The Local Monodromy as a Generalized Algebraic Correspondence

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with an Appendix by
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Abstract. For an algebraic, normal-crossings degeneration over a local field the local monodromy operator and its powers naturally define Galois equivariant classes in the ℓ-adic (middle dimensional) cohomology groups of some precise strata of the special fiber of a normal-crossings model associated to the fiber product degeneration. The paper addresses the question whether these classes are algebraic. It is shown that the answer is positive for any degeneration whose special fiber has (locally) at worst triple points singularities. These algebraic cycles are responsible for and they explain geometrically the presence of poles of local Euler L-factors at integers on the left of the left-central point.

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Introduction

Let X be a proper and smooth variety over a local field K and let $\mathcal{X}$ be a regular model of X defined over the ring of integers $\mathcal{O}_K$ of K. When $\mathcal{X}$ is smooth over $\mathcal{O}_K$, the Tate conjecture equates the ℓ–adic Chow groups of algebraic cycles on the geometric special fiber $X_{\bar{k}}$ of $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ with the Galois invariants in $H^{2*}(X_{\bar{K}}, \mathbf{Q}_\ell(*))$. One of the results proved in [2] (*cf.* Corollary 3.6) shows that the Tate conjecture for smooth and proper varieties over finite fields together with the monodromy–weight conjecture imply a generalization of the above result in the case of semistable reduction. Namely, let $\wp \in \mathrm{Spec}(\mathcal{O}_K)$ be a

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prime over which the special fiber $\mathcal{X} \times \mathrm{Spec}(k(\wp)) = Y$ is a reduced divisor with normal crossings in $\mathcal{X}$ (*i.e.* semistable fiber). Then, assuming the above two conjectures, the ℓ–adic groups of algebraic cycles modulo rational equivalence on the r–fold intersections of components of Y ($r \geq 1$) are related with Galois invariant classes on the Tate twists $H^{2*-(r-1)}(X_{\bar{K}}, \mathbf{Q}_\ell(* - (r-1)))$.
An interesting case is when one replaces X by $X \times_K X$, so that Galois invariant cycles may be identified with Galois equivariant maps $H^*(X_{\bar{K}}, \mathbf{Q}_\ell) \to H^*(X_{\bar{K}}, \mathbf{Q}_\ell(\cdot))$. Examples of such maps are the powers N^i of the logarithm of the local monodromy around $\wp$. The operators $N^i : H^*(X_{\bar{K}}, \mathbf{Q}_\ell) \to H^*(X_{\bar{K}}, \mathbf{Q}_\ell(-i))$ determine classes $[N^i] \in H^{2d}((X \times X)_{\bar{K}}, \mathbf{Q}_\ell(d-i))$ ($d = \dim X_{\bar{K}}$) invariant under the decomposition group. In this paper we study in detail the structure of $[N^i]$ when the special fiber Y of $\mathcal{X}$ has at worst triple points as singularities. That is, we exhibit the corresponding algebraic cycles on the (normal crossings) special fiber $T = \cup_i T_i$ of a resolution $\mathcal{Z}$ of $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{X}$. Denote by $\tilde{N} = 1 \otimes N + N \otimes 1$ the monodromy on the product, and let F be the geometric Frobenius. Then the classes $[N^i]$ naturally determine elements in $\mathrm{Ker}(\tilde{N}) \cap H^{2d}((X \times X)_{\bar{K}}, \mathbf{Q}_\ell(d-i))^{F=1}$. Assuming the monodromy–weight conjecture on the product (*i.e.* the monodromy filtration $L.$ on $H^*((X \times X)_{\bar{K}}, \mathbf{Q}_\ell)$ coincides–up to a shift–with the filtration by the weights of the Frobenius *cf.* [16]) and the semisimplicity of the action of the Frobenius on the inertia invariants, the following identifications hold

$$
\begin{aligned}
(0.1)\quad & \mathrm{Ker}(\tilde{N}) \cap H^{2d}((X \times X)_{\bar{K}}, \mathbf{Q}_\ell(d-i))^{F=1} \\
& \simeq \left((gr^L_{2(d-i)} H^{2d}(T, \mathbf{Q}_\ell))(d-i) \right)^{F=1} \\
& \simeq \\
& \left[\frac{\mathrm{Ker}(\rho^{(2(i+1))} : H^{2(d-i)}(\tilde{T}^{(2i+1)}, \mathbf{Q}_\ell)(d-i) \to H^{2(d-i)}(\tilde{T}^{(2(i+1))}, \mathbf{Q}_\ell)(d-i))}{\mathrm{Image}\ \rho^{(2i+1)}} \right]^{F=1}.
\end{aligned}
$$

Here $\tilde{T}^{(j)}$ denotes the normalization of the j–fold intersection on the closed fiber T. These isomorphisms show that the classes $[N^i]$ have representatives in the cohomology groups of some precise strata of T. Moreover, the Tate conjecture and the semisimplicity of the action of the Frobenius on the smooth schemes $\tilde{T}^{(j)}$ would imply that these classes are algebraic. We refer to § 1, (1.6) for the description of the restriction maps ρ in (0.1).
To better understand the geometry related to the desingulatization process $\mathcal{Z} \to \mathcal{X} \times_{\mathcal{O}_K} \mathcal{X}$, and to avoid at first, some technical complications connected to the theory of the nearby cycles in mixed characteristic, we start by investigating this problem in equal characteristic zero (*i.e.* for semistable degenerations over a disk). There, one can take full advantage of many geometric results based on the theory of the mixed Hodge structures. Under the assumption of the monodromy–weight conjecture and using some techniques of [16], our results generalize to mixed characteristic. The cycles we exhibit on $\tilde{T}^{(2i+1)}$ explain geometrically the presence of poles on specific local factors of the L–function

related to the fiber product $X \times X$. In fact, theorem 6.2 equates, under the assumption of the semisimplicity of the action of the Frobenius F on the inertia invariants $H^*((X \times X)_{\bar{K}}, \mathbf{Q}_\ell)^I$, the rank of any of the groups in (0.1) with $\mathrm{ord}_{s=d-i} \det(Id - FN(\wp)^{-s} | H^{2d}((X \times X)_{\bar{K}}, \mathbf{Q}_\ell)^I)$. Here, $N(\wp)$ denotes the number of elements of the finite field $k(\wp)$.
A study of the local geometry of the normal–crossings special fiber T shows that $[N^i]$ are represented by certain natural "diagonal cycles" on $\tilde{T}^{(2i+1)}$ together with a cycle supported on the exceptional part of the stratum that arises because the classes $[N^i]$ must belong to the kernel of the restriction map $\rho^{(2(i+1))}$ (*cf.* (0.1)). This result is obtained via the introduction of a generalized correspondence diagram for the map

$$(0.2) \qquad N^i : \mathbf{H}^*(Y, gr^L_{r+i}\mathbf{R}\Psi(\mathbf{Q}_{\mathcal{X}})) \to \mathbf{H}^*(Y, (gr^L_{r-i}\mathbf{R}\Psi(\mathbf{Q}_{\mathcal{X}}))(-i)).$$

This morphism describes the monodromy action on the E_1–term of the spectral sequence of weights for the filtered complex of the nearby cycles $(\mathbf{R}\Psi(\mathbf{Q}_{\mathcal{X}}), L.)$ (*cf.* § 2, (2.1)). For $i > 0$, the classes $[N^i]$ do not describe an algebraic correspondence in the classical sense. In fact, the algebraic cycles representing them are only supported on higher strata of the special fiber T (*i.e.* on $\tilde{T}^{(2i+1)}$) and they do not naturally determine classes in the cohomology of T. This is a consequence of the fact that for $i > 0$, the cocycle $[N^i]$ does not have weight zero in the ℓ–adic cohomology of the fiber product $(X \times X)_{\bar{K}}$, as one can easily check from (0.1). Nonetheless, we expect that each of these classes supplies a refined information on the degeneration. Namely, we conjecture that the geometric description that we obtain up to triple points can be generalized to any kind of semistable singularity via a thorough combinatoric study of the toric singularities of the special fiber of the fiber product resolution $\mathcal{Z}$.
The correspondence diagram related to the map (0.2) is built up from the hypercohomology of the Steenbrink filtered resolution $(A^\bullet_{\mathcal{X}}, L.)$ of $\mathbf{R}\Psi(\mathbf{Q}_{\mathcal{X}})$. In § 3 we establish the necessary functoriality properties of the Steenbrink complex and its $L.$–filtration. A difficult point in the description of the correspondence diagram is related to the definition of a product structure on the E_1–terms of the spectral sequence of weights. Example 3.1 points out a problem related to a canonical definition of a product structure for $(A^\bullet_{\mathcal{X}}, L.)$ in the filtered category. It comes out that the monodromy filtration $L.$ is not multiplicative on the level of the filtered complexes. A partial product, canonical only on the $E_2 = E_\infty$-terms is provided in the Appendix. This suffices for purposes of our paper.

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1. Notations and techniques from mixed Hodge theory

In this paragraph we introduce the main notations and recall some results on the mixed Hodge theory of a degeneration.
We denote by X a connected, smooth, complex analytic manifold and we let S be the unit disk. We write $f : X \to S$ for a proper, surjective morphism and we let $Y = f^{-1}(0)$ be its special fiber. We assume that f is *smooth* at every point of $X^* = X \smallsetminus Y$ and that the special fiber Y is an algebraic divisor with *normal–crossings*. The local description of f near a closed point $y \in Y$ is given by:

$$f(z_1, \ldots, z_m) = z_1^{e_1} \cdots z_k^{e_k}$$

for $k \leq m = \dim X$ and $\{z_1, \ldots, z_m\}$ a local coordinate system on a neighborhood of y in X centered at y and $e_i \in \mathbf{Z},\ e_i \geq 1$. The fibers of f have then dimension $d = m - 1$.
A normal–crossings divisor as above is said to have *semistable* reduction (strict normal–crossings) if one has: $e_i = 1\ \forall i$, in the local description of f.
We *fix a parameter* $t \in S$. For $t \neq 0$, let $f^{-1}(t) = X_t$ be the fiber at t. Because the restriction of f at $S^* = S \smallsetminus \{0\}$ is a C^∞, locally trivial fiber bundle, the positive generator of $\pi_1(S^*, t) \simeq \mathbf{Z}$ induces an automorphism T_t of $H^*(X_t, \mathbf{Z})$, called *the local monodromy*. We will always suppose throughout the paper that T_t is *unipotent*. This assumption, together with the local monodromy theorem (*cf.* [7], Theorem 2.1.2), implies that $(T_t - 1)^{i+1} = 0$, on $H^i(X_t, \mathbf{Z})$. The unipotency condition of the local monodromy is for example verified when g.c.d.$(e_i,\ i \in [1, k]) = 1,\ \forall y \in Y$ (*cf. op.cit.*). Under these conditions, the logarithm of the local monodromy is defined to be the finite sum:

$$N_t := \log T_t = (T_t - 1) - \frac{1}{2}(T_t - 1)^2 + \frac{1}{3}(T_t - 1)^3 - \cdots$$

It is known (*cf.* [5]) that the automorphisms T_t of $H^i(X_t, \mathbf{C})$ $(t \in S^*)$, are the fibers of an automorphism T of the fiber bundle $\mathbf{R}^i f_*(\Omega^\bullet_{X/S}(\log Y))$ over S, whose fiber at 0 is described as $T_0 = \exp(-2\pi i N_0)$. By definition, the endomorphism N_0 is the residue at 0 of the Gauss-Manin connection ∇ on the "canonical prolongation" $\mathbf{R}^i f_*(\Omega^\bullet_{X/S}(\log Y))$ of the locally free sheaf $\mathbf{R}^i f_*(\Omega^\bullet_{X^*/S^*})$. Because of the definition of T_0, it makes sense to think of a nilpotent map $N := -\frac{1}{2\pi i} \log T$ as the monodromy operator on the degeneration $f : X \to S$. Via the canonical isomorphism (*cf.* [11], Thm. 2.18)$(t \in S)$:

$$\mathbf{R}^i f_*(\Omega^\bullet_{X/S}(\log Y)) \otimes_{\mathcal{O}_S} k(t) \stackrel{\simeq}{\to} \mathbf{H}^i(X_t, \Omega^\bullet_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_t})$$

where $k(t)$ is the residue field of $\mathcal{O}_S$ at t, we can see the map N_0 as an endomorphism of the hypercohomology of the relative de Rham complex $\Omega^\bullet_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. This complex represents in the derived category $D^+(Y, \mathbf{C})$ of the abelian category of sheaves of $\mathbf{C}$–vector spaces on Y, the complex of the nearby cycles $\mathbf{R}\Psi(\mathbf{C})$. Namely, there exists a non-canonical quasi-isomorphism (*i.e.* depending on the choice of the parameter t on S) $\Omega^\bullet_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \simeq \mathbf{R}\Psi(\mathbf{C}_{\tilde{X}^*}) := i^{-1}\mathbf{R}k_*\mathbf{C}_{\tilde{X}^*}$ (*cf.* [11], § 2). This isomorphism, composed with the canonical map $\Omega^\bullet_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \Omega^\bullet_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y^{\mathrm{red}}}$ (Y^{red} = reduced, induced structure scheme on Y), induces a quasi-isomorphism $(i^{-1}\mathbf{R}k_*\mathbf{C}_{\tilde{X}^*})_{\mathrm{un}} \simeq \Omega^\bullet_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y^{\mathrm{red}}}$ (*cf.* *op.cit.* § 4). Here, we denote by $(i^{-1}\mathbf{R}k_*\mathbf{C}_{\tilde{X}^*})_{\mathrm{un}}$ the maximal subobject of $i^{-1}\mathbf{R}k_*\mathbf{C}_{\tilde{X}^*}$ on which $\pi_1(S^*)$ acts with unipotent automorphisms. We refer to the following commutative diagram for the description of the maps:

$$\begin{tikzcd}
\tilde{X}^* \arrow[r, "k"] \arrow[d] & X \arrow[d, "f"] & Y \arrow[l, "i"'] \arrow[d] \\
\tilde{S}^* \arrow[r, "p"] & S & \{0\}. \arrow[l]
\end{tikzcd}$$

The space $\tilde{S}^* = \{u \in \mathbf{C} \,|\mathrm{Im}\ u > 0\}$ is the upper half plane, the map $p : \tilde{S}^* \to S$ $p(u) = \exp(2\pi i u) = t$, makes $\tilde{S}^*$ in a universal covering of S^* and $\tilde{X}^*$ is the pullback $X \times_S \tilde{S}^*$ of X along p. The morphism k is the natural projection. It factorizes through X^* by means of the injection $j : X^* \to X$. Finally, i is the closed embedding.

Steenbrink, Guillen and Navarro Aznar and Masaiko Saito (*cf.* [11], [6], [12]) defined a mixed Hodge structure on the hypercohomology of the unipotent factor of the complex of the nearby cycles $\mathbf{H}^*(X, \Omega^\bullet_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y^{\mathrm{red}}})$. This is frequently referred as the limiting mixed Hodge structure.

