model for X.

Any level fibre sequence

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

can be functorially replaced up to level equivalence by a fibre sequence in which all objects are level fibrant. Suppose that this has been done — then the induced maps of S^1-spectra

$$QF \xrightarrow{Qi} QX \xrightarrow{Qp} QY$$

forms a level fibre sequence of spectra

$$QF(U) \xrightarrow{Qi} QX(U) \xrightarrow{Qp} QY(U)$$

in each section, and therefore determines a long exact sequence

$$\cdots \xrightarrow{p_*} \pi_{n+1} QY(U) \xrightarrow{\partial} \pi_n QF(U) \xrightarrow{i_*} \pi_n QX(U) \xrightarrow{p_*} \pi_n QY(U) \xrightarrow{\partial} \cdots$$

of presheaves of stable homotopy groups. It follows that there is a natural long exact sequence

$$\cdots \xrightarrow{p_*} \pi_{n+1} Y \xrightarrow{\partial} \pi_n F \xrightarrow{i_*} \pi_n X \xrightarrow{i_*} \pi_n Y \xrightarrow{\partial} \cdots$$

of presheaves of groups associated to a level fibre sequence.

Suppose given a level cofibre sequence

$$A \xrightarrow{i} B \xrightarrow{\pi} B/A, \tag{3.1}$$

and replace the map π up to motivic weak equivalence by a level motivic fibration by taking a factorization

$$\begin{tikzcd} B \arrow[r, "\pi"] \arrow[d, "j"'] & B/A \\ X \arrow[ur, "q"'] & \end{tikzcd}$$

where q is a level motivic fibration and j is a cofibration and a level motivic equivalence. Let F be the fibre of q. Then the cofibre sequence (3.1) is a fibre sequence in the standard way in the motivic setting, in the sense that we can prove

Lemma 3.5. *The cofibration j induces a motivic stable equivalence $j_* : A \to F$.*

Proof. There is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[r, "\pi"] \arrow[d, "j"] & B/A \arrow[d, "j_*"] \\ F \arrow[r] & X \arrow[r, "\pi"'] & X/F \end{tikzcd}$$

The map $q : X \to B/A$ factors through $\pi : X \to X/F$ in that there is a map $q_* : X/F \to B/A$ such that $q_* \cdot \pi = q$. The map q_* is a stable equivalence by Corollary 3.2. One also checks that $q_* j_* \pi = \pi$ so that $q_* j_* = 1$ on B/A, and so j_* is a stable equivalence. Now use Lemma 3.3 to conclude that the induced map $j : A \to F$ of S^1-spectra is a stable equivalence. □

Corollary 3.6. *Any cofibre sequence*

$$A \xrightarrow{i} B \xrightarrow{\pi} B/A$$

induces a natural long exact sequence

$$\cdots \xrightarrow{\pi_*} \pi_{i+1} B/A \xrightarrow{\partial} \pi_i A \xrightarrow{i_*} \pi_i B \xrightarrow{\pi_*} \pi_i B/A \xrightarrow{\partial} \cdots$$

Proof. The sequence is the long exact sequence for the corresponding fibre sequence arising from the construction of Lemma 3.5. □

3.2 Weighted stable homotopy groups

The presheaf $T = \mathbb{A}^1/(\mathbb{A}^1 - 0)$ sits in a pushout square of presheaves

$$\begin{tikzcd} \mathbb{A}^1 - 0 \arrow[r, "i"] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ * \arrow[r] & T, \end{tikzcd}$$

and $\mathbb{A}^1$ is contractible in the motivic homotopy category. A standard argument on mapping cones (which uses properness) implies that there are motivic equivalences

$$T = \mathbb{A}^1/(\mathbb{A}^1 - 0) \xleftarrow{\simeq} M_i/(\mathbb{A}^1 - 0) \xrightarrow{\simeq} S^1 \wedge (\mathbb{A}^1 - 0)$$

involving the mapping cylinder M_i of the inclusion i. All of these objects are compact, by Lemma 2.2, and Proposition 2.13 implies that the displayed equivalences induces equivalences of the stable categories associated to the various suspensions.

For convenience, write $\mathbb{G}_m = \mathbb{A}^1 - 0$, pointed by the global section given by the identity element e (Voevodsky denotes this object by S^1_t [16]). This is the underlying scheme of the multiplicative group, but the group structure is never used.

Recall that a map $g : X \to Y$ of spectra is an stable equivalence if and only if the induced map $g_* : Q_T JX \to Q_T JY$ is a pointwise level equivalence. Recall further that the object $Q_T Y$ for a level fibrant spectrum Y has object at level n given by the filtered colimit

$$Y^n \xrightarrow{\sigma_*} \Omega_T Y^{n+1} \xrightarrow{\Omega_T \sigma_*} \Omega_T^2 Y^{n+2} \to \dots.$$

The homotopy group $\pi_r Q_T Y^n(U)$ in U-sections is isomorphic to the filtered colimit of the diagram

$$\pi_r Y^n(U) \xrightarrow{\sigma_*} \pi_r \Omega_T Y^{n+1}(U) \xrightarrow{\Omega_T \sigma_*} \pi_r \Omega_T^2 Y^{n+2}(U) \to \dots,$$

which can be identified with a filtered colimit of maps in the motivic homotopy category over the scheme U of the form

$$[S^r, Y^n|_U] \to [S^r \wedge T, Y^{n+1}|_U] \to [S^r \wedge T^2, Y^{n+2}|_U] \to \dots$$

Here, T^r denotes an r-fold wedge product of copies of the simplicial presheaf T, and S^r is the r-fold wedge product of copies of S^1. The equivalence $T \simeq S^1 \wedge \mathbb{G}_m$ further implies that this last inductive system can be rewritten as

$$[S^r, Y^n|_U] \to [S^{r+1} \wedge \mathbb{G}_m, Y^{n+1}|_U] \to [S^{r+2} \wedge \mathbb{G}_m^2, Y^{n+2}|_U] \to \dots$$

Write $\pi_{t,s} Y(U)$ for the colimit of the sequence

$$[S^{t+n} \wedge \mathbb{G}_m^{s+n}, Y^n|_U] \to [S^{t+n+1} \wedge \mathbb{G}_m^{s+n+1}, Y^{n+1}|_U] \to \dots$$

The variable t in $\pi_{t,s} Y$ is usually called the *degree*, while s is called the *weight*. The presheaves of groups $\pi_{t,s} Y$ are called the *weighted stable homotopy groups* of the T-spectrum Y.

This last definition of the presheaf $U \mapsto \pi_{t,s} Y(U)$ makes sense for any T-spectrum Y, and there is an isomorphism

$$\pi_r Q_T JY^n(U) \cong \pi_{r-n,-n} Y(U).$$

From a different point of view, if $t \leq s$, then there are isomorphisms

$$\begin{aligned} \varinjlim{}_n [S^{t+n} \wedge \mathbb{G}_m^{s+n}, Y^n|_U] &\cong \varinjlim{}_n [S^n \wedge \mathbb{G}_m^{s-t+n}, Y[-t]^n|_U] \\ &\cong \varinjlim{}_n [S^n \wedge \mathbb{G}_m^n, \Omega_{\mathbb{G}_m}^{s-t} JY[-t]^n|_U], \end{aligned}$$

where $Y[k]^n = Y^{n+k}$ defines the shifted T-spectrum object $Y[k]$ in the standard way for all $k \in \mathbb{Z}$. It follows that there is an isomorphism

$$\pi_{t,s}Y \cong \pi_0 \Omega^{s-t}_{\mathbb{G}_m} Q_T(JY[-t])^0$$

if $t \geq s$. Similarly, if $s \geq t$, there is an isomorphism

$$\pi_{t,s}Y \cong \pi_0 \Omega^{t-s} Q_T(JY[-s])^0.$$

If $g : X \to Y$ is an stable equivalence, then $g_* : Q_T(JX[k]) \to Q_T(JY[k])$ is a pointwise level equivalence for all $k \in \mathbb{Z}$, so that all induced maps

$$g_* : \pi_{t,s}X \to \pi_{t,s}Y$$

are isomorphisms of presheaves. Conversely, if g induces isomorphisms in all bigraded stable homotopy group presheaves, then g induces isomorphisms $\pi_{t,s}X \cong \pi_{t,s}Y$ for $s \leq 0$ and $t \geq s$. In that case

$$\pi_{t,s}Y = \pi_{(t-s)+s,s}Y \cong \pi_{t-s}Q_T Y^{-s},$$

so that $g_* : Q_T JX \to Q_T JY$ is a pointwise level equivalence. We have proved

Lemma 3.7. *A map $g : X \to Y$ of T-spectra is an stable equivalence if and only if g induces isomorphisms*

$$\pi_{t,s}X \cong \pi_{t,s}Y$$

of presheaves of groups for all $t, s \in \mathbb{Z}$.

Given Proposition 2.13, we can assume that T is identically $S^1 \wedge \mathbb{G}_m$, so a T-spectrum consists of pointed simplicial presheaves Y^n and bonding maps

$$S^1 \wedge \mathbb{G}_m \wedge Y^n \to Y^{n+1}.$$

An $S^1/\mathbb{G}_m$-*bispectrum* consists of pointed simplicial presheaves $X^{m,n}$, $m, n \geq 0$, together with bonding maps $\sigma_h : S^1 \wedge X^{m,n} \to X^{m+1,n}$ and $\sigma_v : \mathbb{G}_m \wedge X^{m,n} \to X^{m,n+1}$, such that the diagram

$$\begin{tikzcd}
S^1 \wedge X^{m,n+1} \arrow[rr, "\sigma_h"] & & X^{m+1,n+1} \\
S^1 \wedge \mathbb{G}_m \wedge X^{m,n} \arrow[u, "S^1 \wedge \sigma_v"] \arrow[dr, "\cong", "t \wedge 1"'] & & \\
& \mathbb{G}_m \wedge S^1 \wedge X^{m,n} \arrow[r, "\mathbb{G}_m \wedge \sigma_h"'] & \mathbb{G}_m \wedge X^{m+1,n} \arrow[uu, "\sigma_v"']
\end{tikzcd}$$

commutes, where $t : S^1 \wedge \mathbb{G}_m \to \mathbb{G}_m \wedge S^1$ is the canonical isomorphism which flips smash factors. The maps σ_v and σ_h are called *vertical* and *horizontal* *bonding maps* respectively. Such a gadget may alternatively be viewed as a collection of S^1-spectra

$$X^n = X^{*,n},$$

together with maps of S^1-spectra $X^n \wedge \mathbb{G}_m \to X^{n+1}$ induced by the vertical bonding maps.

For us, the key example arises from a T-spectrum Y, in that it functorially determines an array $Y^{*,*}$

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\mathbb{G}_m^{\wedge 2} \wedge Y^0 & \mathbb{G}_m \wedge Y^1 & Y^2 & \cdots \\
\mathbb{G}_m \wedge Y^0 & Y^1 & S^1 \wedge Y^1 & \cdots \\
Y^0 & S^1 \wedge Y^0 & S^2 \wedge Y^0 & \cdots
\end{array}$$

which has the structure of an $S^1/\mathbb{G}_m$-bispectrum. In effect, the horizontal bonding map $\sigma_h : S^1 \wedge \mathbb{G}_m^k \wedge Y^n \to \mathbb{G}_m^{k-1} \wedge Y^{n+1}$ is defined to be the composite

$$S^1 \wedge \mathbb{G}_m^{k-1} \wedge \mathbb{G}_m \wedge Y^n \xrightarrow{t \wedge 1} \mathbb{G}_m^{k-1} \wedge S^1 \wedge \mathbb{G}_m \wedge Y^n \xrightarrow{1 \wedge \sigma} \mathbb{G}_m^{k-1} \wedge Y^{n+1},$$

and the vertical bonding maps arise from the maps of S^1-spectra $Y^{*,n} \wedge \mathbb{G}_m \to Y^{*,n+1}$ which are canonically determined by the twist isomorphisms

$$(\mathbb{G}_m^k \wedge Y^{n-k}) \wedge \mathbb{G}_m \xrightarrow{t} \mathbb{G}_m \wedge (\mathbb{G}_m^k \wedge Y^{n-k}).$$

for $0 \leq k \leq n$.

An $S^1/\mathbb{G}_m$-bispectrum X has presheaves of *bigraded stable homotopy groups* $\pi_{t,s}X$ defined in bidegree (t,s) and in U-sections to be the colimit of the system

$$\begin{tikzcd}
\vdots & \vdots & \\
{[S^{t+k} \wedge \mathbb{G}_m^{s+l+1}, X^{k,l+1}|_U]} \arrow[u] \arrow[r, "\sigma_{h*}"] & {[S^{t+k+1} \wedge \mathbb{G}_m^{s+l+1}, X^{k+1,l+1}|_U]} \arrow[u] \arrow[r] & \cdots \\
{[S^{t+k} \wedge \mathbb{G}_m^{s+l}, X^{k,l}|_U]} \arrow[u, "\sigma_{v*}"] \arrow[r, "\sigma_{h*}"'] & {[S^{t+k+1} \wedge \mathbb{G}_m^{s+l}, X^{k+1,l}|_U]} \arrow[u, "\sigma_{v*}"'] \arrow[r] & \cdots
\end{tikzcd}$$

Here (presuming that all $X^{k,l}$ are fibrant, which is harmless), the map σ_{h*} takes a representative $\theta : S^r \wedge \mathbb{G}_m^s \to X^{k,l}$ to the composite

$$S^1 \wedge S^r \wedge \mathbb{G}_m^s \xrightarrow{S^1 \wedge \theta} S^1 \wedge X^{k,l} \xrightarrow{\sigma_h} X^{k+1,l},$$

while σ_{v*} takes θ to the composite

$$S^r \wedge \mathbb{G}_m \wedge \mathbb{G}_m^s \xrightarrow{t \wedge \mathbb{G}_m^s} \mathbb{G}_m \wedge S^r \wedge \mathbb{G}_m^s \xrightarrow{\mathbb{G}_m \wedge \theta} \mathbb{G}_m \wedge X^{k,l} \xrightarrow{\sigma_v} X^{k,l+1}.$$

The bispectrum object X determines a sequence of maps of S^1-spectra

$$X^0 \xrightarrow{\sigma_{v*}} \Omega_{\mathbb{G}_m} X^1 \xrightarrow{\Omega_{\mathbb{G}_m}(\sigma_{v*})} \Omega^2_{\mathbb{G}_m} X^2 \to \cdots,$$

where $\Omega_{\mathbb{G}_m}$ is the functor $\mathbf{Hom}_*(\mathbb{G}_m, \ )$. Then the presheaf $\pi_{t,s}X$ is the filtered colimit of the presheaves of stable homotopy groups

$$\pi_t \Omega^{s+l}_{\mathbb{G}_m} JX^l \to \pi_t \Omega^{s+l+1}_{\mathbb{G}_m} JX^{l+1} \to \cdots$$

once X has been replaced up to levelwise equivalence by a levelwise fibrant object JX so that the "loop" constructions make sense.

In particular, starting with a T-spectrum X, a cofinality argument shows that the presheaves of weighted stable homotopy groups $\pi_{t,s}X$ for X as defined above coincide up to natural isomorphism with the presheaves $\pi_{t,s}X^{*,*}$ of bigraded stable homotopy groups for the associated bispectrum object $X^{*,*}$.

3.3 Fibre and cofibre sequences

A *level fibration* $p : X \to Y$ of $S^1/\mathbb{G}_m$-bispectra is a map which consists of fibrations $p : X^{m,n} \to Y^{m,n}$ for all $m, n \geq 0$. Level equivalences and level cofibrations have analogous definitions. One can use standard techniques to show that any map $f : X \to Y$ of $S^1/\mathbb{G}_m$-bispectra has a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & Z \arrow[ur, "p"'] & \end{tikzcd}$$

where p is a level fibration and j is a level cofibration and a level equivalence.

Suppose that

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a level fibre sequence of $S^1/\mathbb{G}_m$-bispectra, and suppose that Y (and hence X) is level fibrant. Then there are fibre sequences of S^1-spectra

$$\Omega^{s+r}_{\mathbb{G}_m} F^r \xrightarrow{i_*} \Omega^{s+r}_{\mathbb{G}_m} X^r \xrightarrow{p_*} \Omega^{s+r}_{\mathbb{G}_m} Y^r$$

and hence long exact sequences in stable homotopy group presheaves

$$\cdots \xrightarrow{p_*} \pi_{t+1}\Omega^{s+r}_{\mathbb{G}_m}Y^r \xrightarrow{\partial} \pi_t\Omega^{s+r}_{\mathbb{G}_m}F^r \xrightarrow{i_*} \pi_t\Omega^{s+r}_{\mathbb{G}_m}X^r \xrightarrow{p_*} \pi_t\Omega^{s+r}_{\mathbb{G}_m}Y^r \xrightarrow{\partial} \cdots$$

Taking a filtered colimit in r gives a long exact sequence

$$\cdots \xrightarrow{p_*} \pi_{t+1,s}Y \xrightarrow{\partial} \pi_{t,s}F \xrightarrow{i_*} \pi_{t,s}X \xrightarrow{p_*} \pi_{t,s}Y \xrightarrow{\partial} \cdots \tag{3.2}$$

for each s. One can remove the condition that Y is level fibrant by using factorization tricks from the previous paragraph.

If

$$A \xrightarrow{i} B \xrightarrow{\pi} B/A$$

is a level cofibre sequence of $S^1/\mathbb{G}_m$-bispectra, then replacing the map π up to level equivalence by a fibration p as above gives a diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j_*"'] & B \arrow[r, "\pi"] \arrow[d, "j"] & B/A \arrow[d, "1_{B/A}"] \\ F \arrow[r] & X \arrow[r, "p"'] & B/A \end{tikzcd}$$

in which p is a level fibration and j is a level equivalence. It follows from Lemma 3.5 that the induced maps $j_* : A^n \to F^n$ are stable equivalences of S^1-spectra. But then the induced maps

$$\pi_{t,s}A \xrightarrow{j_*} \pi_{t,s}F$$

are isomorphisms in all bidegrees. This implies that there is a natural long exact sequence

$$\cdots \xrightarrow{\pi_*} \pi_{t+1,s}B/A \xrightarrow{\partial} \pi_{t,s}A \xrightarrow{i_*} \pi_{t,s}B \xrightarrow{\pi_*} \pi_{t,s}B/A \xrightarrow{\partial} \cdots \tag{3.3}$$

associated to a cofibre sequence of $S^1/\mathbb{G}_m$-bispectra in each s. As a corollary of the construction we have

Corollary 3.8. *There are natural isomorphisms*

$$\pi_{t+1,s}(Y \wedge S^1) \cong \pi_{t,s}Y$$

for all bidegrees (t,s) *and* $S^1/\mathbb{G}_m$*-bispectra* Y.

Lemma 3.9. *Suppose that*

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a level fibre sequence of T*-spectra. Then the induced map* $X/F \to Y$ *is a stable equivalence.*

Proof. The idea is to show that the map $X/F \to Y$ induces isomorphisms

$$\pi_{t,s}(X/F)^{*,*} \cong \pi_{t,s}Y^{*,*}.$$

Form the diagram of maps of $S^1/\mathbb{G}_m$-bispectra

$$\begin{tikzcd} F^{*,*} \arrow[r, "i_*"] \arrow[d, "j_*"'] & X^{*,*} \arrow[r, "p_*"] \arrow[d, "j"] & Y^{*,*} \arrow[d, "1_{Y^{*,*}}"] \\ \overline{F} \arrow[r] & Z \arrow[r, "q"'] & Y^{*,*} \end{tikzcd}$$

where q is a level fibration, j is a level equivalence, and $\overline{F}$ is the fibre of the map q. The map $j_* : F^{*,*} \to \overline{F}$ consists in part of equivalences $F^n \to \overline{F}^{n,n}$ in bidegree (n,n) for all $n \geq 0$, since the sequence

$$F^{*,*} \xrightarrow{i_*} X^{*,*} \xrightarrow{p_*} Y^{*,*}$$

is already an fibre sequence in those bidegrees. A cofinality argument therefore implies that the map $j_* : F^{*,*} \to \overline{F}$ induces isomorphisms

$$j_* : \pi_{t,s}F^{*,*} \xrightarrow{\cong} \pi_{t,s}\overline{F}$$

for all t and s.

The map $Z/\overline{F} \to Y^{*,*}$ of $S^1/\mathbb{G}_m$-bispectra induces isomorphisms in all $\pi_{t,s}$, since it consists of maps $Z^n/\overline{F}^n \to Y^{*,n}$ of S^1-spectra which are stable equivalences by Lemma 3.2.

A long exact sequence argument arising from the comparison of cofibre sequences

$$\begin{tikzcd} F^{*,*} \arrow[r, "i_*"] \arrow[d, "j_*"'] & X^{*,*} \arrow[r, "\pi_*"] \arrow[d, "j"] & (X/F)^{*,*} \arrow[d, "j_*"] \\ \overline{F} \arrow[r] & Z \arrow[r, "\pi"'] & Z/\overline{F} \end{tikzcd}$$

shows that the map $j_* : (X/F)^{*,*} \to Z/\overline{F}$ induces an isomorphism in all $\pi_{t,s}$. The result follows. □

Corollary 3.10. *Suppose that*

$$A \xrightarrow{i} B \xrightarrow{\pi} B/A$$

is a level cofibre sequence of T-spectra, and take a factorization

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[dr, "\pi"'] & X \arrow[d, "p"] \\ & B/A \end{tikzcd}$$

of the map π such that j is a level equivalence and p is a level fibration. Let F be the fibre of the map p. Then the induced map $j_* : A \to F$ is a stable equivalence.

