The Automorphism Group of Linear Sections of the Grassmannians $\mathbb{G}(1, N)$

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Abstract. The Grassmannians of lines in projective N-space, $\mathbb{G}(1, N)$, are embedded by way of the Plücker embedding in the projective space $\mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})$. Let H^l be a general l-codimensional linear subspace in this projective space.

We examine the geometry of the linear sections $\mathbb{G}(1, N) \cap H^l$ by studying their automorphism groups and list those which are homogeneous or quasihomogeneous.

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0 Introduction

Complete intersections in projective space have been studied extensively from many points of view. A natural generalisation is the study of complete intersections in Grassmannians. The first case that presents itself is the case of intersections with linear spaces. Indeed, there is an extensive literature on the simplest case, the Grassmannian of lines in the 3-space, where intersections are known as linear complexes and congruences of lines. L. Roth has studied the rationality of linear sections of Grassmannians of lines in general. If they are smooth and if the dimension of the intersection is greater than half the dimension of the Grassmannian, then they are rational. R. Donagi determined the cohomology and the intermediate Jacobian of some linear sections of Grassmannians of lines.

In this paper we study the linear sections from the point of automorphism groups. Let $\mathbb{G}(1, N)$ be the Grassmann variety of lines in projective N-space, canonically embedded in $\mathbb{P}(\bigwedge^2\mathbb{C}^{N+1})$ and let H^l be an l-codimensional linear subspace in this space. For general H^l we determine the automorphism groups for $\mathbb{G}(1,N)\cap H$, $\mathbb{G}(1,N)\cap H^2$, $\mathbb{G}(1,4)\cap H^3$, and $\mathbb{G}(1,5)\cap H^3$. In the second case we find for example:

Theorem 3.5 *For $N = 2n-1 \geq 5$ the automorphism group of $\mathbb{G}(1,N)\cap H^2$ has $\mathrm{SL}(2,\mathbb{C})^n/\{1,-1\}$ as a normal subgroup and the quotient group is isomorphic to the permutation group $\mathrm{S}(3)$ for $n = 3$, to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ for $n = 4$, and trivial otherwise.*

We believe that apart from trivial cases these are the only general linear sections where automorphism groups of positive dimension appear. Extensive computer checks seem to confirm this.

In particular we prove that the automorphism groups of $\mathbb{G}(1,2n)\cap H$, $\mathbb{G}(1,4)\cap H^2$, $\mathbb{G}(1,5)\cap H^2$, $\mathbb{G}(1,6)\cap H^2$, and $\mathbb{G}(1,4)\cap H^3$ are quasihomogeneous – those of $\mathbb{G}(1,2n-1)\cap H$ and $\mathbb{G}(1,3)\cap H^2$ are even homogenous – whereas all others are not.

As to our methods, in our proofs the rich geometry of the Grassmannian plays a decisive role. Otherwise, we mainly use well known tools like multilinear algebra, Lefschetz theorems, vanishing theorems etc.

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1 Preliminary

The Grassmannian $\mathbb{G}(1,N)$ of lines in $\mathbb{P}_N$ is embedded by way of the Plücker embedding into $\mathbb{P}(\bigwedge^2\mathbb{C}^{N+1})$

$$\begin{array}{ccc} \mathbb{G}(1,N) & \longrightarrow & \mathbb{P}(\bigwedge^2\mathbb{C}^{N+1}) \\ \operatorname{span}\{v,w\} & \longmapsto & \mathbb{P}(v\wedge w). \end{array}$$

We denote by H^l an l-codimensional linear subspace of $\mathbb{P}(\bigwedge^2\mathbb{C}^{N+1})$. Roth [R] examined the geometry of the general linear sections of the Grassmannians and found

Theorem 1.1 *For a general H^l with $0 \leq l \leq 1/2\dim\mathbb{G}(1,N) = N-1$ the intersection with the Grassmannians, $\mathbb{G}(1,N)\cap H^l$, is rational.*

In this article we continue this study by describing the automorphism groups of these sections. As for the notation, given a subvariety Y of a variety X we define $\mathrm{Aut}(Y,X)$ to be the automorphisms of X that induce automorphisms of Y, i.e.

$$\mathrm{Aut}(Y,X) = \{\varphi\in\mathrm{Aut}(X) \mid \varphi(Y)\subseteq Y\}.$$

Recall that the automorphism group of the Grassmannian itself is computed in two steps, see e.g. [H, 10.19]. First one shows that all automorphisms are induced by automorphisms of $\mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})$, i.e.

$$\mathrm{Aut}(\mathbb{G}(1,N)) \cong \mathrm{Aut}(\mathbb{G}(1,N), \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})).$$

Then one proves that for $N \neq 3$ the right hand side group is isomorphic to $\mathbb{P}\mathrm{GL}(N+1,\mathbb{C})$ via

$$\begin{array}{ccc} \mathbb{P}\mathrm{GL}(N+1,\mathbb{C}) & \longrightarrow & \mathrm{Aut}(\mathbb{G}(1,N), \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})) \\ \mathbb{P}(T) & \longmapsto & (\mathbb{P}(\sum v_i \wedge w_i) \mapsto \mathbb{P}(\sum T v_i \wedge T w_i))\,. \end{array}$$

For the linear sections of the Grassmannians we follow the same outline. The first step is the following theorem and its corollary; the second step will be done separately for the different cases in the next sections.

Theorem 1.2 *For $N \geq 4$ and a general linear subspace $H^l \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})$ of codimension $l \leq 2N-5$ the linear section $\mathbb{G}(1,N) \cap H^l$ spans H^l and its automorphisms are induced by automorphisms of H^l, i.e.*

$$\mathrm{Aut}(\mathbb{G}(1,N) \cap H^l) = \mathrm{Aut}(\mathbb{G}(1,N) \cap H^l, H^l)$$

Proof. We will abbreviate $\mathbb{G}(1,N)$ by $\mathbb{G}$. We want to prove that $\mathbb{G} \cap H^l$ spans H^l, i.e. for $\mathbb{G} \cap H^l \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})$

$$\mathrm{h}^0(\mathbb{G} \cap H^l, \mathcal{O}(H)) = \dim \textstyle\bigwedge^2 \mathbb{C}^{N+1} - l.$$

This is known for $l = 0$. For $l \geq 1$ we take the long exact sequence associated to the restriction sequence tensored by $\mathcal{O}(H)$

$$0 \to \mathrm{H}^0(\mathbb{G} \cap H^{l-1}, \mathcal{O}) = \mathbb{C} \to \mathrm{H}^0(\mathbb{G} \cap \mathrm{H}^{l-1}, \mathcal{O}(H)) \to \mathrm{H}^0(\mathbb{G} \cap \mathrm{H}^l, \mathcal{O}(H)) \to$$
$$\to \mathrm{H}^1(\mathbb{G} \cap \mathrm{H}^{l-1}, \mathcal{O}) = 0.$$

Looking at the dimensions we get

$$\mathrm{h}^0(\mathbb{G} \cap H^l, \mathcal{O}(H)) = \mathrm{h}^0(\mathbb{G} \cap H^{l-1}, \mathcal{O}(H)) - 1,$$

and the claim follows by induction.
Now we show that all automorphisms of $\mathbb{G} \cap H^l$ are induced by automorphisms of H^l. This follows if we can show that all divisors of $\mathbb{G} \cap H^l$ are induced by divisors of H^l, i.e.

$$\mathrm{Pic}(\mathbb{G} \cap H^l) = \mathrm{Pic}(H^l) = \mathbb{Z} \cdot H,$$

because then the projective embedding of $\mathbb{G} \cap H^l$ given by the sections in the line bundle $\mathcal{O}(H)$ is equivariant for all automorphisms of $\mathbb{G} \cap H^l$
To see this, note that by the Lefschetz hyperplane section theorem

$$\mathbb{Z} \cdot H = \mathrm{H}^2(\mathbb{G}, \mathbb{Z}) = \mathrm{H}^2(\mathbb{G} \cap H, \mathbb{Z}) = \ldots = \mathrm{H}^2(\mathbb{G} \cap H^l, \mathbb{Z})$$

for $0 \leq l \leq 2N-5$. From the exponential sequence

$$0 \to \mathbb{Z}_{\mathbb{G}\cap H^l} \to \mathcal{O}_{\mathbb{G}\cap H^l} \to \mathcal{O}^*_{\mathbb{G}\cap H^l} \to 0$$

we get as a part of the associated long exact sequence

$$\ldots \to \mathrm{H}^1(\mathbb{G}\cap H^l, \mathcal{O}) \to \mathrm{H}^1(\mathbb{G}\cap H^l, \mathcal{O}^*) \to \mathrm{H}^2(\mathbb{G}\cap H^l, \mathbb{Z}) = \mathbb{Z}\cdot H \to 0$$

and therefore

$$\mathrm{Pic}(\mathbb{G}\cap H^l) = \mathrm{H}^1(\mathbb{G}\cap H^l, \mathcal{O}^*) = \mathbb{Z}\cdot H$$

as soon as we know that $\mathrm{H}^1(\mathbb{G}\cap H^l, \mathcal{O}) = 0$.

This is well known for $l=0$. For $l \geq 1$ we look at the restriction sequence

$$0 \to \mathcal{O}_{\mathbb{G}\cap H^{l-1}}(-H) \to \mathcal{O}_{\mathbb{G}\cap H^{l-1}} \to \mathcal{O}_{\mathbb{G}\cap H^l} \to 0$$

and take its associated long exact sequence

$$\ldots \to \mathrm{H}^1(\mathbb{G}\cap H^{l-1}, \mathcal{O}(-H)) \to \mathrm{H}^1(\mathbb{G}\cap H^{l-1}, \mathcal{O}) \to \mathrm{H}^1(\mathbb{G}\cap H^l, \mathcal{O}) \to$$
$$\to \mathrm{H}^2(\mathbb{G}\cap H^{l-1}, \mathcal{O}(-H)) \to \ldots$$

The right and left cohomology groups are trivial for $l \leq 2N-4$ by Kodaira's vanishing theorem, so

$$0 = \mathrm{H}^1(\mathbb{G}, \mathcal{O}) = \mathrm{H}^1(\mathbb{G}\cap H, \mathcal{O}) = \ldots = \mathrm{H}^1(\mathbb{G}\cap H^l, \mathcal{O}). \qquad \square$$

Corollary 1.3 *For $N \geq 4$ and a general linear subspace $H^l \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})$ of codimension $l \leq N-2$*

$$\mathrm{Aut}(\mathbb{G}(1,N)\cap H^l) = \mathrm{Aut}(\mathbb{G}(1,N)\cap H^l, \mathbb{P}(\textstyle\bigwedge^2\mathbb{C}^{N+1})) \cap \mathrm{Aut}(H^l, \mathbb{P}(\textstyle\bigwedge^2\mathbb{C}^{N+1})).$$

This is also true for $\mathbb{G}(1,4)\cap H^3$.

Proof. The special case of $\mathbb{G}(1,4)\cap H^3$ will be dealt with in Section 7.
In view of the theorem we need only to show that an automorphism of $\mathbb{G}(1,N)\cap H^l$ can be extended to an automorphism of $\mathbb{G}(1,N)$, which is always linear and fixes the linear subspace H^l because $\mathbb{G}(1,N)\cap H^l$ spans H^l. To do so we will study the linear subspaces of $\mathbb{G}(1,N)\cap H^l$.
The linear subspaces of the Grassmannian $\mathbb{G}(1,N)$ are the following Schubert cycles:

1. Let $p \in \mathbb{P}_N$ be a point and $K \subset \mathbb{P}_N$ a linear subspace of dimension k that contains the point p then

   $$\{L \in \mathbb{G}(1,N) \mid p \in L \subseteq K\} \subset \mathbb{G}(1,N)$$

   is a $(k-1)$-dimensional subspace and these form a

   $$\dim \mathbb{P}_N + \dim \mathbb{G}(k-1, N-1) = N + k(N-k)$$

   dimensional family F_{k-1}.

2. Let $E \subset \mathbb{P}_N$ be a plane then

$$\{L \in \mathbb{G}(1,N) \mid L \subseteq K\} \subset \mathbb{G}(1,N)$$

is a plane in $\mathbb{G}(1,N)$ and these form a

$$\dim \mathbb{G}(1,N) = 3(N-2) = 3N-6$$

dimensional family F_2'.

Let $d = \dim \bigwedge^2 \mathbb{C}^{N+1} - 1$. For the variety of 2-planes F_2 (and analogously for F_2') we consider the incidence correspondence

$$\{(E,h) \in F_2 \times \mathbb{G}(d-l,d) \mid E \subset h\} \subset F_2 \times \mathbb{G}(d-l,d).$$

The fibre above any point $h \in \mathbb{G}(d-l,d)$ is exactly the set of 2-planes of F_2 on $\mathbb{G}(1,N) \cap h$, which we denote by $\widetilde{F}_2$. From this correspondence we see immediately that for general H^l either $\widetilde{F}_2$ is empty or $\widetilde{F}_2$ consists of some components of the same dimension (Stein factorisation). The same argument applied to F_2' yields $\widetilde{F}_2'$. Since for $N \geq 4$ $\dim F_2 \neq \dim F_2'$ we thus find that if $\widetilde{F}_2$ and $\widetilde{F}_2'$ are not both empty, then $\dim \widetilde{F}_2 \neq \dim \widetilde{F}_2'$.

Now any automorphism of $\mathbb{G}(1,N) \cap H^l$ is linear, so we conclude that any automorphism of $\mathbb{G}(1,N) \cap H^l$ transforms $\widetilde{F}_2$ resp. $\widetilde{F}_2'$ onto itself. For dimension $\neq 2$ there are only linear spaces of type 1, so we can state:

Any automorphism φ of $\mathbb{G}(1,N) \cap H^l$ permutes the linear spaces of type 1.

Next, let $p \in \mathbb{P}_N$ and L_p the $(N-1)$-dimensional linear subspace of $\mathbb{G}(1,n)$ consisting of the lines through p. From $l \leq N-2$ we see $e = \dim(L_p \cap H^l) \geq 1$. Since H^l is general, for almost all points of $\mathbb{P}_N$ this dimension is exactly e. Now, and this is the crucial remark, $\varphi(L_p \cap H^l)$ being of type 1, is contained in exactly one L_q, $q \in \mathbb{P}_N$. Attaching q to p we obtain for almost all $p \in \mathbb{P}_N$ a map to $\mathbb{P}_N$. If the dimension of $L_{p_0} \cap H^l$ is bigger than the minimum, then this map can still be defined at p_0 in exactly the same way. We claim that this is continuous at p_0. This can be seen by considering a general sequence of points on $\mathbb{P}_N$, say $p_1, p_2, \ldots$, converging to p_0 with $\dim(L_{p_i} \cap H^l)$ minimal and applying the crucial remark twice. The map from $\mathbb{P}_N$ to $\mathbb{P}_N$ thus obtained has an inverse and is holomorphic, so it is a linear automorphism of $\mathbb{P}_N$. This automorphism induces an automorphism of $\mathbb{G}(1,N)$ for which it is easily verified that it coincides with φ on $\mathbb{G}(1,N) \cap H^l$. □

It is tempting to assume that the groups $\mathrm{Aut}(\mathbb{G}(1,N), \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1}))$ and $\mathrm{Aut}(H^l, \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1}))$ in $\mathrm{Aut}(\mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1}))$ intersect transversally. Then the dimension of $\mathrm{Aut}(\mathbb{G}(1,N) \cap H^l)$ could be computed as

$$\begin{aligned}\dim \mathrm{Aut}(\mathbb{G}(1,N) \cap H^l) &= \dim \mathrm{Aut}(\mathbb{G}(1,N)) - \operatorname{codim} \mathrm{Aut}(H^l, \mathbb{P}(\textstyle\bigwedge^2 \mathbb{C}^{N+1})) \\ &= (N+1)^2 - 1 - l\left(\binom{N+1}{2} - l\right).\end{aligned}$$

And we would find the following non-finite groups:

$$\begin{aligned}
\dim \operatorname{Aut}(\mathbb{G}(1,N)\cap H) &= (N^2+3N+2)/2\\
\dim \operatorname{Aut}(\mathbb{G}(1,N)\cap H^2) &= N+4\\
\dim \operatorname{Aut}(\mathbb{G}(1,4)\cap H^3) &= 3.
\end{aligned}$$

Unfortunately, the intersection is not always transversal. Our computation of the automorphism groups will show the following dimensions for $N \geq 4$:

$$\begin{aligned}
\dim \operatorname{Aut}(\mathbb{G}(1,N)\cap H) &= (N^2+3N+2)/2\\
\dim \operatorname{Aut}(\mathbb{G}(1,N)\cap H^2) &= \begin{cases} N+4 & \text{for } N \text{ even}\\ 3(N+1)/2 & \text{for } N \text{ odd}\end{cases}\\
\dim \operatorname{Aut}(\mathbb{G}(1,4)\cap H^3) &= 3\\
\dim \operatorname{Aut}(\mathbb{G}(1,5)\cap H^3) &= 1.
\end{aligned}$$

We conjecture that these are the only non-finite groups. For $N+2 \leq l$ the canonical bundle $K = \mathcal{O}(-N-1+l)$ is positive on $\mathbb{G}(1,N)\cap H^l$, and this conjecture can be proved by Serre's duality theorem and Kodaira's vanishing theorem:

$$\dim \operatorname{Aut}(\mathbb{G}\cap H^l) = h^0(\mathbb{G}\cap H^l, \Theta) = h^{2N-2-l}(\mathbb{G}\cap H^l, K\Omega^1) = 0$$

A proof for the remaining cases $3 \leq l \leq N+1$ seems difficult. For $N \leq 10$ and all l we verified the conjecture for the automorphisms induced by automorphisms of the Grassmannian by computer computations.