We will assume from now on that f is *projective*. Then, the weight filtration on the limiting mixed Hodge structure is the one induced by the nilpotent endomorphism N, namely by the logarithm of the unipotent Picard-Lefschetz transformation T that is already defined at the $\mathbf{Q}$-level. This filtration, which one usually refers to as the monodromy–weight filtration $L.$, is defined inductively. On the limiting cohomology $H^i(\tilde{X}^*, \mathbf{Q})$, it is increasing and has lenght at most $2i$. By the local monodromy theorem $N^{i+1} = 0$, hence one sets $L_0 = \mathrm{Im}\ N^i$ and $L_{2i-1} = \mathrm{Ker}\ N^i$. The monodromy filtration $L.$ becomes a convolution product of the kernel and the image filtration relative to the endomorphism N. These filtrations are defined as

$$K_l\ H^i(\tilde{X}^*, \mathbf{Q}) := \mathrm{Ker}\ N^{l+1}, \qquad I^j\ H^i(\tilde{X}^*, \mathbf{Q}) := \mathrm{Im}\ N^j$$

and their convolution is

$$L = K * I, \qquad L_k := \sum_{l-j=k} K_l \cap I^j. \tag{1.1}$$

It is a very interesting fact that there is no explicit construction of the monodromy-weight filtration $L.$ on $\Omega^\bullet_{X/S}(\log Y)\otimes_{\mathcal{O}_X}\mathcal{O}_Y$ itself. The filtration $L.$ is defined on a complex $A^\bullet_{\mathbf{C}}$ which is a resolution of $\Omega^\bullet_{X/S}(\log Y)\otimes_{\mathcal{O}_X}\mathcal{O}_{Y^{\mathrm{red}}}$. More precisely, the complex $\Omega^\bullet_{X/S}(\log Y)\otimes_{\mathcal{O}_X}\mathcal{O}_{Y^{\mathrm{red}}}$ is isomorphic, in the derived category $D^+(Y,\mathbf{C})$, to the complex $A^\bullet_{\mathbf{C}}$ of $\mathcal{O}_X$–modules supported on Y. The complex $A^\bullet_{\mathbf{C}}$ is the simple complex associated to the double complex $(p,\ q\geq 0)$:

$$A^{p,q}_{\mathbf{C}} := \Omega^{p+q+1}_X(\log Y)/W_q\Omega^{p+q+1}_X(\log Y)$$

where $W_*\Omega^\bullet_X(\log Y)$ is the weight filtration by the order of log-poles (*cf.* [3], § 3). The differentials on it are defined as follows

$$d' : A^{p,q}_{\mathbf{C}} \to A^{p+1,q}_{\mathbf{C}}, \qquad d'(\omega) = d\omega$$

is induced by the differentiation on the complex $\Omega^\bullet_X(\log Y)$ and

$$d'' : A^{p,q}_{\mathbf{C}} \to A^{p,q+1}_{\mathbf{C}}, \qquad d''(\omega) = \theta\wedge\omega$$

where $\theta := f^*(\frac{dt}{t}) = \sum_{i=1}^k e_i\frac{dz_i}{z_i}$ is the form defining the quasi-isomorphism we mentioned before (*cf.* [11], § 4)

$$\Omega^\bullet_{X/S}(\log Y)\otimes_{\mathcal{O}_X}\mathcal{O}_{Y^{\mathrm{red}}} \overset{\wedge\theta}{\to} A^\bullet_{\mathbf{C}}.$$

The total differential on $A^\bullet_{\mathbf{C}}$ is $d = d' + d''$. The weight filtration $W_*\Omega^\bullet_X(\log Y)$ induces a corresponding filtration on $A^\bullet_{\mathbf{C}}$ $(r\in\mathbf{Z})$:

$$W_rA^{p,q}_{X,\mathbf{C}} =: W_{r+q+1}\Omega^{p+q+1}_X(\log Y)/W_q\Omega^{p+q+1}_X(\log Y). \tag{1.2}$$

The filtration that $W_rA^\bullet_{\mathbf{C}}$ induces on $\mathbf{H}^*(Y,A^\bullet_{\mathbf{C}})\simeq\mathbf{H}^*(\tilde{X}^*,\mathbf{C})$ is the kernel filtration K (*cf.* (1.1))

$$K_rH^*(\tilde{X}^*,\mathbf{C}) = W_r\mathbf{H}^*(Y,A^\bullet_{\mathbf{C}}) =: \mathrm{Im}\left(\mathbf{H}^*(Y,W_rA^\bullet_{\mathbf{C}})\to\mathbf{H}^*(Y,A^\bullet_{\mathbf{C}})\right) = \mathrm{Ker}\ N^{r+1}.$$

The monodromy-weight filtration is then defined as

$$L_rA^{p,q} := W_{2q+r+1}\Omega^{p+q+1}_X(\log Y)/W_q\Omega^{p+q+1}_X(\log Y).$$

Via Poincaré residues, the related graded pieces have the following description

$$gr^L_rA^\bullet_{\mathbf{C}} \simeq \bigoplus_{k\geq\max(0,-r)}(a_{2k+r+1})_*\Omega^\bullet_{\tilde{Y}^{(2k+r+1)}}[-r-2k]. \tag{1.3}$$

Here, we have denoted by $\tilde{Y}^{(m)}$ the disjoint union of all intersections $Y_{i_1}\cap\ldots\cap Y_{i_m}$ for $1\leq i_1<\ldots<i_m\leq n$ $(Y = Y_1\cup\ldots\cup Y_n)$. We write $(a_m)_* : \tilde{Y}^{(m)}\to X$ for the natural projection.
The monodromy operator N is induced by an endomorphism $\tilde{\nu}$ of $A^\bullet_{\mathbf{C}}$ which is defined as $(-1)^{p+q+1}$ times the natural projection

$$\nu : A^{p,q}_{\mathbf{C}} \to A^{p-1,q+1}_{\mathbf{C}}.$$

The endomorphism $\tilde{\nu}$ is characterized by its behavior on the L-filtration, namely

$$\tilde{\nu}(L_r A^\bullet_{\mathbf{C}}) \subset L_{r-2} A^\bullet_{\mathbf{C}}$$

and the induced map

$$\tilde{\nu}^r : gr^L_r A^\bullet_{\mathbf{C}} \to gr^L_{-r} A^\bullet_{\mathbf{C}} \tag{1.4}$$

is an isomorphism for all $r \geq 0$. The complex $A^\bullet_{\mathbf{C}}$ contains the subcomplex $W_0 A^\bullet_{\mathbf{C}} = \mathrm{Ker}(\tilde{\nu})$ that is known to be a resolution of $\mathbf{C}_Y$. The filtration L and the Hodge filtration F on $A^\bullet_{\mathbf{C}}$ induce *resp.* the kernel and F filtration on $W_0 A^\bullet_{\mathbf{C}}$. The resulting mixed Hodge structure on $H^*(Y, \mathbf{C})$ is the canonical one. Similarly, the homology $H_*(Y, \mathbf{C})$ (*i.e.* $H^*_Y(X, \mathbf{C})$) with its mixed Hodge structure is calculated by the hypercohomology of the complex $\mathrm{Coker}(\tilde{\nu})$.
Because of the description given in (1.3), the spectral sequence of hypercohomology of the filtered complex $(A^\bullet_{\mathbf{C}}, L)$ (frequently referred as the *weight spectral sequence* of $\mathbf{R}\Psi(\mathbf{C})$) has the E_1 term given by

$$\begin{aligned} E_1^{-r,n+r} &= \bigoplus_{k \geq \max(0,-r)} H^{n-r-2k}(\tilde{Y}^{(2k+r+1)}, \mathbf{C}) \\ d_1 &= \sum_k ((-1)^{r+k} d_1' + (-1)^{k-r} d_1''). \end{aligned} \tag{1.5}$$

The explicit definition of the differentials, in the strict normal–crossings case (*i.e.* semistable degeneration), is the following:

$$\begin{aligned} d_1' &= \rho^{(r+2k+2)} = \sum_{u=1}^{r+2k+2} (-1)^{u-1} \rho_u^{(r+2k+2)} \\ d_1'' &= -\gamma^{(r+2k+1)} = \sum_{u=1}^{r+2k+1} (-1)^u \gamma_u^{(r+2k+1)} \end{aligned} \tag{1.6}$$

where

$$\rho_u^{(r+2k+2)} = (\delta_u^{(r+2k+2)})^* : H^{n-r-2k}(\tilde{Y}^{(2k+r+1)}, \mathbf{C}) \to H^{n-r-2k}(\tilde{Y}^{(2k+r+2)}, \mathbf{C})$$
$$\gamma_u^{(r+2k+1)} = (\delta_u^{(r+2k+1)})_! : H^{n-r-2k}(\tilde{Y}^{(2k+r+1)}, \mathbf{C}) \to H^{n-r-2k+2}(\tilde{Y}^{(2k+r)}, \mathbf{C})$$

are the restrictions, *resp.* the Gysin maps, induced by the inclusions ($u, t \in \mathbf{Z}$)

$$\delta_u^{(t)} : Y_{i_1} \cap \cdots \cap Y_{i_t} \to Y_{i_1} \cap \cdots \cap \widehat{(Y_{i_u})} \cap \cdots \cap Y_{i_t}.$$

In the general normal–crossings case (*i.e.* fibrations locally described by $f(z_1, \ldots, z_m) = z_1^{e_1} \cdots z_k^{e_k}$, $e_i \geq 1$), the definition of d_1' has to take into account multiplicity factors $\pm e_{i_j}$ before each map $(\delta_j^{(t)})^*$. The map d_1' is infact induced from a "wedging" operation with the form $\theta = \sum_{i=1}^k e_i \frac{dz_i}{z_i}$ (*cf.* last page). The definition of d_1'' is analogous to the one given in the strict normal–crossings case.
Notice that the weight spectral sequence (1.5) is built up from a filtered *double* complex. This property distinguishes this weight spectral sequence from others

as *e.g.* the spectral sequence of weights which defines the mixed Hodge structure on a quasi–projective smooth complex variety (*cf.* [3]).
The complex $A^\bullet_{\mathbf{C}}$ is the complex part of a cohomological mixed Hodge complex $A^\bullet_{\mathbf{Q}}$ whose definition is less explicit than $A^\bullet_{\mathbf{C}}$ and for which we refer to [7]. This rational complex induces on $H^\cdot(\tilde{X}^*,\mathbf{Q})$ a rational mixed Hodge structure. The rational representative of the above spectral sequence (1.5) is

$$E_1^{-r,n+r} = \bigoplus_{k\geq \max(0,-r)} H^{n-r-2k}(\tilde{Y}^{(2k+r+1)},\mathbf{Q})(-r-k). \tag{1.7}$$

The index in the round brackets outside the cohomology refers to the Tate twist. Both these spectral sequences degenerate at $E_2 = E_\infty$ and they converge to $H^n(\tilde{X}^*,\mathbf{C})$ and $H^n(\tilde{X}^*,\mathbf{Q})$ respectively.
For curves (*i.e.* $d=1$), the degeneration of the weight spectral sequence provides the exact sequences

$$0 \to E_2^{-1,2} \to H^0(\tilde{Y}^{(2)},\mathbf{Q})(-1) \stackrel{d_1^{-1,2}}{\to} H^2(\tilde{Y}^{(1)},\mathbf{Q}) \to H^2(\tilde{X}^*,\mathbf{Q}) \to 0$$

and

$$0 \to H^0(\tilde{X}^*,\mathbf{Q}) \to H^0(\tilde{Y}^{(1)},\mathbf{Q}) \stackrel{d_1^{0,0}}{\to} H^0(\tilde{Y}^{(2)},\mathbf{Q}) \stackrel{\alpha}{\to} H^1(\tilde{X}^*,\mathbf{Q}). \tag{1.8}$$

The differentials $d_1^{-1,2}$ and $d_1^{0,0}$ are defined as in (1.6) and the map α in (1.8) is the edge map in the spectral sequence. We also have a non canonical decomposition

$$H^1(\tilde{X}^*,\mathbf{Q}) = H^1(\tilde{Y}^{(1)},\mathbf{Q}) \oplus E_2^{-1,2} \oplus E_2^{1,0}.$$

with $E_2^{1,0} = \mathrm{Im}(\alpha)$.
Steenbrink proved that the L–filtration induced on the abutment of the spectral sequence of the nearby cycles is the Picard-Lefschetz filtration, hence it is uniquely described by the following properties

$$N(L_{n+r}H^n(\tilde{X}^*,\mathbf{Q})) \subset (L_{n+r-2}H^n(\tilde{X}^*,\mathbf{Q}))(-1)$$

and

$$N^r : gr^L_{n+r}H^n(\tilde{X}^*,\mathbf{Q}) \stackrel{\simeq}{\to} (gr^L_{n-r}H^n(\tilde{X}^*,\mathbf{Q}))(-r)$$

for $r>0$. In the rest of the paper we will refer to it as the *monodromy filtration.*

2. The monodromy operator as algebraic cocycle

We keep the notations introduced in the last paragraph. As n varies in $[0,2d]$ ($d =$ dimension of the fiber of $f: X \to S$) and $i \geq 0$, the power maps

$$N^i : H^n(\tilde{X}^*,\mathbf{Q}) \to H^n(\tilde{X}^*,\mathbf{Q})(-i)$$

induced by the endomorphism $N : \mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log Y)) \to \mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log Y))$, define elements

$$N^i \in \mathrm{Hom}(H^\cdot(\tilde{X}^*,\mathbf{Q}), H^\cdot(\tilde{X}^*,\mathbf{Q})(-i))$$

which are invariant for the action of the local monodromy group π_1. They can be naturally identified with

$$N^i \in \bigoplus_{n\geq 0}\Big[H^{2d-n}(\tilde{X}^*,\mathbf{Q})(d)\otimes H^n(\tilde{X}^*,\mathbf{Q})(-i)\Big]^{\pi_1} = \Big[H^{2d}(\tilde{X}^*\times\tilde{X}^*,\mathbf{Q})(d-i)\Big]^{\pi_1}.$$

The space $\tilde{X}^*\times_S\tilde{X}^*$ is the generic fiber of the product degeneration $X\times_S X\to S$. After a suitable sequence of blow-ups along $\mathrm{Sing}(Y\times Y)\supset \mathrm{Sing}(X\times_S X)$:

$$Z\to\cdots\to X\times_S X\to S$$

we obtain a normal–crossings degeneration $h:Z\to S$ with Z non singular and whose generic fiber is still $\tilde{X}^*\times\tilde{X}^*$. Its special fiber $T=h^{-1}(0)=T_1\cup\cdots\cup T_N$ has normal crossings singularities. The local description of h along T looks like:

$$h(w_1,\dots,w_{2m})=w_1^{e_1}\cdots w_r^{e_r}$$

for $\{w_1,\dots,w_{2m}\}$ a set of local parameters on Z and $e_1,\dots,e_r$ non-negative integers.
The semistable reduction theorem (*cf.* [9]) assures that modulo extensions of the basis S and up to a suitable sequence of blow-ups and down along subvarieties of the special fiber T, we may eventually obtain from h a semistable degeneration $W\to S$ with $W_0=W_{0_1}\cup\ldots\cup W_{0_M}$ as special fiber.
Because of the assumption of the unipotency of the local monodromy on $H^*(X_t,\mathbf{C})$ (*cf.* § 1), the local monodromy σ of h will be also unipotent. We then call $\tilde{N}=\log(\sigma)$. By the Künneth decomposition it results: $\tilde{N}=1\otimes N+N\otimes 1$ and we have:

$$N^i\in\Big(H^{2d}(\tilde{X}^*\times\tilde{X}^*,\mathbf{Q}(d-i))\Big)^{\pi_1}=\mathrm{Ker}(\tilde{N})\cap H^{2d}(\tilde{X}^*\times\tilde{X}^*,\mathbf{Q}(d-i)).$$

Let consider the monodromy filtration $L.$ relative to the degeneration h. We denote by $Hom_{MH}(\mathbf{Q}(0),V)$ ($Hom(\mathbf{Q},V)$ shortly) the subgroup of Hodge cycles of pure weight $(0,0)$ of a bifiltered $\mathbf{Q}$–vector space V: (V,L,F), endowed with the corresponding mixed Hodge structure. Then, we have the following

PROPOSITION 2.1. *For $i\geq 1$*

$$\begin{aligned} N^i &\in Hom_{MH}\Big(\mathbf{Q}(0),Ker(\tilde{N})\cap H^{2d}(\tilde{X}^*\times\tilde{X}^*,\mathbf{Q}(d-i))\Big)\subseteq\\ &\subseteq Hom_{MH}\Big(\mathbf{Q}(0),Ker(\tilde{N})\cap(gr^L_{2(d-i)}H^{2d}(\tilde{X}^*\times\tilde{X}^*,\mathbf{Q}))(d-i)\Big)\simeq\\ &\simeq Hom_{MH}\Big(\mathbf{Q}(0),(gr^L_{2(d-i)}H^{2d}(T,\mathbf{Q}))(d-i)\Big)\simeq Hom(\mathbf{Q},A),\end{aligned}$$

$$A:=\frac{Ker(\rho^{2(i+1)}:H^{2(d-i)}(\tilde{T}^{(2i+1)},\mathbf{Q})(d-i)\to H^{2(d-i)}(\tilde{T}^{(2(i+1))},\mathbf{Q})(d-i))}{Image\ \rho^{(2i+1)}}.$$

Here ρ is the restriction map on cohomology and by $\tilde{T}^{(j)}$ we mean the disjoint union of all ordered j-fold intersections of the components of T (cf. §1).

Proof. The identification of N^i with a Hodge cycle is a consequence of N being a morphism in the category of Hodge structures. The first inclusion derives from the well known facts that $\mathrm{Ker}(\tilde{N})$ has monodromic weight at most zero and that its Hodge cycles are included (Hom being a functor left exact on the second place) in the corresponding ones for the graded piece $(gr^L_{2(d-i)}H^{2d}(\tilde{X}^*\times\tilde{X}^*,\mathbf{Q}))(d-i)$ of $\mathrm{Ker}(\tilde{N})\cap\bigoplus_j(gr^L_j H^{2d}(\tilde{X}^*\times\tilde{X}^*,\mathbf{Q}))(d-i)$. The second isomorphism comes from the local invariant cycle theorem, namely from the following exact sequence of pure Hodge structures (*cf.* [2], lemma 3.3 and corollary 3.4)

$$0 \to gr^L_{2(d-i)}H^{2d}(T,\mathbf{Q}) \to gr^L_{2(d-i)}H^{2d}(\tilde{X}^*\times\tilde{X}^*,\mathbf{Q}) \overset{N}{\twoheadrightarrow} gr^L_{2(d-i-1)}H^{2d}(\tilde{X}^*\times\tilde{X}^*,\mathbf{Q})(-1)$$

Finally, the last isomorphism is a consequence of the description of the graded piece $(gr^L_{2(d-i)}H^{2d}(T,\mathbf{Q}))(d-i)$ as sub–Hodge structure of $(gr^L_{2(d-i)}H^{2d}(\tilde{X}^*\times\tilde{X}^*,\mathbf{Q}))(d-i)$ (*cf. op.cit.* lemma 3.3). □

Proposition 2.1 shows how the operators N^i can be detected by classes $[N^i]$ in the cohomology of a fixed stratum of the special fiber T. Equivalently, we can say that N^i determine classes $[N^i]\in\mathbf{H}^{2d}(T,(gr^L_{-2i}\mathbf{R}\Psi_h(\mathbf{Q}))(d-i))$ in the $(E_1^{2i,2(d-i)})(d-i)$-term of the spectral sequence of weights for the degeneration h. Here we write $gr^L_{-2i}\mathbf{R}\Psi_h(\mathbf{Q})$ for $gr^L_{-2i}A^\bullet_{W,\mathbf{Q}}$.