Proof. The induced map $X/F \to B/A$ is a stable equivalence by Lemma 3.9. The map $j_* : B/A \to X/F$ is therefore a stable equivalence, so a comparison of long exact sequences argument shows that $j_* : A \to F$ is a stable equivalence. □

3.4 T-suspensions and T-loops

Write $j_X : X \to X_s$ for a natural choice of stably fibrant model X_s for a T-spectrum X, where j_X is a cofibration and a stable equivalence. The aim of this section is to prove and discuss the consequences of the following result:

Theorem 3.11. *The composition*

$$X \xrightarrow{\eta_X} \Omega_T(X \wedge T) \xrightarrow{\Omega j_{X \wedge T}} \Omega_T(X \wedge T)_s$$

arising from the adjunction map η_X is a stable equivalence for all T-spectra X.

The proof of this result is a bit delicate, and will be accomplished with the help of a series of lemmas. We begin with something which is quite general:

Lemma 3.12. *Suppose that the comparison diagram of inductive systems*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"'] & X_1 \arrow[r] \arrow[d, "f_1"] & X_2 \arrow[r] \arrow[d, "f_2"] & \cdots \\ Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \end{tikzcd}$$

consists of stable equivalences $f_i : X_i \to Y_i$. Then the induced map

$$\varinjlim f_i : \varinjlim X_i \to \varinjlim Y_i$$

is a stable equivalence.

Proof. The idea of the proof is to show that we can assume that the spectra X_i and Y_i are stably fibrant.

In effect, suppose that there are trivial cofibrations $j_i : X_i \to (X_i)_s$ and $j_i : Y_i \to (Y_i)_s$ and maps $(f_i)_* : (X_i)_s \to (Y_i)_s$ such that $(X_i)_s$ and $(Y_i)_s$ are stably fibrant for $i \leq n$, and such that the diagrams

$$\begin{tikzcd}
X_{i-1} \arrow[rr] \arrow[dr, "j_{i-1}"] \arrow[dd, "f_{i-1}"'] & & X_i \arrow[dr, "j_i"] \arrow[dd, "f_i" near end] & \\
& (X_{i-1})_s \arrow[rr, crossing over] & & (X_i)_s \arrow[dd, "(f_i)_*"] \\
Y_{i-1} \arrow[rr, "(f_{i-1})_*" near start] \arrow[dr, "j_{i-1}"'] & & Y_i \arrow[dr, "j_i"] & \\
& (Y_{i-1})_s \arrow[rr] \arrow[from=uu, crossing over] & & (Y_i)_s
\end{tikzcd}$$

commute. Now form the commutative diagram

$$\begin{tikzcd}
X_n \arrow[rr] \arrow[dr, "j_n"] \arrow[dd, "f_n"'] & & X_{n+1} \arrow[dr, "j_{n*}"] \arrow[dd, "f_{n+1}" near end] & & \\
& (X_n)_s \arrow[rr, crossing over] & & (X_n)_s \cup_{X_n} X_{n+1} \arrow[r, "j"] \arrow[dd, "f_*"] & (X_{n+1})_s \arrow[dd, dotted, "(f_{n+1})_*"] \\
Y_n \arrow[rr, "(f_n)_*" near start] \arrow[dr, "j_n"'] & & Y_{n+1} \arrow[dr, "j_{n*}"] & & \\
& (Y_n)_s \arrow[rr] \arrow[from=uu, crossing over] & & (Y_n)_s \cup_{Y_n} Y_{n+1} \arrow[r, "j"'] & (Y_{n+1})_s
\end{tikzcd}$$

where both instances of j are trivial cofibrations, and $(X_{n+1})_s$ and $(Y_{n+1})_s$ are stably fibrant. The dotted arrow $(f_{n+1})_*$ exists by the closed model axioms, and the instances of the compositions $j_{n+1} = j \cdot j_{n*}$ are both trivial cofibrations.

In the resulting diagram

$$\begin{tikzcd}
\varinjlim X_n \arrow[r, "j_*"] \arrow[d, "f_*"'] & \varinjlim (X_n)_s \arrow[d, "f_*"] \\
\varinjlim Y_n \arrow[r, "j_*"'] & \varinjlim (Y_n)_s
\end{tikzcd}$$

both instances of j_* are trivial cofibrations by construction, and the map $f_* : \varinjlim (X_n)_s \to \varinjlim (Y_n)_s$ is a filtered colimit of maps which are pointwise weak equivalences in each level, and therefore shares this property. In particular, f_* is a stable equivalence. $\square$

We're going to need the following:

Lemma 3.13. *(Voevodsky) The cyclic permutation $c_{1,2} = (3,2,1) \in \Sigma_3$ induces the identity morphism on $T^{\wedge 3} = T^3$ in the pointed motivic homotopy category.*

For the record (this comes up later), the element $c_{p,q}$ in the symmetric group Σ_{p+q} is the shuffle which moves the first p elements past the last q elements, in order. Explicitly

$$c_{p,q}(i) = \begin{cases} q+i & \text{if } i \leq p, \\ i-p & \text{if } i \geq p+1. \end{cases}$$

Proof of Lemma 3.13. For the purposes of this proof, we shall notationally confuse T^3 with its associated sheaf, and prove the result on the sheaf level. This is harmless, since the canonical map $\eta : X \to \tilde{X}$ taking values in the associated sheaf $\tilde{X}$ is a weak equivalence for any presheaf X.

There is an isomorphism of pointed sheaves

$$\mathbb{A}^n/(\mathbb{A}^n - 0) \wedge \mathbb{A}^1/(\mathbb{A}^1 - 0) \cong \mathbb{A}^{n+1}/(\mathbb{A}^{n+1} - 0),$$

since

$$((\mathbb{A}^n - 0) \times \mathbb{A}^1) \cup (\mathbb{A}^n \times (\mathbb{A}^1 - 0)) = \mathbb{A}^{n+1} - 0$$

inside $\mathbb{A}^{n+1}$. It follows that there is an isomorphism

$$T^n \cong \mathbb{A}^n/(\mathbb{A}^n - 0).$$

There is a pointed algebraic group action

$$Gl_n \times T^n \to T^n$$

in the sheaf category which is induced by the standard action $Gl_n \times \mathbb{A}^n \to \mathbb{A}^n$. It follows that any rational point $g \in Gl_n(\mathbb{Z})$ induces a morphism of sheaves

$$g : T^n \to T^n.$$

In particular, the permutation matrix corresponding to the element $c_{1,2} = (3,2,1)$ induces the map

$$c_{1,2} : T^3 \to T^3$$

in the statement of the lemma.

The element

$$c_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

is a product of elementary transformations in $Gl_3(\mathbb{Z})$, and so there is an algebraic path $\omega : \mathbb{A}^1 \to Gl_3$ such that $\omega(1) = c_{1,2}$ and $\omega(0) = e$. It follows that there is a composite pointed algebraic homotopy

$$\mathbb{A}^1 \times T^3 \xrightarrow{1\times\omega} Gl_3 \times T^3 \to T^3$$

from $c_{1,2} : T^3 \to T^3$ to the identity on T^3 (see also Theorem 1.1 of [8]). The maps $c_{1,2}$ and e therefore coincide in the motivic homotopy category. $\square$

Observe that a T-spectrum X has a natural filtration

$$X \cong \varinjlim L_n X,$$

where $L_n X$ is the spectrum

$$X^0, X^1, \dots, X^n, T \wedge X^n, T^{\wedge 2} \wedge X^n, \dots$$

There is a natural pushout diagram

$$\begin{tikzcd}
\Sigma_T^\infty (T \wedge X^n)[-(n+1)] \arrow[r] \arrow[d] & L_n X \arrow[d] \\
\Sigma^\infty X^{n+1}[-(n+1)] \arrow[r] & L_{n+1} X
\end{tikzcd}$$

Note further that the canonical map $\Sigma_T^\infty X^n[-n] \to L_n X$ is a stable equivalence. The filtration $\{L_n X\}$ is called the *layer filtration* of X.

Lemma 3.14. *Suppose that K is a pointed simplicial presheaf. Then the composition*

$$\Sigma_T^\infty K \xrightarrow{\eta} \Omega_T((\Sigma_T^\infty K) \wedge T) \xrightarrow{\Omega j} \Omega_T((\Sigma_T^\infty K) \wedge T)_s$$

is a stable equivalence.

Proof. Recall that if Y is a spectrum, then the homotopy group presheaves $\pi_r Y_s^n(U)$ of the stably fibrant model $Y_s = IQ_T JY$ are computed by the filtered colimits

$$[S^r, Y^n]_U \xrightarrow{\Sigma} [T \wedge S^r, Y^{n+1}]_U \xrightarrow{\Sigma} \cdots$$

where $[K, X]_U = [K|_U, X|_U]$ means homotopy classes of maps of the restrictions to the site over U. The suspension homomorphism Σ takes a morphism $\theta : T^k \wedge S^r \to Y^{n+k}$ to the composite

$$T \wedge T^k \wedge S^r \xrightarrow{T\wedge\theta} T \wedge Y^{n+k} \xrightarrow{\sigma} Y^{n+k+1}$$

Practically speaking, the suspension morphism Σ is induced by smashing with T on the left.

Observe as well that if Y is level fibrant, then the adjunction isomorphisms

$$[T^k \wedge S^r, \Omega_T Y^{n+k}]_U \cong [T^k \wedge S^r \wedge T, Y^{n+k}]_U$$

fit into commutative diagrams

$$\begin{tikzcd}
{[T^k \wedge S^r, \Omega_T Y^{n+k}]_U} \arrow[r, "\cong"] \arrow[d, "\Sigma"'] & {[T^k \wedge S^r \wedge T, Y^{n+k}]_U} \arrow[d, "\Sigma"] \\
{[T^{k+1} \wedge S^r, \Omega_T Y^{n+k+1}]_U} \arrow[r, "\cong"'] & {[T^{k+1} \wedge S^r \wedge T, Y^{n+k+1}]_U}
\end{tikzcd}$$

It follows that the map in presheaves of stable homotopy groups induced by the composite

$$\Sigma_T^\infty K \xrightarrow{\eta} \Omega_T((\Sigma_T^\infty K) \wedge T) \xrightarrow{\Omega j} \Omega_T((\Sigma_T^\infty K) \wedge T)_s$$

is isomorphic to the filtered colimit of the maps

$$[T^k \wedge S^r, T^{n+k} \wedge K]_U \xrightarrow{\wedge T} [T^k \wedge S^r \wedge T, T^{n+k} \wedge K \wedge T]_U$$

which are induced by smashing with T on the right.

Suppose that $\phi : K \wedge T \to X \wedge T$ is a map of pointed simplicial presheaves, and write $c_t(\phi)$ for the map $T \wedge K \to T \wedge X$ arises from ϕ by conjugation with the twist of smash factors. There is a commutative diagram

$$\begin{tikzcd}
K \wedge T \arrow[r, "t", "\cong"'] \arrow[d, "\phi"'] & T \wedge K \arrow[d, "c_t(\phi)"] \\
X \wedge T \arrow[r, "\cong", "t"'] & T \wedge X
\end{tikzcd}$$

Then there is a diagram

$$\begin{tikzcd}
T \wedge T^2 \wedge K \arrow[d, "c_t(T^2 \wedge \phi)"'] & T^2 \wedge K \wedge T \arrow[l, "t"'] \arrow[r, "T^2 \wedge t"] \arrow[d, "T^2 \wedge \phi"] & T^2 \wedge T \wedge K \arrow[d, "T^2 \wedge c_t(\phi)"] \\
T \wedge T^2 \wedge X & T^2 \wedge X \wedge T \arrow[l, "t"] \arrow[r, "T^2 \wedge t"'] & T^2 \wedge T \wedge X
\end{tikzcd}$$

and hence a diagram

$$\begin{tikzcd}
T^3 \wedge K \arrow[r, "c_{1,2} \wedge K"] \arrow[d, "c_t(T^2 \wedge \phi)"'] & T^3 \wedge K \arrow[d, "T^2 \wedge c_t(\phi)"] \\
T^3 \wedge K \arrow[r, "c_{1,2} \wedge K"'] & T^3 \wedge X
\end{tikzcd}$$

It follows from Lemma 3.13 that the maps in the homotopy category represented by $T^2 \wedge c_t(\phi)$ and $c_t(T^2 \wedge \phi)$ coincide.

As a consequence, there are commutative diagrams

$$\begin{tikzcd}
{[T^k \wedge S^r, T^{n+k} \wedge K]_U} \arrow[r, "T^2\wedge"] \arrow[d, "\wedge T"'] & {[T^2 \wedge T^k \wedge S^r, T^2 \wedge T^{n+k} \wedge K]_U} \arrow[d, "\wedge T"] \\
{[T^k \wedge S^r \wedge T, T^{n+k} \wedge K \wedge T]_U} \arrow[r, "T^2\wedge"] \arrow[d, "c_t"', "\cong"] & {[T^2 \wedge T^k \wedge S^r \wedge T, T^2 \wedge T^{n+k} \wedge K \wedge T]_U} \arrow[d, "\cong"', "c_t"] \\
{[T \wedge T^k \wedge S^r, T \wedge T^{n+k} \wedge K]_U} \arrow[r, "T^2\wedge"'] & {[T^3 \wedge T^k \wedge S^r, T^3 \wedge T^{n+k} \wedge K]_U}
\end{tikzcd}$$

The vertical composites coincide with the map $T\wedge$ induced by smashing on the left with T, so a cofinality argument says that the induced map on the filtered colimits is an isomorphism. $\square$

Proof of Theorem 3.11. It is a consequence of Lemma 2.11 that the functor $X \mapsto X \wedge T$ preserves stable equivalences. It follows that the functors $X \mapsto X$ and $X \mapsto \Omega_T(X \wedge T)_s$ both preserve stable equivalences. The T-spectrum X is a filtered colimit of its layers $L_n X$, and there is a stable equivalence

$$\Sigma_T^\infty X^n[-n] \to L_n X$$

for $n \geq 0$. Write $\eta_* : X \to \Omega_T(X \wedge T)_s$ for the composite in the statement of Theorem 3.11. The proof consists of showing that all maps

$$\Sigma_T^\infty K[-n] \xrightarrow{\eta_*} \Omega_T(\Sigma_T^\infty K[-n] \wedge T)_s \tag{3.4}$$

are stable equivalences. Then we show that these equivalences pass appropriately to filtered colimits.

Shifts preserve stable equivalence, so it suffices to consider the case of the map (3.4) corresponding to $n = 0$, but this is Lemma 3.14.

Suppose given a system

$$X_0 \to X_1 \to \cdots$$

of T-spectra such that all maps

$$\eta_* : X_i \to \Omega_T(X_i \wedge T)_s$$

are stable equivalences. I claim that the induced map

$$\eta_* : \varinjlim X_i \to \Omega_T((\varinjlim X_i) \wedge T)_s$$

is a stable equivalence. The composite

$$\varinjlim X_i \xrightarrow{\varinjlim \eta} \varinjlim \Omega_T(X_i \wedge T) \xrightarrow{\varinjlim \Omega_T j} \varinjlim \Omega_T(X_i \wedge S)_s$$

is a stable equivalence by Lemma 3.12. There is a commutative diagram

$$\begin{tikzcd}
\varinjlim (X_i \wedge T) \arrow[r, "\varinjlim j"] \arrow[d, "\cong"'] & \varinjlim (X_i \wedge T)_s \arrow[d, "c"] \\
(\varinjlim X_i) \wedge T \arrow[r, "j"'] & ((\varinjlim X_i) \wedge T)_s
\end{tikzcd}$$

The map $\varinjlim j$ is a stable equivalence by Lemma 3.12, and so the map c is a pointwise weak equivalence of motivic flasque objects in all levels by a Nisnevich descent argument (Corollary 1.7). There is also a commutative diagram

$$\begin{tikzcd}
\varinjlim X_i \arrow[r, "\eta"] \arrow[ddr, "\varinjlim \eta"'] & \Omega_T((\varinjlim X_i) \wedge T) \arrow[r, "\Omega_T j"] & \Omega_T((\varinjlim X_i) \wedge T)_s \\
& \Omega_T \varinjlim (X_i \wedge T) \arrow[r, "\Omega_T j"'] \arrow[u, "\cong"] & \Omega_T \varinjlim (X_i \wedge T)_s \arrow[u, "\Omega_T c"'] \\
& \varinjlim \Omega_T(X_i \wedge T) \arrow[r, "\varinjlim \Omega_T j"'] \arrow[u, "\cong"] & \varinjlim \Omega_T(X_i \wedge T)_s \arrow[u, "\cong"']
\end{tikzcd}$$

The map $\Omega_T c$ is a pointwise weak equivalence in all levels, so the composite

$$\varinjlim X_i \xrightarrow{\eta} \Omega_T((\varinjlim X_i) \wedge T) \xrightarrow{\Omega_T j} \Omega_T((\varinjlim X_i) \wedge T)_s$$

is a stable equivalence. □

Lemma 3.15. *Suppose that X is level fibrant. Then there is an isomorphism*

$$Q_T(\Omega_T X)^n \cong \Omega_T(Q_T X)^n.$$

In particular, the loop functor $X \mapsto \Omega_T X$ preserves stable equivalences of level fibrant objects.

Proof. Recall that $\Omega_T X$ has bonding map $\sigma : T \wedge \Omega_T X^n \to \Omega_T X^{n+1}$ adjoint to the composite

$$T \wedge \Omega_T X^n \wedge T \xrightarrow{T \wedge ev} T \wedge X^n \xrightarrow{\sigma} X^{n+1}$$

It follows that there is a commutative diagram

$$\begin{tikzcd}
\Omega_T X^n \arrow[r, "\sigma_*"] \arrow[dr, "\Omega_T \sigma_*"'] & \Omega_T^2 X^{n+1} \arrow[d, "\cong"', "t^*"] \\
& \Omega_T^2 X^{n+1}
\end{tikzcd}$$

where t^* is the map which flips loop factors. Inductively, one finds diagrams

$$\begin{tikzcd} \Omega_T^{k+1} X^{n+k} \arrow[r, "\Omega_T^k \sigma_*"] \arrow[d, "c_{k,1}^*"', "\cong"] & \Omega_T^{k+2} X^{n+k+1} \arrow[d, "\cong"', "c_{k+1,1}^*"] \\ \Omega_T^{k+1} X^{n+k} \arrow[r, "\Omega_T^{k+1} \sigma_*"'] & \Omega_T^{k+2} X^{n+k+1} \end{tikzcd}$$

where $c_{k,1}^*$ is precomposition with the map which is induced by the shuffle $c_{k,1}$ in the loop factors. The maps $c_{k,1}^*$ therefore induce the desired isomorphism. □

Corollary 3.16. *Suppose that Y is level fibrant. Then the evaluation map*

$$ev : \Omega_T Y \wedge T \to Y$$

is a stable equivalence.

Proof. The functor $Y \mapsto Y \wedge T$ preserves stable equivalences, so Lemma 3.15 implies that it suffices to assume that Y is stably fibrant.

Take a stably fibrant model $j : \Omega_T Y \wedge T \to (\Omega_T Y \wedge T)_s$ (j is a cofibration as well as a stable equivalence), and form the diagram

$$\begin{tikzcd} \Omega_T Y \wedge T \arrow[r, "j"] \arrow[d, "ev"'] & (\Omega_T Y \wedge T)_s \arrow[dl, "\tilde{ev}"] \\ Y & \end{tikzcd}$$

The idea is to show that $\tilde{ev}$ is a stable equivalence by showing that $\Omega_T \tilde{ev}$ is a stable equivalence. This works, on account of the natural isomorphism

$$\pi_{t,s} \Omega_T X \cong \pi_{t+1,s+1} X$$

for level fibrant objects X — this isomorphism is another consequence of Lemma 3.15. There is a diagram

$$\begin{tikzcd} \Omega_T Y \arrow[d, "\Omega_T \eta"'] \arrow[dr, "\Omega_T \eta_*"] & \\ \Omega_T(\Omega_T Y \wedge T) \arrow[r, "\Omega_T j"] \arrow[d, "\Omega_T ev"'] & \Omega_T(\Omega_T Y \wedge T)_s \arrow[dl, "\Omega_T \tilde{ev}"] \\ \Omega_T Y & \end{tikzcd}$$

The map $\Omega_T \eta_*$ is a pointwise equivalence by Theorem 3.11, and so $\Omega_T \tilde{ev}$ is a stable equivalence. □

Corollary 3.17. *Let $j : Y \to Y_s$ be a choice of stably fibrant model for Y. Then a map $g : X \wedge T \to Y$ is a stable equivalence if and only if the composite*

$$X \xrightarrow{g_*} \Omega_T Y \xrightarrow{\Omega_T j} \Omega_T Y_s$$

is a stable equivalence, where g_* is the adjoint of g.

