Twistor Spaces With a Pencil of Fundamental Divisors

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Abstract. In this paper simply connected twistor spaces Z containing a pencil of fundamental divisors are studied. The Riemannian base for such spaces is diffeomorphic to the connected sum $n\mathbb{CP}^2$. We obtain for $n \geq 5$ a complete description of the set of curves intersecting the fundamental line bundle $K^{-\frac{1}{2}}$ negatively. For this purpose we introduce a combinatorial structure, called *blow-up graph*. We show that for generic $S \in \mid -\frac{1}{2}K \mid$ the algebraic dimension can be computed by the formula $a(Z) = 1 + \kappa^{-1}(S)$. A detailed study of the anti Kodaira dimension $\kappa^{-1}(S)$ of rational surfaces permits to read off the algebraic dimension from the blow-up graphs. This gives a characterisation of Moishezon twistor spaces by the structure of the corresponding blow-up graphs. We study the behaviour of these graphs under small deformations. The results are applied to prove the main existence result, which states that every blow-up graph belongs to a fundamental divisor of a twistor space. We show, furthermore, that a twistor space with $\dim \mid -\frac{1}{2}K \mid = 3$ is a LeBrun space [LeB2]. We characterise such spaces also by the property to contain a smooth rational non-real curve C with $C.(-\frac{1}{2}K) = 2 - n$.

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1 Introduction

For a complex manifold with non-positive Kodaira dimension and zero dimensional Albanese torus, the algebraic dimension is the most basic birational invariant. By definition it is the transcendence degree over $\mathbb{C}$ of the field of meromorphic functions on the manifold. Because it is often a difficult task to compute this invariant in explicit examples, it is interesting to study the algebraic dimension in special classes of manifolds. A class where we can find interesting phenomena is the class of twistor spaces. From our point of view, a *twistor space* is a compact complex three-manifold Z equipped with

- a proper differentiable submersion $\pi : Z \longrightarrow M$ onto a real differentiable four-manifold M (called the base), whose fibres are holomorphic curves in Z which are isomorphic to the complex projective line and have normal bundle in Z isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$ and
- an anti-holomorphic fixed point free involution $\sigma : Z \longrightarrow Z$ with $\pi\sigma = \pi$.

Usually, such spaces arise in 4-dimensional conformal geometry. The points of Z correspond to complex structures on the tangent spaces at M, compatible with the conformal structure. The idea for such a construction traces back to F. Hirzebruch, H. Hopf [HH] and R. Penrose [Pe]. The twistor construction in the context of Riemannian geometry was first developed by M. Atiyah, N. Hitchin, I. Singer [AHS]. It plays an important role as a bridge between conformal Riemannian geometry and complex geometry. Twistor spaces have always negative Kodaira dimension and trivial Albanese torus [H2]. If a twistor space has the maximal possible algebraic dimension $a(Z) = 3$, then it must be simply connected with base homeomorphic to either S^4 or a connected sum of $\mathbb{CP}^2$'s [C2]. Compare with Proposition 2.4 below.
The involution σ is called a real structure and we designate any σ-invariant geometric object as being "real". For example, the fibres of π are called "real twistor fibres", a line bundle $\mathcal{L} \in \operatorname{Pic} Z$ is called real if $\sigma^*\overline{\mathcal{L}} \cong \mathcal{L}$ and a subvariety $D \subset Z$ is called real if $\overline{D} := \sigma(D) = D$. The degree $\deg(\mathcal{L})$ of a line bundle $\mathcal{L} \in \operatorname{Pic} Z$ is by definition the degree of the restriction $\mathcal{L} \otimes \mathcal{O}_F$ to a real twistor fibre $F \subset Z$. The "type" of a twistor space is by definition the sign of the scalar curvature of a metric with constant scalar curvature in the conformal class of M. On every twistor space there exists a distinguished square root $K^{-\frac{1}{2}}$ of the anti-canonical line bundle of Z. This bundle is called the *fundamental* line bundle. The divisors in $| -\frac{1}{2}K |$ are called fundamental divisors. The study of the structure of these divisors and of their linear system played a fundamental role in the study of twistor spaces.
In this paper we study simply connected twistor spaces containing irreducible fundamental divisors. Some authors start with the assumption that the twistor space is of positive type, but we don't here. We show in Section 2 that a simply connected twistor space containing an irreducible fundamental divisor must necessarily have positive type. For a collection of the basic properties of such twistor spaces and appropriate references, the reader is referred to [K1, Sections 2 and 3]. In the final three sections of the paper [K1], the case $c_1^3 = 0$ is studied, whereas the case $c_1^3 > 0$ is fairly well understood (see [H2], [FK], [Po1], [KK], [Po4]). Here we focus on the general case: $c_1^3 < 0$.
The goal of this paper is an understanding of the relationship between the algebraic dimension $a(Z)$, the structure of fundamental divisors and the base locus

and dimension of the fundamental linear system on Z. The results show that a finite set of curves with certain numerical properties contains very important information on the structure of the twistor space. We study the interplay between curves and surfaces, not merely divisors inside our three-manifolds. The basic assumption for our study will be $\dim |-\frac{1}{2}K| \geq 1$. Under this assumption, we develop in Section 3 a clear picture of the possibilities for the base locus and dimension of the fundamental linear system. We also give a new characterisation of LeBrun twistor spaces (Theorem 3.6). For LeBrun twistor spaces and the twistor spaces studied in [CK2] the place among all twistor spaces becomes quite clear by Theorems 3.6 and 3.7. Curves with certain numerical properties play an important role for these results.
To compute the algebraic dimension of a simply connected twistor space one relies on the observation of Y.S. Poon [Po2] that one can compute $a(Z)$ by the Iitaka dimension of the anti-canonical line bundle $\kappa(Z, K^{-1})$. This can be deduced from the fact that K^{-1} generates the unique one-dimensional subspace in $\mathrm{Pic}\, Z \otimes \mathbb{R}$ which is invariant under the involution, induced by the real structure on Z. To compute $a(Z)$ one can use the inequality $a(Z) \leq 1 + \kappa(S, K_S^{-1})$. But in many cases this is not enough for computing the algebraic dimension. In Section 4 we improve it (under our assumption $\dim |-\frac{1}{2}K| \geq 1$) to the equality

$$a(Z) = 1 + \kappa(S, K_S^{-1})$$

for generic fundamental divisors S.
This motivates the study of the anti Kodaira dimension $\kappa^{-1}(S) := \kappa(S, K_S^{-1})$ of rational surfaces in Section 5.
To handle the structure of the base locus of the fundamental system (which is also related to the number of divisors of degree one, see [K1, Proposition 3.7]) we define the notion of a *blow-up graph* (Section 6). This is a combinatorial structure which reflects numerical properties of the components of anti-canonical divisors on rational surfaces. These graphs contain also information on the anti Kodaira dimension.
The existence of new twistor spaces can be shown with the aid of deformation theory [DonF], [C1], [LeBP], [PP2], [C3]. To be able to state interesting results on the structure of twistor spaces constructed in such an indirect manner, we study the behaviour of the blow-up graphs under small deformations in Section 7. These results will then be used in Section 8, where the relationship between $a(Z)$, $\dim |-\frac{1}{2}K|$ and the structure of anti-canonical divisors on fundamental divisors is studied. As a result we see that basic information on the structure of twistor spaces is already contained in a finite set of curves in such a space. We prove in this section a vanishing theorem for the second cohomology of the tangent sheaf:

$$H^2(Z, \Theta_Z) = 0,$$

which is necessary to show the existence of twistor spaces related to arbitrarily prescribed blow-up graphs. Our main existence result (Theorem 8.8) states that every blow-up graph appears as a blow-up graph of a fundamental divisor contained in a twistor space. To prove this we rely on recent results of N.Honda [Ho], who studies the twistor spaces constructed in [PP3].

2 Consequences of the existence of fundamental divisors

In this section we show that the existence of an irreducible fundamental divisor in a simply connected twistor space has strong consequences. We see, for example, that there is no need to assume the twistor space to be of positive type, because we obtain this from our assumption. As a consequence, we have Hitchin's vanishing theorem at our disposal. This states for simply connected twistor spaces of positive type the vanishing of $H^1(Z,\mathcal{L})$ for any line bundle $\mathcal{L}$ with $\deg(\mathcal{L}) \leq -2$ [H1].
In fact, the topology of simply connected twistor spaces containing an effective divisor can be restricted to a few cases by results of P. Gauduchon [Gau] and C. LeBrun [LeB1].
First of all, we cite the following lemma from [PP1, Lemma 2.1], which will be useful in the following.

Lemma 2.1. *Let Z be a compact twistor space and $S \subset Z$ an effective divisor of degree 2 which is irreducible and real, then S is smooth.* □

This implies, in particular, that each real irreducible fundamental divisor $S \in |-\frac{1}{2}K|$ is smooth.
From here on, we are only concerned with *simply connected* twistor spaces. Without assuming Hitchin's vanishing theorem or the twistor space to be of positive type, we can study the structure of irreducible fundamental divisors.

Lemma 2.2. *Let Z be a compact simply–connected twistor space and $S \in |-\frac{1}{2}K|$ be real and irreducible. Then there exists a real twistor fibre $F \subset S$ and $\dim|F| = 1$. The surface S is smooth and rational.*

Proof: From Lemma 2.1 we know smoothness of S. If S would not contain a real twistor fibre, the twistor fibration would give an unramified covering $S \to M$ of degree two, since Z does not contain real points. This is in contradiction with $\pi_1(M) = \pi_1(Z) = 0$. Similarly, if $\dim|F| = 0$, we obtain an unramified covering $S \setminus F \longrightarrow M \setminus \{pt\}$ of degree two. Again, we obtain a contradiction to $\pi_1(M\setminus\{pt\}) = \pi_1(M) = 0$ because $S \setminus F$ is irreducible (being open in the irreducible surface S). This implies $h^0(\mathcal{O}_S(F)) \geq 2$. The adjunction formula on S yields $(F^2)_S = (F.(-K_S))_S - 2 = F.(-\frac{1}{2}K) - 2 = 0$. Hence, we have an exact sequence $0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(F) \longrightarrow \mathcal{O}_F \longrightarrow 0$, implying $h^0(\mathcal{O}_S(F)) \leq h^0(\mathcal{O}_S) + h^0(\mathcal{O}_F) = 2$. Thus $|F|$ is a pencil. On the other hand, $((mF - K_S)^2)_S = 2m(F.(-K_S))_S + ((-K_S)^2)_S = 4m + ((-K_S)^2)_S > 0$ for large positive m. Therefore, S is a projective algebraic surface ([BPV, IV (5.2)]). By Noether's lemma ([GH, IV§3]) the existence of the pencil $|F|$ implies the rationality of S. □

Lemma 2.3. *If Z is a compact, simply connected twistor space containing an irreducible fundamental divisor, then $h^i(K^{\frac{1}{2}}) = 0$ for all i and $h^i(\mathcal{O}_Z) = 0$ for $i > 0$.*

Proof: By assumption $|-\frac{1}{2}K|$ contains an irreducible member, hence, the generic member of this linear system is irreducible. Therefore, we can choose an irreducible real $S \in |-\frac{1}{2}K|$, which is smooth and rational by Lemma 2.2. In particular, we have $h^1(\mathcal{O}_S) = h^2(\mathcal{O}_S) = 0$. Because the restriction defines an isomorphism $H^0(\mathcal{O}_Z) \xrightarrow{\sim} H^0(\mathcal{O}_S)$, the exact sequence $0 \longrightarrow K^{\frac{1}{2}} \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_S \longrightarrow 0$ implies $h^0(K^{\frac{1}{2}}) = 0$ and $h^i(K^{\frac{1}{2}}) = h^i(\mathcal{O}_Z)$, if $i > 0$. Using the Serre duality, this gives the desired vanishing for $i \in \{0,3\}$ and $h^1(\mathcal{O}_Z) = h^1(K^{\frac{1}{2}}) = h^2(K^{\frac{1}{2}}) = h^2(\mathcal{O}_Z)$. The simply connectedness of Z implies $0 = b_1(M) = h^1(\mathcal{O}_Z)$ (see [ES]) finishing the proof. □

Proposition 2.4. *If $Z \longrightarrow M$ is a compact, simply connected twistor space containing an irreducible fundamental divisor, then M is diffeomorphic to the connected sum $n\mathbb{CP}^2$ and M is of positive type.*

Proof: Because Z is compact and M self–dual, we obtain (see e.g. [ES, Cor. 3.2]) $b_-(M) = h^2(\mathcal{O}_Z)$ which vanishes under our assumptions by Lemma 2.3. Therefore, the intersection form on $H^2(M, \mathbb{R})$ is positive definite. To see that the type of M is positive, we recall a theorem of Gauduchon [Gau] stating that a twistor space of negative type does not contain effective divisors. Hence, in our case, the type is non–negative. If the type would be zero, we would obtain (using $\pi_1(M) = 0$) from [Pon, Cor. 4.3], that $\bar{M}$ is a Kähler surface. But, what we have seen above, implies then that the intersection form on $H^2(\bar{M}, \mathbb{R})$ would be negative definite. But for a simply connected complex surface this is impossible by the signature theorem [BPV, IV (2.13.)]. Therefore, M has positive type. In this situation a theorem of Pedersen and Poon [PP1] states that M is diffeomorphic to $n\mathbb{CP}^2$. □

From [Gau] and [LeB1] we obtain that a simply connected twistor space, which contains an effective divisor, can only be built over a self-dual four manifold M having one of the following properties ($\overline{M}$ denotes the anti-self-dual manifold obtained by reversing the orientation of M):

(a) $\overline{M}$ is a blow–up of $\mathbb{P}^2$ at $m > 9$ points or

(b) $\overline{M}$ is a K3–surface with a Ricci–flat metric or

(c) M is homeomorphic to $n\mathbb{CP}^2$ with $n \geq 0$.

From [Po3] we obtain that in case (a) $a(Z) = 0$ and in case (b) $a(Z) = 1$. The goal of the following sections is to gain more knowledge on the algebraic dimension and their relation to the geometry of Z in case (c).

3 The fundamental linear system of a twistor space

We consider a simply connected twistor space Z. In this section we study the fundamental linear system $|-\frac{1}{2}K|$. Under the assumption that it is at least a

pencil, we obtain information on its dimension and the base locus. In Section 8 we study the algebraic dimension $a(Z)$ in more detail.
In the case where an irreducible fundamental divisor exists, Proposition 2.4 shows that the Riemannian base of such a twistor space is diffeomorphic to the connected sum $n\mathbb{CP}^2$ and the conformal class contains a metric with positive scalar curvature. If $n \leq 3$ it is well-known (and follows easily from the Riemann-Roch formula and Hitchin's vanishing theorem) that we have $a(Z) = 3$. The case $n = 4$ was studied in [K1]. Since a twistor space of positive type over $n\mathbb{CP}^2$ with $n \leq 4$ contains always a pencil of fundamental divisors, the picture is in this case fairly satisfactory. If, however, $n > 4$ (which is equivalent to $c_1(Z)^3 < 0$) the situation is much more rich and less understood.
In the rest of this section we denote by Z a twistor space fulfilling:

(3.0) It is simply connected, contains an irreducible fundamental divisor and satisfies $h^0(K^{-\frac{1}{2}}) \geq 2$ and $c_1(Z)^3 < 0$.

We have seen in Section 2 that such a twistor space is of positive type and is built over $n\mathbb{CP}^2$ with $n > 4$. Furthermore, $\mathrm{Pic}(Z)$ is a free abelian group of rank $n+1$ and $(-\frac{1}{2}K)^3 = 2(4-n)$ (see [K1, Section 2]).

Lemma 3.1. *Let $D \subset Z$ be an effective divisor of degree one.*
(a) If $D \cap \overline{D} \neq \emptyset$ then $D.\overline{D} = F$ is a real twistor fibre.
(b) If $h^0(D) \geq 2$, then $\dim \mid D \mid = 1$, $\dim \mid -\frac{1}{2}K \mid = 3$, the base locus of the pencil $\mid D \mid$ is a smooth rational curve B which is disjoint to its conjugate $\overline{B}$. The surface D is rational and intersects the conjugate surface $\overline{D}$. The base locus of the fundamental linear system $\mid -\frac{1}{2}K \mid$ is the curve $B \cup \overline{B}$ and we have $B.(-\frac{1}{2}K) = 2 - n$.