The goal of this paper is to identify the class $[N^i]$ with an algebraic cocycle related to the degeneration $f: X\to S$. In all those cases that we will consider in the paper, this identification is obtained via a "correspondence-type" map $(i\geq 0)$

$$N^i: \mathbf{H}^*(Y, gr^L_r A^\bullet_{X,\mathbf{Q}}) \to \mathbf{H}^*(Y,(gr^L_{r-2i}A^\bullet_{X,\mathbf{Q}})(-i)) = \mathbf{H}^*(Y, gr^L_r(A^\bullet_{X,\mathbf{Q}}(-i)))$$

which makes the following diagram commute

$$\begin{tikzcd}
 & E_1^{-r+2i,*+r+2(d-i)} \arrow[d, equal] \\
\mathbf{H}^*(T, gr^L_r A^\bullet_{Z,\mathbf{Q}}) \arrow[r, "{[N^i]\cdot}"] & \mathbf{H}^{2d+*}(T,(gr^L_{r-2i}A^\bullet_{Z,\mathbf{Q}})(d-i)) \arrow[d, "(p_2)_*"] \\
\mathbf{H}^*(Y, gr^L_r A^\bullet_{X,\mathbf{Q}}) \arrow[u, "(p_1)^*"] \arrow[r, "N^i"] \arrow[d, equal] & \mathbf{H}^*(Y, gr^L_r(A^\bullet_{X,\mathbf{Q}}(-i))) \arrow[d, equal] \\
E_1^{-r,*+r} & E_1^{-r+2i,*+r-2i}
\end{tikzcd} \tag{2.1}$$

The projections $p_1,\ p_2 : \tilde{X}^* \times \tilde{X}^* \to \tilde{X}^*$ on the first and second factor, determine pullbacks and pushforwards on the hypercohomology as we shall describe in § 3.

From the theory we will explain in the next paragraphs and in the Appendix it will follow that N^i has the expected shape. Namely, it is zero when $N^i = 0$ and it is the identity when N^i induces an isomorphism on $E_2^{-r,*+r}$. Also, it will result that p_1^*, $(p_2)_*$ and $[N^i]\ \cdot$ all commute with the differential on E_1. That will imply an induced commutative diagram on E_2.

For $i = 0$, *i.e.* when the correspondence map is the identity, proposition 2.1 can be slightly generalized, using the theory developed in [2] (*cf.* lemma 3.3 and corollary 3.4) and in [1] so that the identity operator is seen as an element in

$$Hom_{MH}\left(\mathbf{Q}, \frac{\mathrm{Ker}(\rho^{(2)} : H^{2d}(\tilde{T}^{(1)}, \mathbf{Q})(d) \to H^{2d}(\tilde{T}^{(2)}, \mathbf{Q})(d))}{\mathrm{Im}\ (-i^* \cdot i_* : H_{2(d-1)}(T^{(1)}, \mathbf{Q})(d-1) \to H^{2d}(\tilde{T}^{(1)}, \mathbf{Q})(d)}\right) \simeq$$

$$\simeq Hom_{MH}\left(\mathbf{Q}, \frac{\mathrm{Im}(i^* : H^{2d}(T, \mathbf{Q})(d) \to H^{2d}(\tilde{T}^{(1)}, \mathbf{Q})(d))}{\mathrm{Im}\ (-i^* \cdot i_* : H_{2(d-1)}(T^{(1)}, \mathbf{Q})(d-1) \to H^{2d}(\tilde{T}^{(1)}, \mathbf{Q})(d)}\right).$$

Here the map i^* (*resp.* i_*) represents the pullback (*resp.* pushforward) relative to the embedding $T^{(1)} \to T$. Proposition 2.1 shows this class as a Hodge cocycle in $H^{2d}(\tilde{X}^* \times \tilde{X}^*, \mathbf{Q}(d))$. That agrees with the classical theory of algebraic correspondences describing the identity map via an algebraic correspondence with the cycle diagonal. Namely, the identity is determined by the diagonal $\Delta_{\tilde{X}^*} \subset \tilde{X}^* \times \tilde{X}^*$ seen as specialization of the cycle diagonal on $\mathcal{X} \times \mathcal{X}$ on the fiber product $\tilde{X}^* \times \tilde{X}^*$. (*cf.* [8]).

The cases described in the next paragraphs will also supply some evidence for our expectation that $[N^i]$ can be always described by an algebraic (motivic) cocycle. Finally, notice that the calculation on the E_1 involves the cohomology of individual components of the strata and it is therefore in some sense local, whereas E_2 introduces relations among components of strata, so that any calculation on it becomes of global nature. That is the reason why the description of the monodromy cycle is carried out mainly at a local level in this paper.

3. Functoriality of the Steenbrink complex and remarks on products

Let $g : Z \to X$ be a morphism between two connected, complex analytic manifolds over a disk S. Let $f : X \to S$ and $h : Z \to S$ be the degeneration maps. Let assume that both Z and X are smooth over $\mathbf{C}$ and they have algebraic special fibers $f^{-1}(0) = Y$ and $h^{-1}(0) = T$ with normal crossings. We have the following commutative diagram

$$\begin{array}{ccccc} T & & \longrightarrow & & Y \\ \downarrow i' & & & & \downarrow i \\ Z & & \xrightarrow{g} & & X \\ & h \searrow & & \swarrow f & \\ & & S & & . \end{array}$$

Locally on the special fibers, f and h have the following description

$$f(z_1,\ldots,z_m)=z_1^{e_1}\cdots z_k^{e_k};\qquad h(w_1,\ldots,w_M)=w_1^{e'_1}\cdots w_K^{e'_K}$$

for $\{z_1,\ldots,z_m\}$ and $\{w_1,\ldots,w_M\}$ local parameters *resp.* on X and Z, $1\le k\le m$, $1\le K\le M$ and $e_1,\ldots,e_k;\ e'_1,\ldots,e'_K$ integers.
Because $g^{-1}(Y)=T$, at any point $y\in g(T)\subset Y$ ($y=g(t)$, for some $t\in T$) where the local description of Y is $z_1^{e_1}\cdots z_k^{e_k}=0$, the pullback sections $g^*(z_{i_j})$ ($\forall\ 1\le i_j\le k$) define divisors on Z supported on T (not necessarily reduced or irreducible).
Let order the components of Y as $Y=Y_1\cup\ldots\cup Y_k$ and let denote by $\tilde{Y}^{(r)}$ the disjoint union of all intersections $Y_{i_1}\cap\ldots\cap Y_{i_r}$ for $1\le i_1<\cdots<i_r\le k$. There is a local system ϵ of rank one on $\tilde{Y}^{(r)}$ of standard orientations of r elements (*cf.* [3]). The canonical morphism

$$g^*\Omega^\bullet_X(\log Y)\to\Omega^\bullet_Z(\log T)$$

is a map of bifiltered complexes with respect to the weight and the Hodge filtrations on X and Z *resp.* (*cf. op.cit.*). In particular it induces the following map of bicomplexes of sheaves supported on the special fibers ($r\ge 0$)

$$g^*(W_rA^\bullet_{X,\mathbf{C}})\to W_rA^\bullet_{Z,\mathbf{C}}$$

where $A^\bullet_{\mathbf{C}}$ is the Steenbrink complex which represents in the derived category the maximal subobject of the complex of nearby cycles where the action of the monodromy is unipotent (*cf.* § 1). $W_rA^\bullet_{\mathbf{C}}$ is the induced weight filtration on $A^\bullet_{\mathbf{C}}$ (*cf.* (1.2)). Because the weight filtration on the complex $A^\bullet_{\mathbf{C}}$ is induced by the weight filtration on the de Rham complex with log-poles, g induces a map in the derived category

$$g^*(W_r\mathbf{R}\Psi_f(\mathbf{Q}_X))\to W_r\mathbf{R}\Psi_h(\mathbf{Q}_Z).$$

Notice that $g^*(\frac{dz_{i_j}}{z_{i_j}})\in W_1\Omega^1_Z(\log T)$, *i.e.* pullbacks preserve poles. Hence, we deduce the functoriality of the monodromy filtration

$$g^*(L_rA^\bullet_{X,\mathbf{C}})\to L_rA^\bullet_{Z,\mathbf{C}}.$$

Because g^{-1} is an exact functor, g determines on the graded pieces a pullback map

$$g^*: gr^L_rA^\bullet_{X,\mathbf{C}}\to gr^L_rA^\bullet_{Z,\mathbf{C}}$$

where

$$gr^L_rA^\bullet_{Z,\mathbf{C}}\simeq\bigoplus_{k\ge\max(0,-r)}(a_{2k+r+1})_*\Omega^\bullet_{\tilde{T}^{(2k+r+1)}}(\epsilon^{2k+r+1})[-r-2k].$$

The functor g^{-1} is also compatible with both differentials d' and d'' on $A^\bullet_{\mathbf{C}}$. Hence, g^* induces a morphism of bifiltered mixed Hodge complexes ($F^\cdot=$ Hodge filtration *cf.* [3])

$$g^*:(A^\bullet_{X,\mathbf{C}},L,F)\to(A^\bullet_{Z,\mathbf{C}},L,F)$$

which in turn induces a map between the spectral sequences of weights

$$g^*: E_1^{-r,q+r}(X) = \mathbf{H}^q(Y, gr_r^L A_X^\bullet) \to \mathbf{H}^q(T, gr_r^L A_Z^\bullet) = E_1^{-r,q+r}(Z).$$

On the rational level this morphism between spectral sequences is described by a direct sum of maps as

(3.1)

$$g^*: H^{q-r-2k}(\tilde{Y}^{(2k+r+1)}, \mathbf{Q})(-r-k) \to H^{q-r-2k}(\tilde{T}^{(2k+r+1)}, \mathbf{Q})(-r-k).$$

Both spectral sequences degenerate at $E_2 = E_\infty$. Keeping track of the multiplicities and the signs for these pullbacks can be rather hard. Let suppose that locally the defining equations for Y and T are $t = \prod_i z_i^{e_i}$ and $t = \prod_j w_j^{e'_j}$ respectively, and we are given strata $Y_I = Y_{i_1} \cap \ldots \cap Y_{i_p}$ $(i_1 < \ldots < i_p)$ and $Y_J = Y_{j_1} \cap \ldots \cap Y_{j_p}$. Then the computation of the multiplicities involved in $g^*: H^*(Y_I, \mathbf{Q}) \to H^*(T_J, \mathbf{Q})$ essentially amounts to determine the coefficients of $\frac{dw_{j_1}}{w_{j_1}} \wedge \ldots \wedge \frac{dw_{j_p}}{w_{j_p}}$ in $g^*(\frac{dz_{j_1}}{z_{j_1}} \wedge \ldots \wedge \frac{dz_{j_p}}{z_{j_p}})$. This technique will be frequently used in the paper.

As an example, we describe the map (3.1) when $f: X \to S$ is a degeneration of curves with normal crossings singularities on its special fiber Y and Z is the blow-up of X at a closed point $P \in Y$. Let $g: Z \to X$ be the blowing up map. If P is a regular point in the special fiber, the number of components of the special fiber T of Z will simply increase by one (the exceptional divisor E) and the remaining components are the same as for Y. Hence $g^*: H^0(\tilde{Y}^{(1)}, \mathbf{C}) \to H^0(\tilde{T}^{(1)}, \mathbf{C})$ is simply the map $g^*(1_{Y_i}) = 1_{T_i} + 1_E$ on the components.

Let suppose instead that P is singular. Since the description of g^* is local around each closed point, we may assume that the degeneration f is given, in a neighborhood of P, by the equation $z_1^{e_1} z_2^{e_2} = t$, being t a chosen parameter on the disk S and e_1, e_2 positive integers. Let assume that $e_1 \leq e_2$. Then, locally around P: $\tilde{Y}^{(1)} = Y_1 \coprod Y_2$. Set-theoretically one has $Y_i = \{z_i = 0\}$ $(i = 1, 2)$ and $\tilde{Y}^{(2)} = Y_1 \cap Y_2 = \{P\}$. Then, $\tilde{T}^{(1)} = T_1 \coprod T_2 \coprod T_3$ where T_1 and T_2 are the strict transforms of the two components Y_i, while T_3 represents the exceptional divisor. We implicitly have fixed the standard orientation on $\tilde{Y}^{(r)}$ (*e.g.* $\tilde{Y}^{(2)} = Y_1 \cap Y_2 = Y_{12}$). On $\tilde{T}^{(r)}$, we choose the orientation for which the exceptional component T_3 is always considered as the last one.

There are only three graded complexes $gr_*^L A_{\mathbf{C}}^\bullet$ non zero both on X and Z. On X they have the following description

$$gr_{-1}^L A_{X,\mathbf{C}}^\bullet \simeq (a_2)_* \Omega^\bullet_{\tilde{Y}^{(2)}}[-1]$$

$$gr_0^L A_{X,\mathbf{C}}^\bullet \simeq (a_1)_* \Omega^\bullet_{\tilde{Y}^{(1)}}$$

and via the isomorphism (1.4) one has:

$$\tilde{\nu}: gr_1^L A_{X,\mathbf{C}}^\bullet \stackrel{\simeq}{\to} gr_{-1}^L A_{X,\mathbf{C}}^\bullet.$$

Hence $E_1^{1,q-1} = \mathbf{H}^q(Y, gr_{-1}^L A_{X,\mathbf{C}}^\bullet) = 0$ unless $q = 1$, in which case we get

$$g^*: H^0(\tilde{Y}^{(2)}, \mathbf{C}) \to H^0(\tilde{T}^{(2)}, \mathbf{C}).$$

To understand the description of this map, one has to look at the local geometry of the blow-up at P. It is quite easy to check that Z is covered by two open sets, say $Z = U \cup V$. To make the notations easier, let call $t_1 = \frac{z_1}{z_2}$ and $t_2 = \frac{z_2}{z_1}$. On U, described by $t_2^{e_2} = \frac{t}{z_1^{e_1+e_2}}$, one has coordinates $\{t_2, z_1\}$, $T_2^{\rm red} = \{t_2 = 0\}$ and $T_3^{\rm red} = \{z_1 = 0\}$. On V, described by $t_1^{e_1} = \frac{t}{z_2^{e_1+e_2}}$, one has coordinates $\{t_1, z_2\}$, $T_1^{\rm red} = \{t_1 = 0\}$ and $T_3^{\rm red} = \{z_2 = 0\}$. Then $\tilde{T}^{(2)} = T_{13} \coprod T_{23}$, here we denote $T_{ij} = T_i \cap T_j$.
On U we have $g^*(\frac{dz_1}{z_1} \wedge \frac{dz_2}{dz_2}) = \frac{dz_1}{dz_1} \wedge \frac{dt_2}{t_2}$, whereas on V one gets $g^*(\frac{dz_1}{z_1} \wedge \frac{dz_2}{dz_2}) = \frac{dt_1}{dt_1} \wedge \frac{dz_2}{z_2}$. Hence, keeping in account the fixed orientation among the components of T, the description of the pullback $g^*(1_{\tilde{Y}^{(2)}}) = g^*(1_{Y_{12}})$ is given by

$$g^*(1_{Y_{12}}) = 1_{T_{13}} - 1_{T_{23}}.$$

The presence of a negative sign is due to the change of orientation. This description defines the above map g^* on H^0. Similarly, we find that

$$g^* : H^0(\tilde{Y}^{(1)}, \mathbf{C}) \to H^0(\tilde{T}^{(1)}, \mathbf{C})$$

is given by $g^*(1_{Y_1}) = 1_{T_1} + 1_{T_3}$ and $g^*(1_{Y_2}) = 1_{T_2} + 1_{T_3}$. The description of g^* on the terms H^1 goes in parallel.
Let now consider the proper map that g induces on the closed fibers. For simplicity of notations we call it $g : T \to Y$. Let $d = (\dim\ T - \dim\ Y)$. The above arguments have shown that g induces a pullback map g^* between the cohomologies of the strata: *cf.* (3.1). Since each stratum is a smooth projective complex variety (not connected), we can use the Poicaré duality to associate to each pullback in (3.1) that contributes to the definition of the map g^* its dual so that we naturally obtain a dual pushforward on the E_1–terms of the spectral sequence of weights that is described by a direct sum of maps as

$$\begin{aligned}(3.2)\quad g_! : H^{q-r-2(k-d)}(\tilde{T}^{(2k+r+1)}, \mathbf{Q})(-r-k+d) \\ \to H^{q-r-2k}(\tilde{Y}^{(2k+r+1)}, \mathbf{Q})(-r-k).\end{aligned}$$

On each stratum $g_!$ is defined by the following formula

$$(\frac{1}{2\pi\sqrt{-1}})^{d-2k-r} \int_{\tilde{Y}^{(2k+r+1)}} g_!(\alpha) \cup \beta = (\frac{1}{2\pi\sqrt{-1}})^{2d-2k-r} \int_{\tilde{T}^{(2k+r+1)}} \alpha \cup g^*(\beta)$$

where $\int$ denotes the morphism trace described by the cap–product with the fundamental class of each component of the stratum, for any chosen couple of elements $\alpha \in H^{q+2(2d-2k-r)}(\tilde{T}^{(2k+r+1)}, \mathbf{Q}(2d-2k-r))$ and $\beta \in H^{-q}(\tilde{Y}^{(2k+r+1)}, \mathbf{Q})$, $q \in Z,\ q \geq 0$.
Notice that although we have a notion of bifiltered pullback

$$g^* : (A^\bullet_X, L, F) \to (A^\bullet_Z, L, F)$$

this does not imply a canonical definition of a product structure on $A^\bullet_{\mathbf{C}}$ obtained via pullback along the diagonal map $\Delta : X \to X \times_S X$. In fact, the property of $f : X \to S$ to have normal crossings reduction is not preserved by the product map $f \times f : X \times_S X \to S$. The space $X \times_S X$ is in general not even smooth over $\mathbf{C}$! Finally, we remark that although the monodromy filtration is not multiplicative on the level of the filtered complexes $(A^\bullet_{\mathbf{C}}, L)$ (the simple example shown below will motivate this claim), it becomes multiplicative on the limiting cohomology with its mixed Hodge structure.