With this Theorem and its Corollary our task of determining the automorphisms of $\mathbb{G}(1,N)\cap H^l$ has been immensely simplified. All we need to do is to find the projective transformations of $\operatorname{Aut}(\mathbb{G}(1,N), \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})) = \mathbb{P}\mathrm{GL}(N+1,\mathbb{C})$ such that their induced action on $\mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})^*$ preserves H^l. To express this in algebraic terms we identify $(\bigwedge^2 \mathbb{C}^{N+1})^*$ with $\bigwedge^2(\mathbb{C}^{N+1})^*$. If a particular basis of $\mathbb{C}^{N+1}$ is chosen, $\bigwedge^2(\mathbb{C}^{N+1})^*$ as antisymmetric forms on $\mathbb{C}^{N+1}$ can also be identified with the antisymmetric matrices of size $N+1$. In concrete terms, if $(e_0,\ldots,e_N)$ is a basis of $\mathbb{C}^{N+1}$ and $E_{ij} \in \mathrm{M}(N+1,\mathbb{C})$ the matrix, which has a 1 in the position (i,j) but is otherwise zero, then

$$\begin{array}{ccc}
\left(\bigwedge^2 \mathbb{C}^{N+1}\right)^* & \longrightarrow & \operatorname{Antisym}(N+1,\mathbb{C})\\
\sum_{i,j} \lambda_{ij}(e_i\wedge e_j)^* & \longmapsto & \frac{1}{2}\sum_{i,j}\lambda_{ij}(E_{ij}-E_{ji}).
\end{array}$$

In these terms a line $l = p\wedge q \in \mathbb{G}(1,N)$ is in the hyperplane $H \in \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})$ iff for a corresponding antisymmetric matrix $A \in \operatorname{Antisym}(N+1,\mathbb{C})$ with $\mathbb{P}(A) = H$ we have ${}^t\!pAq = 0$.

Further, the action of $\mathbb{P}\mathrm{GL}(N+1,\mathbb{C})$ on $\mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})$, which was given for $\mathbb{P}(T) \in \mathbb{P}\mathrm{GL}(N+1,\mathbb{C})$ by

$$\begin{array}{ccc} \mathbb{P}\left(\bigwedge^2\mathbb{C}^{N+1}\right) & \longrightarrow & \mathbb{P}\left(\bigwedge^2\mathbb{C}^{N+1}\right) \\ \mathbb{P}(\sum v_i \wedge w_i) & \longmapsto & \mathbb{P}(\sum Tv_i \wedge Tw_i), \end{array}$$

induces the following action on the dual space

$$\begin{array}{ccc} \mathbb{P}(\mathrm{Antisym}(N+1,\mathbb{C})) & \longrightarrow & \mathbb{P}(\mathrm{Antisym}(N+1,\mathbb{C})) \\ \mathbb{P}(A) & \longmapsto & \mathbb{P}({}^tT^{-1}AT^{-1}). \end{array}$$

Hence an l-codimensional linear subspace $H^l \subseteq \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})$ which is dually given by $\mathbb{P}(\mathrm{span}\,\{A_1,\ldots,A_l\})$ is preserved under T iff every hyperplane containing H^l is mapped to another hyperplane containing H^l, i.e.

$$\begin{array}{ll} {}^tT^{-1}\left(\sum \lambda_i A_i\right) T^{-1} \in \mathrm{span}\,\{A_1,\ldots,A_l\} & \text{for all } \lambda_i \in \mathbb{C} \\ \Longleftrightarrow {}^tT^{-1}A_iT^{-1} \in \mathrm{span}\,\{A_1,\ldots,A_l\} & \text{for } i = 1\ldots l. \end{array}$$

We conclude

Corollary 1.4 *For $N \geq 4$, $0 \leq l \leq N-2$ and a general $H^l \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})$ given by $\mathbb{P}(\mathrm{span}\,\{A_1,\ldots,A_l\}) \subset \mathbb{P}(\mathrm{Antisym}(N+1,\mathbb{C}))$ the automorphism group of $\mathbb{G}(1,N) \cap H^l$ is*

$$\{\mathbb{P}(T) \in \mathbb{P}\mathrm{GL}(N+1,\mathbb{C}) \mid {}^tT^{-1}A_iT^{-1} \in \mathrm{span}\,\{A_1,\ldots,A_l\}\,\forall i\}.$$

In the following sections we will compute the automorphism groups using this Corollary. In the course of the computations we will use geometric arguments for which it is essential to know if a hyperplane $H \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})$ is tangent to $\mathbb{G}(1,N)$ or not. We recall the basic facts together with their short proofs.

Proposition 1.5 *For any line $l_0 \in \mathbb{G}(1,N)$ the Schubert cycle*

$$\sigma := \{l \in \mathbb{G}(1,N) \mid l \cap l_0 \neq \emptyset\} \subseteq \mathbb{G}(1,N)$$

lies inside the tangent space $\mathbb{T}_{l_0}\mathbb{G}(1,N) \subseteq \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})$ and spans it.

Proof. Let $l \in \sigma$, $p \in l \cap l_0$, $q \in l_0 \setminus \{p\}$ and $r \in l \setminus \{p\}$ then

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{G}(1,N) \\ \lambda & \longmapsto & p \wedge (q + \lambda r) \end{array}$$

is a line in $\sigma \subset \mathbb{G}(1,N)$ through l_0 and l. Therefore it is contained in the tangent space $\mathbb{T}_{l_0}\mathbb{G}(1,N)$, in particular $l \in \mathbb{T}_{l_0}\mathbb{G}(1,N)$.
We choose a basis $(e_0,\ldots,e_N)$ of $\mathbb{C}^{N+1}$ such that $l_0 = \mathbb{P}(e_0 \wedge e_1)$. The $2N-1$ points $\mathbb{P}(e_0 \wedge e_1)$, $\mathbb{P}(e_0 \wedge e_i)$, $\mathbb{P}(e_1 \wedge e_i)$ for $i = 2\ldots N$ lie in $\sigma \subset \mathbb{T}_{l_0}\mathbb{G}(1,N)$ and are projectively independent, hence they span $\mathbb{T}_{l_0}\mathbb{G}(1,N)$. □

Corollary 1.6 *Let* $H = \mathbb{P}(A) \in \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})^*$ *be a hyperplane and* $l_0 \in \mathbb{G}(1,N)$ *a line then*

$$\mathbb{T}_{l_0}\mathbb{G}(1,N) \subseteq H \Longleftrightarrow l_0 \subseteq \ker A.$$

Proof. By the Proposition $\mathbb{T}_{l_0}\mathbb{G}(1,N) \subseteq H$ is equivalent to $\sigma \subseteq H$. If we use the same basis of $\mathbb{C}^{N+1}$ as in the proof of the proposition, this means that

$$\begin{aligned}
&\mathbb{P}((\lambda e_0 + \mu e_1) \wedge v) \in H \quad \text{for all } (\lambda\colon \mu) \in \mathbb{P}_1,\ v \in \mathbb{C}^{N+1} \\
\Longleftrightarrow\ &{}^t(\lambda e_0 + \mu e_1) A v = 0 \quad \text{for all } (\lambda\colon \mu) \in \mathbb{P}_1,\ v \in \mathbb{C}^{N+1} \\
\Longleftrightarrow\ &l_0 \subseteq \ker A.
\end{aligned}$$

□

Corollary 1.7 *The dual variety* $\mathbb{G}(1,N)^* \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})^*$ *of the Grassmannian variety* $\mathbb{G}(1,N)$ *consists of matrices of corank* ≥ 2 *for* N *odd resp. corank* ≥ 3 *for* N *even.*

For N *odd it is an irreducible hypersurface of degree* $(N+1)/2$*; for* N *even it is a 3-codimensional subvariety.*

Proof. By the last corollary $H = \mathbb{P}(A) \in \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})^*$ is tangential to $\mathbb{G}(1,N)$ iff $\operatorname{corank} A \geq 2$. Recall that an antisymmetric matrix has even rank. So, for N odd the matrix $A \in \operatorname{Antisym}(N+1, \mathbb{C})$ has corank ≥ 2 iff $\det A = 0$. But again since A is antisymmetric, $\det A$ is the square of the irreducible Pfaffian polynomial $\operatorname{Pf} A$ [B, 5.2], which therefore defines $\mathbb{G}(1,N)^*$.

For N even $\operatorname{corank} A \geq 2$ is equivalent to $\operatorname{corank} A \geq 3$. We compute the dimension of $\mathbb{G}(1,N)^*$ following Mumford [M] and find

$$\begin{aligned}
\dim \begin{pmatrix} \text{space of } A \text{ with} \\ \dim\ker A = 3 \end{pmatrix} &= \dim \mathrm{G}(3, N+1) + \dim {\textstyle\bigwedge^2} \mathbb{C}^{N+1}/\mathbb{C}^3 \\
&= 3(N-2) + (N-2)(N-3)/2 \\
&= (N^2 + N - 6)/2
\end{aligned}$$

$$\Longrightarrow \operatorname{codim} \mathbb{G}(1,N)^* = (N+1)N/2 - (N^2+N-6)/2 = 3.$$

□

2 $\mathbb{G}(1, 2n-1) \cap H$

Let the hyperplane $H \in \mathbb{P}(\bigwedge^2 \mathbb{C}^{2n})$ be given by an element $A \in (\bigwedge^2 \mathbb{C}^{2n})^*$, which we identify with its corresponding antisymmetric matrix. If H is general, A will be a matrix of full rank. This may be taken as the definition of a general H. We will assume from now on that H is general.

The line system $\mathbb{G}(1, 2n-1) \cap H$ in $\mathbb{P}_{2n-1}$ does not lead to obvious special points in the $\mathbb{P}_{2n-1}$. Through every point $p \in \mathbb{P}_{2n-1}$ passes a $\mathbb{P}_{2n-3}$ of lines, namely

$$p \wedge q \in \mathbb{G}(1, 2n-1) \text{ with } q \in \ker {}^t\!pA.$$

For $n \geq 3$ we can compute the automorphism group of $\mathbb{G}(1,2n-1)\cap H$ with the help of Theorem 1.2 and its Corollaries. It consists of elements $\mathbb{P}(T) \in \mathbb{P}\mathrm{GL}(2n,\mathbb{C}) = \mathrm{Aut}(\mathbb{G}(1,2n-1))$ such that $\mathbb{P}(T)$ as an element of $\mathbb{P}\mathrm{GL}(\bigwedge^2 \mathbb{C}^{2n})$ preserves H, i.e.

$$^tT^{-1}AT^{-1} = \lambda A \quad \text{for suitable } \lambda \in \mathbb{C}^*.$$

We may choose coordinates on $\mathbb{P}_{2n-1}$ such that

$$A = \begin{pmatrix} 0 & -\mathrm{E}_n \\ \mathrm{E}_n & 0 \end{pmatrix}.$$

Then by definition

$$\mathrm{Sp}(2n,\mathbb{C}) = \{T \in \mathrm{GL}(2n,\mathbb{C}) \mid {}^tT^{-1}AT^{-1} = A\},$$

and we have an isomorphism

$$\begin{array}{ccc} \{T \in \mathrm{GL}(2n,\mathbb{C}) \mid \exists \lambda_T \in \mathbb{C}^* : {}^tT^{-1}AT^{-1} = \lambda_T A\}/\mathbb{C}^* & \longrightarrow & \mathrm{Sp}(2n,\mathbb{C})/\{1,-1\} \\ \mathbb{C}^* \cdot T & \longmapsto & \pm \frac{1}{\sqrt{\lambda_T}} T. \end{array}$$

Therefore we see

Proposition 2.1 *The automorphism group of* $\mathbb{G}(1,2n-1)\cap H$ *for a general* $H \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})$ *is* $\mathrm{Sp}(2n,\mathbb{C})/\{1,-1\}$. *Its action on* $\mathbb{G}(1,2n-1)\cap H$ *is homogeneous.*

Proof. The missing case of $\mathbb{G}(1,3)\cap H$ can be found in [FH, p. 278]. The transitivity of the action follows from Witt's theorem [Br, 12.31]. □

3 $\mathbb{G}(1,2n-1)\cap H^2$

A 2-codimensional linear subspace $L = H^2$ of $\mathbb{P}(\bigwedge^2 \mathbb{C}^{2n})$ can be thought of as the pencil of hyperplanes containing it. So it gives a line $L^* = \mathbb{P}(\lambda A - \mu B) \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{2n})^*$. We identify again $(\bigwedge^2 \mathbb{C}^{2n})^*$ with the antisymmetric matrices of size $2n$. The line L^* intersects the dual Grassmannian $\mathbb{G}(1,2n-1)^*$, which consists of antisymmetric matrices of rank $\leq 2n-2$ and is a hypersurface of degree n by Corollary 1.7, in at most n points. For the moment a line L^*, and hence L, will be called general if it has n points of intersection, $H_i = \mathbb{P}(\lambda_i A - \mu_i B) \in L^*$, $i = 1 \ldots n$, with the dual Grassmannian. These hyperplanes H_i are tangent to the Grassmannian $\mathbb{G}(1,2n-1)$ at the points $l_i := \ker(\lambda_i A - \mu_i B) \in \mathbb{G}(1,2n-1)$ by Corollary 1.6. Therefore we get n exceptional lines $l_1,\ldots,l_n$ in $\mathbb{P}_{2n-1}$.

The intersection of the Grassmannian with its tangent hyperplane H_i contains all lines that intersect l_i, because these lines are already contained in the intersection $\mathbb{G}(1,2n-1)\cap \mathbb{T}_{l_i}\mathbb{G}(1,2n-1)$ by Proposition 1.5.

So, any line through a point $p \in l_i$ will be in the subspace $L \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{2n})$ as soon as it is contained in any other hyperplane $H \in L^* \setminus \{H_i\}$. This gives one linear restriction to lines through p, so that there is at least a $\mathbb{P}_{2n-3}$ of lines through the points of the lines l_i. In contrast, through a general point of $\mathbb{P}_{2n-1} \setminus \bigcup l_i$ there is only a $\mathbb{P}_{2n-4}$ of lines. In fact, we have

Proposition 3.1 *The points of the lines $l_1, \dots, l_n$ are characterized by the property that through each of them passes a $\mathbb{P}_{2n-3}$ of lines, i.e.*

$$\left\{ p \in \mathbb{P}_{2n-1} \;\middle|\; \begin{array}{l} \text{through } p \text{ passes a } \mathbb{P}_{2n-3} \\ \text{of lines of } \mathbb{G}(1, 2n-1) \cap L \end{array} \right\} = \bigcup l_i .$$

Furthermore, the lines $l_1, \dots, l_n$ span the whole $\mathbb{P}_{2n-1}$.

This may easily be seen if we write the pencil of hyperplanes L^* in its normal form.

Proposition 3.2 (Donagi[D]) *Given a pencil of hyperplanes $L^* = \mathbb{P}(\lambda A - \mu B) \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{2n})^*$ such that the line L^* intersects the Pfaffian hypersurface in n different points. Then there is a basis of $\mathbb{C}^{2n}$ such that*

$$A = \begin{pmatrix} J & & 0 \\ & \ddots & \\ 0 & & J \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \lambda_1 J & & 0 \\ & \ddots & \\ 0 & & \lambda_n J \end{pmatrix} \quad \text{with } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The points $(\lambda_1 : 1), \dots, (\lambda_n : 1) \in \mathbb{P}_1 \cong L^*$ are unique up to a projective transformation of $\mathbb{P}_1$.

