# Permanence Properties of C*-exact Groups

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Abstract. It is shown that the class of exact groups, as defined in a previous paper, is closed under various operations, such as passing to a closed subgroup and taking extensions. Taken together, these results imply, in particular, that all almost-connected locally compact groups are exact. The proofs of the permanence properties use a result relating the exactness of sequences of maps in which corresponding algebras are strongly Morita equivalent. The statement of this result is based on a notion of reduced twisted crossed product for covariant systems which are twisted in the sense of Green. The theory of these reduced twisted crossed products and the proof of the exactness result are given in the first part of the paper.

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## 1. Introduction.

Given a locally compact group $G$, let $\mathcal{C}^*_G$ be the category whose objects are the pairs $(A,\alpha)$ consisting of a C*-algebra $A$ and a continuous action $\alpha$ of $G$ on $A$, and whose maps are the $G$-equivariant *-homomorphisms between C*-algebras with continuous $G$-actions. Following [KW], the group $G$ is said to be *C*-exact* (or just *exact*) if the reduced crossed product functor $A \to A \rtimes_{\alpha,r} G$, for $(A,\alpha) \in \mathcal{C}^*_G$, is short-exact. To be more precise, $G$ is exact if and only if, whenever $(I,\alpha),(A,\beta)$ and $(B,\gamma)$ are elements of $\mathcal{C}^*_G$ and there is a $G$-equivariant short exact sequence

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{q} B \longrightarrow 0,$$

of maps, the corresponding sequence

$$0 \longrightarrow I \rtimes_{\alpha,r} G \xrightarrow{\iota_r} A \rtimes_{\beta,r} G \xrightarrow{q_r} B \rtimes_{\gamma,r} G \longrightarrow 0$$

of reduced crossed products is exact. This is equivalent to saying that for $(A,\alpha) \in \mathcal{C}^*_G$, if $I$ is an $\alpha_G$-invariant ideal of $A$, then the quotient $(A \rtimes_{\alpha,r} G)/(I \rtimes_{\alpha|,r} G)$ is canonically isomorphic to $(A/I) \rtimes_{\dot{\alpha},r} G$, where $\alpha|$ and $\dot{\alpha}$ are the restriction and quotient actions of $G$ on $I$ and $A/I$, respectively.

We introduced group exactness in [KW], primarily as a criterion for the continuity of crossed products of continuous bundles of C*-algebras. Given a continuous bundle $\mathcal{A} = \{A, X, A_x\}$ over a locally compact Hausdorff space $X$ with a continuous fibre-preserving action $\alpha$ of a group $G$ on the bundle C*-algebra $A$, it is not in general clear that the reduced crossed product bundle $\mathcal{A} \rtimes_{\alpha,r} G = \{A \rtimes_{\alpha,r} G, X, A_x \rtimes_{\alpha_x,r} G\}$ is continuous, though we know of no instance where continuity fails. One of the main results in [KW] is that, for a given $G$, $\mathcal{A} \rtimes_{\alpha,r} G$ is continuous for all pairs $(\mathcal{A}, \alpha)$ if and only if $G$ is exact. It is thus of some importance to know which groups are exact, and it is this problem which is addressed in this paper.

The most basic question is whether, in fact, all locally compact groups are exact. We have so far not been able to resolve this question even in the discrete case, and to the best of our knowledge the exactness of arbitrary discrete groups remains a significant open problem. What we are able to show is that the class of exact groups is closed under various operations such as passing to closed subgroups and taking extensions. Moreover groups possessing closed exact subgroups of finite covolume or which are cocompact are themselves exact. Using these permanence results we can show that groups from a wide class, including, in particular, all connected groups, are exact.

To prove these results we use adaptations of a number of techniques from the theory of induced representations of C*-algebras. Originally formulated by Rieffel to give an interpretation of Mackey's theory of induced representations of groups in terms of C*-algebras, this theory has been developed by P. Green [Gr] and others to give powerful techniques for handling crossed products of C*-algebras. The main tools that we use to prove the permanence results are imprimitivity theorems asserting strong Morita equivalences between various C*-crossed products by a group $G$ on the one hand and by a closed subgroup $H$ of $G$ on the other. These results all follow either from Green's generalisation to crossed products of Rieffel's imprimitivity theorem [Gr, §2], or from Raeburn's symmetric generalisation of Green's theorem [Rae, Theorem 1.1]. We shall use Green's notion of a twisted action of a group $G$ on a C*-algebra [Gr] to prove that exactness is preserved on taking extensions.

Let $N$ be a closed normal subgroup of $G$ and suppose that $(A,\alpha) \in \mathcal{C}^*_G$. Using $\alpha$ to denote also the restriction $\alpha|N$, $G$ has a canonical continuous action $\gamma$ on $A \rtimes_\alpha N$, and there is a canonical homomorphic embedding $\tau : N \to \mathcal{U}(A \rtimes_\alpha N)$, where $\mathcal{U}(A \rtimes_\alpha N)$ is the unitary group of the multiplier algebra $M(A \rtimes_\alpha N)$. The map $\tau$, which is an example of a twisting map, satisfies various compatibility conditions relative to $\gamma$ (see §2). The system $\{A \rtimes_\alpha N, G, \gamma, \tau\}$ is an example of a twisted covariant system in the sense of Green.

In general a twisted covariant system $\{A, G, \alpha, \tau\}$ consists of a continuous action $\alpha$ of $G$ on $A$ and a continuous group monomorphism $\tau$ from a closed normal subgroup $N$ of $G$ into the unitary group $\mathcal{U}(A)$ of the multiplier algebra $M(A)$ satisfying the aforementioned compatibility conditions. There is a natural idea of a twist-preserving covariant pair of representations of a twisted covariant system and the full twisted crossed product $A \rtimes_{\alpha,\tau} G$ is defined as the unique quotient of the usual full crossed product $A \rtimes_\alpha G$ which is universal for the representations obtained as the integrated forms of the twist-preserving covariant pairs of representations of $\{A, G\}$. If $(\gamma, \tau)$ is the twisted action of the previous paragraph, the crossed products $(A \rtimes_{\alpha|N} N) \rtimes_{\gamma,\tau} G$ and $A \rtimes_\alpha G$ are canonically *-isomorphic, by [Gr, §1]. By a result of Echterhoff [Ech, Theorem 1], if $\{B, G, N, \alpha, \tau\}$ is a twisted covariant system, there is an associated covariant system $\{C, G/N, \beta\}$ such that the twisted crossed product $B \rtimes_{\alpha,\tau} G$ is strongly Morita equivalent to the ordinary crossed product $C \rtimes_\beta (G/N)$. Combining these results, one finds that, with $A$, $G$, $\alpha$ and $N$ as above, $A \rtimes_{\alpha|N} N$ is strongly Morita equivalent to $C \rtimes_\beta (G/N)$, where $C = (C_0(G/N) \otimes A) \rtimes_{\Delta^\alpha} G$, $\Delta^\alpha$ being the diagonal action of $G$ on $C_0(G/N) \otimes A \cong C_0(G/N, A)$, and $\beta$ is a certain action of $G/N$ on $C$.

An analogous result for reduced crossed products is used in §5 to show that an extension of an exact group by an exact group is exact. This requires the definition, for a given a twisted covariant system $\{A, G, \alpha, \tau\}$, of a reduced twisted crossed product $A \rtimes_{\alpha,\tau,r} G$, which reduces to the ordinary reduced crossed product if the twisting is trivial, that is, if $N = \{1\}$. Although there are definitions of twisted reduced crossed products in the literature for twisted actions coming from cocycles, so far as we are aware none has been given hitherto for twisted actions in the sense of Green. Using the reduced twisted crossed product we show that $A \rtimes_{\alpha,r} G$ is strongly Morita equivalent to $(A \rtimes_{\alpha,r} N) \rtimes_{\gamma,\tau,r} G$ for a suitable twisted action $(\gamma, \tau)$ of $G$ on $A \rtimes_{\alpha,r} N$. In fact our result is sharper, in that the Morita equivalence we establish is functorial in $A$ in a certain sense.

The other permanence results are also proved using analogues for reduced crossed products of known imprimitivity theorems for full crossed products. In order to unify our techniques as much as possible, we give a general imprimitivity theorem in §2 which covers all the cases we need. This section also contains a brief review of twisted covariant systems, full twisted crossed products and other relevant background material. We define the reduced twisted crossed product in §3, and deduce imprimitivity results for reduced crossed products that parallel those for full crossed products in §2. These results are the basis of the proofs of the permanence properties mentioned above, which are established in §§4 and 5. The permanence properties are applied in §6 to show that groups of various types are exact.

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## 2. Imprimitivity results for full twisted crossed products.

In this section we recall the basic ideas of Green's theory of twisted group actions and state some of the imprimitivity results which will be used in later sections to prove the permanence results. Most of this material is a straightforward generalisation of [Gr,§§1,2], but we have found it necessary to make some aspects of the theory which are not immediately accessible in Green's treatment more explicit. Throughout the paper all groups will be assumed locally compact. Our notation follows that of [KW], for the most part. For each locally compact group $G$, $m_G$ will denote a particular left Haar measure on $G$ and $\Delta_G$ the modular function. For $(A, \alpha) \in \mathcal{C}^*_G$, the full and reduced crossed products of $A$ by $G$ are denoted by $A \rtimes_\alpha G$ and $A \rtimes_{\alpha,r} G$, respectively. If $(B, \beta) \in \mathcal{C}^*_G$, and $\theta : A \to B$ is a completely positive $G$-equivariant map, then $\theta_u$ and $\theta_r$ will denote the canonical extensions of the map $f \to \theta(f); C_c(G, A) \to C_c(G, B)$, where $(\theta(f))(s) = \theta(f(s))$ for $s \in G$, to completely positive maps

$$A \rtimes_\alpha G \to B \rtimes_\beta G$$

and

$$A \rtimes_{\alpha,r} G \to B \rtimes_{\beta,r} G,$$

respectively. If $\{\pi, V\}$ is a covariant pair of representations of the covariant system $\{A, G, \alpha\}$, then $\pi \rtimes V$ will denote the corresponding integrated form representation of the full crossed product $A \rtimes_\alpha G$.

Let $G$ be a locally compact group, let $H$ be a closed subgroup of $G$ and let $(A, \alpha) \in \mathcal{C}^*_H$. Recall that the C*-algebra $\mathrm{Ind}(A, \alpha)$ is the *-subalgebra of the C*-algebra $C_b(G, A)$ of bounded continuous $A$-valued functions on $G$ consisting of those functions $f$ such that

$$\alpha_h(f(xh)) = f(x)$$

for $h, x \in H$, so that the function $x \to ||f(x)||$ is constant on left cosets of $H$, and such that the associated continuous function on $G/H$ given by

$$xH \to ||f(x)||$$

is in $C_0(G/H)$. As is easily seen, $\mathrm{Ind}(A, \alpha)$ is closed in $C_b(G, A)$, and is, moreover, the bundle algebra of a continuous bundle of C*-algebras on $G/H$ with constant fibre $A$. In general this bundle is nontrivial, though if the action $\alpha$

extends to an action of $G$ on $A$, then the bundle is isomorphic to the trivial bundle on $G/H$ with fibre $A$. In fact when $\alpha$ is defined on all of $G$, an automorphism $\nu$ of $C_b(G,A)$ is defined by

$$(\nu(f))(s) = \alpha_s(f(s)).$$

For $f \in \mathrm{Ind}(A,\alpha)$, $x \in G$ and $h \in H$, $(\nu(f))(xh) = \alpha_x((f)(x))$, so that $\nu(f)$ is constant on left cosets of $H$. If we identify $\nu(f)$ with the corresponding function in $C_0(G/H,A)$, the restriction of $\nu$ to $\mathrm{Ind}(A,\alpha)$ gives an isomorphism $\nu_A$ of $\mathrm{Ind}(A,\alpha)$ onto $C_0(G/H,A)$.

A continuous action $\tilde{\alpha}$ of $G$ on $\mathrm{Ind}(A,\alpha)$ is given by

$$(\tilde{\alpha}_g(\psi))(s) = \psi(g^{-1}s).$$

If $\alpha$ is defined on all of $G$, for $\psi \in \mathrm{Ind}(A,\alpha)$, $g, s \in G$,

$$(\nu(\tilde{\alpha}_g(\psi)))(s) = \alpha_s((\tilde{\alpha}_g(\psi))(s)) = \alpha_s(\psi(g^{-1}s)) = (\Delta^\alpha_g(\nu(\psi)))(s),$$

where $\Delta^\alpha$ is the diagonal action of $G$ on $C_0(G/H,A)$ given by

$$(\Delta^\alpha_g(f))(s) = \alpha_g(f(g^{-1}s))$$

for $f \in C_0(G/H,A)$. Thus $\nu_A$ is an equivariant isomorphism between the covariant systems $\{\mathrm{Ind}(A,\alpha), G, \tilde{\alpha}\}$ and $\{C_0(G/H,A), G, \Delta^\alpha\}$.

Let $E_0$ and $B_0$ be the *-algebras $C_c(G,\mathrm{Ind}(A,\alpha))$ and $B_0 = C_c(H,A)$, with the convolution products relative to the actions $\tilde{\alpha}$ and $\alpha$, respectively, and let $X_0 = C_c(G,A)$. The algebras $E_0$ and $B_0$ are taken to have the C*-norms and positive cones resulting from their canonical embeddings in $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha}} G$ and $A \rtimes_\alpha H$, respectively. The linear space $X_0$ is given an $E_0$–$B_0$ bimodule structure and $E_0$- and $B_0$-valued inner products as follows. For $f \in E_0$, $g \in B_0$, and $x, y \in X_0$, $fx$, $xg$, $\langle x,y\rangle_{E_0}$ and $\langle x,y\rangle_{B_0}$ are defined by

$$\begin{aligned}
(fx)(r) &= \int_G f(s,r)x(s^{-1}r)dm_G(s)\\
(xg)(r) &= \int_H \delta(t)\alpha_t(x(rt)g(t^{-1}))dm_H(t)\\
\langle x,y\rangle_{E_0}(s,r) &= \int_H \Delta_G(rs^{-1}t)\alpha_t(x(rt)y(s^{-1}rt)^*)dm_H(t)\\
\langle x,y\rangle_{B_0}(t) &= \delta(t)\int_G x(s)^*\alpha_t(y(st))dm_G(s),
\end{aligned}$$

where $\delta(t) = \Delta_G(t)^{1/2}/\Delta_H(t)^{1/2}$. It is easily checked that $fx, xg \in X_0$, $\langle x,y\rangle_{E_0} \in E_0$ and $\langle x,y\rangle_{B_0} \in B_0$. The map $(f,x) \to fx$ is a left action of $E_0$ on $X_0$ and is the integrated form of the covariant pair of left actions of $\mathrm{Ind}(A,\alpha)$ and $G$ on $X_0$ given by

$$\begin{aligned}
(\psi x)(r) &= \psi(r)x(r)\\
(sx)(r) &= x(s^{-1}r)
\end{aligned}$$

($\psi \in \mathrm{Ind}(A,\alpha),\ s \in G,\ x \in X_0$). The map $(x,g) \to xg$ is a right action of $B_0$ on $X_0$ and is the integrated form of the covariant pair of right actions of $A$ and $H$ on $X_0$ given by

$$\begin{aligned}(xa)(r) &= x(r)a \\ (xt)(r) &= \Delta_G(t)^{-1/2}\Delta_H(t)^{-1/2}\alpha_{t^{-1}}(x(rt^{-1}))\end{aligned}$$

($a \in A,\ t \in H,\ x \in X_0$).

The following theorem generalises [Gr, Proposition 3] and [Rie1, §7]. It is straightforward, if rather tedious, to write out a proof along the lines of those of [Gr] and [Rie1], though, as Siegfried Echterhoff has pointed out to us, the result is a corollary of Raeburn's more general symmetric imprimitivity theorem [Rae, Theorem 1.1, special case 1.5]. We are grateful to Echterhoff for drawing our attention to the latter, and also for showing us how the result can, alternatively, be deduced directly from Green's original imprimitivity theorem.

THEOREM 2.1 *With the structure defined above, $X_0$ is an $E_0$–$B_0$ equivalence (or imprimitivity) bimodule.*

REMARK If the action $\alpha$ is actually defined on the whole of $G$, so that the covariant systems $\{\mathrm{Ind}(A,\alpha), G, \tilde{\alpha}\}$ and $\{C_0(G/H, A), G, \Delta^\alpha\}$ are equivariantly isomorphic, it is straightforward to verify that the $E_0$–$B_0$ equivalence bimodule $X_0$ is isomorphic to the $C_c(G, C_0(G/H, A))$–$C_c(H, A)$ equivalence bimodule constructed by Green in [Gr], and that Theorem 2.1 reduces to [Gr,Proposition 3].

Let $|| \ ||_u$ be the universal C*-norm on $C_c(H,A)$. If $X_A$ is the completion of $X_0$ with respect to the norm $x \to ||\langle x, x\rangle_{B_0}||_u^{1/2}$, the action of $B_0$ extends canonically to a right action of its completion $A \rtimes_\alpha H$ on $X_A$. Moreover the left action of $E_0$ on $X_0$ extends to a left action on $X_A$ by bounded operators, the operator norm on $E_0$ being a C*-norm, generally incomplete. If $E$ is the completion of $E_0$ with respect to this norm, there is a canonical left action of $E$ on $X_A$ extending that of $E_0$, and $X_A$ is an $E$–$(A \rtimes_\alpha H)$ equivalence bimodule [Rie2]. The C*-algebra $E$ is a quotient of the full crossed product $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha}} G$. In Corollary 2.2 we shall show that the kernel of the quotient map is trivial, so that $E \cong \mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha}} G$ canonically. Since the proofs of the corollary and other later results use induced representations, we review the inducing process briefly.

Let $A$ and $B$ be strongly Morita equivalent pre-C*-algebras (i.e. normed *-algebras whose norms satisfy the C*-condition but are not necessarily complete), and let $X$ be an $A$–$B$ equivalence bimodule. If $\pi$ is a contractive *-representation of $B$ on a Hilbert space $\mathcal{H}$, then the corresponding induced representation ${}^X\pi$ acts contractively and non-degenerately on the Hilbert space ${}^X\mathcal{H}$ obtained by completing $X \otimes_B \mathcal{H}$ with respect to the semi-norm $\sum x_i \otimes \xi_i \to ||(\pi(\langle x_i, x_i\rangle_B)\xi_i|\xi_i)||^{1/2}$ and for $a \in A$,

$${}^X\pi(a)(x \otimes \xi) = ax \otimes \xi \qquad (x \in X,\ \xi \in \mathcal{H}).$$

Let $X^*$ be the $B$–$A$ equivalence bimodule dual to $X$. Thus $X^*$ is the image of $X$ by an antilinear bijection $x \to x^*$ such that

$$\bar{\lambda} x^* = (\lambda x)^*, \quad b x^* = (x b^*)^*, \quad x^* a = (a^* x)^*$$

for $\lambda \in \mathbb{C}$, $a \in A$, $b \in B$ and $x \in X$, and the $B$- and $A$-valued inner products on $X^*$ are given by

$$\langle x^*, y^* \rangle_B = \langle y, x \rangle_B, \quad \langle x^*, y^* \rangle_A = \langle y, x \rangle_A$$

for $x, y \in X$. If $\sigma$ is a contractive representation of $A$ on a Hilbert space $\mathcal{K}$, then ${}^{X^*}\sigma$ is a contractive representation of $B$. The $A$–$A$ equivalence bimodules $X \otimes_B X^*$ and $A$ are isomorphic, and likewise there is an isomorphism between the $B$–$B$ equivalence bimodules $X^* \otimes_A X$ and $B$. It follows that there are unitary equivalences ${}^{X^*}({}^X\pi) \cong \pi$ and ${}^X({}^{X^*}\sigma) \cong \sigma$ for any non-degenerate representations $\pi$ and $\sigma$ of $B$ and $A$, respectively, so that there is a bijective correspondence between the equivalence classes of non-degenerate representations of $A$ and $B$. In the rest of the paper all representations will be assumed non-degenerate.