Proof: This lemma can be deduced from [Ku] and [Po4] but we prefer to give a direct proof here.
(a) Assume $D \cap \overline{D} \neq \emptyset$. Consider a point $z \in D \cap \overline{D}$ and denote by F the real twistor fibre containing z. But F and $D \cap \overline{D}$ are real, hence $\overline{z} \in F \cap D$. Using that D is of degree one we conclude $F \subset D \cap \overline{D}$. As D is smooth and irreducible we have for every real twistor fibre $F \subset D$ an exact normal bundle sequence:

$$0 \longrightarrow N_{F|D} \longrightarrow N_{F|Z} \longrightarrow \mathcal{O}_Z(D) \otimes \mathcal{O}_F \longrightarrow 0.$$

Using $N_{F|Z} \cong \mathcal{O}_F(1)^{\oplus 2}$ and $\mathcal{O}_Z(D) \otimes \mathcal{O}_F \cong \mathcal{O}_F(1)$, we obtain $N_{F|D} \cong \mathcal{O}_F(1)$, which means $(F^2)_D = 1$. As in the proof of Lemma 2.2 this implies that D is algebraic and rational. Since any two real twistor lines are disjoint we infer from the Hodge index theorem that F is the unique real twistor fibre contained in D. This implies $D \cap \overline{D} = F$ because the intersection $D \cap \overline{D}$ would contain a second twistor fibre if it contains a point outside F. We even have $D.\overline{D} = F$, since $D.\overline{D} = rF$ implies $r = ((D.\overline{D}).F)_D = \overline{D}.F = 1$.
(b) If we have $\mathcal{O}_Z(D - \overline{D}) \cong \mathcal{O}_Z$, then $c_1(\mathcal{O}_Z(D))$ would be invariant under the involution on $H^2(Z, \mathbb{Z})$. This would imply $\mathcal{O}_Z(4D) \cong K_Z^{-1}$, which is only possible if $n = 0$. In this case $Z = \mathbb{P}^3$ by [H2] and [FK]. But we assumed $n > 4$.

Hence, $\mathcal{O}_Z(D-\overline{D})$ is not the trivial line bundle. This implies $H^0(\mathcal{O}_Z(D-\overline{D})) = 0$ because there is no effective divisor of degree zero. If D and $\overline{D}$ would be disjoint, we would have $\mathcal{O}_Z(D) \otimes \mathcal{O}_{\overline{D}} \cong \mathcal{O}_{\overline{D}}$. Considering the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(D-\overline{D}) \longrightarrow \mathcal{O}_Z(D) \longrightarrow \mathcal{O}_Z(D) \otimes \mathcal{O}_{\overline{D}} \longrightarrow 0$$

we would obtain a contradiction to the assumption $h^0(\mathcal{O}_Z(D)) \geq 2$. Therefore, we must have $D \cap \overline{D} \neq \emptyset$, hence $D.\overline{D} = F$.
Let now $D' \in \mid D \mid \setminus \{D\}$ and define $B := D.D'$. We obtain $B.\overline{D} = D.D'.\overline{D} = (D.\overline{D}).D' = F.D' = 1$. Since $\mid D \mid$ is at least a pencil, this implies B is smooth. This computation shows furthermore $(B.F)_D = B.\overline{D} = 1$. In particular $B \cap \overline{B} = \emptyset$, because $\overline{B} \subset \overline{D}$ and Z does not contain a real point.
Let $\mathcal{I}_B \subset \mathcal{O}_Z$ be the ideal of $B \subset Z$ and denote by $s, s' \in H^0(\mathcal{O}_Z(D))$ sections defining the divisors D, D'. By $V \subset H^0(\mathcal{O}_Z(D))$ we denote the vector space generated by s, s'. This pair of sections defines the exact Koszul complex

$$0 \longrightarrow \mathcal{O}_Z(-2D) \longrightarrow V \otimes \mathcal{O}_Z(-D) \longrightarrow \mathcal{I}_B \longrightarrow 0. \tag{1}$$

Since $\deg(-2D) = -2$ we obtain from Hitchin's vanishing $h^1(\mathcal{O}_Z(-2D)) = 0$ and $h^2(\mathcal{O}_Z(-2D)) = 0$. Hence, $H^1(\mathcal{O}_Z(-D)) \otimes V \cong H^1(\mathcal{I}_B)$. But the exact sequence $0 \longrightarrow \mathcal{O}_Z(-D) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_D \longrightarrow 0$ and $H^1(\mathcal{O}_Z) = 0$ imply $H^1(\mathcal{O}_Z(-D)) = 0$. Hence, $H^1(\mathcal{I}_B) = 0$, showing that the restriction $H^0(\mathcal{O}_Z) \longrightarrow H^0(\mathcal{O}_B)$ is surjective. Thus, B is connected, hence irreducible.
The linear system $\mid F \mid$ on D is of dimension two and does not have base points which can be seen from the exact sequence $0 \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_D(F) \longrightarrow \mathcal{O}_F(F) \longrightarrow 0$. Since $(B.F)_D = 1$ we see that B is the strict transform of a line in $\mathbb{CP}^2$ under the blow up $D \longrightarrow \mathbb{CP}^2$, defined by $\mid F \mid$. In particular, B is smooth and rational.
Now we can compute $B.(-\frac{1}{2}K) = B.(D+\overline{D}) = D.D'.D+B.\overline{D} = D^3+1 = 2-n$, and this is negative since we assumed $n > 4$. In particular B and $\overline{B}$ are contained in the base locus of $\mid -\frac{1}{2}K \mid$. By a lemma of Poon [Po4, Lemma 1.4], we can conclude $h^0(K^{-\frac{1}{2}}) \leq 4$.
To determine the base locus of $\mid -\frac{1}{2}K \mid$ consider a base point z of this linear system. This point is contained in every divisor of the form $D+\overline{D}$ of which there exist an infinite number. Thus there exists a pair of effective linearly equivalent divisors of degree one D, D' such that $z \in D \cap D' = B$ or $z \in \overline{D} \cap \overline{D}' = \overline{B}$. This shows that the base locus of $\mid -\frac{1}{2}K \mid$ is contained in $B \cup \overline{B}$, hence $B \cup \overline{B}$ is the base locus.
Finally, we have to compute the dimension of the fundamental linear system. For this purpose we tensor the exact sequence (1) with $\mathcal{O}_Z(D+\overline{D}) \cong K^{-\frac{1}{2}}$ and obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_Z(\overline{D}-D) \longrightarrow V \otimes \mathcal{O}_Z(\overline{D}) \longrightarrow \mathcal{I}_B \otimes \mathcal{O}_Z(D+\overline{D}) \longrightarrow 0.$$

If we use $H^0(\mathcal{O}_Z(\overline{D}-D)) = 0$ we obtain an injection $H^0(V \otimes \mathcal{O}_Z(\overline{D})) = V \otimes H^0(\mathcal{O}_Z(\overline{D})) \subset H^0(\mathcal{I}_B \otimes K^{-\frac{1}{2}}) \subset H^0(K^{-\frac{1}{2}})$. Since $V \otimes \overline{V} \subset V \otimes H^0(\mathcal{O}_Z(\overline{D}))$

we obtain $4 \le h^0(K^{-\frac{1}{2}})$ which implies by the above inequality $\dim \mid K^{-\frac{1}{2}} \mid = 3$ and hence $\dim \mid D \mid = 1$. $\square$

Lemma 3.2. *Assume Z contains only finitely many divisors of degree one. If $A \subset Z$ is an irreducible and reduced curve with $A.(-\frac{1}{2}K) < 0$, then there exists a smooth real fundamental divisor $S \in \mid -\frac{1}{2}K \mid$ containing a real twistor fibre F with $2 \ge F.A \ge 1$. We have $A.F = 2$ if and only if A is real.*

Proof: Let $x \in A$ be a point and $x \in F \subset Z$ the real twistor fibre containing this point. Since $\mid -\frac{1}{2}K \mid$ is at least a pencil, there exists a divisor $S \in |-\frac{1}{2}K|$ containing a given point $y \in F \setminus \{x, \bar{x}\}$. Because $F.S = 2$ and $S \cap F \supset \{y, x, \bar{x}\}$ the twistor fibre F is contained in S. So the real subsystem $|-\frac{1}{2}K|_F \subset |-\frac{1}{2}K|$ of divisors containing F is not empty. Because S contains at most a real one-parameter family of real twistor fibres, the intersection points of A with real twistor fibres contained in S form at most a real one-dimensional subset of points z on A. Therefore, we obtain at least a one-parameter family of surfaces S containing a real twistor fibre F with $F.A \ge 1$. Since we assumed that there are only finitely many divisors of degree one, we can choose an irreducible real fundamental divisor among them, which is smooth by Lemma 2.1. Since $(A.(-K_S))_S = A.(-\frac{1}{2}K) < 0$ each real anti-canonical divisor $C \in \mid -K_S \mid$ contains A and $\overline{A}$. Since F is nef this implies $(F.A)_S \le (F.(-K_S))_S = F.(-\frac{1}{2}K) = 2$. If $A \ne \overline{A}$ there even holds $(F.(A + \overline{A}))_S \le (F.(-K_S))_S = 2$ implying $(F.A)_S = (F.\overline{A})_S = 1$. If $A = \overline{A}$, we must have $(F.A)_S \ne 1$ since S does not contain real points, hence $(F.A)_S = 2$ in this case. $\square$

Lemma 3.3. *If Z is a twistor space satisfying condition (3.0), then:*

(a) There exists a reduced irreducible curve $A \subset Z$ with $A.(-\frac{1}{2}K) < 0$.

(b) If we have $A.(-\frac{1}{2}K) > 2 - n$ for every reduced irreducible curve $A \subset Z$, then $\dim \mid -\frac{1}{2}K \mid = 1$.

Proof: Let S be a smooth real fundamental divisor. We have $K_S^{-1} \cong K^{-\frac{1}{2}} \otimes \mathcal{O}_S$ and $\dim \mid -\frac{1}{2}K \mid = \dim \mid -K_S \mid +1$, hence $\mid -K_S \mid \ne \emptyset$. Since $((-K_S)^2)_S = (-\frac{1}{2}K)^3 = 2(4-n) < 0$ we obtain (a). To show (b) we recall from [K1, Prop. 3.6] that there exists a succession of blow-ups $\sigma : S \to \mathbb{P}^1 \times \mathbb{P}^1$ such that the anticanonical system $|-K_S|$ contains a real member C mapped onto a curve C' on $\mathbb{P}^1 \times \mathbb{P}^1$ having one of the following four types :

(0) $C' \in |\mathcal{O}(2,2)|$ is a smooth elliptic curve,

(1) C' has four components $C' = F' + \overline{F'} + G' + \overline{G'}$ where $F' \in |\mathcal{O}(0,1)|$ and $G' \in |\mathcal{O}(1,0)|$ are not real,

(2) C' has two components $C' = F' + C'_0$ where $F' \in |\mathcal{O}(0,1)|$ is real and $C'_0 \in |\mathcal{O}(2,1)|$ is real, smooth and rational,

(3) C' has two distinct components $C' = A' + \overline{A'}$ where $A', \overline{A'} \in |\mathcal{O}(1,1)|$.

At each step of blow-up a conjugate pair of points, lying on the image of C, is blown up. The pencil $\mid F \mid$ generated by a real twistor fibre $F \subset S$ is mapped to the pencil $|\mathcal{O}(0,1)|$ on $\mathbb{P}^1 \times \mathbb{P}^1$. It was shown in [K1, Prop. 3.3] that none of

the blown up points lies over a real member of $|\mathcal{O}(0,1)|$. This implies in case (2) that there is a component C_0 of C with $(C_0.(-K_S))_S = 6-2n < 2-n$. In case (0) we have $(C^2)_S = (C.(-K_S))_S = 8-2n < 0$ and C is irreducible, hence $|-K_S| = \{C\}$. If in case (3) all the blown up points lie over smooth points of $C' = A' + \overline{A'}$, then $C = A + \overline{A}$ with $(A.(-K_S))_S = (\overline{A}.(-K_S))_S = 4-n < 0$. Hence, $|-K_S| = \{C\}$. If, however, in case (3) the conjugate pair of singular points $A' \cap \overline{A'}$ is blown up, then we can choose the succession of blow-ups σ such that C is mapped to a curve of type (1) in $\mathbb{P}^1 \times \mathbb{P}^1$. This is done by an elementary transformation (see [K1, Cor. 4.3]).
To deal with case (1) we choose an irreducible reduced curve $G \subset Z$ with $G.(-\frac{1}{2}K) < 0$. By Lemma 3.1 the assumption of (b) implies that there does not exist a pencil of divisors of degree one. Hence we can apply Lemma 3.2 and can find a smooth real fundamental divisor S with $(F.(G+\overline{G}))_S = 2$ for twistor fibres $F \subset S$. Take $C \subset S$ as above in the description of type (1), then G is a component of C, hence smooth rational and not real. This implies $(F.G)_S = (F.\overline{G})_S = 1$. Thus, the curves G and $\overline{G}$ are mapped to the components G' and $\overline{G'}$ of C'. Since $G.(-\frac{1}{2}K) = \overline{G}(-\frac{1}{2}K) < 0$, at least three of the blown up points are lying over G' and their conjugates over $\overline{G'}$. Since we assumed $G.(-\frac{1}{2}K) > 2-n$, at most $n-1$ blown-up points lie over G'. Hence, a nonempty set of blown-up points lies over a conjugate pair F', $\overline{F'}$ of members of $|\mathcal{O}(0,1)|$. This implies that these curves are not movable, hence $|-K_S| = \{C\}$. $\square$

PROPOSITION 3.4. *Let Z be a twistor space satisfying condition (3.0) and let $A \subset Z$ be an irreducible reduced curve.*

- *(a) If A is not real, then $A.(-\frac{1}{2}K) \geq 2-n$.*
- *(b) If $A.(-\frac{1}{2}K) < 2-n$, then A is real (i.e. $A = \overline{A}$) and it is the unique irreducible reduced curve having negative intersection number with $K^{-\frac{1}{2}}$. Only the following two cases are possible:*
  - *(i) $A.(-\frac{1}{2}K) = 8-2n$, $n > 6$ and A is smooth elliptic. In this case* $\dim|-\frac{1}{2}K| = 1$.
  - *(ii) $A.(-\frac{1}{2}K) = 6-2n$ and A is smooth rational. In this case* $\dim|-\frac{1}{2}K| = 2$.
- *(c) If $A.(-\frac{1}{2}K) = 2-n$, then $\{A, \overline{A}\}$ is the set of all irreducible reduced curves in Z with negative intersection number with $K^{-\frac{1}{2}}$ and either*
  - *(i) $n = 6$ and $A = \overline{A}$ smooth and elliptic and* $\dim|-\frac{1}{2}K| = 1$, *or*
  - *(ii) A is smooth rational and not real. In this case* $\dim|-\frac{1}{2}K| = 3$.

PROOF: If a pencil of divisors of degree one exists, all the statements are clear by Lemma 3.1. Therefore, we assume that there exists only a finite number of divisors of degree one. This allows us to apply Lemma 3.2.

If $A.(-\frac{1}{2}K) \geq 0$ nothing is to prove. Assume $A.(-\frac{1}{2}K) < 0$ and choose a smooth fundamental divisor S as in Lemma 3.2. Let, furthermore, be $\sigma : S \to \mathbb{P}^1 \times \mathbb{P}^1$ a succession of blow-ups as in the proof of Lemma 3.3. Using the notation of that proof, we obtain that A must be a component of C. Since every curve on $\mathbb{P}^1 \times \mathbb{P}^1$ has nonnegative self-intersection number, (a) is clear from the description of the types (0) – (3).
The assumption of (b) implies that we are in types (0) or (2) which correspond to the cases (i) and (ii) respectively. In type (0), the irreducible curve C has negative intersection number with $-K_S$, hence $\mid -K_S \mid = \{C\}$, implying $\dim \mid -\frac{1}{2}K \mid = 1$ in case (i) of (b). In type (2) we obtain $\mid -K_S \mid = C_0 + |F|$ yielding $\dim \mid -\frac{1}{2}K \mid = 2$ in case (ii) of (b). Finally, if $A.(-\frac{1}{2}K) = 2 - n$ we can have type (0) only if $n = 6$, giving the case (i) of (c). Otherwise, we are in type (1) and n blown-up points are over G'. The conjugate set of blown-up points lies over $\overline{G'}$, hence the components F' and $\overline{F'}$ of C' are movable. This yields $\dim \mid -\frac{1}{2}K \mid = 3$ and $A + \overline{A}$ is mapped to $G' + \overline{G'}$ giving the statement (ii) of (c). □

Now we are ready to give new characterisations of the Moishezon twistor spaces introduced by LeBrun [LeB2] and studied by Kurke [Ku]. Recall the following result of Kurke [Ku] and Poon [Po4]:

Theorem 3.5. *If Z contains a pencil of divisors of degree one, then it is one of the Moishezon twistor spaces introduced by LeBrun [LeB2] and studied by Kurke [Ku].*

The following theorem provides new characterisations for these twistor spaces.