Proof of Proposition 3.1. The hyperplane $H_i = \mathbb{P}(\lambda_i A - \mu_i B)$ has, written as an antisymmetric matrix, the kernel $l_i = \operatorname{span}\{e_{2i-1}, e_{2i}\}$ which means it is tangent to $\mathbb{G}(1, 2n-1)$ at l_i. All lines of $\mathbb{G}(1, 2n-1) \cap L$ through the point p are given by $p \wedge q$ with ${}^t\!pAq = {}^t\!pBq = 0$. In order to have a $\mathbb{P}_{2n-2}$ of lines through p, the linear forms ${}^t\!pA$ and ${}^t\!pB$ must be linear dependent, i.e. there are $\lambda, \mu \in \mathbb{C}$ with

$$0 = \lambda\, {}^t\!pA - \mu\, {}^t\!pB = {}^t\!p(\lambda A - \mu B).$$

Therefore p is in the kernel of a matrix of the pencil, but these kernels are the lines $l_1, \dots, l_n$, so p is contained in one of them. □

Knowing the exceptional lines $l_1, \dots, l_n$, one can immediately give some lines which are in the line system.

Proposition 3.3 *Any line in $\mathbb{P}_{2n-1}$ which intersects two exceptional lines is an element of the line system $\mathbb{G}(1, 2n-1) \cap L$.*

The exceptional lines themselves are not in the line system.

Proof. If a line l intersects l_i and l_j, it lies – as a point of the Grassmannian $\mathbb{G}(1,2n-1)$ – in H_i and H_j, hence in $L = H_i \cap H_j$.

Assume that the exceptional line l_1 is an element of $\mathbb{G}(1,2n-1)$. By Proposition 3.1 the lines through a point $p \in l_1$ sweep out a hyperplane. This hyperplane contains the line l_1 by assumption and the other exceptional lines $l_2,\dots,l_n$ by the first part of this proposition. But this contradicts the second statement of Proposition 3.1. □

Remark 3.4 *From Proposition 3.2 we also see that any position of the n points of the line L^* is possible. In particular, we may call a line general if the position of the points is general in the sense needed below.*

Using this geometric description we can determine the automorphisms of $\mathbb{G}(1,2n-1)\cap L$. For the moment we restrict ourselves to $n \geq 3$ in order to be able to use Theorem 1.2. By this Theorem and its Corollaries we can view an automorphism of $\mathbb{G}(1,2n-1)\cap L$ as an element $\mathbb{P}(T)$ of $\mathbb{P}\mathrm{GL}(2n,\mathbb{C})$. To make the notation simpler, we will write only T for $\mathbb{P}(T)$ if no confusion can result. Since the points of the exceptional lines are characterized by the property of Proposition 3.1, T must map the union of the lines $l_i \subset \mathbb{P}_{2n-1}$ onto itself. Permutations of the lines may occur, but – as we will presently see – not all permutations are possible.

If we view the automorphism T as an element of $\mathrm{Aut}(L,\mathbb{P}(\bigwedge^2\mathbb{C}^{2n}))$, it interchanges the hyperplanes containing L, i.e. it induces a projective transformation of the line $L^* \subset \mathbb{P}(\bigwedge^2\mathbb{C}^{2n})$. Naturally, the transformation of L^* must preserve the union of points of intersection of L^* with the dual Grassmannian, which determine the lines l_i. Now, if a transformation of $\mathbb{P}_{2n-1}$ permutes the lines l_i, then the induced transformation of L^* must permute the corresponding points of L^* in the same way.

Since not every permutation of four or more points on a line can be induced by a projective transformation, not all permutations are possible. In fact, if the points are in general position, we get the following subgroups of the permutation groups:

n	subgroup of $\mathrm{S}(n)$
3	$\mathrm{S}(3)$
4	$\{(1\,2\,3\,4),(2\,1\,4\,3),(3\,4\,1\,2),(4\,3\,2\,1)\}\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
≥ 5	$\{\mathrm{id}\}$

On the other hand, any permutation $\sigma \in \mathrm{S}(n)$ of the points on L^* that is induced by a projective transformation φ of L^* can be induced by an automorphism of $\mathbb{G}(1,2n-1)\cap L$. To see this, let us write L^* in its normal form and define $T \in \mathrm{GL}(2n,\mathbb{C})$ as

$$T(e_{2i}) := e_{2\sigma(i)} \quad \text{and} \quad T(e_{2i-1}) := e_{2\sigma(i)-1}.$$

This transformation permutes the lines in the prescribed way, and as an automorphism of $\mathbb{P}(\bigwedge^2 \mathbb{C}^{2n})$ it fixes L since the transformed line L^* is

$$
{}^tT^{-1}\left(\lambda\begin{pmatrix} J & & 0 \\ & \ddots & \\ 0 & & J \end{pmatrix} - \mu\begin{pmatrix} \lambda_1 J & & 0 \\ & \ddots & \\ 0 & & \lambda_n J \end{pmatrix}\right)T^{-1}
$$

$$
= \lambda\begin{pmatrix} J & & 0 \\ & \ddots & \\ 0 & & J \end{pmatrix} - \mu\begin{pmatrix} \lambda_{\sigma^{-1}(1)} J & & 0 \\ & \ddots & \\ 0 & & \lambda_{\sigma^{-1}(n)} J \end{pmatrix}.
$$

Changing the parametrisation of the line by φ we get back the old parametrisation of the line L^* by the definition of φ. So this T is an automorphism of $\mathbb{G}(1, 2n-1) \cap L$ that induces the permutation of lines we started with.

Now we can restrict our attention to transformations that do not permute the lines since we can obtain every permutation by composing with one of the transformations from above. A transformation leaving all the lines individually fixed has the form

$$
T = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \quad \text{with } t_1, \ldots, t_n \in \mathrm{GL}(2, \mathbb{C}).
$$

This T will fix the line system $\mathbb{G}(1, 2n-1) \cap L$ in $\mathbb{P}_{2n-1}$ iff it preserves L^*, i.e. for all $\lambda, \mu \in \mathbb{C}$ there exists $\alpha, \beta \in \mathbb{C}$ such that

$$
{}^tT^{-1}(\lambda A - \mu B)T^{-1} = \alpha A + \beta B.
$$

It is sufficient to check this for $(\lambda, \mu) = (1, 0)$ and $(0, -1)$. Since

$$
{}^tT^{-1}AT^{-1} = \begin{pmatrix} \det t_1^{-1} J & & 0 \\ & \ddots & \\ 0 & & \det t_n^{-1} J \end{pmatrix}
$$

$$
{}^tT^{-1}BT^{-1} = \begin{pmatrix} \lambda_1 \det t_1^{-1} J & & 0 \\ & \ddots & \\ 0 & & \lambda_n \det t_n^{-1} J \end{pmatrix}
$$

this is equivalent to the question if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with

$$
\begin{aligned}
(\det t_1^{-1}, \ldots, \det t_n^{-1}) &= \alpha(1, \ldots, 1) + \beta(\lambda_1, \ldots, \lambda_n) \\
(\lambda_1 \det t_1^{-1}, \ldots, \lambda_n \det t_n^{-1}) &= \gamma(1, \ldots, 1) + \delta(\lambda_1, \ldots, \lambda_n).
\end{aligned}
$$

It follows

$$
\begin{aligned}
&-\gamma(1,\ldots,1)+(\alpha-\delta)(\lambda_1,\ldots,\lambda_n)+\beta(\lambda_1^2,\ldots,\lambda_n^2)=0\\
&\Longrightarrow \alpha=\delta,\ \beta=\gamma=0\\
&\Longrightarrow \det t_1=\ldots=\det t_n.
\end{aligned}
$$

We normalize by $\det t_1 = 1$, i.e. $t_1,\ldots,t_n \in \mathrm{SL}(2,\mathbb{C})$. Then only T and $-T \in \mathrm{GL}(2n,\mathbb{C})$ give the same element in $\mathbb{P}\mathrm{GL}(2n,\mathbb{C})$. So that as a group the automorphisms of $\mathbb{G}(1,2n-1)\cap L$ that do not permute the exceptional lines are isomorphic to $\mathrm{SL}(2,\mathbb{C})^n/\{1,-1\}$.

Altogether we get

Theorem 3.5 *For $n\geq 3$ the automorphism group of the intersection of $\mathbb{G}(1,2n-1)$ with a general 2-codimensional linear subspace of $\mathbb{P}(\bigwedge^2\mathbb{C}^{2n})$ has $\mathrm{SL}(2,\mathbb{C})^n/\{1,-1\}$ as a normal subgroup and the quotient group is isomorphic to the permutation group $\mathrm{S}(3)$ for $n=3$, to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ for $n=4$, and trivial otherwise.*

The automorphism group is isomorphic to the subgroup of $\mathbb{P}\mathrm{GL}(2n,\mathbb{C})$ that consists of the elements

$$
P_\sigma \cdot \begin{pmatrix} t_1 & & 0\\ & \ddots & \\ 0 & & t_n\end{pmatrix} \quad \text{with } t_1,\ldots,t_n\in\mathrm{SL}(2,\mathbb{C})
$$

where P_σ is the identity for $n\geq 5$ and otherwise defined by

$$
\begin{aligned}
P_\sigma(e_{2i}) &= e_{2\sigma(i)}\\
P_\sigma(e_{2i-1}) &= e_{2\sigma(i)-1}
\end{aligned}
$$

$$
\text{for } \sigma\in\begin{cases}\mathrm{S}(n) & \text{if } n=3\\ \{(1\,2\,3\,4),(2\,1\,4\,3),(3\,4\,1\,2),(4\,3\,2\,1)\} & \text{if } n=4.\end{cases}
$$

For the sake of completeness we recall the classical case of $\mathbb{G}(1,3)\cap H^2$.

Remark 3.6 *The automorphism group of $\mathbb{G}(1,3)\cap H^2$ is an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{P}\mathrm{GL}(2,\mathbb{C})\times\mathbb{P}\mathrm{GL}(2,\mathbb{C})$. It acts homogeneously on $\mathbb{G}(1,3)\cap H^2$.*

Proof. The Grassmannian $\mathbb{G}(1,3)$ is a smooth quadric in $\mathbb{P}(\bigwedge^2\mathbb{C}^4)\cong\mathbb{P}_5$. Therefore $\mathbb{G}(1,3)\cap H^2$ is a smooth quadric in $\mathbb{P}_3$. Hence it is isomorphic to the Segre variety $\mathbb{P}_1\times\mathbb{P}_1$ in $\mathbb{P}_3$. The automorphism group of $\mathbb{P}_1\times\mathbb{P}_1$ is generated by $\mathbb{P}\mathrm{GL}(2,\mathbb{C})\times\mathbb{P}\mathrm{GL}(2,\mathbb{C})$ together with the automorphism that exchanges the $\mathbb{P}_1$s. All the automorphisms extend to $\mathbb{P}_3$. Obviously, the group acts transitively on $\mathbb{P}_1\times\mathbb{P}_1$. $\square$

For the rest of this section we consider the question if the action of the other automorphism groups is quasihomogeous on the corresponding line system, i.e. if there is an open orbit.

This cannot be the case for $n \geq 7$ since then the dimension of the line system $\mathbb{G}(1, 2n-1) \cap H^2$, $2(2n-2)-2 = 4n-6$, is larger than the dimension of the automorphism group, $3n$.

For $n = 3$ the action is quasihomogeneous. To see that one can adjust the $(\lambda_1, \lambda_2, \lambda_3)$ in the normal form of the line system to $(1, 0, -1)$ by a projective transformation and compute the stabiliser of the line $(1:0:1:0:1:0) \wedge (1:1:1:-2:1:1)$ by hand or computer and see that it is 3-dimensional. So the dimension of its orbit is $3 \cdot 3 - 3 = 6$, which is just the dimension of the line system.

For $n = 4, 5, 6$ the group does not act quasihomogenously anymore. For this one computes again the dimension of the stabiliser of a general line. Since the group acts transitively on $\mathbb{P}_{2n-1} \setminus \bigcup L_i$, we may restrict our attention to lines through one of those points, e.g. $(1:0:\ldots:1:0)$. Using a computer one sees that the stabilizer of a general line through this point has again dimension 3. Hence the orbit has dimension $3n-3$, which is less then the dimension of the line system, $4n-6$.

4 $\mathbb{G}(1, 5) \cap H^3$

Let $L = H^3 \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^6)$ be a general 3-codimensional subspace. With our usual identification of $(\bigwedge^2 \mathbb{C}^6)^*$ with the antisymmetric matrices $\mathrm{Antisym}(6, \mathbb{C})$ its dual plane $L^* = \mathbb{P}(\lambda A + \mu B + vC) \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^6)^*$ intersects the dual Grassmannian $\mathbb{G}(1, 5)^*$, which consists of matrices of rank ≤ 4 and is a hypersurface of degree 3 by Corollary 1.7, in an irreducible cubic C^*. By Corollary 1.6 a point $(\lambda : \mu : \nu) \in C^*$ corresponds to the hyperplane $h_{(\lambda:\mu:\nu)} = \mathbb{P}(\lambda A + \mu B + \nu C)$ that is tangent to the Grassmannian at the point

$$l_{(\lambda:\mu:\nu)} := \ker(\lambda A + \mu B + \nu C) \subset \mathbb{P}_5.$$

In analogy to the former case we have

Lemma 4.1

$$\left\{ p \in \mathbb{P}_5 \;\middle|\; \begin{array}{l} \textit{through } p \textit{ passes a } \mathbb{P}_2 \\ \textit{of lines of } \mathbb{G}(1,5) \cap L \end{array} \right\} = \bigcup_{(\lambda:\mu:\nu) \in C^*} l_{(\lambda:\mu:\nu)} \subset \mathbb{P}_5$$

Proof. Since by definition the lines in $\mathbb{G}(1, 5) \cap L$ that contain p are $p \wedge q$ with ${}^t\!pAq = {}^t\!pBq = {}^t\!pCq = 0$, we see that

$$\begin{aligned} &\text{through } p \text{ passes at least a } \mathbb{P}_2 \text{ of lines of } \mathbb{G}(1,5) \cap L \\ \iff &{}^t\!pA, {}^t\!pB, {}^t\!pC \text{ are linear dependent} \\ \iff &\exists (\lambda : \mu : \nu) \in \mathbb{P}_2 \text{ with } {}^t\!p(\lambda A + \mu B + \nu C) = 0 \\ \iff &p \in \ker(\lambda A + \mu B + \nu C) = l_{(\lambda:\mu:\nu)}. \end{aligned}$$

We also note that there cannot be a $\mathbb{P}_3$ of lines of $\mathbb{G}(1,5) \cap L$ through a point p. Because if there were one, then $\dim \operatorname{span}\{{}^t\!pA, {}^t\!pB, {}^t\!pC\} = 1$, i.e. there exist two points $(\lambda : \mu : \nu), (\lambda' : \mu' : \nu') \in \mathbb{P}_2$ with

$$ {}^t\!p(\lambda A + \mu B + \nu C) = {}^t\!p(\lambda' A + \mu' B + \nu' C) = 0. $$

It follows that all the matrices

$$ (\alpha\lambda + \beta\lambda')A + (\alpha\mu + \beta\mu')B + (\alpha\nu + \beta\nu')C \quad \text{for all } (\alpha : \beta) \in \mathbb{P}_1 $$

have a non-trivial kernel. Hence the line $(\alpha\lambda + \beta\lambda' : \alpha\mu + \beta\mu' : \alpha\nu + \beta\nu')$ must lie in $L^* \cap \mathbb{G}(1,5)^* = C^*$. But this is a contradiction since the cubic C^* is irreducible. □

PROPOSITION 4.2 *The lines $l_{(\lambda:\mu:\nu)} \subset \mathbb{P}_5$ with $(\lambda : \mu : \nu) \in C^*$ do not intersect each other.*

Proof. Assume that the line $l_{(\lambda:\mu:\nu)}$ intersects the line $l_{(\lambda':\mu':\nu')}$ in the point p, i.e.

$$ p \in \ker(\lambda A + \mu B + \nu C) \cap \ker(\lambda' A + \mu' B + \nu' C) \neq 0. $$

Then

$$ p \in \ker((\alpha\lambda + \beta\lambda')A + (\alpha\mu + \beta\mu')B + (\alpha\nu + \beta\nu')C) \neq 0 \quad \text{for all } (\alpha : \beta) \in \mathbb{P}_1, $$

and the line $(\alpha\lambda + \beta\lambda' : \alpha\mu + \beta\mu' : \alpha\nu + \beta\nu')$ must be contained in the irreducible cubic C^*, which is a contradiction. □

Let us again derive a normal form:

PROPOSITION 4.3 *For a general plane $L^* = \mathbb{P}(\lambda A + \mu B + \nu C) \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^6)^*$ there exists a choice of bases of L^* and $\mathbb{C}^6$ such that*

$$ A = \begin{pmatrix} 0 & -1 & & & & \\ 1 & 0 & & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & 0 & 0 \\ & & & & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & & & & \\ 0 & 0 & & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & 0 & -1 \\ & & & & 1 & 0 \end{pmatrix} $$

$$ C = \left(\begin{array}{cc|cc|cc} 0 & 0 & -\alpha & 0 & -\gamma & 0 \\ 0 & 0 & 0 & -\alpha & -\delta & -\gamma \\ \hline \alpha & 0 & 0 & -1 & -\beta & 0 \\ 0 & \alpha & 1 & 0 & 0 & -\beta \\ \hline \gamma & \delta & \beta & 0 & 0 & 0 \\ 0 & \gamma & 0 & \beta & 0 & 0 \end{array}\right). $$

Remark 4.4 *It is also possible to derive a more symmetric normal form where all three matrices look like C only with the $\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}$ block moved along the diagonal, but this is not more useful for our computations.*

Proof of proposition 4.3. We may assume that the line $\mathbb{P}(\lambda A + \mu B) \subset L^*$ is a general line. By Proposition 3.2 there exists a choice of coordinates (corresponding to $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = \infty$) such that A and B are of the required form. Further, if we change the coordinates of $\mathbb{C}^6$ by transformations of the type

$$T = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \quad t_1, t_2, t_3 \in \mathrm{SL}(2, \mathbb{C}),$$

then A and B will stay the same by Theorem 3.5.