If $A$ and $B$ are actually C*-algebras, by [Rie1] there are bijective correspondences between (i) ideals of $A$, (ii) closed $A$–$B$-invariant subspaces of $X$ and (iii) ideals of $B$. If $Y$ is a closed $A$–$B$-invariant subspace of $X$, the corresponding ideals of $A$ and $B$ are

$$\begin{aligned} A_Y &= \overline{\operatorname{span}}\{< y, x >_A : x \in X, y \in Y\} \\ B_Y &= \overline{\operatorname{span}}\{< x, y >_B : x \in X, y \in Y\}, \end{aligned}$$

respectively. In the opposite direction, if $I$ and $J$ are ideals in $B$ and $A$, respectively, then the corresponding $A$–$B$-invariant subspaces of $X$ are

$$Y_I = XI = \overline{\operatorname{span}}\{xz : x \in X, z \in I\}$$

and

$${}_JY = JX = \overline{\operatorname{span}}\{zx : z \in J, x \in X\}.$$

These correspondences clearly respect inclusion. When necessary we shall say that $I$ and $J$ correspond *via* $X$. It is straightforward to verify that if $\pi$ is a representation of $B$, then the ideals $\ker^X \pi$ and $\ker \pi$ of $A$ and $B$, respectively, correspond via X. In particular, ${}^X\pi$ is faithful if and only if $\pi$ is faithful.

Corollary 2.2 *The operator norm on $E_0$ is the the universal C*-norm coming from the canonical embedding of $E_0$ in* $\operatorname{Ind}(A, \alpha) \rtimes_{\tilde{\alpha}} G$, *and $X_A$ is canonically an* $(\operatorname{Ind}(A, \alpha) \rtimes_{\tilde{\alpha}} G)$–$(A \rtimes_\alpha H)$ *equivalence bimodule.*

*Proof:* Let $\{\pi, U\}$ be a covariant pair of representations of the system $\{A, H, \alpha\}$ on a Hilbert space $\mathcal{H}$. Writing $X$ for $X_A$, let ${}^X\pi$ and ${}^XU$ denote the restrictions of the induced representation ${}^X(\pi \rtimes U)$ to $\operatorname{Ind}(A, \alpha)$ and $G$, respectively. We shall say that the covariant pair $\{^X\pi, ^X U\}$ is induced from the pair $\{\pi, U\}$. Similarly, given a covariant pair $\{\sigma, V\}$ of representations of $\{\mathrm{Ind}(A,\alpha), G, \tilde{\alpha}\}$, we obtain a covariant pair $\{^{X^*}\sigma, ^{X^*}U\}$ of representations of $\{A, H\}$. By the above discussion, the pairs $\{\pi, U\}$ and $\{^{X^*}(^X\pi), ^{X^*}(^X U)\}$ are unitarily equivalent, as are the pairs $\{\sigma, V\}$ and $\{^X(^{X^*}\sigma), ^X(^{X^*}V)\}$.

Now let $\{\pi, U\}$ and $\{\sigma, V\}$ be universal covariant pairs for $\{A, H, \alpha\}$ and $\{\mathrm{Ind}(A,\alpha), G, \tilde{\alpha}\}$, respectively. Replacing $\{\pi, U\}$ and $\{\sigma, V\}$ by the pairs $\{\pi \oplus^{X^*} \sigma, U \oplus^{X^*} V\}$ and $\{\sigma \oplus^X \pi, V \oplus^X U\}$, respectively, we can assume, since the inducing process respects direct sums, that

$$\sigma =^X \pi, \quad V =^X U, \quad \pi =^{X^*} \sigma, \quad U =^{X^*} V.$$

Now the representation $\sigma \rtimes V$ of $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha}} G$ is universal, hence faithful. Also, the representation $^X(\pi \rtimes U)$ of $E$ has restrictions $\sigma$ and $V$ to $\mathrm{Ind}(A,\alpha)$ and $G$, respectively. This implies that $\sigma \rtimes V$ factorises via the quotient map $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha}} G \to E$, which implies that the quotient map is injective, so that $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha}} G \cong E$ as required. □

Remark If $\{\pi, U\}$ is any universal covariant pair of representations of $\{A, H, \alpha\}$, then $\pi \rtimes U$ is a faithful representation of $A \rtimes_\alpha H$, and so $^{X_A}(\pi \rtimes U)$ is a faithful representation of $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha}} G$. Thus $\{^{X_A}\pi, ^{X_A}U\}$ is universal for $\{\mathrm{Ind}(A,\alpha), G, \tilde{\alpha}\}$.

Let $G$ be a locally compact group with closed normal subgroup $N$. For $(A,\alpha) \in \mathcal{C}^*_G$ a *twisting map* for $N$ is a strictly continuous homomorphism $\tau$ of $N$ into the unitary group $\mathcal{U}(A)$ of $M(A)$ such that for $n \in N$, $s \in G$,

$$\tau(n) a \tau(n)^{-1} = \alpha_n(a)$$

and

$$\tau(sns^{-1}) = \alpha_s(\tau(n)).$$

The pair $(\alpha, \tau)$ is called a *twisted action* of of $G$ on $A$ relative to $N$, and, provided $A$ is nonzero, $\{A, G, \alpha, \tau\}$ is referred to as a twisted covariant system. A covariant pair of representations $\{\pi, V\}$ of $\{A, G\}$ on a Hilbert space $\mathcal{H}$ is *$\tau$-covariant* or *twist-preserving* if

$$\bar{\pi}(\tau(n)) = V_n$$

for $n \in N$, where $\bar{\pi}$ denotes the canonical extension of $\pi$ to the multiplier algebra $M(A)$.

Let $I_\tau$ be the closed, two-sided ideal $\bigcap_{\{\pi,V\}} \ker(\pi \rtimes V)$ of the full crossed product $A \rtimes_\alpha G$, where the supremum is over all $\tau$-covariant pairs of representations of $\{A, G, \alpha\}$. The *full twisted crossed product* $A \rtimes_{\alpha,\tau} G$ is the C*-algebra $(A \rtimes_\alpha G)/I_\tau$. It has the universal property that if $\{\pi, V\}$ is a $\tau$-covariant pair of representations of $\{A, G, \alpha\}$, then $I_\tau \subseteq \ker(\pi \rtimes V)$, so that $\pi \rtimes V$ is the composition of a representation $\pi \rtimes_\tau V$ of $A \rtimes_{\alpha,\tau} G$ with the quotient map $A \rtimes_\alpha G \to A \rtimes_{\alpha,\tau} G$.

Although it is not made explicit in [Gr], for a given twisted covariant system $\{A, G, \alpha, \tau\}$ it is always possible to find a twist-preserving covariant pair of representations $\{\pi, V\}$ with $\pi$ faithful. In §3 we shall construct for a given faithful representation $\pi$ of $A$ on a Hilbert space $\mathcal{H}$ a $\tau$-covariant pair of representations $\{\pi_{\alpha,\tau}, \lambda_\tau\}$ of $\{A, G, \alpha, \tau\}$ on a Hilbert space $L^2_\tau(G, \mathcal{H})$ canonically associated with $\pi$ with $\pi_{\alpha,\tau}$ faithful. In the case when $N$ is trivial, this pair reduces to the usual regular pair $\{\pi_\alpha, \lambda^G\}$. It follows that we can find a faithful representation $\pi$ of $A \rtimes_{\alpha,\tau} G$ on a Hilbert space $\mathcal{H}$, with restrictions $\{\pi_A, \pi_G\}$ to $\{A, G\}$ such that $\pi_A$ is injective and for $a \in A$, $g \in G$, $\pi_A(a)$ and $\pi_G(g)$ are multipliers of $\pi(A \rtimes_{\alpha,\tau} G)$. If we identify $A \rtimes_{\alpha,\tau} G$ with its image under $\pi$ and the multiplier algebra $M(A \rtimes_{\alpha,\tau} G)$ with a *-subalgebra of the weak closure of this image, $\pi_A$ and $\pi_G$ are respectively a *-monomorphism of $A$ and a group homomorphism of $G$ with kernel contained in $N$ into $M(A \rtimes_{\alpha,\tau} G)$. With these identifications, $\pi_A$ and $\pi_G$ are independent of $\pi$, and will be referred to as the *canonical morphisms*. It then follows that for any faithful representation of $A \rtimes_{\alpha,\tau} G$, $\pi_A$ is injective. A twisted covariant pair $\{\pi, V\}$ will be called *universal* if the representation $\pi \rtimes_\tau V$ of $A \rtimes_{\alpha,\tau} G$ is faithful.

Now let $G$ be a locally compact group with a closed normal subgroup $N$ and let $H$ be a closed subgroup of $G$ containing $N$. If $(A, \alpha) \in \mathcal{C}^*_H$, let $\tau : N \to M(A)$ be a twisting map for $N$. A homomorphism $\tilde{\tau} : N \to \mathcal{U}(\mathrm{Ind}(A, \alpha))$ is defined by

$$(\tilde{\tau}(n)\psi)(s) = \tau(s^{-1}ns)\psi(s) \qquad (\psi \in \mathrm{Ind}(A, \alpha)).$$

It is straightforward to verify that $\{\mathrm{Ind}(A, \alpha), G, \tilde{\alpha}, \tilde{\tau}\}$ is a twisted covariant system relative to $N$.

Proposition 2.3 *Let $\{\pi, V\}$ be a covariant pair of representations of the covariant system $\{A, H, \alpha, \tau\}$ on a Hilbert space $\mathcal{H}$. The pair $\{^{X_A}\pi, ^{X_A}V\}$ is $\tilde{\tau}$-covariant if and only if the pair $\{\pi, V\}$ is $\tau$-covariant.*

*Proof:* Let $\tilde{\pi} = ^{X_A}\pi$ and $U = ^{X_A}V$. For $n \in N$, $f, g \in C_c(G, A) \subseteq X_A$ and $\xi, \eta \in \mathcal{H}$,

$$\begin{aligned}
&\big(U_n(f \otimes \xi)\big| g \otimes \eta\big) \\
&\quad = \big(\pi(\langle g, nf\rangle_B)\xi\big|\eta\big) \\
&\quad = \int_H \big(\pi(\langle g, nf\rangle_B(t))V_t\xi\big|\eta\big)\, dm_H(t) \\
&\quad = \int_H \int_G \delta(t)\big(\pi(g(s)^*\alpha_t(f(n^{-1}st)))V_t\xi\big|\eta\big)\, dm_G(s)\, dm_H(t) \\
&\quad = \int_G \int_H \delta(t)\big(\pi(g(s)^*\alpha_t(f(s(s^{-1}n^{-1}st))))V_t\xi\big|\eta\big)\, dm_H(t)\, dm_G(s)
\end{aligned}$$

$$\overset{t\to s^{-1}nst}{=} \int_G\int_H \delta(t)\big(\pi(g(s)^*\alpha_{s^{-1}nst}(f(st)))V_{s^{-1}ns}V_t\xi\big|\eta\big)\,dm_H(t)\,dm_G(s)$$

(Since $\Delta_G|N = \Delta_H|N = \Delta_N, N$ being a normal subgroup)

$$= \int_G\int_H \delta(t)\big(\pi(g(s)^*)V_{s^{-1}ns}\pi(\alpha_t(f(st)))V_t\xi\big|\eta\big)\,dm_H(t)\,dm_G(s)$$

and

$$\begin{aligned}&\big(\tilde\tau(n)f\otimes\xi\big|g\otimes\eta\big)\\ &= \int_G\int_H \delta(t)\big(\pi(g(s)^*\alpha_t(\tau(t^{-1}s^{-1}nst)f(st)))V_t\xi\big|\eta\big)\,dm_H(t)\,dm_G(s)\\ &= \int_G\int_H \delta(t)\big(\pi(g(s)^*)\bar\pi(\tau(s^{-1}ns))\pi(\alpha_t(f(st)))V_t\xi\big|\eta\big)\,dm_H(t)\,dm_G(s).\end{aligned}$$

If the pair $\{\pi, V\}$ is $\tau$-covariant, it follows that

$$\big(U_n(f\otimes\xi)\big|g\otimes\eta\big) = \big(\tilde\tau(n)f\otimes\xi\big|g\otimes\eta\big),$$

so that $U_n = \tilde\pi(\tilde\tau(n))$, which means that the pair $\{\tilde\pi, U\}$ is $\tilde\tau$-covariant. If, conversely, $\{\tilde\pi, U\}$ is $\tilde\tau$-covariant, then

$$\big(U_n(f\otimes\xi)\big|g\otimes\eta\big) = \big(\tilde\tau(n)f\otimes\xi\big|g\otimes\eta\big),$$

and, by the above calculations,

$$\int_G\int_H\delta(t)\big(\pi(g(s)^*)(V_{s^{-1}ns} - \bar\pi(\tau(s^{-1}ns)))\pi(\alpha_t(f(st)))V_t\xi\big|\eta\big)\,dm_H(t)\,dm_G(s)=0 \qquad (*).$$

Let $a, b \in A$, let $\varepsilon > 0$ and let $\mathcal{V}$ be a symmetric compact neighbourhood of the identity in $G$ such that for $s, t \in \mathcal{V}^2$

$$\Big|\big(\pi(b^*)(V_{s^{-1}ns} - \bar\pi(\tau(s^{-1}ns)))\pi(\alpha_t(a))V_t\xi\big|\eta\big) - \big(\pi(b^*)(V_n - \bar\pi(\tau(n)))\pi(a)\xi\big|\eta\big)\Big| \le \varepsilon.$$

Letting $h$ be a continuous positive function with support in $\mathcal{V}$ such that

$$\int_G\int_H \delta(t)h(s)h(st)\,dm_H(t)\,dm_G(s) = 1$$

and taking $f$ and $g$ to be the functions $s \to h(s)a$ and $s \to h(s)b$, respectively, a simple calculation shows that the difference between $\pi(b^*)(V_n - \bar\pi(\tau(n)))\pi(a)$ and the integral on the left-hand side of $(*)$ has modulus less than or equal to $\varepsilon$. Since $\varepsilon$ is arbitrary, this implies that

$$\pi(b^*)(V_n - \bar\pi(\tau(n)))\pi(a) = 0$$

for $a, b \in A$, so that $\bar\pi(\tau(n)) = V_n$, by the nondegeneracy assumption on $\pi$, which implies that the pair $\{\pi, V\}$ is $\tau$-preserving. $\square$

Let $I_\tau$ be the kernel of the canonical quotient map $A \rtimes_\alpha H \to A \rtimes_{\alpha,\tau} H$. If $\tilde{I}$ is the ideal of $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha}} G$ corresponding to $I_\tau$ via $X_A$, let $E_\tau$ be the quotient $(\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha}} G)/\tilde{I}$. Then $X_{A,\tau} = X_A/X_A I_\tau$ is an $E_\tau$–$(A \rtimes_{\alpha,\tau} H)$ equivalence bimodule. The following theorem generalises [Gr,Corollary 5].

Theorem 2.4 *The C*-algebra $E_\tau$ is canonically isomorphic to* $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha},\tilde{\tau}} G$ *and $X_{A,\tau}$ is an* $(\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha},\tilde{\tau}} G)$–$(A \rtimes_{\alpha,\tau} H)$ *equivalence bimodule.*

*Proof:* The proof is very similar to that of Corollary 2.2. Let $\{\pi, V\}$ and $\{\sigma, U\}$ be universal twist-covariant pairs of representations of $\{A, H, \alpha, \tau\}$ and $\{\mathrm{Ind}(A,\alpha), G, \tilde{\alpha}, \tilde{\tau}\}$, respectively. Then $\{\sigma, U\}$ is unitarily equivalent to the pair $\{{}^{X_A}({}^{X_A^*}\sigma), {}^{X_A}({}^{X_A^*}U)\}$ and, by Proposition 2.3, the pairs $\{{}^{X_A}\pi, {}^{X_A}V\}$ and $\{{}^{X_A^*}\sigma, {}^{X_A^*}U\}$ are $\tilde{\tau}$- and $\tau$-covariant, respectively. Replacing $\{\pi, V\}$ and $\{\sigma, U\}$ by $\{\pi \oplus {}^{X_A^*}\sigma, V \oplus {}^{X_A^*}U\}$ and $\{\sigma \oplus {}^{X_A}\pi, U \oplus {}^{X_A}V\}$, respectively, we can assume that the pair of representations of $\{\mathrm{Ind}(A,\alpha), G, \tilde{\alpha}, \tilde{\tau}\}$ induced from $\{\pi, V\}$ is universal. By our earlier discussion the ideal $\tilde{I}$ is the kernel of the representation $({}^{X_A}\pi) \rtimes ({}^{X_A}V)$, which is the kernel of the canonical quotient map $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha}} G \to \mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha},\tilde{\tau}} G$. It follows that $X_{A,\tau}$ is an $(\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha},\tilde{\tau}} G)$–$(A \rtimes_{\alpha,\tau} H)$ equivalence bimodule. □

Remarks 2.5 1. It follows, by reasoning similar to that of the remark following the proof of Corollary 2.2, that if $\{\pi, V\}$ is any universal $\tau$-covariant pair of representations of $\{A, H, \alpha, \tau\}$, then the $\tilde{\tau}$-covariant pair $\{{}^{X_A}\pi, {}^{X_A}V\}$ is universal for $\{\mathrm{Ind}(A,\alpha), G, \tilde{\alpha}, \tilde{\tau}\}$.

2. Suppose that $H = N$, so that $\alpha$ is a continuous action of $N$ on $A$, and that $\tau : N \to \mathcal{U}(A)$ is a twisting map. The pair of homomorphisms $\{id, \tau\}$ of $\{A, N\}$, where $id : A \to M(A)$ is the canonical embedding, is $\tau$-covariant. If we represent $M(A)$ faithfully on a Hilbert space in such a way that the restriction of the representation to $A$ is non-degenerate, we can regard this pair of maps as a $\tau$-covariant pair of representations, and the integrated form of $\{id, \tau\}$ is a *-homomorphism $\Phi$ of $A \rtimes_{\alpha,\tau} N$ into $M(A)$. For $f \in C_c(N, A)$,

$$\Phi(f) = \int_N f(n)\tau(n)\, dm_N(n) \in A.$$

It follows, by taking limits, that $\Phi(A \rtimes_{\alpha,\tau} N) \subseteq A$, and, by considering $f$ with suitably small support containing the identity of $N$, that the image of $\Phi$ is dense in $A$. Thus $\Phi(A \rtimes_{\alpha,\tau} N) = A$.

Let $\{\pi, V\}$ be a $\tau$-covariant pair of representations of the twisted system $\{A, N, \alpha, \tau\}$ on a Hilbert space $\mathcal{H}$. Then $\bar{\pi}(\tau(n)) = V_n$ for $n \in N$ and for $f \in C_c(N, A)$,

$$\begin{aligned} (\pi \rtimes V)(f) &= \int_N \pi(f(n))\bar{\pi}(\tau(n))\, dm_N(n) \\ &= \pi\left(\int_N f(n)\tau(n)\, dm_N(n)\right) \\ &= \pi(\Phi(f)). \end{aligned}$$

Thus $\pi \rtimes V = \pi \circ \Phi$. If $\{\pi, V\}$ is a universal pair for $\{A, N, \alpha\}$, then $\pi \rtimes_\tau V$ and $\pi$ are faithful. This implies that $\Phi$ is an isomorphism, i.e. $A \rtimes_{\alpha,\tau} N \cong A$. If $\pi$ is a representation of $A$, then $\{\pi, \bar{\pi} \circ \tau\}$ is a $\tau$-covariant pair for $\{A, N\}$ and $\pi \rtimes (\bar{\pi} \circ \tau) = \pi \circ \Phi$. If $\pi$ is faithful, this implies that the pair $\{\pi, \bar{\pi} \circ \tau\}$ is universal for $\{A, N, \alpha\}$.