EXAMPLE 3.1.

Let $f : \mathbf{P}^1_S \to S$ be a $\mathbf{P}^1$-fibration over a disk S. We blow a closed point $P \in \mathbf{P}^1_0 = Y$ in the fiber $\mathbf{P}^1_0$ over the origin $\{0\}$. The resulting map $h : Z \to S$ has a normal crossings special fiber $h^{-1}(0) = T = T_1 \cup T_2$, where T_1 is the strict transform of Y and T_2 is the exceptional component (*i.e.* $\mathbf{P}^1$). The intersection $Q = T_1 \cap T_2 = T_{12}$ is transverse. Locally around Q, h has the following description

$$h(z_1, z_2) = z_1 z_2.$$

Consider the subcomplex $W_0(A^\bullet_{Z,\mathbf{C}})$ of $A^\bullet_{Z,\mathbf{C}}$ filtered by the monodromy filtration L induced on it by the one on $A^\bullet_{Z,\mathbf{C}}$ (*cf.* § 1, (1.2)). Its hypercohomology computes $H^*(T, \mathbf{C})$ and it can be determined in terms of the homology of the complex

$$\begin{aligned}\{\mathcal{C}^\bullet :\ H^{\cdot}(\tilde{T}^{(1)}, \mathbf{C}) \xrightarrow{d} H^{\cdot}(\tilde{T}^{(2)}, \mathbf{C})\} \quad &= \\ &\{\mathcal{C}^\bullet :\ H^{\cdot}(T_1, \mathbf{C}) \oplus H^{\cdot}(T_2, \mathbf{C}) \xrightarrow{d} H^{\cdot}(T_{12}, \mathbf{C})\}\end{aligned}$$

where $\mathcal{C}^\bullet$ sits in degrees zero and one. The differential d on $\mathcal{C}^\bullet$ is of "Čech type" *i.e.* it is an alternate sum of pullback maps as defined in (1.6). A product in the filtered derived category $(A^\bullet_{Z,\mathbf{C}}, L)$ if any exists, should induce a product on $\mathcal{C}^\bullet$. The tensor product $\mathcal{C}^\bullet \otimes \mathcal{C}^\bullet$ is a complex sitting in degrees zero, one and two and it has the following description

$$\begin{aligned}\{\mathcal{C}^\bullet \otimes \mathcal{C}^\bullet :\ \bigoplus_{i,j\in[1,2]} (H^{\cdot}(T_i, \mathbf{C}) \otimes H^{\cdot}(T_j, \mathbf{C})) \xrightarrow{d\otimes d} \bigoplus_{i=1}^{2} \{(H^{\cdot}(T_i, \mathbf{C}) \otimes H^{\cdot}(T_{12}, \mathbf{C})) \oplus \\ \oplus (H^{\cdot}(T_{12}, \mathbf{C}) \otimes H^{\cdot}(T_i, \mathbf{C}))\} \xrightarrow{d\otimes d} (H^{\cdot}(T_{12}, \mathbf{C}) \otimes H^{\cdot}(T_{12}, \mathbf{C}))\}.\end{aligned}$$

However, there is no way to define canonically the product

$$\mu : \mathcal{C}^\bullet \otimes \mathcal{C}^\bullet \to \mathcal{C}^\bullet.$$

In fact, let's look for a possible description of it in each degree. In degree zero a product should satisfy

$$\begin{aligned}&H^{\cdot}(T_1, \mathbf{C}) \otimes H^{\cdot}(T_1, \mathbf{C}) \mapsto H^{\cdot}(T_1, \mathbf{C}), \\ &H^{\cdot}(T_2, \mathbf{C}) \otimes H^{\cdot}(T_2, \mathbf{C}) \mapsto H^{\cdot}(T_2, \mathbf{C}), \\ &H^{\cdot}(T_i, \mathbf{C}) \otimes H^{\cdot}(T_j, \mathbf{C}) \mapsto 0, \quad i, j = 1, 2.\end{aligned}$$

In degree one, one could start by setting

$$
\begin{aligned}
&H^\cdot(T_1,\mathbf{C})\otimes H^\cdot(T_{12},\mathbf{C})\mapsto H^\cdot(T_{12},\mathbf{C}),\\
&H^\cdot(T_{12},\mathbf{C})\otimes H^\cdot(T_2,\mathbf{C})\mapsto H^\cdot(T_{12},\mathbf{C}),\\
&H^\cdot(T_2,\mathbf{C})\otimes H^\cdot(T_{12},\mathbf{C})\mapsto 0,\\
&H^\cdot(T_{12},\mathbf{C})\otimes H^\cdot(T_1,\mathbf{C})\mapsto 0.
\end{aligned}
$$

Notice however, that this definition is not at all canonical, as one could alternatively set

$$
\begin{aligned}
&H^\cdot(T_2,\mathbf{C})\otimes H^\cdot(T_{12},\mathbf{C})\mapsto H^\cdot(T_{12},\mathbf{C}),\\
&H^\cdot(T_{12},\mathbf{C})\otimes H^\cdot(T_1,\mathbf{C})\mapsto H^\cdot(T_{12},\mathbf{C}),\\
&H^\cdot(T_1,\mathbf{C})\otimes H^\cdot(T_{12},\mathbf{C})\mapsto 0,\\
&H^\cdot(T_{12},\mathbf{C})\otimes H^\cdot(T_2,\mathbf{C})\mapsto 0.
\end{aligned}
$$

Finally, in degree two one would have

$$H^\cdot(T_{12},\mathbf{C})\otimes H^\cdot(T_{12},\mathbf{C})\mapsto H^\cdot(T_{12},\mathbf{C})$$

4. Semistable degenerations with double points

This section is mainly devoted to the determination of $[N]$ for one–dimensional semistable fibrations with at worst double points as singularities. The description of $[N]$ is obtained via the introduction of the algebraic correspondence-type square on the cohomology groups of the special fiber as described in (2.1). A one–dimensional double point degeneration is the simplest example of a normal crossings fibration. The generalization of these results to double points semistable degenerations of arbitrary dimension is done at the end of this paragraph where we also report as an example of application of these results the case of a Lefschetz pencil.

We keep the same notations as in § 3, in particular we denote by $f: X\to S$ a *semistable* fibration of fiber dimension one. Its special fiber is denoted by Y. By definition, locally around a double point $P\in Y$ the description of f looks like

$$f(z_1,z_2)=z_1z_2$$

for $\{z_1,z_2\}$ local parameters on X at P. For one dimensional fiberings, the only group where the local monodromy may act non trivially is $gr_2^LH^1(\tilde{X}^*,\mathbf{Q})$, in which case the identity map on the E_1-terms of the weight spectral sequence (1.5)

$$E_1^{-1,2}=H^0(\tilde{Y}^{(2)},\mathbf{Q})(-1)\stackrel{\mathrm{Id}}{\to}H^0(\tilde{Y}^{(2)},\mathbf{Q})(-1)=E_1^{1,0}(-1)$$

determines an isomorphism of rational Hodge structures of weight two on the related graded groups $E_2=E_\infty$. This isomorphism is induced by the action of the local monodromy N around the origin:

$$N : gr_2^L H^1(\tilde{X}^*, \mathbf{Q}) \overset{\simeq}{\to} (gr_0^L H^1(\tilde{X}^*, \mathbf{Q}))(-1)$$

It is a well known consequence of the Clemens-Schmid exact sequence (considered as a sequence of mixed Hodge structures) that (*cf.* [9])

$$gr_2^L H^1(\tilde{X}^*, \mathbf{Q}) \neq 0 \;\Leftrightarrow$$
$$\Leftrightarrow\; \mathrm{Ker}(\rho^{(2)} : H^1(\tilde{Y}^{(1)}, \mathbf{Q}) \to H^1(\tilde{Y}^{(2)}, \mathbf{Q})) \neq 0 \;\Leftrightarrow\; h^1(|\Gamma|) \neq 0$$

where $h^1(|\Gamma|)$ is the dimension of the first rational cohomology group of the geometric realization of the dual graph of Y. It follows from proposition 2.1 that $[N] \in \mathbf{H}^2(T, (gr_{-2}^L A^\bullet_{Z,\mathbf{Q}})) = H^0(\tilde{T}^{(3)}, \mathbf{Q})$ determines a Hodge class

$$(4.1) \quad [N] \in Hom_{MH}(\mathbf{Q}(0), gr_0^L H^2(T, \mathbf{Q})) \simeq$$
$$\simeq Hom_{MH}\left(\mathbf{Q}(0), \frac{H^0(\tilde{T}^{(3)}, \mathbf{Q})}{\mathrm{Image}(\rho^{(3)} : H^0(\tilde{T}^{(2)}, \mathbf{Q}) \to H^0(\tilde{T}^{(3)}, \mathbf{Q}))}\right).$$

Here T is the special fiber of a normal–crossings degeneration $h : Z \to S$. The variety Z is a smooth threefold over $\mathbf{C}$ obtained via resolution of the singularities of $X \times_S X$. Notice that no more than three components of T intersect at the same closed point since $\dim Z = 3$.
We shall determine the Hodge cycle $[N] \in E_1^{2,0}(Z) = H^0(\tilde{T}^{(3)}, \mathbf{Q})$ by means of a "correspondence type" map

$$N : \mathbf{H}^*(Y, gr_r^L A^\bullet_{X,\mathbf{Q}}) \to \mathbf{H}^*(Y, (gr_{r-2}^L A^\bullet_{X,\mathbf{Q}})(-1)) = \mathbf{H}^*(Y, gr_r^L (A^\bullet_{X,\mathbf{Q}}(-1)))$$

as we explained in (2.1). From the proof it will easily follow that the map N is zero for $* \neq 1$ and is the identity for $* = 1 = r$. On the E_2-level it will induce (for $* = 1 = r$) a commutative diagram

$$\begin{tikzcd}
gr_2^L H^1(\tilde{X}^* \times \tilde{X}^*, \mathbf{Q}) \arrow[r, "{[N]\,\cdot}"] & gr_2^L H^3(\tilde{X}^* \times \tilde{X}^*, \mathbf{Q}) = E_2^{1,2} \arrow[d, "(p_2)_*"] \\
E_2^{-1,2} = gr_2^L H^1(\tilde{X}^*, \mathbf{Q}) \arrow[u, "(p_1)^*"] \arrow[r, "N"] & (gr_0^L H^1(\tilde{X}^*, \mathbf{Q}))(-1) = (E_2^{1,0})(-1)
\end{tikzcd}$$

The pullback p_1^* and pushforward $(p_2)_*$ are defined as in § 3. The above diagram will determine uniquely both $[N] \in Hom_{MH}(\mathbf{Q}(0), gr_0^L H^2(T, \mathbf{Q}))$ and the product $[N]\,\cdot$.
The following result defines the geometry of the model Z and the special fiber T after resolving the singularities of $X \times_S X$ and $Y \times Y$.

LEMMA 4.1. *Let $z_1 z_2 = w_1 w_2$ be a local description of $X \times_S X$ around the point (P, P), with $P \in Y = Y_1 \cup Y_2$ a double point of f and $\{w_1,\ w_2\}$ a second set of regular parameters on X at P. After a blow-up of $X \times_S X$ with center at the origin (z_1, z_2, w_1, w_2), the resulting degeneration $h : Z \to S$ is normal–crossings. Its special fiber T is the union of five irreducible components: $T = \cup_{i=1}^5 T_i$. We number them so that the first four are the strict transforms of the irreducible components $Y_i \times Y_j$ of $Y \times Y$, namely $T_1 = (Y_1 \times Y_1)\tilde{}$, $T_2 = (Y_1 \times Y_2)\tilde{}$,*

$T_3 = (Y_2 \times Y_1)\tilde{}$, $T_4 = (Y_2 \times Y_2)\tilde{}$. The last one T_5 represents the exceptional divisor of the blow-up. We have $\tilde{T}^{(1)} = \coprod_i T_i$. The scheme Z is covered by four affine charts $\mathcal{U}_j$. On each of them there are three non empty components T_k. The scheme $\tilde{T}^{(3)}$ is the disjoint union of four zero dimensional schemes (closed points): $T_{125} \in \mathcal{U}_2$, $T_{135} \in \mathcal{U}_4$, $T_{245} \in \mathcal{U}_3$ and $T_{345} \in \mathcal{U}_1$, each of whose supports projects isomorphically onto the diagonal $\Delta_{12} : Y_{12} \to Y_{12} \times Y_{12}$.

Proof. The local description of $X \times_S X$ around (P,P) is given by the equations $z_1 z_2 = w_1 w_2$ and $z_1 z_2 = t$, for $t \in S$ a fixed parameter on the disk. We choose the standard orientation of the sets $\{z_1, z_2\}$ and $\{w_1, w_2\}$ and we write $w'_{i_1} = \frac{w_i}{z_1}$, $w'_{i_2} = \frac{w_i}{z_2}$, $w_{ij} = \frac{w_i}{w_j}$, $z'_{i_1} = \frac{z_i}{w_1}$, $z'_{i_2} = \frac{z_i}{w_2}$ and $z_{ij} = \frac{z_i}{z_j}$, for $i,j = 1,2$. After a single blow-up of $X \times_S X$ at the origin (z_1, z_2, w_1, w_2), the resulting model Z is non singular as one can see by looking at the first of the following tables which describes Z on each of the four charts $\mathcal{U}_j$ who cover it. In the second table, we have collected for each $\mathcal{U}_j$, the description of the non empty divisors $T_k \in T^{(1)}$ there. We use the pullbacks $p_1^*(\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2})$ and $p_2^*(\frac{dw_1}{w_1} \wedge \frac{dw_2}{w_2})$ to define in the third table the pullbacks $p_i^*(1_{Y_{12}}) \in H^0(\tilde{T}^{(2)}, \mathbf{Q})$.

Open sets	Loc. coordinates and relations
$\mathcal{U}_1$	$\{w'_{1_1}, w'_{2_1}, z_1\},\ w'_{1_1} w'_{2_1} = z_{21}$
$\mathcal{U}_2$	$\{w'_{1_2}, w'_{2_2}, z_2\},\ w'_{1_2} w'_{2_2} = z_{12}$
$\mathcal{U}_3$	$\{z'_{1_1}, z'_{2_1}, w_1\},\ z'_{1_1} z'_{2_1} = w_{21}$
$\mathcal{U}_4$	$\{z'_{1_2}, z'_{2_2}, w_2\},\ z'_{1_2} z'_{2_2} = w_{12}$

Open sets	Divisors
$\mathcal{U}_1$	$T_3 = \{w'_{1_1} = 0\},\ T_4 = \{w'_{2_1} = 0\},\ T_5 = \{z_1 = 0\}$
$\mathcal{U}_2$	$T_1 = \{w'_{1_2} = 0\},\ T_2 = \{w'_{2_2} = 0\},\ T_5 = \{z_2 = 0\}$
$\mathcal{U}_3$	$T_2 = \{z'_{1_1} = 0\},\ T_4 = \{z'_{2_1} = 0\},\ T_5 = \{w_1 = 0\}$
$\mathcal{U}_4$	$T_1 = \{z'_{1_2} = 0\},\ T_3 = \{z'_{2_2} = 0\},\ T_5 = \{w_2 = 0\}$

Open sets	$p_1^*(1_{Y_{12}})$	$p_2^*(1_{Y_{12}})$
$\mathcal{U}_1$	$-1_{T_{35}} - 1_{T_{45}}$	$-1_{T_{45}} + 1_{T_{35}} + 1_{T_{34}}$
$\mathcal{U}_2$	$1_{T_{15}} + 1_{T_{25}}$	$-1_{T_{25}} + 1_{T_{15}} + 1_{T_{12}}$
$\mathcal{U}_3$	$-1_{T_{45}} + 1_{T_{25}} + 1_{T_{24}}$	$-1_{T_{25}} - 1_{T_{45}}$
$\mathcal{U}_4$	$-1_{T_{35}} + 1_{T_{15}} + 1_{T_{13}}$	$1_{T_{15}} + 1_{T_{35}}$

The global description of the pullbacks $p_1^*(1_{Y_{12}})$ and $p_2^*(1_{Y_{12}})$ is

$$p_1^*(1_{Y_{12}}) = (1_{T_{15}} + 1_{T_{25}} - 1_{T_{35}} - 1_{T_{45}}) + 1_{T_{13}} + 1_{T_{24}}$$

$$p_2^*(1_{Y_{12}}) = (1_{T_{15}} - 1_{T_{25}} + 1_{T_{35}} - 1_{T_{45}}) + 1_{T_{12}} + 1_{T_{34}}.$$

Finally, notice that each $\mathcal{U}_j$ is isomorphic to $\mathbf{A}^3$ and in each of them one has three non empty components T_k. □

The following result holds

THEOREM 4.2. *Let $f : X \to S$ be the semistable degeneration of curves as described above. Then, the following description of $[N] \in H^0(\tilde{T}^{(3)}, \mathbf{Q})$ (cf. (4.1)) holds:*

$$[N] = a_{125}1_{T_{125}} + a_{135}1_{T_{135}} + a_{245}1_{T_{245}} + a_{345}1_{T_{345}}$$

where the (rational) numbers a's are subject to the following requirement:

$$-2a_{125} + 2a_{135} - 2a_{245} + 2a_{345} = 1.$$

The induced class $[N]$ in $gr_0^L H^2(T, \mathbf{Q})$ (i.e. modulo boundary relations via the restriction map $\rho^{(3)}$ cf. (1.6)) determines a unique zero-cycle.