Theorem 3.6. *If Z is a twistor space satisfying condition (3.0) then the following properties are equivalent:*

- *(a) Z contains a pencil of divisors of degree one.*
- *(b)* $\dim \mid -\frac{1}{2}K \mid = 3$.
- *(c)* $\dim \mid -\frac{1}{2}K \mid \geq 3$.
- *(d) There exist exactly two reduced irreducible curves in Z having negative intersection number with $K^{-\frac{1}{2}}$. These two curves are smooth rational, form a conjugate pair $\{A, \overline{A}\}$ and $A.(-\frac{1}{2}K) = 2 - n$.*
- *(e) There exists a smooth rational curve $A \subset Z$ with $A.(-\frac{1}{2}K) = 2 - n$.*

Proof: The implication (a) $\Rightarrow$ (b) was shown in Lemma 3.1. (c) $\Rightarrow$ (d) follows from Lemma 3.3(a) and Proposition 3.4. (d) $\Rightarrow$ (e) is obvious. (e) $\Rightarrow$ (a) can be shown by the same proof as in the case $n = 4$ given in [K1, Prop. 5.3.], therefore we omit it here. □

Theorem 3.7. *If Z is a twistor space satisfying condition (3.0) then the following properties are equivalent:*

(a) $\dim \mid -\frac{1}{2}K \mid = 2$.

(b) There exists a smooth irreducible real rational curve $C_0 \subset Z$ with the property $C_0.(-\frac{1}{2}K) = 2(3-n)$. This is the unique irreducible reduced curve in Z having negative intersection number with $K^{-\frac{1}{2}}$.

(c) There exists a smooth real rational curve $C_0 \subset Z$ with $C_0.(-\frac{1}{2}K) < 0$.

PROOF: This follows from Lemma 3.3(a) and Proposition 3.4. For (c) $\Rightarrow$ (a) see also [K2, Thm. 2.1]. □

4 COMPUTATION OF THE ALGEBRAIC DIMENSION

The computation of the algebraic dimension of a specific compact complex manifold Z is often a very difficult task. It is known that in general there exists a line bundle $\mathcal{A} \in \operatorname{Pic} Z$ whose Iitaka dimension $\kappa(Z, \mathcal{A})$ is equal to $a(Z)$. It is an observation of Y.S. Poon [Po2], [Po3] that we can choose $\mathcal{A} = K^{-\frac{1}{2}}$ if Z is a simply connected twistor space and $\kappa(Z, K^{-\frac{1}{2}}) \neq -\infty$. If $S \in \mid -\frac{1}{2}K \mid$ is an irreducible smooth fundamental divisor on a twistor space, then the inequality $a(Z) \leq 1 + \kappa(S, K_S^{-1})$ is easy to see. But this will in general not suffice to compute $a(Z)$. The following theorem improves this situation a lot.

THEOREM 4.1. *Let Z be a compact complex manifold, $\mathcal{F}$ and $\mathcal{A}$ line bundles on Z, $\Lambda \subseteq \mid \mathcal{F} \mid$ a one-dimensional linear system. Assume $a(Z) = \kappa(Z, \mathcal{A})$ and that the general member of Λ is irreducible and reduced. Then, for general $S \in \Lambda$, the following formula holds:*

$$a(Z) = 1 + \kappa(S, \mathcal{A} \otimes \mathcal{O}_S).$$

PROOF: The linear system Λ does not have a fixed component since it contains an irreducible reduced member. Let $\varphi : Z \dashrightarrow \mathbb{P}^1$ be the meromorphic map defined by the pencil $\Lambda \subset \mid \mathcal{F} \mid$. By $B \subset Z$ we denote the set of indeterminacy of φ, that is the base locus of Λ. Using Hironaka's theorem on resolutions of singularities in the complex analytic case (see [AHV]), we can resolve the singularities of the graph space of φ to obtain a proper modification $\sigma : \widetilde{Z} \longrightarrow Z$ and a holomorphic map $\widetilde{\varphi} : \widetilde{Z} \longrightarrow \mathbb{P}^1$ such that: $\widetilde{Z}$ is a smooth compact complex manifold, $\widetilde{\varphi}$ is proper and surjective, σ induces an isomorphism $\widetilde{Z} \setminus \sigma^{-1}(B) \longrightarrow Z \setminus B$ and $\widetilde{\varphi} = \varphi \circ \sigma$ on $\widetilde{Z} \setminus \sigma^{-1}(B)$.
In particular, the generic fibre of $\widetilde{\varphi}$ is smooth (see [U, Cor. 1.8]). Since $\widetilde{Z}$ is irreducible and reduced and $\widetilde{\varphi}$ maps $\widetilde{Z}$ onto a smooth curve, the map $\widetilde{\varphi}$ is flat. But $\sigma^{-1}(B)$ has at least codimension one in Z and the general member S of Λ is, by assumption, an irreducible smooth divisor in Z, hence the generic fibre of $\widetilde{\varphi}$ is connected. This implies, the general fibre $\widetilde{S}$ of $\widetilde{\varphi}$ is smooth and irreducible and σ induces a proper modification $\sigma : \widetilde{S} \longrightarrow S = \sigma(\widetilde{S}) \in \Lambda$. By [U, 5.13] we obtain:

$$\kappa(\widetilde{S}, \sigma^*(\mathcal{A} \otimes \mathcal{O}_S)) = \kappa(S, \mathcal{A} \otimes \mathcal{O}_S)$$

and $\kappa(\widetilde{Z}, \sigma^*\mathcal{A}) = \kappa(Z, \mathcal{A})$.
Let m be a positive integer and consider the projective fibre space $\mathbb{P}(\widetilde{\varphi}_*\sigma^*\mathcal{A}^{\otimes m})$ over $\mathbb{P}^1$. We have meromorphic maps $\Phi_m : \widetilde{Z} \longrightarrow \mathbb{P}(\widetilde{\varphi}_*\sigma^*\mathcal{A}^{\otimes m})$ compatible with the maps to $\mathbb{P}^1$. The restriction of Φ_m to a generic fibre $\widetilde{S} \subset \widetilde{Z}$ of $\widetilde{\varphi}$ is the map given by the line bundle $(\sigma^*\mathcal{A}^{\otimes m}) \otimes \mathcal{O}_{\widetilde{S}}$ (see [U, (2.8)–(2.10)]). This implies for $m \gg 0$:

$$\dim \Phi_m(\widetilde{Z}) = 1 + \kappa(\widetilde{S}, \sigma^*\mathcal{A} \otimes \mathcal{O}_{\widetilde{S}}).$$

Since $\mathbb{P}^1$ and hence $\mathbb{P}(\widetilde{\varphi}_*\sigma^*\mathcal{A}^{\otimes m})$ are projective, we have $a(\Phi_m(\widetilde{Z})) = \dim \Phi_m(\widetilde{Z})$ and obtain

$$a(Z) = a(\widetilde{Z}) \geq a(\Phi_m(\widetilde{Z})) = 1 + \kappa(\widetilde{S}, \sigma^*\mathcal{A} \otimes \mathcal{O}_{\widetilde{S}}).$$

Finally, since we assumed $\kappa(Z, \mathcal{A}) = a(Z) \geq 0$, implying $h^0(\mathcal{A}^{\otimes m}) > 0$ for $m \gg 0$, we can apply [U, Thm. 5.11] to the proper holomorphic map $\widetilde{\varphi} : \widetilde{Z} \longrightarrow \mathbb{P}^1$ to obtain

$$\kappa(\widetilde{Z}, \sigma^*\mathcal{A}) \leq \kappa(\widetilde{S}, \sigma^*\mathcal{A} \otimes \mathcal{O}_{\widetilde{S}}) + 1.$$

All the inequalities together yield

$$a(Z) \geq 1 + \kappa(\widetilde{S}, \sigma^*\mathcal{A} \otimes \mathcal{O}_{\widetilde{S}}) \geq \kappa(\widetilde{Z}, \sigma^*\mathcal{A}) = \kappa(Z, \mathcal{A}) = a(Z)$$

which gives the claim. □

Definition 4.2. The *anti Kodaira dimension* of a compact complex variety X is the number $\kappa^{-1}(X) := \kappa(X, K_X^{-1})$.

Corollary 4.3. *Let Z be a compact, simply connected twistor space containing an irreducible fundamental divisor. If $h^0(K^{-\frac{1}{2}}) \geq 2$ and $S \in \mid -\frac{1}{2}K \mid$ is generic, then:*

$$a(Z) = 1 + \kappa^{-1}(S).$$

Proof: Our assumptions imply that there exists a pencil $\Lambda \subseteq \mid -\frac{1}{2}K \mid$ whose general member is irreducible and reduced. The general fundamental divisor of Z is contained in such a pencil. By Poon's theorem we have $a(Z) = \kappa(Z, K^{-\frac{1}{2}})$ and by the adjunction formula we obtain $K_S^{-1} \cong K^{-\frac{1}{2}} \otimes \mathcal{O}_S$. Application of Theorem 4.1 gives the result. □

5 Anti Kodaira Dimension of Rational Surfaces

The results of the previous section motivate the study of the anti Kodaira dimension of rational surfaces. Such studies were made by Sakai [Sa] but we are interested in a more detailed knowledge on the relationship between the anti Kodaira dimension and the numerical properties of the components of anti-canonical divisors. The desired results can also not be found in the papers

of E. Looijenga [Lo] and B. Harbourne [Hb] who studied surfaces containing effective anti-canonical divisors.
In contrast to the Kodaira dimension, the anti Kodaira dimension is not a birational invariant. Its behaviour under blow-ups becomes more transparent by the following results.

LEMMA 5.1. *Let S' be a smooth surface, $P' \in S'$ a point and $C' \in |-K_{S'}|$ an anti-canonical divisor. By $\sigma : S \longrightarrow S'$ we denote the blow–up with centre P'. Then we have:*

(a) $\kappa^{-1}(S) \leq \kappa^{-1}(S')$ *and*

(b) if $\mathrm{mult}_{P'}(C') \geq 2$*, then* $\kappa^{-1}(S) = \kappa^{-1}(S')$.

PROOF: Let $E \subset S$ be the exceptional divisor of σ. Then $\sigma^* K_{S'}^{-1} \cong K_S^{-1} \otimes \mathcal{O}_S(E)$. Because E is effective and $\sigma_* \mathcal{O}_S \cong \mathcal{O}_{S'}$ we obtain with $m \geq 1$: $h^0(S, K_S^{-m}) \leq h^0(S, K_S^{-m} \otimes \mathcal{O}_S(mE)) = h^0(S, \sigma^* K_{S'}^{-m}) = h^0(S', K_{S'}^{-m})$. This proves (a).
Assume now $\mathrm{mult}_{P'}(C') \geq 2$, then $\tilde{C} := \sigma^* C' - 2E$ is effective. This is true for non-reduced C'. Using $K_S^{-2} \cong \sigma^* K_{S'}^{-2} \otimes \mathcal{O}_S(-2E) \cong \sigma^* K_{S'}^{-1} \otimes \mathcal{O}_S(\tilde{C})$, we obtain $h^0(S', K_{S'}^{-m}) = h^0(S, \sigma^* K_{S'}^{-m}) \leq h^0(S, \sigma^* K_{S'}^{-m} \otimes \mathcal{O}_S(m\tilde{C})) = h^0(S, K_S^{-2m})$. This implies $\kappa^{-1}(S') \leq \kappa(S, K_S^{-2}) = \kappa^{-1}(S)$ and we obtain (b). □

THEOREM 5.2. *Let S' be a smooth rational surface and $C' \in |-K_{S'}|$ an anti-canonical divisor. The irreducible components (with reduced structure) are denoted by $C'_1, \ldots, C'_r$. Assume that among the C'_ν there is no smooth rational (-1)–curve. Then we have:*
(a) $\exists \nu : (C'_\nu.(-K_{S'}))_{S'} > 0 \Rightarrow \kappa^{-1}(S') = 2$
(b) $\forall \nu : (C'_\nu.(-K_{S'}))_{S'} = 0 \Rightarrow \kappa^{-1}(S') \in \{0, 1\}$
(c) $\forall \mu : (C'_\mu.(-K_{S'}))_{S'} \leq 0$ and $\exists \nu : (C'_\nu.(-K_{S'}))_{S'} < 0 \Rightarrow \kappa^{-1}(S') = 0$
In the case (b) we have $\kappa^{-1}(S') = 0 \iff \forall m \geq 1 : h^0(C', N^{\otimes m}) = 0$*, with the abbreviation* $N := K_{S'}^{-1} \otimes \mathcal{O}_{C'}$.

PROOF: We start with the observation that the exact sequence $0 \longrightarrow K_{S'} \longrightarrow \mathcal{O}_{S'} \longrightarrow \mathcal{O}_{C'} \longrightarrow 0$ and the rationality of S' imply $h^0(\mathcal{O}_{C'}) = 1$. As a consequence we obtain that C' is connected.
Recall that for arbitrary $D \in \mathrm{Pic}(S')$ and effective $D' \in \mathrm{Pic}(S')$ one always has $\kappa(S', D) \leq \kappa(S', D + D')$. If D is nef (i.e. for each effective divisor D' one has $(D.D')_{S'} \geq 0$), then $(D^2)_{S'} > 0$ if and only if $\kappa(S', D) = 2$.
To show (a) we assume first the existence of a component C'_ν with $({C'_\nu}^2)_{S'} > 0$. Such a divisor is nef and $\kappa(S', C'_\nu) = 2$, but $-K_{S'} - C'_\nu$ is effective, hence $\kappa^{-1}(S') = 2$.
Assume now $({C'_\mu}^2)_{S'} \leq 0$ for all μ. By assumption we have one component C'_ν with $(C'_\nu.(-K_{S'}))_{S'} > 0$. We show $({C'_\nu}^2)_{S'} = 0$ as follows: The genus formula gives $2p_a(C'_\nu) - 2 < 2p_a(C_\nu) - 2 + (C'_\nu.(-K_{S'}))_{S'} = ({C'_\nu}^2)_{S'} \leq 0$, hence the arithmetic genus $p_a(C'_\nu)$ vanishes and $C'_\nu \cong \mathbb{P}^1$. In turn, this implies

$0 \geq ({C'_\nu}^2)_{S'} = (C'_\nu.(-K_{S'}))_{S'} - 2 > -2$. By assumption, we have $({C'_\nu}^2)_{S'} \neq -1$ and conclude $({C'_\nu}^2)_{S'} = 0$. In particular, $-K_{S'}$ is not a multiple of C'_ν.
Because C' is connected, we can choose a component $C'_\mu \neq C'_\nu$ with $c := (C'_\mu.C'_\nu)_{S'} > 0$. We define $D := c \cdot C'_\mu + (1 - ({C'_\mu}^2)_{S'}) \cdot C'_\nu$ which is an effective divisor. Since $(D.C'_\mu)_{S'} = c \cdot ({C'_\mu}^2)_{S'} + (1 - ({C'_\mu}^2)_{S'}) \cdot c = c > 0$ and $(D.C'_\nu)_{S'} = c^2 + (1 - ({C'_\mu}^2)_{S'}) \cdot ({C'_\nu}^2)_{S'} = c^2 > 0$, we obtain $(D^2)_{S'} > 0$ and D is nef. If we choose $m = \max\{c, 1 - ({C'_\mu}^2)_{S'}\}$, then $\kappa^{-1}(S') = \kappa(S', K_{S'}^{-m}) \geq \kappa(S', D) = 2$ and (a) is proved.
If we have $(C'_\nu.(-K_{S'}))_{S'} = 0$ for all components of C', then $-K_{S'}$ is nef and $((-K_{S'})^2)_{S'} = 0$, hence $\kappa^{-1}(S') < 2$. Because we assumed that $\mid -K_{S'} \mid$ is non–empty, we have $\kappa^{-1}(S') \geq 0$ and (b) is shown.
To show (c) we can apply [Lo, (1.3)] which proves that the matrix $((C'_i.C'_j)_{S'})_{i,j}$ is negative definite. Hence, in the Zariski decomposition $C' = P' + N'$ of the divisor C' we have $P' = 0$ (see [Sa]). This implies $\kappa^{-1}(S') = \kappa(S', C') = \kappa(S', P') = 0$, hence (c).
To distinguish, in the case (b), anti Kodaira dimensions zero and one, we consider the exact sequence ($m \geq 1$):

$$0 \longrightarrow K_{S'}^{-(m-1)} \longrightarrow K_{S'}^{-m} \longrightarrow N^{\otimes m} \longrightarrow 0 \tag{2}$$

If $h^0(C', N^{\otimes m}) = 0$ for all $m \geq 1$, we obtain $h^0(K_{S'}^{-m}) = 1$ for $m \geq 1$ and $\kappa^{-1}(S') = 0$. On the other hand, if there exists some $m \geq 1$ with $h^0(C', N^{\otimes m}) > 0$, then we let m_0 be the smallest one with this property. From the sequence (2) we obtain $h^0(S', K_{S'}^{-m}) = 1$ for $0 \leq m < m_0$. We have $h^2(S', K_{S'}^{-m}) = h^0(S', K_{S'}^{m+1}) = 0$ for $m \geq 0$ (because S' is rational) and $((-K_{S'})^2)_{S'} = 0$ (in case (b)) and obtain from the Riemann–Roch formula $h^0(K_{S'}^{-m}) - h^1(K_{S'}^{-m}) = 1$. Therefore, $h^1(K_{S'}^{-m}) = 0$ for $0 \leq m < m_0$. The exact sequence (2) with $m = m_0$ implies now $h^0(K_{S'}^{-m_0}) > 1$, thus $\kappa^{-1}(S') > 0$. □

Remark 5.3. It is a remarkable fact that the numerical information contained in an anti-canonical divisor is not sufficient for the computation of the anti Kodaira dimension, if its components are orthogonal to the canonical class. This phenomenon also appears in the paper [Sa]. It is the reason that it is difficult to construct simply connected twistor spaces of algebraic dimension two (see [CK1]).