## 3. The reduced twisted crossed product.

Let $G$ be a locally compact group and let $(A, \alpha) \in \mathcal{C}^*_G$. Let $N$ be a closed normal subgroup of $G$ and let $\{\pi, V\}$ be a covariant pair of representations of $\{A, N, \alpha|N\}$ on a Hilbert space $\mathcal{H}$ such that $\pi \rtimes V$ is a faithful representation of the full crossed product $A \rtimes_{\alpha|N} N$. If we identify $A \rtimes_{\alpha|N} N$ with its image under $\pi \rtimes V$, a twisted action $(\gamma, \tau)$ of $G$ on $A \times_\alpha N$ relative to $N$ is defined by

$$\begin{aligned}\gamma_s\left(\int_N \pi(f(t))V_t\, dm_N(t)\right) &= V_s\left(\int_N \pi(f(t))V_t\, dm_N(t)\right)V_s^{-1}\\ &= \int_N \frac{\Delta_G(s)}{\Delta_{G/N}(sN)}\pi(\alpha_s(f(s^{-1}ts))V_t\, dm_N(t)\end{aligned}$$

for $f \in C_c(N, A)$ and $s \in G$, and

$$\tau(n) = V_n \qquad (n \in N)$$

(cf. [Ech, proof of Theorem 1, *et seq.*]). This twisted action has the fundamental property that there is a natural isomorphism

$$A \rtimes_\alpha G \cong (A \rtimes_\alpha N) \rtimes_{\gamma,\tau} G$$

[Gr]. In §5 we shall need an analogous isomorphism with $A \rtimes_{\alpha,r} G$ on the left and $A \rtimes_{\alpha,r} N$ on the right. To formulate such a result we give a definition of reduced twisted crossed product appropriate to the present context. Although there are various definitions of reduced twisted crossed product in the literature for cocycle twistings, our definition seems to be new.

For the definition we require a twisted version of the left regular representation of a crossed product, which we construct as follows. Let $\pi$ be a not necessarily faithful representation of $A$ on a Hilbert space $\mathcal{H}$ and let $C_c(G, \mathcal{H}, \tau)$ be the set of those continuous $\mathcal{H}$-valued functions $f$ on $G$ whose supports have relatively compact image in $G/N$ and which satisfy

$$\bar{\pi}(\tau(n))f(s) = f(sn^{-1}) \qquad (*)$$

(or, equivalently, $\bar{\pi}(\tau(s^{-1}ns))(f(ns)) = f(s)$) for $s \in G, n \in N$. For $f \in C_c(G, \mathcal{H}, \tau)$ the nonnegative-valued real function $s \to ||f(s)||$ is constant on each coset of $N$, and if we denote the common value on the coset $sN$ by $||f(sN)||$, the function $sN \to ||f(sN)||$ is in $C_c(G/N, \mathbb{R})$. Let

$$||f||_2^\tau = \left(\int_{G/N} ||f(sN)||^2 dm_{G/N}(sN)\right)^{1/2}.$$

Then $|| \ ||_2^\tau$ is a norm on $C_c(G,\mathcal{H},\tau)$ and the completion $L^2_\tau(G,\mathcal{H})$ is a Hilbert space. It is not difficult to see that $L^2_\tau(G,\mathcal{H})$ is precisely the family of equivalence classes modulo null sets of $\mathcal{H}$-valued measurable functions $f$ on $G$ satisfying $(*)$ and such that

$$\left(\int_{G/N} ||f(sN)||^2 dm_{G/N}(sN)\right)^{1/2} < \infty.$$

For $a \in A$, $\xi \in L^2_\tau(G,\mathcal{H})$ and $g, s \in G$ let

$$(\pi_{\alpha,\tau}(a)\xi)(s) = \pi(\alpha_{s^{-1}}(a))\xi(s)$$

and

$$(\lambda_{\tau,g}\xi)(s) = \xi(g^{-1}s).$$

It is readily checked that $\pi_{\alpha,\tau}(a)\xi$ and $\lambda_{\tau,g}\xi$ are in $L^2_\tau(G,\mathcal{H})$, so that $\pi_{\alpha,\tau}$ and $\lambda_\tau$ are representations of $A$ and $G$, respectively. When necessary we shall write $\lambda_\tau^G$ to make it clear which group is involved.

Lemma 3.1 *1. The pair $\{\pi_{\alpha,\tau},\lambda_\tau\}$ is $\tau$-covariant. If $\pi$ is faithful, so is $\pi_{\alpha,\tau}$.*

*2. If $\dot\lambda$ denotes the representation of $G$ on $L^2(G/N)$ obtained by composing the quotient map $G \to G/N$ with the left regular representation of $G/N$, then $\{\pi_{\alpha,\tau}\otimes 1_{L^2(G/N)}, \lambda_\tau\otimes\dot\lambda\}$ is a $\tau$-covariant pair of representations of $\{A,G,\alpha,\tau\}$ which is unitarily equivalent to the pair $\{\pi_{\alpha,\tau}\otimes 1_{L^2(G/N)}, \lambda_\tau\otimes 1_{L^2(G/N)}\}$.*

*3. If $\{\pi, V\}$ is a $\tau$-covariant pair of representations of $\{A,G,\alpha,\tau\}$ on a Hilbert space $\mathcal{H}$, then $\{\pi\otimes 1_{L^2(G/N)}, V\otimes\dot\lambda\}$ is a $\tau$-covariant pair which is unitarily equivalent to the pair $\{\pi_{\alpha,\tau},\lambda_\tau\}$.*

*Proof:* 1. It follows readily from the definitions that, for $a \in A$ and $g \in G$,

$$\lambda_{\tau,g}\pi_{\alpha,\tau}(a)\lambda_{\tau,g}^{-1} = \pi_{\alpha,\tau}(\alpha_g(a)),$$

so that $\{\pi_{\alpha,\tau},\lambda_\tau\}$ is a covariant pair for $(A,\alpha)$. Also, if $n, s \in N$, $\xi \in C_c(G,\mathcal{H},\tau)$,

$$\begin{aligned}(\lambda_{\tau,n}\xi)(s) &= \xi(n^{-1}s)\\ &= \bar\pi(\tau(s^{-1}ns))\xi(s)\\ &= \bar\pi(\alpha_{s^{-1}}(\tau(n)))\xi(s)\\ &= (\bar\pi_{\alpha,\tau}(\tau(n))\xi)(s),\end{aligned}$$

i.e. $\{\pi_{\alpha,\tau},\lambda_\tau\}$ is $\tau$-preserving.

For $f \in C_c(G,\mathcal{H})$ let $\bar f$ be given by

$$\bar f(s) = \int_N \bar\pi(\tau(m))f(sm)dm_N(m).$$

Then

$$\begin{aligned}\bar{f}(sn^{-1}) &= \int_N \bar{\pi}(\tau(m))f(sn^{-1}m)dm_N(m)\\ &= \int_N \bar{\pi}(\tau(n))\bar{\pi}(\tau(n^{-1}m))f(sn^{-1}m)dm_N(m)\\ &= \bar{\pi}(\tau(n))\bar{f}(s).\end{aligned}$$

Also, if $\operatorname{supp} f \subseteq C$ for some compact subset $C$ of $G$, then $\operatorname{supp} \bar{f} \subseteq CN$, so that $\bar{f}$ is in $L^2_\tau(G,\mathcal{H})$.

Suppose that $\pi$ is faithful. Let $a$ be a nonzero element of $A$ and let $\xi \in \mathcal{H}$ such that $\pi(a)\xi \neq 0$. For $\varepsilon > 0$ let $C$ be a compact neighbourhood of the identity $e$ in $G$ such that $||\tau(m^{-1})\xi - \xi|| \leq \varepsilon$ for $m \in C \cap N$. Taking $f$ a continuous nonnegative-valued real function on $G$ with support in $C^{-1}$ such that $\int_N f(n)dm_N(n) = 1$ and defining $F \in C_c(G,\mathcal{H})$ by $F(s) = f(s)\xi$,

$$\begin{aligned}||(\pi_{\alpha,\tau}(a)\bar{F})(e) - \pi(a)\xi|| &= ||\int_N f(m)\pi(a)(\pi(\tau(m))\xi - \xi)dm_N(m)||\\ &\leq \sup_{n\in C\cap N} ||\pi(a)(\pi(\tau(n))\xi - \xi)|| \int_N f(m^{-1})dm_N(m)\\ &\leq \varepsilon||a||\end{aligned}$$

which implies, since $\varepsilon$ is arbitrary and $\pi$ is faithful, that $\pi_{\alpha,\tau}(a) \neq 0$, i.e. $\pi_{\alpha,\tau}$ is faithful.

2. Regarding elements of $L^2_\tau(G,\mathcal{H}) \otimes L^2(G/N)$ as equivalence classes of $\mathcal{H}$-valued functions on $G \times (G/N)$, it is straightforward to show that a unitary operator $U$ on $L^2_\tau(G,\mathcal{H}) \otimes L^2(G/N)$ is defined by

$$(U\xi)(r, sN) = \xi(r, r^{-1}sN).$$

Then

$$U(\pi_{\alpha,\tau}(a) \otimes 1)U^* = \pi_{\alpha,\tau}(a) \otimes 1$$

and

$$U(\lambda_{\tau,g} \otimes 1)U^* = \lambda_{\tau,g} \otimes \dot{\lambda}_{gN},$$

for $a \in A$ and $g \in G$, i.e. $U$ implements the stated equivalence.

3. For $\xi \in C_c(G/N,\mathcal{H})$ let $W\xi$ be the $\mathcal{H}$-valued function on $G$ given by

$$(W\xi)(s) = V_{s^{-1}}\xi(sN).$$

Then for $n \in N$

$$\begin{aligned}(W\xi)(sn^{-1}) &= V_{ns^{-1}}\xi(sN)\\ &= (\bar{\pi}(\tau(n))W\xi)(s),\end{aligned}$$

so that $W\xi \in L^2_\tau(G,\mathcal{H})$. It is also immediate that $||W\xi||^\tau_2 = ||\xi||_2$, the latter norm being that on $L^2(G/N,\mathcal{H})$. Thus $W$ extends to an isometry of $L^2(G/N,\mathcal{H})$ into $L^2_\tau(G,\mathcal{H})$. For $\xi \in L^2_\tau(G,\mathcal{H})$ the $\mathcal{H}$-valued function $s \to V_s\xi(s)$ on $G$ is constant on each coset of $N$. Letting $W_1\xi$ be the $\mathcal{H}$-valued function on $G/N$ given by

$$(W_1\xi)(sN) = V_s\xi(s),$$

$W_1$ is an isometry from $L^2_\tau(G,\mathcal{H})$ to $L^2(G/N,\mathcal{H})$, and $W_1 = W^{-1}$. Hence $W$ is bijective. Moreover

$$\begin{aligned}(\lambda_{\tau,g}W\xi)(s) &= (W\xi)(g^{-1}s)\\ &= V_{s^{-1}}V_g\xi(g^{-1}sN)\\ &= V_{s^{-1}}((V_g\otimes\dot{\lambda}_g)\xi)(sN)\\ &= (W(V_g\otimes\dot{\lambda}_g)\xi)(s)\end{aligned}$$

and

$$\begin{aligned}(\pi_{\alpha,\tau}(a)W\xi)(s) &= \pi(\alpha_{s^{-1}}(a))V_{s^{-1}}\xi(sN)\\ &= V_{s^{-1}}\pi(a)\xi(sN)\\ &= (W(\pi(a)\otimes 1)\xi)(s).\end{aligned}$$

This shows that the pairs $\{\pi\otimes 1_{L^2(G/N)}, V\otimes\dot{\lambda}\}$ and $\{\pi_{\alpha,\tau},\lambda_\tau\}$ are unitarily equivalent.

□

By analogy with the untwisted case, we would like to define the reduced twisted crossed product $A\rtimes_{\alpha,\tau,r}G$ to be the image of $A\rtimes_\alpha G$ under the representation $\pi_{\alpha,\tau}\rtimes\lambda_\tau$, where $\pi$ is some faithful representation of $A$. First, however, it is necessary to show that the resulting quotient of $A\rtimes_\alpha G$ does not depend on the choice of $\pi$. This is achieved in what follows by showing that, for a given $\pi$, the $\tau$-covariant pair $\{\pi_{\alpha,\tau},\lambda^G_\tau\}$ is obtained by inducing the representation $\pi$ of $A\cong A\rtimes_{\alpha,\tau}N$ to $M(\mathrm{Ind}(A,\alpha|N)\rtimes_{\tilde{\alpha},\tilde{\tau}}G)$ via the equivalence bimodule $X_A$ of §2 and composing the induced representation with canonical morphisms from $A$ and $G$ into this multiplier algebra.

Let $H$ be a closed subgroup of $G$ containing $N$, let $\{A,\alpha\}\in\mathcal{C}^*_H$ and let $\mathrm{Ind}(A,\alpha)$ be the associated C*-algebra defined in §2. Let $\pi$ be a faithful representation of $A$ on $\mathcal{H}$ and for $\psi\in\mathrm{Ind}(A,\alpha)$ and $\xi\in L^2_\tau(G,\mathcal{H})$ let

$$(\tilde{\pi}_{\alpha,\tau}(\psi)\xi)(s) = \pi(\psi(s))\xi(s).$$

Then

$$\begin{aligned}((\tilde{\pi}_{\alpha,\tau}(\psi)\xi)(sn^{-1}) &= \pi(\psi(sn^{-1}))\bar{\pi}(\tau(n))\xi(s)\\ &= \bar{\pi}(\tau(n))\pi(\alpha_{n^{-1}}(\psi(sn^{-1})))\xi(s)\\ &= \bar{\pi}(\tau(n))\pi(\psi(s))\xi(s)\\ &= \bar{\pi}(\tau(n))((\tilde{\pi}_{\alpha,\tau}(\psi)\xi)(s),\end{aligned}$$

so that $\tilde{\pi}_{\alpha,\tau}(\psi)\xi \in L^2_\tau(G,\mathcal{H})$, and $\tilde{\pi}_{\alpha,\tau}$ is a representation of $\mathrm{Ind}(A,\alpha)$ on $L^2_\tau(G,\mathcal{H})$. For $g \in G$,

$$\begin{aligned}(\lambda^G_{\tau,g}\tilde{\pi}_{\alpha,\tau}(\psi)\lambda^G_{\tau,g^{-1}}\xi)(s) &= \pi(\psi(g^{-1}s))\xi(s)\\ &= (\tilde{\pi}_{\alpha,\tau}(\tilde{\alpha}_g(\psi))\xi)(s),\end{aligned}$$

so that $\{\tilde{\pi}_{\alpha,\tau}, \lambda^G_\tau\}$ is a covariant pair of representations of the covariant system $\{\mathrm{Ind}(A,\alpha), G, \tilde{\alpha}\}$. The proof of Lemma 3.1 (1) shows that

$$\lambda^G_{\tau,n} = \bar{\tilde{\pi}}_{\alpha,\tau}(\tilde{\tau}(n))$$

for $n \in N$, which implies that the pair $\{\tilde{\pi}_{\alpha,\tau}, \lambda^G_\tau\}$ is $\tilde{\tau}$-covariant.

Proposition 3.2 *The $\tilde{\tau}$-covariant pair of representations of $\{\mathrm{Ind}(A,\alpha), G, \tilde{\alpha}, \tilde{\tau}\}$ induced from the $\tau$-covariant pair of representations $\{\pi_{\alpha,\tau}, \lambda^H_\tau\}$ of $\{A, H, \alpha, \tau\}$ via the equivalence bimodule $X_A$ of §2 is unitarily equivalent to the pair $\{\tilde{\pi}_{\alpha,\tau}, \lambda^G_\tau\}$.*

*Proof:* We shall also assume, as we may, that the left Haar measures $m_G$, $m_H$, $m_N$ and $m_{G/N}$ have been chosen so that

$$\int_G f(s)\,dm_G(s) = \int_{G/N}\int_N f(sn)\,dm_N(n)dm_{G/N}(sN) \tag{1}$$

and

$$\int_H g(t)\,dm_H(t) = \int_{H/N}\int_N g(tn)\,dm_N(n)dm_{H/N}(sN) \tag{2}$$

for $f \in C_c(G)$, $g \in C_c(H)$.

For $f, g \in X_0 = C_c(G,A)$ and $\xi, \eta \in L^2_\tau(H,\mathcal{H})$, we calculate the inner product $(f\otimes\xi|g\otimes\eta)$ in ${}^{X_A}L^2_\tau(H,\mathcal{H})$. To prevent the notation becoming too cumbersome, we regard $\mathcal{H}$ as an $M(A)$-module via $\bar{\pi}$, so that, for $a \in M(A)$ and $\zeta \in \mathcal{H}$, $a\zeta$ will denote $(\bar{\pi}(a))\zeta$, and similarly regard $L^2_\tau(H,\mathcal{H})$ as an $M(A\rtimes_\alpha H)$-module via $\overline{\pi_{\alpha,\tau}\rtimes\lambda^H_\tau}$. Then

$$\begin{aligned}&(f\otimes\xi|g\otimes\eta)\\ &= (\langle g,f\rangle_{B_0}\xi|\eta)\\ &= \int_{H/N}\int_H\int_G \delta(t)\Delta_G(s)^{-1}\big(\alpha_{r^{-1}}(g(s^{-1})^*)\alpha_{r^{-1}t}(f(s^{-1}t))\xi(t^{-1}r)\big|\eta(r)\big)\\ &\qquad\qquad \times dm_G(s)dm_H(t)dm_{H/N}(rN)\\ &= \int_{H/N}\int_H\int_G \delta(t)\Delta_G(s)^{-1}\big(\alpha_{r^{-1}t}(f(s^{-1}t))\xi(t^{-1}r)\big|\alpha_{r^{-1}}(g(s^{-1}))\eta(r)\big)\end{aligned}$$

$$\times dm_G(s)dm_H(t)dm_{H/N}(rN)$$

$$\overset{\substack{t\to rt\\ s\to rs}}{=} \int\limits_{H/N}\int\limits_{H}\int\limits_{G} \delta(t)\delta(r)\Delta_G(r)^{-1}\Delta_G(s)^{-1}$$
$$\times\big(\alpha_t(f(s^{-1}t))\xi(t^{-1})\big|\alpha_{r^{-1}}(g(s^{-1}r^{-1}))\eta(r)\big)$$
$$\times dm_G(s)dm_H(t)dm_{H/N}(rN)$$

$$\overset{s\to s^{-1}}{=} \int\limits_{H/N}\int\limits_{H}\int\limits_{G} \delta(t)\delta(r)\Delta_G(r)^{-1}\big(\alpha_t(f(st))\xi(t^{-1})\big|\alpha_{r^{-1}}(g(sr^{-1}))\eta(r)\big)$$
$$\times dm_G(s)dm_H(t)dm_{H/N}(rN)$$

$$= \int\limits_{H/N}\int\limits_{H}\int\limits_{G/N}\int\limits_{N} \delta(t)\delta(r)\Delta_G(r)^{-1}$$
$$\times\big(\alpha_t(f(smt))\xi(t^{-1})\big|\alpha_{r^{-1}}(g(smr^{-1}))\eta(r)\big)$$
$$\times dm_N(m)dm_{G/N}(sN)dm_H(t)dm_{H/N}(rN)$$