Proof. We determine $[N]$ as a cocycle making the following square commute

$$\begin{array}{lccc} & gr_2^L H^1(\tilde{X}^* \times \tilde{X}^*, \mathbf{Q}) & \xrightarrow{[N]\,\cdot} & gr_2^L H^3(\tilde{X}^* \times \tilde{X}^*, \mathbf{Q}) = E_2^{1,2} \\ (4.2) & {\scriptstyle (p_1)^*}\uparrow & & \downarrow{\scriptstyle (p_2)_*} \\ & E_2^{-1,2} = gr_2^L H^1(\tilde{X}^*, \mathbf{Q}) & \xrightarrow{N} & (gr_0^L H^1(\tilde{X}^*, \mathbf{Q}))(-1) = (E_2^{1,0})(-1) \end{array}$$

In terms of cohomologies of strata, we have to describe explicitly a representative of $[N]$ in $E_1^{2,0}(Z)$ that satisfies the commutativity of

$$\begin{array}{lccc} & H^0(\tilde{T}^{(2)}, \mathbf{Q})(-1) & \xrightarrow{[N]\,\cdot} & H^2(\tilde{T}^{(2)}, \mathbf{Q}) \\ (4.3) & {\scriptstyle p_1^*}\uparrow & & \downarrow{\scriptstyle (p_2)_*} \\ & E_1^{-1,2} = H^0(\tilde{Y}^{(2)}, \mathbf{Q})(-1) & = & H^0(\tilde{Y}^{(2)}, \mathbf{Q})(-1) = E_1^{1,0}(-1). \end{array}$$

With the notations used in lemma 4.1 the description of $[N]$ is given by

$$[N] = a_{125}1_{T_{125}} + a_{135}1_{T_{135}} + a_{245}1_{T_{245}} + a_{345}1_{T_{345}}.$$

For the standard choice of the orientations of $\{z_1, z_2\}$ and $\{w_1, w_2\}$ and the numbering of the T_i's defined in lemma 4.1, the local description of the pull-backs $p_i^*(1_{Y_{12}})$ for $i = 1, 2$ is given in the third table of the above lemma. Following the definition described in the Appendix (*cf.* (7.6)), the product $[N] \cdot p_1^*(1_{Y_{12}}(-1))$ is then the following

$$
\begin{aligned}
(4.4) \qquad & [N] \cdot p_1^*(1_{Y_{12}}(-1)) = \\
&= [N] \cdot (1_{T_{15}}(-1) + 1_{T_{15}}(-1) - 1_{T_{35}}(-1) - 1_{T_{45}}(-1)) \\
&= a_{125}(g_1(1_{T_{125}} \cdot 1_{T_{15}}(-1)) - g_2(1_{T_{125}} \cdot 1_{T_{25}}(-1))) \\
&\quad + a_{135}(g_1(1_{T_{135}} \cdot 1_{T_{15}}(-1)) + g_3(1_{T_{135}} \cdot 1_{T_{35}}(-1))) \\
&\quad + a_{245}(g_2(1_{T_{245}} \cdot 1_{T_{25}}(-1)) + g_4(1_{T_{245}} \cdot 1_{T_{45}}(-1))) \\
&\quad + a_{345}(-g_3(1_{T_{345}} \cdot 1_{T_{35}}(-1)) + g_4(1_{T_{345}} \cdot 1_{T_{45}}(-1))) \\
&= a_{125}(1_{T_{25}} - 1_{T_{15}}) + a_{135}(1_{T_{35}} + 1_{T_{15}}) \\
&\quad + a_{245}(1_{T_{45}} + 1_{T_{25}}) + a_{345}(-1_{T_{45}} + 1_{T_{35}}).
\end{aligned}
$$

The maps g_1, g_2, g_3 and g_4 are the pushforwards as introduced in the Appendix. The following formula illustrates the product $1_{T_{ijk}} \cdot \sum_{l,m} 1_{T_{lm}}(-1)$ following the definition of it given in the Appendix:

$$
\begin{aligned}
1_{T_{ijk}} \cdot \sum_{l,m} 1_{T_{lm}}(-1) &= 1_{T_{ijk}} \cdot (1_{T_{ik}}(-1) + 1_{T_{jk}}(-1)) \\
&= g_i(1_{T_{ijk}} \cdot 1_{T_{ik}}(-1)) - g_j(1_{T_{ijk}} \cdot 1_{T_{jk}}(-1)) = g_i(1_{T_{ijk}}) - g_j(1_{T_{ijk}}) \\
&\qquad \in \operatorname{Image}\big(\bigoplus_t g_t : H^0(\tilde{T}^{(3)}, \mathbf{Q})(-1) \to H^2(\tilde{T}^{(2)}, \mathbf{Q})\big).
\end{aligned}
$$

In (4.4), we have denoted, for simplicity of notations, the difference $g_i(1_{T_{ijk}}) - g_j(1_{T_{ijk}})$ with $1_{T_{jk}} - 1_{T_{ik}}$. The map g_i represents the pushforward on cycles deduced from the embedding $g_i : T_{ijk} \to T_{jk}$. The definition of g_j is similar. Therefore, via the local definition of the pushforward $(p_2)_*$ along the affine charts (*cf.* §3 and third table in lemma 4.1), we obtain:

$$
(p_2)_*([N] \cdot p_1^*(1_{Y_{12}}(-1))) = (-2a_{125} + 2a_{135} - 2a_{245} + 2a_{345})1_{Y_{12}}(-1).
$$

The commutativity of (4.3) and hence of (4.2) is then equivalent to the requirement

$$
-2a_{125} + 2a_{135} - 2a_{245} + 2a_{345} = 1.
$$

Hence, the operator $[N]$ is determined as a cocycle in $H^0(\tilde{T}^{(3)}, \mathbf{Q})$ by the setting

$$
\begin{aligned}
(4.5) \qquad [N] &= a_{125}1_{T_{125}} + a_{135}1_{T_{135}} + a_{245}1_{T_{245}} + a_{345}1_{T_{345}}; \\
&- 2a_{125} + 2a_{135} - 2a_{245} + 2a_{345} = 1.
\end{aligned}
$$

Up to boundary relations by means of the restriction map $\rho^{(3)}$ which connects the elements $1_{T_{125}}$ with $1_{T_{245}}$ and $1_{T_{135}}$ with $1_{T_{345}}$, (4.5) determines a unique zero–cycle in the quotient $E_2^{2,0}(Z)$ (*cf.* (4.1)). Of course, if $N = 0$, this class may be trivial. □

Remark 4.3.

The description of $[N] \in E_1^{2,0}(Z)$ as well as the relation among the coefficients a_{ijk} in (4.5) is not unique in E_1. In fact, it depends on the choice of the desingularization process, as well as on the ordering of the components $T_k \in \tilde{T}^{(1)}$. For example, for the ordering of T_k for which T_1 represents in each chart the exceptional divisor of the blow-up ($T_2 = (Y_1 \times Y_1)\tilde{}$, $T_3 = (Y_1 \times Y_2)\tilde{}$, $T_4 = (Y_2 \times Y_1)\tilde{}$, $T_5 = (Y_2 \times Y_2)\tilde{}$), the setting (4.5) becomes

$$\begin{aligned} [N] &= a_{123} 1_{T_{123}} + a_{124} 1_{T_{124}} + a_{135} 1_{T_{135}} + a_{145} 1_{T_{145}}; \\ &- a_{123} + a_{124} - a_{135} + a_{145} = 1. \end{aligned}$$

If instead we choose to desingularize $X \times_S X$ via a blowing-up along $z_1 = w_1 = 0$ and we set the order among the T_k's so that the exceptional divisor is represented in each chart by the last component (*i.e.* $T_1 = (Y_1 \times Y_2)\tilde{}$, $T_2 = (Y_2 \times Y_1)\tilde{}$, $T_3 = (Y_2 \times Y_2)\tilde{}$, $T_4 = (Y_1 \times Y_1)\tilde{}$,), then we would get

$$\begin{aligned} [N] &= a_{134} 1_{T_{134}} + a_{234} 1_{T_{234}}; \\ &- a_{134} + a_{234} = 1. \end{aligned}$$

It is a consequence of the uniqueness of the product structure on the corresponding E_2–terms that all these different settings determine a unique description of $[N] \in E_2^{2,0}(Z)$.

In what it follows we support some evidence for our belief that the description of $[N]$ for a double points degeneration of higher fiber dimension (*i.e.* locally described by $f(z_1, \dots, z_n) = z_i z_j$, *cf.* below) is deducible from the case worked out for curves. As already remarked, the description of $[N]$ in the cohomology of the strata of the special fiber of the fiber product resolution is of local nature, *i.e.* it can be described locally around each double point. For a higher dimensional double points degeneration $[N]$ should be again described in terms of a "diagonal" cocycle whose support projects isomorphically onto the diagonal $\Delta_{12} \in Y_{12} \times Y_{12}$ as it was shown in theorem 4.2. In general, that "diagonal" cocycle would be formally locally a bundle over the corresponding diagonal cocycle which comes up for a degeneration of curves. This is a consequence of the local description of the degeneration map around a double point. We give now some details for these ideas.

Let $f : X \to S$ be a semistable degeneration with double points of fiber dimension d over the disk S. Then, locally in a neighborhood of a double point P on Y, f has the following description

$$f(z_1, \dots, z_n) = z_i z_j$$

for $\{z_1, \dots, z_n\}$ a set of regular parameters on X at P and suitable indices $i < j$ in $I = \{1, \dots, n\}$. Let $Y = Y_1 \cup Y_2$ be the local description of Y in a neighborhood of $P \in Y_1 \cap Y_2 = Y_{12}$. Locally around P, $\{z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_n\}$ are free parameters for this description. Hence, the special fiber is locally around the point, formally isomorphic to $\mathbf{A}^{d-1} \times \hat{Y}$ with $\hat{Y} = \hat{Y_1} \cup \hat{Y_2}$ of dimension 1. In a formal neighborhood of P, Y is defined by $\mathrm{Spec}\big(\mathbf{C}\{\{z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots z_n\}\}[z_i, z_j]/z_i z_j\big)$. The model X

is formally locally isomorphic to $\mathbf{A}^{d-1} \times \hat{X}$, with $\hat{X}$ of fiber dimension 1 and special fiber $\hat{Y}$. The formal description of $X \times_S X$ is similar, namely $X \times_S X \simeq \mathbf{A}^{d-1} \times \mathbf{A}^{d-1} \times (\hat{X} \times_S \hat{X})$. Keeping the same notations introduced before, we get a formal local description of the stratum $\tilde{T}^{(3)}$ (containing the cocycle $[N]$) as $\mathbf{A}^{d-1} \times \mathbf{A}^{d-1} \times \hat{\tilde{T}}^{(3)}$, with $\hat{\tilde{T}}^{(3)}$ collection of points. $[N]$ is a cycle (of dimension $d-1$) in $\tilde{T}^{(3)}$ formally, locally described by $\Delta_{\mathbf{A}^{d-1}} \times \hat{\tilde{T}}^{(3)}$. This scheme is isomorphic to the formal completion of the diagonal $\Delta_{Y_{12}} \subset Y_{12} \times Y_{12}$, *i.e.* $\hat{\Delta}_{Y_{12}} \simeq \mathbf{A}^{d-1} \times \hat{Y}_{12}$, $(\dim\ \hat{Y}_{12} = 0)$.

In this way, the description of $[N]$ would be deduced from a formal local description of the Lefschetz pencil of fiber dimension one $f : \hat{X} \to \mathbf{C}\{\{t\}\}$. Hence, one would get a formal local class representative of N as a bundle over the diagonal cocycle which describes $[N]$ in theorem 4.2. What said so far supports evidence for the following

Conjecture 4.4. *Let $f : X \to S$ be a semistable double points degeneration of fiber dimension d. Then, the local monodromy operator is described by a unique algebraic cocycle of codimension d-1 in the stratum $\tilde{T}^{(3)}$ $(\dim\ \tilde{T}^{(3)} = 2(d-1))$ i.e.*

$$[N] \in \frac{CH^{d-1}(\tilde{T}^{(3)})}{Image\big(\rho^{(3)} : CH^{d-1}(\tilde{T}^{(2)}) \to CH^{d-1}(\tilde{T}^{(3)})\big)}.$$

The formal local description of $[N]$ is given by the algebraic cycle $\Delta_{\mathbf{A}^{d-1}} \times \hat{\tilde{T}}^{(3)}$.

Notice that for a double point degeneration of fiber dimension $d > 1$, $[N]$ may represent the monodromy map acting non trivially on different graded pieces of the limiting cohomology. However, they are all of type $gr^L_{q+1} H^q(\tilde{X}^*, \mathbf{Q}) = E_2^{-1,q+1}(X)$ for $q \in [0, d]$. In fact, for double point degenerations we have always $N = 0$ on $gr^L_q H^q(\tilde{X}^*, \mathbf{Q})$, and $\mathbf{H}^*(Y, gr^L_i A^\bullet_{X,\mathbf{C}}) = 0$ for $i \neq -1,\ 0,\ 1$ because no more than two components of Y intersect simultaneusly at the same closed point.

As an example of application of these results we consider the case of a Lefschetz pencil of fiber dimension at least three. The description of $[N]$ is the same to the one just described for a degeneration with double points. We will only show how to reduce in this case the study of $[N]$ to the previous one. A Lefschetz pencil of fiber dimension greater than one is not even normal-crossings because the special fiber is irreducible and singular. We will only consider the case of odd fiber dimension since Lefschetz pencils of even fiber dimension have trivial monodromy always.

Let $f' : \mathcal{X} \to S$ be such a pencil and let $n = 2m+1$ be the dimension of its fiber. Locally, in a neighborhood of the singular point of the special fiber $\mathcal{Y}$, the pencil f' is described by

$$f'(z_0, \ldots, z_n) = \sum_{\nu=0}^{m} z_\nu z_{\nu+1+m}$$

where as usual $\{z_0, \dots, z_n\}$ represents a set of regular parameters on $\mathcal{X}$. It is clear from the definition that the special fiber $\mathcal{Y}$ is irreducible and singular at the origin $(z_0, \dots, z_n)$. However, after a single blow-up at that point we get a normal-crossings degeneration $f : X \to S$ with special fiber locally described by $Y = Y_1 \cup Y_2$. The component Y_1 is the exceptional divisor of the blow-up, a projective space of dimension n which intersects the strict transform Y_2 of Y along a quadric hypersurface Y_{12} of dimension $2m$. The component Y_1 appears with multiplicity $e_1 = 2$ whereas Y_2 is reduced (*i.e.* $e_2 = 1$). Let $h : X \to \mathcal{X}$ be the blow-up map. It is a (proper) map of S-schemes, therefore it induces a morphism

$$g^*\mathbf{R}\Psi_{f'}(\mathbf{Q}_{\mathcal{X}}) \to \mathbf{R}\Psi_f(\mathbf{Q}_X)$$

of complexes of nearby cycles. This morphism induces in turn a homomorphism between the corresponding hypercohomologies

$$g^* : \mathbf{H}^i(\mathcal{Y}, \mathbf{R}\Psi_{f'}(\mathbf{Q})) \to \mathbf{H}^i(Y, \mathbf{R}\Psi_f(\mathbf{Q}))$$

In order to work with the resolution $A^\bullet_{\mathbf{Q}}$ of $\mathbf{R}\Psi_f(\mathbf{Q})$ which carries the monodromy filtration, we have to consider Y with its reduced structure (the exceptional divisor has multiplicity $e_1 = 2$ as algebraic cycle on X). Because $g.c.d.(e_1, e_2) = 1\ \forall y \in Y$ the action of the local monodromy on the complex of sheaves $\mathbf{R}\Psi_f(Y, \mathbf{Q})$ is unipotent (*cf.* § 1). That implies that the monodromy operator acts unipotently on cohomology.
Because f' is a Lefschetz pencil of fiber dimension n, the only group where N acts non trivially is $H^n(\tilde{X}^*, \mathbf{Q})$. Also, $[N]$ determines an element in $(H^{2n}(\tilde{X}^* \times \tilde{X}^*, \mathbf{Q}(n-1)))^{\pi_1}$ and because the generic fibers of f' and f are the same, we may as well consider $[N] \in \mathbf{H}^{2n}(Y \times Y, \mathbf{R}\Psi_f(\mathbf{Q}))^{\pi_1}$.
The map f is locally described by $z_i^2 q(z_0, \dots, \hat{z}_i, \dots, z_n) = t$ for some $i \in [0, n]$, t being a local parameter on S and $q(z_0, \dots, \hat{z}_i, \dots, z_n)$ an irreducible quadratic polynomial. Via the extension of the basis $S' \to S\ \tau \mapsto \sqrt{t}$, the degeneration f is deformed to $w_i z_i = \tau$, with $w_i = \frac{\tau}{z_i}$ and $w_i^2 = h$. It is clear that this procedure does not affect the special fibers (*i.e.* the reduced closed fibers are the same). Hence, after a possible normalization of the resulting model, we obtain a double point semistable degeneration $h : Z \to S$. Let $T = T_1 \cup T_2$ be its special fiber. Then $[N]$ can be seen as a Hodge cycle in $H^{2n}(T \times T, \mathbf{R}\Psi_h(\mathbf{Q}))^{\pi_1} = \mathrm{Ker}(\tilde{N}) \cap H^{2n}(\tilde{X}^* \times \tilde{X}^*, \mathbf{Q})$, for $\tilde{N} = 1 \otimes N + N \otimes 1$. The geometric description of $[N]$ is then the same as the one we have shown before. The class $[N]$ represents the monodromy operator acting non trivially only on $gr^L_{n+1} H^n(\tilde{X}^*, \mathbf{Q})$.