We write the matrix C as

$$C = \begin{pmatrix} c_1 J & -{}^tC_{21} & -{}^tC_{31} \\ C_{21} & c_2 J & -{}^tC_{32} \\ C_{31} & C_{32} & c_3 J \end{pmatrix} \quad \text{with} \quad \begin{array}{l} J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};\ c_1, c_2, c_3 \in \mathbb{C} \\ C_{21}, C_{31}, C_{32} \in \mathrm{M}(2, \mathbb{C}). \end{array}$$

We may assume that $c_1 = c_3 = 0$, $c_2 = 1$, otherwise we replace C by the matrix $1/(c_2 - c_1 - c_3)(C - c_1 A - c_3 B)$. This is possible since $c_2 - c_1 - c_3 \neq 0$, because C is general. So C looks like

$$C = \begin{pmatrix} 0 & -{}^tC_{21} & -{}^tC_{31} \\ C_{21} & J & -{}^tC_{32} \\ C_{31} & C_{32} & 0 \end{pmatrix}.$$

The generality of C ensures that the matrices C_{21} and C_{32} are invertible, so

$$T = \begin{pmatrix} \frac{1}{\alpha} C_{21} & 0 & 0 \\ 0 & \mathrm{E}_2 & 0 \\ 0 & 0 & \frac{1}{\beta} {}^tC_{32} \end{pmatrix} \quad \text{with} \quad \begin{array}{l} \alpha = \sqrt{\det C_{21}} \\ \beta = \sqrt{\det C_{32}} \end{array}$$

is of the above mentioned type and transforms C into

$$C' := {}^tT^{-1} C T^{-1} = \begin{pmatrix} 0 & -\alpha \mathrm{E}_2 & -{}^t\overline{C} \\ \alpha \mathrm{E}_2 & J & -\beta \mathrm{E}_2 \\ \overline{C} & \beta \mathrm{E}_2 & 0 \end{pmatrix} \quad \text{with } \overline{C} := \alpha\beta C_{32}^{-1} C_{31} C_{21}^{-1}.$$

This matrix will be transformed under

$$T = \begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & {}^tt & 0 \\ 0 & 0 & t^{-1} \end{pmatrix} \quad \text{with } t \in \mathrm{SL}(2, \mathbb{C})$$

into

$$ {}^tT^{-1}C'T^{-1} = \begin{pmatrix} 0 & -\alpha \mathrm{E}_2 & -{}^tt^t\overline{C}t \\ \alpha \mathrm{E}_2 & J & -\beta \mathrm{E}_2 \\ {}^tt\overline{C}t & \beta \mathrm{E}_2 & 0 \end{pmatrix}. $$

So, all that remains to show is: Given a general matrix $\overline{C} \in \mathrm{M}(2,\mathbb{C})$ there is a matrix $t \in \mathrm{SL}(2,\mathbb{C})$ such that

$$ {}^tt\overline{C}t = \begin{pmatrix} \gamma & \delta \\ 0 & \gamma \end{pmatrix}. $$

If

$$ \overline{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & -\frac{c_{21}}{c_{11}} \\ 0 & 1 \end{pmatrix} $$

then

$$ \overline{C}' = {}^tt\overline{C}t = \begin{pmatrix} c_{11} & c_{12} - c_{21} \\ 0 & \frac{\det \overline{C}}{c_{11}} \end{pmatrix} $$

and an additional transformation by

$$ t = \begin{pmatrix} \frac{\sqrt[4]{\det \overline{C}}}{\sqrt{c_{11}}} & 0 \\ 0 & \frac{\sqrt{c_{11}}}{\sqrt[4]{\det \overline{C}}} \end{pmatrix} $$

takes $\overline{C}$ into the desired form

$$ {}^tt\overline{C}'t = \begin{pmatrix} \sqrt{\det \overline{C}} & c_{12} - c_{21} \\ 0 & \sqrt{\det \overline{C}} \end{pmatrix}. \qquad \square $$

Remark 4.5 *In terms of this coordinates the cubic $C^* \subset L^*$ is given as*

$$ \lambda^2\mu + \mu^2\lambda + \lambda\mu\nu - (\gamma^2+\beta^2)\lambda\nu^2 - (\alpha^2+\gamma^2)\mu\nu^2 + (\alpha\beta\delta - \gamma^2)\nu^3. $$

One checks that the cubic is smooth for general $\alpha, \beta, \gamma, \delta$.

Now we start to determine the automorphism group of $\mathbb{G}(1,5)\cap L$. A given automorphism

$$ \varphi \in \mathrm{Aut}(\mathbb{G}(1,5)\cap L) = \mathrm{Aut}(L, \mathbb{P}(\bigwedge^2\mathbb{C}^6)) \cap \mathrm{Aut}(\mathbb{G}(1,5), \mathbb{P}(\bigwedge^2\mathbb{C}^6)) $$

induces a dual automorphism φ^* on the dual projective space $\mathbb{P}(\bigwedge^2\mathbb{C}^6))^*$ that preserves L^* and the dual Grassmannian $\mathbb{G}(1,5)^*$, i.e.

$$ \varphi^* \in \mathrm{Aut}(L^*, \mathbb{P}(\bigwedge^2\mathbb{C}^6)^*) \cap \mathrm{Aut}(\mathbb{G}(1,5)^*, \mathbb{P}(\bigwedge^2\mathbb{C}^6)^*). $$

In particular, φ^* induces a projective transformation of L^* preserving C^*. But a smooth cubic has only finitely many automorphisms that are induced by a projective linear transformation [BK, 7.3].

To find all automorphisms of $\mathbb{G}(1,5) \cap L$ that induce the identity on L^*, we look for the $T \in \mathbb{P}\mathrm{GL}(6,\mathbb{C})$ such that

$$^tT^{-1}(\lambda A + \mu B + \nu C)T^{-1} \in \mathbb{C} \cdot (\lambda A + \mu B + \nu C) \quad \text{for all } \lambda, \mu, \nu \in \mathbb{C}.$$

It suffices to check this for $(\lambda,\mu,\nu) = (1,0,0)$, $(0,1,0)$, and $(0,0,1)$. If we normalize the representation of T in $\mathrm{GL}(6,\mathbb{C})$ by $\det T = 1$, we know from the previous section that $^tT^{-1}AT^{-1} = \mathbb{C} \cdot A$ and $^tT^{-1}BT = \mathbb{C} \cdot B$ is equivalent to

$$T = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \quad \text{with } t_1, t_2, t_3 \in \mathrm{SL}(2,\mathbb{C}).$$

Furthermore, we compute

$$^tT^{-1}CT^{-1} = \begin{pmatrix} 0 & -\alpha\, {}^tt_1^{-1}t_2^{-1} & -{}^tt_1^{-1}\left(\begin{smallmatrix}\gamma & 0\\ \delta & \gamma\end{smallmatrix}\right)t_3^{-1} \\ \alpha\, {}^tt_2^{-1}t_1^{-1} & 0 & -\beta\, {}^tt_2^{-1}t_3^{-1} \\ {}^tt_3^{-1}\left(\begin{smallmatrix}\gamma & \delta\\ 0 & \gamma\end{smallmatrix}\right)t_1^{-1} & \beta\, {}^tt_3^{-1}t_2^{-1} & 0 \end{pmatrix},$$

so that $^tT^{-1}CT^{-1} = \vartheta \cdot C$ iff $t_1 = \frac{1}{\vartheta}{}^tt_2^{-1} = t_3 =: t$ and

$$^tt^{-1}\begin{pmatrix} \gamma & \delta \\ 0 & \gamma \end{pmatrix} t^{-1} = \vartheta \begin{pmatrix} \gamma & \delta \\ 0 & \gamma \end{pmatrix}.$$

Because of $\det t_1 = \det t_2 = 1$, ϑ must be either 1 or -1. Setting

$$t = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow t^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

the last condition together with $\det t = 1$ requires that the following polynomials vanish:

$$\begin{gathered} (d^2 + c^2 - \vartheta)\gamma - dc\delta,\ (db + ac)\gamma - (\vartheta - ad)\delta \\ (db + ac)\gamma - bc\delta,\ (b^2 + a^2 - \vartheta)\gamma - ba\delta,\ ad - bc - 1 \end{gathered}$$

The Gröbner basis of the ideal generated by these polynomials with respect to the lexicographical order $\gamma > \delta > a > b > c > d$ can be computed for $\vartheta = 1$ as

$$\gamma a + \delta c - \gamma d,\ b + c,\ ad + c^2 - 1,$$

so that

$$t = \begin{pmatrix} a & \frac{\gamma}{\delta}(a-d) \\ \frac{\gamma}{\delta}(d-a) & d \end{pmatrix} \quad \text{with } \det t = 1.$$

For $\vartheta = -1$ we get as the Gröbner basis

$$\delta, a+d, -c+b, d^2+c^2+1.$$

Since in the general case $\delta \neq 0$, this gives no further automorphisms.

The one-dimensional subgroup of $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$ consisting of elements like t above acts on $\mathbb{P}_1$ with the two fixed points $(-\delta \pm \sqrt{\delta^2 - 4\gamma^2} : 2\gamma)$. Hence it is conjugate to the one-dimensional subgroup of $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$ that acts on $\mathbb{P}_1$ with the fixed points 0 and ∞. Now this subgroup consists of the invertible diagonal matrices of $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$, so it is isomorphic to $\mathbb{C}^*$. Therefore we have shown

Theorem 4.6 *The component of the automorphism group of* $\mathbb{G}(1,5) \cap H^3$ *containing the identity is isomorphic to* $\mathbb{C}^*$. *The quotient of* $\mathrm{Aut}(\mathbb{G}(1,5) \cap H^3)$ *by this component is a subgroup of the finite group of projective automorphisms of a smooth cubic in* $\mathbb{P}_2$.

5 $\mathbb{G}(1,2n) \cap H$

The hyperplane H is given by an element $A \in (\bigwedge^2 \mathbb{C}^{2n+1})^*$ which can be thought of as an antisymmetric matrix of size $2n+1$. Since antisymmetric matrices have an even rank, the general H corresponds to an A of rank $2n$. The one dimensional kernel of A as a point of $\mathbb{P}_{2n}$ is called the center c of H.

The center plays a special role in the geometry of the line system $\mathbb{G}(1,2n) \cap H$ in $\mathbb{P}_{2n}$.

Proposition 5.1 *Every line through the center of the line system* $\mathbb{G}(1,2n) \cap H$ *is in the line system. The center is the only point with this property.*

Moreover, if the line $l \not\ni c$ *belongs to the line system, so does every line in the plane spanned by the line* l *and the center* c.

Proof. The line $c \wedge p$ through the center will be in the line system if ${}^t cAp = 0$. But c is the kernel of A, so this is true. On the other hand, if $\overline{c}$ is a point such that every line through it belongs to the line system, then ${}^t\overline{c}Ap = 0$ for all $p \in \mathbb{P}_{2n}$. Hence $\overline{c}$ must be in the kernel of A, and therefore $\overline{c} = c$.

Let the line $l = p \wedge q$ be in the line system. All the lines in the plane spanned by l and c – except the lines through c itself – can be written as

$$(\alpha p + \beta c) \wedge (\lambda q + \mu c) \quad \text{for } (\alpha : \beta), (\lambda : \mu) \in \mathbb{P}_1.$$

These will be in the line system since

$$\begin{aligned}(\alpha {}^t p + \beta {}^t c) A (\lambda q + \mu c) &= \alpha\lambda {}^t pAq + \alpha\mu {}^t pAc + \beta\lambda {}^t cAq + \beta\mu {}^t cAc \\ &= \alpha\lambda {}^t pAq = 0.\end{aligned}$$

□

Let us for a moment look at the projection $\mathbb{P}(\mathbb{C}^{2n+1}/c)$ of $\mathbb{P}_{2n}$ from the center c. This projection maps all lines in a plane through the center – except the lines through c itself – to only one line. Hence we get a codimension one line system inside $\mathbb{P}_{2n-1}$. In fact, it is of the form $\mathbb{G}(1, 2n-1) \cap \overline{H}$, which is most easily seen in coordinates. We choose a basis $(e_0, \ldots, e_{2n})$ of $\mathbb{C}^{2n+1}$ such that the hyperplane H is given by the matrix

$$A = \left(\begin{array}{cc|c} 0 & -\mathrm{E}_n & 0 \\ \mathrm{E}_n & 0 & \vdots \\ & & 0 \\ \hline 0 \cdots 0 & & 0 \end{array}\right) \in \mathrm{Antisym}(2n+1, \mathbb{C}).$$

The center of H is $c = \mathbb{P}(e_{2n})$. So the projected line system is $\mathbb{G}(1, 2n-1) \cap \overline{H}$, where $\overline{H}$ is given by the matrix A with the last row and column deleted.

This description helps to determine the automorphism group of $\mathbb{G}(1, 2n) \cap H$.

First of all, any of the automorphism must – as a transformation $T \in \mathbb{P}\mathrm{GL}(2n+1, \mathbb{C})$ – preserve the center, i.e. $Tc = c$. Therefore it induces a transformation $\overline{T}$ of the projected space $\mathbb{P}(\mathbb{C}^{2n+1}/c)$. This induced transformation $\overline{T}$ has to preserve the projected line system $\mathbb{G}(1, 2n-1) \cap \overline{H}$. Since this case has been treated in Section 2, we know that if we normalize $\overline{T}$ by $\det \overline{T} = 1$, then $\overline{T} \in \mathrm{Sp}(2n, \mathbb{C})$. Therefore T must have been of the form

$$T = \left(\begin{array}{c|c} & 0 \\ \overline{T} & \vdots \\ & 0 \\ \hline a_0 \ \cdots \ a_{2n-1} & b \end{array}\right) \quad \text{with} \quad \begin{array}{l} \overline{T} \in \mathrm{Sp}(2n, \mathbb{C}) \\ a_i \in \mathbb{C} \\ b \in \mathbb{C}^*. \end{array}$$

One immediately checks that ${}^tT^{-1}AT = A$, so that the automorphism group as a subset of $\mathbb{P}\mathrm{GL}(2n+1, \mathbb{C})$ consists of all elements of the above type. Since we normalized $\overline{T}$, we have up to multiplication by -1 an unique representative in the class of $\mathbb{P}\mathrm{GL}(2n+1, \mathbb{C})$.

A small computation shows that

$$N := \left\{ \left(\begin{array}{c|c} & 0 \\ \mathrm{E}_{2n} & \vdots \\ & 0 \\ \hline a_0 \ \cdots \ a_{2n-1} & 1 \end{array}\right) \mid a_i \in \mathbb{C} \right\} \subset \mathrm{Aut}(\mathbb{G}(1, 2n) \cap H)$$

is a normal subgroup which is isomorphic to $(\mathbb{C}^{2n}, +)$.

Collecting everything together we have

Proposition 5.2 *The automorphism group of* $\mathbb{G}(1, 2n) \cap H$ *for a general hyperplane* $H \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{2n+1})$ *is an extention of* $\mathrm{Sp}(2n, \mathbb{C}) \times \mathbb{C}^*/\{1, -1\}$ *by*

$(\mathbb{C}^{2n}, +)$ *and is isomorphic to the group*

$$\left\{ \left(\begin{array}{ccc|c} & & & 0 \\ & T & & \vdots \\ & & & 0 \\ \hline a_0 & \cdots & a_{2n-1} & b \end{array} \right) \middle| \begin{array}{l} T \in \mathrm{Sp}(2n, \mathbb{C}) \\ a_i \in \mathbb{C} \\ b \in \mathbb{C}^* \end{array} \right\} / \{1, -1\}.$$

The action of the automorphism group on the line system is described by the following

Proposition 5.3 *The action of the automorphism group of* $\mathbb{G}(1, 2n) \cap H$ *on the lines of* $\mathbb{G}(1, 2n) \cap H$ *has two orbits:*

1. *the lines containing the center* c
2. *the lines that do not.*

Proof. Since all the automorphisms preserve the center any orbit will be contained in these two sets.