(by (1))

$$\overset{t\to m^{-1}t}{=} \int\limits_{H/N}\int\limits_{H}\int\limits_{G/N}\int\limits_{N} \delta(m)^{-1}\delta(t)\delta(r)\Delta_G(r)^{-1}$$
$$\times\big(\alpha_{m^{-1}t}(f(st))\xi(t^{-1}m)\big|\alpha_{r^{-1}}(g(smr^{-1}))\eta(r)\big)$$
$$\times dm_N(m)dm_{G/N}(sN)dm_H(t)dm_{H/N}(rN)$$

$$\overset{m\to m^{-1}}{=} \int\limits_{H/N}\int\limits_{H}\int\limits_{G/N}\int\limits_{N} \delta(t)\delta(r)\Delta_G(r)^{-1}\Delta_N(m)^{-1}$$
$$\times\big(\alpha_{mt}(f(st))\xi(t^{-1}m^{-1})\big|\alpha_{r^{-1}}(g(sm^{-1}r^{-1}))\eta(r)\big)$$
$$\times dm_N(m)dm_{G/N}(sN)dm_H(t)dm_{H/N}(rN)$$

(since $\delta(m)=1$ for $m\in N$)

$$= \int\limits_{H/N}\int\limits_{H}\int\limits_{G/N}\int\limits_{N} \delta(t)\delta(r)\Delta_G(r)^{-1}\Delta_N(m)^{-1}$$
$$\times\big(\tau(m)\alpha_t(f(st))\tau(m^{-1})\xi(t^{-1}m^{-1})\big|\alpha_{r^{-1}}(g(sm^{-1}r^{-1}))\tau(m)\eta(rm)\big)$$

$$\times dm_N(m)dm_{G/N}(sN)dm_H(t)dm_{H/N}(rN)$$

$$= \int\limits_{H/N}\int\limits_{H}\int\limits_{G/N}\int\limits_{N} \delta(t)\delta(r)\Delta_G(rm)^{-1} \times\big(\alpha_t(f(st))\xi(t^{-1})\big|\alpha_{m^{-1}r^{-1}}(g(sm^{-1}r^{-1}))\eta(rm)\big) \times dm_N(m)dm_{G/N}(sN)dm_H(t)dm_{H/N}(rN)$$

$$= \int\limits_{G/N}\int\limits_{H}\int\limits_{H} \delta(t)\delta(r)\Delta_G(r)^{-1}\big(\alpha_t(f(st))\xi(t^{-1})\big|\alpha_{r^{-1}}(g(sr^{-1}))\eta(r)\big) \times dm_H(t)dm_H(r)dm_{G/N}(sN)$$

(by (2))

$$\overset{r\to r^{-1}}{=} \int\limits_{G/N}\int\limits_{H}\int\limits_{H} \delta(t)\delta(r)\big(\alpha_t(f(st))\xi(t^{-1})\big|\alpha_r(g(sr))\eta(r^{-1})\big) \times dm_H(t)dm_H(r)dm_{G/N}(sN).$$

Let $T(f\otimes\xi)$ be the $A$-valued function on $G$ given by

$$(T(f\otimes\xi))(s) = \int_H \delta(t)\alpha_t(f(st))\xi(t^{-1})\, dm_H(t).$$

Then

$$\begin{aligned} T(f\otimes\xi)(sn^{-1}) &= \int_H \delta(t)\alpha_t(f(sn^{-1}t))\xi(t^{-1})\, dm_H(t) \\ &= \int_H \delta(t)\alpha_{nt}(f(st))\xi(t^{-1}n^{-1})\, dm_H(t) \\ &= \tau(n)((T(f\otimes\xi))(s)), \end{aligned}$$

since $\delta(n)=1$ for $n\in N$, and if $K_1$ is the support of $f$ in $G$ and $K_2$ is the support of $\xi$ in $H$, then the support of $T(f\otimes\xi)$ is contained in the set $K_1K_2$. The latter set has relatively compact image in $G/N$ since the same is true of $K_2$ and $K_1$ is compact. It follows that $T(f\otimes\xi)\in C_c(G,\mathcal{H},\tau)$. By the above calculation, $T$ is an isometric linear map from a dense subspace of ${}^{X_A}L^2_\tau(H,\mathcal{H})$ into $L^2_\tau(G,\mathcal{H})$. Standard arguments involving partitions of unity show that the image of $T$ is dense in $L^2_\tau(G,\mathcal{H})$. Thus $T$ has an extension to an isometry U from ${}^{X_A}L^2(H,\mathcal{H})$ onto $L^2_\tau(G,\mathcal{H})$.

For $\psi \in \mathrm{Ind}(A, \alpha)$, $f \in X_0$, $\xi \in \mathcal{H}$ and $g, s \in G$,

$$\begin{aligned}[T(\psi f \otimes \xi)](s) &= \int_H \gamma(t)\alpha_t(\psi(st)f(st))\xi(t^{-1})\, dm_H(t) \\ &= \psi(s)(T(f \otimes \xi))(s) \\ &= [\tilde{\pi}_{\alpha,\tau}(\psi)(T(f \otimes \xi))](s)\end{aligned}$$

and

$$\begin{aligned}[T(gf \otimes \xi)](s) &= \int_H \gamma(t\alpha_t(f(g^{-1}st))\xi(t^{-1})\, dm_H(t) \\ &= [\lambda_{\tau,g}(T(f \otimes \xi))](s),\end{aligned}$$

from which it follows that $U$ implements the desired unitary equivalence. □

In the following corollary we assume that $H = N$, so that $\alpha$ is an action of $N$ on $A$ by inner automorphisms.

Corollary 3.3 *If the representation $\pi$ is faithful, then the integrated form representation $\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau}} \lambda_\tau$ of $\mathrm{Ind}(A, \alpha) \rtimes_{\tilde{\alpha},\tilde{\tau}} G$ is faithful.*

*Proof:* By Proposition 3.2, $\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau}} \lambda_\tau$ is the representation of $\mathrm{Ind}(A, \alpha) \rtimes_{\tilde{\alpha},\tilde{\tau}} G$ induced from the representation $\pi \rtimes_\tau \lambda_\tau^N$ of $A \rtimes_{\alpha,\tau} N$ via $X_A$. The Hilbert space $L^2_\tau(N, \mathcal{H})$ is just the space of continuous $\mathcal{H}$-valued functions $f$ such that

$$f(n) = \bar{\pi}(\tau(n^{-1}))f(e)$$

for $n \in N$, with norm $||f(e)||$, and the map $f \to f(e)$ is an isometry of $L^2_\tau(N, \mathcal{H})$ onto $\mathcal{H}$. This map implements a unitary equivalence between the $\tau$-covariant pairs $\{\pi_{\alpha,\tau}, \lambda_\tau^N\}$ and $\{\pi, \bar{\pi} \circ \tau\}$. By Remark 2.5 (2), the latter pair is universal for $\{A, N, \alpha, \tau\}$, since $\pi$ is faithful. Hence $\{\pi_{\alpha,\tau}, \lambda_\tau^N\}$ is universal for $\{A, N, \alpha, \tau\}$, so that $\pi_{\alpha,\tau} \rtimes_\tau \lambda_\tau^N$ is a faithful representation of $A \rtimes_\tau N$ ($\cong A$, by Remark 2.5 (2)). Since faithful representations induce faithful representations, the result follows. □

We are now ready to define the reduced twisted crossed product. Let $G$ be a locally compact group with a closed normal subgroup $N$. Let $(A, \alpha) \in \mathcal{C}^*_G$ and let $\tau : N \to \mathcal{U}(A)$ be a twisting map relative to $\alpha$. Let $\pi$ be a faithful representation of $A$ on a Hilbert space $\mathcal{H}$. Letting $E = \mathrm{Ind}(A, \alpha|N)$, we note in passing that by the discussion of §2 the pair $(E, \tilde{\alpha})$ is $G$-equivariantly isomorphic to $C_0(G/N, A)$ with $G$ acting by left translation. By Corollary 3.3, the representation $\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau}} \lambda_\tau$ of $E \rtimes_{\tilde{\alpha},\tilde{\tau}} G$ on $L^2_\tau(G, \mathcal{H})$ is faithful, and hence so is its canonical extension $\overline{\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau}} \lambda_\tau}$ to $M(E \rtimes_{\tilde{\alpha},\tilde{\tau}} G)$. Identifying $M(E \rtimes_{\tilde{\alpha},\tilde{\tau}} G)$ with its image under $\overline{\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau}} \lambda_\tau}$, for $a \in A$, $\psi \in E$ and $\xi \in L^2_\tau(G, \mathcal{H})$,

$$\begin{aligned}(\pi_{\alpha,\tau}(a)\tilde{\pi}_{\alpha,\tau}(\psi)\xi)(s) &= \pi(\alpha_{s^{-1}}(a))\pi(\psi(s))\xi(s) \\ &= (\tilde{\pi}_{\alpha,\tau}(a\psi)\xi)(s),\end{aligned}$$

where $a\psi$ is the element of $E$ given by

$$(a\psi)(s) = \alpha_{s^{-1}}(a)\psi(s).$$

For $f \in C_c(G,E)$ and $a \in A$, let $af$ be the element of $C_c(G,E)$ given by $(af)(s) = af(s)$. Then

$$\begin{aligned}
\pi_{\alpha,\tau}(a)(\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau}} \lambda_\tau)(f) &= \pi_{\alpha,\tau}(a) \int_G \tilde{\pi}_{\alpha,\tau}(f(s))\lambda_{\tau,s}\, dm_G(s) \\
&= \int_G \tilde{\pi}_{\alpha,\tau}(af(s))\lambda_{\tau,s}\, dm_G(s) \\
&= (\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau}} \lambda_\tau)(af) \\
&\in \mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha},\tilde{\tau}} G,
\end{aligned}$$

and, by taking limits of sequences of such $f$, it follows that $\pi_{\alpha,\tau}(a)(\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau}} \lambda_\tau)(x) \in E \rtimes_{\tilde{\alpha},\tilde{\tau}} G$ for all $x \in E \rtimes_{\tilde{\alpha},\tilde{\tau}} G$. Similarly, if $g \in G$ and $\psi \in E$, let $g\psi$ be the element of $E$ given by

$$(g\psi)(s) = \alpha_g(\psi(g^{-1}s)),$$

and for $f \in C_c(G,E)$ let $gf$ be the element of $C_c(G,E)$ such that $(gf)(s) = gf(s)$. A similar calculation shows that $\lambda_{\tau,g}$ multiplies $E \rtimes_{\tilde{\alpha},\tilde{\tau}} G$. There are thus canonical homomorphisms $\pi_0$ and $\lambda_0$ from $A$ and $G$ into $M(E \rtimes_{\tilde{\alpha},\tilde{\tau}} G)$ given by

$$\pi_0(a)f = af, \qquad \lambda_0(g)g = gf$$

for $f \in C_c(G,E)$. Moreover $\pi_{\alpha,\tau} = (\overline{\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau}} \lambda_\tau}) \circ \pi_0$ and $\lambda_\tau = (\overline{\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau}} \lambda_\tau}) \circ \lambda_0$, from which it follows that $\pi_0$ is an isomorphism, and $\{\pi_0, \lambda_0\}$ is a $\tau$-covariant pair.

DEFINITION 3.4 The *reduced twisted crossed product* $A \rtimes_{\alpha,\tau,r} G$ is the image of $A \rtimes_\alpha G$ in $M(\mathrm{Ind}(A,\alpha|N) \rtimes_{\tilde{\alpha},\tilde{\tau}} G)$ under the *-homomorphism $\pi_0 \rtimes \lambda_0$.

In the next proposition we consider the natural class of mappings between twisted covariant systems with respect to given $G$ and $N$. Let $\{A, G, \alpha, \tau\}$ and $\{B, G, \beta, \tau'\}$ be two such systems and let $\theta : A \to B$ be a $G$-equivariant *-homomorphism. We shall say that $\theta$ is *twist-equivariant* (with respect to $\tau$ and $\tau'$) if $\theta(\tau(n)a) = \tau'(n)\theta(a)$ for $n \in N$ and $a \in A$.

PROPOSITION 3.5 *1. Let $\pi$ be a representation of $A$ on a Hilbert space $\mathcal{H}$. Then the representation $\pi_{\alpha,\tau} \rtimes_\tau \lambda_\tau$ of $A \rtimes_{\alpha,\tau} G$ is the composition of a representation $\pi_{\alpha,\tau} \rtimes_{\tau,r} \lambda_\tau$ of $A \rtimes_{\alpha,\tau,r} G$ with the canonical quotient map $A \rtimes_{\alpha,\tau} G \to A \rtimes_{\alpha,\tau,r} G$. If $\pi$ is faithful, then so is $\pi_{\alpha,\tau} \rtimes_{\tau,r} \lambda_\tau$.*

*2. Let $\{A, G, \alpha, \tau\}$ and $\{B, G, \beta, \tau'\}$ be twisted covariant systems with respect to the closed normal subgroup $N$ of $G$, and let $\theta : A \to B$ be a *-homomorphism which is twist-equivariant with respect to the given actions and* *twisting maps. Then there is a unique *-homomorphism* $\theta_{N,r} : A \rtimes_{\alpha,\tau,r} G \to B \rtimes_{\beta,\tau',r} G$ *such that the diagram*

$$\begin{tikzcd} A \rtimes_{\alpha} G \arrow[r, "\theta_u"] \arrow[d] & B \rtimes_{\beta} G \arrow[d] \\ A \rtimes_{\alpha,\tau,r} G \arrow[r, "\theta_{N,r}"] & B \rtimes_{\alpha,\tau',r} G \end{tikzcd}$$

*commutes, the vertical arrows denoting the canonical *-homomorphisms. The morphism* $\theta_{N,r}$ *is injective (resp. surjective) if and only if* $\theta$ *is injective (resp. surjective). If* $\operatorname{Im}\theta$ *is an ideal of* $B$*, then* $\operatorname{Im}\theta_{N,r}$ *is an ideal of* $B \rtimes_{\beta,\tau',r} G$.

*Proof:* 1. This follows immediately from the factorisations $\pi_{\alpha,\tau} = (\overline{\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau}} \lambda_\tau}) \circ \pi_0$ and $\lambda_\tau = (\overline{\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau}} \lambda_\tau}) \circ \lambda_0$, and the fact that, if $\pi$ is faithful, then $\overline{\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau}} \lambda_\tau}$ is a faithful representation of $M(\operatorname{Ind}(A,\alpha) \rtimes_{\tilde{\alpha},\tilde{\tau}} G)$.

2. If $\{\pi, V\}$ is a $\tau'$-covariant pair of representations of $\{B, G, \beta\}$, then $\{\pi \circ \theta, V\}$ is a covariant pair of representations of $\{A, G, \alpha\}$ on a Hilbert space $\mathcal{H}$, and for $n \in N$ and $a \in A$,

$$\begin{aligned} \overline{(\pi \circ \theta)}(\tau(n))(\pi \circ \theta)(a) &= \pi(\theta(\tau(n)a)) \\ &= \overline{\pi}(\tau'(n))(\pi \circ \theta)(a). \end{aligned}$$

If $\theta$ is surjective, this shows that $\overline{(\pi \circ \theta)}(\tau(n)) = V_n$, so that the pair $\{\pi \circ \theta, V\}$ is $\tau$-covariant. By part 1, there is a canonical *-epimorphism $\theta_{N,r} : A \rtimes_{\alpha,\tau,r} G \to B \rtimes_{\beta,\tau',r} G$ such that

$$(\pi \rtimes_{\tau',r} V) \circ \theta_{N,r} = (\pi \circ \theta) \rtimes_{\tau,r} V, \tag{$*$}$$

where $(\pi \circ \theta) \rtimes_{\tau,r} V$ and $\pi \rtimes_{\tau',r} V$ are the representations of $A \rtimes_{\alpha,\tau,r} G$ and $B \rtimes_{\beta,\tau',r} G$ associated with $\pi \circ \theta$ and $\pi$, respectively.

If, on the other hand, $\theta$ is injective, let $\mathcal{H}_1$ be the closure in $\mathcal{H}$ of $\pi(\theta(A))\mathcal{H}$ and let $E$ be the projection onto $\mathcal{H}_1$. Then by the covariance condition, $EV_g = V_g E$ for $g \in G$. Letting $W_g = EV_g|\mathcal{H}_1$, the above identity implies that

$$\overline{(\pi \circ \theta)}(\tau(n)) = W_n E$$

for $n \in N$. Defining $\sigma$ by $\sigma(a) = \pi(\theta(a))|\mathcal{H}_1$, it follows that $\{\sigma, W\}$ is a $\tau$-covariant pair for $\{A, G, \alpha, \tau\}$. It is easily seen that the images of $A \rtimes_\alpha G$ under the representations $(\pi \circ \theta) \rtimes V$ and $\sigma \rtimes W$ are isomorphic. If $\pi$ is faithful, so is $\sigma$, and the latter image is canonically isomorphic to $A \rtimes_{\alpha,\tau,r} G$ by part 1. This implies that there is a canonical *-monomorphism $\theta_{N,r} : A \rtimes_{\alpha,\tau,r} G \to B \rtimes_{\beta,\tau',r} G$ for which $(*)$ holds.

Combining these two cases, the existence of $\theta_{N,r}$ satisfying $(*)$ for arbitrary $\theta$ follows. The commutativity of the given diagram is a simple consequence of $(*)$. □

We return now to the general situation where $N$ is a closed normal subgroup of $G$, $H$ is a closed subgroup of $G$ containing $N$, $(A,\alpha) \in \mathcal{C}_H^*$ and $\tau : N \to \mathcal{U}(A)$ is a twisting map relative to $\alpha$. Let $X_A$ be the $(\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha}} G)$–$(A \rtimes_\alpha H)$ equivalence bimodule constructed in §2, and let $\pi$ be a faithful representation of $A$ on a Hilbert space $\mathcal{H}$. The kernel $J_{\tau,r}$ of the canonical quotient map $A \rtimes_\alpha H \to A \rtimes_{\alpha,\tau,r} H$ is the kernel of the representation $\pi_{\alpha,\tau} \rtimes \lambda_\tau^H$. By Proposition 3.2 the representation of $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha}} G$ induced from this representation via $X_A$ is the integrated form of the pair $\{\tilde{\pi}_{\alpha,\tau}, \lambda_\tau^G\}$. Let $I$ be the kernel of this representation. By Lemma 3.1, the $\tilde{\tau}$-covariant pair $\{\tilde{\pi}_{\alpha,\tau} \otimes 1_{L^2(G/N)}, \lambda_\tau^G \otimes 1_{L^2(G/N)}\}$ is unitarily equivalent to the pair $\{\tilde{\pi}_{\alpha,\tau} \otimes 1_{L^2(G/N)}, \lambda_\tau^G \otimes \dot{\lambda}\}$, which is in turn unitarily equivalent to the pair $\{(\tilde{\pi}_{\alpha,\tau})_{\tilde{\alpha},\tilde{\tau}}, \lambda_\tau\}$. Since $\pi$, and hence $\tilde{\pi}_{\alpha,\tau}$, are faithful, the kernel of the integrated form of the pair $\{(\tilde{\pi}_{\alpha,\tau})_{\tilde{\alpha},\tilde{\tau}}, \lambda_\tau\}$ is the ideal $I_{\tilde{\tau},r}$, the kernel of the canonical *-homomorphism from $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha}} G$ to $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha},\tilde{\tau},r} G$, by Proposition 3.5 (1). It follows that that $I$ coincides with $I_{\tilde{\tau},r}$ and corresponds to $J_{\tau,r}$ via $X_A$. By [Rie2] $X_{A,\tau,r} = X_A/X_A J_{\tau,r}$ is an $(\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha},\tilde{\tau},r} G)$–$(A \rtimes_{\alpha,\tau,r} H)$ equivalence bimodule. In fact, $X_{A,\tau,r}$ is obtained from the $E_0$–$B_0$ equivalence bimodule $X_0 = C_c(G,A)$ of §2 by completing with respect to the semi-norm $f \to ||(\pi_{\alpha,\tau} \rtimes \lambda_\tau^H)(< f, f >_B)||^{1/2}$. This proves

Theorem 3.6 *The $C^*$-algebras $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha},\tilde{\tau},r} G$ and $A \rtimes_{\alpha,\tau,r} H$ are strongly Morita equivalent via the equivalence bimodule $X_{A,\tau,r}$.*

Remark 3.7 When $H = N$, it follows by Remark 2.5 (2) that if $\pi$ is a faithful representation of $A$, the pair $\{\pi_{\alpha,\tau}, \lambda_\tau^N\}$ is universal for $\{A, N\}$, and the kernels of the canonical quotient maps of $A \rtimes_\alpha N$ onto $A \rtimes_{\alpha,\tau} N$ and $A \rtimes_{\alpha,\tau,r} N$ coincide. Since these kernels correspond via $X_A$ to the kernels of the canonical quotient maps of $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha}} G$ onto $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha},\tilde{\tau}} G$ and $\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha},\tilde{\tau},r} G$, respectively, the latter kernels coincide, and there is a canonical isomorphism

$$\mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha},\tilde{\tau}} G \cong \mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha},\tilde{\tau},r} G.$$

Specialising to the case where $A = \mathbb{C}$, $N = \{1\}$, we recover the well known isomorphism

$$C_0(G) \rtimes_\lambda G \cong C_0(G) \rtimes_{\lambda,r} G,$$

where $\lambda$ is the action of $G$ on $C_0(G)$ by left translation. As is well-known, the reduced crossed product on the right-hand side is isomorphic to the space of compact operators on $L^2(G)$.