5. Semistable degenerations with triple points

A semistable degeneration with triple points is the first case where both the operators N and N^2 may be non trivial. In this paragraph we will mainly consider a triple point degeneration of surfaces. The description of $[N]$ and $[N^2]$ for higher dimensional triple points degenerations can be deduced from

the one for surfaces using the same kind of arguments described in the last paragraph for double points degenerations of higher fiber dimension.
Let $f: X \to S$ be a surfaces degeneration with reduced normal crossings and with triple points on its special fiber Y. We keep the basic notations as in the previous sections. Then, locally around a triple point $P \in Y$ we may assume that f has the following description:

$$f(z_1, z_2, z_3) = z_1 z_2 z_3.$$

As usual, $\{z_1, z_2, z_3\}$ is a regular set of parameters on X at P. Globally on X, the special fiber can be the union of more than three components *i.e.* $Y = Y_1 \cup \ldots \cup Y_N$, but at most three of them intersect at the same closed point. The Clemens–Schmid exact sequence of mixed Hodge structures describes the behavior of the operators N and N^2 in terms of some invariants on the special fiber. Namely

LEMMA 5.1. *(Monodromy criteria) Let $f: X \to S$ be a semistable degeneration of surfaces, then*

$$N = 0 \text{ on } H^1(\tilde{X}^*, \mathbf{Q}) \Leftrightarrow h^1(|\Gamma|) = 0$$

$$N = 0 \text{ on } H^2(\tilde{X}^*, \mathbf{Q}) \Leftrightarrow h^2(|\Gamma|) = 0 \text{ and } \rho^{(2)}: H^1(\tilde{Y}^{(1)}, \mathbf{Q}) \twoheadrightarrow H^1(\tilde{Y}^{(2)}, \mathbf{Q})$$

$$N^2 = 0 \text{ on } H^2(\tilde{X}^*, \mathbf{Q}) \Leftrightarrow h^2(|\Gamma|) = 0.$$

Here $h^i(|\Gamma|)$ denotes the dimension of the ith-cohomology group of the geometric realization of the dual graph of Y.

Proof. *cf.* [9]. □

A degeneration of K-3 surfaces with special fiber made by rational surfaces intersecting along a cycle of rational curves, is an example for which both N and N^2 are non zero (*cf.* [9]).
Let us suppose that at least one of the groups $gr_2^L H^1(\tilde{X}^*, Q)$ and $gr_3^L H^2(\tilde{X}^*, \mathbf{Q})$ is non zero (for the above example it is well known that $gr_2^L H^1(\tilde{X}^*, \mathbf{Q}) = 0$, as $H^1(\tilde{X}^*, \mathbf{Q}) = 0$). The map N acts on them as an isomorphism of pure Hodge structures

$$N: gr_2^L H^1(\tilde{X}^*, Q) \overset{\simeq}{\to} (gr_0^L H^1(\tilde{X}^*, \mathbf{Q}))(-1)$$

$$N: gr_3^L H^2(\tilde{X}^*, \mathbf{Q}) \overset{\simeq}{\to} (gr_1^L H^2(\tilde{X}^*, \mathbf{Q}))(-1).$$

The only group where N^2 behaves as an isomorphism is $gr_4^L H^2(\tilde{X}^*, \mathbf{Q})$. The map N^2 is defined by the composition

$$gr_4^L H^2(\tilde{X}^*, \mathbf{Q}) \overset{N}{\to} (gr_2^L H^2(\tilde{X}^*, \mathbf{Q}))(-1) \overset{N}{\to} (gr_0^L H^2(\tilde{X}^*, \mathbf{Q}))(-2).$$

The sequence is not exact in the middle. The map N on the left is injective and the one on the right surjects $(gr_2^L H^2(\tilde{X}^*, \mathbf{Q}))(-1)$ onto $(gr_0^L H^2(\tilde{X}^*, \mathbf{Q}))(-2)$. Its kernel, in term of the spectral sequence of weights is

$$\big(\mathrm{Im}(\mathbf{H}^2(Y, gr_1^W \Omega_X^{\bullet+1}(\log Y)) \otimes \mathbf{Q} \to \mathbf{H}^2(Y, A^{\bullet}_{X,\mathbf{Q}}))\big)(-1) \simeq$$

$$\simeq \frac{\mathrm{Ker}(\rho^{(2)} : H^2(\tilde{Y}^{(1)}, \mathbf{Q})(-1) \to H^2(\tilde{Y}^{(2)}, \mathbf{Q})(-1))}{\mathrm{Im}(\gamma^{(2)} : H^0(\tilde{Y}^{(2)}, \mathbf{Q})(-2) \to H^2(\tilde{Y}^{(1)}, \mathbf{Q})(-1))}.$$

We first consider N and its related class $[N]$. Both $gr_2^L H^1(\tilde{X}^*, \mathbf{Q})$ and $gr_3^L H^2(\tilde{X}^*, \mathbf{Q})$ are described in terms of cohomology classes on $\tilde{Y}^{(2)}$ (*cf.* (1.7)). The study of the correspondence-diagram (2.1) is similar for them. Namely, once one has found an algebraic cycle representing $[N]$, it certainly makes both the correspondence diagrams commute. For degenerations of surfaces it follows from proposition 2.1 that

(5.1)

$$[N] \in (gr_2^L H^4(T, \mathbf{Q}))(1) \simeq \frac{\mathrm{Ker}(\rho^{(4)} : H^2(\tilde{T}^{(3)}, \mathbf{Q})(1) \to H^2(\tilde{T}^{(4)}, \mathbf{Q})(1))}{\mathrm{Im}(\rho^{(3)} : H^2(\tilde{T}^{(2)}, \mathbf{Q})(1) \to H^2(\tilde{T}^{(3)}, \mathbf{Q})(1))}$$

where $h : Z \to S$ is a normal crossings degeneration with special fiber T and generic fiber $\tilde{X}^* \times \tilde{X}^*$ obtained via resolution of the singularities of $X \times_S X$. Similarly, one has

$$(5.2) \qquad [N^2] \in gr_0^L H^4(T, \mathbf{Q}) \simeq \frac{H^0(\tilde{T}^{(5)}, \mathbf{Q})}{\mathrm{Im}(\rho^{(5)} : H^0(\tilde{T}^{(4)}, \mathbf{Q}) \to H^0(\tilde{T}^{(5)}, \mathbf{Q}))}.$$

Both $[N]$ and $[N^2]$ have the further property to be Hodge cycles in the cohomologies of the corresponding strata. The following lemma determines the geometry of the model Z and the special fiber T after resolving the singularities of $X \times_S X$ and $Y \times Y$.

LEMMA 5.2. *Let $z_1 z_2 z_3 = w_1 w_2 w_3$ be a local description of $X \times_S X$ around the point (P, P), being $P \in Y = \cup_{i=1}^3 Y_i$ a triple point of f and $\{w_1,\ w_2,\ w_3\}$ a second set of regular parameters on X at P. After three blows-up of $X \times_S X$ with centers at $z_i = 0 = w_i$ ($i = 1, 2, 3$) the resulting degeneration $h : Z \to S$ is normal–crossings. Its special fiber T is the union of nine irreducible components: $T = \cup_{i=1}^9 T_i$. We number them so that the first six are the strict transforms of the irreducible components $Y_i \times Y_j$ of $Y \times Y$: $T_1 = (Y_1 \times Y_2)\tilde{}$, $T_2 = (Y_1 \times Y_3)\tilde{}$, $T_3 = (Y_2 \times Y_1)\tilde{}$, $T_4 = (Y_2 \times Y_3)\tilde{}$, $T_5 = (Y_3 \times Y_1)\tilde{}$, $T_6 = (Y_3 \times Y_2)\tilde{}$. The last three components are the exceptional divisors of the three blows-up: $T_7 = (Y_1 \times Y_1)\tilde{}$, $T_8 = (Y_2 \times Y_2)\tilde{}$, $T_9 = (Y_3 \times Y_3)\tilde{}$. We have $\tilde{T}^{(1)} = \coprod_i T_i$. The scheme Z is covered by eight affine charts, on each of them there are at most five non empty components T_i. Among the components T_{ijk} whose disjoint union defines the scheme $\tilde{T}^{(3)}$, T_{178} and T_{378} contain resp. the curves "diagonal" $\tilde{\delta}_{12}$ and δ_{12} whose supports project isomorphically onto the diagonal $\Delta_{12} : Y_{12} \to Y_{12} \times Y_{12}$. Similarly, T_{279} and T_{579} contain resp. $\tilde{\delta}_{13}$ and δ_{13} whose support projects isomorphically onto $\Delta_{13} : Y_{13} \to Y_{13} \times Y_{13}$. Finally, T_{489} and T_{689} contain $\tilde{\delta}_{23}$ and δ_{23} whose support is isomorphic to Δ_{23}. The exceptional surface T_{789}–intersection of the three exceptional divisors of h–is isomorphic to the blow-up Bl of $\mathbf{P}^1 \times \mathbf{P}^1$ at the points $\{(0,1) \times (1,0)\}$ and $\{(1,0) \times (0,1)\}$. Finally, the scheme $\tilde{T}^{(5)}$ is the disjoint union of six irreducible components (points). They are: $T_{12789}, T_{16789}, T_{24789}, T_{34789}, T_{35789}, T_{56789}$. Their support maps isomorphically onto the (point) diagonal $\Delta_{123} : Y_{123} \to Y_{123} \times Y_{123}$.*

Proof. The local description of $X \times_S X$ at (P, P) is given by the equations $z_1 z_2 z_3 = w_1 w_2 w_3$ and $z_1 z_2 z_3 = t$, for $t \in S$ a fixed parameter on the disk. We choose the standard orientation of the sets $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ and we write $w_i' = \frac{w_i}{z_i}$, $z_i' = \frac{z_i}{w_i}$ for $i = 1, 2, 3$. After three blows-up of $X \times_S X$ along the subvarieties $z_i = 0 = w_i$, the resulting model Z is non singular as one can see by looking at the first of the following tables which describes Z on each of the eight charts $\mathcal{U}_j$ who cover it. In the second table, we have collected for each $\mathcal{U}_j$, the description of the non empty divisors $T_k \in T^{(1)}$ there and the third table shows the "diagonal" curves δ and $\tilde{\delta}$ defined in each chart. The remaining charts describe the pullbacks $p_1^*(\frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j})$, $p_2^*(\frac{dw_i}{w_i} \wedge \frac{dw_j}{w_j})$, $p_1^*(\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \frac{dz_3}{z_3})$ and $p_2^*(\frac{dw_1}{w_1} \wedge \frac{dw_2}{w_2} \wedge \frac{dw_3}{w_3})$ in terms of the related descriptions by cocycles classes in the corresponding cohomologies.

Open sets	Loc. coordinates and relations
$\mathcal{U}_1$	$\{w_1', w_2', w_3', z_1, z_2, z_3\},\ w_1' w_2' w_3' = 1$
$\mathcal{U}_2$	$\{w_1', w_2', z_1, z_2, w_3\},\ w_1' w_2' = z_3'$
$\mathcal{U}_3$	$\{w_1', w_3', z_1, z_3, w_2\},\ w_1' w_3' = z_2'$
$\mathcal{U}_4$	$\{z_2', z_3', z_1, w_2, w_3\},\ z_2' z_3' = w_1'$
$\mathcal{U}_5$	$\{w_2', w_3', z_2, z_3, w_1\},\ w_2' w_3' = z_1'$
$\mathcal{U}_6$	$\{z_1', z_3', z_2, w_1, w_3\},\ z_1' z_3' = w_2'$
$\mathcal{U}_7$	$\{z_1', z_2', z_3, w_1, w_2\},\ z_1' z_2' = w_3'$
$\mathcal{U}_8$	$\{z_1', z_2', z_3', w_1, w_2, w_3\},\ z_1' z_2' z_3' = 1$

Open sets	Divisors
$\mathcal{U}_1$	$T_7 = \{z_1 = 0\},\ T_8 = \{z_2 = 0\},\ T_9 = \{z_3 = 0\}$
$\mathcal{U}_2$	$T_5 = \{w_1' = 0\},\ T_6 = \{w_2' = 0\},\ T_7 = \{z_1 = 0\},$ $T_8 = \{z_2 = 0\},\ T_9 = \{w_3 = 0\}$
$\mathcal{U}_3$	$T_3 = \{w_1' = 0\},\ T_4 = \{w_3' = 0\},\ T_7 = \{z_1 = 0\},$ $T_8 = \{w_2 = 0\},\ T_9 = \{z_3 = 0\}$
$\mathcal{U}_4$	$T_3 = \{z_2' = 0\},\ T_5 = \{z_3' = 0\},\ T_7 = \{z_1 = 0\},$ $T_8 = \{w_2 = 0\},\ T_9 = \{w_3 = 0\}$
$\mathcal{U}_5$	$T_1 = \{w_2' = 0\},\ T_2 = \{w_3' = 0\},\ T_7 = \{w_1 = 0\},$ $T_8 = \{z_2 = 0\},\ T_9 = \{z_3 = 0\}$
$\mathcal{U}_6$	$T_1 = \{z_1' = 0\},\ T_6 = \{z_3' = 0\},\ T_7 = \{w_1 = 0\},$ $T_8 = \{z_2 = 0\},\ T_9 = \{w_3 = 0\}$
$\mathcal{U}_7$	$T_2 = \{z_1' = 0\},\ T_4 = \{z_2' = 0\},\ T_7 = \{w_1 = 0\},$ $T_8 = \{w_2 = 0\},\ T_9 = \{z_3 = 0\}$
$\mathcal{U}_8$	$T_7 = \{w_1 = 0\},\ T_8 = \{w_2 = 0\},\ T_9 = \{w_3 = 0\}$

Open sets	"Diagonal" curves
$\mathcal{U}_1$	none
$\mathcal{U}_2$	$\delta_{13} = \{w'_1 = z_1 = w_3 = 0,\ w'_2 = 1\} \subset T_{579},\ \delta_{13} \cap T_8 \neq \emptyset$ $\delta_{23} = \{w'_2 = z_2 = w_3 = 0,\ w'_1 = 1\} \subset T_{689},\ \delta_{23} \cap T_7 \neq \emptyset$
$\mathcal{U}_3$	$\delta_{12} = \{w'_1 = z_1 = w_2 = 0,\ w'_3 = 1\} \subset T_{378},\ \delta_{12} \cap T_9 \neq \emptyset$ $\tilde{\delta}_{23} = \{w'_3 = z_3 = w_2 = 0,\ w'_1 = 1\} \subset T_{489},\ \tilde{\delta}_{23} \cap T_7 \neq \emptyset$
$\mathcal{U}_4$	$\delta_{12} = \{z'_2 = z_1 = w_2 = 0,\ z'_3 = 1\} \subset T_{378},\ \delta_{12} \cap T_9 \neq \emptyset$ $\delta_{13} = \{z'_3 = z_1 = w_3 = 0,\ z'_2 = 1\} \subset T_{579},\ \delta_{13} \cap T_8 \neq \emptyset$
$\mathcal{U}_5$	$\tilde{\delta}_{12} = \{w'_2 = z_2 = w_1 = 0,\ w'_3 = 1\} \subset T_{178},\ \tilde{\delta}_{12} \cap T_9 \neq \emptyset$ $\tilde{\delta}_{13} = \{w'_3 = z_3 = w_1 = 0,\ w'_2 = 1\} \subset T_{279},\ \tilde{\delta}_{13} \cap T_8 \neq \emptyset$
$\mathcal{U}_6$	$\tilde{\delta}_{12} = \{z'_1 = z_2 = w_1 = 0,\ z'_3 = 1\} \subset T_{178},\ \tilde{\delta}_{12} \cap T_9 \neq \emptyset$ $\delta_{23} = \{z'_3 = z_2 = w_3 = 0,\ z'_1 = 1\} \subset T_{689},\ \tilde{\delta}_{23} \cap T_7 \neq \emptyset$
$\mathcal{U}_7$	$\tilde{\delta}_{13} = \{z'_1 = z_3 = w_1 = 0,\ z'_2 = 1\} \subset T_{279},\ \tilde{\delta}_{13} \cap T_8 \neq \emptyset$ $\tilde{\delta}_{23} = \{z'_2 = z_3 = w_2 = 0,\ z'_1 = 1\} \subset T_{489},\ \tilde{\delta}_{23} \cap T_7 \neq \emptyset$
$\mathcal{U}_8$	none

Denote by $v_{Y_{ij}}$ a class in $H^*(Y_{ij}, \mathbf{C})$ and by $v_{T_{lk}}$ a class in $H^*(\tilde{T}^{(2)}, \mathbf{C})$. Then we have