First we show that the lines containing c form one orbit. For two lines $c \wedge p$ and $c \wedge q$, we may assume $p, q \in \mathbb{P}(\mathbb{C}^{2n} \times 0)$. Take a $\overline{T} \in \mathrm{Sp}(2n, \mathbb{C})$ that maps p to q. The trivial extention of $\overline{T}$ to $T \in \mathrm{SL}(2n+1, \mathbb{C})$ will take $c \wedge p$ to $c \wedge q$.

The other lines will form the second orbit since any line not containing the center can be pushed into the hyperplane $\mathbb{P}(\mathbb{C}^{2n} \times 0)$ by a transformation with an element of the normal subgroup N. There one can use the transitive action of the $\mathrm{Aut}(\mathbb{G}(1, 2n-1) \cap H)$ subgroup to show that all these lines can be mapped onto each other. □

6 $\mathbb{G}(1, 2n) \cap H^2$

Let $L = H^2$ be a 2-codimensional linear subspace of $\mathbb{P}(\bigwedge^2 \mathbb{C}^{2n+1})$. We want to study the linear line system $\mathbb{G}(1, 2n) \cap L$. To L corresponds the line $L^* = \mathbb{P}(\lambda A - \mu B) \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{2n+1})^*$ of the hyperplanes $H_{(\lambda:\mu)} = \mathbb{P}(\lambda A - \mu B)$ containing L. We identify as always $(\bigwedge^2 \mathbb{C}^{2n+1})^*$ with the antisymmetric matrices $\mathrm{Antisym}(2n+1, \mathbb{C})$. The locus of antisymmetric matrices of corank 3 in $\mathrm{Antisym}(2n+1, \mathbb{C})$ is 3-codimensional by Corollary 1.7. Therefore a line L^* may be called general if it does not intersect it. Hence for the general line L^* the antisymmetric matrices $\lambda A - \mu B$ corresponding to the hyperplanes $H_{(\lambda:\mu)}$ have all corank 1. So each of the hyperplane sections $\mathbb{G}(1, 2n) \cap H_{(\lambda:\mu)}$ has a unique center $c_{(\lambda:\mu)} \in \mathbb{P}_{2n}$ by Proposition 5.1. These centers play an important role in the geometry of $\mathbb{G}(1, 2n) \cap L$.

Proposition 6.1 *The centers* $c_{(\lambda:\mu)}$ *are those points of* $\mathbb{P}_{2n}$ *through which there passes a* $\mathbb{P}_{2n-2}$ *of lines of the line system* $\mathbb{G}(1, 2n) \cap L$*. Through all the other points of* $\mathbb{P}_{2n}$ *passes only a* $\mathbb{P}_{2n-3}$ *of lines.*

Proof. The lines of the line system through a point $p \in \mathbb{P}_{2n}$ are $p \wedge q$ with ${}^t pAq = {}^t pBq = 0$. So we need to show that ${}^t pA$ and ${}^t pB$ are linear dependent iff p is a center of a hyperplane $H_{(\lambda:\mu)}$. Now ${}^t pA$ and ${}^t pB$ are linear dependent precisely if there exists a $(\lambda:\mu) \in \mathbb{P}_1$ with $0 = \lambda {}^t pA - \mu {}^t pB = {}^t p(\lambda A - \mu B)$, i.e. p is the kernel of $\lambda A - \mu B$, which is by definition the center of $H_{(\lambda:\mu)}$. □

Remark 6.2 *Any line that contains two centers is a member of the line system* $\mathbb{G}(1, N) \cap L$.

Proof. If the line contains the centers $c_{(\alpha:\beta)}$ and $c_{(\lambda:\mu)}$, it is contained in the hyperplanes $H_{(\alpha:\beta)}$ and $H_{(\lambda:\mu)}$ by Proposition 5.1 and therefore in their intersection $L = H_{(\alpha:\beta)} \cap H_{(\lambda:\mu)}$. □

Next we want to know more about the curve $c_{(\lambda:\mu)}$.

Proposition 6.3 *Let A, B be two antisymmetric matrices of size $2n+1$ such that every non-zero linear combination of them has corank 1. Then the map*

$$\begin{array}{rccc} c: & \mathbb{P}_1 & \longrightarrow & \mathbb{P}_{2n} \\ & (\lambda:\mu) & \longmapsto & \ker(\lambda A - \mu B) \end{array}$$

is a parametrisation of a rational normal curve of degree n.

Proof. (compare [SR, X,4.3] for $n = 2$.) First we show that the map is injective. If it is not, there are two points of $\mathbb{P}_1$ with the same image. We may assume that this is the case for $(1:0)$ and $(0:1)$, i.e. A and B have the same kernel, say e_0. Writing A and B in a basis with e_0 as first element, we have

$$A = \begin{pmatrix} 0 & \cdots\ 0 \\ \vdots & \widetilde{A} \\ 0 & \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & \cdots\ 0 \\ \vdots & \widetilde{B} \\ 0 & \end{pmatrix} \quad \text{with } \widetilde{A}, \widetilde{B} \in \mathrm{Antisym}(2n, \mathbb{C}).$$

Since $\det(\lambda\widetilde{A} - \mu\widetilde{B})$ is a homogeneous polynomial of degree $2n$, there exist a $(\lambda' : \mu') \in \mathbb{P}_1$ with $\det(\lambda'\widetilde{A} - \mu'\widetilde{B}) = 0$. But then $\lambda' A - \mu' B$ has corank at least two, which contradicts our assumption.

Secondly, we proof that the map is of maximal rank everywhere. If it is not, we may assume that it is not maximal at $(1:0)$. Restricting to the chart $\lambda = 1$, this means $c'(0) = 0$. Now from

$$\begin{aligned} &(A - \mu B)c(\mu) = 0 \\ \Longrightarrow\ &Ac'(\mu) - Bc(\mu) - \mu Bc'(\mu) = 0 \\ \Longrightarrow\ &Ac'(0) - Bc(0) = 0 \end{aligned}$$

$Bc(0) = 0$ follows. Therefore A and B have the same kernel $c(0)$, and we are back in the above chain of arguments.

Finally, we have to show that the embedding c is of degree n. For this we give an explicit form of the map. Recall [B, 5.2] that the determinant of an antisymmetric matrix $C = (c_{ij})$ of size $2n$ is the square of the irreducible Pfaffian polynomial $\operatorname{Pf} C$,

$$\operatorname{Pf} C := \sum_{\sigma} \operatorname{sgn}(\sigma) c_{\sigma(1)\sigma(2)} \dots c_{\sigma(2n-1)\sigma(2n)},$$

where σ runs through all permutations $\mathrm{S}(2n)$ with $\sigma(2i-1) < \sigma(2i)$ for $i = 1 \dots n$ and $\sigma(2i) < \sigma(2i+2)$ for $i = 1 \dots n-1$.

Let $c_i(\lambda : \mu)$ denote $(-1)^i$-times the Pfaffian of the matrix $\lambda A - \mu B$ with the i-th row and column deleted. Then the $c_i(\lambda : \mu)$ are irreducible polynomials of degree n and by a straightforward but messy computation one can check that $(c_0(\lambda : \mu) : \dots : c_{2n}(\lambda : \mu))$ is the kernel of $\lambda A - \mu B$. Therefore $c = (c_0 : \dots : c_{2n})$, which shows that c is a degree n embedding of $\mathbb{P}_1$. $\square$

After we have determined the special points in $\mathbb{P}_{2n}$ of the line system $\mathbb{G}(1, 2n) \cap L$, we are nearly ready to compute its automorphism group. It remains to give a normal form for the line $L^* \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{2n+1})$ to make computations easier. This normal form was found by Donagi [D, 2.2]. But he did not give a proof for it since his main interest was lines in $\mathbb{P}_{2n-1}$ and not in $\mathbb{P}_{2n}$. So we give the proof here.

PROPOSITION 6.4 *Let L^* be a line in $\mathbb{P}(\bigwedge^2 \mathbb{C}^{2n+1})^*$ such that the antisymmetric matrices corresponding to the points of L^* have all corank 1. Then there exists a basis $(e_0, \dots, e_{2n})$ of $\mathbb{C}^{2n+1}$ such that the line can be taken as $L^* = \mathbb{P}(\lambda A - \mu B)$ with the matrices*

$$A = \left(\begin{array}{cc|c} 0 & -\mathrm{E}_n & 0 \\ \mathrm{E}_n & 0 & \vdots \\ & & 0 \\ \hline \multicolumn{2}{c|}{0 \cdots 0} & 0 \end{array}\right) \quad \textit{and} \quad B = \left(\begin{array}{c|c|c} 0 & \begin{matrix} 0 \\ \vdots \end{matrix} & -\mathrm{E}_n \\ \hline 0 \cdots & 0 & \cdots 0 \\ \hline \mathrm{E}_n & \begin{matrix} \vdots \\ 0 \end{matrix} & 0 \end{array}\right).$$

Proof. Let A and B any two matrices of L^* in an arbitrary basis. We will adjust the basis in three steps to achieve the required form for A and B.

1^{st} *Step:* We know that the map

$$\begin{array}{rccc} c: & \mathbb{P}_1 & \longrightarrow & \mathbb{P}_{2n} \\ & (\lambda : \mu) & \longmapsto & \ker(\lambda A - \mu B) \end{array}$$

is a parametrisation of a rational normal curve of degree n. Modulo projective transformations of $\mathbb{P}_1$ and $\mathbb{P}_{2n}$ such parametrisations are all the same. So we can pick a basis of $\mathbb{P}_1$ and $n+1$ linear independent vectors $e_n, \ldots, e_{2n}$ of $\mathbb{C}^{2n+1}$ such that

$$\begin{array}{rccc} c: & \mathbb{P}_1 & \longrightarrow & \mathbb{P}_{2n} \\ & (\lambda:\mu) & \longmapsto & \mathbb{P}\left(\sum\limits_{i=0}^{n} \lambda^i \mu^{n-i} e_{n+i}\right). \end{array}$$

Extending $(e_n, \ldots, e_{2n})$ to a basis $(e_0, \ldots, e_{2n})$ of $\mathbb{C}^{2n+1}$ and denoting by $a_0, \ldots, a_{2n}$ resp. $b_0, \ldots, b_{2n}$ the columns of A resp. B, the fact that $c(\lambda:\mu)$ is the kernel of $\lambda A - \mu B$ for all $(\lambda:\mu) \in \mathbb{P}_1$ has the following consequences for A and B:

$$\begin{aligned} & (\lambda A - \mu B)\left(\sum_{i=0}^{n} \lambda^i \mu^{n-i} e_{n+i}\right) = 0 \\ \Longrightarrow & \sum_{i=0}^{n} \lambda^{i+1}\mu^{n-i} a_{n+i} - \sum_{i=0}^{n} \lambda^i \mu^{n+1-i} b_{n+i} = 0 \\ \Longrightarrow & -\mu^{n+1} b_n + \sum_{i=0}^{n-1} \lambda^{i+1}\mu^{n-i}(a_{n+1} - b_{n+i+1}) + \lambda^{n+1} a_{2n} = 0 \\ \Longrightarrow & \, b_n = 0,\ a_{2n} = 0,\ a_{n+i} = b_{n+i+1} \quad \text{for } i = 0 \ldots n-1. \end{aligned} \tag{$*$}$$

We claim that this implies:

$$a_{n+i,n+j} = 0 \qquad \text{for } i,j = 0 \ldots n.$$

Indeed, for $1 \le i \le n$, $0 \le j \le n-1$ we have using $(*)$

$$a_{n+i,n+j} = b_{n+i,n+j+1} = -b_{n+j+1,n+i} = -a_{n+j+1,n+i-1} = a_{n+i-1,n+j+1}.$$

This shows that the $a_{n+i,n+j}$ are all the same for $i+j = \mathit{const}$, in particular $a_{n+i,n+j} = a_{n+j,n+i}$. On the other hand, by the antisymmetricity of A we have $a_{n+i,n+j} = -a_{n+j,n+i}$, and the claim follows.

Using $(*)$ again we know that A and B look in our basis like

$$A = \left(\begin{array}{cc|c} \widetilde{A} & -{}^tM & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ M & 0 & \\ \hline 0 \cdots 0 & & 0 \end{array}\right) \quad \text{and} \quad B = \left(\begin{array}{c|c|c} \widetilde{B} & \begin{matrix}0\\ \vdots\end{matrix} & -{}^tM \\ \hline 0 \cdots & 0 & \cdots 0 \\ \hline M & \begin{matrix}\vdots\\ 0\end{matrix} & 0 \end{array}\right)$$

with $\widetilde{A}, \widetilde{B} \in \mathrm{Antisym}(n, \mathbb{C})$ and $M \in \mathrm{GL}(n, \mathbb{C})$.

2^{nd} *Step:* Here we will improve the choice of $(e_0, \ldots, e_{n-1})$ to achieve $\widetilde{A} = 0$ and $M = \mathrm{E}_n$. We claim:

Let an antisymmetric matrix $A \in \mathrm{Antisym}(2n+1, \mathbb{C})$ of rank $2n$ and linear independent vectors $e_n, \ldots, e_{2n} \in \mathbb{C}^{2n+1}$ with ${}^te_i A e_j = 0$ for $n \le i, j \le 2n$ and

$Ae_{2n}=0$ be given. Then $(e_n,\dots,e_{2n})$ can be extended to a basis $(e_0,\dots,e_{2n})$ of $\mathbb{C}^{2n+1}$ such that in this basis A is given as

$$A=\left(\begin{array}{cc|c} 0 & -\mathrm{E}_n & 0 \\ & & \vdots \\ \mathrm{E}_n & 0 & 0 \\ \hline \multicolumn{2}{c|}{0\cdots 0} & 0 \end{array}\right).$$

The proof is by induction. The statement is trivial for $n=0$. Assuming the claim for $n-1$, we prove it for n. Let $W:=\bigcap_{i=1}^{n}\ker{}^te_{n+i}A$, then there exists an $e_0\in W$ with ${}^te_nAe_0=1$. If not, we would have $W=W\cap\ker{}^te_nA$ and with $e_{2n}\in\ker A$

$$\dim\operatorname{span}\{{}^te_nA,\dots,{}^te_{2n}A\}=\dim\operatorname{span}\{{}^te_{n+1}A,\dots,{}^te_{2n-1}A\}\le n-1,$$

which contradicts $\operatorname{rank}A=2n$.

Set $V:=\ker{}^te_0A\cap\ker{}^te_nA$, then $\dim V=2(n-1)+1$ and $e_{n+1},\dots,e_{2n}\in V$. Therefore the induction hypothesis can be applied to $A|_V$. Together with ${}^te_0Ae_n=1$ and ${}^te_0Av={}^te_nAv=0$ for $v\in V$ this implies the stated form of the matrix.

So up to now A and B look like

$$A=\left(\begin{array}{cc|c} 0 & -\mathrm{E}_n & 0 \\ & & \vdots \\ \mathrm{E}_n & 0 & 0 \\ \hline \multicolumn{2}{c|}{0\cdots 0} & 0 \end{array}\right)\qquad B=\left(\begin{array}{c|c|c} \widetilde{B} & \begin{matrix}0\\ \vdots\end{matrix} & -\mathrm{E}_n \\ \hline 0\cdots & 0 & \cdots 0 \\ \hline \mathrm{E}_n & \begin{matrix}\vdots\\ 0\end{matrix} & 0 \end{array}\right).$$

3^{rd} *Step:* We adjust the vectors $(e_0,\dots,e_{n-1})$ so that $\widetilde{B}=0$ and A stays the same.