Proof. There is a diagram

$$\begin{tikzcd} X \wedge T \arrow[r, "j"] \arrow[d, "g"] & (X \wedge T)_s \arrow[d, "\tilde{g}"'] \\ Y \arrow[r, "j"'] & Y_s \end{tikzcd}$$

where both maps labeled j are stably fibrant models. Then g is a stable equivalence if and only if $\tilde{g}$ is a stable equivalence if and only if the composite

$$X \xrightarrow{\eta_*} \Omega_T (X \wedge T)_s \xrightarrow{\Omega_T \tilde{g}} \Omega_T Y_s$$

is a stable equivalence. □

Corollary 3.18. *A map $g : X \to Y$ is a stable equivalence if and only if the suspension $g \wedge T : X \wedge T \to Y \wedge T$ is a stable equivalence.*

In the final part of this section we show that all of the usual candidates for suspension functors on T-spectra are naturally equivalent in the motivic stable category. This is the content of the next two lemmas. As a corollary, all of the corresponding loop functors are naturally stably equivalent.

Lemma 3.19. *The canonical map $\sigma : \Sigma_T^\ell X \to X[1]$ from the fake suspension $\Sigma_T^\ell X$ to the shift $X[1]$ is a natural stable equivalence.*

Proof. The map

$$\sigma : \Sigma_T^\ell (\Sigma_T^\infty K[-n]) \to (\Sigma_T^\infty K[-n])[1]$$

is an isomorphism in level p for $p \geq n$ and for all $n \geq 0$. The fake suspension $X \mapsto \Sigma_T^\ell X$ and shift $X \mapsto X[1]$ functors preserve colimits, so we can argue along the layer filtration using Lemma 3.12. It therefore suffices to show that both functors preserve stable equivalence.

In order to see that the shift functor $X \mapsto X[1]$ preserves stable equivalences, it suffices to show that the shift $X[1] \to (IQ_T JX)[1]$ of the canonical stable equivalence is a stable equivalence. For this, it enough to show that the shifted map $(JX)[1] \to (Q_T JX)[1]$ is a stable equivalence, but this is a consequence of the isomorphism $(Q_T JX)[1] \cong Q_T (JX[1])$.

The fake loop functor $X \mapsto \Omega_T^\ell X$ preserves stably fibrant objects, according to the characterization given by Lemma 2.7 and Lemma 2.8. The fake suspension functor $Y \mapsto \Sigma_T^\ell Y$ preserves level cofibrations and level weak equivalences, so that the fake loop functor preserves injective fibrations by adjointness. It follows that the fake loop functor preserves the class of stably fibrant injective objects.

We know from Corollary 2.12 that a map $f : X \to Y$ is a stable equivalence if and only if it induces a weak equivalence

$$f^* : \mathbf{hom}(Y, W) \to \mathbf{hom}(X, W)$$

for all stably fibrant injective W. If $f : X \to Y$ is a stable equivalence of T-spectra and W is stably fibrant and injective, then the map

$$(\Sigma_T^\ell f)^* : \mathbf{hom}(\Sigma_T^\ell Y, W) \to \mathbf{hom}(\Sigma_T^\ell X, W)$$

is isomorphic to the map

$$f^* : \mathbf{hom}(Y, \Omega_T^\ell W) \to \mathbf{hom}(X, \Omega_T^\ell W),$$

and is therefore a weak equivalence since $\Omega_T^\ell W$ is stably fibrant and injective. If follows that $\Sigma_T^\ell f : \Sigma_T^\ell X \to \Sigma_T^\ell Y$ is a stable equivalence. □

Lemma 3.20. *The fake suspension functor $X \mapsto \Sigma_T^\ell X$ is naturally stably equivalent to the functor $X \mapsto X \wedge T$.*

Proof. Both functors preserve level equivalences, so it suffices to assume that X (by taking associated sheaves) is a sheaf of T-spectra, where T and all of its smash powers are notationally confused with their associated sheaves. We do this so that we can use the explicit pointed algebraic homotopy $h : T^3 \times \mathbb{A}^1 \to T^3$ from $c_{1,2}$ to the identity which appears in the proof of Lemma 3.13. Write $d^a : T^3 \to T^3 \times \mathbb{A}^1$ for the map which is induced by the rational point $a : * \to \mathbb{A}^1$. Then hd^0 is the identity map on T^3 and $hd^1 = c_{1,2} : T^3 \to T^3$.

Recall that the fake suspension $\Sigma_T^\ell X$ consists of the objects $T \wedge X^n$ and bonding maps $T \wedge \sigma : T^2 \wedge X^n \to T \wedge X^{n+1}$. The object $X \wedge T$ consists of the pointed simplicial presheaves $X^n \wedge T$ and bonding maps $\sigma \wedge T : T \wedge X^n \wedge T \to X^{n+1} \wedge T$. After twisting along the isomorphisms $t : X^n \wedge T \cong T \wedge X^n$, we can identify $X \wedge T$ up to isomorphism with a spectrum consisting of objects $T \wedge X^n$ and having bonding maps σ given by the composites

$$T^2 \wedge X^n \xrightarrow{t \wedge X^n} T^2 \wedge X^n \xrightarrow{T \wedge \sigma} T \wedge X^{n+1}$$

It follows that there are commutative diagrams

$$\begin{tikzcd}
T^3 \wedge X^n \arrow[r, "T^2 \wedge \sigma"] \arrow[d, "c_{1,2} \wedge X^n"'] & T^2 \wedge X^{n+1} \arrow[r, "T \wedge \sigma"] & T \wedge X^{n+2} \\
T^3 \wedge X^n \arrow[r, "T \wedge \sigma"'] & T^2 \wedge X^{n+1} \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

The method of proof is to show that the "partial spectrum" objects X_1 and X_2, having constituent simplicial presheaves

$$X_1^n = X_2^n = T \wedge X^{2n}$$

and bonding maps $T^2 \wedge X_i^n \to X_i^{n+1}$ defined by the composites

$$\sigma_1 = (T \wedge \sigma)(T^2 \wedge \sigma)$$

and $\sigma_2 = \sigma(T \wedge \sigma)$ respectively (as in the diagram) are naturally stably equivalent.

The idea is to use the natural algebraic homotopies $h : T^3 \wedge X^{2n} \ltimes \mathbb{A}^1 \to T \wedge X^{2n+2}$ from σ_1 to σ_2 and the constant algebraic homotopies c on σ_1 to define natural level weak equivalences

$$X_2 \xleftarrow{h_*} \mathrm{Tel}(X_1) \xrightarrow{c_*} X_1$$

where $\mathrm{Tel}(X_1)$ is the *algebraic telescope*. The construction is by exact analogy with that of the ordinary mapping telescope given in [11, pp.11–15]. To summarize, one inductively constructs a sequence of trivial cofibrations

$$X_1^n \xrightarrow{j_n} CX_1^n \xrightarrow{\alpha_n} \mathrm{Tel}(X_1)^n,$$

where j_n is the inclusion of X_1^n in the algebraic mapping cylinder CX_1^n given by the pushout diagram

$$\begin{tikzcd} T^2 \wedge X_1^{n-1} \arrow[r, "\sigma_1"] \arrow[d, "d^0"'] & X_1^n \arrow[d, "j_n"] \\ T^2 \wedge X_1^{n-1} \ltimes \mathbb{A}^1 \arrow[r, "\zeta_n"'] & CX_1^n \end{tikzcd}$$

and α_n is inductively defined by the pushout diagram

$$\begin{tikzcd} T^2 \wedge X_1^{n-1} \arrow[r, "d^1"] \arrow[d, "j_{n-1}"'] & T^2 \wedge X_1^{n-1} \ltimes \mathbb{A}^1 \arrow[r, "\zeta_n"] & CX_1^n \arrow[dd, "\alpha_n"] \\ T^2 \wedge CX_1^{n-1} \arrow[d, "T^2 \wedge \alpha_{n-1}"'] & & \\ T^2 \wedge \mathrm{Tel}(X_1)^{n-1} \arrow[rr, "\sigma"'] & & \mathrm{Tel}(X_1)^n \end{tikzcd}$$

The bonding maps $\sigma : T^2 \wedge \mathrm{Tel}(X_1)^{n-1} \to \mathrm{Tel}(X_1)^n$ are also defined by this construction. The identity on X_1^n and $h : T^2 \wedge X_1^n \ltimes \mathbb{A}^1 \to T \wedge X_1^{n+1}$ together determine a weak equivalence $\hat{h} : CX_1^n \to X_2^n$ and the map $\hat{h}$ extends levelwise along the trivial cofibrations $\alpha_n : CX_1^n \to \mathrm{Tel}(X_1)^n$ to a natural map of partial spectra $h_* : \mathrm{Tel}(X_1) \to X_2$. The map h_* is a levelwise weak equivalence. □

Corollary 3.21. *Suppose that X is a level fibrant spectrum. Then the spectra $\Omega^{\ell}_T X$, $\Omega_T X$ and $X[-1]$ are naturally stably equivalent.*

Remark 3.22. A statement analogous to Theorem 3.11 is true for S^1-spectra, in that the composite

$$X \xrightarrow{\eta_X} \Omega(X \wedge S^1) \xrightarrow{\Omega(j_X \wedge T)} \Omega(X \wedge S^1)_s$$

is a natural weak equivalence in the motivic stable model structure for S^1-spectra. The proof is formally the same as that displayed for Theorem 3.11, with T replaced by S^1. The key is that it is well known that the cyclic permutation $c_{1,2}$ acts trivially in the ordinary homotopy category on S^3. With suitable modifications, the rest of the statements up to Corollary 3.18 also hold formally for S^1-spectra, so that the suspension and loop functors determine a self equivalence of categories for the motivic stable category of S^1-spectra, as one would expect. The analogues of Lemma 3.19 and Lemma 3.20 for S^1-spectra follow from standard results of stable homotopy theory, along with Lemma 3.1.

4 Motivic symmetric spectra

We continue to work within motivic homotopy theory on the smooth Nisnevich site $(Sm|_S)_{Nis}$, meaning that we formally contract the affine line onto a rational point within the associated category of simplicial presheaves. As before, T denotes either the quotient $\mathbb{A}^1/(\mathbb{A}^1 - 0)$ or the equivalent object $S^1 \wedge \mathbb{G}_m$. As in all discussions of geometric theories, one tacitly assumes that all objects in $(Sm|_S)_{Nis}$ (including the base scheme S) are bounded above by a fixed large cardinal, and that the category itself is a skeleton. This means that the site is small, and so its morphisms form a set. We shall assume that α is an infinite cardinal which is an upper bound for the cardinality of the set of morphisms of this site.

A *symmetric T-spectrum* X on the Nisnevich site $(Sm|_S)_{Nis}$ is a T-spectrum together with symmetric group actions $\Sigma_n \times X^n \to X^n$ such that the composite bonding maps $T^p \wedge X^n \to X^{p+n}$ is $(\Sigma_p \times \Sigma_n)$-equivariant. A map $f : X \to Y$ of such objects is a map of T-spectra which is equivariant in each level for the ambient symmetric group action. The resulting category will be denoted by $\mathbf{Spt}^{\Sigma}_T(Sm|_S)_{Nis}$. This category is complete and cocomplete.

The most primitive example of a symmetric T-spectrum is the *sphere T-spectrum*, which will be denoted by T. Explicitly,

$$T^n = \begin{cases} S^0 & \text{if } n = 0, \\ T^{\wedge n} & \text{if } n > 0. \end{cases}$$

with the obvious isomorphisms $T \wedge T^n \cong T^{n+1}$ as bonding maps.

4.1 The level structure

Say that a map $f : X \to Y$ of symmetric T-spectra is a *level equivalence* if each of the component maps $f : X^n \to Y^n$ is a motivic equivalence. The map f is said to be a *level cofibration* if each of the maps $X^n \to Y^n$ is a cofibration of simplicial presheaves. Write $\mathbf{sE}$ for the class of level equivalences in the category of symmetric T-spectra.

Proposition 4.1. *The class $\mathbf{sE}$ of level equivalences and the class of level cofibrations of symmetric T-spectra together satisfy the following properties:*

sE1*: The class of morphisms $\mathbf{sE}$ is closed under retracts.*

sE2*: Given a composable pair of morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

if any two of f, g and gf are in the class $\mathbf{sE}$, then so is the third.

sE3*: Every pointwise level equivalence is in $\mathbf{sE}$.*

sE4*: The class of $\mathbf{sE}$-trivial cofibrations is closed under pushout.*

sE5*: Suppose that γ is a limit ordinal, and there is a functor*

$$X : \gamma \to \mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis}$$

such that for each morphism $i \leq j$ of γ, the induced map $X(i) \to X(j)$ is an $\mathbf{sE}$-trivial cofibration. Then the canonical maps

$$X(i) \xrightarrow{\tau_i} \varinjlim_{j\in\gamma} X(j)$$

are $\mathbf{sE}$-trivial cofibrations.

sE6*: Suppose that the morphisms $f_i : X_i \to Y_i$ are $\mathbf{sE}$-trivial cofibrations for $i \in I$. Then the morphism*

$$\bigvee_{i\in I} f_i : \bigvee_{i\in I} X_i \to \bigvee_{i\in I} Y_i$$

is an $\mathbf{E}$-trivial cofibration.

sE7*: There is an infinite cardinal λ which is at least as large as the cardinality of the set of morphisms of $(Sm|_S)_{Nis}$, such that for every diagram*

$$\begin{tikzcd} & X \arrow[d, "i"] \\ A \arrow[r] & Y \end{tikzcd}$$

of maps of T-spectra with i a $\mathbf{sE}$-trivial cofibration, and A α-bounded, there is an α-bounded subobject $B \subset Y$ such that $A \subset B$, and such that the inclusion $B \cap X \hookrightarrow B$ is an $\mathbf{sE}$-trivial cofibration.

A *pointwise level equivalence* is a map $f : X \to Y$ of symmetric T-spectra such that all maps $f : X^n(U) \to Y^n(U)$ are weak equivalences of simplicial sets in all sections and levels. An **sE**-trivial level cofibration is a map of symmetric T-spectra which is both a level equivalence and a level cofibration.

Proof. Every weak equivalence of simplicial presheaves is a motivic equivalence, giving sE3. With the exception of sE7, the remaining statements are due to the existence of the motivic closed model structure for the category of simplicial presheaves on $(Sm|_S)_{Nis}$.

The proof of sE7 is analogous to the proof of Proposition 2.15. One begins by showing, using the method of proof of Lemma 1 of [13], that the class of maps which are level local weak equivalences and level cofibrations has the bounded cofibration property with respect to the cardinal λ. The argument is then completed just as in the last paragraph of the proof of Proposition 2.15 by using the controlled level fibrant model construction $X \mapsto \mathcal{L}X$ in place of the functor F. □

A *symmetric sequence* X consists of pointed simplicial presheaves X^n, $n \geq o$, each of which carries a symmetric group action $\Sigma_n \times X^n \to X^n$. There is an obvious category of such things. The *product* $X \otimes Y$ in the category of symmetric sequences is defined by

$$(X \otimes Y)^n = \bigvee_{p+q=n} \Sigma_n \otimes_{(\Sigma_p \times \Sigma_q)} X^p \wedge Y^q.$$

A symmetric sequence map $f : X \otimes Y \to Z$ therefore consists of $(\Sigma_p \times \Sigma_q)$-equivariant pointed maps $f : X^p \wedge Y^q \to Z^{p+q}$, so that a symmetric T-spectrum Z can be identified with a symmetric sequence carrying a T-module structure, or a symmetric sequence map $\sigma : T \otimes Z \to Z$. Note that there is a canonical twist isomorphism $\tau : X \otimes Y \to Y \otimes X$ which is determined by the composites

$$X^p \wedge Y^q \xrightarrow{t} Y^q \wedge X^p \xrightarrow{in_e} (Y \otimes X)^{q+p} \xrightarrow{c_{q,p}} (Y \otimes X)^{p+q}.$$

Here, t is the canonical twist of smash factors and in_e is the inclusion corresponding to the coset of the identity e in

$$(Y \otimes X)^{q+p} \cong \bigvee_{\Sigma_{q+p}/(\Sigma_q \times \Sigma_p)} Y^q \wedge X^p.$$

Following [7] and [13], given a pointed simplicial presheaf K, the *free symmetric sequence* $G_n K$ consists of the simplicial presheaf

$$\Sigma_n \otimes K = \bigvee_{\sigma \in \Sigma_n} K,$$

concentrated in level n, and the *free symmetric T-spectrum* $F_n(K) = T \otimes G_n K$ is defined at level p by

$$F_n(K)^p = (T \otimes G_n K)^p = \Sigma_p \otimes_{\Sigma_{p-n} \times \Sigma_n} (T^{p-n} \wedge (\bigvee_{\sigma \in \Sigma_n} K)).$$

The object $F_n(K)$ is free in the sense that morphisms $F_n(K) \to X$ in the category of symmetric T-spectra are in one to one correspondence with pointed simplicial presheaf maps $K \to X^n$.

An *injective fibration* in the category of symmetric T-spectra is a map which has the right lifting property with respect to all morphisms which are both level cofibrations and level equivalences. It follows from the existence of the free object functors $K \mapsto F_n(K)$ that every injective fibration $p : X \to Y$ is a level fibration in the sense that it consists of fibrations $p : X^n \to Y^n$ in all levels.

Theorem 4.2. *The category*

$$\mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis}$$

of symmetric T-spectra on the smooth Nisnevich site, together with the classes of level cofibrations, level equivalences and injective fibrations, satisfies the axioms for a proper closed simplicial model category.

Proof. The proof proceeds just like that of Theorem 3 of [13], using the method of [4] and Proposition 4.1. The function complex $\mathbf{hom}(X, Y)$ giving the simplicial structure is defined in n-simplices in the usual way by

$$\mathbf{hom}(X, Y)_n = \hom(X \wedge \Delta^n_+, Y),$$

where the pointed simplicial set Δ^n_+ is the result of attaching a disjoint base point to the standard n-simplex. □

The functor

$$U : \mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis} \to \mathbf{Spt}_T(Sm|_S)_{Nis}$$

taking values in T-spectra forgets the symmetric group actions. The functor U has a left adjoint *symmetrization functor* V such that for $n \geq 0$

$$V(\Sigma_T^{\infty} K[-n]) = F_n(K),$$

where $\Sigma_T^{\infty} K$ is the suspension T-spectrum

$$K,\ T \wedge K,\ T^2 \wedge K,\ \dots$$

and $\Sigma_T^{\infty} K[r]$ is the result of shifting in the usual way:

$$\Sigma_T^{\infty} K[r]^p = (\Sigma_T^{\infty} K)^{r+p}.$$

As in Section 3.4, every T-spectrum X has a layer filtration

$$X = \varinjlim L_n X$$

and V preserves colimits, so that

$$VX = \varinjlim VL_n X,$$

and there are pushouts

$$\begin{tikzcd} F_{n+1}(T \wedge X^n) \arrow[r] \arrow[d, "\sigma_*"'] & VL_n X \arrow[d] \\ F_{n+1}(X^{n+1}) \arrow[r] & VL_{n+1}X. \end{tikzcd}$$

in the category of symmetric T-spectra.

There is a natural isomorphism of T-spectra

$$(UW)^K \cong U(W^K),$$

which induces a simplicial adjunction isomorphism

$$\mathbf{hom}(VA, W) \cong \mathbf{hom}(A, UW).$$

We shall also need the following:

Lemma 4.3. *The functor V takes cofibrations of T-spectra to level cofibrations of symmetric T-spectra.*

Proof. The proof is just like that of Lemma 5 of [13], and begins with the observation that the functor

$$K \mapsto V(\Sigma_T^\infty K[-n]) = F_n(K)$$

takes cofibrations of pointed simplicial presheaves to level cofibrations of symmetric T-spectra for $n \geq 0$. □

4.2 The stable structure

Say that a map $p : X \to Y$ of symmetric T-spectra is a *stable fibration* if the underlying map $p_* : UX \to UY$ is a stable fibration of T-spectra.

Proposition 4.4. *Every map $f : X \to Y$ of symmetric T-spectra has a natural factorization*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & Y \end{tikzcd}$$

such that p is a stable fibration, and j is a level cofibration which has the left lifting property with respect to all stable fibrations.

Proof. By the methods of [4] and Corollary 2.16, a map of symmetric T-spectra is a stable fibration if and only if it has the right lifting property with respect to all maps $i_* : VA \to VB$ induced by λ-bounded cofibrations $i : A \to B$ which are stable equivalences. The factorization is constructed by a transfinite small object argument of size $\beta > 2^\lambda$, as in the proof of Lemma 6 of [13]. □

Observe that if j is a level cofibration which has the left lifting property with respect to all stable fibrations, then j induces a trivial fibration

$$j^* : \mathbf{hom}(Z, W) \to \mathbf{hom}(X, W)$$

of simplicial sets for each stably fibrant object W, by appropriate use of Quillen's axiom SM7 for the motivic stable closed model structure on the category of T-spectra.