Corollary 5.4. *Let S be a smooth rational surface, $C \in \mid -K_S \mid$ an effective anti-canonical divisor with components $C_1, \ldots, C_r$ and denote $N := K_S^{-1} \otimes \mathcal{O}_C$. Assume that among the C_ν there is no smooth rational (-1)-curve or that S cannot be blown-down to a surface with the properties of Theorem 5.2 (b). Then, the anti–Kodaira dimension $\kappa^{-1}(S)$ is determined by the pair (C, N).*

Proof: Since $(C_\nu^2)_S = 2p_a(C_\nu) - 2 + (C_\nu.(-K_S))_S$ and $(C_\nu.(-K_S))_S = \deg(N \otimes \mathcal{O}_{C_\nu})$, this follows from Theorem 5.2. □

6 Blow-up graphs

In this section we develop a method to handle the numerical information of an anti-canonical divisor on a surface obtained by a sequence of blow-ups from $\mathbb{P}^1 \times \mathbb{P}^1$ at points lying over an anti-canonical curve with four irreducible components.

In view of our application to twistor spaces, we are only interested in blow-ups of conjugate pairs of points to have real structures on all the blown-up surfaces. We equip $S = \mathbb{P}^1 \times \mathbb{P}^1$ with the real structure given by the antipodal map on the first factor and the usual real structure on the second (cf. [K1, Ch. 3]). Choose an anti-canonical curve $C = F + \overline{F} + G + \overline{G}$ with $\overline{F} \neq F \in \mid \mathcal{O}(0,1) \mid$ and $G \in \mid \mathcal{O}(1,0) \mid$.

Let $S^{(k)} \longrightarrow S^{(k-1)} \longrightarrow \dots \longrightarrow S^{(0)} = S$ be a sequence of blow-ups at each step of which we blow up a conjugate pair of points lying on the effective anti-canonical divisor $C^{(i)}$ which is mapped to $C \subset S$. Denote by $\sigma_i : S^{(k)} \longrightarrow S^{(i)}$ the partial blow-up $(0 \leq i \leq k)$. The curve $C^{(k)} \subset S^{(k)}$ is a "cycle of rational curves" as defined in [K1, Def. 3.5] with an even number of irreducible components. Denote its components by $C_1, C_2, \dots, C_{2m}$ such that C_i intersects C_{i-1} and C_{i+1} (we consider indices modulo $2m$). We have $2 \leq m \leq k+2$. We associate the following graph to the given sequence of blow-ups:

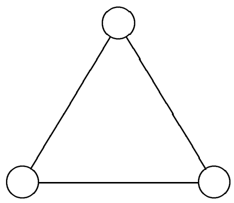
The graph contains $k+2$ vertices, some of which are possibly marked. We let m of these vertices correspond to the pairs of conjugate curves in $S^{(k)}$: (C_1, C_{m+1}), (C_2, C_{m+2}), $\dots, (C_m, C_{2m})$. We denote these vertices by $v_1, \dots, v_m$ and call them *internal* vertices of the graph. Two of these vertices v_i and v_j are joined by one edge if and only if there is an integer $0 \leq r \leq k$ such that $\sigma_r(C_i)$ and $\sigma_r(C_j)$ are curves and $\sigma_r(C_i \cup C_{m+i}) \cap \sigma_r(C_j \cup C_{m+j}) \neq \emptyset$.

The graph can also contain *external* vertices. These vertices $v_{m+1}, \dots, v_k$ correspond bijectively to conjugate pairs of irreducible smooth rational curves contracted under $\sigma_0 : S^{(k)} \longrightarrow S$. These are those strict transforms in $S^{(k)}$ of exceptional curves of the blow-ups which are not components of $C^{(k)}$. Hence, for every external vertex v there exists an integer $1 \leq r(v) \leq k-1$ such that the curves corresponding to v are the strict transforms of the exceptional curves of the blow-up $S^{(r(v)+1)} \longrightarrow S^{(r(v))}$. The number of components of $C^{(r(v))}$ and of $C^{(r(v)+1)}$ are equal and the blown-up points lie on $\sigma_{r(v)}(C_i \cup C_{m+i})$ for precisely one i. We denote this index i by $i(v)$. An external vertex is connected with an other external vertex by an edge if and only if the corresponding pairs of conjugate curves in $S^{(k)}$ have nonempty intersection. Every external vertex v is connected by an edge with precisely one internal vertex, namely with $v_{i(v)}$. Finally, we equip an external vertex v with a marking if and only if for every external vertex w connected with v by an edge we have $r(v) \leq r(w)$. Internal vertices are never marked. In our pictures we shall draw the vertices as circles and indicate the marked vertices by an asterisk inside this circle.

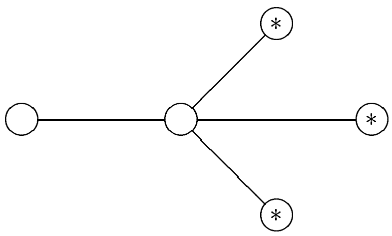
In the description of the following examples the reader should keep in mind that we consider only blow-ups of conjugate pairs of points. Thus, if there is written: "if we blow up $P \in F$, then ... ", one should read: "if we blow up

$P \in F$ and the conjugate point $\overline{P} \in \overline{F}$, then ... ".

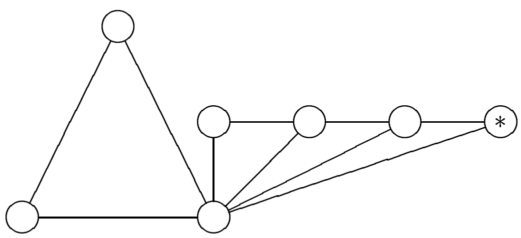
Example 6.1. If $k = 1$ and we blow up the point $F \cap G$ then the graph is the following:



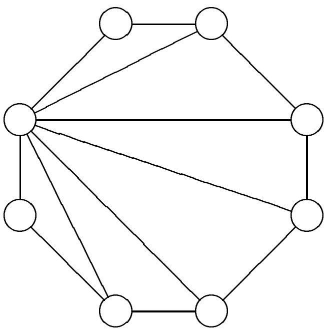
Example 6.2. If we blow up k distinct points on F, which are not contained in $G \cup \overline{G}$, then we have $m = 2$ and the graph contains k marked external vertices which are joined with one of the internal vertices. If $k = 3$, the graph looks like:



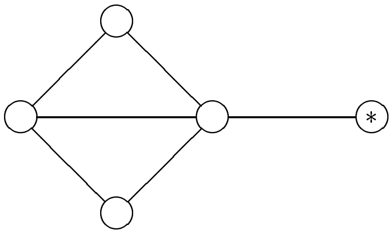
Example 6.3. If in the previous example we blow up four times the same point on the strict transforms of F, the resulting graph can be drawn as follows:



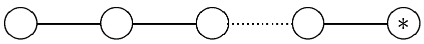
Example 6.4. If $k \geq 2$ and we always blow up the unique point over the point $F \cap G$ lying on the strict transform of F, we obtain a $(k + 2)$-gon divided into triangles by the diagonals from one vertex to all other vertices. All vertices are internal in this case. If $k = 6$ the graph looks like:



Example 6.5. If $k = 3$ and we blow up the two points $F \cap (G \cup \overline{G})$ and a third point on F, the graph is the following:



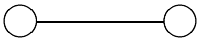
DEFINITION 6.6. A *blow-up graph* is a graph consisting of a finite number of vertices, and edges connecting distinct vertices. Some of the vertices are marked. Between two vertices there exists at most one edge. This graph can be drawn in the real plane such that a subset of at least two non-marked vertices (called internal vertices) form a regular m-gon such that the edges connecting them are represented by mutually disjoint diagonals giving a triangulation of this m-gon. (If $m = 2$ this means that the two vertices are connected by one edge.) The remaining vertices are called external vertices. The subgraph formed by these vertices and the edges among them is the disjoint union of chains like this:



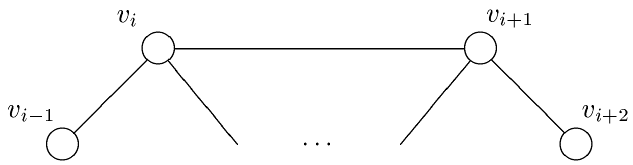
such that each chain contains precisely one marked vertex. The marked vertex of such a chain is an endpoint, i.e. is not connected with two other vertices in that chain. Finally, every external vertex is connected with precisely one internal vertex in such a way that the vertices of one chain are connected with the same internal vertex.
If v is a vertex of such a graph, we denote by $n(v)$ the number of edges adjacent to this vertex v plus the number of its markings (which is zero or one).

PROPOSITION 6.7. *The graph associated to a blow-up in the way defined above is always a blow-up graph in the sense of Definition 6.6. Moreover, the internal vertices $v_1, \ldots, v_m$ form the vertices of the m-gon of the blow-up graph such that v_i and v_{i+1} are neighbours along the boundary of the m-gon.*

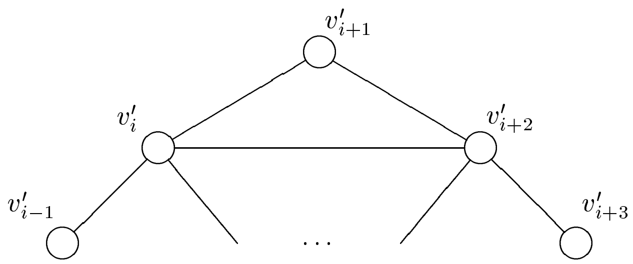
The self-intersection number of each of the curves corresponding to a vertex v is equal to $1-n(v)$.
Every blow-up graph appears as a graph associated to a sequence of blow-ups.

Proof: We prove the proposition by induction on $k \geq 0$. If $k=0$ we obtain $m=2$ and the graph is a 2-gon:  consisting of 2 internal vertices. In this case the proposition is clear, because $(F^2)_S = (G^2)_S = 0$ on S.
For the inductive step let Γ be the graph associated to $S^{(k)} \longrightarrow \ldots \longrightarrow S^{(0)}$. Let $S^{(k+1)} \longrightarrow S^{(k)}$ be a further blow-up of a conjugate pair of points $\{P, \overline{P}\}$ lying on $C^{(k)}$ and denote by Γ' the graph associated to the sequence of blow-ups $S^{(k+1)} \longrightarrow S^{(k)} \longrightarrow \ldots \longrightarrow S^{(0)}$. Assume that Γ is a blow-up graph and the self-intersection numbers in $S^{(k)}$ are those given by the claim. Then there are three possibilities:

(1) P is a singular point of $C^{(k)}$, or equivalently, P is contained in two components of $C^{(k)}$. The corresponding internal vertices v_i and v_{i+1} are neighbours in Γ along the boundary of the m-gon of internal vertices. The exceptional curves of $S^{(k+1)} \longrightarrow S^{(k)}$ are components of $C^{(k+1)}$, hence correspond to a new internal vertex of Γ'. Therefore, the graph Γ' contains an $(m+1)$-gon of internal vertices $\{v'_1, \ldots, v'_{m+1}\}$ and is obtained from Γ by adding a new internal vertex, which is connected with v_i and v_{i+1}. The numbering of the vertices in Γ' can be chosen such that $v'_j = v_j$ if $1 \leq j \leq i$, v'_{i+1} is the new vertex and $v'_{j+1} = v_j$ if $i+1 \leq j \leq m$. If part of Γ looks like the following picture:

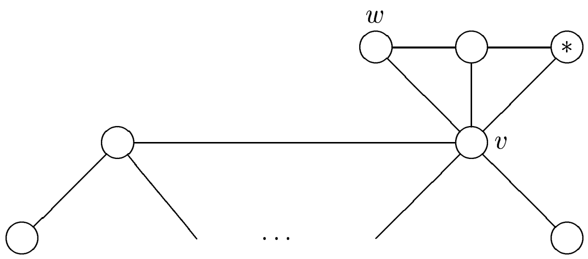


the graph Γ' is of the following kind:

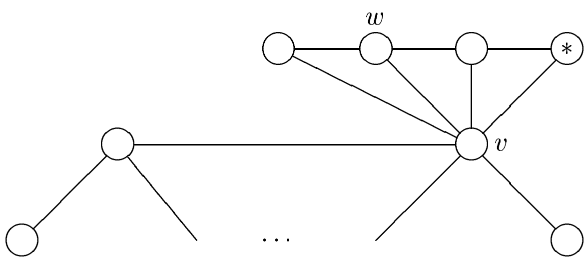


This procedure will be recalled by saying "we added an internal triangle".

(2) P is a smooth point on $C^{(k)}$. In this case, the conjugate pair of exceptional curves of the blow-up $S^{(k+1)} \longrightarrow S^{(k)}$ is not contained in $C^{(k+1)}$. It corresponds, therefore, to a new external vertex of Γ'. Assume P lies on the strict transform E of an exceptional curve of one of the previous blow-ups, which is not a component of $C^{(k)}$. Since E intersects $C^{(k)}$ we must have $E^2 = -1$. Hence, by the inductive hypothesis, the corresponding external vertex w is one end of its chain of external vertices. Moreover, if this chain consists of more that one vertex, it is the non-marked end, because this is the only vertex on this chain having $n(w) = 2$. Let v be the internal vertex being connected with w. The graph Γ' is obtained from Γ by adding a new external vertex which is connected with v and w. It, therefore, becomes the unmarked end of its chain of external vertices. For a graph Γ containing:



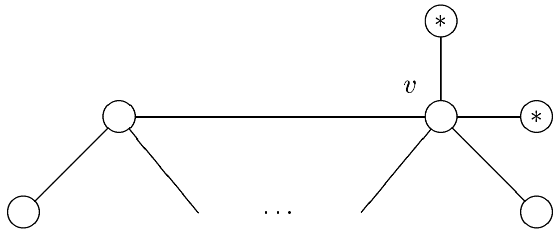
we obtain a graph Γ' like the following:



We call this procedure "adding an external triangle".

(3) P is a smooth point on $C^{(k)}$ not lying on a curve corresponding to an external vertex. Let v be the internal vertex of Γ corresponding to the components of $C^{(k)}$ containing P. Then we obtain Γ' by adding a marked external vertex to Γ and connect it with v. For example, from a graph Γ containing:

we obtain Γ' with:



We shall say, we "added a marked (external) vertex".

In each of these three cases it is clear that Γ' is again a blow-up graph and the self-intersection numbers of the curves corresponding to the vertices decrease by the number of additional edges at such a vertex. The self-intersection number of the curves corresponding to a new vertex is -1 since these are the exceptional curves of the blow-up. Hence, by the inductive hypothesis we obtain that the self-intersection numbers can be computed as $1 - n(v)$.
To show that every blow-up graph is associated to a sequence of blow-ups, we first observe that we can construct every blow-up graph Γ in the following way:

- We start with the 2-gon.
- We carry out $(m-2)$ steps of "adding an internal triangle" and obtain a triangulated m-gon.
- We add the necessary number of marked vertices.
- We add "external triangles".

As seen above, each step of this procedure corresponds to a blow-up of a conjugate pair of points, such that there exists a sequence of blow-ups determining the given graph Γ. □

Remark 6.8. Observe that every vertex of a blow-up graph is connected with at least one internal vertex by an edge. For every marked vertex v we have $n(v) \in \{2, 3\}$. Every vertex v with $n(v) > 3$ is an internal vertex. The set of internal vertices is determined by the vertices, their edges and markings. Therefore, we don't need a special marking for them.

Next we give the interpretation of the results of Section 5 in terms of our graphs associated to blow-ups of surfaces.
Let Γ and Γ' be blow-up graphs such that Γ is obtained from Γ' by adding an internal triangle. This corresponds to a blow up $S \longrightarrow S'$ of a singular point (more precisely a conjugate pair of such points) on an anti-canonical divisor of a rational surface S'. By Lemma 5.1(b) these surfaces have the same anti Kodaira dimension. If for every sequence of blow-ups $S \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with

associated blow-up graph Γ the anti Kodaira dimension $\kappa^{-1}(S)$ is the same, we define the anti Kodaira dimension of the graph Γ by $\kappa^{-1}(\Gamma) := \kappa^{-1}(S)$ and say that the *graph determines the anti Kodaira dimension*. This property of a graph is not changed by adding an internal triangle.