We note that a transformation of $\mathbb{C}^{2n+1}$ by

$$T=\left(\begin{array}{cc|c} \mathrm{E}_n & 0 & 0 \\ & & \vdots \\ t & \mathrm{E}_n & 0 \\ \hline \multicolumn{2}{c|}{0\cdots 0} & 1 \end{array}\right)^{-1}\quad \text{with } t\in\mathrm{Sym}(n,\mathbb{C})$$

does not change A since

$${}^tT^{-1}AT^{-1}={}^tT^{-1}(AT^{-1})=\left(\begin{array}{cc|c} \mathrm{E}_n & t & 0 \\ & & \vdots \\ 0 & \mathrm{E}_n & 0 \\ \hline \multicolumn{2}{c|}{0\cdots 0} & 1 \end{array}\right)\left(\begin{array}{cc|c} -t & -\mathrm{E}_n & 0 \\ & & \vdots \\ \mathrm{E}_n & 0 & 0 \\ \hline \multicolumn{2}{c|}{0\cdots 0} & 0 \end{array}\right)=A.$$

If we denote by $\overline{t}$ resp. $|t\in\mathrm{M}(n\times n,\mathbb{C})$ the matrix that we obtain by deleting the first row resp. column of t and adding a row of zeroes below resp. a column of zeroes on the right side, we can write down the transformation of B as follows:

$$
{}^tT^{-1}BT^{-1} = {}^tT^{-1}(BT^{-1}) = \left(\begin{array}{cc|c} \mathrm{E}_n & t & 0 \\ & & \vdots \\ 0 & \mathrm{E}_n & 0 \\ \hline 0\cdots 0 & & 1 \end{array}\right) \left(\begin{array}{c|c|c} \widetilde{B}-\overline{t} & \begin{matrix}0\\ \vdots\end{matrix} & -\mathrm{E}_n \\ \hline 0 \cdots & 0 & \cdots 0 \\ \hline \mathrm{E}_n & \begin{matrix}\vdots \\ 0\end{matrix} & 0 \end{array}\right)
$$

$$
= \left(\begin{array}{c|c|c} \widetilde{B}-\overline{t}+|t & \begin{matrix}0\\ \vdots\end{matrix} & -\mathrm{E}_n \\ \hline 0 \ \cdots & 0 & \cdots \ 0 \\ \hline \mathrm{E}_n & \begin{matrix}\vdots \\ 0\end{matrix} & 0 \end{array}\right).
$$

So to finish this step, we need to show that every antisymmetric matrix $\widetilde{B} = (b_{ij}) \in \mathrm{Antisym}(n, \mathbb{C})$ can be written as $\overline{t} - |t$ for a symmetric matrix $t \in \mathrm{Sym}(n, \mathbb{C})$. The entries of $\overline{t} - |t$ are

$$(t_{i+1,j} - t_{i,j+1})_{ij}$$

where $t_{n+1,i} := t_{i,n+1} := 0$ for all $i = 1 \dots n$. Obviously, $\overline{t} - |t$ is antisymmetric. We set $t_{1i} := t_{i1} := 0$ for $i = 1 \dots n$ and define recursively for j from n down to 2

$$t_{i+1,j} := t_{i,j+1} + b_{ij} \quad \text{for } i = 1 \dots j-1.$$

Then by the symmetry of t the whole matrix t is defined and $\overline{t} - |t = B$ $\square$

For the linear system in normal form the rational curve of centers has the parametrisation

$$
c(\lambda:\mu) = \ker(\lambda A - \mu B) = \ker \left(\begin{array}{c|c} \Large 0 & \begin{matrix} -\lambda & \mu & & \\ & \ddots & \ddots & \\ & & -\lambda & \mu \end{matrix} \\ \hline \begin{matrix} \lambda & & \\ -\mu & \ddots & \\ & \ddots & \lambda \\ & & -\mu \end{matrix} & \Large 0 \end{array}\right)
$$

$$= (0 : \dots : 0 : \mu^n : \mu^{n-1}\lambda : \dots : \lambda^n).$$

The $\mathbb{P}_{2n-2}$ of lines through a center $c_{(\lambda:\mu)}$ is given by

$$c_{(\lambda:\mu)} \wedge q \quad \text{where } q \in \mathbb{P}_{2n} \text{ with } {}^tc_{(\lambda:\mu)}Aq = {}^tc_{(\lambda:\mu)}Bq = 0,$$

i.e. q must be an element of the hyperplane $h_{(\lambda:\mu)} \in \mathbb{P}_{2n}^*$

$$\begin{aligned} h_{(\lambda:\mu)} &= \ker {}^t c_{(\lambda:\mu)} A \cap {}^t c_{(\lambda:\mu)} B \\ &= \ker \begin{pmatrix} \mu^n & \mu^{n-1}\lambda & \cdots & \mu\lambda^{n-1} & 0 & \cdots & 0 \\ \mu^{n-1}\lambda & \mu^{n-2}\lambda^2 & \cdots & \lambda^n & 0 & \cdots & 0 \end{pmatrix} \\ &= \ker \begin{pmatrix} \mu^{n-1} & \mu^{n-2}\lambda & \cdots & \lambda^{n-1} & 0 & \cdots & 0 \end{pmatrix}. \end{aligned}$$

So the hyperplanes $h_{(\lambda:\mu)}$, which are traced out by the $\mathbb{P}_{2n-2}$ of lines through the centers, give rise to a rational normal curve of degree $n-1$ in the space of hyperplanes containing the center curve. That the hyperplanes $h_{(\lambda:\mu)}$ contain the center curve could already be seen from the Remark 6.2, by which $h_{(\lambda:\mu)}$ must contain any line connecting $c_{(\lambda:\mu)}$ with any other point of the center curve.

Now we are ready to study the automorphism group of $\mathbb{G}(1,2n)\cap L$.

Any automorphism $T \in \mathrm{Aut}(\mathbb{G}(1,2n)\cap L) \subseteq \mathbb{P}\mathrm{GL}(2n+1,\mathbb{C})$ has to map the center curve onto itself and also the projective space $P \cong \mathbb{P}_n$ spanned by the center curve onto itself. It is known [H, 10.12] that the group of automorphisms of $\mathbb{P}_n$ fixing a rational normal curve of degree n is isomorphic to $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$. If the rational normal curve is given by

$$\begin{array}{cccc} c: & \mathbb{P}_1 & \longrightarrow & \mathbb{P}_n \\ & (\lambda:\mu) & \longmapsto & (\mu^n : \mu^{n-1}\lambda : \ldots : \lambda^n), \end{array}$$

this isomorphism $\mathbb{P}\mathrm{GL}(2,\mathbb{C}) \cong \mathrm{Aut}(c,\mathbb{P}_n)$ maps

$$t = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{P}\mathrm{GL}(2,\mathbb{C})$$

to $t_{n+1} \in \mathbb{P}\mathrm{GL}(n+1,\mathbb{C})$ where t_{n+1} is the unique matrix such that

$$t_{n+1} \begin{pmatrix} \mu^n \\ \mu^{n-1}\lambda \\ \vdots \\ \lambda^n \end{pmatrix} = \begin{pmatrix} (d\mu + c\lambda)^n \\ (d\mu + c\lambda)^{n-1}(b\mu + a\lambda) \\ \vdots \\ (b\mu + a\lambda)^n \end{pmatrix};$$

for example

$$t_2 = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \qquad t_3 = \begin{pmatrix} d^2 & 2cd & c^2 \\ bd & ad+bc & ac \\ b^2 & 2ab & a^2 \end{pmatrix}.$$

Applying this to the center curve restricts the form of the transformation T to

$$T = \left(\begin{array}{c|c} * & 0 \\ \hline * & t_{n+1} \end{array}\right).$$

We know further that if T maps $c_{(\lambda:\mu)}$ to $c_{(a\lambda+b\mu:c\lambda+d\mu)}$, then it must map the hyperplane $h_{(\lambda:\mu)}$ to $h_{(a\lambda+b\mu:c\lambda+d\mu)}$. Therefore it induces also an automorphism on the rational curve h of degree $n-1$ in the dual projective space $(\mathbb{P}_{2n}/P)^*$ of hyperplanes containing P. Hence T must be of the form

$$T = \left(\begin{array}{c|c} \alpha\,{}^t t_n^{-1} & 0 \\ \hline * & t_{n+1} \end{array}\right) \quad \text{with } \alpha \in \mathbb{C}^*.$$

We make the following *claim*:

$$T = \left(\begin{array}{c|c} {}^t t_n^{-1} & 0 \\ \hline 0 & t_{n+1} \end{array}\right) \in \mathbb{P}\mathrm{GL}(2n+1,\mathbb{C})$$

is an automorphism of the linear system $\mathbb{G}(1,2n)\cap L$.

Proof. We need to check that for every $t \in \mathbb{P}\mathrm{GL}(2,\mathbb{C})$

$${}^t T^{-1}(\lambda A - \mu B)T^{-1} \in \operatorname{span}\{A,B\}$$

for all $\lambda,\mu \in \mathbb{C}$. Since $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$ is a group, this is equivalent to the statement that for every $t \in \mathbb{P}\mathrm{GL}(2,\mathbb{C})$

$${}^t T(\lambda A - \mu B)T \in \operatorname{span}\{A,B\}$$

for all $\lambda,\mu \in \mathbb{C}$. Because of the linearity it is enough to do this for $(\lambda,\mu)=(1,0)$ and $(0,-1)$. Denoting by $\overline{t_{n+1}}$ resp. $\underline{t_{n+1}}$ the matrix t_{n+1} with the first resp. last row deleted, we compute:

$${}^t T(AT) = \left(\begin{array}{c|c} t_n^{-1} & 0 \\ \hline 0 & {}^t t_{n+1} \end{array}\right)\left(\begin{array}{c|c} 0 & -\underline{t_{n+1}} \\ \hline \begin{array}{c} {}^t t_n^{-1} \\ 0\cdots 0 \end{array} & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & -t_n^{-1}\underline{t_{n+1}} \\ \hline {}^t(t_n^{-1}\underline{t_{n+1}}) & 0 \end{array}\right)$$

$${}^t T(BT) = \left(\begin{array}{c|c} t_n^{-1} & 0 \\ \hline 0 & {}^t t_{n+1} \end{array}\right)\left(\begin{array}{c|c} 0 & -\overline{t_{n+1}} \\ \hline \begin{array}{c} 0\cdots 0 \\ {}^t t_n^{-1} \end{array} & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & -t_n^{-1}\overline{t_{n+1}} \\ \hline {}^t(t_n^{-1}\overline{t_{n+1}}) & 0 \end{array}\right).$$

So, if we show

$$\underline{t_{n+1}} = d\left(t_n{}^{0}_{0}\right) + c\left({}^{0}_{0}t_n\right) \Longrightarrow t_n^{-1}\underline{t_{n+1}} = d\left(\mathrm{E}_n{}^{0}_{0}\right) + c\left({}^{0}_{0}\mathrm{E}_n\right)$$

$$\overline{t_{n+1}} = b\left(t_n{}^{0}_{0}\right) + a\left({}^{0}_{0}t_n\right) \Longrightarrow t_n^{-1}\overline{t_{n+1}} = b\left(\mathrm{E}_n{}^{0}_{0}\right) + a\left({}^{0}_{0}\mathrm{E}_n\right),$$

where ${}^0_0$ stands for adding a column of zeroes, then

$$
\begin{aligned}
{}^tTAT &= dA + cB \\
{}^tTBT &= bA + aB.
\end{aligned}
$$

To show the equality for $\underline{t_{n+1}}$ note that on the one hand $\underline{t_{n+1}}$ is the unique matrix with

$$
\underline{t_{n+1}} \begin{pmatrix} \mu^n \\ \mu^{n-1}\lambda \\ \vdots \\ \lambda^n \end{pmatrix} = \begin{pmatrix} (d\mu + c\lambda)^n \\ (d\mu + c\lambda)^{n-1}(b\mu + a\lambda) \\ \vdots \\ (d\mu + c\lambda)(b\mu + a\lambda)^{n-1} \end{pmatrix}
$$

and on the other hand

$$
\begin{pmatrix} (d\mu + c\lambda)^n \\ (d\mu + c\lambda)^{n-1}(b\mu + a\lambda) \\ \vdots \\ (d\mu + c\lambda)(b\mu + a\lambda)^{n-1} \end{pmatrix} = (d\mu + c\lambda) \begin{pmatrix} (d\mu + c\lambda)^{n-1} \\ (d\mu + c\lambda)^{n-2}(b\mu + a\lambda) \\ \vdots \\ (b\mu + a\lambda)^{n-1} \end{pmatrix}
$$

$$
= (d\mu + c\lambda) t_n \begin{pmatrix} \mu^{n-1} \\ \mu^{n-2}\lambda \\ \vdots \\ \lambda^{n-1} \end{pmatrix} = dt_n \begin{pmatrix} \mu^n \\ \mu^{n-1}\lambda \\ \vdots \\ \mu\lambda^{n-1} \end{pmatrix} + ct_n \begin{pmatrix} \mu^{n-1}\lambda \\ \mu^{n-2}\lambda^2 \\ \vdots \\ \lambda^n \end{pmatrix}
$$

$$
= \left(d \left(t_n {}^0_0 \right) + c \left({}^0_0 t_n \right) \right) \begin{pmatrix} \mu^n \\ \mu^{n-1}\lambda \\ \vdots \\ \lambda^n \end{pmatrix}.
$$

Of course, the proof for $\overline{t_{n+1}} = b\left(t_n {}^0_0\right) + a\left({}^0_0 t_n\right)$ is analogous. □

Given any automorphism of the line system $\mathbb{G}(1,2n)\cap L$ we can compose it with one of the above automorphisms such that the composition fixes the center curve pointwise. So, we can focus our attention to automorphisms of the last type.

Lemma 6.5 *All automorphisms of $\mathbb{G}(1,2n)\cap L$ that fix the center curve pointwise are of the form*

$$
T = \begin{pmatrix} \alpha \mathrm{E}_n & 0 \\ S & \mathrm{E}_{n+1} \end{pmatrix} \text{ with } \alpha \in \mathbb{C}^*,\ S \in \mathrm{M}((n+1)\times n, \mathbb{C}),
$$

where the matrix $S \in \mathrm{M}((n+1)\times n, \mathbb{C})$ has the same entries along the minor diagonals, i.e. $s_{ij} = s_{kl}$ for $i+j = k+l$.

As a group these matrices are isomorphic to the semi direct product $\mathbb{C}^{2n} \ltimes \mathbb{C}^*$, $(s,\alpha)\cdot(s',\alpha') = (\alpha' s + s', \alpha\alpha')$.

Proof. We need only to check the property of S and the group structure. T is an automorphism iff

$$ {}^tT^{-1}AT^{-1}, {}^tT^{-1}BT^{-1} \in \operatorname{span}\{A, B\}. $$

The inverse of T is

$$ T^{-1} = \begin{pmatrix} \frac{1}{\alpha}\mathrm{E}_n & 0 \\ -\frac{1}{\alpha}S & \mathrm{E}_{n+1} \end{pmatrix}. $$

Now if $\overline{S}$ resp. $\underline{S} \in \mathrm{M}(n \times n, \mathbb{C})$ denote the matrix S with the first resp. last row deleted, then

$$ {}^tT^{-1}(AT^{-1}) = \begin{pmatrix} \frac{1}{\alpha}\mathrm{E}_n & -\frac{1}{\alpha}{}^tS \\ 0 & \mathrm{E}_{n+1} \end{pmatrix} \left(\begin{array}{cc|c} \frac{1}{\alpha}\underline{S} & -\mathrm{E}_n & 0 \\ & & \vdots \\ \frac{1}{\alpha}\mathrm{E}_n & 0 & 0 \\ \hline 0 \cdots 0 & & 0 \end{array}\right) $$

$$ = \left(\begin{array}{cc|c} \frac{1}{\alpha^2}(\underline{S} - {}^t\underline{S}) & -\frac{1}{\alpha}\mathrm{E}_n & 0 \\ & & \vdots \\ \frac{1}{\alpha}\mathrm{E}_n & 0 & 0 \\ \hline 0 \cdots 0 & & 0 \end{array}\right) $$

$$ {}^tT^{-1}(BT^{-1}) = \begin{pmatrix} \frac{1}{\alpha}\mathrm{E}_n & -\frac{1}{\alpha}{}^tS \\ 0 & \mathrm{E}_{n+1} \end{pmatrix} \left(\begin{array}{c|c|c} \frac{1}{\alpha}\overline{S} & \begin{matrix} 0 \\ \vdots \end{matrix} & -\mathrm{E}_n \\ \hline 0 \cdots & 0 & \cdots 0 \\ \hline \frac{1}{\alpha}\mathrm{E}_n & \begin{matrix} \vdots \\ 0 \end{matrix} & 0 \end{array}\right) $$

$$ = \left(\begin{array}{c|c|c} \frac{1}{\alpha^2}(\overline{S} - {}^t\overline{S}) & \begin{matrix} 0 \\ \vdots \end{matrix} & -\frac{1}{\alpha}\mathrm{E}_n \\ \hline 0 \quad \cdots & 0 & \cdots \quad 0 \\ \hline \frac{1}{\alpha}\mathrm{E}_n & \begin{matrix} \vdots \\ 0 \end{matrix} & 0 \end{array}\right). $$

Therefore T is an automorphism iff $\underline{S} = {}^t\underline{S}$ and $\overline{S} = {}^t\overline{S}$. In other words

$$ \begin{array}{rcl} s_{ij} & = & s_{ji} \\ s_{i+1,j} & = & s_{j+1,i} \end{array} \qquad \text{for } 1 \le i, j \le n, $$

so

$$ s_{ij} = s_{ji} = s_{(j-1)+1,i} = s_{i+1,j-1} $$

for $j > 1$ and $i < n$, hence $s_{ij} = s_{kl}$ for $i + j = k + l$.