It follows from Theorem 4.2 and Proposition 4.4 that any morphism $f : X \to Y$ of symmetric T-spectra may be successively factored

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[drr, "f"'] & X_s \arrow[r, "i_2"] \arrow[dr, "p_1"] & X_{si} \arrow[d, "p_2"] \\ & & Y \end{tikzcd}$$

where

1) i_1 is a level cofibration which has the left lifting property with respect to all stable fibrations, and p_1 is a stable fibration;

2) i_2 is a level cofibration and a level equivalence, and p_2 is an injective fibration.

In particular, Up_2 is a level fibration, which is level equivalent to the stable fibration Up_1, so that p_2 is a stable fibration by Lemma 2.7 as well as an injective fibration of symmetric T-spectra. By specializing to $Y = *$, we obtain a natural construction

$$X \xrightarrow{i_1} X_s \xrightarrow{i_2} X_{si}$$

of an *injective stably fibrant model* X_{si} for a given symmetric T-spectrum X.

Say that a map $f : X \to Y$ of symmetric T-spectra is a *stable equivalence* if it induces a weak equivalence of Kan complexes

$$f^* : \mathbf{hom}(Y, W) \to \mathbf{hom}(X, W)$$

for each injective stably fibrant object W. The maps i_1 and i_2 above are both stable equivalences. Following [13] we can also show

Lemma 4.5. *Suppose that the objects X and Y are stably fibrant and injective. Then a map $g : X \to Y$ is a stable equivalence if and only if it is a level equivalence.*

Proof. If g is a stable equivalence, then the map

$$g^* : \mathbf{hom}(Y, X) \to \mathbf{hom}(X, X)$$

is a weak equivalence of Kan complexes, since X is stably fibrant and injective. It follows that g is a homotopy equivalence.

The converse follows from the closed simplicial model structure for level cofibrations and level weak equivalences for symmetric T-spectra, since all symmetric T-spectra are cofibrant and all stably fibrant injective objects W are fibrant for that theory. □

Corollary 4.6. *Suppose that X and Y are stably fibrant objects. Then a map $g : X \to Y$ is a stable equivalence if and only if it is a level equivalence.*

Suppose that Z is a symmetric T-spectrum and that K is a pointed simplicial presheaf. The symmetric T-spectrum

$$Z^K = \mathbf{Hom}_*(K, Z)$$

is defined in levels by

$$\mathbf{Hom}_*(K, Z)^n = \mathbf{Hom}_*(K, Z^n),$$

where $\mathbf{Hom}_*$ denotes the pointed internal hom functor, as in Section 1.1. The structure map

$$T^p \wedge \mathbf{Hom}_*(K, Z^n) \xrightarrow{\sigma} \mathbf{Hom}_*(K, Z^{p+n})$$

is the unique pointed simplicial set map making the diagram

$$\begin{tikzcd}
T^p \wedge \mathbf{Hom}_*(K, Z^n) \wedge K \arrow[r, "\sigma \wedge K"] \arrow[d, "T^p \wedge \mathrm{ev}"'] & \mathbf{Hom}_*(K, Z^{p+n}) \wedge K \arrow[d, "\mathrm{ev}"] \\
T^p \wedge Z^n \arrow[r, "\sigma"'] & Z^{p+n}
\end{tikzcd}$$

commute, where ev is the evaluation map wherever it appears. This construction is natural in K and Z, and there are natural isomorphisms

$$\mathbf{Hom}_*(K \wedge L, Z) \cong \mathbf{Hom}_*(K, \mathbf{Hom}_*(L, Z))$$

for all pointed simplicial presheaves K, L, and symmetric T-spectra Z.

We shall write $\Omega_T X$ for the symmetric T-spectrum $\mathbf{Hom}_*(T, X)$, in order to simplify notation.

Following [13], define a natural *shift functor* $Z \mapsto Z[1]$ for symmetric T-spectra Z by setting $Z[1]^m = Z^{1+m}$, where $\sigma \in \Sigma_m$ acts on $Z[1]^m$ as $1 \oplus \sigma \in \Sigma_{m+1}$. The structure map $\sigma_* : T^p \wedge Z[1]^m \to Z[1]^{p+m}$ is defined to be the composite

$$T^p \wedge Z^{1+m} \xrightarrow{\sigma} Z^{p+1+m} \xrightarrow{c(p,1)\oplus 1} Z^{1+p+m},$$

where $c(p,1) \in \Sigma_{p+1}$ is the cyclic permutation of order $p+1$. One checks that σ_* is $\Sigma_p \times \Sigma_m$-equivariant. Define the shifted symmetric T-spectrum $Z[s]$ inductively by $Z[s] = Z[s-1][1]$, or directly.

The standard maps $\sigma_* : Z^n \to \mathbf{Hom}_*(T, Z^{1+n})$ which are adjoint to the composites

$$Z^n \wedge T \xrightarrow{t} T \wedge Z^n \xrightarrow{\sigma} Z^{1+n}$$

together determine a natural map of of symmetric T-spectra

$$\sigma_* : Z \to \mathbf{Hom}_*(T, Z[1]) \cong \mathbf{Hom}_*(T, Z)[1],$$

or equivalently a map

$$\sigma_* : Z \to \Omega_T(Z[1]) \cong (\Omega_T Z)[1]. \tag{4.1}$$

Suppose that Z is a symmetric T-spectrum which is level fibrant. Flipping loop factors defines a natural isomorphism

$$t^* : \Omega_T^2 Z[2] \to \Omega_T^2 Z[2],$$

and there is an isomorphism $(1,2) : Z[2] \to Z[2]$ which consists of maps $(1,2) : Z^{2+n} \to Z^{2+n}$ induced by the transposition $(1,2) \in \Sigma_{2+n}$. Write $\tilde{\sigma}$ for the bonding maps of $\Omega_T Z[1]$. Then there is a natural commutative diagram

$$\begin{tikzcd}
\Omega_T Z[1] \arrow[r, "\Omega_T \sigma_*[1]"] \arrow[dr, "\tilde{\sigma}_*"'] & \Omega_T^2 Z[2] \arrow[d, "(1,2)_* t^*"] \\
 & \Omega_T^2 Z[2]
\end{tikzcd}$$

which translates into a diagram of simplicial presheaves

$$\begin{tikzcd}
\Omega_T Z^{n+1} \arrow[r, "\Omega_T \sigma_*"] \arrow[dr, "\tilde{\sigma}_*"'] & \Omega_T^2 Z^{n+2} \arrow[d, "(1,2)_* t^*"] \\
 & \Omega_T^2 Z^{n+2}
\end{tikzcd} \tag{4.2}$$

for each n.

For a level fibrant object Z, define the symmetric T-spectrum $Q_T^{\Sigma} Z$ to be the filtered colimit of the system

$$Z \xrightarrow{\sigma_*} \Omega_T Z[1] \xrightarrow{\tilde{\sigma}_*} \Omega_T^2 Z[2] \xrightarrow{\tilde{\tilde{\sigma}}_*} \cdots \tag{4.3}$$

Lemma 4.7. *Suppose that Z is a level fibrant symmetric T-spectrum. Then there is a natural isomorphism*

$$Q_T^{\Sigma} Z^n \cong Q_T (UZ)^n.$$

Warning: Lemma 4.7 only says that the T-spectra $U(Q_T^{\Sigma}Z)$ and $Q_T(UZ)$ are isomorphic in each level. The assertion that these are isomorphic spectrum objects is one of the canonical mistakes in the theory of symmetric spectra.

Proof of Lemma 4.7. To extend the notation for the bonding map $\tilde{\sigma}$ of $\Omega_T Z[1]$ given above, write

$$\sigma_*^{\sim(n)} = \widetilde{\sigma^{\sim(n-1)}}_* : \Omega_T^n Z[n] \to \Omega_T^{n+1} Z[n+1],$$

so that $\tilde{\sigma}_* = \sigma_*^{\sim(1)}$ and $\tilde{\tilde{\sigma}}_* = \sigma_*^{\sim(2)}$ in the sequence (4.3). Repeated instances of the diagram (4.2) paste together to give a natural commutative diagram

$$\begin{tikzcd}
\Omega_T^k Z^{n+k} \arrow[r, "\sigma_*^{\sim(k)}"] \arrow[dr, "\Omega_T^k \sigma_*"'] & \Omega_T^{k+1} Z^{n+k+1} \arrow[d, "\theta_{k+1}"] \\
& \Omega_T^{k+1} Z^{n+k+1}
\end{tikzcd}$$

where θ_{k+1} is a composite of isomorphisms $\Omega_T^i t^*$ or $(1,2)_*$.

Now suppose given natural isomorphisms $\gamma_i : \Omega_T^i Z^{n+i} \to \Omega_T^i Z^{n+i}$ such that the diagram

$$\begin{tikzcd}
\Omega_T Z^{n+1} \arrow[r, "\sigma_*^{\sim(1)}"] \arrow[d, "1"'] & \Omega_T^2 Z^{n+2} \arrow[r, "\sigma_*^{\sim(2)}"] \arrow[d, "\gamma_2"'] & \cdots \arrow[r, "\sigma_*^{\sim(k-1)}"] & \Omega_T^k Z^{n+k} \arrow[d, "\gamma_k"] \\
\Omega_T Z^{n+1} \arrow[r, "\Omega_T \sigma_*"'] & \Omega_T^2 Z^{n+2} \arrow[r, "\Omega_T^2 \sigma_*"'] & \cdots \arrow[r, "\Omega_T^{k-1} \sigma_*"'] & \Omega_T^k Z^{n+k}
\end{tikzcd}$$

commutes, and all isomorphisms γ_i are compositions of of $\Omega_T^j t^*$ or $(i,i+1)_*$. In particular, presume that $\gamma_2 = t^*(1,2)_* : \Omega_T^2 Z^{n+2} \to \Omega_T^2 Z^{n+2}$. Then the isomorphism $\Omega_T^j t^*$ commutes with $\Omega_T^k \sigma_* : \Omega_T^k Z^{n+k} \to \Omega_T^{k+1} Z^{n+k+1}$, and

$$\sigma_*(i,i+1)_* = (i+1,i+2)_* \sigma_*$$

so there is an isomorphism $\overline{\gamma}_{k+1}$ composed of maps $\Omega_T^j t^*$ and $(i,i+1)_*$ such that the diagram

$$\begin{tikzcd}
\Omega_T^k Z^{n+k} \arrow[r, "\sigma_*^{\sim(k)}"] \arrow[dr, "\Omega_T^k \sigma_*"'] \arrow[dd, "\gamma_k"'] & \Omega_T^{k+1} Z^{n+k+1} \arrow[d, "\theta_{k+1}"] \\
& \Omega_T^{k+1} Z^{n+k+1} \arrow[d, "\overline{\gamma}_{k+1}"] \\
\Omega_T^k Z^{n+k} \arrow[r, "\Omega_T^k \sigma_*"'] & \Omega_T^{k+1} Z^{n+k+1}
\end{tikzcd}$$

commutes. The natural isomorphism γ_{k+1} is defined by $\gamma_{k+1} = \overline{\gamma}_{k+1}\theta_{k+1}$. $\square$

Formally, there is a map $c : Q_T^\Sigma X \wedge K \to Q_T^\Sigma(X \wedge K)$ which fits into a natural commutative diagram

$$\begin{tikzcd} Q_T^\Sigma X \wedge K \arrow[r, "c"] & Q_T^\Sigma(X \wedge K), \\ X \wedge K \arrow[u, "\gamma_X \wedge K"] \arrow[ur, "\gamma_{X \wedge K}"'] & \end{tikzcd}$$

for all symmetric T-spectra X and pointed simplicial sets K. It follows that the functor Q_T^Σ prolongs to a simplicial functor

$$Q_T^\Sigma : \mathbf{hom}(X, Y) \to \mathbf{hom}(Q_T^\Sigma X, Q_T^\Sigma Y).$$

Proposition 4.8. *Suppose that $\alpha : X \to Y$ is a map of symmetric T-spectra such that $U\alpha : UX \to UY$ is a stable equivalence of T-spectra. Then α is a stable equivalence of symmetric T-spectra.*

Proof. We can assume that X and Y are level fibrant. If W is a stably fibrant and injective object, then the canonical map $\gamma_W : W \to Q_T^\Sigma W$ is a level equivalence, and hence induces a weak equivalence

$$\gamma_W^* : \mathbf{hom}(Q_T^\Sigma W, W) \to \mathbf{hom}(W, W).$$

It follows that there is a map $g_W : Q_T^\Sigma W \to W$ such that the composite $g_W \gamma_W$ is simplicially homotopic to the identity 1_W on W.

The composite

$$\mathbf{hom}(X, W) \xrightarrow{Q_T^\Sigma} \mathbf{hom}(Q_T^\Sigma X, Q_T^\Sigma W) \xrightarrow{g_{W*}} \mathbf{hom}(Q_T^\Sigma X, W) \xrightarrow{\gamma_X^*} \mathbf{hom}(X, W)$$

is induced by composition with $g_W \gamma_W$, and is therefore homotopic to the identity on $\mathbf{hom}(X, W)$. The composition and the homotopy are natural in X. If $\alpha : X \to Y$ induces a stable equivalence $U\alpha : UX \to UY$, then the induced map $Q_T^\Sigma \alpha : Q_T^\Sigma X \to Q_T^\Sigma Y$ is a level equivalence by Lemma 4.7, and so the maps

$$Q_T^\Sigma \alpha^* : \mathbf{hom}(Q_T^\Sigma Y, W) \to \mathbf{hom}(Q_T^\Sigma X, W)$$

and hence the morphisms

$$\alpha^* : \mathbf{hom}(Y, W) \to \mathbf{hom}(X, W)$$

are weak equivalences of pointed simplicial sets. $\square$

Recall that if Y is a symmetric T-spectrum and $n \geq 0$, then the shift $Y[n]$ is defined by $Y[n]^p = Y^{n+p}$, with $\alpha \in \Sigma_p$ acting as $1_n \oplus \alpha$. The bonding map $T^q \wedge Y[n]^p \to Y[n]^{q+p}$ for $Y[n]$ is the composite

$$T^q \wedge Y^{n+p} \xrightarrow{\sigma} Y^{q+n+p} \xrightarrow{c_{q,n} \oplus 1} Y^{n+q+p}$$

where σ is the original bonding map for Y.

Suppose that X is a symmetric T-spectrum, with T-module structure map $\sigma : T \otimes X \to X$. Then the symmetric sequence $G_n(S^0) \otimes X$ has a symmetric T-spectrum structure, with T-structure given by the composite

$$T \otimes G_n(S^0) \otimes X \xrightarrow{\tau \otimes 1} G_n(S^0) \otimes T \otimes X \xrightarrow{1 \otimes \sigma} G_n(S^0) \otimes X.$$

A symmetric T-spectrum map $f : G_n(S^0) \otimes X \to Y$ consists of pointed simplicial presheaf maps $f : X^p \to Y^{n+p}$ which are equivariant for the homomorphisms $\Sigma_p \to \Sigma_{n+p}$ defined by $\alpha \mapsto 1_n \oplus \alpha$, and such that the diagrams

$$\begin{tikzcd}
T^q \wedge X^p \arrow[r, "1 \wedge f"] \arrow[dd, "\sigma"'] & T^q \wedge Y^{n+p} \arrow[d, "\sigma"] \\
 & Y^{q+n+p} \arrow[d, "c_{q,n} \oplus 1"] \\
X^{q+p} \arrow[r, "f"'] & Y^{n+q+p}
\end{tikzcd}$$

commute. It follows that the symmetric T-spectrum map $f : G_n(S^0) \otimes X \to Y$ can be identified with a symmetric T-spectrum map $X \to Y[n]$, and we have proved

Lemma 4.9. *The functor $X \mapsto G_n(S^0) \otimes X$ is left adjoint to the shift functor $Y \mapsto Y[n]$ for $n \geq 0$.*

The functor $X \mapsto G_n(S^0) \otimes X$ preserves level cofibrations and level equivalences, so we have

Corollary 4.10. *The shift functor $Y \mapsto Y[n]$ preserves injective fibrations and level trivial injective fibrations. In particular, if Y is an injective symmetric T-spectrum, then $Y[n]$ is an injective symmetric T-spectrum for $n \geq 0$.*

Lemma 4.11. *Suppose that the commutative diagram*

$$\begin{tikzcd}
A_1 \arrow[r, "i_1"] \arrow[d, "f_1"'] & B_1 \arrow[r, "\pi_1"] \arrow[d, "f_2"] & B_1/A_1 \arrow[d, "f_3"] \\
A_2 \arrow[r, "i_2"'] & B_2 \arrow[r, "\pi_2"'] & B_2/A_2
\end{tikzcd}$$

is a comparison diagram of level cofibre sequences. Then if any two of the maps f_1, f_2 and f_3 are stable equivalences of symmetric T-spectra, then so is the third.

Proof. We shall show that f_1 is a stable equivalence if f_2 and f_3 are stable equivalences. This amounts to showing that the map f_1^* in the comparison diagram of fibrations

$$\begin{tikzcd}
\mathbf{hom}(B_2/A_2,W) \arrow[r, "\pi_2^*"] \arrow[d, "f_3^*"'] & \mathbf{hom}(B_2,W) \arrow[r, "i_2^*"] \arrow[d, "f_2^*"] & \mathbf{hom}(A_2,W) \arrow[d, "f_1^*"] \\
\mathbf{hom}(B_1/A_1,W) \arrow[r, "\pi_1^*"'] & \mathbf{hom}(B_1,W) \arrow[r, "i_1^*"'] & \mathbf{hom}(A_1,W)
\end{tikzcd}$$

is a weak equivalence for any choice of stably fibrant injective object W, in the presence of knowing that the simplicial set maps f_2^* and f_3^* are weak equivalences.

There is a levelwise equivalence

$$W \to \Omega_T W[1] \simeq \Omega\Omega_{\mathbb{G}_m} W[1],$$

of stably fibrant injective objects, where $W[1]$ is injective by Corollary 4.10. It is also the case that $\Omega_{\mathbb{G}_m} W[1]$ is stably fibrant and injective. It follows that the comparison diagram of fibrations can be delooped infinitely often. In particular, f_1^* is part of a stable weak equivalence of infinite loop spaces, and is therefore a weak equivalence of simplicial sets. □

Corollary 4.12. *Suppose that the commutative diagram*

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d, "f_1"'] & X_1 \arrow[r, "p_1"] \arrow[d, "f_2"] & Y_1 \arrow[d, "f_3"] \\
F_2 \arrow[r] & X_2 \arrow[r, "p_2"'] & Y_2
\end{tikzcd}$$

is a comparison diagram of level fibre sequences of symmetric T-spectra. Then if any two of f_1, f_2 and f_3 are stable equivalences, then so is the third.

Proof. Use Lemma 3.9 to replace the comparison of fibre sequences by the a comparison of level cofibre sequences

$$F_i \xrightarrow{i} X_i \xrightarrow{\pi} X_i/F_i. \tag{4.4}$$

More precisely, Lemma 3.9 guarantees that the map of T-spectra underlying $p_{i*} : X_i/F_i \to Y_i$ is a stable equivalence, so that p_{i*} is a stable equivalence of symmetric T-spectra by Proposition 4.8. Now use Lemma 4.11. □

We are now ready to prove the following:

Proposition 4.13. *Suppose that $p : X \to Y$ is a map of symmetric T-spectra which is both a stable fibration and a stable equivalence. Then p is a level equivalence.*

Proof. It suffices to show that the fibre F of p is level contractible. If so, the underlying map Up of T-spectra is a stable fibration and a stable equivalence by a long exact sequence argument in bigraded stable homotopy groups (3.2), and is therefore a level equivalence.

The comparison map

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[r, "p"] \arrow[d, "\simeq"', "p"] & Y \arrow[d, "1"] \\ * \arrow[r] & Y \arrow[r, "1"'] & Y \end{tikzcd}$$

of level fibre sequences and Corollary 4.12 together imply that the map $F \to *$ is a stable equivalence of stably fibrant objects, so it is a level weak equivalence by Corollary 4.6. □

Corollary 4.14. *A map $p : X \to Y$ of symmetric T-spectra is a stable fibration and a stable equivalence if and only if it is both a level fibration and a level equivalence.*

Proof. One direction is Proposition 4.13; the other follows from the definition of stable equivalence of symmetric T-spectra and Lemma 2.7. □

Say that a map $i : A \to B$ of symmetric T-spectra is a *stable cofibration* if it has the left lifting property with respect to all morphisms $p : X \to Y$ which are simultaneously stable fibrations and stable equivalences. In view of Corollary 4.14, the maps

$$F_n(A_+) \to F_n(L_U \Delta^r_+)$$

induced by the inclusions $A \subset L_U \Delta^r$ are stable cofibrations for all r and objects U. Here, L_U denotes the left adjoint to the U-sections functor for simplicial presheaves.

Theorem 4.15. *The category $\mathbf{Spt}^{\Sigma}_T(Sm|_S)_{Nis}$ of symmetric T-spectra on the smooth Nisnevich site, and the classes of stable equivalences, stable fibrations and stable cofibrations, together satisfy the axioms for a proper closed simplicial model category.*

Proof. On account of Proposition 4.4, every map $g : X \to Y$ of symmetric T-spectra has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "g"'] & Z \arrow[d, "p"] \\ & Y \end{tikzcd} \tag{4.5}$$

such that p is a stable fibration, and j has the left lifting property with respect to all stable fibrations and induces trivial fibrations $j^* : \mathbf{hom}(Z, W) \to$

$\mathbf{hom}(X, W)$ for all stably fibrant objects W. In particular, j is a stable equivalence and a stable cofibration. The map j is a level cofibration, by Lemma 4.3.