Theorem 6.9. *Let $S \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be a sequence of blow-ups of conjugate pairs of points as before with anti-canonical divisor $C = \sum_{i=1}^{2m} C_i$ and Γ the associated blow-up graph. Assume that $(C_i^2)_S \neq -1$ for all i. Then:*

(a) The graph Γ cannot be obtained from an other blow-up graph by adding an internal triangle.

(b) The graph Γ determines the anti Kodaira dimension if and only if it contains an internal vertex v with $n(v) \neq 3$.

(c) If Γ contains an internal vertex v with $n(v) \leq 2$ then $\kappa^{-1}(\Gamma) = 2$.

(d) If for all internal vertices v of Γ we have $n(v) \geq 3$ and for at least one of these vertices this inequality is strict, then $\kappa^{-1}(\Gamma) = 0$.

If $m = 2$ and $n(v_1) = 2$ (that is $(C_1^2)_S = -1$), then

(b') The graph Γ determines the anti Kodaira dimension if and only if $n(v_2) \neq 5$.

(c') If $n(v_2) \leq 4$, then $\kappa^{-1}(\Gamma) = 2$.

(d') If $n(v_2) \geq 6$, then $\kappa^{-1}(\Gamma) = 0$.

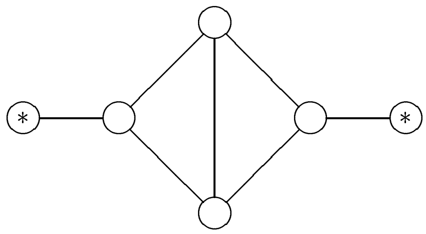
Proof: To be able to apply Theorem 5.2 we recall that the conjugate pairs of components of the anti-canonical divisor on the surface S correspond to the internal points of the associated blow-up graph. These components are irreducible smooth rational curves. By the adjunction formula for such a component C_i we obtain $(C_i.(-K_S))_S = 3 - n(v_i)$. (We keep denoting the internal vertex of Γ corresponding to C_i, $1 \leq i \leq m$ by v_i.) Therefore almost all statements are purely a translation of the statements of Theorem 5.2. We have to prove only two things.
First, the assertions (c') and (d'), if $m = 2$ and $(C_1^2)_S = (C_3^2)_S = -1$. This correspond to $n(v_1) = 2$. This case is not covered by Theorem 5.2. Let $-l = (C_2^2)_S = (C_4^2)_S$, then $n(v_2) = l + 1$. We can contract C_1 and C_3 to obtain a smooth rational surface S' with $C' = C_2' + C_4' \in \mid -K_{S'} \mid$ being the image of C. Then we have $(C_2'^2)_{S'} = (C_4'^2)_{S'} = 2 - l$ and $\kappa^{-1}(S) = \kappa^{-1}(S')$ by Lemma 5.1 (b). If $l = 3$ it is easy to see that C' is nef and big, hence $\kappa^{-1}(S') = 2$. If $l \neq 3$ we can apply Theorem 5.2 and obtain: $\kappa^{-1}(S) = \kappa^{-1}(S') = 2$ if $l < 4$, $\kappa^{-1}(S) = \kappa^{-1}(S') \in \{0, 1\}$ if $l = 4$ and $\kappa^{-1}(S) = \kappa^{-1}(S') = 0$ if $l > 4$. This shows (c') and (d').
Second, we have to prove (b) and (b'). So, we are looking for two sequences of blow-ups $S_0 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and $S_1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with same associated graph Γ but with $\kappa^{-1}(S_j) = j$. The graph Γ is required to fulfill $m = 2, n(v_1) = 2, n(v_2) = 5$

or should be a blow-up graph whose internal vertices v all have $n(v) = 3$. Using Remark 6.11 below this follows by a similar argumentation as in [CK1, Section 4]. □

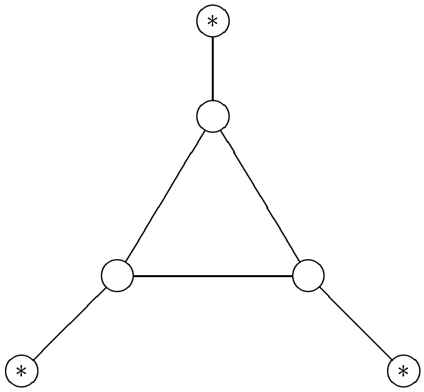
Remark 6.10. A blow-up graph Γ contains more information than necessary for computing $\kappa^{-1}(\Gamma)$. The values for all $n(v)$ at internal vertices would suffice. We shall see later (Sections 7, 8) the reason for using such graphs.

Remark 6.11. It is an easy observation that every triangulated m-gon contains at least one vertex with more than three incident edges, provided $m \geq 5$. This implies, together with Theorem 6.9 (b), that a blow-up graph determines the anti Kodaira dimension, provided it contains at least 5 internal vertices and it cannot be obtained from an other blow-up graph by adding an internal triangle. The following five blow-up graphs are the only ones with the property that each internal vertex v has $n(v) = 3$.

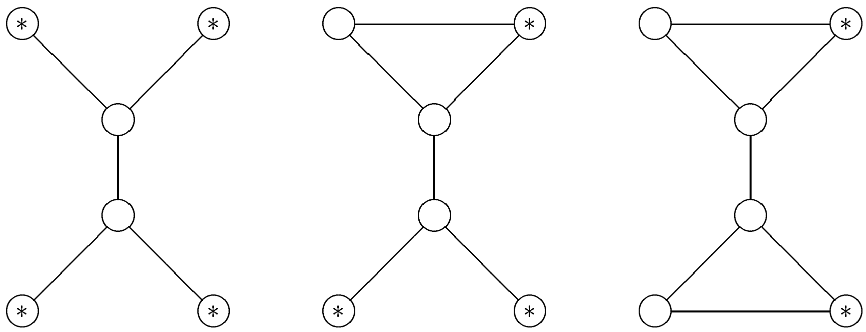
If m (the number of internal vertices) is four, there is only one possibility:



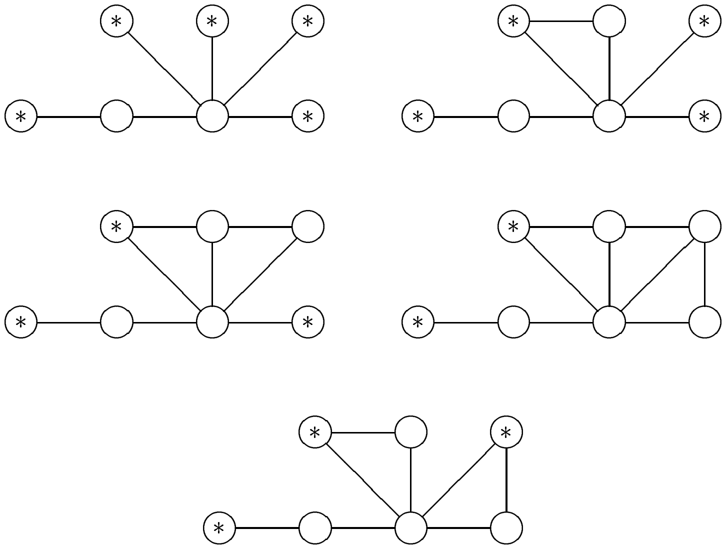
If $m = 3$ the following graph is the unique blow-up graph with precisely three edges starting at each internal vertex:



If $m = 2$ there exist three possibilities, whose differences concern only the markings and the edges between external vertices:

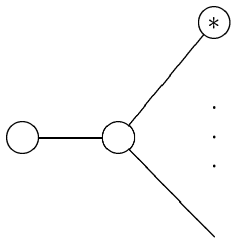


All these five graphs have six vertices. Such graphs are obtained by blowing up four pairs of conjugate points.
To obtain a complete understanding of all blow-up graphs not determining the anti Kodaira dimension, we describe below the graphs mentioned in item (b') of the above theorem. There are only five blow-up graphs with $m = 2$, $n(v_1) = 2$ and $n(v_2) = 5$. Again, they differ only in the markings and edges between the external vertices:

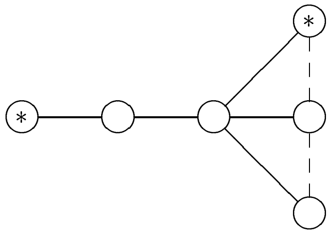


These graphs appear by blowing up five conjugate pairs of points, starting with $\mathbb{P}^1 \times \mathbb{P}^1$. But, as seen in the proof of Theorem 6.9, on such a surface we can contract a pair of (-1)-curves to arrive at a smooth rational surface, having by Lemma 5.1 (b) the same anti-Kodaira dimension as the surface we started with. The blown-down surface is obtained by blowing-up four conjugate pairs of points which are sitting on a conjugate pair of curves of type $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. This situation appears as type (3) at the beginning of Section 8. In the paper [CK1] we studied a similar situation and showed how to construct twistor spaces of algebraic dimension one and two by moving the blown-up points a bit.

Remark 6.12. If Γ is a graph as in Theorem 6.9 (c) or (c'), then $m = 2$ or it contains an internal vertex v with $n(v) = 1$, since a vertex with $n(v) = 2$ corresponds to a (-1)-curve. Such a graph contains exactly two internal vertices (i.e. $m = 2$) and one of them is connected with at most one external vertex. If one internal vertex is not connected with an external vertex, we have no further restrictions:



If both vertices are connected with an external vertex, then the number of external vertices is at most four, one of them is connected with one internal vertex, the remaining at most three with the other internal vertex:



According to Theorems 5.2 and 6.9 the blow-up graphs associated to a sequence of blow ups resulting in a surface with anti Kodaira dimension two are precisely those which are obtained by adding a finite number of internal triangles to one of the graphs described in this remark. In particular, we find among them all blow-up graphs having no external vertex.

7 Small deformations of blow-up graphs

In this section we study small deformations of rational surfaces obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$. The results will be used in Section 8 to show the existence of twistor spaces containing fundamental divisors with certain properties. We study the behaviour of blow-up graphs under small deformations, so that we can apply the results of the previous sections to the deformed surfaces.

Definition 7.1. Let

$$S^{(k)} \xrightarrow{\sigma_k} S^{(k-1)} \xrightarrow{\sigma_{k-1}} \dots \xrightarrow{\sigma_2} S^{(1)} \xrightarrow{\sigma_1} S^{(0)} = S$$

be a sequence of blow-ups of points $P^{(i)} \in S^{(i)}$ on surfaces. We call a flat family of surfaces $\mathcal{S} \longrightarrow \mathcal{T}$ together with a $\mathcal{T}$-morphism $\mathcal{S} \longrightarrow S \times \mathcal{T}$ *a family of blow-ups* of S, if we are given $\mathcal{T}$-flat families $\mathcal{S}_i \longrightarrow \mathcal{T}$ with sections $\varphi_i : \mathcal{T} \longrightarrow \mathcal{S}_i$

$(0 \leq i \leq k-1)$ such that $\mathcal{S}_{i+1} \longrightarrow \mathcal{T}$ is obtained by blowing up $\mathcal{S}_i \longrightarrow \mathcal{T}$ along $\varphi_i(\mathcal{T}) \subset \mathcal{S}_i$ and $\mathcal{S}_0 = S \times \mathcal{T}$, $\mathcal{S} = \mathcal{S}_k$. We say that this family is a *deformation of the given sequence of blow-ups*, if there is a point $0 \in \mathcal{T}$ such that for $1 \leq i \leq k$ the fibre of the blow-up morphism $\mathcal{S}_i \longrightarrow \mathcal{S}_{i-1}$ over $0 \in \mathcal{T}$ is isomorphic to the given blow-up $S^{(i)} \longrightarrow S^{(i-1)}$.

PROPOSITION 7.2. *Let S be a smooth surface and $A, B \subset S$ smooth curves intersecting transversally at $P \in S$. Consider a sequence of morphisms*

$$S^{(k)} \xrightarrow{\sigma_k} S^{(k-1)} \xrightarrow{\sigma_{k-1}} \ldots \xrightarrow{\sigma_2} S^{(1)} \xrightarrow{\sigma_1} S^{(0)} = S$$

where σ_{i+1} is the blow-up of a point $P^{(i)} \in S^{(i)}$. Denote by $\sigma^{(i)} : S^{(i)} \longrightarrow S$ the composition $\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_i$ and define inductively $A^{(0)} = A$, $B^{(0)} = B$ and $A^{(i)}, B^{(i)} \subseteq S^{(i)}$ to be the strict transforms of $A^{(i-1)}, B^{(i-1)} \subseteq S^{(i-1)}$.
Assume: $P^{(0)} = P \in A^{(0)} \cap B^{(0)}$ and $\sigma^{(i)}(P^{(i)}) = P$ for all $1 \leq i \leq k-1$. $P^{(i)} \in A^{(i)}$ if $0 \leq i \leq a$ and $P^{(i)} \in B^{(i)}$ if $a+1 \leq i \leq k-1$ for an integer $0 \leq a \leq k-1$.
Let $\pi : \{0, 1, \ldots, k-1\} \longrightarrow \{1, 2, \ldots, \beta\}$ be a monotone partition of $\{0, 1, \ldots, a\}$ and $\{a+1, \ldots, k-1\}$. This means β is a positive integer and π is a surjective map with the properties $i \leq j \Rightarrow \pi(i) \leq \pi(j)$ and $\alpha := \pi(a) < \pi(a+1)$. The fibres of π form then the usual partition sets $\pi_i := \pi^{-1}(i) \subseteq \{0, 1, \ldots, k-1\}$.
Then there exists a deformation $\mathcal{S}_\pi \longrightarrow \mathcal{T}_\pi$ of the given sequence of blow-ups, such that every neighbourhood of the special point $0 \in \mathcal{T}_\pi$ contains a point $t \in \mathcal{T}_\pi$ whose fibre $S_t := (\mathcal{S}_\pi)_t$ is isomorphic to a sequence of blow-ups

$$S_t \cong S_t^{(k)} \longrightarrow S_t^{(k-1)} \longrightarrow \ldots \longrightarrow S_t^{(1)} \longrightarrow S_t^{(0)} = S$$

at points $Q^{(i)} \in S_t^{(i)}$ with the following property (where we defined $A_t^{(i)}, B_t^{(i)}$ and $\sigma_t^{(i)}$ in the same way as $A^{(i)}, B^{(i)}$ and $\sigma^{(i)}$):
$Q^{(i)} \in A_t^{(i)}$ if $0 \leq i \leq a$,
$Q^{(i)} \in B_t^{(i)}$ if $a < i \leq k-1$,
$\sigma_t^{(i)}(Q^{(i)}) \neq P$ for all i and
$\sigma_t^{(i)}(Q^{(i)}) = \sigma_t^{(j)}(Q^{(j)})$ if and only if $\pi(i) = \pi(j)$.
In particular, there exists a deformation $\mathcal{S} \longrightarrow \mathcal{T}$ of the given sequence of blow-ups, such that every neighbourhood of the special point $0 \in \mathcal{T}$ contains a point $t \in \mathcal{T}$ whose fibre $\mathcal{S}_t$ is isomorphic to a blow-up of S at k distinct points $Q^{(i)}$ $(0 \leq i \leq k-1)$ with the property $Q^{(i)} \in A \setminus \{P\}$ for $0 \leq i \leq a$ and $Q^{(i)} \in B \setminus \{P\}$ for $a < i \leq k-1$.

The proof requires some preparation and will be given after Lemma 7.5.

DEFINITION 7.3. We say that a quadruple (S, A, B, P) is *admissible* with parameters in T, if $S \longrightarrow T$ is a flat family of smooth projective surfaces, $A, B \subset S$ are flat sub-families of smooth curves and $P = A \cap B$ is a section of S over T.

LEMMA 7.4. *Let (S, A, B, P) be admissible with parameters in T. We define $\widetilde{S} \longrightarrow S \times_T A$ to be the blow-up along the graph $\Gamma_A \subset S \times_T A$ of the embedding*

$A \subset S$, i.e. Γ_A is the intersection of $S \times_T A$ with the diagonal of $S \times_T S$. By $\widetilde{A} \subset \widetilde{S}$ and $\widetilde{B} \subset \widetilde{S}$ we denote the strict transform of $A \times_T A$ and $B \times_T A$ respectively. Let finally $\widetilde{P} := \widetilde{A} \cap \widetilde{B}$ and $\widetilde{S} \longrightarrow \widetilde{T} := A$ be the morphism induced by the projection $S \times_T A \longrightarrow A$, then $(\widetilde{S}, \widetilde{A}, \widetilde{B}, \widetilde{P})$ is admissible with parameters in $\widetilde{T}$. Furthermore, $\widetilde{A} \longrightarrow A \times_T A$ and $\widetilde{P} \longrightarrow P \times_T A$ are isomorphisms and $\widetilde{B} \longrightarrow B \times_T A$ is the blow-up of $P \times_T P$, where the morphisms are those induced by the blow-up $\widetilde{S} \longrightarrow S \times_T A$.