The statement about the group action follows from

$$ \begin{pmatrix} \alpha\mathrm{E}_n & 0 \\ S & \mathrm{E}_{n+1} \end{pmatrix} \begin{pmatrix} \alpha'\mathrm{E}_n & 0 \\ S' & \mathrm{E}_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha\alpha'\mathrm{E}_n & 0 \\ \alpha' S + S' & \mathrm{E}_{n+1} \end{pmatrix}. \qquad \square $$

Collecting the results we have

Theorem 6.6 *The automorphism group of $\mathbb{G}(1,2n) \cap L$ is an extention of $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$ by the semi direct product $\mathbb{C}^{2n} \ltimes \mathbb{C}^*$.*

It is isomorphic to the matrix subgroup of $\mathbb{P}\mathrm{GL}(2n+1,\mathbb{C})$ given by

$$\begin{pmatrix} \alpha \mathrm{E}_n & 0 \\ S & \mathrm{E}_{n+1} \end{pmatrix} \begin{pmatrix} {}^t t_n^{-1} & 0 \\ 0 & t_{n+1} \end{pmatrix}$$

where $\alpha \in \mathbb{C}^*$, $S \in \mathrm{M}((n+1) \times n, \mathbb{C})$ with $s_{ij} = s_{kl}$ for $i+j = k+l$ and $t_n \in \mathrm{Aut}(h, \mathbb{P}_{n-1})$ resp. $t_{n+1} \in \mathrm{Aut}(c, \mathbb{P}_n)$ are the transformations that are induced by the $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$ action on the rational normal curve $h \subset \mathbb{P}_{n-1}$ resp. $c \subset \mathbb{P}_n$.

Proof. It remains to show that the automorphism fixing the center curve pointwise form a normal subgroup, but that can be easily computed. □

Remark 6.7 *An automorphism of $\mathbb{G}(1,2n) \cap L$ is determined by its action on the lines intersecting the center curve.*

In contrast to that, the line system, i.e. the position of the line $L^* \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{2n+1})^*$, is not determined by these lines, as a simple dimension count shows. Giving these lines is equivalent to giving the two rational curves $c \subset \mathbb{P}_{2n}$ and $h \subset \mathbb{P}_{2n}/P \cong \mathbb{P}_{n-1}$ and a correspondence between them, so that we have the following dimension count

$$(2(2n+1)-4) + (2n-4) + 3 < \dim \mathbb{G}(1, \mathbb{P}(\textstyle\bigwedge^2 \mathbb{C}^{2n+1})) = 2\left(\binom{2n+1}{2} - 2\right).$$

Proof of the remark. We need to show that only the identity fixes these lines one by one. First a transformation T that fixes the lines must fix the center curve, hence by the Lemma 6.5 it is of the form

$$T = \begin{pmatrix} \alpha \mathrm{E}_n & 0 \\ S & \mathrm{E}_{n+1} \end{pmatrix}.$$

We compute the induced action $\widetilde{T}$ of T on $\{l \in \mathbb{G}(1,2n) \cap L \mid c_{(0:1)} \in l\}$, the $\mathbb{P}_{2n-2}$ of lines through $c_{(0:1)} = e_n$. A line $l \in \mathbb{G}(1,2n)$ through $c_{(0:1)}$ will be in the line system $\mathbb{G}(1,2n) \cap L$ iff it lies in the hyperplane $h_{(0:1)} = \ker(1:0:\ldots:0)$. Therefore the $\mathbb{P}_{2n-2}$ of lines through e_n is given by

$$e_n \wedge x \quad \text{with } x \in \mathbb{P}(\mathrm{span}\,\{e_1, \ldots, e_{n-1}, e_{n+1}, \ldots, e_{2n}\}).$$

Using $(e_1 \wedge e_n, \ldots, e_{n-1} \wedge e_n, e_{n+1} \wedge e_n, \ldots, e_{2n} \wedge e_n)$ as a basis, the induced action $\widetilde{T}$ is

$$\widetilde{T} = \begin{pmatrix} \alpha \mathrm{E}_{n-1} & 0 \\ |\overline{S} & \mathrm{E}_n \end{pmatrix}.$$

Here $|\overline{S}$ denotes the matrix S with the first row and column deleted. In order to have $\widetilde{T} = \mathrm{E}_{2n-1}$, we must have $\alpha = 1$ and $|\overline{S} = 0$.

The same computation for the lines through $c_{(1:0)} = e_{2n}$ yields $\alpha = 1$ and $\underline{S}| = 0$ from which $S = 0$ and the remark follow. □

For the rest of the section we analyze the action of the automorphism group on the line system $\mathbb{G}(1, 2n) \cap L$. We start with $\mathbb{G}(1, 4) \cap L$.

Proposition 6.8 *The action of* $\mathrm{Aut}(\mathbb{G}(1, 4) \cap L)$ *on the lines has four orbits:*

1. *tangents of the center conic*
2. *secants of the center conic*
3. *lines through the center conic that do not lie in the plane of the center conic*
4. *lines that do not intersect the plane of the center curve.*

Proof. Since any automorphism maps the center conic onto itself, it is clear by the geometric description that all the mentioned lines lie in different orbits.

Any line in the plane P of the center conic intersects the conic twice, so by the Remark 6.2 it is a member of the line system. Since the automorphism group acts like $\mathrm{Aut}(c, P) \cong \mathbb{P}\mathrm{GL}(2, \mathbb{C})$ on the plane P, the first two orbits are obvious.

To see that the lines of 3) form one orbit, we have to exhibit an automorphism that given two lines of 3) maps one onto the other. Since the $\mathbb{P}\mathrm{GL}(2, \mathbb{C})$ part of the automorphism group acts transitively on the center conic, we may assume that both lines pass through the same point of the center conic, say $e_2 = c_{(0:1)}$. Now the induced action $\widetilde{T}$ of an automorphism T fixing the center conic pointwise on the $\mathbb{P}_2$ of lines through e_2 was computed in the proof of the Remark 6.7 as

$$\widetilde{T} = \begin{pmatrix} \alpha & 0 & 0 \\ f & 1 & 0 \\ g & 0 & 1 \end{pmatrix} \quad \text{with } \alpha \in \mathbb{C}^*;\ f, g \in \mathbb{C}.$$

These transformations act transitively on $\mathbb{P}_2 \setminus \mathbb{P}(\mathrm{span}\,\{\widetilde{e_1}, \widetilde{e_2}\})$, where the line $\mathbb{P}(\mathrm{span}\,\{\widetilde{e_1}, \widetilde{e_2}\})$ corresponds to the lines through e_2 that lie in the plane of the center conic.

The lines of 4) are all the remaining lines since there are no lines that intersect the plane P of the center conic but not the conic c itself. This is clear because the $\mathbb{P}_1$ of lines through a point $p \in P \setminus c$ is formed by the lines through p in the plane P, so there can be no other line.

Finally, we have to show that the lines of 4) form one orbit. By a small computation one checks that $\mathrm{Aut}(\mathbb{G}(1, 4) \cap L) \subset \mathbb{P}\mathrm{GL}(5, \mathbb{C})$ acts transitively on

$\mathbb{P}_4 \setminus P$. So, it suffices to show that the line $e_0 \wedge e_1$ can be mapped to any other line through e_0 by an automorphism. Any of these lines can be written as

$$e_0 \wedge (e_1 + \beta e_4) \quad \text{with } \beta \in \mathbb{C},$$

and the automorphism

$$T = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \beta & 0 & 0 & 1 \end{array}\right)$$

will take $e_0 \wedge e_1$ to it. □

Proposition 6.9 *The automorphism group acts quasihomogeneously on* $\mathbb{G}(1,6) \cap H^2$.

Proof. For this it is enough to show that the stabiliser of the line $l = e_0 \wedge e_2$ is a 2-dimensional subgroup since then

$$\begin{aligned} \dim \operatorname{Orbit}(l) &= \dim \operatorname{Aut}(\mathbb{G}(1,6) \cap H^2) - \dim \operatorname{Stab}(l) = 10 - 2 = 8 \\ &= \dim \mathbb{G}(1,6) \cap H^2. \end{aligned}$$

If we normalize the $t \in \mathbb{P}\mathrm{GL}(2,\mathbb{C})$ by $\det t = 1$, every $T \in \operatorname{Aut}(\mathbb{G}(1,6) \cap H^2)$ can be written by the Theorem 6.6 as

$$T = \left(\begin{array}{ccc|cccc} \alpha & & 0 & & & & \\ & \alpha & & & 0 & & \\ 0 & & \alpha & & & & \\ \hline e & f & g & 1 & & & \\ f & g & h & & 1 & 0 & \\ g & h & i & & & 1 & \\ h & i & j & 0 & & & 1 \end{array}\right) \begin{pmatrix} {}^t t_3^{-1} & 0 \\ 0 & t_4 \end{pmatrix}$$

$$\text{with } {}^t t_3^{-1} = \begin{pmatrix} a^2 & -ab & b^2 \\ -2ac & ab+cd & -2bd \\ c^2 & -cd & d^2 \end{pmatrix}.$$

To compute the stabilizer we start by looking only at the first three entries of

$$\begin{aligned} Te_0 &= (\alpha a^2, -2\alpha ac, \alpha c^2, \ldots) \\ Te_2 &= (\alpha b^2, -2\alpha bd, \alpha d^2, \ldots). \end{aligned}$$

Since we must have $Te_0, Te_2 \in l$, $ac = bd = 0$ follows. By $\det t = ad - bc = 1$ we have the two possibilities $b = c = 0$, $d = a^{-1}$ and $a = d = 0$, $c = -b^{-1}$. We examine only the first case, the second being similar. Now we have

$$\begin{aligned} Te_0 &= a^2(\alpha, 0, 0, e, f, g, h) \\ Te_2 &= a^2(0, 0, \alpha, g, h, i, j). \end{aligned}$$

From $Te_0, Te_2 \in l$ we conclude $e = f = g = h = 0$ resp. $g = h = i = j = 0$. Therefore, including the case $(a = d = 0,\ c = -b^{-1})$, the stabilizer of l is

$$\mathrm{Stab}(l) = \left\{ \begin{pmatrix} \alpha a^2 & & & & & & \\ & 0 & & & & 0 & \\ & & \alpha a^{-2} & & & & \\ & & & a^{-3} & & & \\ & & & & a^{-1} & & \\ & 0 & & & & a & \\ & & & & & & a^3 \end{pmatrix}, \begin{pmatrix} 0 & \alpha b^2 & & & & & \\ & 0 & & & & 0 & \\ \alpha b^{-2} & 0 & & & & & \\ & & & & & & -b^{-3} \\ & & & & & b^{-1} & \\ & 0 & & & -b & & \\ & & & b^3 & & & 0 \end{pmatrix} \right\}. \qquad \square$$

Proposition 6.10 *For $n \geq 4$ the action of the automorphism group on $\mathbb{G}(1, 2n) \cap H^2$ is not quasihomogeneous.*

Proof. We project the $\mathbb{P}_{2n}$ from the space P of the center curve onto $\mathbb{P}_{2n}/P \cong \mathbb{P}_{n-1}$. This projects the lines of $\mathbb{G}(1, 2n) \cap H^2$ not intersecting P surjectively onto the lines $\mathbb{G}(1, \mathbb{P}_{2n}/P)$ of $\mathbb{P}_{2n}/P$. The automorphisms of $\mathbb{G}(1, 2n) \cap H^2$ induce automorphisms of $\mathbb{P}_{2n}/P$. As matrices these are the upper left $n \times n$ matrices of the matrices of Theorem 6.6, i.e. they are of the form ${}^t t_n^{-1}$. So, as a group this induced automorphism group is isomorphic to $\mathrm{Aut}(h^*, \mathbb{P}_{2n}/P) \cong \mathbb{P}\mathrm{GL}(2, \mathbb{C})$. If $\mathrm{Aut}(\mathbb{G}(1, 2n) \cap H^2)$ acts quasihomogeneously, then $\mathrm{Aut}(h^*, \mathbb{P}_{2n}/P)$ would have to act quasihomogeneously on $\mathbb{G}(1, \mathbb{P}_{2n}/P) \cong \mathbb{G}(1, n-1)$, but this contradicts

$$\dim \mathbb{P}\mathrm{GL}(2, \mathbb{C}) = 3 < \dim \mathbb{G}(1, n-1) = 2n - 4. \qquad \square$$

7 $\mathbb{G}(1, 4) \cap H^3$

Let $L = H^3$ be a general 3-codimensional subspace of $\mathbb{P}(\bigwedge^2 \mathbb{C}^5) \cong \mathbb{P}_9$. To L corresponds the plane $L^* = \mathbb{P}(\lambda A + \mu B + \nu C) \subseteq \mathbb{P}(\bigwedge^2 \mathbb{C}^5)^*$ of hyperplanes containing L. Since the locus of antisymmetric matrices of corank 3 is 3-codimensional in $\mathbb{P}(\bigwedge^2 \mathbb{C}^5)^*$ by Corollary 1.7, L^* does not contain any. Therefore to each of the hyperplanes $H_{(\lambda:\mu:\nu)} \subset L$ corresponds a unique center $c_{(\lambda:\mu:\nu)} \in \mathbb{P}_4$. In complete analogy to the last case we get

Lemma 7.1 *The centers $c_{(\lambda:\mu:\nu)}$ are those points of $\mathbb{P}_4$ through which there passes a $\mathbb{P}_1$ of lines of the line system $\mathbb{G}(1, 4) \cap L$. Through all the other points passes a unique line.*

Proposition 7.2 *The map of centers*

$$\begin{array}{rccc} c: & L^* \cong \mathbb{P}_2 & \longrightarrow & \mathbb{P}_4 \\ & (\lambda : \mu : \nu) & \longmapsto & c_{(\lambda:\mu:\nu)} = \ker(\lambda A + \mu B + \nu C) \end{array}$$

is an embedding of $\mathbb{P}_2$ in $\mathbb{P}_4$ of degree 2, i.e. its image is a smooth projected Veronese surface.

REMARK 7.3 *Any line that contains three centers is in the line system.*

Proof. Let $c(p_0)$, $c(p_1)$ and $c(p_2)$ with $p_0,p_1,p_2 \in L^*$ be the three centers on the line l. By the definition of the centers we have $l \in H_{p_i}$. Since c maps lines in L^* onto conics in $\mathbb{P}_4$, the three points p_0,p_1,p_2 do not lie on a line, hence they span L^*. So $l \in H_{p_0}\cap H_{p_1}\cap H_{p_2} = L$. □

From the statements we get a complete picture of the lines of $\mathbb{G}(1,4)\cap H^3$ in $\mathbb{P}_4$. We define the trisecant variety $\mathrm{Tri}(X)$ of a variety $X \subseteq \mathbb{P}_N$ by:

$$\mathrm{Tri}(X) := \overline{\{l \in \mathbb{G}(1,N) \mid \#(X\cap l) \geq 3\}}$$

Then we have

COROLLARY 7.4 (CASTELNUOVO) $\mathbb{G}(1,4)\cap L$ *is the trisecant variety of the smooth projected Veronese surface* $\mathrm{Im}\, c \subset \mathbb{P}_4$*.*

Proof.(see [C] or [SR, X, 4.4]) By the remark above the trisecant variety is contained in the irreducible variety $\mathbb{G}(1,4)\cap L$. So it is enough to show that both varieties have the same dimension. The Lemma 7.1 together with the Proposition 7.2 shows that there is an unique line of $\mathbb{G}(1,4)\cap L$ through a general point of $\mathbb{P}_4$ and that $\mathbb{G}(1,4)\cap L$ is the closure of such lines. The same statement for the trisecant variety is classical [SR, VII,3.2]. Hence both varieties have dimension three. □

The general trisecant intersects the projected Veronese surface $\mathrm{Im}\, c$ in three different points. Their inverse image under c are triples of points in $\mathbb{P}_2$ that have to fulfill some conditions since there is only a 3-dimensional family of these triples. To see what these conditions are, we recall some facts about the Veronese surface [H].

The Veronese surface V is the image of the embedding

$$\begin{array}{rccc} \upsilon: & \mathbb{P}_2 = \mathbb{P}(\mathbb{C}^3) & \longrightarrow & \mathbb{P}(\mathrm{Sym}^2\mathbb{C}^3) \\ & \mathbb{P}(v) & \longmapsto & \mathbb{P}(v\cdot v)\,. \end{array}$$

Its secant variety consists of the points of $\mathbb{P}(\mathrm{Sym}^2\mathbb{C}^3)$ that are the product of two vectors of $\mathbb{C}^3$,

$$\mathrm{Sec}(V) = \left\{\mathbb{P}(v\cdot w) \mid v,w \in \mathbb{C}^3\setminus\{0\}\right\}.$$

The projected Veronese surface will be smooth – like in our case – iff the center of projection P is not in the secant variety.