If $(A,\alpha), (B,\beta) \in \mathcal{C}_H^*$, and $\theta : A \to B$ is an $H$-equivariant *-homomorphism, an associated *-homomorphism $\tilde{\theta}$ :$\mathrm{Ind}(A,\alpha) \to \mathrm{Ind}(B,\beta)$ is defined by

$$(\tilde{\theta}(f))(h) = \theta(f(h)).$$

Let $(A,\alpha) \in \mathcal{C}_H^*$ and let $I$ be an $\alpha_H$-invariant ideal of $A$. If $\iota : I \to A$ and $q : A \to A/I$ denote the inclusion and quotient maps, respectively, then $\iota$ and

$q$ are equivariant when $I$ has the restriction action $\alpha|I$ and $A/I$ the quotient action $\dot{\alpha}$.

Lemma 3.8 *The sequence*

$$0 \longrightarrow \text{Ind}\,(I,\alpha|) \xrightarrow{\tilde{\iota}} \text{Ind}\,(A,\alpha) \xrightarrow{\tilde{q}} \text{Ind}\,(A/I,\dot{\alpha}) \longrightarrow 0$$

*is exact.*

*Proof*: If $f \in \ker \tilde{q}$, then

$$0 = (\tilde{q}(f))(h) = q(f(h)) \Rightarrow f(h) \in I$$

for $h \in H$, i.e. $f \in$Ind$(I,\alpha|)$. □

Let $(A,\alpha) \in \mathcal{C}^*_H$ and let $\tau : N \to \mathcal{U}(A)$ be a twisting map relative to $\alpha$. If $I$ is an $\alpha_H$-invariant ideal of $A$, there are unital *-homomorphisms $\bar{\iota} : M(I) \to M(A)$ and $\bar{q} : M(A) \to M(A/I)$ given by $\bar{\iota}(u)x = ux$ and $\bar{q}(u)q(a) = q(ua)$ which extend $\iota$ and $q$, respecively. Twisting maps $\tau_I$ and $\tau_{A/I}$ relative to the restriction action $\alpha|I$ and the quotient action $\dot{\alpha}$ are given by $\tau_I = \bar{\iota} \circ \tau$ and $\tau_{A/I} = \bar{q} \circ \tau$, respectively, relative to which $\iota$ and $q$ are twist-equivariant. To simplify the notation, we write $\tau$ for both $\tau_I$ and $\tau_{A/I}$. By the same token, there are twisting maps relative to $N$, which will be denoted by $\tilde{\tau}$, on Ind$(I,\alpha|)$ and Ind$(A/I,\dot{\alpha})$, relative to which the induced morphisms $\tilde{\iota}$ and $\tilde{q}$ are twist-equivariant.

The following theorem is the main technical result for §§4 and 5.

Theorem 3.9 *The sequence*

$$0 \longrightarrow I \rtimes_{\alpha|,\tau,r} H \xrightarrow{\iota_{N,r}} A \rtimes_{\alpha,\tau,r} H \xrightarrow{q_{N,r}} (A/I) \rtimes_{\dot{\alpha},\tau,r} H \longrightarrow 0 \qquad (*)$$

*is exact if and only if the sequence*

$$0 \longrightarrow \text{Ind}(I,\alpha|) \rtimes_{\tilde{\alpha}|,\tilde{\tau},r} G \xrightarrow{\tilde{\iota}_{N,r}} \text{Ind}(A,\alpha) \rtimes_{\tilde{\alpha},\tilde{\tau},r} G$$

$$\xrightarrow{\tilde{q}_{N,r}} \text{Ind}(A/I,\dot{\alpha}) \rtimes_{\tilde{\dot{\alpha}},\tilde{\tau},r} G \longrightarrow 0 \qquad (**)$$

*is exact.*

*Proof*: Let $B_I$, $B_A$ and $B_{A/I}$ denote the three C*-algebras in $(*)$ and $E_I$, $E_A$ and $E_{A/I}$ the three C*-algebras in $(**)$. Let $J$ and $\tilde{J}$ be the kernels of the homomorphisms $q_{N,\tau} : B_A \to B_{A/I}$ and $\tilde{q}_{N,\tilde{\tau}} : E_A \to E_{A/I}$. Let $J = \ker q_{N,\tau}$ and $\tilde{J} = \ker \tilde{q}_{N,\tilde{\tau}}$. Identifying $B_I$ and $E_I$ with their images in $B_A$ and $E_A$ under the embeddings $\iota_{N,\tau}$ and $\tilde{\iota}_{N,\tilde{\tau}}$, respectively, it follows by Proposition 3.5

that $B_I \subseteq J$ and $E_I \subseteq \tilde{J}$. To prove the proposition it suffices to show that the ideals $B_I$ and $J$ of $B_A$ correspond to the ideals $E_I$ and $\tilde{J}$, respectively, of $E_A$ via the equivalence bimodule $X_{A,\tau,r}$.

The natural embedding of $C_c(G, I)$ in $C_c(G, A)$ extends to an embedding of the $E_I$–$B_I$ equivalence bimodule $X_{I,\tau,r}$ as an $E_I$–$B_I$ submodule $X$ of $X_{A,\tau,r}$. It follows readily from the definitions that $X$ is the norm closed linear span of both $X_{A,\tau,r}B_I$ and $E_I X_{A,\tau,r}$, from which it follows that $E_I$ and $B_I$ correspond via $X_{A,\tau,r}$.

Let $\sigma$ be a faithful representation of $A/I$ on a Hilbert space $\mathcal{H}$, and let $\pi = \sigma \circ q$. By Propositions 3.2 and 3.5, the representation of $E_A$ induced from the representation $\pi \rtimes_{\tau,r} \lambda_\tau^H$ of $B_A$ via $X_{A,\tau,r}$ is unitarily equivalent to $\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau},r} \lambda_{\tilde{\tau}}^G$. Since

$$\pi \rtimes_{\tau,r} \lambda_\tau^H = (\sigma \rtimes_{\tau,r} \lambda_\tau^H) \circ q_{N,r}$$

and

$$\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau},r} \lambda_{\tilde{\tau}}^G = (\tilde{\sigma}_{\dot{\alpha},\tau} \rtimes_{\tilde{\tau},r} \lambda_{\tilde{\tau}}^G) \circ \tilde{q}_{N,r}$$

the images of $\pi \rtimes_{\tau,r} \lambda_\tau^H$ and $\tilde{\pi}_{\alpha,\tau} \rtimes_{\tilde{\tau},r} \lambda_{\tilde{\tau}}^G$ are canonically isomorphic to $B_{A/I}$ and $E_{A/I}$ and their kernels are $J$ and $\tilde{J}$, respectively. By the discussion in §2 it follows that $J$ and $\tilde{J}$ correspond via $X_{A\tau,r}$. Since correspondence of ideals respects inclusion, this implies that $J = B_I$ if and only if $\tilde{J} = E_I$, i.e. $(*)$ is exact if and only if $(**)$ is exact. □

In §5 we shall consider a continuous action $\alpha$ of $H$ on $A$ which extends to a continuous action, also denoted by $\alpha$, of $G$ on $A$, and such that the ideal $I$ is $\alpha_G$-invariant, so that $\alpha|$ and $\dot{\alpha}$ also extend to continuous actions $\alpha|$ and $\dot{\alpha}$ of $G$ on $I$ and $A/I$, respectively. As noted in §2, there are then natural isomorphism

$$\theta_I : \operatorname{Ind}(I, \alpha|) \to C_0(G/H, I),$$

$$\theta_A : \operatorname{Ind}(A, \alpha) \to C_0(G/H, A),$$

and

$$\theta_{A/I} : \operatorname{Ind}(A/I, \dot{\alpha}) \to C_0(G/H, A/I).$$

A twisting map $\tilde{\tau} : N \to M(C_0(G/H, A))$ is defined by

$$(\tilde{\tau}(n)f)(sH) = \tau(n)f(gH) \qquad (f \in C_0(G/H, A)).$$

Relative to this twisting map and that on $\operatorname{Ind}(A, \alpha)$, the map $\theta_A$ is twist equivariant. Defining twisting maps $\tilde{\tau} : N \to M(C_0(G/H, I))$ and $\tilde{\tau} : N \to M(C_0(G/H, A/I))$ similarly, the maps $\theta_I$ and $\theta_{A/I}$ are twist-equivariant, and it follows straightforwardly that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ind}(I, \alpha|) \arrow[r, "\tilde{\iota}"] \arrow[d, "\theta_I"] & \operatorname{Ind}(A, \alpha) \arrow[r, "\tilde{q}"] \arrow[d, "\theta_A"] & \operatorname{Ind}(A/I, \dot{\alpha}) \arrow[r] \arrow[d, "\theta_{A/I}"] & 0 \\
0 \arrow[r] & C_0(G/H, I) \arrow[r, "id\otimes\iota"] & C_0(G/H, A) \arrow[r, "id\otimes q"] & C_0(G/H, A/I) \arrow[r] & 0
\end{tikzcd}$$

commutes (we have identified $C_0(G/H, A)$ with $C_0(G/H) \otimes A$, etc., in the bottom row to define the horizontal maps). Taking reduced twisted crossed products by $G$, we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Ind}(I, \alpha|) \rtimes_{\tilde{\alpha},\tilde{\tau},r} G \arrow[r] \arrow[d] & C_0(G/H, I) \rtimes_{\Delta^\alpha,\tilde{\tau},r} G \arrow[d] \\
\mathrm{Ind}(A, \alpha) \rtimes_{\tilde{\alpha},\tilde{\tau},r} G \arrow[r] \arrow[d] & C_0(G/H, A) \rtimes_{\Delta^\alpha,\tilde{\tau},r} G \arrow[d] \\
\mathrm{Ind}(A/I, \dot{\alpha}) \rtimes_{\tilde{\dot{\alpha}},\tilde{\tau},r} G \arrow[r] \arrow[d] & C_0(G/H, A/I) \rtimes_{\Delta^{\dot{\alpha}},\tilde{\tau},r} G \arrow[d] \\
0 & 0
\end{tikzcd}$$

Since the maps denoted by the horizontal arrows are bijections, by Proposition 3.5 (2), the left-hand column is exact if and only if the same is true of the right-hand column. The following corollary is now an immediate consequence of Theorem 3.9.

COROLLARY 3.10 *The sequence*

$$0 \longrightarrow I \rtimes_{\alpha|,\tau,r} H \xrightarrow{\iota_{N,r}} A \rtimes_{\alpha,\tau,r} H \xrightarrow{q_{N,r}} (A/I) \rtimes_{\dot{\alpha},\tau,r} H \longrightarrow 0$$

*is exact if and only if the sequence*

$$0 \longrightarrow (C_0(G/H, I) \rtimes_{\Delta^{\alpha|},\tilde{\tau},r} G \xrightarrow{(id\otimes\iota)_{N,r}} (C_0(G/H, A) \rtimes_{\Delta^\alpha,\tilde{\tau},r} G$$

$$\xrightarrow{(id\otimes q)_{N,r}} (C_0(G/H, A/I)) \rtimes_{\Delta^{\dot{\alpha}},\tilde{\tau},r} G \longrightarrow 0$$

*is exact.*

## 4. CLOSED SUBGROUPS.

Throughout this section $H$ will denote a closed subgroup of the locally compact group $G$. The following theorem is the first of the two main results of this section.

THEOREM 4.1 *If $G$ is exact, then so is $H$.*

*Proof*: Let $(A,\alpha) \in \mathcal{C}_H^*$ and let $I$ be an $\alpha_H$-invariant ideal of $A$. By Lemma 3.8, the associated sequence

$$0 \to \mathrm{Ind}(I,\alpha|) \to \mathrm{Ind}(A,\alpha) \to \mathrm{Ind}(A/I,\dot{\alpha}) \to 0$$

is exact. Since $G$ is exact, the corresponding sequence

$$0 \to \mathrm{Ind}(I,\alpha|) \rtimes_{\tilde{\alpha},r} G \to \mathrm{Ind}(A,\alpha) \rtimes_{\tilde{\alpha},r} G \to \mathrm{Ind}(A/I,\dot{\alpha}) \rtimes_{\tilde{\dot{\alpha}},r} G \to 0$$

is exact. By Theorem 3.9, the sequence

$$0 \longrightarrow I \rtimes_{\alpha|,r} H \xrightarrow{\iota_r} A \rtimes_{\alpha,r} H \xrightarrow{q_r} (A/I) \rtimes_{\dot{\alpha},r} H \longrightarrow 0$$

is exact. Since $(A,\alpha)$ and $I$ are arbitrary, this implies the exactness of $H$. $\square$

For the rest of the section we assume that $G$ is $\sigma$-compact and $H$ is exact. By [Bo, Chap. VII, §2, Th. 2] there is a Borel measure on $G/H$ which is quasi-invariant for the action of $G$ on $H$ given by left translation, though in general it is not possible to find a measure which is actually invariant for this action. If an invariant Borel probability measure on $G/H$ exists, $H$ is said to have *finite covolume* in $G$. We shall assume that this is the case for the rest of this section.

Let $(A,\alpha) \in \mathcal{G}_G^*$ and let $\bar{\alpha}$ denote the canonical extension of $\alpha$ to a continuous action of $G$ on the multiplier algebra $M(A)$. The natural embedding of $A$ in $M(A)$ is $G$-equivariant relative to this action and the corresponding crossed product map is an embedding of $A \rtimes_{\alpha,r} G$ as an ideal of $M(A) \rtimes_{\bar{\alpha},r} G$. It follows that there is a canonical *-homomorphism

$$M(A) \rtimes_{\bar{\alpha},r} G \to M(A \rtimes_{\alpha,r} G)$$

extending the natural embedding of $A \rtimes_{\alpha,r} G$ in $M(A \rtimes_{\alpha,r} G)$. This *-homomorphism is, in fact, an embedding. To see this, let $\pi$ be a faithful representation of $A$ on a Hilbert space $\mathcal{H}$. Then $\pi_\alpha \rtimes_r \lambda$ is a faithful representation of $A \rtimes_{\alpha,r} G$ on $\mathcal{H} \otimes L^2(G)$ which extends to a faithful representation of $M(A \rtimes_{\alpha,r} G)$ on $\mathcal{H} \otimes L^2(G)$. Let $\bar{\pi}$ be the canonical extension of $\pi$ to $M(A)$. Then $\bar{\pi}$ is a faithful representation of $M(A)$ on $\mathcal{H}$ and $\bar{\pi}_{\bar{\alpha}} \rtimes_r \lambda$ is a faithful representation of $M(A) \rtimes_{\bar{\alpha},r} G$ on $\mathcal{H} \otimes L^2(G)$. If we identify $M(A \rtimes_{\alpha,r} G)$ with its image on $\mathcal{H} \otimes L^2(G)$, the above *-homomorphism $M(A) \rtimes_{\bar{\alpha},r} G \to M(A \rtimes_{\alpha,r} G)$ is just the *-monomorphism $\bar{\pi}_{\bar{\alpha}} \rtimes_r \lambda$.

Let $(A,\alpha) \in \mathcal{C}_G^*$ and let $E_A = (C_0(G/H) \otimes A) \rtimes_{\Delta^\alpha,r} G$. Replacing $(A,\alpha)$ by the pair $(C_0(G/H) \otimes A, \Delta^\alpha)$ in the argument of the previous paragraph, we get a canonical embedding

$$\kappa : M(C_0(G/H) \otimes A) \rtimes_{\bar{\Delta}^\alpha,r} G \to M(E_A),$$

where $\bar{\Delta}^\alpha$ is the canonical extension of the diagonal action $\Delta^\alpha$ to $M(C_0(G/H) \otimes A)$. If $\pi$ is the embedding $a \to 1 \otimes a$ of $A$ in $M(C_0(G/H) \otimes A)$, $\pi$ is $G$-equivariant and the corresponding crossed product map

$$\pi_r : A \rtimes_{\alpha,r} G \to M(C_0(G/H) \otimes A) \rtimes_{\bar{\Delta}^\alpha,r} G$$

is an embedding. Let $\Phi_A$ denote the embedding

$$\kappa\pi_r : A \rtimes_{\alpha,r} G \to M(E_A).$$

If $I$ is an $\alpha_G$-invariant ideal of $A$, and $q : A \to A/I$ is the quotient map, associated with the *-homomorphism $\tilde{q} = id \otimes q : C_0(G/H) \otimes A \to C_0(G/H) \otimes (A/I)$ is the surjective crossed product *-homomorphism

$$\tilde{q}_r : E_A \to E_{A/I}.$$

This map extends to a *-homomorphism $M(E_A) \to M(E_{A/I})$, also denoted by $\tilde{q}_r$, which need not be surjective.

LEMMA 4.2 $\tilde{q}_r \Phi_A = \Phi_{A/I} q_r$.

*Proof*: Let $\pi$ and $\lambda$ be the canonical embeddings of $C_0(G/H) \otimes A$ and $G$ in $M(E_A)$, respectively, and let $\bar{\pi}$ be the embedding of $M(C_0(G/H) \otimes A)$ in $M(E_A)$ obtained by extending $\pi$ as above. Then for $a \in C_c(G, A)$,

$$\Phi_A(a) = \int_G \bar{\pi}(1 \otimes a(s))\lambda_s \, ds.$$

The linear span of the subset

$$\left\{ \int_G \pi(f \otimes b(t))\lambda_s \, ds : f \in C_0(G/H), b \in C_c(G, A) \right\}$$

of $E_A$ is dense in $E_A$, and for $a, b \in C_c(G, A)$, $f \in C_0(G/H)$,

$$\begin{aligned}
\tilde{q}_r(\Phi_A(a))\bar{q}_r(\int_G \pi(f \otimes b(t)\lambda_t \, dt) &= \tilde{q}_r(\Phi_A(a) \int_G \pi(f \otimes b(t)\lambda_t \, dt) \\
&= \bar{q}_r(\int_G \int_G \pi(f \otimes a(s)\alpha_s(b(t)))\lambda_{st} \, ds \, dt) \\
&= \int_G \int_G \pi(f \otimes q(a(s))q(\alpha_s(b(t))))\lambda_s t \, ds \, dt \\
&= \Phi_{A/I}(q(a))\bar{q}_r(\int_G \pi(f \otimes b(t)\lambda_t \, dt).
\end{aligned}$$

Thus $\tilde{q}_r \Phi_A = \Phi_{A/I} q$. □

LEMMA 4.3 For $f \in C_0(G/H)$ and $a \in A \rtimes_{\alpha,r} G$, $\bar{\pi}(f \otimes 1)\Phi_A(a) \in E_A$.