A transfinite small object argument says that $g : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "g"'] & U \arrow[d, "q"] \\ & Y \end{tikzcd}$$

such that i has the left lifting property with respect to all maps which are simultaneously level fibrations and level weak equivalences, and q has the right lifting property with respect to all morphisms $F_n(A_+) \to F_n(L_U \Delta^r_+)$ corresponding to cofibrations $A \hookrightarrow L_U \Delta^n$ of simplicial presheaves for all n and objects $U \in \mathcal{C}$. In particular, q is a level trivial fibration and hence a stable fibration as well as a stable equivalence by Corollary 4.14. The map i has the left lifting property with respect to all maps which are stable fibrations and stable equivalences, also by Corollary 4.14, so that i is a stable cofibration. It is a consequence of the small object argument that the map i is a level cofibration.

The factorization axiom CM5 has therefore been demonstrated. The existence of the factorization (4.5) implies that every map which is a stable cofibration and a stable equivalence has the left lifting property with respect to all stable fibrations and is a level cofibration, by a standard argument. We have proved CM4, and the axioms CM1 − CM3 are obvious.

If $i : K \hookrightarrow L$ is an inclusion of simplicial sets and $p : X \to Y$ is a stable fibration of symmetric T-spectra, then the induced map

$$(i^*, p_*) : \mathbf{hom}_*(L_+, X) \to \mathbf{hom}_*(K_+, X) \times_{\mathbf{hom}_*(K_+, Y)} \mathbf{hom}_*(L_+, Y)$$

is a stable fibration, which is trivial if i is a weak equivalence or p is a stable equivalence. This is on account of the corresponding statement for T-spectra and Corollary 4.14, and implies the axiom SM7 for $\mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis}$.

The properness assertion is a consequence of Lemma 4.11 and Corollary 4.12. □

4.3 The smash product

The *smash product* $X \wedge Y$ of the symmetric T-spectra X and Y is defined by the symmetric sequence coequalizer

$$T \otimes X \otimes Y \rightrightarrows X \otimes Y \to X \wedge Y$$

of the map $m \otimes 1 : T \otimes X \otimes Y \to X \otimes Y$ with the composite

$$T \otimes X \otimes Y \xrightarrow{\tau \otimes 1} X \otimes T \otimes Y \xrightarrow{1 \otimes m} X \otimes Y$$

where m denotes the T-module structure for each of X and Y. The T-module structure on $X \wedge Y$ is induced by the map $m \otimes 1 : T \otimes X \otimes Y \to X \otimes Y$.

The smash product gives the category $\mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis}$ of symmetric T-spectra the structure of a symmetric monoidal category. This is a formal consequence of the fact that the symmetric T-spectrum T is a commutative monoid in the category of symmetric sequences, just as in [7].

A map $h : X \wedge Y \to Z$ of symmetric T-spectra can be characterized as a collection of $(\Sigma_p \times \Sigma_q)$-equivariant maps $h_{p,q} : X^p \wedge Y^q \to Z^{p+q}$, $p, q \geq 0$, which are T-linear in the sense that the diagram

$$\begin{tikzcd} T^r \wedge X^p \wedge Y^q \arrow[r, "\sigma\wedge 1"] \arrow[d, "1\wedge h_{p,q}"'] & X^{r+p} \wedge Y^q \arrow[d, "h_{r+p,q}"] \\ T^r \wedge Z^{p+q} \arrow[r, "\sigma"'] & Z^{r+p+q} \end{tikzcd} \tag{4.6}$$

commutes, and are T-bilinear, meaning that the following diagram commutes

$$\begin{tikzcd} T^r \wedge X^p \wedge Y^q \arrow[r, "t\wedge 1"] \arrow[d, "\sigma\wedge 1"'] & X^p \wedge T^r \wedge Y^q \arrow[r, "1\wedge\sigma"] & X^p \wedge Y^{r+q} \arrow[d, "h_{p,r+q}"] \\ X^{r+p} \wedge Y^q \arrow[r, "h_{r+p,q}"'] & Z^{r+p+q} \arrow[r, "c_{r,p}\oplus 1"'] & Z^{p+r+q} \end{tikzcd} \tag{4.7}$$

for each $p, q, r \geq 0$.

Lemma 4.16. *There is a natural isomorphism*

$$\hom(F_n(S^0) \wedge A, X) \cong \hom(A, X[n])$$

for symmetric T-spectra A and X.

Proof. Recall that the symmetric T-spectrum $F_n(S^0) \cong T \otimes G_n(S^0)$ has the form

$$F_n(S^0)_j = \begin{cases} * & j < n \\ \Sigma_j \otimes_{\Sigma_{j-n}\times\Sigma_n} (T^{j-n} \wedge \Sigma_{n+}) & j \geq n \end{cases}$$

and has the obvious T-action. Here, Σ_{n+} denotes the set $\Sigma_n \sqcup \{*\}$, pointed by the terminal object $*$.

A map $h : F_n(S^0) \wedge X \to Y$ is therefore determined by $(\Sigma_p \times \Sigma_n \times \Sigma_q)$-equivariant maps $h_{p+n,q} : T^p \wedge \Sigma_{n+} \wedge X^q \to Y^{n+p+q}$ for $p, q \geq 0$, which satisfy compatibility conditions given by diagrams (4.6) and (4.7) above. In particular the maps

$$h_{n,q} : \Sigma_{n+} \wedge X^q \to Y^{n+q}$$

are completely determined by the Σ_q-equivariant composites

$$X^q \xrightarrow{in_e} \Sigma_{n+} \wedge X^q \xrightarrow{h_{n,q}} Y^{n+q},$$

where $\sigma \in \Sigma_q$ acts on Y^{n+q} via $1_n \oplus \sigma$ and in_e is the inclusion of the wedge summand corresponding to the identity element $e \in \Sigma_n$. Then the Σ_q-equivariant maps

$$h_q = h_{n,q} in_e : X^q \to Y^{n+q}$$

define a map of symmetric T-spectra $h_* : X \to Y[n]$ — seeing this is a matter of chasing the definitions through instances of the diagrams (4.6) and (4.7).

For the converse, suppose given a map $h : X \to Y[n]$ of symmetric T-spectra, which is defined by Σ_q-equivariant maps $h_q : X^q \to Y^{n+q}$. Then h_q uniquely extends to a $(\Sigma_n \times \Sigma_q)$-equivariant map $h_{n,q} : \Sigma_{n+} \wedge X^q \to Y^{n+q}$. Define the map $h_{p+n,q} : T^p \wedge \Sigma_{n+} \wedge X^q \to Y^{p+n+q}$ to be the composite

$$T^p \wedge \Sigma_{n+} \wedge X^q \xrightarrow{1 \wedge h_{n,q}} T^p \wedge Y^{n+q} \xrightarrow{\sigma} X^{p+n+q}.$$

This description of the maps $h_{n,q}$ is determined by h and the T-linearity requirement. For the T-bilinearity, it suffices to show that the diagram

$$\begin{tikzcd}
T^p \wedge \Sigma_{n+} \wedge X^q \arrow[r, "t \wedge 1"] \arrow[d, "1 \wedge h_{n,q}"'] & \Sigma_{n+} \wedge T^p \wedge X^q \arrow[d, "1 \wedge \sigma"] \\
T^p \wedge Y^{n+q} \arrow[d, "\sigma"'] & \Sigma_{n+} \wedge X^{p+q} \arrow[d, "h_{n,p+q}"] \\
Y^{p+n+q} \arrow[r, "c_{p,n} \oplus 1"'] & Y^{n+p+q}
\end{tikzcd}$$

commutes, but this follows from the commutativity of the diagram

$$\begin{tikzcd}
T^p \wedge X^q \arrow[r, "1 \wedge h_q"] \arrow[dd, "\sigma"'] & T^p \wedge Y^{n+q} \arrow[d, "\sigma"] \\
& Y^{p+n+q} \arrow[d, "c_{p,n} \oplus 1"] \\
X^{p+q} \arrow[r, "h_{p+q}"'] & Y^{n+p+q}
\end{tikzcd}$$

that arises from the symmetric T-spectrum map $h : X \to Y[n]$. □

Corollary 4.17. *There is a natural isomorphism of symmetric T-spectra*

$$F_n(S^0) \wedge X \cong G_n(S^0) \otimes X.$$

Proof. Both functors are left adjoint to the shift functor $X \mapsto X[n]$ — see Lemma 4.9. $\square$

Corollary 4.18. *There are isomorphisms*

$$F_n(A) \wedge F_m(B) \cong F_{n+m}(A \wedge B),$$

and these isomorphisms are natural in pointed simplicial presheaves A and B.

Proposition 4.19. *Suppose that $i : A \to B$ is a stable cofibration and that $j : C \to D$ is a level cofibration. Then the map*

$$(i,j)_* : (B \wedge C) \cup_{(A \wedge C)} (A \wedge D) \to B \wedge D$$

is a level cofibration. If i and j are both cofibrations, then $(i,j)_*$ is a cofibration. If either i or j is a stable equivalence, then $(i,j)_*$ is a stable equivalence.

Proof. We shall begin with the statements on stable cofibrations. The map (i_*, j_*) induced by the cofibrations $i_* : F_n(A) \to F_n(B)$ and $j_* : F_m(C) \to F_m(D)$ is isomorphic to the map obtained from the pointed simplicial set cofibration

$$(B \wedge C) \cup_{(A \wedge C)} (A \wedge D) \to B \wedge D$$

by applying the functor F_{n+m}, so (i_*, j_*) is a cofibration.

Suppose that we fix a choice of cofibration $j : C \to D$. Then the collection of level cofibrations $i : A \to B$ for which the map (i,j) is a cofibration (respectively a trivial cofibration) is saturated; this means that the collection is closed under pushouts, filtered colimits over ordinals, and retracts. It follows that all maps of the form (i, j_*) and hence all maps (i,j) are cofibrations, for all cofibrations i, and then for all cofibrations j.

The cofibre of the cofibration (i,j) is $B/A \wedge D/C$, and both factors are cofibrant. To show that $(i,j)_*$ is a trivial cofibration if either i or j is a stable equivalence, it suffices to show that, given cofibrant objects A and B, the symmetric T-spectrum $A \wedge B$ is trivially cofibrant if either A or B is trivially cofibrant. For this, it is enough to show that the map $1 \wedge i_* : A \wedge F_n(K) \to A \wedge F_n(L)$ is a trivial cofibration if A is trivially cofibrant and $i : K \to L$ is a cofibration of pointed simplicial presheaves.

We have natural isomorphisms

$$F_n(K) \cong F_n(S^0) \wedge K$$

and we also know from Lemma 4.16 that there is an isomorphism

$$\hom(A \wedge F_n(S^0), X) \cong \hom(A, X[n]) \qquad (4.8)$$

It follows that the diagram

$$\begin{tikzcd} A \wedge F_n(K) \arrow[r] \arrow[d, "1\wedge i_*"'] & X \arrow[d, "p"] \\ A \wedge F_n(L) \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

is adjoint to a diagram

$$\begin{tikzcd} & X^L[n] \arrow[d] \\ A \arrow[r] \arrow[ur, dotted] & X^K[n] \times_{Y^K[n]} Y^L[n] \end{tikzcd}$$

and the dotted arrow exists by axiom SM7 and the fact that stable fibrations shift in the category of T-spectra. In particular, $1 \wedge i_*$ is a trivial cofibration.

Suppose more generally that i is a stable cofibration and that j is a level cofibration. To show that $(i,j)_*$ is a level cofibration, it suffices by a saturation argument to show that the map

$$(F_n(L) \wedge C) \cup_{(F_n(K)\wedge C)} (F_n(K) \wedge D) \to F_n(L) \wedge D$$

is a level cofibration for all cofibrations $K \to L$ of pointed simplicial presheaves. This amounts to showing that the dotted arrow exists in all diagrams

$$\begin{tikzcd} C \arrow[r] \arrow[d, "j"'] & X^L[n] \arrow[d] \\ D \arrow[r] \arrow[ur, dotted] & (Y^L \times_{Y^K} X^K)[n] \end{tikzcd}$$

arising from all trivial injective fibrations p, but this is a consequence of the Corollary 4.10 and the properness property for the level model structure on symmetric T-spectra (Theorem 4.2).

The same argument implies that any trivial level cofibration $j : C \to D$ induces a trivial level cofibration $(i,j)_*$ for any stable cofibration i. It follows that a level weak equivalence $f : E \to F$ induces a level weak equivalence $1 \wedge f : A \wedge E \to A \wedge F$ for any cofibrant symmetric T-spectrum A.

The map $(i,j)_*$ is a level cofibration with cofibre $B/A \wedge D/C$, where B/A is cofibrant. To show that $(i,j)_*$ is stably trivial if either i or j is stably trivial, it suffices once again to show that if B is cofibrant, then $A \wedge B$ is stably equivalent to a point if this is so for either A or B. But there is a level weak equivalence $\overline{A} \to A$ where $\overline{A}$ is cofibrant by Corollary 4.14 and Theorem 4.15, and the induced map $\overline{A} \wedge B \to A \wedge B$ is a level equivalence by the argument above. The result is therefore a consequence of the cofibration case. □

Write $\mathbf{Map}_\Sigma(X,Y)$ for the *mapping symmetric T-spectrum* object associated to symmetric T-spectra X and Y. This object exists formally in the category of symmetric T-spectra, just as in [7, Lemma 2.2.2]. In particular, there are natural adjunction isomorphisms

$$\mathbf{hom}(Z \wedge X, Y) \cong \mathbf{hom}(Z, \mathbf{Map}_\Sigma(X,Y))$$

Every symmetric T-spectrum X functorially determines a symmetric T-spectrum object $X[*]$ in the category of symmetric T-spectra, with objects $X[n]$, $n \geq 0$ and having bonding maps $T^p \wedge X[n] \to X[p+n]$. Each $X[n]$ carries a canonical Σ_n-action, and the maps $\sigma : T^p \wedge X[n] \to X[p+n]$ are $(\Sigma_p \times \Sigma_n)$-equivariant. The map σ is defined in level r by the bonding map $T^p \wedge X^{n+r} \to X^{p+n+r}$ of the original symmetric T-spectrum X.

The point of the remainder of this section is to characterize the levels $\mathbf{Map}_\Sigma(X,Y)^n$ in terms of the internal function spaces $\mathbf{Hom}(X, Y[n])$ arising from shifts of Y.

Write $\alpha : F_{n+p}(S^0) \wedge T^p \to F_n(S^0)$ for the map of symmetric T-spectra which picks out the copy of T^p corresponding to the identity $e \in \Sigma_n$ in

$$T^p \wedge \Sigma_{n+} \subset F_n(S^0)^{n+p}.$$

Then $\mathbf{Hom}(F_n(S^0), X) \cong X^n$ and precomposition with the map α induces the adjoint $X^n \to \Omega_T^p X^{n+p}$ of the bonding map $T^p \wedge X^n \to X^{n+p}$.

It follows that there are isomorphisms

$$\begin{aligned}\mathbf{Map}_\Sigma(X,Y)^n &= \mathbf{Hom}(F_n(S^0), \mathbf{Map}_\Sigma(X,Y)) \\ &\cong \mathbf{Hom}(F_n(S^0) \wedge X, Y) \\ &\cong \mathbf{Hom}(X, Y[n])\end{aligned}$$

by Lemma 4.16. One sees further that the adjoint bonding map

$$\mathbf{Map}_\Sigma(X,Y)^n \xrightarrow{\sigma_*} \Omega_T^p \mathbf{Map}_\Sigma(X,Y)^{n+p}$$

is determined by precomposition with α.

There is a commutative diagram

$$\begin{tikzcd}
\hom(F_n(S^0) \wedge X, Y) \arrow[r, "\cong"] \arrow[d, "\alpha^*"'] & \hom(X, Y[n]) \arrow[d, "\sigma^*"] \\
\hom(F_{n+p}(S^0) \wedge T^p \wedge X, Y) \arrow[r, "\cong"'] & \hom(X, \Omega_T^p Y[n+p])
\end{tikzcd}$$

involving canonical isomorphisms and the adjoint $Y[n] \to \Omega_T^p Y[n+p]$ of the map $\sigma : T^p \wedge Y[n] \to Y[n+p]$. This, in turn, is a consequence of the commutativity of the diagram

$$\begin{tikzcd} F_{n+p}(S^0) \wedge T^p \wedge Y[n] \arrow[r, "\alpha\wedge 1"] \arrow[d, "1\wedge\sigma"'] & F_n(S^0) \wedge Y[n] \arrow[d, "ev_n"] \\ F_{n+p}(S^0) \wedge Y[n+p] \arrow[r, "ev_{n+p}"'] & Y \end{tikzcd}$$

where $ev_n : F_n(S^0) \wedge Y[n] \to Y$ is adjoint to the identity map $Y[n] \to Y[n]$. One uses the concrete description of ev_n given by proof of the Lemma 4.16 to show that this diagram commutes.

We have shown the following:

Proposition 4.20. *There is a natural isomorphism*

$$\mathbf{Map}_\Sigma(X,Y)^n \cong \mathbf{Hom}(X,Y[n]),$$

and the bonding maps of $\mathbf{Map}_\Sigma(X,Y)$ *are induced by composition with the adjoints* $Y[n] \to \Omega_T^p Y[p+n]$ *of the maps* $\sigma : T^p \wedge Y[n] \to Y[p+n]$

4.4 Equivalence of stable categories

The purpose of this section is to show that the stable closed model structure on the category $\mathbf{Spt}_T^\Sigma(Sm|_S)_{Nis}$ of symmetric T-spectra has associated homotopy category equivalent to the motivic stable category arising from the category $\mathbf{Spt}_T(Sm|_S)_{Nis}$ of T-spectra.

The equivalence of homotopy categories is induced by the functors U (which forgets the symmetry) and its left adjoint V. As in [7] and [13], the proof of the equivalence of homotopy categories boils down to showing that any stably fibrant model $j : VX \to (VX)_s$ associated to a cofibrant T-spectrum X induces a stable equivalence given by the composite

$$X \xrightarrow{\eta} UVX \xrightarrow{Uj} U(VX)_s.$$

The idea of proof is to use a layer filtration for X, and then show that the result for all of the layers implies the statement for X.

The functor V preserves stably trivial cofibrations and level equivalences, and hence preserves stable equivalences. It follows that the functor $X \mapsto U(VX)_s$ preserves stable equivalences. Each of the layers of X is a shifted suspension object up to stable equivalence, so we inductively prove the claim for shifted suspensions, beginning with suspension T-spectra $\Sigma_T^\infty K$ associated to pointed simplicial presheaves K.

The canonical map $\eta : \Sigma_T^\infty K \to UV(\Sigma_T^\infty K)$ is an isomorphism, so it suffices to find a stably fibrant model

$$V(\Sigma_T^\infty K) \cong T \otimes K \xrightarrow{j} (T \otimes K)_s$$

for the symmetric T-spectrum $T \otimes K$ such that the map j induces a stable equivalence $Uj : U(T \otimes K) \to U(T \otimes K)_s$ of T-spectra — this is Lemma 4.23 below.

The construction that we use involves T-bispectra. A *T-bispectrum* X consists of pointed simplicial presheaves $X^{r,s}$, $r, s \geq 0$, together with bonding maps

$$\sigma_h : T \wedge X^{r,s} \to X^{r+1,s} \text{ and } \sigma_v : T \wedge X^{r,s} \to X^{r,s+1},$$

such that the diagram

$$\begin{tikzcd}
T \wedge X^{r,s+1} \arrow[rr, "\sigma_h"] & & X^{r+1,s+1} \\
T \wedge T \wedge X^{r,s} \arrow[u, "T\wedge\sigma_v"] \arrow[dr, "t\wedge 1"', "\cong"] & & \\
& T \wedge T \wedge X^{r,s} \arrow[r, "T\wedge\sigma_h"'] & T \wedge X^{r+1,s} \arrow[uu, "\sigma_v"']
\end{tikzcd}$$

commutes, where $t : T \wedge T \to T \wedge T$ is the isomorphism which flips smash factors. A T-bispectrum may alternatively be viewed as a T-spectrum object in the category of T-spectra, in the sense that the collections of objects $X^{r,*}$ form T-spectra for all $r \geq 0$, and the horizontal bonding maps σ_h determine morphisms $\sigma_{h*} : X^{r,*} \wedge T \to X^{r+1,*}$ given in levels by the composites

$$X^{r,s} \wedge T \xrightarrow[\cong]{t} T \wedge X^{r,s} \xrightarrow{\sigma_h} X^{r+1,s}.$$

There is of course another way to interpret X as a T-spectrum object, by starting with the T-spectra $X^{*,s}$ and taking bonding maps $X^{*,s} \wedge T \to X^{*,s+1}$ induced by the maps σ_v.