An intersection of the Veronese surface V with a hyperplane $H \in \mathbb{P}(\mathrm{Sym}^2\mathbb{C}^3)^*$ gives the conic $\upsilon^{-1}(V\cap H) \subset \mathbb{P}_2$ which is described by the equation H if we identify $\mathbb{P}(\mathrm{Sym}^2\mathbb{C}^3)^*$ with the polynomials of degree 2 modulo $\mathbb{C}^*$. The conics that we get as hyperplane sections of the projected Veronese surface are precisely the conics that we get as hyperplane sections of the Veronese surface by hyperplanes that contain the projection center P. So these conics fulfill one linear condition given by P. We view $P \in \mathbb{P}(\mathrm{Sym}^2\mathbb{C}^3)$ as a conic C_P^* in $\mathbb{P}_2^*$. Since P does not lie in the secant variety of the Veronese surface, P is not the product of two elements of $\mathbb{C}^3$, hence C_P^* is not the union of two lines. Therefore it is smooth. We denote the dual conic of C_P^* by $C_P \subset \mathbb{P}_2$.

Now, three different points of the projected Veronese surface $\operatorname{Im} c \subset \mathbb{P}_4$ lie on a line, the trisecant, iff any hyperplane that contains two of them contains all three. Under the inverse of the embedding c that means the following on the $\mathbb{P}_2$:

Three different points of $\mathbb{P}_2$ are the inverse image $c^{-1}(l)$ of a trisecant l of the projected Veronese surface $\operatorname{Im} c$ iff all conics that fulfill the linear condition given by P (or equivalently by C_P) and pass through two of the points pass through all three of them.

The propositions in the appendix tell us that these triples of points are the vertices of the non-degenerated polar triangles of the conic C_P. By a continuity argument the trisecants that are tangent to $\operatorname{Im} c$ at one point and intersect it in another correspond to the degenerated polar triangles, and the trisecants that intersect $\operatorname{Im} c$ in only one point "with multiplicity three" correspond to a triple point on the conic C_P. We also see that there are no 4-secants. Since if there is one, there would be four points in $\mathbb{P}_2$ such that any three of them build a different polar triangle. But this is impossible because a polar triangle is already determined by two of its vertices.

With this geometric description it is easy to compute the automorphism group of $\mathbb{G}(1,4) \cap H^3$. By Theorem 1.2 any automorphism is induced by a projective linear transformation of H^3. Hence it maps the $\mathbb{P}_1$'s of lines of Lemma 7.1 onto themselves and induces thereby an automorphism of the set of centers, the projected Veronese surface $\operatorname{Im} c$. Since the $\mathbb{G}(1,4) \cap H^3$ consists of the trisecants of $\operatorname{Im} c$, this automorphism determines the original automorphism of $\mathbb{G}(1,4) \cap H^3$.

In order to complete the proof of Corollary 1.3 we have to show that the automorphism of the projected Veronese surface $\operatorname{Im} c \subset \mathbb{P}_4$ is induced by one of $\mathbb{P}_4$. Since c is the composition of the Veronese map v and the projection $\pi_P : \mathbb{P}_5 \to \mathbb{P}_4$, we consider the situation in the $\mathbb{P}_5$. Since $\operatorname{Im} c \cong \mathbb{P}_2 \cong v(\mathbb{P}_2)$ and $\operatorname{Aut}(\mathbb{P}_2) \cong \operatorname{Aut}(v(\mathbb{P}_2), \mathbb{P}_5)$ [H, 10.9], the automorphism of $\operatorname{Im} c$ induces an automorphism of $\mathbb{P}_5$. We know further that the automorphism of $\operatorname{Im} c$ maps the trisecants of $\operatorname{Im} c$ onto themselves. In the $\mathbb{P}_5$ this means that it preserves the planes that contain the projection center P and three points of the Veronese surface, so it must fix their common intersection, the projection center P. Hence the automorphism descends onto the $\mathbb{P}_4$.

Now under the inverse of the embedding c, the automorphism of $\operatorname{Im} c$ preserving the trisecants corresponds to an automorphism of $\mathbb{P}_2$ preserving the polar triangles of the conic $C_P \subset \mathbb{P}_2$. Such an automorphism of $\mathbb{P}_2$ maps the degenerated polar triangles onto themselves. In particular, it maps the tangents to the conic C_P onto themselves. Therefore it has to fix the conic C_P.

So we have seen how an automorphism of $\mathbb{G}(1,4)\cap L$ induces an unique automorphism of $\mathbb{P}_2$ fixing C_P, hence an automorphism of C_P since $\operatorname{Aut}(C_P,\mathbb{P}_2)\cong \operatorname{Aut}(C_P)\cong \mathbb{P}\mathrm{GL}(2,\mathbb{C})$.

On the other hand, any projective linear transformation of $\mathbb{P}_2$ that fixes the conic C_P preserves the polar triangles of C_P. Therefore it induces via the embedding c an automorphism of the projected Veronese surface $\operatorname{Im} c$ that preserves triples of points that lie on a line. So it defines an automorphism of the trisecants of $\operatorname{Im} c$, which is the same as an automorphism of $\mathbb{G}(1,4)\cap L$.

We summarize:

Theorem 7.5 *The automorphism group of* $\mathbb{G}(1,4)\cap H^3$ *is isomorphic to* $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$.

The description of the orbits of this automorphism group follows immediately.

Proposition 7.6 *The action of* $\operatorname{Aut}(\mathbb{G}(1,4)\cap H^3)$ *on the linear system* $\mathbb{G}(1,4)\cap H^3$ *has three orbits:*

1. *trisecants of the projected Veronese surface that intersect it in three points*
2. *trisecants that are tangent to the projected Veronese surface at one point and intersect it in another*
3. *trisecants that intersect the projected Veronese surface in only one point "with multiplicity three".*

Proof. By what was said above, this is equivalent to the classical statement that the action of group $\operatorname{Aut}(C_P,\mathbb{P}_2)$ on the polar triangles has three orbits: the non-degenerated triangles, the degenerated ones and the triple points on C_P. □

8 Appendix: Polar Triangles

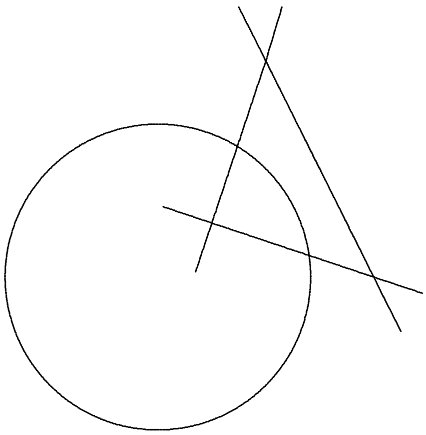
Here we prove the needed propositions about polar triangles. The whole appendix may be seen as a modern exposition of [SF, 348]. First we recall the basic definitions.

Let C_A be a smooth conic in $\mathbb{P}_2$, which is given by the quadratic equation ${}^t xAx = 0$, where $A \in \mathrm{GL}(3,\mathbb{C})$ is a symmetric, invertible matrix. Then C_A induces a polarity P by

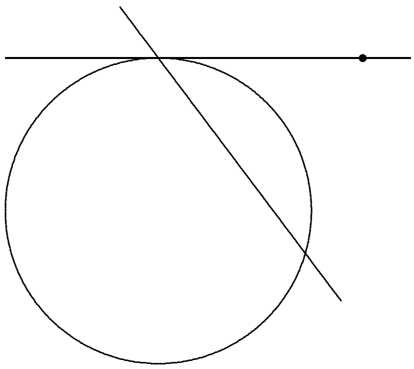
$$\begin{array}{cccc} P: & \mathbb{P}_2 & \longrightarrow & \mathbb{P}_2^* \\ & \mathbb{P}(x) & \longmapsto & \mathbb{P}({}^t xA)\,. \end{array}$$

For a point $p \in \mathbb{P}_2$ the line $P(p)$ is called the polar of p and p the pole of $P(p)$.

A polar triangle is given by three points, at least two of which are different, such that the polar of each point contains the other two points, i.e. (p, q, r) is a polar triangle if ${}^t\!pAq = {}^t\!qAr = {}^t\!rAp = 0$. The sides of the triangle are the polars of the points. In the non-degenerated case when all three points are different, the three points cannot lie on a line and therefore span the whole $\mathbb{P}_2$. In the degenerated case, (p, p, q), p lies on the conic and q on the tangent to the conic at the point p. The sides are the polar of q and twice the tangent.



Non-degenerated polar triangle



Degenerated polar triangle

Proposition 8.1 *Let $C_A = \{x \in \mathbb{P}_2 \mid {}^t\!xAx = 0\}$ be a smooth conic and $C_A^* = \{x \in \mathbb{P}_2^* \mid {}^t\!xA^{-1}x = 0\}$ its dual conic. Further, let $C_B = \{x \in \mathbb{P}_2 \mid {}^t\!xBx = 0\}$ be a conic such that*

$$\sum_{i,j=0}^{2} a^{ij} b_{ij} = 0 \tag{$*$}$$

where $A^{-1} = (a^{ij}), B = (b_{ij}) \in \mathrm{Sym}(3, \mathbb{C})$. Finally, let (p, q, r) be a polar triangle of C_A then:

If two of the three points p, q, r lie on the conic C_B, then also the third.

In the case of a degenerate polar triangle, (p, p, q), the condition that C_B contains p twice means that C_B contains p and C_B is either singular in p or its tangent in b is the polar of q.

Proof. One can show that all the properties in the statement of the proposition are independent of the choice of coordinates, so we may pick nice ones. We have to distinguish between the two cases of the polar triangle being degenerated or not. We treat the case of the non-degenerated polar triangle first.

By a suitable choice of coordinates we may assume that $p=(1:0:0)$, $q=(0:1:0)$ and $r=(0:0:1)$. Then the assumption that (p,q,r) is a polar triangle of C_A translates into

$$ {}^tpAq = {}^tqAr = {}^trAp = 0 \Longleftrightarrow a_{01}=a_{12}=a_{02}=0. $$

By a scaling of the coordinates we can achieve that $a_{00}=a_{11}=a_{22}=1$, so that $C_A=\{x\in\mathbb{P}_2 \mid x_0^2+x_1^2+x_2^2=0\}$. Then the condition $(*)$ reads $b_{00}+b_{11}+b_{22}=0$. If the two points p and q are on the conic C_B, we have ${}^tpBp=b_{00}=0$ and ${}^tqBq=b_{11}=0$. By $(*)$ we see $0=b_{22}={}^trBr$, i.e. the third point r lies also on the conic C_B.

Now we treat the case of the degenerated polar triangle (p,p,q). We choose coordinates such that $C_A=\{x\in\mathbb{P}_2 \mid x_0^2+x_1^2+x_2^2=0\}$ and $p=(1:i:0)$. The point $q\neq p$ must lie on the tangent to C_A. So it has coordinates $q=(\lambda:i\lambda:1)$, and its polar is spanned by p and $(1:0:-\lambda)$. Now using the assumptions

$$ b_{00}+b_{11}+b_{22}=0 \qquad (*) $$

$$ p\in C_B \Longleftrightarrow {}^tpBp=0 \Longleftrightarrow b_{00}+2ib_{01}-b_{11}=0, \qquad (**) $$

we have to show

$$ q\in B \Longleftrightarrow C_B \text{ singular in } p \quad \text{or} \quad \mathbb{T}_pC_B = \text{ polar of } q. $$

We rewrite this as

$$ {}^tqBq = {}^t\!\begin{pmatrix}\lambda\\ i\lambda\\ 1\end{pmatrix} B \begin{pmatrix}\lambda\\ i\lambda\\ 1\end{pmatrix} = 0 \Longleftrightarrow {}^tpB\begin{pmatrix}1\\ 0\\ -\lambda\end{pmatrix} = {}^t\!\begin{pmatrix}1\\ i\\ 0\end{pmatrix} B \begin{pmatrix}1\\ 0\\ -\lambda\end{pmatrix} = 0. $$

But this is true since -2 times the left hand side plus (λ^2+1) times $(**)$ plus $(*)$ gives the right hand side. $\square$

Now we prove the converse of the last proposition.

PROPOSITION 8.2 *Given a smooth conic $C_A=\{x\in\mathbb{P}_2 \mid {}^txAx=0\}$*

$$ \mathcal{B} := \left\{ C_B=\{x\in\mathbb{P}_2 \mid {}^txBx=0\} \,\middle|\, \sum_{i,j=0}^{2} a^{ij}b_{ij}=0 \right\} $$

is a four dimensional family of conics. Let $p,q,r\in\mathbb{P}_2$ be three points, at least two of which are different, with the property that if two of them lie on a conic $C_B\in\mathcal{B}$ then also the third.

Then (p,q,r) is a polar triangle of C_A.

Proof. We will show that if (p,q,r) is not a polar triangle then there exits a $C_A \in \mathcal{B}$ for which this property is violated. We have to treat several cases.

First let the three points be all different, then they cannot lie on a line. Because if they would, the conics in the at least two dimensional family

$$\mathcal{B}_{p,q} := \{C_B \in \mathcal{B} \mid p,q \in C_B\}$$

of conics of $\mathcal{B}$ passing through p and q must split off the line through the three points. If we pick coordinates such that this line is given by $\{x_2 = 0\}$, then $\mathcal{B}_{p,q}$ must be

$$\mathcal{B}_{p,q} = \{\mathrm{V}(x_2(\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2)) \mid (\lambda_0 : \lambda_1 : \lambda_2) \in \mathbb{P}_2\}\,.$$

This means that the conics C_B with the matrices

$$B = \begin{pmatrix} 0 & 0 & b_{02} \\ 0 & 0 & b_{12} \\ b_{02} & b_{12} & b_{22} \end{pmatrix} \quad \text{with } b_{02}, b_{12}, b_{22} \in \mathbb{C}$$

are all in $\mathcal{B}$. Hence the matrix A^{-1} must be of the type

$$A^{-1} = \begin{pmatrix} a^{00} & a^{01} & 0 \\ a^{01} & a^{11} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{with } a^{00}, a^{01}, a^{11} \in \mathbb{C},$$

but this contradicts the invertibility of A^{-1}.

Now since p,q,r span the $\mathbb{P}_2$ we may pick coordinates such that $p = (1:0:0)$, $q = (0:1:0)$ and $r = (0:0:1)$. That (p,q,r) is not a polar triangle of C_A means that ${}^t\!pAq = a_{01} \neq 0$, ${}^t\!qAr = a_{12} \neq 0$ or ${}^t\!rAp = a_{02} \neq 0$. Assuming $\det A = 1$ we conclude that not all of the $a^{01} = a_{02}a_{12} - a_{01}a_{22}$, $a^{02} = a_{01}a_{12} - a_{02}a_{11}$ and $a^{12} = a_{01}a_{02} - a_{00}a_{12}$ can be zero. If for example $a^{02} \neq 0$, then

$$B = \begin{pmatrix} 0 & 0 & -a^{11} \\ 0 & 2a^{02} & 0 \\ -a^{11} & 0 & 0 \end{pmatrix}$$

gives a conic $C_B \in \mathcal{B}$ that contains the points p and q, but not r.

Now let us look at the case where two of the points p,q,r are the same. The points (p,p,q) will not form a polar triangle if ${}^t\!pAp \neq 0$ or ${}^t\!pAq \neq 0$.

For the case ${}^t\!pAp \neq 0$ we pick coordinates such that A is the identity matrix and $p = (1:0:0)$. Let

$$B = \begin{cases} \begin{pmatrix} 0 & -q_2 & q_1 \\ -q_2 & 0 & 0 \\ q_1 & 0 & 0 \end{pmatrix} & \text{for } q_0 \neq 0 \text{ or } q_1^2 + q_2^2 \neq 0 \\[2ex] \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & \pm i \\ 1 & \pm i & -1 \end{pmatrix} & \text{for } q = (0:1:\pm i), \end{cases}$$

then C_B is a conic of $\mathcal{B}$ that contains p and q, but is smooth in p, and its tangent in p is not the polar of q, so it does not contain p twice.

Finally, if ${}^t\!pAp = 0$ and ${}^t\!pAq \neq 0$, we pick coordinates such that A is the identity matrix and $p = (1:i:0)$. Let

$$B = \begin{cases} \begin{pmatrix} -2i & 2 & iq_0 - 2q_1 \\ 2 & 2i & -iq_1 \\ iq_0 - 2q_1 & -iq_1 & 0 \end{pmatrix} & \text{for } q = (q_0 : q_1 : 1) \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \text{for } q_2 = 0, \end{cases}$$

then we are in the same situation as above. □

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