*Proof*: By the continuity of $\Phi_A$, it is enough to check this for $a$ in the dense subset $C_c(G, A)$ of $A \rtimes_{\alpha,r} G$. Then

$$\bar{\pi}(f \otimes 1)\Phi_A(a) = \int_G \pi(f \otimes a(s))\lambda_s \, ds \in E_A.$$

□

Fix an invariant probability measure $\mu$ on $G/H$. The map $P : C_0(G/H) \otimes A \to A$ given by

$$P(f \otimes a) = \left( \int_{G/H} f \, d\mu \right) a$$

is completely positive, contractive and $G$-equivariant. The corresponding complete contraction $P_r : E_A \to A \rtimes_{\alpha,r} G$ is given by

$$P_r( \int_G \pi(f \otimes a(s))\lambda_s \, ds) = \left( \int_G f \, d\mu \right) a$$

for $f \in C_0(G/H), a \in C_c(G, A)$.

LEMMA 4.4 *(i) For $f \in C_0(G/H)$ and $a \in A \rtimes_{\alpha,r} G$,*

$$P_r(\bar{\pi}(f \otimes 1)\Phi_A(a)) = \left( \int_G f \, d\mu \right) a.$$

*(ii) $P_r(E_I) \subseteq I \rtimes_{\alpha|,r} G$.*

*Proof*: (i) If $a \in C_c(G, A)$,

$$\bar{\pi}((f \otimes 1)\Phi_A(a)) = \int_G \pi(f \otimes a(s))\lambda_s \, ds$$

and

$$P_r(\bar{\pi}((f \otimes 1)\Phi_A(a)) = \left( \int_G f \, d\mu \right) a,$$

by the definition of $P_r$. The identity for general $a$ now follows by the continuity of $\Phi_A$ and $P_r$.

(ii) This is immediate from the definition of $P_r$.

THEOREM 4.5 *Let $G$ be a $\sigma$-compact group. If $G$ has a closed exact subgroup $H$ which has finite covolume in $G$, then $G$ is exact.*

*Proof*: Let $\mu$ be an invariant probability measure on $G/H$. We must show that if $A$ is a C*-algebra with an action $\alpha$ of $G$ and $I$ is an $\alpha_G$-invariant ideal of $A$ with quotient map $q : A \to A/I$, then $\ker q_r \subseteq I \rtimes_{\alpha,r} G$.

Let $x \in \ker q_r$. By Lemma 4.2, if $f \in C_0(G/H)$,

$$\bar{q}_r(\bar{\pi}(f \otimes 1)\Phi_A(x)) = \bar{\pi}(f \otimes 1)\bar{q}_r(\Phi_A(x)) = \bar{\Phi}_{A/I}(q_r(x)) = 0,$$

and so $\bar{\pi}(f \otimes 1)\Phi_A(x) \in \ker \bar{q}_r$. Since $H$ is exact, the sequence

$$0 \to I \rtimes_{\alpha|,r} H \to A \rtimes_{\alpha,r} H \to (A/I) \rtimes_{\dot{\alpha},r} H \to 0$$

is exact. By Corollary 3.10, the sequence

$$0 \to E_I \to E_A \to E_{A/I} \to 0$$

is exact, so that $\ker \bar{q}_r = E_I$. Thus $\bar{\pi}(f \otimes 1)\Phi_A(x) \in E_I$ and

$$\left(\int_G f\, d\mu\right) x = P_r(\bar{\pi}(f \otimes 1)\Phi_A(x)) \in I \rtimes_{\alpha|,r} G,$$

by Lemma 4.4 (ii). Since $G$ is $\sigma$-compact, there is a compact subset $K$ of $G$ such that $\mu(K) \neq 0$. Choosing $f \in C_0(G/H)$ such that $f(g) = 1$ for $g \in K$, it follows that $\int_G f\, d\mu \neq 0$, which implies that $x \in I \rtimes_{\alpha|,r} G$, as required. □

## 5. Extension of an exact group by an exact group.

The main result of this section is

Theorem 5.1 *Let $G$ be a locally compact group and let $N$ be a closed normal subgroup of $G$. If $N$ and $G/N$ are exact, then $G$ is exact.*

Let $N$ be a closed normal subgroup of $G$ and let $(A, \alpha) \in \mathcal{C}_G^*$. As indicated at the beginning of §3, there are a twisted action $(\gamma^\alpha, \tau)$ of $G$ on $A \rtimes_\alpha N$ relative to $N$ canonically associated with $\alpha$ and a natural isomorphism $A \rtimes_\alpha G \cong (A \rtimes_\alpha N) \rtimes_{\gamma^\alpha,\tau} G$. If $\{\pi, U\}$ is a universal covariant pair of representations of the system $\{A, G, \alpha\}$ on a Hilbert space $\mathcal{H}$ and $A \rtimes_\alpha N$ is identified with its image under the representation $\pi \rtimes U$ to $B(\mathcal{H})$, $\gamma^\alpha$ and $\tau$ are given by

$$\gamma_s^\alpha(x) = U_s x U_{s^{-1}}$$

and

$$\tau_n = U_n$$

for $x \in A \rtimes_\alpha N$, $s \in G$ and $n \in N$. That $\gamma^\alpha$ and $\tau$ have the stated properties follows from the proof of the reduced case, which is given in the following proposition. We define a twisted action of $G$ on the reduced crossed product $A \rtimes_{\alpha,r} N$ with analogous properties.

By Lemma 3.1 (3), the restriction of the representation $(\pi \otimes 1) \rtimes_r (U \otimes \lambda^G)$ on $\mathcal{H} \otimes L^2(G)$ to $A \rtimes_{\alpha,r} N$ is faithful. Identifying $A \rtimes_{\alpha,r} N$ with its image under this representation, an action $\gamma^{\alpha,r}$ of $G$ on $A \rtimes_{\alpha,r} N$ is given by

$$\gamma_s^{\alpha,r}(x) = (U_s \otimes \lambda_s) x (U_{s^{-1}} \otimes \lambda_{s^{-1}})$$

for $x$ in $A \rtimes_{\alpha,r} N$ and $s \in G$, and a twisting map $\tau'$ is given by

$$\tau'(t) = U_t \otimes \lambda_t$$

for $t \in N$. It is immediate that the canonical quotient map $A \rtimes_\alpha N \to A \rtimes_{\alpha,r} N$ is twist-equivariant relative to this action and twisting.

Proposition 5.2 *There is an isomorphism* $\Phi_A : (A \rtimes_{\alpha,r} N) \rtimes_{\gamma^{\alpha,r},\tau',r} G \to A \rtimes_{\alpha,r} G$ *which is natural in the sense that if* $\{B, G, \beta\}$ *is another* $G$*-covariant system and* $\theta : A \to B$ *is a* $G$*-equivariant *-homomorphism with associated homomorphisms*

$$\theta_r : A \rtimes_{\alpha,r} G \to B \rtimes_{\beta,r} G$$

*and*

$$\tilde{\theta} : (A \rtimes_{\alpha,r} N) \rtimes_{\gamma^{\alpha,r},\tau',r} G \to (B \rtimes_{\beta,r} N) \rtimes_{\gamma^{\beta,r},\tau',r} G,$$

*then* $\Phi_B \tilde{\theta} = \theta_r \Phi_A$*, i.e. the diagram*

$$\begin{tikzcd}
(A \rtimes_{\alpha,r} N) \rtimes_{\gamma^{\alpha,r},\tau',r} G \arrow[r, "\tilde{\theta}"] \arrow[d, "\Phi_A"'] & (B \rtimes_{\beta,r} N) \rtimes_{\gamma^{\beta,r},\tau',r} G \arrow[d, "\Phi_B"'] \\
A \rtimes_{\alpha,r} G \arrow[r, "\theta_r"] & B \rtimes_{\beta,r} G
\end{tikzcd}$$

*commutes.*

*Proof:* Let$\{\pi, U\}$ be a covariant pair of representations of $\{A, G, \alpha\}$ on the Hilbert space $\mathcal{H}$ with $\pi$ faithful. The crossed product $A \rtimes_{\alpha,r} N$ can be identified with the C*-algebra generated by the operators on $\mathcal{H} \otimes L^2(G)$ of form

$$\int_G (\pi(a(n)) \otimes 1)(U_n \otimes \lambda_n) dm_N(n)$$

for $a \in C_c(N, A)$. By Lemma 3.1 (3), $(A \rtimes_{\alpha,r} N) \rtimes_{\alpha,\tau',r} G$ can be identified with the C*-algebra (in fact the closed linear span) generated by the set $\{T_f : f \in C_c(G \times N, A)\}$ of operators on $\mathcal{H} \otimes L^2(G) \otimes L^2(G/N)$, where

$$\begin{aligned}
T_f &= \int_G \int_N (\pi(f(s,n)) \otimes 1 \otimes 1)(U_n \otimes \lambda_n \otimes 1)(U_s \otimes \lambda_s \otimes \dot{\lambda}_{sN}) dm_N(n) dm_G(s) \\
&= \int_G \int_N (\pi(f(n^{-1}s, n)) \otimes 1 \otimes 1)(U_s \otimes \lambda_s \otimes \dot{\lambda}_{sN}) dm_N(n) dm_G(s) \\
&\in A \rtimes_{\alpha,r} G
\end{aligned}$$

for $f \in C_c(G \times N, A)$, $A \rtimes_{\alpha,r} G$ being identified here with its image under the integrated form representation of the pair $\{\pi \otimes 1 \otimes 1, U \otimes \lambda \otimes \dot{\lambda}\}$ (which is unitarily equivalent to the pair $\{(\pi \otimes 1) \otimes 1, (U \otimes \dot{\lambda}) \otimes \lambda\}$).

Let $g \in C_c(N)$ be a nonnegative real-valued function such that

$$\int_N g(n) dm_N(n) = 1,$$

let $a \in C_c(G, A)$ and let $f \in C_c(G \times N, A)$ be given by

$$f(s, n) = a(s) g(n).$$

Choosing $g$ with support in a suitable neighbourhood of the identity of $N$, the operator of the above form corresponding to this $f$ can be made to approximate

$$\int_G (\pi(a(s)) \otimes 1 \otimes 1)(U_s \otimes \lambda_s \otimes \dot{\lambda}_{sN}) dm_G(s)$$

arbitrarily closely in norm. Thus if we define $\Phi_A$ on $\{T_f : f \in C_c(G \times N, A)\}$ by

$$\Phi_A(T_f) = \int_G \int_N (\pi(f(n^{-1}s, n)) \otimes 1)(U_s \otimes \lambda_s) dm_N(n) dm_G(s),$$

$\Phi_A$ extends uniquely to a *-isomorphism of $(A \rtimes_{\alpha,r} N) \rtimes_{\gamma^{\alpha,r},\tau',r} G$ onto $A \rtimes_{\alpha,r} G$. The naturalness of $\Phi_A$ is a straightforward consequence of this definition. □

Combes [Co] introduced a notion of Morita equivalence for actions of a locally compact group $G$. Let $(A, \alpha), (B, \beta) \in \mathcal{C}_G^*$ and let $X$ be a $B$–$A$ equivalence bimodule. A continuous action $u$ of $G$ on $X$ is a set of bijective linear isometries $\{u_s : s \in G\}$ of $X$ such that the map $s \to u_s$ is strongly continuous and for each $s$

$$u_s(x < y, z >_A) = u_s(x) < u_s(y), u_s(z) >_A$$

for $x, y, z \in X$. The actions $\beta$ and $\alpha$, or more accurately the pairs $(B, \beta)$ and $(A, \alpha)$, are *Morita equivalent* if there is a continuous action $u$ of $G$ on $X$ such that for each $s$,

$$\alpha_s(< x, y >_A) =< u_s(x), u_s(y) >_A$$

and

$$\beta_s(< x, y >_B) =< u_s(x), u_s(y) >_B$$

for $x, y \in X$. When $(A, \alpha)$ and $(B, \beta)$ are Morita equivalent, there are Morita equivalences between $A \rtimes_\alpha G$ and $B \rtimes_\beta G$, and between $A \rtimes_{\alpha,r} G$ and $B \rtimes_{\beta,r} G$, by [Co, §3].

Echterhoff [Ech] has extended this idea to twisted actions as follows. Let $N$ be a closed normal subgroup of $G$ and let $(\alpha, \tau)$ and $(\beta, \sigma)$ be twisted actions of $G$ on $A$ and $B$, respectively, relative to $N$. Then $(\beta, \sigma)$ and $(\alpha, \tau)$ are Morita equivalent relative to the pair $(X, u)$ if $u$ satisfies the above identities and also

$$u_n x = \sigma_n x \tau_n^{-1}$$

for $n \in N$ and $x \in X$. Moreover, if $\{A, G, \alpha, \tau\}$ and $\{B, G, \beta, \sigma\}$ are Morita equivalent, then $A \rtimes_{\alpha,\tau} G$ is Morita equivalent to $B \rtimes_{\beta,\sigma} G$ [Ech, p.174, Remark 2]. We shall show the corresponding result for the reduced crossed products.

Although the arguments that follow involving Morita equivalence could be expressed solely in terms of equivalence bimodules, we have found it easier to bring out some of the functorial aspects of the proof using the equivalent idea of a linking algebra, due to Brown, Green and Rieffel [BGR], which we now briefly recall. If $C$ is a nonzero C*-algebra, a projection $p \in M(C)$ is *full* if the linear span of the set $CpC = \{xpy : x, y \in C\}$ is dense in $C$. Suppose that $p, q \in M(C)$ are full projections such that $p + q = 1$, and let $A = pCp$, $B = qCq$ and $X = pCq$. If we define $A$- and $B$- valued inner products on $X$ by

$$< x, y >_A = xy^*,\ < x, y >_B = x^* y$$

and let $A$ and $B$ act on $X$ by left and right multiplication, respectively, then $X$ becomes an $A$–$B$ equivalence bimodule. Conversely, if $A$ and $B$ are C*-algebras and $X$ is an $A$–$B$ equivalence bimodule, then we can find a C*-algebra $C$, known as a linking algebra for $A$ and $B$, and full projections $p, q \in M(C)$ such that $p+q = 1$, $A \cong pCp$, $B \cong qCq$ and such that $X$ and $pCq$ are isomorphic as $A$–$B$ equivalence modules. Passage from $X$ to the corresponding $C$ is functorial, in a sense to be made precise in what follows.

Let IMP be the category whose objects are triples $\{A, B, X\}$ consisting of a Morita equivalent pair of C*-algebras $A, B$ and an $A$–$B$ equivalence bimodule $X$. Given two such triples $\{A, B, X\}$ and $\{A_1, B_1, X_1\}$, a map between them is a triple $\{\phi, \psi, \omega\}$, consisting of *-homomorphisms $\phi : A \to A_1$, $\psi : B \to B_1$, and a linear map $\omega : X \to X_1$ satisfying

$$\omega(ax) = \psi(a)\omega(x),\ \omega(xb) = \omega(x)\phi(b),$$

$$\omega(x) < \omega(y), \omega(z) >_{B_1} = < \omega(x), \omega(y) >_{A_1} \omega(z),$$

$$< \omega(x), \omega(y) >_{A_1} = \phi(< x, y >_A),\ < \omega(x), \omega(y) >_{B_1} = \psi(< x, y >_B)$$

for $a \in A$, $b \in B$ and $x, y, z \in X$. Given two pairs $A, A_1$ and $B, B_1$ of C*-algebras, *-homomorphisms $\phi : A \to A_1$ and $\psi : B \to B_1$ will be said to be *Morita equivalent* if there is an $\omega : X \to X_1$ such that $\{\phi, \psi, \omega\}$ is a map in IMP. This is consistent with the definition of Morita equivalence for actions when $\phi$ and $\psi$ are automorphisms of $A$ and $B$, respectively.

Let $A$ and $B$ be Morita equivalent C*-algebras and let $X$ be an $A$–$B$ equivalence bimodule. To see that the associated linking algebra is related to $X$ functorially, we recall its construction (cf. [BGR, proof of Theorem 1]). Let $X^*$ be the $B$–$A$ equivalence bimodule conjugate to $X$. There is a conjugate linear bijection $x \to x^*; X \to X^*$ such that

$$bx^* = (xb^*)^*,\ x^*a = (a^*x)^*,\ < x^*, y^* >_A = < y, x >_A,\ < x^*, y^* >_B = < y, x >_B$$

for $a \in A$, $b \in B$ and $x, y \in X$. For $x, y \in X$ let $x^*y$ and $xy^*$ be the elements $< x, y >_A$ and $< x, y >_B$ of $A$ and $B$, respectively. The set of matrices

$$C = \{ \begin{bmatrix} a & x \\ y^* & b \end{bmatrix} : a \in A,\ b \in B,\ x, y \in X\},$$

with matrix addition and multiplication, is a *-algebra. Moreover $X \oplus B$ has left and right actions of $C$ and $B$, respectively, and $C$- and $B$-valued inner products can be defined in such a way that $X \oplus B$ becomes a $C$–$B$ equivalence bimodule. A norm on $X \oplus B$ is defined in terms of the $B$-valued inner product on $X$ and the norm of $B$ relative to which the associated C*-norm on $C$ is complete. The projections

$$p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are in $M(C)$, are both full, and $A \cong pCp$ and $B \cong qCq$. Moreover $pCq$ and $X$ are isomorphic as $A$–$B$ equivalence bimodules. Let $\{A, B, X\}, \{A_1, B_1, X_1\}$ be objects of IMP and let $\{\phi, \psi, \omega\} : \{A, B, X\} \to \{A_1, B_1, X_1\}$ be a morphism. If $C$ and $C_1$ are the linking algebras constructed from $\{A, B, X\}$ and $\{A_1, B_1 X_1\}$, respectively, by this procedure, a *-homomorphsim $\phi = \Phi_{\{}\phi, \psi, \omega\} : C \to C_1$ is defined by

$$\Phi_{\{}\phi, \psi, \omega\}(\begin{bmatrix} a & x \\ y^* & b \end{bmatrix}) = \begin{bmatrix} \phi(a) & \omega(x) \\ (\omega(y))^* & \psi(b) \end{bmatrix}.$$

Then $\Phi(p) = p$, $\Phi(q) = q$, $\Phi|pCp = \psi$ and $\Phi|qCq = \phi$. Let LINK be the category whose objects are triples $\{C, p, q\}$ consisting of a C*-algebra $C$ with full projections $p, q \in M(C)$ such that $p + q = 1$. If $\{C, p, q\}$ and $\{C_1, p_1, q_1\}$ are objects of LINK, a morphism $\theta$ from $\{C, p, q\} \to \{C_1, p_1, q_1\}$ is a a *-homomorphism $C \to C_1$ such that $\theta(px) = p_1\theta(x)$ for $x \in C$. Writing $C_{\{A,B,X\}}$ for the linking algebra constructed from the triple $\{A, B, X\}$, the map $\{A, B, X\} \to C_{\{A,B,X\}}$ is a functor from IMP to LINK giving an equivalence of categories.

For strongly Morita equivalent C*-algebras $A$ and $B$, the order-preserving correspondence between the ideals of $A$ and $B$ described in §2 can be expressed elegantly in terms of $C$. If $I$ is an ideal of $A$ or $B$, then $I_C$, the ideal of $C$ generated by $I$, is just the closure of the linear span of the set $CIC$. If $I$ is an ideal of $C$, let

$$I_A = pIp, \ I_B = qIq.$$

Then $I_A = I \cap pCp = I \cap A$ and $I_B = I \cap qCq = I \cap B$, and it follows easily from the fullness of $p$ and $q$ that for any ideal $I$ of $C$, $I = (I_A)_C = (I_B)_C$. For any ideal $I$ of $A$, $I = (I_C)_A$, and similarly for $B$. The map $I \to (I_C)_B$ is thus an order-preserving bijection from the ideals of $A$ to the ideals of $B$.