Open sets	$p_1^*(v_{Y_{12}})$	$p_2^*(v_{Y_{12}})$
$\mathcal{U}_1$	$v_{T_{78}}$	$v_{T_{78}}$
$\mathcal{U}_2$	$v_{T_{78}}$	$v_{T_{56}} + v_{T_{58}} - v_{T_{67}} + v_{T_{78}}$
$\mathcal{U}_3$	$-v_{T_{37}} - v_{T_{47}} + v_{T_{78}}$	$v_{T_{38}} + v_{T_{78}}$
$\mathcal{U}_4$	$-v_{T_{37}} + v_{T_{78}}$	$v_{T_{78}} + v_{T_{38}} + v_{T_{58}}$
$\mathcal{U}_5$	$v_{T_{18}} + v_{T_{28}} + v_{T_{78}}$	$v_{T_{78}} - v_{T_{17}}$
$\mathcal{U}_6$	$v_{T_{18}} + v_{T_{78}}$	$v_{T_{78}} - v_{T_{17}} - v_{T_{67}}$
$\mathcal{U}_7$	$v_{T_{24}} + v_{T_{28}} - v_{T_{47}} + v_{T_{78}}$	$v_{T_{78}}$
$\mathcal{U}_8$	$v_{T_{78}}$	$v_{T_{78}}$

Hence, the global description of the pullbacks $p_1^*(v_{Y_{12}})$ and $p_2^*(v_{Y_{12}})$ are

$$p_1^*(v_{Y_{12}}) = (v_{T_{18}} + v_{T_{28}} - v_{T_{37}} - v_{T_{47}} + v_{T_{78}}) + v_{T_{24}}$$
$$p_2^*(v_{Y_{12}}) = (-v_{T_{17}} + v_{T_{38}} + v_{T_{58}} - v_{T_{67}} + v_{T_{78}}) + v_{T_{56}}\,.$$

Open sets	$p_1^*(v_{Y_{13}})$	$p_2^*(v_{Y_{13}})$
$\mathcal{U}_1$	$v_{T_{79}}$	$v_{T_{79}}$
$\mathcal{U}_2$	$-v_{T_{57}} - v_{T_{67}} + v_{T_{79}}$	$v_{T_{79}} + v_{T_{59}}$
$\mathcal{U}_3$	$v_{T_{79}}$	$v_{T_{79}} - v_{T_{47}} + v_{T_{39}} + v_{T_{34}}$
$\mathcal{U}_4$	$-v_{T_{57}} + v_{T_{79}}$	$v_{T_{79}} + v_{T_{39}} + v_{T_{59}}$
$\mathcal{U}_5$	$v_{T_{19}} + v_{T_{29}} + v_{T_{79}}$	$v_{T_{79}} - v_{T_{27}}$
$\mathcal{U}_6$	$v_{T_{16}} + v_{T_{19}} - v_{T_{67}} + v_{T_{79}}$	$v_{T_{79}}$
$\mathcal{U}_7$	$v_{T_{29}} + v_{T_{79}}$	$v_{T_{79}} - v_{T_{27}} - v_{T_{47}}$
$\mathcal{U}_8$	$v_{T_{79}}$	$v_{T_{79}}$

Hence we have the global descriptions

$$p_1^*(v_{Y_{13}}) = (v_{T_{19}} + v_{T_{29}} - v_{T_{57}} - v_{T_{67}} + v_{T_{79}}) + v_{T_{16}}$$
$$p_2^*(v_{Y_{13}}) = (-v_{T_{27}} + v_{T_{39}} - v_{T_{47}} + v_{T_{59}} + v_{T_{79}}) + v_{T_{34}}\,.$$

Open sets	$p_1^*(v_{Y_{23}})$	$p_2^*(v_{Y_{23}})$
$\mathcal{U}_1$	$v_{T_{89}}$	$v_{T_{89}}$
$\mathcal{U}_2$	$-v_{T_{58}} - v_{T_{68}} + v_{T_{89}}$	$v_{T_{89}} + v_{T_{69}}$
$\mathcal{U}_3$	$v_{T_{39}} + v_{T_{49}} + v_{T_{89}}$	$v_{T_{89}} - v_{T_{48}}$
$\mathcal{U}_4$	$v_{T_{35}} + v_{T_{39}} - v_{T_{58}} + v_{T_{89}}$	$v_{T_{89}}$
$\mathcal{U}_5$	$v_{T_{89}}$	$v_{T_{89}} - v_{T_{28}} + v_{T_{19}} + v_{T_{12}}$
$\mathcal{U}_6$	$-v_{T_{68}} + v_{T_{89}}$	$v_{T_{89}} + v_{T_{19}} + v_{T_{69}}$
$\mathcal{U}_7$	$v_{T_{49}} + v_{T_{89}}$	$v_{T_{89}} - v_{T_{28}} - v_{T_{48}}$
$\mathcal{U}_8$	$v_{T_{89}}$	$v_{T_{89}}$

Finally we have

$$p_1^*(v_{Y_{23}}) = (v_{T_{39}} + v_{T_{49}} - v_{T_{58}} - v_{T_{68}} + v_{T_{89}}) + v_{T_{35}}$$
$$p_2^*(v_{Y_{23}}) = (v_{T_{19}} - v_{T_{28}} - v_{T_{48}} + v_{T_{69}} + v_{T_{89}}) + v_{T_{12}}.$$

Using the above tables we deduce the following

Open sets	$p_1^*(1_{Y_{123}})$	$p_2^*(1_{Y_{123}})$
$\mathcal{U}_1$	$1_{T_{789}}$	$1_{T_{789}}$
$\mathcal{U}_1$	$1_{T_{578}} + 1_{T_{678}} + 1_{T_{789}}$	$1_{T_{569}} + 1_{T_{589}} - 1_{T_{679}} + 1_{T_{789}}$
$\mathcal{U}_3$	$-1_{T_{379}} - 1_{T_{479}} + 1_{T_{789}}$	$-1_{T_{348}} + 1_{T_{389}} + 1_{T_{478}} + 1_{T_{789}}$
$\mathcal{U}_4$	$1_{T_{357}} - 1_{T_{379}} + 1_{T_{578}} + v_{T_{789}}$	$1_{T_{389}} + 1_{T_{589}} + 1_{T_{789}}$
$\mathcal{U}_5$	$1_{T_{189}} + 1_{T_{289}} + 1_{T_{789}}$	$1_{T_{127}} - 1_{T_{179}} + 1_{T_{278}} + v_{T_{789}}$
$\mathcal{U}_6$	$-1_{T_{168}} + 1_{T_{189}} + 1_{T_{678}} + 1_{T_{789}}$	$-1_{T_{179}} - 1_{T_{679}} + v_{T_{789}}$
$\mathcal{U}_7$	$1_{T_{249}} + 1_{T_{289}} - 1_{T_{479}} + 1_{T_{789}}$	$1_{T_{278}} + 1_{T_{478}} - v_{T_{789}}$
$\mathcal{U}_8$	$1_{T_{789}}$	$1_{T_{789}}$

We then obtain

$$\begin{aligned} p_1^*(1_{Y_{123}}) &= (1_{T_{189}} + 1_{T_{249}} + 1_{T_{289}} - 1_{T_{379}} - 1_{T_{479}} + 1_{T_{789}}) \\ &\quad - 1_{T_{168}} + 1_{T_{357}} + 1_{T_{578}} + 1_{T_{678}} \\ p_2^*(1_{Y_{123}}) &= (-1_{T_{179}} + 1_{T_{389}} + 1_{T_{569}} + 1_{T_{589}} - 1_{T_{679}} + 1_{T_{789}}) \\ &\quad + 1_{T_{127}} + 1_{T_{278}} - 1_{T_{348}} + 1_{T_{478}}. \end{aligned}$$

Notice that with the exception of $\mathcal{U}_1$ and $\mathcal{U}_8$ that are open sets in $\mathbf{A}^5$ on which only the exceptional components T_7, T_8 and T_9 are non empty, all the remaining charts $\mathcal{U}_j$ are isomorphic to $\mathbf{A}^5$ and in each of them one has five components T_k non empty.
On $\mathcal{U}_3 \cap \mathcal{U}_4$ the surface T_{378} contains the curve δ_{12}, and on $\mathcal{U}_5 \cap \mathcal{U}_6$, T_{178} contains the curve $\tilde{\delta}_{12}$. The curves δ_{12} and $\tilde{\delta}_{12}$ are different: *i.e.* $T_1 = \emptyset$ on $\mathcal{U}_3$ and $\mathcal{U}_4$, but their supports map isomorphically onto the same diagonal $\Delta_{12} : Y_{12} \to Y_{12} \times Y_{12}$.
Similarly, $\mathcal{U}_2 \cap \mathcal{U}_4$ contains δ_{13} whose support maps isomorphically onto Δ_{13}, whereas $\mathcal{U}_5 \cap \mathcal{U}_7$ contains $\tilde{\delta}_{13}$, whose support maps still isomorphically onto Δ_{13}: $\delta_{13} \cap \tilde{\delta}_{13} = \emptyset$.
Finally, $\delta_{23} \subset \mathcal{U}_2 \cap \mathcal{U}_6$, $\delta_{23} \simeq \Delta_{23}$, while $\tilde{\delta}_{23} \subset \mathcal{U}_3 \cap \mathcal{U}_7$, $\tilde{\delta}_{23} \simeq \Delta_{23}$ and $\delta_{23} \cap \tilde{\delta}_{23} = \emptyset$.
The blow-up Z_1 of $X \times_S X$ at $z_1 = 0 = w_1$ is the strict transform of $X \times_S X$ in the blow-up of $\mathbf{A}^6$ along the corresponding linear subvariety. Let $(\tilde{z}_1, \tilde{w}_1)$ be a

couple of homogeneus coordinates. The exceptional divisor, say $E_1^{(1)}$, is locally a $\mathbf{P}^1_{(\tilde{z}_1,\tilde{w}_1)}$–bundle over $\{z_1 = 0 = w_1\}$. Then, the intersection $E_1^{(1)} \cap Z_1$ is locally defined on $E_1^{(1)}$ by $z_2 z_3 \tilde{z}_1 - w_2 w_3 \tilde{w}_1 = 0$. The blow $E_1^{(2)}$ of $\mathbf{P}^1 \times \{z_1 = 0 = w_1\}$ on $\mathbf{P}^1 \times \{z_1 = z_2 = w_1 = w_2 = 0\}$ defines the strict transform of $E_1^{(1)}$ after the second blow-up along $\{z_2 = w_2\}$. Said $E_2^{(2)}$ the exceptional divisor of the second blow-up and $(\tilde{z}_2, \tilde{w}_2)$ another couple of homogeneus coordinates, one has $E_1^{(2)} \cap E_2^{(2)} = \mathbf{P}^1_{(\tilde{z}_1,\tilde{w}_1)} \times \mathbf{P}^1_{(\tilde{z}_2,\tilde{w}_2)} \times \{z_1 = z_2 = w_1 = w_2 = 0\}$. Finally, after the third blowing at $\{z_3 = 0 = w_3\}$ the three exceptional divisors $E_1^{(3)}$, $E_2^{(3)}$ and $E_3^{(3)}$ will intersect the strict transform Z of $X \times_S X$ along the exceptional surface T_{789}. This surface is described by the equation $\tilde{z}_1\tilde{z}_2\tilde{z}_3 - \tilde{w}_1\tilde{w}_2\tilde{w}_3 = 0$ in $E_1^{(3)} \cap E_2^{(3)} \cap E_3^{(3)} = \mathbf{P}^1_{(\tilde{z}_1,\tilde{w}_1)} \times \mathbf{P}^1_{(\tilde{z}_2,\tilde{w}_2)} \times \mathbf{P}^1_{(\tilde{z}_3,\tilde{w}_3)} \times \{z_1 = z_2 = z_3 = w_1 = w_2 = w_3 = 0\} = (\mathbf{P}^1)^3$, $(\tilde{z}_3, \tilde{w}_3)$ being a third couple of homogeneus coordinates. Let consider the projection $T_{789} \to \mathbf{P}^1_{(\tilde{z}_2,\tilde{w}_2)} \times \mathbf{P}^1_{(\tilde{z}_3,\tilde{w}_3)}$. The fiber of this map over a given point in the base $(\mathbf{P}^1)^2$ is defined by a linear equation as $\alpha z_1 - \beta w_1 = 0$. If either α or β (or both) is not zero, then this fiber is reduced to a single point, so the projection map is locally an isomorphism. On the other hand, $\alpha = 0 = \beta$ happens over the two points $(1,0) \times (0,1)$ and $(0,1) \times (1,0)$, where the fiber is a $\mathbf{P}^1$. Since T_{789} is non singular, these two copies of $\mathbf{P}^1$ are Cartier divisors, so by the universal property of blow-ups the map factors through the blow-up Bl of $(\mathbf{P}^1)^2$ at the two points (*i.e.* $T_{789} \to Bl \to (\mathbf{P}^1)^2$). It is easy to see from this description that $T_{789} \simeq Bl$.
It is straighforward to verify from the second table the description of $\tilde{T}^{(5)}$ on each chart $\mathcal{U}_j$ and the statement concerning its support. □

The following result generalizes the description of $[N]$ given in theorem 4.2 for double points degenerations.

THEOREM 5.3. *Let $f : X \to S$ be a semistable degeneration of surfaces as we have considered above. With the same notations as in lemma 5.2, let $\pi : Bl \to \mathbf{P}^1 \times \mathbf{P}^1$ be the morphism definying the blow-up of $\mathbf{P}^1 \times \mathbf{P}^1$ at the points $\{(0,1) \times (1,0)\}$ and $\{(1,0) \times (0,1)\}$, being $Bl \simeq T_{789}$. Let $F_1 = \pi^*(\{pt\} \times \mathbf{P}^1)$ and $F_2 = \pi^*(\mathbf{P}^1 \times \{pt\})$ be the two fundamental fibers and let E_1 and E_2 be the two exceptional divisors of π. The following description of $[N] \in Ker\ \rho^{(4)}$ (cf. (5.1)) holds:*

$$[N] = a_{178}\tilde{\delta}_{12} + a_{279}\tilde{\delta}_{13} + a_{378}\delta_{12} + a_{489}\tilde{\delta}_{23} + a_{579}\delta_{13} + a_{689}\delta_{23} + \Gamma.$$

The 1-cycle $\Gamma \subset Bl$ and the (rational) numbers a's are subject to the following requirements:

$$\Gamma = xF_1 + yF_2 + zE_1 + wE_2, \qquad \textit{with} \quad w = z - 1, \quad x, y, z, w \in \mathbf{Q}$$

$$a_{178} - a_{378} = a_{279} - a_{579} = a_{489} - a_{689} = 1$$

and the relations among them are given by the following set of equalities

$$a_{178} = -w, \qquad a_{279} = -(y+w), \qquad a_{378} = -z,$$
$$a_{489} = x+z, \qquad a_{579} = -(y+z), \qquad a_{689} = x+w.$$

Furthermore, for those degenerations with $N^2 \neq 0$, the class $[N^2] \in E_1^{0,4}(Z) = H^0(\tilde{T}^{(5)}, \mathbf{Q})$ (cf. (5.2)) can be exhibited as:

$$[N^2] = b_{12789}T_{12789} + b_{16789}T_{16789} + b_{24789}T_{24789}$$
$$+ b_{34789}T_{34789} + b_{35789}T_{35789} + b_{56789}T_{56789}.$$

The (rational) numbers b's must satisfy the following equation:

$$-b_{12789} + b_{16789} - b_{24789} + b_{34789} - b_{35789} - b_{56789} = 1.$$

Hence, the induced classes of $[N]$ in $gr_2^L H^4(T, \mathbf{Q})(1)$ and of $[N^2]$ in $gr_0^L H^4(T, \mathbf{Q})$ (i.e. modulo boundary relations via the restriction maps $\rho^{(3)}$ and $\rho^{(5)}$ cf. (1.6)) determine algebraic cocycles of dimension one and zero respectively.

Proof. We will determine $[N]$ as a cocycle making the following square commute (*i.e.* this is the one one has to study for a degeneration of K-3 surfaces of the type mentioned above)

$$\begin{tikzcd} gr_3^L H^2(\tilde{X}^* \times \tilde{X}^*, \mathbf{Q}) \arrow[r, "{[N]\cdot}"] & gr_5^L H^6(\tilde{X}^* \times \tilde{X}^*, \mathbf{Q})(1) = (E_2^{1,5})(1) \arrow[d, "(p_2)_*"] \\ E_2^{-1,3} = gr_3^L H^2(\tilde{X}^*, \mathbf{Q}) \arrow[u, "(p_1)^*"] \arrow[r, "N"] & (gr_1^L H^2(\tilde{X}^*, \mathbf{Q}))(-1) = (E_2^{1,1})(-1) \end{tikzcd}$$

Note that besides the commutativity of the square, one has to impose another condition on $[N]$ in order for it to represent the operator N. That arises from (5.1). Namely, the representative of N in $(E_1^{2,2})(1) = H^2(\tilde{T}^{(3)}, \mathbf{Q})(1)$ must belong to the kernel of the related restriction map $\rho^{(4)}$. This condition was automatically satisfied for double point degenerations since $T^{(4)} = \emptyset$ always in that case. We will explicitly describe a representative $[N]$ of N in $(E_1^{2,2})(1)$ that satisfies the commutativity of the following square

$$(5.3) \qquad \begin{tikzcd} H^1(\tilde{T}^{(2)}, \mathbf{Q})(-1) \arrow[r, "{[N]\cdot}"] & H^5(\tilde{T}^{(2)}, \mathbf{Q})(1) \arrow[d, "(p_2)_*"] \\ H^1(\tilde{Y}^{(2)}, \mathbf{Q})(-1) \arrow[u, "p_1^*"] \arrow[r, equal] & H^1(\tilde{Y}^{(2)}, \mathbf{Q})(-1). \end{tikzcd}$$

With the notations introduced in lemma 5.2 we first remark that the cocycles $[\delta_{ij}] = (\Delta_{ij})_*(1_{Y_{ij}})$ ($i, j = 1, 2, 3$, $i \neq j$), $\Delta_{ij} : Y_{ij} \to Y_{ij} \times Y_{ij}$ being the diagonal embedding, evidently satisfy the cohomological equality

$$(p_2)_*(\Delta_*(1_{Y_{ij}}) \cdot (p_1)^*(v)) = (p_2)_*(\Delta_*\Delta^* p_1^*(v)) = (p_2)_*(\Delta_*(v)) = v$$

for $1_{Y_{ij}} \in H^0(Y_{ij}, \mathbf{Q})$ and any element $v \in H^1(\tilde{Y}^{(2)}, \mathbf{Q})(-1)$. However, since a simple linear combination as $a_{178}\tilde{\delta}_{12} + a_{279}\tilde{\delta}_{13} + a_{378}\delta_{12} + a_{489}\tilde{\delta}_{23} + a_{579}\delta_{13} + a_{689}\delta_{23}$ (the coefficients a's are integers) does not satisfy the requirement of being in the kernel of the restriction map $\rho^{(4)}$ (*cf.* (5.1) and (1.6)), we have to add to the above "diagonal" definition a 1-cocycle $\Gamma \subset T_{789}$, so that the completed linear combination defines an element in $(E_2^{2,2})(1)$ representing N. Notice that since the exceptional surface T_{789} projects down via p_2, onto the triple point P, this modification by Γ does not spoil the commutativity of (5.3), once we have checked it for the partial representative of $[N]$ given in terms of the above diagonals.