These definitions are analogous to those for ordinary bispectra [11]. Perhaps much of that machinery can be pushed through for T-bispectra — the trick for the moment is to avoid doing so.

A morphism $g : X \to Y$ of T-bispectra is a collection of maps

$$g : X^{r,s} \to Y^{r,s}$$

which preserve all structure. A map $g : X \to Y$ is said to be a *level equivalence* (respectively *fibration*) if each of the component maps $g : X^{r,s} \to Y^{r,s}$ is an equivalence (respectively fibration). It is an easy exercise, using the level model structure for T-spectra, to show that there is a level equivalence $i : X \to Y$ for every object X, such that Y is level fibrant.

Suppose that X is level fibrant. Then the map $\sigma_{h*} : X^{r,*} \wedge T \to X^{r+1,*}$ of T-spectra has an adjoint $\sigma_{h*} : X^{r,*} \to \Omega_T X^{r+1,*}$, and so there are commutative diagrams

$$\begin{tikzcd}
X^{r,s} \arrow[r, "\sigma_{h*}"] \arrow[d, "\sigma_{v*}"'] & \Omega_T X^{r+1,s} \arrow[d, "(\sigma_v)_*"] \\
\Omega_T X^{r,s+1} \arrow[r, "\Omega_T \sigma_{h*}"'] & \Omega_T^2 X^{r+1,s+1}
\end{tikzcd}$$

One has to be careful here: the map $(\sigma_v)_*$ is the adjoint of the canonical choice of bonding map $\sigma_v : T \wedge \Omega_T X^{r+1,s} \to \Omega_T X^{r+1,s+1}$ for the T-spectrum $\Omega_T X^{r+1,s}$, and a calculation shows that there is a commutative diagram

$$\begin{tikzcd}
\Omega_T X^{r+1,s} \arrow[r, "\Omega_T \sigma_{v*}"] \arrow[dr, "(\sigma_v)_*"'] & \Omega_T^2 X^{r+1,s+1} \arrow[d, "t^*"] \\
& \Omega_T^2 X^{r+1,s+1}
\end{tikzcd}$$

where t^* is induced by flipping the loop factors. It follows that composing two instances of these diagrams give a picture

$$\begin{tikzcd}
X^{r,s} \arrow[r, "\sigma_{h*}"] \arrow[dd, "\Omega_T(\sigma_{v*})\sigma_{v*}"'] & \Omega_T X^{r+1,s} \arrow[d, "\Omega_T(\Omega_T(\sigma_{v*})\sigma_{v*})"] \\
& \Omega_T^3 X^{r+1,s+2} \arrow[d, "c_{2,1}^*"] \\
\Omega_T^2 X^{r,s+2} \arrow[r, "\Omega_T^2 \sigma_{h*}"'] & \Omega_T^3 X^{r+1,s+2}
\end{tikzcd}$$

where $c_{2,1}^* = \Omega_T(t^*)t^*$ is induced in loop factors by the cyclic permutation $c_{2,1}$ of order 3.

Lemma 3.13 implies that the map c^* induces the identity in presheaves of homotopy groups. We therefore have a commutative diagram of presheaves of groups

$$\begin{tikzcd}
\pi_j X^{r,s} \arrow[r] \arrow[d] & \pi_j \Omega_T^2 X^{r+2,s} \arrow[r] \arrow[d] & \cdots \\
\pi_j \Omega_T^2 X^{r,s+2} \arrow[r] \arrow[d] & \pi_j \Omega_T^4 X^{r+2,s+2} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots &
\end{tikzcd} \tag{4.9}$$

in which the horizontal morphisms induced by maps $\Omega_T^{2n}(\Omega_T(\sigma_h)\sigma_h)$ and the vertical maps are induced by maps $\Omega_T^{2n}(\Omega_T(\sigma_v)\sigma_v)$

Write $\pi_j QX^{r,s}$ for the filtered colimit of the diagram (4.9), and say that a map $g : X \to Y$ of level fibrant T-bispectra is a *stable equivalence* if it induces isomorphisms of presheaves of groups

$$\pi_j QX^{r,s} \xrightarrow[\cong]{g_*} \pi_j QY^{r,s}$$

for all j, r and s. One expands the definition of stable equivalence to arbitrary T-bispectra by declaring a map to be a stable equivalence if the induced map on level fibrant models is a stable equivalence.

The presheaves of groups $\pi_j QX^{r,s}$ are filtered colimits of presheaves of stable homotopy groups corresponding to both horizontal and vertical choices of T-spectra. This leads immediately to the following

Lemma 4.21. *Suppose that $g : X \to Y$ is a map of T-bispectra such that either all maps $g : X^{r,*} \to Y^{r,*}$, $r \geq 0$, or all maps $g : X^{*,s} \to Y^{*,s}$, $s \geq 0$, are stable equivalences of T-spectra. Then g is a stable equivalence of T-bispectra.*

A T-bispectrum Y is said to be *stably fibrant* if it is level fibrant and all bonding maps $\sigma_h : Y^{r,s} \to \Omega_T Y^{r+1,s}$ and $\sigma_v : Y^{r,s} \to \Omega_T Y^{r,s+1}$ are equivalences (hence pointwise equivalences).

Every T-spectrum Z has an associated *suspension T-bispectrum* $\Sigma_T^\infty Z$ consisting of the objects

$$Z,\ Z \wedge T,\ Z \wedge T^2, \dots$$

The technical device that begins the proof of the main result of this section is the following:

Lemma 4.22. *Let Z be a T-spectrum and suppose that Y is a stably fibrant T-bispectrum. Suppose that the morphism $g : \Sigma_T^\infty Z \to Y$ is a stable equivalence of T-bispectra. Then the map $g : Z \to Y^0$ at level 0 is a stable equivalence of T-spectra, and Y^0 is a stably fibrant T-spectrum.*

Proof. We can suppose that there is a level fibrant model $j : \Sigma_T^\infty Z \to X$ for $\Sigma_T^\infty Z$ such that the map g factors through j. Make the suspension index of $\Sigma_T^\infty Z$ the horizontal index, so that

$$(\Sigma_T^\infty Z)^{r,s} = Z^s \wedge T^r.$$

The map of T-spectra

$$X^{r,*} \xrightarrow{\Omega_T(\sigma_{h*})\sigma_{h*}} \Omega_T^2 X^{r+2,*}$$

is a stable equivalence by Theorem 3.11 and Lemma 3.15, and so there is an isomorphism

$$\pi_j(Q_T X^{r,*})^s \cong \varinjlim \pi_j \Omega_T^{2n} X^{r,s+2n} \cong \pi_j QX^{r,s}.$$

There is a similar isomorphism

$$\pi_j(Q_T Y^{r,*})^s \cong \varinjlim \pi_j \Omega_T^{2n} Y^{r,s+2n} \cong \pi_j Q Y^{r,s}.$$

since Y is stably fibrant. The morphisms

$$\pi_j Q X^{r,s} \to \pi_j Q Y^{r,s}$$

are isomorphisms of presheaves of groups by assumption, so in particular the map

$$\pi_j(Q_T X^{0,*})^s \to \pi_j(Q_T Y^{0,*})^s$$

is an isomorphism as well. □

Lemma 4.23. *Suppose that K is a pointed simplicial presheaf, and let $i : T \otimes K \to (T \otimes K)_s$ be a stably fibrant model for the symmetric T-spectrum $T \otimes K$. Then i induces a stable equivalence $Ui : U(T \otimes K) \to U(T \otimes K)_s$ of T-spectra.*

Corollary 4.24. *Suppose that K is a pointed simplicial presheaf. Then the map*

$$\Sigma_T^\infty K \xrightarrow{\eta_*} UV(\Sigma_T^\infty K)_s$$

is a stable equivalence.

Proof of Lemma 4.23. It suffices to find just one stably fibrant model for $T \otimes K$ which satisfies the statement of the lemma.

There is a T-spectrum object $\Sigma_T^\infty(T \otimes K)$ in the category of symmetric T-spectra, given by

$$\Sigma_T^\infty(T \otimes K)^n = (T \otimes K) \wedge T^n.$$

Suppose that n is the horizontal degree, so that the T-bispectrum underlying this object is specified in bidegrees by

$$U(\Sigma_T^\infty(T \otimes K))^{r,s} = T^s \wedge K \wedge T^r.$$

The functor Q_T and the level fibrant model functor $\mathcal{L}$ are both simplicial functors, so the maps of T-spectra

$$T^s \wedge K \wedge T^* \to \mathcal{L} Q_T \mathcal{L}(T^s \wedge K \wedge T^*)$$

determine a map

$$\Sigma_T^\infty(T \otimes K) \to \mathcal{L} Q_T \mathcal{L}(\Sigma_T^\infty(T \otimes K))$$

of T-spectrum objects in the category of symmetric T-spectra whose underlying map of T-bispectra consists of stably fibrant models in each vertical degree. By Theorem 3.11, the vertical bonding map

$$\mathcal{L} Q_T \mathcal{L}(T^s \wedge K \wedge T^*) \to \Omega_T \mathcal{L} Q_T \mathcal{L}(T^{s+1} \wedge K \wedge T^*)$$

is a stable equivalence and hence a level equivalence, so that the T-bispectrum $U(\mathcal{L}Q_T\mathcal{L}(\Sigma_T^\infty(T \otimes K)))$ is stably fibrant. Thus, the symmetric T-spectrum $\mathcal{L}Q_T\mathcal{L}((T \otimes K) \wedge S^0)$ is stably fibrant, as is its underlying T-spectrum. Finally, Lemma 4.22 implies that the map of T-spectra

$$U((T \otimes K) \wedge S^0) \to U(\mathcal{L}Q_T\mathcal{L}((T \otimes K) \wedge S^0))$$

is a stable equivalence. $\square$

Lemma 4.25. *A map $g : X \to Y$ of symmetric T-spectra is a stable equivalence if and only if the suspension $g \wedge T : X \wedge T \to Y \wedge T$ is a stable equivalence.*

Proof. If g is a stable equivalence, then $g \wedge T$ is a stable equivalence, on account of the isomorphisms

$$\mathbf{hom}(X \wedge T, W) \cong \mathbf{hom}(X, \Omega_T W)$$

and the fact that the functor Ω_T preserves stably fibrant injective objects.

If $g \wedge T$ is a stable equivalence, then the natural stable equivalence $\sigma_* : W \to \Omega_T W[1]$ of (4.1) (see also Corollary 4.10) induces a diagram

$$\begin{tikzcd}
\mathbf{hom}(Y, W) \arrow[r, "g^*"] \arrow[d, "\simeq"'] & \mathbf{hom}(X, W) \arrow[d, "\simeq"] \\
\mathbf{hom}(Y, \Omega_T W[1]) \arrow[r, "g^*"] \arrow[d, "\cong"'] & \mathbf{hom}(X, \Omega_T W[1]) \arrow[d, "\cong"] \\
\mathbf{hom}(Y \wedge T, W[1]) \arrow[r, "(g \wedge T)^*"'] & \mathbf{hom}(X \wedge T, W[1])
\end{tikzcd}$$

If $g \wedge T$ is a stable equivalence, then $(g \wedge T)^*$ is a weak equivalence for all stably fibrant injective W, so g^* is a weak equivalence for all such W. $\square$

Corollary 4.26. *The composite*

$$\eta_* : X \xrightarrow{\eta} \Omega_T(X \wedge T) \xrightarrow{\Omega_T j} \Omega_T(X \wedge T)_s$$

is a stable equivalence of symmetric T-spectra, for any choice of stably fibrant model j for $X \wedge T$.

Proof. There is a diagram

$$\begin{tikzcd}
X \wedge T \arrow[r, "\eta_* \wedge T"] \arrow[dr, "j"'] & \Omega_T(X \wedge T)_s \wedge T \arrow[d, "ev"] \\
& (X \wedge T)_s
\end{tikzcd}$$

and the evaluation map ev is a stable equivalence of the underlying T-spectra by Corollary 3.16. Now use the Lemma 4.25. $\square$

Lemma 4.27. *The natural map $\eta_* : X \to U(VX)_s$ is a stable equivalence if and only if the map $\eta_* : X \wedge T \to U(V(X \wedge T))_s$ is a stable equivalence.*

Proof. There is a commutative diagram

$$\begin{tikzcd}
X \wedge T \arrow[r, "\eta\wedge T"] \arrow[drr, "\eta"'] & UV(X) \wedge T \arrow[r, "Uj\wedge T"] \arrow[d, "\cong"'] & U(VX)_s \wedge T \arrow[d, "\cong"] \\
& U(V(X) \wedge T) \arrow[r, "U(j\wedge T)"] \arrow[d, "\cong"'] & U((VX)_s \wedge T) \arrow[d, "U\tilde{j}"] \\
& UV(X \wedge T) \arrow[r, "Uj"'] & U(V(X \wedge T)_s)
\end{tikzcd}$$

Here $\tilde{j} : (VX)_s \wedge T \to (V(X \wedge T)_s$ is a map of symmetric T-spectra which makes the diagram

$$\begin{tikzcd}
V(X) \wedge T \arrow[r, "j\wedge T"] \arrow[d, "\cong"'] & V(X)_s \wedge T \arrow[d, "\tilde{j}"] \\
V(X \wedge T) \arrow[r, "j"'] & (V(X \wedge T))_s
\end{tikzcd}$$

commute — it exists since $j \wedge T$ is a trivial cofibration if $j : V(X) \to V(X)_s$ is a trivial cofibration. It therefore suffices to show that $U\tilde{j}$ is a stable equivalence of T-spectra.

It further suffices to show that the composite

$$UY \wedge T \xrightarrow{\cong} U(Y \wedge T) \xrightarrow{Uj} U(Y \wedge T)_s \tag{4.10}$$

is a stable equivalence if Y is a stably fibrant symmetric T-spectrum and $j : Y \wedge T \to (Y \wedge T)_s$ is a stably fibrant model. This, however, is a consequence of the commutativity of the diagram

$$\begin{tikzcd}
UY \arrow[r, "\eta"] \arrow[drr, "U\eta"'] & \Omega_T(UY \wedge T) \arrow[r, "\cong"] & \Omega_T U(Y \wedge T) \arrow[r, "\Omega_T Uj"] \arrow[d, "\cong"'] & \Omega_T U(Y \wedge T)_s \arrow[d, "\cong"] \\
& & U\Omega_T(Y \wedge T) \arrow[r, "U\Omega_T j"'] & U\Omega_T(Y \wedge T)_s
\end{tikzcd}$$

The top horizontal composite in this diagram is the adjoint of the composite (4.10), while the composite

$$Y \xrightarrow{\eta} \Omega_T(Y \wedge T) \xrightarrow{\Omega_T j} \Omega_T(Y \wedge T)_s$$

is a levelwise equivalence of stably fibrant symmetric T-spectra, by Corollary 4.26. □

There are canonical stable equivalences

$$\Sigma_T^\infty K[-n] \wedge T^n \to \Sigma_T^\infty K$$

and

$$\Sigma_T^\infty X^n[-n] \to L_n X$$

where $L_n X$ is the n^{th} stage of the layer filtration for a T-spectrum X. The following is then a consequence of Corollary 4.24 and Lemma 4.27:

Corollary 4.28. *1) Suppose that K is a pointed simplicial presheaf. Then the map*

$$\eta_* : \Sigma_T^\infty K[n] \to UV(\Sigma_T^\infty K[n])_s$$

is a stable equivalence for all $n \in \mathbb{Z}$.

2) Suppose that X is a T-spectrum. Then the map

$$\eta_* : L_n X \to UV(L_n X)_s$$

is a stable equivalence for all $n \geq 0$.

Proof. For part 2), recall that the functor V preserves stably trivial cofibrations and level equivalences, and hence preserves stable equivalences, so that the functor $X \mapsto U(VX)_s$ preserves stable equivalences. Part 2) is therefore a consequence of part 1), while 1) follows from Lemma 4.27. $\square$

Lemma 4.29. *Suppose that*

$$X_0 \to X_1 \to \cdots$$

is an inductive system of T-spectra such that all maps $\eta_* : X_n \to U(VX_n)_s$ are stable equivalences. Then the map

$$\eta_* : \varinjlim X_n \to UV(\varinjlim X_n)_s$$

is a stable equivalence.

Proof. There is a commutative diagram

$$\begin{tikzcd}
 & \varinjlim UV(X_n) \arrow[rr, "\varinjlim Uj"] \arrow[dr, "\cong"] & & \varinjlim U(V(X_n))_s \arrow[dr, "\cong"] & \\
\varinjlim X_n \arrow[ur, "\varinjlim \eta"] \arrow[dr, "\eta"'] & & U(\varinjlim V(X_n)) \arrow[rr, "U(\varinjlim j)"] \arrow[dl, "\cong"] & & U(\varinjlim V(X_n)_s) \arrow[dl, "U\tilde{j}"] \\
 & UV(\varinjlim X_n) \arrow[rr, "Uj"'] & & UV(\varinjlim X_n)_s &
\end{tikzcd}$$

where the displayed isomorphisms are canonical and $\tilde{j}$ make the following diagram commute:

$$\begin{tikzcd} \varinjlim V(X_n) \arrow[r, "\varinjlim j"] \arrow[d, "\cong"'] & \varinjlim V(X_n)_s \arrow[d, "\tilde{j}"] \\ V(\varinjlim X_n) \arrow[r, "j"'] & V(\varinjlim X_n)_s \end{tikzcd}$$

Note that we can presume that the stably trivial cofibrations $j : V(X_n) \to V(X_n)_s$ of symmetric T-spectra can be chosen so that the induced map $\varinjlim j : \varinjlim V(X_n) \to \varinjlim V(X_n)_s$ is a stably trivial cofibration, so that the existence of the map $\tilde{j}$ makes sense. This is the analogue of a step in the proof of Lemma 3.12 (a corresponding result, namely that stable equivalences are closed under filtered colimits, holds for symmetric T-spectra, via the same proof). It follows that $\tilde{j}$ is a stable equivalence, but then Corollary 1.7 implies that $\tilde{j}$ is a level equivalence, and so $U\tilde{j}$ is a level equivalence as well. Observe finally that Lemma 3.12 implies that the composite

$$\varinjlim X_n \xrightarrow{\varinjlim \eta} \varinjlim UV(X_n) \xrightarrow{\varinjlim Uj} \varinjlim U(V(X_n)_s)$$

is a stable equivalence. $\square$

Corollary 4.28 and Lemma 4.29 together imply the following:

Proposition 4.30. *The natural map $\eta_* : X \to U(V(X))_s$ is a stable equivalence for all T-spectra X.*

Theorem 4.31. *The functors U and V induce an adjoint equivalence of stable homotopy categories*

$$\mathrm{Ho}(\mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis}) \leftrightarrows \mathrm{Ho}(\mathbf{Spt}_T(Sm|_S)_{Nis})$$

Proof. We show that the adjoint pair of functors (U, V) is a Quillen equivalence.

Suppose that W is a stably fibrant symmetric T-spectrum. Then the canonical map $\epsilon : VU(W) \to W$ is a stable equivalence. To see this, take a factorization

$$\begin{tikzcd} VU(W) \arrow[r, "j"] \arrow[d, "\epsilon"'] & (VU(W))_s \arrow[dl, "\tilde{j}"] \\ W & \end{tikzcd}$$

and apply the functor U to obtain the diagram

$$\begin{tikzcd} U(W) \arrow[r, "\eta"] \arrow[dr, "1"'] & U(VU(W)) \arrow[r, "Uj"] \arrow[d, "U\epsilon"] & U(VU(W))_s \arrow[dl, "U\tilde{j}"] \\ & U(W) & \end{tikzcd}$$

The composite $Uj \cdot \eta : U(W) \to U(VU(W))_s$ is a stable equivalence by Proposition 4.30, so that $U\tilde{j}$ is a stable equivalence of T-spectra. But then $\tilde{j}$ is a stable equivalence of symmetric T-spectra by Proposition 4.8, and so $\epsilon : VU(W) \to W$ is a stable equivalence.

Proposition 4.13 implies that U preserves stable trivial fibrations, while it preserves stable fibrations by definition.

Suppose that X is a cofibrant T-spectrum and W is a stably fibrant symmetric T-spectrum. We have seen that $\epsilon : VU(W) \to W$ is a stable equivalence, and we also know that V preserves stable equivalences — see the proof of Corollary 4.28. Thus, if $f : X \to U(W)$ is a stable equivalence then the adjoint $f_* : V(X) \to W$ is a stable equivalence.

Conversely, if f_* is a stable equivalence, then f_* factors through a level equivalence $\tilde{f} : (V(X))_s \to W$, and there is a diagram

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "\eta_*"'] & UV(X) \arrow[r, "Uf_*"] \arrow[d, "j"] & U(W) \\ & U(V(X))_s \arrow[ur, "U\tilde{f}"'] & \end{tikzcd}$$

The map η_* is a stable equivalence by Proposition 4.30 and $U\tilde{f}$ is a level equivalence, so that f is a stable equivalence. $\square$

4.5 Symmetric S^1-spectra

The results proved above for symmetric T-spectra have analogues for symmetric S^1-spectra, with proofs that are formally the same in many cases. These results will be summarized here.