Proof: Since $\Gamma_A \subset S \times_T A$ is a section of the projection $S \times_T A \longrightarrow A$ we obtain flatness of $\widetilde{S} \longrightarrow \widetilde{T} = A$. Since $(A \times_T A) \cap (B \times_T A) = P \times_T A$ and $\Gamma_A \cap (P \times_T A) = P \times_T P$ is a divisor in $P \times_T A$, we obtain an isomorphism $\widetilde{P} = \widetilde{A} \cap \widetilde{B} \longrightarrow P \times_T A$, hence $\widetilde{P} \subset \widetilde{S}$ is a section of $\widetilde{S} \longrightarrow \widetilde{T}$. Because $(A \times_T A) \cap \Gamma_A$ is the diagonal in $A \times_T A$ and A has relative dimension one over T, we obtain an isomorphism $\widetilde{A} \longrightarrow A \times_T A$, which is, hence, a flat family of smooth curves.

On the other hand, $\Gamma_A \cap (B \times_T A) = P \times_T P$, hence $\widetilde{B} \longrightarrow B \times_T A$ is the blow-up of the sub-scheme of codimension two $P \times_T P \subset B \times_T A$. Since $\widetilde{B}, A$ are smooth we obtain flatness of $\widetilde{B} \longrightarrow A$ as soon as we have shown that all fibres are one-dimensional. But this is clear since the fibres of $\widetilde{S}$ over $\widetilde{T} = A$ are surfaces which are obtained by the blow-up of precisely one point of the corresponding fibre of S over T. □

In the following we denote the admissible quadruple $(\widetilde{S}, \widetilde{A}, \widetilde{B}, \widetilde{P})$ constructed in the lemma by $\mathbb{B}_A(S, A, B, P)$. Interchanging the role of A and B we obtain $\mathbb{B}_B(S, A, B, P)$ with parameters in $\widetilde{T} = B$.

We use this construction to define recursively the deformation which will be used in the proof of the proposition.

Let $\mathcal{T}^{(0)}$ be a point, $\mathcal{S}^{(0)} := S, \mathcal{A}^{(0)} := A, \mathcal{B}^{(0)} := B$ and $\mathcal{P}^{(0)} = \mathcal{A}^{(0)} \cap \mathcal{B}^{(0)} = P$. Then $(\mathcal{S}^{(0)}, \mathcal{A}^{(0)}, \mathcal{B}^{(0)}, \mathcal{P}^{(0)})$ is admissible with parameters in $\mathcal{T}^{(0)}$. We define

$$(\mathcal{S}^{(i+1)}, \mathcal{A}^{(i+1)}, \mathcal{B}^{(i+1)}, \mathcal{P}^{(i+1)}) := \begin{cases} \mathbb{B}_{\mathcal{A}^{(i)}}(\mathcal{S}^{(i)}, \mathcal{A}^{(i)}, \mathcal{B}^{(i)}, \mathcal{P}^{(i)}) & \text{if } 0 \le i \le a, \\ \mathbb{B}_{\mathcal{B}^{(i)}}(\mathcal{S}^{(i)}, \mathcal{A}^{(i)}, \mathcal{B}^{(i)}, \mathcal{P}^{(i)}) & \text{if } a < i < k. \end{cases}$$

The following lemma provides more information on the parameter spaces $\mathcal{T}^{(i+1)} = \mathcal{A}^{(i)}$ if $i \le a$ and $\mathcal{T}^{(i+1)} = \mathcal{B}^{(i)}$ if $a < i \le k-1$.

The careful reader will observe that we abuse notation a bit by using P to denote on one hand the point $P \in S$ and on the other hand the reduced closed sub-scheme $P \subset S$ supported by this point. This allows us to write $P \times A^i$ instead of $\{P\} \times A^i$ and will not cause confusion.

Lemma 7.5. *If $0 \le i \le a+1$ we have:*

(a) $\mathcal{A}^{(i)} \cong A \times A^i$ and the structure of a family of curves in $\mathcal{S}^{(i)}$ is given by the projection to the last i components (i.e. the first component is omitted) $\mathcal{A}^{(i)} \cong A \times A^i \longrightarrow \mathcal{A}^{(i-1)} \cong A^i = \mathcal{T}^{(i)}$.

(b) Under this isomorphism, $\mathcal{P}^{(i)} \subset \mathcal{A}^{(i)}$ corresponds to $P \times A^i$.

(c) $\mathcal{B}^{(i)} \longrightarrow B \times A^i$ *is obtained by successively blowing up first* $P \times H_i$ *and then the strict transforms of* $P \times H_{i-1}, P \times H_{i-2}, \ldots, P \times H_1$, *where we denote by* $H_m \subset A^i$ *the hyper-surface being the preimage of* $P \in A$ *under the* m*-th projection* $A^i \longrightarrow A$. *Again, the projection* $B \times A^i \longrightarrow A^i$ *gives the structure map* $\mathcal{B}^{(i)} \longrightarrow A^i = \mathcal{T}^{(i)}$.

If $s \geq 2$ *and* $i = a + s \leq k - 1$ *we have:*

(d) $\mathcal{B}^{(i)} \longrightarrow B^s \times A^{a+1}$ *is obtained by successively blowing up the sub-varieties of codimension two* $H'_s \times H_{a+1}$, *then the strict transforms of* $H'_s \times H_a, \ldots, H'_s \times H_1$ *followed by the same sequence with* H'_{s-1} *replacing* H'_s, *etc. up to* $H'_1 \times H_1$. *Here we let* $H'_n \subset B^s$ *be the preimage of* $P \in B$ *under the* n*-th projection* $B^s \longrightarrow B$. *The map* $\mathcal{B}^{(i)} \longrightarrow \mathcal{B}^{(i-1)} = \mathcal{T}^{(i)}$ *is induced by the projection which forgets the first component* $B \times B^{s-1} \times A^{a+1} \longrightarrow B^{s-1} \times A^{a+1}$.

(e) *For all* $1 \leq i \leq k$ *the family* $\mathcal{S}^{(i)} \longrightarrow \mathcal{T}^{(i)}$ *is a family of blow-ups of the surface* S *(see Definition 7.1), which implies in particular that it is obtained from* $S \times \mathcal{T}^{(i)}$ *by a succession of* i *blow ups of one point in each fibre. If we consider the sequence of blow-ups of* S *corresponding to a point* $t \in \mathcal{T}^{(k)}$, *then the images in* S *of the blown-up points are precisely the components of the image of* t *under the blow-up* $\mathcal{T}^{(k)} = \mathcal{B}^{(k-1)} \longrightarrow B^{k-1-a} \times A^{a+1}$ *(if* $a = k - 1$ *one has no blow up, namely* $\mathcal{T}^{(k)} = \mathcal{A}^{(k-1)} \cong A^k$*)*

Proof: Assume $0 \leq i \leq a+1$. Since we use $\mathbb{B}_{\mathcal{A}^*}$ to construct $\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(a+1)}$ the statements (a) and (b) follow by induction from Lemma 7.4, where we use always (for different T') the natural isomorphism $(A \times_T T') \times_{T'} (A \times_T T') \cong A \times_T (A \times_T T')$ which forgets the first T'.
The statement of (c) is clear for $i = 0, 1$ from the same lemma, which also implies, that $\mathcal{B}^{(i)} \longrightarrow \mathcal{B}^{(i-1)} \times_{\mathcal{A}^{(i-2)}} \mathcal{A}^{(i-1)}$ is the blow-up at $\mathcal{P}^{(i-1)} \times_{\mathcal{A}^{(i-2)}} \mathcal{P}^{(i-1)}$. Using (a) and (b) this translates by induction to the statement that $\mathcal{B}^{(i)} \longrightarrow (B \times A^{i-1}) \times_{A^{i-1}} A^i \cong B \times A^i$ is the succession of the blow-ups of (the strict transforms of) $P \times H_i, P \times H_{i-1}, \ldots, P \times H_2$ followed by the blow-up of the strict transforms of $(P \times A^{i-1}) \times_{A^{i-1}} (P \times A^{i-1}) \cong P \times P \times A^{i-1} = P \times H_1$.
We assume now $s \geq 2$ and $i = a + s \leq k - 1$. To prove (d) we first observe (cf. (a)) that the Lemma 7.4 implies that $\mathcal{B}^{(i)}$ is isomorphic to the s-fold fibre product $\mathcal{B}^{(a+1)} \times_{\mathcal{T}^{(a+1)}} \mathcal{B}^{(a+1)} \times_{\mathcal{T}^{(a+1)}} \ldots \times_{\mathcal{T}^{(a+1)}} \mathcal{B}^{(a+1)}$ and the projection to $\mathcal{B}^{(i-1)}$ is by forgetting the first factor. But we know from (a) and (c) that $\mathcal{T}^{(a+1)} = A^{a+1}$ and $\mathcal{B}^{(a+1)} \longrightarrow B \times A^{a+1}$ is the blow-up of $P \times H_{a+1}, P \times H_a, \ldots, P \times H_1$ and the projection to A^{a+1} gives the map $\mathcal{B}^{(a+1)} \longrightarrow \mathcal{T}^{(a+1)}$.
Induction on s implies now easily the claim of (d). The statement (e) is clear by induction since $\mathcal{S}^{(0)} = S$ and $\mathcal{S}^{(i)} \longrightarrow \mathcal{S}^{(i-1)} \times_{\mathcal{T}^{(i-1)}} \mathcal{T}^{(i)}$ is the blow-up of the section of the projection to $\mathcal{T}^{(i)}$ given by the inclusion $\mathcal{T}^{(i)} \subset \mathcal{S}^{(i-1)}$. □
Proof: (of Proposition 7.2)
Let $\mathcal{S} := \mathcal{S}^{(k)}$ and $\mathcal{T} := \mathcal{T}^{(k)}$ with the notation of Lemma 7.5. The assumptions imply that for every $0 \leq i \leq k-1$ the surface $S^{(i)}$ is the fibre of $\mathcal{S}^{(i)} \longrightarrow \mathcal{T}^{(i)}$

over a point $P^{(i-1)} \in \mathcal{T}^{(i)} \subset \mathcal{S}^{(i-1)}$. The point $P^{(i)} \in \mathcal{S}^{(i)}$ is the intersection point of the section $\mathcal{P}^{(i)} \subset \mathcal{S}^{(i)}$ with the fibre $S^{(i)}$.
The special point $0 \in \mathcal{T} = \mathcal{T}^{(k)}$ corresponds to $P^{(k-1)} \in \mathcal{T}^{(k)} \subset \mathcal{S}^{(k-1)}$. Its image under the sequence of blow-ups $\mathcal{T} \longrightarrow B^{k-a-1} \times A^{a+1}$ is the point $P^{k-a-1} \times P^{a+1} = (P, P, \dots, P)$.
Using the partition π we can define an embedding $\delta_\pi : B^{\beta-\alpha} \times A^\alpha \longrightarrow B^{k-a-1} \times A^{a+1}$ by the formula $\delta_\pi(x_\beta, x_{\beta-1}, \dots, x_1) := (x_{\pi(k-1)}, x_{\pi(k-2)}, \dots, x_{\pi(0)})$. This is a kind of diagonal. The fibre product of the sequence of blow-ups $\mathcal{T} \longrightarrow B^{k-1-a} \times A^{a+1}$ with δ_π defines a variety $\mathcal{T}_\pi$ together with a morphism $\mathcal{T}_\pi \longrightarrow \mathcal{T}$ and a sequence of blow-ups $\mathcal{T}_\pi \longrightarrow B^{\beta-\alpha} \times A^\alpha$. $\mathcal{T}_\pi \subset \mathcal{T}$ is the strict transform of $\delta_\pi(B^{\beta-\alpha} \times A^\alpha)$ and, therefore, the morphism $\mathcal{T}_\pi \longrightarrow B^{\beta-\alpha} \times A^\alpha$ is the composition of the blow-ups of $H'_{\beta-\alpha} \times H_\alpha$ followed by the blow-up of the strict transforms of $H'_{\beta-\alpha} \times H_{\alpha-1}, \dots, H'_{\beta-\alpha} \times H_1, H'_{\beta-\alpha-1} \times H_\alpha, H'_{\beta-\alpha-1} \times H_{\alpha-1}, \dots, H'_{\beta-\alpha-1} \times H_1, \dots, H'_1 \times H_\alpha, \dots, H'_1 \times H_1$. The H'_i, H_j have the same meaning as above but now as sub-varieties in $B^{\beta-\alpha}$ and A^α respectively. The assumption that the partition π is monotone ensures that we put the $H'_i \times H_j$ in the right order.
Since A and B are by assumption smooth irreducible curves, $\mathcal{T}_\pi$ is smooth and irreducible. The preimage in $\mathcal{T}_\pi$ of the union of all two-fold diagonals in $B^{\beta-\alpha} \times A^\alpha$ and the set of points with at least one component equal to P is a Zariski-closed subset of $\mathcal{T}_\pi$ containing $P^{(k-1)}$ and has codimension one in $\mathcal{T}_\pi$. On its complement the blow-up $\mathcal{T}_\pi \longrightarrow B^{\beta-\alpha} \times A^\alpha$ is an isomorphism (by Lemma 7.5). Hence, each (analytic) neighbourhood of $P^{(k-1)} \in \mathcal{T}_\pi$ contains a point t, whose image in $B^{\beta-\alpha} \times A^\alpha$ is a point whose components are distinct from each other and from P. Hence, the fibre of $\mathcal{S} \longrightarrow \mathcal{T}$ over the image of t in $\mathcal{T}$ is a sequence of blow-ups $S_t^{(i+1)} \longrightarrow S_t^{(i)}$ of points $Q^{(i)} \in S_t^{(i)}$ with the required properties.
The family $\mathcal{S}_\pi \longrightarrow \mathcal{T}_\pi$ obtained by base change via $\mathcal{T}_\pi \longrightarrow \mathcal{T}$ from $\mathcal{S} \longrightarrow \mathcal{T}$ is the family with the required properties. The particular situation with k distinct points in S corresponds to the partition $\pi : \{0, 1, \dots, k-1\} \longrightarrow \{1, 2, \dots, k\}$ given by $\pi(i) = i + 1$. In this case we have $\mathcal{T}_\pi = \mathcal{T}$. □
Because our main interest is the study of sequences of blow-ups of conjugate pairs, we need an additional result to make Proposition 7.2 applicable. On the other hand, we want to patch together deformations of the kind described in Proposition 7.2 centred around different points $P \in S$. For both purposes, we can apply the following lemma.

Lemma 7.6. *Let S be a smooth surface and $\mathcal{S}' \longrightarrow S \times \mathcal{T}'$ and $\mathcal{S}'' \longrightarrow S \times \mathcal{T}''$ be two flat families of sequences of blow-ups of S. Hence, we are given sections $\varphi'_i : \mathcal{T}' \longrightarrow \mathcal{S}'_i$ $(0 \le i \le k'-1)$ of $\mathcal{T}'$-flat families $\mathcal{S}'_i \longrightarrow \mathcal{T}'$ being the blow-up of $\mathcal{S}'_{i-1} \longrightarrow \mathcal{T}'$ along $\varphi'_{i-1}(\mathcal{T}') \subset \mathcal{S}'_{i-1}$ and $\mathcal{S}'_0 = S \times \mathcal{T}'$, $\mathcal{S}' = \mathcal{S}'_{k'}$. By $\psi'_i : \mathcal{T}' \longrightarrow S$ we denote the composition of φ'_i with the projection to S. Similarly for $\mathcal{S}'' \longrightarrow \mathcal{T}''$. Assume $\bigcup_{i=0}^{k'-1} \psi'_i(\mathcal{T}') \cap \bigcup_{i=0}^{k''-1} \psi''_i(\mathcal{T}'') = \emptyset$.*
Then $\mathcal{S} := \mathcal{S}' \times_S \mathcal{S}'' \longrightarrow \mathcal{T} := \mathcal{T}' \times \mathcal{T}''$ is a flat family of blow-ups of S. The fibre over $(t', t'') \in \mathcal{T}$ is isomorphic to the blow-up of S corresponding to $t' \in \mathcal{T}'$

followed by the sequence of blow-ups corresponding to $t'' \in \mathcal{T}''$.