Lemma 5.3 *Let* $\{A, B, X\}, \{A_1, B_1, X_1\} \in$ IMP, *and let* $\{\phi, \psi, \omega\} : \{A, B, X\} \to \{A_1, B_1, X_1\}$ *be a morphism. Then* $\ker\psi$ *corresponds to* $\ker\phi$ *under the above bijection.*

*Proof:* If $\Phi : C_{\{A,B,X\}} \to C_{\{A_1,B_1,X_1\}}$ is the *-homomorphism corresponding to $\{\phi, \psi, \omega\}$, then $x \in \ker\Phi \cap A \Leftrightarrow \phi(x) = 0 \Leftrightarrow x \in \ker\phi$. Hence $\ker\phi = \ker\Phi \cap A$. Similarly $\ker\psi = \ker\Phi \cap B$. Thus $\ker\phi$, $\ker\Phi$ and $\ker\psi$ are corresponding ideals. □

Let $\{A, G, \alpha, \sigma\}$ and $\{B, G, \beta, \tau\}$ be Morita equivalent twisted covariant systems relative to a pair $(X, u)$ consisting of an $A$–$B$ equivalence bimodule $X$

with an action $u$ of $G$, with twisting relative to the normal subgroup $N$ of $G$. Then for each $s \in G$, $\{\alpha_s, \beta_s, u_s\}$ is a map in IMP, in fact an automorphism of $\{A, B, X\}$. If $\Gamma_s$ denotes the corresponding automorphism of the linking algebra $C = C_{\{A,B,X\}}$, then

$$\Gamma_s(\begin{bmatrix} a & x \\ y^* & b \end{bmatrix}) = \begin{bmatrix} \alpha_s(a) & u_s x \\ (u_s y)^* & \beta_s(b) \end{bmatrix}.$$

It is immediate that $s \to \Gamma_s$ is a continuous action of $G$ on $C_{\{A,B,X\}}$, $\Gamma$ fixes $p$ and $q$, $\Gamma|A = \alpha$ and $\Gamma|B = \beta$. For $n \in N$

$$\kappa_n = \begin{bmatrix} \sigma_n & 0 \\ 0 & \tau_n \end{bmatrix}$$

is in $M(C)$ and the map $\kappa : n \to \kappa_n$ is a twisting map for the action $\Gamma$.

The canonical embedding of $C$ in $M(C \rtimes_\Gamma G)$ extends to an embedding of $M(C)$ in $M(C \rtimes_\Gamma G)$, where $x \in M(C)$ is identified with the element of $M(C \rtimes_\Gamma G)$ which sends $f$ in $C_c(G, C)$ to $xf$. With this identification, $p$ and $q$ are in $M(C \rtimes_\Gamma G)$, are full projections for $C \rtimes_\Gamma G$, and there are canonical isomorphisms

$$p(C \rtimes_\Gamma G)p \cong A \rtimes_\alpha G, \quad q(C \rtimes_\Gamma G)q \cong B \rtimes_\beta G,$$

by [Co, §6]. In fact, if $\pi_0 : C \to M(C \rtimes_\Gamma G)$ and $U_0 : G \to M(C \rtimes_\Gamma G)$ are the canonical embeddings and $A$ is identified with $pCp$, then $\{\pi_0|A, U_0\}$ is a covariant pair of representations of $\{A, G, \beta\}$, and the integrated form of this pair is a *-homomorphism $\theta_A : A \rtimes_\alpha G \to M(C \rtimes_\Gamma G)$. The image of $A \rtimes_\alpha G$ under $\theta_\alpha$ is just $p(C \rtimes_\Gamma G)p$. To see that $\theta_A$ is injective, let $\{\pi, U\}$ be a universal covariant pair of representations of $\{A, G, \alpha\}$ on a Hilbert space $\mathcal{H}$. By the equivariant form of Stinespring's theorem, there are a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a covariant pair of representations $\{\pi_1, U_1\}$ of $\{C, G, \Gamma\}$ on $\mathcal{K}$ such that $\bar{\pi}_1(p)$ is the projection onto $\mathcal{H}$ and the pair $\{\bar{\pi}_1(p)(\pi_1|B)\bar{\pi}_1(p), \bar{\pi}_1(p)U_1\bar{\pi}_1(p)\}$ is a covariant pair of representations of $\{A, G, \alpha\}$ unitarily equivalent to $\pi \rtimes U$. Then $\bar{\pi}_1(p)\,((\pi_1 \rtimes U_1) \circ \theta_A)\,\bar{\pi}_1(p)$ is unitarily equivalent to $\pi \rtimes U$, which is faithful. Thus $\theta_A$ is faithful.

The canonical quotient map $q_C : C \rtimes_\Gamma G \to C \rtimes_{\Gamma,\kappa,r} G$ extends to a *-homomorphism $\bar{q}_C : M(C \rtimes_\Gamma G) \to M(C \rtimes_{\Gamma,\kappa,r} G)$, and the covariant pair $\{\bar{q}_C \circ (\pi_0|B), \bar{q}_C \circ U\}$ of homomorphisms of $\{A, G, \alpha\}$ is twist-preserving. It follows by Lemma 3.1 that $q_C \circ \theta_A$ has a factorisation $q_C \circ \theta_A = \bar{\theta}_A \circ q_A$, where $q_A : A \rtimes_\alpha G \to A \rtimes_{\alpha,\sigma,r} G$ is the quotient map and $\bar{\theta}_A : A \rtimes_{\alpha,\sigma,r} G \to C \rtimes_{\Gamma,\kappa,r} G$ is a *-isomorphism with image $\bar{p}(C \rtimes_{\Gamma,\kappa,r} G)\bar{p}$, where $\bar{p}$ is the projection $\bar{q}_C(p)$. Applying analogous considerations to the crossed products involving $B$, we obtain *-monomorphisms

$$\theta_B : B \rtimes_\beta G \to q(C \rtimes_\Gamma G)q$$

and

$$\bar{\theta}_B : B \rtimes_{\beta,\sigma,r} G \to \bar{q}(C \rtimes_{\Gamma,\kappa,r} G)\bar{q}$$

such that $q_C \circ \theta_B = \bar{\theta}_B \circ q_B$, where $\bar{q} = \bar{q}_C(q)$. Since the fullness of the projections $p$ and $q$ for $C \rtimes_\Gamma G$ implies the fullness of $\bar{p}$ and $\bar{q}$ for $C \rtimes_{\Gamma,\kappa,r} G$, we obtain

Proposition 5.4 *If $\{A, G, \alpha, \tau\}$ and $\{B, G, \beta, \sigma\}$ are Morita equivalent twisted covariant systems relative to $N$, then the C*-algebras $A \rtimes_{\alpha,\tau,r} G$ and $B \rtimes_{\beta,\sigma,r} G$ are strongly Morita equivalent. Moreover the diagram*

$$\begin{tikzcd}
A \rtimes_\alpha G \arrow[r, "q_A"] \arrow[d, "\theta_A"] & A \rtimes_{\alpha,\tau,r} G \arrow[d, "\bar{\theta}_A"] \\
C \rtimes_\Gamma G \arrow[r, "q_C"] & C \rtimes_{\Gamma,\kappa,r} G \\
B \rtimes_\beta G \arrow[r, "q_B"] \arrow[u, "\theta_B"] & B \rtimes_{\beta,\sigma,r} G \arrow[u, "\bar{\theta}_B"]
\end{tikzcd}$$

*commutes, so that the *-homomorphisms $q_A$ and $q_B$ are Morita equivalent.*

If $\{A, G, \alpha, \tau\}$ is a twisted covariant system relative to $N$, then there is an action $\beta$ of $G$ on $E_A = C_0(G/N, A) \rtimes_{\Delta^\alpha, \bar{\tau}} G$ such that the twisted covariant systems $\{A, G, \alpha, \tau\}$ and $\{E_A, G, \beta, 1_N\}$ are Morita equivalent, where $1_N$ is the trivial twisting map $n \to 1 \in M(B)$ [Ech, Theorem 1].

An equivalence bimodule $Y_A$ giving this Morita equivalence is obtained by applying the mapping $\{\nu_A, id, \bar{\alpha}\}$ of IMP to the triple $\{\mathrm{Ind}(A, \alpha) \rtimes_{\bar{\alpha}} G, A, X_A\}$ of §2, with $H = N$, where $\nu_A : \mathrm{Ind}(A, \alpha) \to C_0(G/N, A)$ is the map isomorphism of §2 and $(\bar{\alpha}(x))(s) = \alpha_s(x(s))$ for $x \in C_c(G, A)$. Letting $X_1 = C_c(G, A)$, $E_0 = C_c(G, C_0(G/N, A)))$ and $B_0 = C_c(N, A)$, with the convolution products relative to the actions $\Delta^\alpha$ and $\alpha$, respectively, the algebras $E_0$ and $B_0$ having the C*-norms and positive cones coming from their canonical embeddings in $C_0(G/N, A) \rtimes_{\Delta^\alpha} G$ and $A \rtimes_\alpha N$, respectively, the resulting $E_0$–$B_0$ equivalence bimodule structure on $X_1$ is given by

$$\begin{aligned}
(fx)(r) &= \int_G f(s, rN)\alpha_s(x(s^{-1}r))dm_G(s) \\
(xg)(r) &= \int_H x(rt)\alpha_{rt}(g(t^{-1}))dm_H(t) \\
\langle x, y\rangle_{E_0}(s, rN) &= \int_H \Delta_G(s^{-1}rt)x(rt)\alpha_s(y(s^{-1}rt)^*)dm_H(t) \\
\langle x, y\rangle_{B_0}(t) &= \delta(t)\int_G \alpha_{s^{-1}}(x(s)^*y(st))dm_G(s).
\end{aligned}$$

The action $\beta$ of $G$ on $E_A$ is given on an element $f$ of the dense *-subalgebra $C_c(G, C_0(G/N, A))$ of $E_A$ by

$$(\beta_s(f))(r, tN) = f(r, tsN),$$

where $f$ is considered as a function on $G \times (G/N)$, and an action $u$ of $G$ on $X_1$ is given by

$$(u_s x)(t) = \Delta_G(s)\Delta_{G/N}(s)^{-1/2} x(ts).$$

Recalling that, by Remark 2.5 (2), there is a canonical isomorphism $A \cong A \rtimes_{\alpha,1_n} N$, the equivalence bimodule $Y_A$ is then obtained by completing $X_1$ with respect to the norm $x \to || < x, x >_A ||^{1/2}$ and extending the left and right actions to $E_A$ and $A$, respectively, by continuity.

Before proving Theorem 5.1 we require one further observation. Let $\beta$ be a twisted action of $G$ on a C*-algebra $D$ relative to $N$ with trivial twisting map $1_N$, so that $\beta_n = id_D$ for $n \in N$. Then an action $\bar{\beta}$ of $G/N$ on $D$ is given by

$$\bar{\beta}_{sN}(x) = \beta_s(x).$$

The analogue of the following result for full twisted crossed products is implicit in [Ech].

Lemma 5.5 *There is an isomorphism $\Psi_D : D \rtimes_{\bar{\beta},r} (G/N) \to D \rtimes_{\beta,1_N,r} G$ which is natural in the sense that, if $\{D, G, \beta, 1_N\}$ and $\{D_1, G, \beta_1, 1_N\}$ are twisted covariant systems and $\theta : D_1 \to D$ is a $G$-equivariant twist-preserving *-homomorphism, then the diagram*

$$\begin{tikzcd} D_1 \rtimes_{\bar{\beta}_1,r} (G/N) \arrow[r, "\theta_r"] & D \rtimes_{\bar{\beta},r} (G/N) \\ D_1 \rtimes_{\beta_1,1_N,r} G \arrow[u, "\Psi_{D_1}"] \arrow[r, "\theta_{N,r}"] & D \rtimes_{\beta,1_N,r} G \arrow[u, "\Psi_D"] \end{tikzcd} \qquad (*)$$

*commutes, where the horizontal maps are the *-homomorphisms corresponding canonically to $\theta$.*

*Proof:* Let $\pi_0 : D \to M(D \rtimes_{\bar{\beta},r} (G/N))$ and $\lambda_0 : G/N \to M(D \rtimes_{\bar{\beta},r} (G/N))$ be the canonical monomorphisms. If $\dot{\lambda}_0$ denotes the composition of $\lambda_0$ with the quotient homomorphism $G \to G/N$, and $M(D \rtimes_{\bar{\beta},r} (G/N))$ is represented faithfully on some Hilbert space $\mathcal{H}$, the pair $\{\pi_o, \lambda_0\}$ is a covariant pair of representations of the system $\{D, G, \beta\}$ which is twist-covariant for the twisting map $1_N$. The representation $\pi_0 \rtimes_{1_n} \lambda_0$ of $D \rtimes_{\beta,1_N} G$ in $M(D \rtimes_{\bar{\beta},r} (G/N))$ has image $D \rtimes_{\bar{\beta},r} (G/N)$, and, by Proposition 3.5 (1), has a factorisation $\pi_0 \rtimes_{1_N} \lambda_0 = \psi_D \circ q$, where $\psi_D$ is a *-homomorphism from $D \rtimes_{\beta,1_N,r} G$ onto $D \rtimes_{\bar{\beta},r} (G/N)$ and $q : D \rtimes_{\beta,1_N} G \to D \rtimes_{\beta,1_N,r} G$ is the canonical quotient map. Since $\pi_0$ is faithful, $\psi_D$ is injective.

If $\theta : D_1 \to D$ is a $G$-covariant twist-preserving *-homomorphism, let $\bar{\theta}_{N,r} = \psi_D^{-1}\theta_r\psi_{D_1}$. By the definition of $\psi_D$ and $\psi_{D_1}$, the diagram

$$\begin{tikzcd} D_1 \rtimes_{\beta_1} G \arrow[r, "\theta_u"] \arrow[d] & D \rtimes_\beta G \arrow[d] \\ D_1 \times_{\beta_1,1_N,r} G \arrow[r, "\bar{\theta}_{N,r}"] & D \rtimes_{\beta,1_N,r} G \end{tikzcd}$$

commutes, where the vertical arrows are the canonical quotient maps. By Proposition 3.5 (2), $\bar{\theta}_{N,r} = \theta_{N,r}$, from which the commutativity of $(*)$ is immediate. $\square$

*Proof of Theorem 5.1.* Assume that $N$ and $G/N$ are exact. Let $(I, \alpha|), (A, \alpha), (B, \dot{\alpha}) \in \mathcal{C}^*_G$ and let $\iota : I \to A$ and $q : A \to B$ be $G$-equivariant *-homomorphisms such that $\operatorname{im}\iota = \ker q$, i.e. such that the sequence

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{q} B \longrightarrow 0 \tag{1}$$

is exact. To prove the theorem we must show that the sequence

$$0 \longrightarrow I \rtimes_{\alpha|,r} G \xrightarrow{\iota_r} A \rtimes_{\alpha,r} G \xrightarrow{q_r} B \rtimes_{\dot{\alpha},r} G \longrightarrow 0 \tag{2}$$

is exact. In what follows we shall identify $I$ with its image in $A$.

Since $N$ is exact, the sequence

$$0 \longrightarrow I \rtimes_{\alpha|,r} N \xrightarrow{\iota_r} A \rtimes_{\alpha,r} N \xrightarrow{q_r} B \rtimes_{\dot{\alpha},r} N \longrightarrow 0 \tag{3}$$

is exact. Let $I_N = I \rtimes_{\alpha|,r} N$, $A_N = A \rtimes_{\alpha,r} N$ and $B_N = B \rtimes_{\dot{\alpha},r} N$. By Proposition 5.2 and the discussion preceding it, there are $N$-twisted actions $(\gamma_I, \tau)$, $(\gamma_A, \tau)$ and $(\gamma_B, \tau)$ of $G$ on $I_N$, $A_N$ and $B_N$, respectively, and isomorphisms

$$\Phi_I : I_N \rtimes_{\gamma_I,\tau,r} G \to I \rtimes_{\alpha|,r} G,$$

$$\Phi_A : A_N \rtimes_{\gamma_A,\tau,r} G \to A \rtimes_{\alpha,r} G$$

and

$$\Phi_B : B_N \rtimes_{\gamma_B,\tau,r} G \to B \rtimes_{\dot{\alpha},r} G$$

such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & I_N \rtimes_{\gamma_I,\tau,r} G \arrow[r, "\iota_{N,r}"] \arrow[d, "\Phi_I"'] & A_N \rtimes_{\gamma_A,\tau,r} G \arrow[r, "q_{N,r}"] \arrow[d, "\Phi_A"'] & B_N \rtimes_{\gamma_B,\tau,r} G \arrow[r] \arrow[d, "\Phi_B"'] & 0 \\
0 \arrow[r] & I \rtimes_{\alpha|,r} G \arrow[r, "\iota_r"] & A \rtimes_{\alpha,r} G \arrow[r, "q_r"] & B \rtimes_{\dot{\alpha},r} G \arrow[r] & 0
\end{tikzcd}$$

commutes, where $\iota_{N,r}$ and $q_{N,r}$ are the crossed product homomorphisms corresponding to the *-homomorphisms $\iota_r|I_N$ and $q_r|A_N$, then latter $G$-equivariant and twist preserving relative to the twisted actions on $I_N$, $A_N$ and $B_N$, as follows readily from the definitions. The exactness of (2) is thus equivalent to that of the sequence

$$0 \longrightarrow I_N \rtimes_{\gamma_I,\tau,r} G \xrightarrow{\iota_{\tau,r}} A_N \rtimes_{\gamma_A,\tau,r} N \xrightarrow{q_{\tau,r}} B_N \rtimes_{\gamma_B,\tau,r} N \longrightarrow 0 \tag{4}$$

By Corollary 3.10, $A_N$ and $E_A = (C_0(G/N, A_N) \rtimes_{\Delta\gamma,\tilde{\tau}} G$ are Morita equivalent via the $E_A$–$A_N$ equivalence bimodule obtained by completing $X_0 = C_c(G, A_n)$ with respect to the seminorm $||\ \ ||$ given by

$$||x|| = ||< x, x >_{A_N} ||^{1/2}.$$

Letting $C_A = C_{\{A,B,X\}}$, we identify $E_A$ with $pC_Ap$ and $A_N$ with $qC_Aq$. If $\beta_A$ is the action of $G$ on $E_A$ defined earlier such that $(\beta_A, 1_N)$ is Morita equivalent to $(\gamma_A, \tau)$ via $(X_A, u_A)$, where $u_A$ is the canonically defined action of $G$ on $X_A$, let $(\Gamma_A, \kappa)$ be the corresponding twisted action of $G$ on $C_A$. We define $X_I$, $X_B$, $C_I$, $C_B$, $\beta_I$, $\beta_B$, $\Gamma_I$ and $\Gamma_B$ similarly. If $q_X : C_c(G, A_N) \to C_c(G, B_N)$ is the natural map given by

$$(q_X(f))(s) = q_r(f(s)),$$

for $s \in G$, then

$$||q_X(f)|| = ||< q_X(f), q_X(f) >_{B_N} ||^{1/2} = ||q_r(< f, f >)_{A_N}||^{1/2} \leq ||f||.$$

If $\iota_X : C_c(G, I_N) \to C_c(G, A_N)$ is defined similarly starting from $\iota$, since $\iota_r : I_N \to A_N$ is an isometric embedding, the same is true of $\iota_X$. It follows that $\iota_X$ and $q_X$ extend to an isometric embedding of $X_I$ in $X_A$ and a contraction of $X_A$ onto $X_B$, respectively, which we will still denote by $\iota_X$ and $q_X$. Identifying $X_I$ with its image in $X_A$ we get a corresponding embedding $\iota_C : C_I \to C_A$. It is straightforward to verify that the diagram

$$\begin{tikzcd}
E_I \arrow[r, "\iota_E"] \arrow[d] & E_A \arrow[d] \\
C_I \arrow[r, "\iota_C"] & C_A \\
I_N \arrow[r, "\iota_r"] \arrow[u] & A_N \arrow[u]
\end{tikzcd}$$

commutes. Similarly the surjection $q_X$ gives rise to a *-epimorphism $q_C : C_A \to C_B$ given by

$$q_C(\begin{bmatrix} b & x \\ y^* & a \end{bmatrix}) = \begin{bmatrix} q_E(b) & q_X(x) \\ q_X(y)^* & q_r(a) \end{bmatrix}.$$