The analogy begins with the definition. A *symmetric S^1-spectrum* X is an S^1-spectrum consisting of pointed simplicial presheaves X^n, $n \geq 0$, with bonding maps $\sigma : S^1 \wedge X^n \to X^{n+1}$, with symmetric group actions $\Sigma_n \times X^n \to X^{n+1}$, such that the composite bonding maps $S^p \wedge X^n \to X^{p+n}$ are $(\Sigma_p \times \Sigma_n)$-equivariant. There is an obvious category of such things, which is denoted by $\mathbf{Spt}^{\Sigma}_{S^1}(Sm|_S)_{Nis}$. This category is, in the language of [13], the category of presheaves of symmetric spectra on the smooth Nisnevich site $(Sm|_S)_{Nis}$. We know from [13] that this category carries a well behaved stable closed model structure which is created by the Nisnevich topology. The point of this section is to show that there is an additional motivic stable closed model structure such that the associated homotopy category is equivalent to the motivic stable category for S^1-spectra.

As for symmetric T-spectra, say that a map $f : X \to Y$ is a *level equivalence* if each component map $f : X^n \to Y^n$ is a motivic equivalence. The map f is a *level cofibration* if each $f : X^n \to Y^n$ is a cofibration of simplicial presheaves. Finally, a map $g : Z \to W$ is an *injective fibration* if it has the right lifting property with respect to all maps which are level cofibrations and level weak equivalences. We then have the following:

Theorem 4.32. *The category $\mathbf{Spt}^{\Sigma}_{S^1}(Sm|_S)_{Nis}$ of symmetric S^1-spectra on the smooth Nisnevich site, together with the classes of level cofibrations, level equivalences and injective fibrations, satisfies the axioms for a proper closed simplicial model category.*

The proof of this result is just like that of Theorem 4.2: the controlled level fibrant construction $Y \mapsto \mathcal{L}(Y)$ for simplicial presheaves extends to a functor on symmetric S^1-spectra (diagram (1.2)), and we know from [13] that level cofibrations and level local equivalences of symmetric S^1-spectra satisfy a bounded cofibration condition. These two facts can be used together with the argument in the proof of Proposition 2.15 to show that the level motivic equivalences and level cofibrations of symmetric S^1-spectra satisfy a bounded cofibration condition. The rest of the proof is formal.

The definition and properties of the left adjoint V to the forgetful functor

$$U : \mathbf{Spt}^{\Sigma}_{S^1}(Sm|_S)_{Nis} \to \mathbf{Spt}_{S^1}(Sm|_S)_{Nis}$$

taking values in S^1-spectra are already well known.

We say that a map $p : X \to Y$ of symmetric S^1-spectra is a *stable fibration* if the underlying map $Up : UX \to UY$ of S^1-spectra is a (motivic) stable fibration. Proposition 2.15 has an analogue for S^1-spectra which implies that a map $q : Z \to W$ of S^1-spectra is a stable fibration if and only if it has the right lifting property with respect to all λ-bounded cofibrations $A \to B$ which are stable equivalences. It follows that a map $p : X \to Y$ of symmetric S^1-spectra is a stable fibration if and only if it has the right lifting property with respect to images $V(A) \to V(B)$ of all λ-bounded trivial cofibrations of S^1-spectra under the functor V. This implies the following analogue of Proposition 4.4:

Proposition 4.33. *Every map $f : X \to Y$ of symmetric S^1-spectra has a natural factorization*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & Y \end{tikzcd}$$

such that p is a stable fibration, and j is a level cofibration which has the left lifting property with respect to all stable fibrations.

As before, this result implies the existence of injective stably fibrant models.

Say that a map $f : X \to Y$ of symmetric S^1-spectra is a *stable equivalence* if it induces a weak equivalence

$$g^* : \mathbf{hom}(Y, W) \to \mathbf{hom}(X, W)$$

for all stably fibrant injective objects W.

The shift construction $X \mapsto X[n]$, the natural map $X \to \Omega X[1]$ and the symmetric stabilization functor $X \mapsto Q^{\Sigma}X = Q^{\Sigma}_{S^1}X$ are already well known [7], [13], and the same argument as for Proposition 4.8 gives the following:

Proposition 4.34. *Suppose that $\alpha : X \to Y$ is a map of symmetric S^1-spectra such that $U\alpha : UX \to UY$ is a stable equivalence of S^1-spectra. Then α is a stable equivalence of symmetric S^1-spectra.*

The description $X \mapsto G_n(S^0) \otimes X$ of the left adjoint to the shift functor is also well known. This functor preserves level cofibrations and level weak equivalences by construction and the properness of the unstable motivic closed model structure, so that the adjoint $Y \mapsto Y[n]$ preserves injective fibrations. In particular, if W is stably fibrant and injective, then the canonical map $W \to \Omega W[1]$ is a level equivalence of stably fibrant injective objects. The function complex $\mathbf{hom}(X, W)$ is therefore an infinite loop space for all symmetric S^1-spectra X and stably fibrant injective objects W, so that we can prove

Lemma 4.35. *Suppose that the commutative diagram*

$$\begin{tikzcd} A_1 \arrow[r, "i_1"] \arrow[d, "f_1"'] & B_1 \arrow[r, "\pi_1"] \arrow[d, "f_2"] & B_1/A_1 \arrow[d, "f_3"] \\ A_2 \arrow[r, "i_2"'] & B_2 \arrow[r, "\pi_2"'] & B_2/A_2 \end{tikzcd}$$

is a comparison diagram of level cofibre sequences. Then if any two of the maps f_1, f_2 and f_3 are stable equivalences of symmetric S^1-spectra, then so is the third.

The proof is by analogy with the proof of Lemma 4.11.

Insofar as we know that fibre and cofibre sequences coincide in the motivic stable category of S^1-spectra (Corollary 3.2), we also have the analogue of Corollary 4.12, and this implies

Proposition 4.36. *Suppose that $p : X \to Y$ is a map of symmetric S^1-spectra which is both a stable fibration and a stable equivalence. Then p is a level equivalence.*

Corollary 4.37. *A map $p : X \to Y$ of symmetric S^1-spectra is a stable fibration and a stable equivalence if and only if it is both a level fibration and a level equivalence.*

Say that a map $i : A \to B$ of symmetric S^1-spectra is a *stable cofibration* if it has the left lifting property with respect to all maps $p : X \to Y$ which are stable equivalences and stable fibrations. Then we have

Theorem 4.38. *The category $\mathbf{Spt}^{\Sigma}_{S^1}(Sm|_S)_{Nis}$ of symmetric S^1-spectra on the smooth Nisnevich site, and the classes of stable equivalences, stable fibrations and stable cofibrations, together satisfy the axioms for a proper closed simplicial model category.*

Write η_* for the composite

$$X \xrightarrow{\eta} UVX \xrightarrow{Uj} U(VX)_s$$

where $j : VX \to (VX)_s$ is a stably fibrant model of the symmetric S^1-spectrum VX. Then Proposition 4.30 translates as follows:

Proposition 4.39. *The natural map $\eta_* : X \to U(VX)_s$ is a stable equivalence for all S^1-spectra X.*

Just as before, this is the key step in demonstrating that the category of symmetric spectrum objects is a model for the stable category:

Theorem 4.40. *The functors U and V induce an adjoint equivalence of stable homotopy categories*

$$\mathrm{Ho}(\mathbf{Spt}^{\Sigma}_{S^1}(Sm|_S)_{Nis}) \leftrightarrows \mathrm{Ho}(\mathbf{Spt}_{S^1}(Sm|_S)_{Nis})$$

Again, one shows that the adjoint pair of functors (U, V) is a Quillen equivalence.

The proofs of Proposition 4.30 and Theorem 4.31 occupied all of Section 4.4, and the proofs of Proposition 4.39 and Theorem 4.40 are exactly the same, subject to replacing T by S^1. As before, the interesting part is proving Proposition 4.39 in the case of suspension objects — the analogue is Lemma 4.23. That proof involved T-bispectra, which translates here to S^1-bispectra, or presheaves of bispectra in the sense of [11], but interpreted in motivic homotopy theory.

Finally, the categorical material on smash products in Section 4.3 arises from manipulations of free functors that are well known for ordinary symmetric spectra, and therefore hold for symmetric S^1-spectra. The homotopically significant statement is Proposition 4.19:

Proposition 4.41. *Suppose that $i : A \to B$ is a stable cofibration and that $j : C \to D$ is a level cofibration. Then the map*

$$(i, j)_* : (B \wedge C) \cup_{(A\wedge C)} (A \wedge D) \to B \wedge D$$

is a level cofibration. If i and j are both cofibrations, then $(i, j)_*$ is a cofibration. If either i or j is a stable equivalence, then $(i, j)_*$ is a stable equivalence.

The statement and proof of this result are really quite generic, and hold essentially anywhere that one succeeds in generating the usual machinery of symmetric spectrum objects. This includes the present discussion of symmetric S^1-spectra in the motivic context, and also translates into a statement for presheaves of symmetric spectra in the sense of [13].

Appendices

A Properness

The purpose of this section is to show that the closed model structure that arises from formally collapsing a simplicial presheaf I to a point satisfies the properness axiom. This is true over arbitrary small Grothendieck sites and, more explicitly, for the f-local theory for any rational point $f : * \to I$. This result specializes to properness for motivic homotopy theory: that is the case of a rational point $* \to \mathbb{A}^1$ on the affine line, in the category of simplicial presheaves (or sheaves) for the site $(Sm|_S)_{Nis}$ of smooth k-schemes equipped with the Nisnevich topology. I shall revert to the original homotopy theoretic notation (see also Section 1.2) for the general discussion that follows.

Suppose that $\mathcal{C}$ is a small Grothendieck site, and let α be a cardinal which is an upper bound for the cardinality of the set $\mathrm{Mor}(\mathcal{C})$ of morphisms of $\mathcal{C}$. Suppose that I is a simplicial presheaf on $\mathcal{C}$ having a rational point $f : * \to I$. We will show that the f-local closed model structure on $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ is proper, for any such map $f : * \to I$.

Let D be a simplicial presheaf on the site $\mathcal{C}$, and write $f : D \to D \times I$ for the composite

$$D \cong D \times * \xrightarrow{1_D \times f} D \times I.$$

Lemma A.1. *Suppose given maps*

$$D \xrightarrow{f} D \times I \xrightarrow{g} X$$

and a global fibration $\pi : U \to X$, and suppose that X is f-fibrant. Then the induced map

$$f_* : U \times_X D \to U \times_X (D \times I)$$

is an f-equivalence.

Proof. To make the notation easier, given a map $\alpha : V \to X$, write $V_\alpha = U \times_X V$ for the pullback of α along $\pi : U \to X$. In this notation, the statement of the lemma is the assertion that the induced map

$$f_* : D_{gf} \to (D \times I)_g$$

is an f-equivalence.

The object X is f-fibrant and the projection map $pr : D \times I \to D$ is an f-equivalence, so there is a simplicial homotopy

$$\begin{tikzcd} D \times I \arrow[r, "d^0"] \arrow[d, "pr"'] & (D \times I) \times \Delta^1 \arrow[d, "h"] & D \times I \arrow[l, "d^1"'] \arrow[dl, "g"] \\ D \arrow[r, "gf"'] & X. & \end{tikzcd}$$

Pulling back along the global fibration $\pi : U \to X$ gives a diagram

$$\begin{tikzcd}
D_{gf} \arrow[r, "d^0_*"] \arrow[d, "f_*"'] & (D \times \Delta^1)_{h(f\times 1)} \arrow[d, "(f\times 1)_*"] & D_{gf} \arrow[l, "d^1_*"'] \arrow[d, "f_*"] \\
(D \times I)_{gf\cdot pr} \arrow[r, "d^0_*"'] & (D \times I \times \Delta^1)_h & (D \times I)_g \arrow[l, "d^1_*"]
\end{tikzcd}$$

All of the maps labeled d^ϵ_* are local weak equivalences, since π is a global fibration and the ordinary closed model structure for $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ is proper. It therefore suffices to show that the map $f_* : D_{gf} \to (D \times I)_{gf\cdot pr}$ is an f-equivalence.

But the map $gf \cdot pr$ factors through the projection map pr, so that there is an isomorphism

$$\theta : (D \times I)_{gf\cdot pr} \xrightarrow{\cong} D_{gf} \times I$$

and a commutative diagram

$$\begin{tikzcd}
D_{gf} \arrow[d, "f_*"'] \arrow[dr, "f"] & \\
(D \times I)_{gf\cdot pr} \arrow[r, "\cong", "\theta"'] & D_{gf} \times I
\end{tikzcd}$$

The map f_* is therefore an f-equivalence. □

An *elementary f-trivial cofibration* is a member of the saturation of the family of cofibrations consisting of the maps

$$(* \times L_U\Delta^n) \cup_{(*\times Y)} (I \times Y) \subset I \times L_U\Delta^n,$$

and all maps

$$C \hookrightarrow D$$

which are cofibrations and local weak equivalences, where D is α-bounded. An *f-injective fibration* is a map $p : Z \to W$ which has the right lifting property with respect to all elementary f-trivial cofibrations.

Lemma A.2. *1) An f-injective fibration p is a global fibration.*

2) The class of f-injective fibrations is closed under composition.

3) A simplicial presheaf Z is f-local if and only if the map $Z \to *$ is an f-injective fibration.

4) Every simplicial presheaf map $g : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "j"'] & & Y \\ & W \arrow[ur, "q"'] & \end{tikzcd}$$

where q is an f-injective fibration and j is an elementary f-cofibration and an f-equivalence.

5) Every elementary f-cofibration is an f-equivalence.

Proof. Part 4) is the consequence of a standard transfinite small object argument.

The family of maps having the left lifting property with respect to all f-injective fibrations is a saturated class containing the generating elementary f-cofibrations, so that the elementary f-cofibrations have the left lifting property with respect to all f-injective fibrations. It follows from the factorization statement 4) that every elementary f-cofibration is a retract of an elementary f-cofibration which is an f-equivalence. But then every elementary f-cofibration is an f-equivalence, giving 5). ∎

Now we can list some consequences of Lemmas A.1 and A.2:

Lemma A.3. *Suppose given maps*

$$C \xrightarrow{j} D \xrightarrow{g} X$$

and a global fibration $\pi : U \to X$, and suppose that X is f-fibrant and j is an elementary f-cofibration. Then the induced map

$$j_* : U \times_X C \to U \times_X D$$

is an f-equivalence.

Proof. The class of cofibrations $C \hookrightarrow D \to X$ over X which pull back to f-equivalences $U \times_X C \to U \times_X D$ is saturated by exactness of pullback, and contains all ordinary trivial cofibrations since the standard closed model structure on $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ is proper.

In any diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] & L_U\Delta^n \arrow[d, "f_*"'] \arrow[drr, "f"] & & \\ I \times Y \arrow[r] \arrow[drrr] & (I \times Y) \cup_Y L_U\Delta^n \arrow[drr, "\theta"] & & \\ & & & I \times L_U\Delta^n \arrow[r, "g"'] & X \end{tikzcd}$$

the maps f and f_* pull back to f-equivalences along π by Lemma A.1, and so θ pulls back to an f-equivalence along π. This means that all generators of the family of elementary f-cofibrations pull back to f-equivalences along π, so all elementary f-cofibrations pull back to f-equivalences along π. □

Corollary A.4. *Suppose given a pullback diagram*

$$\begin{tikzcd} A \times_X U \arrow[r, "g_*"] \arrow[d] & U \arrow[d, "\pi"] \\ A \arrow[r, "g"'] & X \end{tikzcd}$$

where X is f-fibrant, g is an f-equivalence and π is a global fibration. Then the induced map g_* is an f-equivalence.

Proof. Find a factorization

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "j"'] & & X \\ & W \arrow[ur, "q"'] & \end{tikzcd}$$

of g, where j is an elementary f-cofibration and q is an f-injective fibration. Then W is f-fibrant by Lemma A.2, and the fact that the classes of f-fibrant objects and f-injective objects coincide [4]. Thus, q is an f-equivalence of f-fibrant objects, and is therefore an ordinary local weak equivalence, and hence pulls back to a local weak equivalence along the global fibration π. But then the elementary f-cofibration j pulls back to an f-equivalence by Lemma A.3. □

Theorem A.5 (Properness). *Suppose given a diagram*

$$\begin{tikzcd} A \times_X U \arrow[r, "g_*"] \arrow[d] & U \arrow[d, "\pi"] \\ A \arrow[r, "g"'] & Z \end{tikzcd}$$

such that π is an f-fibration and g is an f-equivalence. Then the induced map g_* is an f-equivalence.

Proof. Form a diagram

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d, "\pi"'] & V \arrow[d, "p"] \\ Z \arrow[r, "j"'] & \mathcal{L}Z \end{tikzcd}$$

such that i is a cofibration and an f-equivalence, $\mathcal{L}Z$ is f-fibrant, p is an f-fibration, and j is a cofibration and an f-equivalence. Consider the pullback diagram

$$\begin{tikzcd} Z \times_{\mathcal{L}Z} V \arrow[r, "j_*"] \arrow[d, "p_*"'] & V \arrow[d, "p"] \\ Z \arrow[r, "j"'] & \mathcal{L}Z \end{tikzcd}$$

The map $j_* : Z \times_{\mathcal{L}Z} V \to V$ is an f-equivalence by Corollary A.4. The induced comparison

$$\begin{tikzcd} U \arrow[rr, "\theta"] \arrow[dr, "\pi"'] & & Z \times_{\mathcal{L}Z} V \arrow[dl, "p_*"] \\ & Z & \end{tikzcd}$$

is an f-equivalence of f-fibrant objects in $\mathbf{S}\mathrm{Pre}(\mathcal{C}) \downarrow X$, hence a homotopy equivalence, and so the map θ is a local weak equivalence. Properness for the standard closed model structure on $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ implies that the induced map

$$A \times_Z U \xrightarrow{\theta_*} A \times_{\mathcal{L}Z} V$$

is a local weak equivalence. Thus, in the diagram

$$\begin{tikzcd} A \times_Z U \arrow[r, "g_*"] \arrow[d, "\theta_*"'] & U \arrow[d, "\theta"] \\ A \times_{\mathcal{L}Z} V \arrow[r, "g'"'] & Z \times_{\mathcal{L}Z} V \end{tikzcd}$$

the map g_* is an f-equivalence if and only if g' is an f-equivalence. But the maps $j_* g'$ and j_* are f-equivalences by Corollary A.4, so g' is an f-equivalence. $\square$

Theorem A.5 is not the full properness assertion for the f-local theory but it is the heart of the matter. The second half of the properness axiom says that the class of f-equivalences is closed under pushout along cofibrations. This means that, given a pushout diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & C \arrow[d] \\ B \arrow[r, "g_*"'] & B \cup_A C \end{tikzcd}$$

with i a cofibration and g an f-equivalence, the map g_* should be an f-equivalence. This is easily proved: the functor $\mathbf{hom}(\ ,W)$ takes pushouts of simplicial presheaves to pullbacks of simplicial sets, and the map $i^* : \mathbf{hom}(B,W) \to \mathbf{hom}(A,W)$ is a fibration and $g^* : \mathbf{hom}(C,W) \to \mathbf{hom}(A,W)$ is a weak equivalence if W is f-local. Properness for ordinary simplicial sets implies that the induced map

$$g_*^* : \mathbf{hom}(B \cup_A C, W) \to \mathbf{hom}(B,W)$$

is a weak equivalence of simplicial sets. This is true for all f-local objects W, so that g_* is an f-equivalence.

B Motivic homotopy theory of presheaves

Let $\mathbf{S}\,\mathrm{Shv}(Sm|_S)_{Nis}$ (respectively $\mathrm{Shv}(Sm|_S)_{Nis}$) denote the category of simplicial sheaves (respectively sheaves) on the smooth Nisnevich site $(Sm|_S)_{Nis}$ for a Noetherian scheme S of finite dimension. Suppose that $\mathrm{Pre}(Sm|_S)_{Nis}$ and $\mathbf{S}\mathrm{Pre}(Sm|_S)_{Nis}$ denote the corresponding categories of presheaves and simplicial presheaves. We know that the categories of simplicial sheaves and simplicial presheaves carry closed model structures obtained from the local structures for the Nisnevich topology by formally contracting the affine line $\mathbb{A}^1$, and that the resulting homotopy categories are equivalent, and are models for the motivic homotopy category — see Theorem 1.1 and Theorem 1.2.

The purpose of this section is to explain the Morel-Voevodsky result that the sheaf category $\mathrm{Shv}(Sm|_S)_{Nis}$ inherits a closed model structure from the category of simplicial sheaves in such a way that the associated homotopy category is also a model for the motivic homotopy category. We actually do a little more here (Theorem B.4 below), and show that the category of presheaves $\mathrm{Pre}(Sm|_S)_{Nis}$ has a proper closed simplicial model structure, so that there is an adjoint equivalence of the associated homotopy category $\mathrm{Ho}(\mathrm{Pre}(Sm|_S)_{Nis})$ with the motivic homotopy category. The Morel-Voevodsky result for sheaves (Theorem B.6) is a consequence of Theorem B.4, in a way that one has come to expect.