Proof: By the disjointness assumption we can lift φ_i'' to a section $\widetilde{\varphi}_i''$ of $\mathcal{S}_i \longrightarrow \mathcal{T}' \times \mathcal{T}''$ where $\mathcal{S}_0 := \mathcal{S}' \times \mathcal{T}'' = \mathcal{S}' \times_S (S \times \mathcal{T}'')$ and $\mathcal{S}_i \longrightarrow \mathcal{S}_{i-1}$ is the blow-up of $\widetilde{\varphi}_{i-1}''(\mathcal{T}' \times \mathcal{T}'')$, that is $\mathcal{S}_i \cong \mathcal{S}' \times_S \mathcal{S}_i''$. This gives the lemma. $\square$

Remark 7.7. We shall apply this lemma in the following situation to obtain a real structure on the family $\mathcal{S} \longrightarrow \mathcal{T}$. We assume S is a surface with a real structure (without real points) and $\mathcal{S}''$ is the conjugate family to $\mathcal{S}'$, this means $\mathcal{T}'' = \overline{\mathcal{T}'}$, $\mathcal{S}'' = \overline{\mathcal{S}'}$ and φ_i'' is the conjugate section to φ_i'. The projection $\mathcal{S}'' \longrightarrow S$ is the composition of the corresponding projection $\overline{\mathcal{S}''} \longrightarrow \overline{S}$ with the isomorphism $S \longrightarrow \overline{S}$ defining the real structure. The real structures on $\mathcal{S}$ and $\mathcal{T}$ are given by interchanging the components. This is anti-holomorphic since the identity $\mathcal{S}' \longrightarrow \overline{\mathcal{S}'}$ is.

We start now the study of small deformations with the aid of the blow-up graphs of Section 6.

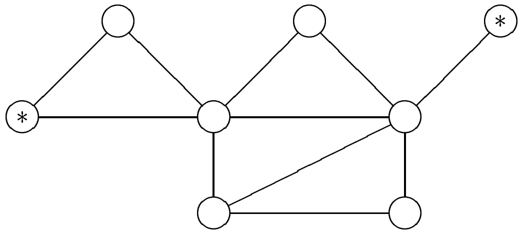
Definition 7.8. We say that a blow-up graph Γ is a *small deformation* of an other blow-up graph Γ_0 if and only if there exists a flat family of surfaces $\mathcal{S} \longrightarrow \mathcal{T}$ with real structures having special fibre S_0 over the real point $0 \in \mathcal{T}$ such that S_0 is isomorphic to a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ with associated graph Γ_0 and every (analytic) neighbourhood of $0 \in \mathcal{T}$ contains a real point $t \in \mathcal{T}(\mathbb{R}) \setminus \{0\}$ whose fibre S_t is isomorphic to a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ with associated graph Γ.

In the following we want to determine blow-up graphs which are small deformations of a given blow-up graph. We shall not solve the problem of determining all graphs being a small deformation of a given one, because this includes the study of different graphs belonging to isomorphic surfaces. The results obtained here are sufficient for our applications.

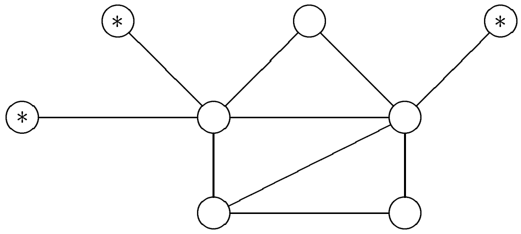
Definition 7.9. Let Γ_0, Γ be blow-up graphs with the same set of vertices. We say Γ is an *elementary deformation* of Γ_0 if we obtain Γ by removing one edge from Γ_0 which connects two internal vertices or two external vertices in Γ_0. We require that one of these vertices, call it v, is marked in Γ but not marked in Γ_0. All other markings of Γ and Γ_0 coincide.

Remark 7.10. Since the number of adjacent edges of the vertex v in Γ is one less than in the graph Γ_0 but it is marked in Γ, the number $n(v)$ must be the same for both graphs. If the removed edge connects two external vertices, the chain of external vertices in Γ_0 containing this edge splits into two chains in Γ. One of these two parts does already contain a marked point. Therefore, the vertex to be marked is in this case already determined by Γ_0. If we remove an edge connecting two internal vertices, the vertex v must fulfill $n(v) = 2$ or $n(v) = 3$, since in Γ it is an external marked vertex. In general, the vertex v which becomes marked in Γ is not determined by the graph Γ_0.

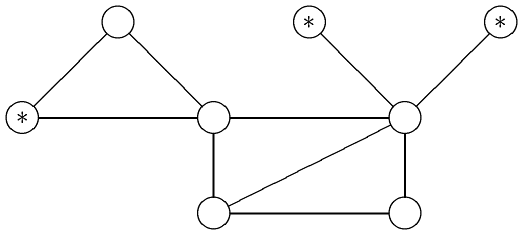
Example 7.11. If Γ_0 is the following graph:



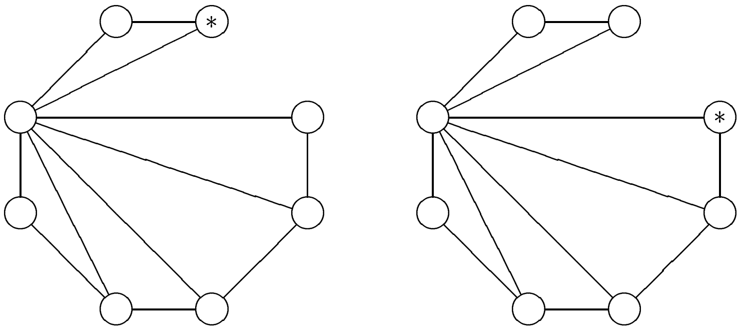
we obtain as an elementary deformation by removing a connection of external vertices the following graph:



and by removing an edge connecting two internal vertices we obtain the elementary deformation:



Example 7.12. The following two graphs are elementary deformations of the graph Γ_0 drawn in Example 6.4. Here we see that we have two possibilities for the additional marking in the graph Γ. In the first example we have $m = 6$ and in the second $m = 3$, whereas for the graph Γ_0 we have $m = 8$.



THEOREM 7.13. *Let Γ_0, Γ be two blow-up graphs such that Γ can be obtained from Γ_0 by a finite number of elementary deformations. Then Γ is a small deformation of Γ_0.*

PROOF: Assume Γ_0 and Γ have the same set of vertices. If $\Gamma_0 \neq \Gamma$ there is a certain set $\mathcal{V}$ of vertices which are marked in Γ and not marked in Γ_0. Every vertex $v \in \mathcal{V}$ is contained in precisely one (maximal) chain of external vertices $\mathcal{C}(v)$ in the graph Γ. By $\mathcal{C}$ we denote the union of these sets of external vertices $\mathcal{C} = \bigcup_{v\in\mathcal{V}} \mathcal{C}(v)$.

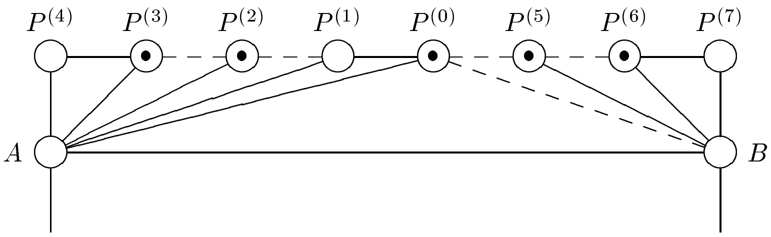
Let Γ' be the graph obtained from Γ by removing all the vertices in $\mathcal{C}$ and the edges connecting them with internal vertices of Γ. This means, Γ can be obtained from Γ' by adding marked vertices and external triangles. As seen in the proof of Proposition 6.7 there exist sequences of blow-ups $S_1 \xrightarrow{\sigma} S_1' \xrightarrow{\sigma'} S = \mathbb{P}^1 \times \mathbb{P}^1$ such that Γ (resp. Γ') is the graph associated to the sequence of blow ups $\sigma' \circ \sigma$ (resp. σ'). Furthermore, it is clear from the definitions, that we obtain Γ_0 from Γ' by adding internal triangles, external triangles and marked vertices. This implies the existence of a sequence of blow-ups $\sigma_0 : S_0 \longrightarrow S_1'$ such that Γ_0 is the graph associated to the composition $\sigma' \circ \sigma_0$.

The graph obtained from Γ_0 by removing all the internal vertices of Γ consists of certain connected components. We denote by $\mathcal{C}_i$ with $1 \leq i \leq c$ the subsets of $\mathcal{C}$ obtained by intersection with these connected components.

Every set $\mathcal{C}_i$ consists entirely of internal or of external vertices of Γ_0, because in a blow-up graph two marked vertices are not contained in the same chain of external vertices. If $\mathcal{C}_i$ contains an internal vertex of Γ_0, then it contains exactly one vertex connected with two internal vertices of Γ. All other vertices of $\mathcal{C}_i$ are connected with precisely one of these internal vertices of Γ. If $\mathcal{C}_i$ consists of external vertices of Γ_0, then all its vertices are connected with the same internal vertex in Γ.

From the relation between blow-ups and the operation of adding a marked vertex or an external triangle to a graph (described in the proof of Proposition 6.7) it is clear that the sets $\mathcal{C}_i$ are precisely the equivalence classes on $\mathcal{C}$ given by the equivalence relation: $w \sim w'$ if and only if the conjugate pairs of curves corresponding to w and w' are mapped under $\sigma_0 : S_0 \longrightarrow S_1'$ to the same conjugate pair of points. These points lie on the curves in S_1', corresponding to the internal vertices of the graph Γ' connected with $\mathcal{C}_i$.

By Lemma 7.6 it is enough to prove the theorem in the case of only one set $\mathcal{C}_i$. But in this case the result is a reformulation of Proposition 7.2 (using Lemma 7.6 to obtain a version of Proposition 7.2 with pairs of blown-up points at each step, see Remark 7.7). The partition of the set $\mathcal{C}_i$ is defined by the chains of external vertices of Γ inside $\mathcal{C}_i$. The following picture gives an example of a part of a graph Γ_0 where the edges which are not edges of Γ are drawn with broken lines. The vertices of the set $\mathcal{V}$ are indicated with bullets.



In this example we have five sets in the corresponding partition (i.e. $\beta = 5$), namely $\{P^{(0)}, P^{(1)}\}, \{P^{(2)}\}, \{P^{(3)}, P^{(4)}\}, \{P^{(5)}\}, \{P^{(6)}, P^{(7)}\}$. □

Corollary 7.14. *Every blow-up graph Γ is a small deformation of a blow-up graph Γ_0 with the same number of vertices, but having no external vertices.*

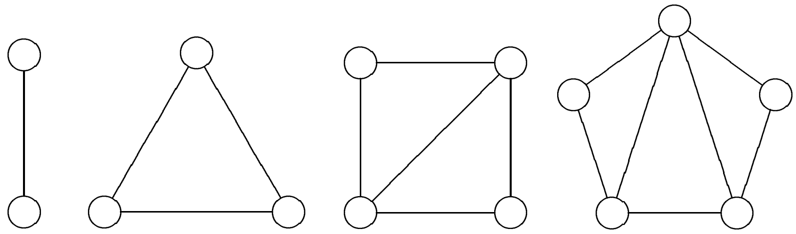
Proof: This follows easily by induction from the observation that we obtain a blow-up graph Γ_0 by the following procedure: In a blow-up graph Γ we unmark a marked external vertex v. Let w be the unique internal vertex connected with v. Then, we connect v by an edge with one of the internal vertices which are neighbours of w along the boundary of the m-gon of internal vertices of the given graph Γ. □

This motivates the following definition.

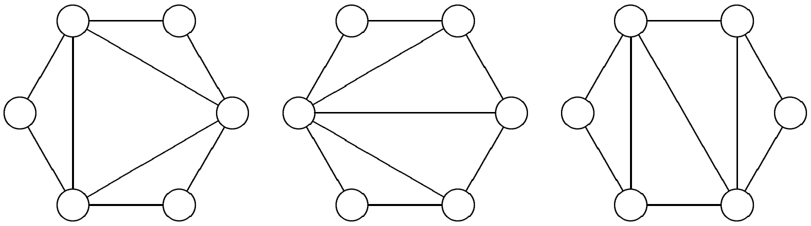
Definition 7.15. A *basic* blow-up graph is a blow-up graph which does not contain external vertices.

Remark 7.16. Let us equip the set of blow-up graphs with the partial ordering generated by the requirement: $\Gamma \geq \Gamma_0$ if Γ is an elementary deformation of Γ_0. Then the basic blow-up graphs are precisely the minimal elements in this PO-set.

Remark 7.17. For every $2 \leq m \leq 5$ there exists precisely one basic blow-up graph with m vertices. They are the following:



All other blow-up graphs with $m \leq 5$ are small deformations of them. If $m = 6$ there exist three different basic blow-up graphs:



8 APPLICATION TO TWISTOR SPACES

We return to the situation of Section 3. Let Z be a compact, simply connected twistor space containing an irreducible fundamental divisor and satisfying $c_1(Z)^3 < 0$ and $h^0(K^{-\frac{1}{2}}) \geq 2$.
By Proposition 2.4 we know that the Riemannian base of such a twistor space is diffeomorphic to the connected sum $n\mathbb{CP}^2$ (with $n > 4$) and the conformal class contains a metric with positive scalar curvature.
The existence of such a pencil implies that the algebraic dimension of Z must be positive. Let S be an irreducible real fundamental divisor, then there exists a sequence of blow-ups of $n \geq 5$ conjugate pairs of points $S \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. We know from [K1, Prop. 3.6] that we can choose this succession of blow-ups such that the anti-canonical system $|-K_S|$ contains a real member C mapped onto a curve C' on $\mathbb{P}^1 \times \mathbb{P}^1$ having one of the following four types :

(0) $C' \in |\mathcal{O}(2,2)|$ is a smooth elliptic curve,

(1) C' has four components $C' = F' + \overline{F'} + G' + \overline{G'}$ where $F' \in |\mathcal{O}(0,1)|$ and $G' \in |\mathcal{O}(1,0)|$ are not real,

(2) C' has two components $C' = F' + C_0'$ where $F' \in |\mathcal{O}(0,1)|$ is real and $C_0' \in |\mathcal{O}(2,1)|$ is real, smooth and rational,

(3) C' has two distinct components $C' = A' + \overline{A'}$ where $A', \overline{A'} \in |\mathcal{O}(1,1)|$.

In the case of type (0) the curve C is smooth elliptic and $(C^2)_S < 0$, hence, by Theorem 5.2 we have $\kappa^{-1}(S) = 0$. Corollary 5.4 implies that we have for generic real fundamental divisors $\kappa^{-1}(S) = 0$. Hence, by Corollary 4.3 we obtain $a(Z) = 1$.
In the type (2) case we always have $\kappa^{-1}(S) = 2$, because there is no point on F' blown up and hence the strict transform F of F' is a curve with $(F.K_S^{-1})_S = 2$. Again, by Corollaries 5.4 and 4.3 we obtain $a(Z) = 3$. This was also obtained in [K2].
The case of type (3) reduces to type (1) using elementary transformations, if the intersection points of A' and $\overline{A'}$ are blown up. Otherwise, we obtain $(A.(-K_S))_S < 0$ and $C = A + \overline{A}$. In this situation, Theorem 5.2 tells us $\kappa^{-1}(S) = 0$ and again we compute $a(Z) = 1$ using the Corollaries 5.4 and 4.3.
It remains to study the situation of type (1). This is precisely the situation where we can associate to the sequence of blow-ups a blow-up graph Γ. If Γ does not contain one of the ten graphs of Remark 6.11 as a subgraph that contains all external vertices and all edges between them, then Γ determines the anti Kodaira dimension of S by Theorem 6.9. If this is the case, the algebraic dimension $a(Z)$ is determined by Γ. This follows from Corollaries 5.4 and 4.3, because the restriction $K_S^{-1} \otimes \mathcal{O}_C \cong K^{-\frac{1}{2}} \otimes \mathcal{O}_C$ does not depend on the chosen fundamental divisor S. For example we can formulate the following theorem:

THEOREM 8.1. *A simply connected twistor space Z containing at least a pencil of fundamental divisors is Moishezon if and only if it fulfills the equivalent* *conditions of Theorem 3.7 or contains a real irreducible fundamental divisor S possessing an associated blow-up graph that either*

- *contains one internal vertex which is connected with all external vertices, or*
- *contains at most four external vertices and a pair of connected internal vertices with the property that one of them is connected with precisely one of the external vertices and the other one with all the remaining external vertices.*

In particular, basic blow-up graphs appear only in Moishezon spaces.