Again it is routine to verify that the diagram

$$\begin{tikzcd}
E_A \arrow[r, "q_E"] \arrow[d] & E_B \arrow[d] \\
C_A \arrow[r, "q_C"] & C_B \\
A_N \arrow[r, "q_r"] \arrow[u] & B_N \arrow[u]
\end{tikzcd}$$

commutes. All the maps in these two diagrams are twist-preserving relative to the respective twistings for each algebra. Hence if we take the twisted crossed products of all the algebras by $G$, we get commuting diagrams

$$\begin{array}{ccc} E_I \rtimes_{\beta_I,1_N,r} G & \xrightarrow{(\iota_E)_r} & E_A \rtimes_{\beta_A,1_N,r} G \\ \downarrow & & \downarrow \\ C_I \rtimes_{\Gamma_I,\kappa,r} G & \xrightarrow{(\iota_C)_r} & C_A \rtimes_{\Gamma_A,\kappa,r} G \\ \uparrow & & \uparrow \\ I_N \rtimes_{\gamma_I,\tau,r} G & \xrightarrow{\iota_r} & A_N \rtimes_{\gamma_A,\tau,r} G \end{array} \quad \text{and} \quad \begin{array}{ccc} E_A \rtimes_{\beta_A,1_N,r} G & \xrightarrow{(q_E)_r} & E_B \rtimes_{\beta_B,1_N,r} G \\ \downarrow & & \downarrow \\ C_A \rtimes_{\Gamma_A,\kappa,r} G & \xrightarrow{(q_C)_r} & C_B \rtimes_{\Gamma_B,\kappa,r} G \\ \uparrow & & \uparrow \\ A_N \rtimes_{\gamma_A,\tau,r} G & \xrightarrow{q_r} & B_N \rtimes_{\gamma_B,\tau,r} G \end{array}$$

In these diagrams the vertical arrows denote the embedding maps resulting from the identifications $E_I \rtimes_{\beta_I,1_N,r} G = p(C_I \rtimes_{\Gamma_I,\kappa,r} G)p$, $I_N \rtimes_{\gamma_I,\tau,r} G = q(C_I \rtimes_{\Gamma_I,\kappa,r} G)q$, etc. From the left-hand diagram it is apparent that the ideals $E_I \rtimes_{\beta_I,1_N,r} G$ of $E_A \rtimes_{\beta_A,1_N,r} G$ and $I_N \rtimes_{\gamma_I,\tau,r} G$ of $A_N \rtimes_{\gamma_A,\tau,r} G$ correspond. From the right-hand diagram the *-homomorphisms $(q_E)_r$ and $q_r$ are seen to be Morita equivalent, so that, by Lemma 5.3, their kernels are corresponding ideals of $E_A \rtimes_{\beta_A,1_N,r} G$ and $A_N \rtimes_{\gamma_A,\tau,r} G$, respectively. Thus $\ker(q_E)_r = E_I \rtimes_{\beta_I,1_N,r} G$ if and only if $\ker q_r = I_N \rtimes_{\gamma_I,\tau,r} G$, that is, the sequence (4) is exact if and only if the sequence

$$0 \longrightarrow E_I \rtimes_{\beta_I,1_N,r} G \xrightarrow{(\iota_E)_r} E_A \rtimes_{\beta_A,1_N,r} G \xrightarrow{(q_E)_r} E_B \rtimes_{\beta_B,1_N,r} G \longrightarrow 0 \tag{5}$$

is exact.

Lemma 5.5 implies that the sequence (5) is exact if and only if the sequence

$$0 \longrightarrow E_I \rtimes_{\bar\beta_I,r} (G/N) \xrightarrow{(\iota_E)_r} E_A \rtimes_{\bar\beta_A,r} (G/N) \xrightarrow{(q_E)_r} E_B \rtimes_{\bar\beta_B,r} (G/N) \longrightarrow 0 \tag{6}$$

is exact.

Since the sequence (3) is exact by assumption, the sequence

$$0 \longrightarrow E_I \xrightarrow{\iota_E} E_A \xrightarrow{q_E} E_B \longrightarrow 0$$

is exact, by Corollary 3.10. Since $G/N$ is exact by assumption, this implies that the sequence (6) is exact, from which the exactness of the sequences (5), (4) and (2) follow successively. Hence $G$ is exact. □

## 6. Examples of exact groups.

a. Amenable Groups. The following result is essentially well-known.

Proposition 6.1 *Let $G$ be an amenable locally compact group. Then $G$ is exact.*

*Proof*: Let $(A, \alpha) \in \mathcal{C}^*_G$, and let $I$ be an $\alpha_G$-invariant ideal of $A$, so that the sequence

$$0 \to I \to A \to (A/I) \to 0$$

is exact. Then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \rtimes_{\alpha|} G & \longrightarrow & A \rtimes_{\alpha} G & \longrightarrow & (A/I) \rtimes_{\dot{\alpha}} G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I \rtimes_{\alpha|,r} G & \longrightarrow & A \rtimes_{\alpha,r} G & \longrightarrow & (A/I) \rtimes_{\dot{\alpha},r} G & \longrightarrow & 0 \end{array}$$

is commutative, where the vertical arrows are the canonical *-homomorphisms, and the top row is exact. Since $G$ is amenable, the canonical *-homomorphisms $A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$, etc., are injective, from which it follows that the lower row of the diagram is exact. Hence $G$ is exact. □

b. Discrete subgroups of semisimple Lie groups. Our goal here is to show that the discrete groups $SL_n(\mathbb{Z})$ are exact for $n = 1, 2, \ldots$, though we shall, in fact, prove a more general result. The following fact is undoubtedly known, although we lack a reference.

Proposition 6.2 *Let $A$ and $B$ be Morita equivalent C*-algebras. Then $A$ is nuclear if and only if $B$ is nuclear.*

*Proof*: Let $(C, e, f)$ be a linking algebra for $A$ and $B$, so that $e, f$ are full projections in $M(C)$ such that $e + f = 1$, $A \cong eCe$ and $B \cong fCf$. Let $C^{**}$ be represented as a von Neumann algebra on a Hilbert space $\mathcal{H}$. If we regard $M(C)$ as canonically embedded in $C^{**}$, the fullness conditions imply that $e$ and $f$ have central support 1 in $C^{**}$, from which it follows that $e(C^{**})'e \cong (C^{**})' \cong f(C^{**})'f$. Now $A^{**} \cong eC^{**}e$ and $A$ is nuclear if and only if $A^{**}$ is injective. Since a von Neumann algebra is injective if and only if its commutant is, it follows that $A$ is nuclear if and only if $(C^{**})'$, and hence $C^{**}$, are injective. Since by the same reasoning $B$ is nuclear if and only if $C^{**}$ is injective, the result follows. □

We believe that the Morita equivalence technique used in the proof of the next proposition is essentially due to Alain Connes (unpublished), though we lack a precise attribution.

Proposition 6.3 *Let $G$ be a locally compact group which has a closed amenable subgroup $H$ such that $G/H$ is compact. Then any closed discrete subgroup of $G$ is exact.*

*Proof*: Assume first that $H$ and $K$ are just closed subgroups of $G$, and let $A = C_0(G/K)$. The continuous action of $G$ on $G/K$ by left multiplication gives rise to a continuous action $\alpha$ of $G$ on $A$. Restricting $\alpha$ to

$H$, the algebras $C_0(G/H, A) \rtimes_{\Delta^\alpha,r} G \cong C_0((G/H) \times (G/K)) \rtimes_{\Delta,r} G$ and $A \rtimes_{\alpha,r} H \cong C_0(G/K) \rtimes_{\alpha,r} H$ are Morita equivalent, by Theorem 3.6, where $\Delta$ is the action of $G$ on $C_0((G/H) \times (G/K))$ arising from the diagonal left action of $G$ on $(G/H) \times (G/K)$. Interchanging $H$ and $K$, it follows that $C_0(G/K) \rtimes_{\alpha,r} H$ and $C_0(G/H) \rtimes_{\beta,r} K$ are Morita equivalent, where $\beta$ is the action of $K$ coming from left multiplication on $G/H$.

Now assume that $H$ is amenable, $G/H$ is compact and $K$ is discrete. Then $C_0(G/H) = C(G/H)$ and $C^*_r(K) \subseteq C(G/H) \rtimes_{\beta,r} K$ canonically. Also $C_0(G/K) \rtimes_{\alpha,r} H$, being a crossed product of a nuclear C*-algebra by an amenable group, is nuclear. This implies, by Proposition 6.2, that $C(G/H) \rtimes_{\beta,r} K$ is nuclear, so that $C^*_r(K)$ is exact, so that, by the equivalence of exactness for $K$ and $C^*_r(K)$ [KW, Theorem 5.2], $K$ is exact. □

COROLLARY 6.4 *Any closed discrete subgroup of a connected semisimple Lie group is exact.*

*Proof*: Let $G$ be a connected semisimple Lie group. The centre $Z$ of $G$ is discrete, and, applying the Iwasawa decomposition [Kn], $G = KAN$, where $K$, $A$ and $N$ are connected closed subgroups of $G$, $Z \subseteq K$, $K/Z$ is compact, $N$ is nilpotent, and $A$ is abelian. Moreover $A$ normalises $N$ and $AN$ is a connected solvable Lie group. Since $AN$ has a composition series with abelian quotients, it follows that $AN$, and hence $ZAN$, are amenable, and $G/ZAN$ is homeomorphic to $K/Z$. The result now follows from Theorem 6.3. □

C. CLOSED LINEAR GROUPS. Since $SL_n(\mathbb{R})$ is semisimple [Kn] and contains $SL_n(\mathbb{Z})$ as a closed discrete subgroup, it is an immediate consequence of Corollary 6.4 that $SL_n(\mathbb{Z})$ is exact for $n = 1, 2, \ldots$. In fact the group $SL_n(\mathbb{Z})$ is a lattice in $SL_n(\mathbb{R})$, i.e. a closed discrete subgroup of finite covolume [Rag,Theorem 10.5], so that $SL_n(\mathbb{R})$ is exact by Theorem 4.5.

PROPOSITION 6.5 *For $n \in \mathbb{N}$ any closed subgroup of $GL_n(\mathbb{R})$ is exact.*

*Proof*: The determinant gives a continuous homomorphisms of $GL_n(\mathbb{R})$ onto the multiplicative groups $\mathbb{R} \setminus \{0\}$ with kernel $SL_n(\mathbb{R})$. Thus $GL_n(\mathbb{R})$ is an extension of an exact group by an abelian group, hence is exact by Proposition 6.1 and Theorem 5.1. The result now follows by Theorem 4.5. □

D. CONNECTED LOCALLY COMPACT GROUPS.

PROPOSITION 6.6 *Any connected real semisimple Lie group is exact.*

*Proof*: Let $G$ be a connected semisimple Lie group with centre $Z$ and Lie algebra $\mathfrak{g}$. If Ad is the adjoint representation of $G$ on $\mathfrak{g}$, we have a Lie group isomorphism $G/Z \cong Ad(G)$, and $Ad(G)$ coincides with $\mathrm{Aut}_0(\mathfrak{g})$, the connected component of the identity of the Lie group $\mathrm{Aut}(\mathfrak{g})$ [Kn]. Since the latter group is a closed subgroup of $GL(\mathfrak{g})$, and $\mathfrak{g}$ is finite-dimensional, it follows by Proposition 6.5 that $G/Z$ is exact. Since $Z$ is abelian, hence exact, the result now follows by Theorem 5.1. □

An alternative proof of this proposition follows by the technique of §4, since, by a deep theorem of Borel [Rag, Theorem 14.1], any connected noncompact semisimple Lie group contains a lattice.

Proposition 6.7 *Any connected real Lie group is exact.*

*Proof*: Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. If $\operatorname{rad}\ \mathfrak{g}$ is the radical of $\mathfrak{g}$, $\operatorname{rad}\mathfrak{g}$ is a solvable ideal of $\mathfrak{g}$ and $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ is semisimple. If $R$ is the closed normal subgroup of $G$ with Lie algebra $\operatorname{rad}\mathfrak{g}$, then $R$ is solvable, hence exact, by earlier discussion, and $G/R$ has Lie algebra $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$. Thus $G/R$ is semisimple, hence exact by Proposition 6.6, and the result follows using Theorem 5.2. □

Theorem 6.8 *Any connected locally compact group is exact.*

*Proof*: Let $G$ be a connected locally compact group. By [MZ, Theorem 4.6] $G$ has a closed normal compact subgroup $K$ such that $G/K$ is a real Lie group. Since $G/K$ is connected, it is exact by Proposition 6.7. The result now follows by Theorem 5.2, since $K$, being amenable, is exact. □

Recall that a locally compact group $G$ is *almost-connected* if the quotient group $G/G_0$ of $G$ by the connected component $G_0$ of the identity is compact. The following corollary is an immediate consequence of Proposition 6.1, Theorem 6.8 and Theorem 5.1.

Corollary 6.9 *Any almost-connected group is exact.*

e. Exactness of certain discrete groups. By [KW, Theorem 5.2], a discrete group $G$ is exact if and only if $C_r^*(G)$ is exact. For certain groups $G$, $C_r^*(G)$ can be explicitly embedded as a C*-subalgebra of a nuclear C*-algebra. For these groups, $C_r^*(G)$, being subnuclear, is exact, so that $G$ is exact. Two classes for which $C_r^*(G)$ is known to be subnuclear are (a) the free groups and (b) the hyperbolic groups. The case of free groups was treated in [KW, Corollary 5.3], where it was shown, using a celebrated construction of Choi, that if $G$ is a free group on at most countably many generators, then $C_r^*(G)$ can be embedded as a C*-subalgebra of the Cuntz algebra $\mathcal{O}_2$. Very recently Dykema [D] has shown that a reduced amalgamated free product of exact C*-algebras is exact. Since $C_r^*(\mathbb{Z})$ is abelian, hence nuclear, and $C_r^*(\mathbb{F}_\Lambda)$, where $\mathbb{F}_\Lambda$ is the free group on the set $\Lambda$, is the reduced free product of copies of $C_r^*(\mathbb{Z})$ indexed by $\Lambda$, Dykema's result together with our result just cited gives a new proof that free groups are exact. If $G$ is a hyperbolic group, Adams [Ad] has shown that the natural action $\alpha$ of $G$ on its Gromov boundary $\partial G$ is amenable, which implies that the crossed product $C(\partial G) \rtimes_{\alpha,r} G$ is nuclear. Since $G$ is discrete, $C^*_r(G)$ is a closed subalgebra of $C(\partial G) \rtimes_{\alpha,r} G$, hence exact. Germain [Ger] has recently given a concise and fairly simple proof of Adams' amenability result.

## 7. Concluding Remarks.

1. After we had completed most of this paper, Georges Skandalis pointed out that Corollary 6.9 can be obtained more directly by a different route. The various structure results used above together imply that if $G$ is an almost connected group, then $G$ contains a closed amenable subgroup $H$ such that $G/H$ with the quotient topology is compact. Corollary 6.9 is then an immediate consequence of Proposition 6.1 and the following theorem, which is closely related to Theorem 4.1, and has a similar, though simpler, proof.

Theorem *Let $G$ be a locally compact group with a closed exact subgroup $H$. If $G/H$ is compact, then $G$ is exact.*

*Proof*: Let $(A,\alpha) \in \mathcal{C}^*_G$ and let $I$ be an $\alpha_G$-invariant ideal of $A$. If $\theta : A \to C_0(G/H) \otimes A \cong C_0(G/H, A)$ is the embedding $a \to 1 \otimes a$, let $\Phi_A$ denote the crossed product map

$$\Phi_A : A \rtimes_\alpha G \to C_0(G/H, A) \rtimes_{\Delta^\alpha,r} G.$$

Then $\Phi_A$ is an embedding, and if corresponding embeddings

$$\Phi_I : I \rtimes_{\alpha|,r} G \to C_0(G/H, I) \rtimes_{\Delta^\alpha,r} G$$

and

$$\Phi_{A/I} : (A/I) \rtimes_{\dot\alpha,r} G \to C_0(G/H, A/I) \rtimes_{\Delta^{\dot\alpha},r} G$$

are defined similarly, then the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
I \rtimes_{\alpha|,r} G \arrow[r, "\phi_I"] \arrow[d] & C_0(G/H, I) \rtimes_{\Delta^\alpha,r} G \arrow[d] \\
A \rtimes_{\alpha,r} G \arrow[r, "\Phi_A"] \arrow[d] & C_0(G/H, A) \rtimes_{\Delta^\alpha,r} G \arrow[d] \\
(A/I) \rtimes_{\dot\alpha,r} G \arrow[r, "\Phi_{A/I}"] \arrow[d] & C_0(G/H, A/I) \rtimes_{\Delta^{\dot\alpha},r} G \arrow[d] \\
0 & 0
\end{tikzcd}$$

commutes. Since $H$ is exact, Corollary 3.10 implies, just as in the proof of Theorem 4.5, that the right-hand column is exact. If $x$ is in the kernel of the quotient map $A \rtimes_{\alpha,r} G \to (A/I) \rtimes_{\dot{\alpha},r} G$, it follows that $\Phi_A(x)$ is in the kernel of the quotient map

$$C_0(G/H, A) \rtimes_{\Delta^\alpha,r} G \to C_0(G/H, A/I) \rtimes_{\Delta^{\dot{\alpha}},r} G,$$

which is $C_0(G/H, I) \rtimes_{\Delta^\alpha,r} G$. Thus

$$\Phi_A(x) \in (A \rtimes_{\alpha,r} G) \cap (C_0(G/H, I) \rtimes_{\Delta^\alpha,r} G).$$

Let $\{e_\mu\}$ be a bounded approximate identity for $I$. If we identify $1 \otimes e_\mu$ with its image in $M((C_0(G/H, I) \rtimes_{\Delta^\alpha,r} G)$ under the canonical embedding of $M(C_0(G/H, A))$ discussed in §4, it is readily checked that for each $\mu$, $(1 \otimes e_\mu)y \in \Phi_A(I \rtimes_{\alpha|,r} G)$ for $y \in \Phi_A(A \rtimes_{\alpha,r} G)$, and $\lim_\mu (1 \otimes e_\mu)z = z$ for $z \in C_0(G/H, I) \rtimes_{\Delta^\alpha,r} G$. Then

$$\Phi_A(x) = \lim_\mu (1 \otimes e_\mu)x \in \Phi_A(I \rtimes_{\alpha|,r} G),$$

which shows that $x \in I \rtimes_{\alpha|,r} G$. Thus the left-hand column of the above diagram is exact, which implies that $G$ is exact. □

2. If $G$ is a locally compact group, the quotient $G/G_0$ by the connected component $G_0$ of the identity is a totally disconnected group. By Corollary 6.9, $G_0$ is exact. If $G/G_0$ is exact, it then follows, by Theorem 5.1, that $G$ is exact. Thus to resolve the question of whether all locally compact groups are exact, it is enough to consider only totally disconnected groups. Our feeling is that if there if there are groups which are not exact, then there will probably be a discrete example.

3. In [KW, Lemma 2.5] we showed that a group which has an increasing family of exact open subgroups with union the whole group is itself exact. We have been unable to show that exactness is preserved under general inductive limits. Likewise, we do not know if exactness is preserved on passing to a quotient. If this were the case, then all discrete groups would be exact, since any discrete group is a quotient of a free group, which is exact, as noted in §5 (e).

*Added note:* The Referee has informed us that, in a recent preprint [Y], Guoliang Yu has studied a combinatorial property, *property A*, of discrete groups which is preserved under semi-direct products. It seems that property A is formally stronger than C*-exactness, but we only became aware of [Y] in September 1999, and have not yet studied all possible connections with our results.

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