Morel and Voevodsky construct a singular functor

$$S = S_{\mathbb{A}^1} : \mathbf{S}\,\mathrm{Shv}(Sm|_S)_{Nis} \to \mathbf{S}\,\mathrm{Shv}(Sm|_S)_{Nis}$$

in terms of the internal hom functor by specifying $S(B) = \mathbf{Hom}(\mathbb{A}^\bullet, B)$ for ordinary sheaves B, and then by defining $S(X)$ for a simplicial sheaf X to be the diagonal of the bisimplicial object

$$\mathbf{Hom}(\mathbb{A}^m, X_n).$$

Here, $\mathbb{A}^\bullet$ refers to the standard cosimplicial k-variety made up of the affine planes $\mathbb{A}^n$. The singular functor specializes, in particular, to a functor

$$S : \mathrm{Shv}(Sm|_S)_{Nis} \to \mathbf{S}\,\mathrm{Shv}(Sm|_S)_{Nis},$$

This last functor has a canonical left adjoint

$$|\cdot| : \mathbf{S}\,\mathrm{Shv}(Sm|_S)_{Nis} \to \mathrm{Shv}(Sm|_S)_{Nis},$$

which is defined by a suitable coend. This means that there is a coequalizer in the sheaf category having the form

$$\bigsqcup_{\theta:\mathbf{m}\to\mathbf{n}} X_n \times \mathbb{A}^m \rightrightarrows \bigsqcup_n X_n \times \mathbb{A}^n \to |X|$$

for a simplicial sheaf X that one expects from the definition of the realization functor from simplicial sets to spaces. Morel and Voevodsky show [14] that, for a suitable closed model structure on the sheaf category $\mathrm{Shv}(Sm|_S)_{Nis}$, these functors define an adjoint equivalence of the associated homotopy categories.

These constructions are easily generalized to simplicial presheaves, with completely analogous definitions. There is a *singular functor*

$$S = S_{\mathbb{A}^1} : \mathbf{S}\mathrm{Pre}(Sm|_S)_{Nis} \to \mathbf{S}\mathrm{Pre}(Sm|_S)_{Nis}$$

which is defined on presheaves C by setting $S(C) = \mathbf{Hom}(\mathbb{A}^\bullet, C)$; then $S(Y)$ is defined for a simplicial presheaf Y by taking the diagonal of the bisimplicial presheaf

$$\mathbf{Hom}(\mathbb{A}^m, Y_n).$$

There is a *realization functor*

$$|\cdot| : \mathbf{S}\mathrm{Pre}(Sm|_S)_{Nis} \to \mathrm{Pre}(Sm|_S)_{Nis},$$

defined by coend, so that there is a coequalizer

$$\bigsqcup_{\theta:\mathbf{m}\to\mathbf{n}} Y_n \times \mathbb{A}^m \rightrightarrows \bigsqcup_n Y_n \times \mathbb{A}^n \to |Y|$$

in the presheaf category, for simplicial presheaves Y. The realization functor is left adjoint to the singular functor, just as before.

We now have the following analogue of a string of results for the singular functor on simplicial sheaves, proved by Morel and Voevodsky in [14]:

LEMMA B.1. *The singular functor*

$$S : \mathbf{S}\mathrm{Pre}(Sm|_S)_{Nis} \to \mathbf{S}\mathrm{Pre}(Sm|_S)_{Nis}$$

has the following properties:

1) *The functor S takes the morphism $f : * \to \mathbb{A}^1$ to a weak equivalence of simplicial sheaves.*

2) *For any simplicial presheaf X, the canonical map $\eta : X \to S(X)$ is a motivic weak equivalence and a cofibration.*

3) The realization functor preserves cofibrations and motivic weak equivalences.

Proof. For 1), it suffices to show that the simplicial set

$$S(\mathbb{A}^1)(Sp(R)) \cong R[t_*]$$

is contractible for affine schemes $Sp(R)$, where $R[t_*]$ is the simplicial R-algebra with n-simplices

$$R[t_*]_n = R[t_0, \dots, t_n]/(\sum t_i = 1),$$

and having face maps defined by

$$d_i(t_j) = \begin{cases} t_j & \text{if } j < i, \\ 0 & \text{if } j = i, \text{ and} \\ t_{j-1} & \text{if } j > i. \end{cases}$$

It is well known (for many years — see [1], for example) that the simplicial set underlying this simplicial R-algebra is contractible, with contracting (chain) homotopy given by

$$f(t_0, \dots t_n) \mapsto (1 - t_0) f(t_1, \dots, t_{n+1}).$$

For 2), the canonical map η for simplicial sets is the diagonal of a corresponding bisimplicial set map made of canonical maps $\eta : B \to \mathbf{Hom}(\mathbb{A}^\bullet, B)$ defined for simplicial presheaves B. This map is a morphism of simplicial presheaves which on n-simplices is the map

$$B \to \mathbf{Hom}(\mathbb{A}^n, B) \tag{B.1}$$

defined by precomposition with the map $\mathbb{A}^n \to *$. There is a contracting homotopy $h : \mathbb{A}^n \times \mathbb{A}^1 \to \mathbb{A}^n$ defined by

$$((t_1, \dots, t_n), s) \mapsto (t_1 s, \dots, t_n s).$$

This contracting homotopy induces an obvious map

$$h_* : \mathbf{Hom}(\mathbb{A}^n, B) \times \mathbb{A}^1 \to \mathbf{Hom}(\mathbb{A}^n, B),$$

and the existence of the homotopy h_* implies that the map (B.1) is an $\mathbb{A}^1$ homotopy equivalence, and hence a motivic weak equivalence. The motivic model structure for the simplicial presheaf category is proper, so that standard techniques imply that the map $\eta : X \to S(X)$ is a motivic weak equivalence for all simplicial presheaves X.

To prove statement 3), observe that any cosimplicial set E determines a set-valued realization functor $X \mapsto |X|_E$ defined on simplicial sets by the coequalizer

$$\bigsqcup_{\theta:\mathbf{m}\to\mathbf{n}} X_n \times E^m \rightrightarrows \bigsqcup_n X_n \times E^n \to |Y|_E.$$

One sees easily that there is a bijection $|\Delta^n|_E \to E^n$ defined by the maps $\Delta^n_k \times D^k \to D^n$ given by $(\theta, x) \mapsto \theta_*(x)$. This bijection is natural in ordinal number maps; in particular, the induced function $|\partial\Delta^1|_E \to |\Delta^1|_E$ is isomorphic to the function

$$(d^0, d^1) : E^0 \sqcup E^0 \to E^1.$$

Also, all diagrams

$$\begin{tikzcd} E^{n-2} \arrow[r, "d^{j-1}"] \arrow[d, "d^i"'] & E^{n-1} \arrow[d, "d^i"] \\ E^{n-1} \arrow[r, "d^j"'] & E^n \end{tikzcd}$$

corresponding to $i < j$ are pullbacks by the cosimplicial identities for $n \geq 2$. Thus, there is an isomorphism

$$|\partial\Delta^n|_E \cong \partial E^n,$$

where ∂E^n denotes the union of the images $d^i(E^{n-1})$ in E^n, and that the induced map $|\partial\Delta^n|_E \to |\Delta^n|_E$ is an injection for $n \geq 2$. It follows that the realization functor $X \mapsto |X|_E$ takes cofibrations to injections if and only if E is *unaugmented* in the (traditional — see [3]) sense that the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & E^0 \arrow[d, "d^0"] \\ E^0 \arrow[r, "d^1"'] & E^1 \end{tikzcd}$$

is a pullback.

Also, if E is unaugmented, one can show that the natural map

$$X_0 \times E^0 \to |X|_E$$

is an inclusion, by induction on the skeleta of X.

Any cosimplicial object D in the category of simplicial presheaves determines a D-realization functor $Y \mapsto |Y|_D$, defined by a coequalizer diagram

$$\bigsqcup_{\theta:\mathbf{m}\to\mathbf{n}} Y_n \times D^m \rightrightarrows \bigsqcup_n Y_n \times D^n \to |Y|_D.$$

as above. Write $|Y|_D^{(p)}$ for the image of

$$\bigsqcup_{0 \leq n \leq p} Y_n \times D^n$$

in $|Y|_D$, and let $s_{[p]}Y_p$ be the degenerate part of Y_{p+1}. Then there is a pushout diagram

$$\begin{tikzcd} (s_{[p]}Y_p \times D^{p+1}) \cup (Y_{p+1} \times \partial D^{p+1}) \arrow[r] \arrow[d] & |Y|_D^{(p)} \arrow[d] \\ Y_{p+1} \times D^{p+1} \arrow[r] & |Y|_D^{(p+1)} \end{tikzcd}$$

The vertical maps are cofibrations, and the canonical map

$$Y_0 \times D^0 \to |Y|_D^{(0)}$$

is an isomorphism if D is unaugmented.

A properness argument therefore implies that any level motivic equivalence $D \to E$ of unaugmented cosimplicial presheaves induces a natural motivic equivalence $|Y|_D \to |Y|_E$. In particular, the maps of cosimplicial objects

$$\mathbb{A}^n \leftarrow \mathbb{A}^n \times \Delta^n \to \Delta^n$$

are level motivic equivalences, and so there are natural motivic equivalences

$$|Y|_{\mathbb{A}^\bullet} \leftarrow |Y|_{\mathbb{A}^\bullet \times \Delta} \to |Y|_\Delta \cong Y.$$

The realization functor $Y \mapsto |Y| = |Y|_{\mathbb{A}^\bullet}$ therefore preserves motivic equivalences. It follows also that this realization functor preserves cofibrations of simplicial presheaves. □

Corollary B.2. *The singular functor preserves fibrations.*

Corollary B.3. *There is a natural motivic weak equivalence $Y \simeq |Y|$, for all simplicial presheaves Y.*

Say that a map $g : X \to Y$ of presheaves on the smooth Nisnevich site $(Sm|_S)_{Nis}$ is a *motivic weak equivalence* if the associated morphism of constant simplicial presheaves is a motivic weak equivalence. A *cofibration* of presheaves is an inclusion, and a *motivic fibration* is a map which has the right lifting property with respect to all maps which are simultaneously cofibrations and motivic weak equivalences.

Given a presheaf X and a simplicial set K, write $X \otimes K$ for the presheaf given by

$$X \otimes K = |X \times K|.$$

There is an isomorphism

$$X \otimes K \cong \varinjlim_{\sigma:\Delta^n \to K} X \otimes \Delta^n,$$

where the colimit is indexed over the simplex category of K, and one checks that there is a natural isomorphism

$$X \otimes \Delta^n \cong X \times \mathbb{A}^n.$$

The category of presheaves on $(Sm|_S)_{Nis}$ acquires a simplicial structure from these definitions: the function complex $\mathbf{hom}(X, Y)$ has n-simplices specified by

$$\mathbf{hom}(X, Y)_n = \hom(X \otimes \Delta^n, Y) \cong \hom(X \times \mathbb{A}^n, Y),$$

while for a simplicial set K and a presheaf X, the mapping presheaf $\mathbf{hom}(K, X)$ is given in terms of the internal hom functor by

$$\mathbf{hom}(K, X) = \varprojlim_{\sigma:\Delta^n \to K} \mathbf{Hom}(\mathbb{A}^n, X).$$

Theorem B.4. *With these definitions, we have the following:*

1) *The category* $\mathrm{Pre}(Sm|_S)_{Nis}$ *of presheaves on the smooth Nisnevich site of a Noetherian scheme S of finite dimension satisfies the axioms for a proper closed simplicial model category.*

2) *The singular and realization functors determine an adjoint equivalence of motivic homotopy categories*

$$\mathrm{Ho}(\mathrm{Pre}(Sm|_S)_{Nis}) \simeq \mathrm{Ho}(\mathbf{S}\mathrm{Pre}(Sm|_S)_{Nis}).$$

Proof. Recall from [4, p.1086] that the category $\mathbf{S}\mathrm{Pre}(Sm|_S)_{Nis}$ of simplicial presheaves on the smooth Nisnevich site and the class $\mathbf{E}$ of motivic weak equivalences together satisfy a list of properties analogous to those appearing in the statement of Proposition 4.1. These include, for example, the bounded cofibration condition:

E7: There is an infinite cardinal λ which is an upper bound for the cardinality of the set of morphisms of $(Sm|_S)_{Nis}$, such that for every simplicial presheaf diagram

$$\begin{tikzcd} & X \arrow[d, "i"] \\ A \arrow[r] & Y \end{tikzcd}$$

with i an $\mathbf{E}$-trivial cofibration and A an λ-bounded subobject of Y, there is a subobject $B \subset Y$ such that $A \subset B$, the object B is λ-bounded, and the inclusion $B \cap X \hookrightarrow B$ is an $\mathbf{E}$-trivial cofibration.

Here, an $\mathbf{E}$-trivial cofibration is a map which is a cofibration and a motivic weak equivalence. We are also tacitly working over a small, full subcategory of

$(Sm|_S)_{Nis}$ consisting of objects of size at most some fixed infinite cardinal, so that the statement of E7 makes sense.

Up to isomorphism, a subobject of a constant simplicial presheaf must be constant, so that the bounded cofibration condition E7 for simplicial presheaves implies a bounded cofibration condition for ordinary presheaves on $(Sm|_S)_{Nis}$. The other axioms E1 – E6 for the class of cofibrations and motivic weak equivalences in the presheaf category are trivial consequences of the corresponding results for simplicial presheaves. It follows that a map $p : X \to Y$ of presheaves is a fibration if and only if it has the right lifting property with respect to all λ-bounded cofibrations which are motivic equivalences — the argument appears in the proof of Theorem 1.1 of [4]. Continuing in that vein, a transfinite small object argument then implies that every map $g : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "g"'] & Z \arrow[d, "p"] \\ & Y \end{tikzcd}$$

such that p is a fibration, and j is a motivic weak equivalence and a cofibration.

Write L_U* for the free presheaf on a section over U. Then I claim that the presheaf map $p : X \to Y$ is a fibration and a motivic weak equivalence if it has the right lifting property with respect to all inclusions $A \subset L_U*$. A map p having this lifting property has the right lifting property with respect to all inclusions, so it is a fibration. The induced map $p_* : S(X) \to S(Y)$ has the right lifting property with respect to all cofibrations, by an adjointness argument and the fact that realization preserves cofibrations. The map p_* is therefore a fibration and a motivic weak equivalence of simplicial presheaves. The canonical map $\eta : X \to S(X)$ is a motivic weak equivalence of simplicial presheaves, so the original map $p : X \to Y$ must also be a motivic weak equivalence of presheaves. A transfinite small object argument then implies that every map $g : X \to Y$ of presheaves has a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "g"'] & W \arrow[d, "q"] \\ & Y \end{tikzcd}$$

where i is a cofibration and q is both a fibration and a motivic weak equivalence.

We have proved the factorization axiom CM5. The style of its proof further implies, in a standard way, that every map which is a fibration and a motivic weak equivalence is a retract of a map which has the right lifting property with respect to all cofibrations, and therefore has the same right lifting property. The axiom CM4 follows. The other closed model axioms are trivial to verify. The simplicial model axiom SM7 is a consequence of the corresponding axiom for simplicial presheaves, together with part 3) of Lemma B.1.

To show that motivic weak equivalences of presheaves are stable under pullback along fibrations, it suffices to observe that the singular functor preserves and reflects motivic weak equivalences, in addition to preserving fibrations. The pullback part of the properness assertion therefore follows from the corresponding assertion for simplicial presheaves. The pushout part is a more direct consequence of the statement for simplicial presheaves.

To prove 2), note that the singular and realization functors both preserve motivic weak equivalences, and hence induce functors

$$S : \mathrm{Ho}(\mathrm{Pre}(Sm|_S)_{Nis}) \leftrightarrows \mathrm{Ho}(\mathbf{S}\mathrm{Pre}(Sm|_S)_{Nis}) : |\cdot|.$$

To show that these functors give an equivalence of categories, it suffices to show that the canonical map $\eta : X \to S|X|$ is a motivic equivalence for all simplicial presheaves X. Then the map $S\epsilon : S|S(Y)| \to S(Y)$ would be a motivic weak equivalence for all presheaves Y by a triangle identity, and so $\epsilon : |S(Y)| \to Y$ would be a motivic weak equivalence since the singular functor reflects motivic weak equivalences.

If X is a constant simplicial presheaf, then the canonical map $\eta : X \to S|X|$ is isomorphic to the map $\eta : X \to S(X)$, since $X \cong |X|$ in this case.

Recall that

$$S(Y)_n(U) = \mathbf{Hom}(\mathbb{A}^n, Y)(U) \cong Y(\mathbb{A}^n \times U)$$

for all presheaves Y. It follows that the singular functor preserves all colimits in presheaves and hence in simplicial presheaves. In other words, the singular functor satisfies a very strong excision property.

Every simplicial presheaf X is a coend for the morphisms $X_n \times \Delta^n \to X$, and the skeletal filtration $\mathrm{sk}_r X$ is defined by pushouts of cofibrations

$$\begin{tikzcd}
(s_{[r]}X_r \times \Delta^{r+1}) \cup (X_{r+1} \times \partial\Delta^{r+1}) \arrow[r] \arrow[d] & \mathrm{sk}_r X \arrow[d] \\
X_{r+1} \times \Delta^{r+1} \arrow[r] & \mathrm{sk}_{r+1} X
\end{tikzcd}$$

Here, $s_{[r]}X_r$ is the degenerate part of X_{r+1}. More generally, define

$$s_{[k]}X_r = \bigcup_{i=0}^{r} s_i(X_r) \subset X_{r+1},$$

and observe that there are pushout of cofibration diagrams

$$\begin{tikzcd}
s_{[k]}X_{p-1} \arrow[r, "s_{k+1}"] \arrow[d] & s_{[k]}X_p \arrow[d] \\
X_p \arrow[r, "s_{k+1}"'] & s_{[k+1]}X_p
\end{tikzcd}$$

for all p and k that make sense. The composite functor $X \mapsto S|X|$ preserves all colimits and hence preserves the skeletal filtration for X in the sense that both of the above species of diagrams are taken to pushouts of cofibrations. On account of properness for the motivic model structure for simplicial presheaves, it therefore suffices to show that the maps

$$X_r \times \Delta^r \to S|X_r \times \Delta^r|$$

are motivic weak equivalences. But the projections $X_r \times \Delta^r \to X_r$ are motivic weak equivalences and the composite $S|\cdot|$ preserves motivic weak equivalences, so the claim reduces to the constant case. ☐

Here is a corollary of the proof of Theorem B.4:

Corollary B.5. *Suppose that $g : X \to Y$ is a map of simplicial presheaves such that every map $g : X_n \to Y_n$ is a motivic weak equivalence of presheaves. Then g is a motivic weak equivalence of simplicial presheaves.*

Say that a map of sheaves $g : X \to Y$ on the Nisnevich site $(Sm|_S)_{Nis}$ is a *motivic weak equivalence* if the associated map of constant simplicial sheaves (or presheaves) is a motivic weak equivalence. A *cofibration* of sheaves is just an inclusion, and a map of simplicial sheaves is a *motivic fibration* if it has the right lifting property with respect to all maps which are simultaneously cofibrations and motivic weak equivalences of simplicial sheaves.

Theorem B.6. *With these definitions, we have the following:*

1) *The category $\mathrm{Shv}(Sm|_S)_{Nis}$ of sheaves on the smooth Nisnevich site of a Noetherian scheme S of finite dimension satisfies the axioms for a proper closed simplicial model category.*

2) *The singular and realization functors determine an adjoint equivalence of motivic homotopy categories*

$$\mathrm{Ho}(\mathrm{Shv}(Sm|_S)_{Nis}) \simeq \mathrm{Ho}(\mathbf{S}\,\mathrm{Shv}(Sm|_S)_{Nis}).$$

3) *The associated sheaf functor determines an adjoint equivalence of motivic homotopy categories*

$$\mathrm{Ho}(\mathrm{Pre}(Sm|_S)_{Nis}) \simeq \mathrm{Ho}(\mathrm{Shv}(Sm|_S)_{Nis}).$$

Proof. For 1), note that a map $g : X \to Y$ of sheaves is a motivic equivalence (respectively cofibration) if and only if is a motivic equivalence (respectively cofibration) of presheaves. The associated sheaf map $A \to \tilde{A}$ is a local isomorphism, and hence a motivic weak equivalence of presheaves. It follows that the classes of motivic weak equivalences and cofibrations of sheaves satisfy the axioms E1 – E7 involved in the proof of Theorem B.4, and then the closed model structure for the sheaf category is a formal consequence, just as before. Properness is an easy consequence of properness for the presheaf category.

In 2), the singular functor

$$S : \mathrm{Shv}(Sm|_S)_{Nis} \to \mathbf{S}\,\mathrm{Shv}(Sm|_S)_{Nis}$$

is defined as for presheaves, and so it preserves and reflects motivic weak equivalences. The simplicial sheaf realization $|X|$ of a simplicial sheaf X is the associated sheaf of the presheaf level realization, so that the map $\eta : X \to S|X|$ is a motivic weak equivalence, on account of the fact that we know the corresponding statement for simplicial presheaves. The adjoint equivalence of homotopy categories then follows just as in the presheaf case.

Statement 3) is obvious. □

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