Proof: The observations at the beginning of this section show for $n \geq 5$ that a Moishezon twistor space, not fulfilling the conditions of Theorem 3.7, contains a real fundamental divisor S possessing a blow-up graph. For $n = 4$ this follows from [K1] and in case $n \leq 3$ every twistor space contains a fundamental divisor possessing an associated blow-up graph.
Observe that a blow-up graph with at most five vertices always fulfills the conditions of the theorem. Since in the case $n \leq 3$ all twistor spaces are Moishezon, nothing is to prove then. In the case $n = 4$ (corresponding to blow-up graphs with six vertices) the result follows from previous work [K1] and the observation, that (in this case) $K^{-\frac{1}{2}}$ is not nef if and only if the corresponding blow-up graph fulfills the conditions of the theorem. By a *nef* line bundle we mean here one which has non-negative intersection number with all curves in Z. Let us, therefore, assume $n \geq 5$.
In the theorem the conditions on the graph are made to match precisely the graphs obtained by adding internal triangles to a graph fulfilling condition (c) or (c') of Theorem 6.9. If $\dim \mid -\frac{1}{2}K \mid = 1$, we can apply Corollary 5.4 and Theorem 6.9 to show that the generic $S \in \mid -\frac{1}{2}K \mid$ has $\kappa^{-1}(S) = 2$. Corollary 4.3 implies that Z is a Moishezon space. If $\dim \mid -\frac{1}{2}K \mid \geq 2$, then the result follows from Theorems 3.6, 3.7 and the observations at the beginning of this section. □

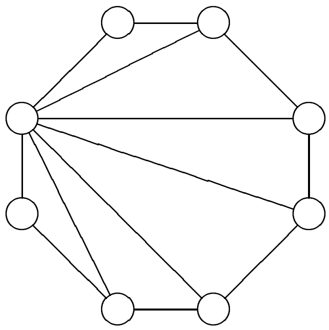
Remark 8.2. A blow-up graph fulfills the properties of the theorem precisely when it contains one of the graphs of Remark 6.12 as a subgraph that contains all external vertices of it.

Remark 8.3. In Corollary 7.14 we saw that every blow-up graph is a small deformation of a basic blow-up graph. By Theorem 8.1 basic blow-up graphs appear only in Moishezon twistor spaces. This suggests that one could hope to be able to construct twistor spaces containing a fundamental divisor associated to an arbitrarily given blow-up graph by studying small deformations of Moishezon twistor spaces. We shall see in Theorem 8.8 that this in fact works. In particular, small deformations of Moishezon twistor spaces need not to be Moishezon [C1], [LeBP].

It would be very interesting to obtain a better understanding of $a(Z)$ and $\kappa^{-1}(S)$ if Γ does not determine the anti Kodaira dimension. This would help to understand the case $a(Z) = 2$.

Definition 8.4. A blow-up graph Γ is called *twistorial* if there exists a twistor space Z containing an irreducible fundamental divisor S which is obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by a sequence of blow-ups whose associated graph is Γ.

Example 8.5. The basic blow-up graphs of Example 6.4 containing one vertex which is connected with all other vertices are twistorial.
By Proposition 6.7, Theorems 3.6 and 3.5 one should search for a corresponding fundamental divisor in a LeBrun twistor space. Such twistor spaces Z are birational to conic bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ whose discriminant is the union of n irreducible divisors in the linear system $\mid \mathcal{O}(1,1) \mid$. The fundamental linear system is isomorphic to $\mid \mathcal{O}(1,1) \mid$ such that every divisor in $\mid \mathcal{O}(1,1) \mid$ corresponds to a fundamental divisor in Z. The most degenerate LeBrun spaces are those where the n components of the discriminant of the conic bundle are contained in one pencil in $\mid \mathcal{O}(1,1) \mid$. Such a pencil has two base-points on $\mathbb{P}^1 \times \mathbb{P}^1$. Every real member of this pencil, which is different from the n components of the discriminant of the conic bundle, has as its associated graph the basic blow-up graph mentioned above. (Below, the picture for the case $n = 6$ is drawn.) This degenerate case was not studied in [LeB2]. The details can be found in [Ku].



In the sequel we want to study the question which blow-up graphs are twistorial. For that purpose we have to show the existence of twistor spaces with certain properties. A very efficient tool for constructing new twistor spaces is the following theorem, which has its origin in the paper [DonF].

Theorem 8.6. *([DonF],[LeB3],[PP2],[C3]) Let Z be a Moishezon twistor space with $H^2(Z, \Theta_Z) = 0$. Then, any real member of a small deformation of Z is again a twistor space. Furthermore, any small deformation of a real irreducible fundamental divisor S with real structure is induced by a deformation of Z in the sense that the deformed surfaces are members of the fundamental system of the deformed twistor spaces.*

For LeBrun twistor spaces the vanishing of $H^2(Z, \Theta_Z)$ was shown in the papers [LeBP], [C1] and [C3]. But the authors of these papers do not take care of the degenerate case. Therefore we need the following theorem, whose proof grew out of a discussion with H. Kurke. The author is grateful to him.

Theorem 8.7. *If Z is a Moishezon twistor space containing an irreducible fundamental divisor, then $H^2(Z,\Theta_Z)=0$.*

Proof: The space Z is simply connected [C2] and of positive type by Proposition 2.4 or [Po2]. Let $S\in\mid -\frac{1}{2}K\mid$ be an irreducible fundamental divisor. By Lemma 2.2 S is a smooth rational surface. The adjunction formula implies $K^{\frac{1}{2}}\otimes\mathcal{O}_S\cong K_S$. The exact sequence

$$0\longrightarrow N^{\vee}_{S|Z}\longrightarrow\Omega^1_Z\otimes\mathcal{O}_S\longrightarrow\Omega^1_S\longrightarrow 0$$

implies $H^0(\Omega^1_Z\otimes\mathcal{O}_S)=0$, because $H^0(N^{\vee}_{S|Z})=H^0(\mathcal{O}_S(-S))=H^0(K_S)=0$ and $H^0(\Omega^1_S)=0$ by the rationality of S.
On the other hand, the restriction map $\operatorname{Pic}Z\longrightarrow\operatorname{Pic}S$ is injective by [K1, Lemma 3.1]. The Fröhlicher spectral sequence (which degenerates for Moishezon varieties [U]) together with the rationality of S, the vanishing of $H^0(Z,\Omega^2_Z)$ ([H2]) and Lemma 2.3 induces natural isomorphisms $\operatorname{Pic}Z\cong H^1(Z,\Omega^1_Z)$ and $\operatorname{Pic}S\cong H^1(S,\Omega^1_S)$. The corresponding natural injective map $H^1(Z,\Omega^1_Z)\longrightarrow H^1(S,\Omega^1_S)$ is the composition of the natural maps $H^1(\Omega^1_Z)\longrightarrow H^1(\Omega^1_Z\otimes\mathcal{O}_S)\longrightarrow H^1(\Omega^1_S)$. The first morphism, which is hence injective, appears in the exact cohomology sequence of

$$0\longrightarrow\Omega^1_Z(-S)\longrightarrow\Omega^1_Z\longrightarrow\Omega^1_Z\otimes\mathcal{O}_S\longrightarrow 0.$$

With the vanishing of $H^0(\Omega^1_Z\otimes\mathcal{O}_S)$, shown above, we obtain now: $H^1(\Omega^1_Z\otimes K^{\frac{1}{2}})=H^1(\Omega^1_Z(-S))=0$.
Using the standard exact sequence $0\longrightarrow N^{\vee}_{S|Z}\longrightarrow\Omega^1_Z\otimes\mathcal{O}_S\longrightarrow\Omega^1_S\longrightarrow 0$ we obtain, using $N^{\vee}_{S|Z}=\mathcal{O}_S(-S)=K^{\frac{1}{2}}\otimes\mathcal{O}_S$, the exact sequence

$$0\longrightarrow K_S^{\otimes 2}\longrightarrow\Omega^1_Z\otimes K^{\frac{1}{2}}\otimes\mathcal{O}_S\longrightarrow\Omega^1_S\otimes K_S\longrightarrow 0.$$

Since S is a rational surface we have $h^0(K_S^{\otimes 2})=0$ and $h^0(\Omega^1_S\otimes K_S)=h^2(\Theta_S)=0$. Hence, we obtain $h^0(\Omega^1_Z\otimes K^{\frac{1}{2}}\otimes\mathcal{O}_S)=0$. Using the exact sequence

$$0\longrightarrow\Omega^1_Z\otimes K\longrightarrow\Omega^1_Z\otimes K^{\frac{1}{2}}\longrightarrow\Omega^1_Z\otimes K^{\frac{1}{2}}\otimes\mathcal{O}_S\longrightarrow 0,$$

and the vanishing of $h^1(\Omega^1_Z\otimes K^{\frac{1}{2}})$, this implies $h^1(\Omega^1_Z\otimes K)=0$. By Serre duality we obtain the desired vanishing. ☐
The following theorem is the main result of this section.

Theorem 8.8. *Every blow-up graph is twistorial.*

Proof: Combining Theorems 8.7, 8.6 with Theorem 7.13 and Corollary 7.14 we see that it is enough to show that every basic blow-up graph is twistorial.

The corresponding twistor spaces are provided by the equivariant version of the method of Donaldson and Friedman [DonF] to construct self-dual structures on the connected sum of two self-dual manifolds. Such a method was developed by Pedersen and Poon in the paper [PP3]. The spaces obtained in the case of the action of a two dimensional torus are investigated in detail in a recent preprint of Honda [Ho].

His main result is that the twistor spaces obtained by the equivariant version of the Donaldson-Friedman construction contain a pencil of fundamental divisors invariant under the action of the two-dimensional torus. The general member of this pencil is a smooth toric surface, which is isomorphic to a successive blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at conjugate pairs of fixed points of the action.

Furthermore, he shows, using the results of Orlik and Raymond [OR], that every such toric surface appears as a fundamental divisor in a twistor space. But the fixed points of the torus action are precisely the singularities of the (unique) torus invariant effective anti-canonical divisor on the toric surface. This means that an arbitrary sequence of blow ups of conjugate singularities of the torus invariant effective anti-canonical divisor, starting at $\mathbb{P}^1 \times \mathbb{P}^1$, leads to a surface which appears as a fundamental divisor in a twistor space. Because every basic blow-up graph can be obtained in this way, the theorem is proven. $\square$

There is another construction of twistor spaces over $n\mathbb{CP}^2$ with the symmetry of the two-torus, introduced by D. Joyce [J]. It seems to be not clear, whether these spaces contain a pencil of fundamental divisors or not. But, observe that D. Joyce associates (in a different way) to each of his spaces one of the basic blow-up graphs [J, p. 541]. These graphs reflect the orbit structure and isotropy groups of the action of $T^2 = S^1 \times S^1$ on $n\mathbb{CP}^2$.

References

[AHV] J. Aroca, H. Hironaka, J. Vicente: Desingularisation theorems, Mem. Math. Inst. Jorge Juan No. 30, Madrid, 1977

[AHS] M.F. Atiyah, N.J. Hitchin, I.M. Singer: Self–duality in four–dimensional Riemannian geometry, *Proc. Roy. Soc. London Ser. A* 362 (1978) 425-461

[BPV] W. Barth, C. Peters, Van de Ven: Compact complex surfaces, Springer, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 4, Berlin 1984

[C1] F. Campana: The class $\mathcal{C}$ is not stable by small deformations, *Math. Ann.* 290 (1991) 19-30

[C2] F. Campana: On Twistor Spaces of the Class $\mathcal{C}$, *J. Diff. Geom.* 33 (1991) 541-549

[C3] F. Campana: The class $\mathcal{C}$ is not stable by small deformations II, *Contemp. Math.* 162 (1994) 65-76

[CK1] F. Campana, B. Kreussler: Existence of twistor spaces of algebraic dimension two over the connected sum of four complex projective planes, to appear in *Proc. of the AMS*

[CK2] F. Campana, B. Kreussler: A Conic Bundle Description of Moishezon Twistor Spaces Without Effective Divisors of Degree One, *Math. Zeit.* 229 (1998) 137-162

[DonF] S. Donaldson, R. Friedman: Connected sums of self-dual manifolds and deformations of singular spaces, *Nonlinearity* 2 (1989) 197-239

[ES] M.G. Eastwood, M.A. Singer: The Fröhlicher Spectral Sequence on a Twistor Space, *J. Diff. Geom.* 38 (1993) 653-669

[FK] T. Friedrich, H. Kurke: Compact four–dimensional self–dual Einstein manifolds with positive scalar curvature, *Math. Nach.* 106 (1982) 271-299

[Gau] P. Gauduchon: Structures de Weyl et théorèmes d'annulation sur une variété conforme autoduale, *Ann. Scuola Norm. Sup. Pisa* 18 (1991) 563-629

[GH] P. Griffiths, J. Harris: Principles of algebraic geometry, New York: John Wiley, 1978

[Hb] B. Harbourne: Anticanonical Rational Surfaces, *Trans. AMS* 349 (1997) 1191-1208

[H] R. Hartshorne: Algebraic geometry, GTM 52, Springer 1977

[HH] F. Hirzebruch, H. Hopf: Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten, *Math. Ann.* 136 (1958) 156-172

[H1] N.J. Hitchin: Linear field equations on self–dual spaces, *Proc. Roy. Soc. London Ser. A* 370 (1980) 173-191

[H2] N.J. Hitchin: Kählerian twistor spaces, *Proc. Lond. Math. Soc., III Ser.* 43 (1981) 133-150

[Ho] N. Honda: On the structure of Pedersen-Poon twistor spaces, preprint, Hiroshima, 1998

[J] D. Joyce: Explicit construction of self-dual 4-manifolds, *Duke Math. Journal* 77 (1995) 519-552

[K1] B. Kreussler: On the algebraic dimension of twistor spaces over the connected sum of four complex projective planes, *Geometriae Dedicata* 71 (1998) 263-285

[K2] B. KREUSSLER: Moishezon twistor spaces without effective divisors of degree one, *J. Algebraic Geometry* 6 (1997) 379-390

[KK] B. KREUSSLER, H. KURKE: Twistor spaces over the connected sum of 3 projective planes, *Compositio Math.* 82 (1992) 25-55

[Ku] H. KURKE: Classification of twistor spaces with a pencil of surfaces of degree 1, Part I, *Math. Nachr.* 158 (1992) 67-85

[LeB1] C. LEBRUN: On the topology of self-dual 4-manifolds, *Proc. AMS* 98 (1986) 637-640

[LeB2] C. LEBRUN: Explicit self–dual metrics on $\mathbb{CP}^2\#\ldots\#\mathbb{CP}^2$, *J. Diff. Geom.* 34 (1991) 223-253

[LeB3] C. LEBRUN: Twistors, Kähler manifolds, and bimeromorphic geometry. I, *J. AMS* 5 (1992) 289-316

[LeBP] C. LEBRUN, Y.S. POON: Twistors, Kähler manifolds, and bimeromorphic geometry. II, *J. AMS* 5 (1992) 317-325

[Lo] E. LOOIJENGA: Rational surfaces with an anti-canonical cycle, *Ann. Math.* 114 (1981) 267-322

[OR] P. ORLIK, F. RAYMOND: Action of the torus on 4-manifolds I, *Trans. AMS* 152 (1970) 531-559

[PP1] H. PEDERSEN, Y.S. POON: Self–duality and differentiable structures on the connected sum of complex projective planes, *Proc. AMS* 121 (1994) 859-864

[PP2] H. PEDERSEN, Y.S. POON: A relative deformation of Moishezon twistor spaces, *J. Alg. Geom.* 3 (1994) 685-701

[PP3] H. PEDERSEN, Y.S. POON: Equivariant connected sums of compact self-dual manifolds, *Math. Ann.* 301 (1995) 717-749

[Pe] R. PENROSE: Nonlinear gravitons and curved twistor theory, *General Relativity and Gravitation* 7 (1976) 31-52

[Pon] M. PONTECORVO: Algebraic dimension of twistor spaces and scalar curvature of anti–self–dual metrics, *Math. Ann.* 291 (1991) 113-122

[Po1] Y.S. POON: Compact self–dual manifolds with positive scalar curvature, *J. Diff. Geom.* 24 (1986) 97-132

[Po2] Y.S. POON: Algebraic dimension of twistor spaces, *Math. Ann.* 282 (1988) 621-627

[Po3] Y.S. POON: Twistor spaces with Meromorphic Functions, *Proc. AMS* 111 (1991) 331-338

[Po4] Y.S. Poon: On the algebraic structure of twistor spaces, *J. Diff. Geom.* 36 (1992) 451-491

[Sa] F. Sakai: D-Dimension of Algebraic Surfaces and Numerically Effective Divisors, *Compositio Math.* 48 (1983) 101-118

[U] K. Ueno: Classification Theory of Algebraic Varieties and Compact Complex Spaces, *Lecture Notes in Math.* 439, Springer, Berlin 1975

[Z] O. Zariski: The Theorem of Riemann-Roch for High Multiples of an Effective Divisor on an Algebraic Surface, *Ann. Math.* 76 (1962) 560-615

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