The 1-cycle Γ will be described as a combination of the generators $F_1,\ F_2,\ E_1,\ E_2$ of the Neron-Severi group $NS(T_{789})$. First of all, let consider the six curves T_{k789} for $k = 1, \ldots, 6$. They are elements of $\tilde{T}^{(4)}$. We describe them using the generators of $NS(T_{789})$. Because $\pi(T_{1789}) = \{(0,1) \times (1,0)\}$, $T_{1789} = E_2$. Similarly, we have $T_{3789} = E_1$, as $\pi(T_{3789}) = \{(1,0) \times (0,1)\}$. The remaining four curves are described using the projection formula. For example, we know that $\pi(T_{2789}) = (0,1) \times \mathbf{P}^1$ and that $\pi^*((0,1) \times \mathbf{P}^1) = F_1 = E_2 + T_{2789}$. Hence we have $T_{2789} = F_1 - E_2$. With a similar procedure we obtain $T_{4789} = F_2 - E_1$, $T_{5789} = F_1 - E_1$ and $T_{6789} = F_2 - E_2$. The geometry of the intersections among the generators of $NS(T_{789})$ is well known, namely $E_1 \cdot E_2 = E_1 \cdot F_2 = E_1 \cdot F_1 = E_2 \cdot F_1 = E_2 \cdot F_2 = F_1 \cdot F_1 = F_2 \cdot F_2 = 0$, $E_1 \cdot E_1 = -1 = E_2 \cdot E_2$ and $F_1 \cdot F_2 = 1$.

Let $\Gamma = xF_1 + yF_2 + zE_1 + wE_2$ be an element of $NS(T_{789})$, with $x, y, z, w \in \mathbf{Q}$. Then, we must solve

$$[N] = a_{178}\tilde{\delta}_{12} + a_{279}\tilde{\delta}_{13} + a_{378}\delta_{12} + a_{489}\tilde{\delta}_{23} + a_{579}\delta_{13} + a_{689}\delta_{23} + \Gamma$$

for Γ subject to the condition that $[N]$ is in ker $\rho^{(4)}$, for $\rho^{(4)} = \sum_{u=1}^{4}(-1)^{u-1}\rho_u^{(4)}$ (*cf.* (1.6)). For example we have $\rho^{(4)}(a_{178}\tilde{\delta}_{12}) = -a_{178}(\tilde{\delta}_{12} \cdot T_9)$, while $\rho^{(4)}(a_{279}\tilde{\delta}_{13}) = a_{279}(\tilde{\delta}_{13} \cdot T_8)$. Following these rules we obtain the system

$$\begin{aligned}
&a_{178} = \Gamma \cdot T_{1789} = -w, && a_{279} = -\Gamma \cdot T_{2789} = -(y+w)\\
(5.4)\qquad &a_{378} = \Gamma \cdot T_{3789} = -z, && a_{489} = \Gamma \cdot T_{4789} = x+z\\
&a_{579} = -\Gamma \cdot T_{5789} = -(y+z), && a_{689} = \Gamma \cdot T_{6789} = x+w.
\end{aligned}$$

For the standard choice of the orientations of $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ and the numbering of the T_i's setted in lemma 5.2, the local description of the pullbacks $\frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j}$ and $\frac{dw_i}{w_i} \wedge \frac{dw_j}{w_j}$ $(i \neq j,\ i,j = 1,2,3)$ in terms of cohomology classes $v_{T_{ij}}$ and $v_{T_{ijk}}$, is given following the tables shown in the proof of lemma 5.2.

Let $v_{ij} \in H^1(\tilde{Y}^{(2)}, \mathbf{Q})(-1)$, then via the multiplicative rule described in the Appendix (*cf.* the similar calculation done in the proof of theorem 4.2) we obtain

$$\begin{gathered}
[N] \cdot p_1^*(v_{12} + v_{13} + v_{23}) = \\
= [N] \cdot (v_{T_{18}} + v_{T_{78}} + v_{T_{29}} + v_{T_{79}} + v_{T_{49}} + v_{T_{89}}) = \\
= a_{178}g_1(\tilde{\delta}_{12} \cdot v_{T_{18}}) - a_{378}g_7(\delta_{12} \cdot v_{T_{78}}) + a_{279}g_2(\tilde{\delta}_{13} \cdot v_{T_{29}}) - a_{579}g_7(\delta_{13} \cdot v_{T_{79}}) + \\
+ a_{489}(g_4(\tilde{\delta}_{23} \cdot v_{T_{49}}) - a_{689}(g_8(\delta_{23} \cdot v_{T_{89}}) = \\
= a_{178}v_{78}(1) - a_{378}v_{38}(1) + a_{279}v_{79}(1) - a_{579}v_{59}(1) + a_{489}v_{89}(1) - a_{689}v_{69}(1)
\end{gathered}$$

where g_j are the pushforward maps defined in the Appendix. Applying the map $(p_2)_*$ we have

$$(p_2)_*([N]\cdot p_1^*(v_{12}+v_{13}+v_{23})) \\ = (a_{178}-a_{378})v_{12}+(a_{279}-a_{579})v_{13}+(a_{489}-a_{689})v_{23}.$$

The commutativity of the diagram (5.3) is then equivalent to the requirement

$$a_{178}-a_{378}=a_{279}-a_{579}=a_{489}-a_{689}=1 \tag{5.5}$$

The linear system (5.4) may be then read as $z-w=1$. Therefore, any curve $\Gamma=xF_1+yF_2+zE_1+wE_2$ satisfying the condition $z-w=1$ can be used in the description of $[N]\in(E_1^{2,2})(1)$.
The description of $[N^2]$ is similar. For instance, from proposition 2.1 we have

$$[N^2]\in gr_0^L H^4(T,\mathbf{Q})\simeq\frac{H^0(\tilde{T}^{(5)},\mathbf{Q})}{\mathrm{Im}(\rho^{(5)}:H^0(\tilde{T}^{(4)},\mathbf{Q})\to H^0(\tilde{T}^{(5)},\mathbf{Q}))}$$

Via the procedure described in (2.1), $[N^2]$ is then determined in terms of the commutativity of the following square

$$\begin{array}{ccc} gr_4^L H^2(\tilde{X}^*\times\tilde{X}^*,\mathbf{Q}) & \xrightarrow{[N^2]\,\cdot} & gr_4^L H^6(\tilde{X}^*\times\tilde{X}^*,\mathbf{Q})=E_2^{2,4} \\ {\scriptstyle (p_1)^*}\uparrow & & \downarrow{\scriptstyle (p_2)_*} \\ E_2^{-2,4}=gr_4^L H^2(\tilde{X}^*,\mathbf{Q}) & \xrightarrow{N^2} & (gr_0^L H^2(\tilde{X}^*,\mathbf{Q}))(-2)=(E_2^{2,0})(-2). \end{array}$$

The related E_1 description is

$$\begin{array}{ccc} H^0(\tilde{T}^{(3)},\mathbf{Q})(-2) & \xrightarrow{[N^2]\,\cdot} & H^4(\tilde{T}^{(3)},\mathbf{Q}) \\ {\scriptstyle p_1^*}\uparrow & & \downarrow{\scriptstyle (p_2)_*} \\ H^0(\tilde{Y}^{(3)},\mathbf{Q})(-2) & = & H^0(\tilde{Y}^{(3)},\mathbf{Q})(-2). \end{array}$$

The scheme $\tilde{T}^{(5)}$ is the disjoint union of the zero dimensional schemes T_{12789}, T_{16789}, T_{35789} and T_{56789}. Their support map all isomorphically onto the diagonal $\Delta_{123}:Y_{123}\to Y_{123}\times Y_{123}$. With a similar procedure as the one used above to describe $[N]$, we write

$$[N^2]=b_{12789}T_{12789}+b_{16789}T_{16789}+b_{24789}T_{24789} \\ +b_{34789}T_{34789}+b_{35789}T_{35789}+b_{56789}T_{56789}$$

for some integers b's. Imposing the commutativity of the above diagram, by means of the description of the pullbacks $p_1^*(1_{Y_{123}})$ and $p_2^*(1_{Y_{123}})$ as shown in the last table appearing in the proof of lemma 5.2, we finally get the condition

$$-b_{12789}+b_{16789}-b_{24789}+b_{34789}-b_{35789}-b_{56789}=1.$$

□

It is straightforward to verify that both $[N]$ and $[N^2]$ make diagrams like (2.1) commute, for any choice of the indices $*$ and r.

Remark 5.4.

It is easy to verify that the description of $[N]$ and $[N^2]$ given in theorem 5.3 holds also for a normal–crossings degeneration (not semistable) like $f(z_1,\dots,z_n) = z_i^2 z_j$, $i,j \in [1,n]$, $i \neq j$. This applies in particular to the case of normal–crossings degenerations of curves with triple points as described above. The desingularization process of the threefold $X \times_S X$ is obtained via two blow-ups along $z_i = 0 = w_i$ and $z_j = 0 = w_j$ by analogy to what we have done in Remark 4.3. For the description of $[N]$ we also refer to the same Remark.

6. An arithmetic interpretation of the monodromy operator in mixed characteristic

The calculations on the geometric description of $[N^i]$ that we have done in the previous sections only involve the (local) geometry of the special fiber of a degeneration. Hence they equally hold in mixed characteristic also, *i.e.* for a degeneration $f : \mathcal{X} \to \mathrm{Spec}(\Lambda) = S$, where Λ is a Henselian discrete valuation ring with η and v as its generic and closed points respectively. In analogy with the classical case, the model $\mathcal{X}$ is assumed to be proper and the map f is supposed to be flat, smooth over the generic point η and with a normal–crossings special fiber Y defined over the *finite* field $k(v)$ of characteristic $p > 0$. Locally, for the étale topology $\mathcal{X}$ is S-isomorphic to $S[x_1,\dots,x_n]/(x_1^{e_1}\cdots x_k^{e_k} - \pi)$, where π is a uniformizing parameter in Λ and $e_i \in \mathbf{Z}$, $\forall i = 1,\dots,k$. For simplicity, we also assume that Λ is a *finite extension* of $\mathbf{Z}_\ell$ or $\mathbf{Q}_\ell$, where $l \neq p$ is a prime number.
The complex of nearby cycles is then defined as $\mathbf{R}\Psi(\Lambda) := \bar{i}^{-1}\mathbf{R}\bar{j}_*\Lambda$. Here $i : Y \to \mathcal{X}$ (*resp.* $j : \mathcal{X}_\eta \to \mathcal{X}$) is the natural closed (*resp.* open) embedding that one "extends" to the algebraic closure $k(\bar{v})$ of $k(v)$ (*resp.* a separable closure $k(\bar{\eta})$ of $k(\eta)$). Assume that the multiplicities e_i are prime to ℓ and $g.c.d.(e_i,p) = 1$. Then, the wild inertia acts trivially on $\mathbf{R}\Psi(\Lambda)$ and the theory exposed in [16] shows that the nearby cycle complex has an abstract description in the derived category $D^+(Y,\Lambda[\mathbf{Z}_\ell(1)])$ of the abelian category of complexes of sheaves of $\Lambda[\mathbf{Z}_\ell(1)]$–modules on Y, by a complex $A^\bullet_{\mathcal{X},\Lambda}$, supported on Y. $A^\bullet_{\mathcal{X},\Lambda}$ can be interpreted as the analogue of the Steenbrink resolution in the classical case. Therefore, the related study of it goes in parallel with the classical one in equal characteristic zero. We refer to *op.cit.* and [7] (*e.g.* Théorème 3.2) for further detail.
The power maps ($n \in [0,2d]$, $i \geq 0$, $d = \dim\ \mathcal{X}_\eta$) $N^i : H^n(\mathcal{X}_{\bar{\eta}},\Lambda) \to H^n(\mathcal{X}_{\bar{\eta}},\Lambda)(-i)$ define elements

$$N^i \in \bigoplus_{n\geq 0}\Big[H^{2d-n}(\mathcal{X}_{\bar{\eta}},\Lambda)(d) \otimes H^n(\mathcal{X}_{\bar{\eta}},\Lambda)(-i)\Big]^G = \Big[H^{2d}(\mathcal{X}_{\bar{\eta}} \times \mathcal{X}_{\bar{\eta}},\Lambda)(d-i)\Big]^G$$

invariant for the action of the Galois group $G = \mathrm{Gal}(\bar{\eta}/\eta)$ on the cohomology of the product $\mathcal{X}_{\bar{\eta}} \times \mathcal{X}_{\bar{\eta}}$. Assume that $f : \mathcal{X} \to S$ has at worst triple points. Then, the singularities of both $\mathcal{X} \times_S \mathcal{X}$ and $Y \times Y$ can be resolved locally around each singular point by a sequence of at most three blows-up, as we described in details in §§ 2,4,5. The resulting degeneration $h : \mathcal{Z} \to S$ is normal–crossings with special fiber $T = T_1 \cup \ldots \cup T_N$. Let $\mathcal{X}_{\bar\eta} \times \mathcal{X}_{\bar\eta} = \mathcal{Z}_{\bar\eta}$ be its geometric generic fiber. Denote by $\tilde N = 1 \otimes N + N \otimes 1$ the logarithm of the local monodromy on the product degeneration h. Then, the analogue of proposition 2.1 is the following

Proposition 6.1. *Assume the monodromy-weight conjecture on $H^*(\mathcal{Z}_{\bar\eta}, \Lambda)$ and the semisimplicity of the Frobenius on the inertia invariants. Then*

$$\begin{aligned} N^i \in & \left[Ker(\tilde N) \cap H^{2d}(\mathcal{Z}_{\bar\eta}, \Lambda(d-i))\right]^{F=1} \\ & \simeq \left[Ker(\tilde N) \cap (gr^L_{2(d-i)} H^{2d}(\mathcal{Z}_{\bar\eta}, \Lambda))(d-i)\right]^{F=1} \\ & \simeq \left((gr^L_{2(d-i)} H^{2d}(T, \Lambda))(d-i)\right)^{F=1} \\ & \simeq \end{aligned}$$

$$\left[\frac{Ker(\rho^{(2(i+1)} : H^{2(d-i)}(\tilde T^{(2i+1)}, \Lambda)(d-i) \to H^{2(d-i)}(\tilde T^{(2(i+1))}, \Lambda)(d-i))}{Image\ \rho}\right]^{F=1}$$

where F is the geometric Frobenius.

The following result shows the relation of proposition 6.1 with the arithmetic of the degeneration h

Theorem 6.2. *Assume the monodromy-weight conjecture on $\mathcal{Z}_{\bar\eta}$ and the semisimplicity of the action of the frobenius F on $H^*(\mathcal{Z}_{\bar\eta}, \Lambda)^I$. Then, for $i > 0$ and $d = \dim\ \mathcal{X}_{\bar\eta}$*

$$\underset{s=d-i}{ord} \det(Id - FN(v)^{-s} | H^{2d}(\mathcal{Z}_{\bar\eta}, \Lambda)^I) =$$

$$rk\left[\frac{Ker(\rho^{2(i+1)} : H^{2(d-i)}(\tilde T^{(2i+1)}, \Lambda)(d-i) \to H^{2(d-i)}(\tilde T^{(2(i+1))}, \Lambda)(d-i))}{Image\ \rho}\right]^{F=1}.$$

$N(v)$ is the number of elements of the finite residue field $k(v)$.

Proof. cf. [2], theorem 3.5. □

This result explains geometrically the pole of the local factor at v of the L-function $L(H^{2d}(\mathcal{Z}_{\bar\eta}, \mathbf{Q}_\ell), s)$ at the points $s = d-1$ and $s = d-2$, with the presence of the "diagonal" cycles representing the monodromy powers on the strata of T as we previously described.

7. Appendix (by Spencer Bloch)

Our objective in this appendix is to define a multiplication between the total complex of E_1-terms of the Steenbrink spectral sequence and the graded complex

$$H^*(Y^{(\bullet)}),\ \rho = \text{restriction} \tag{7.1}$$

which is the E_1 complex converging to the cohomology of the special fiber Y. We order the components $Y = Y_1 \cup \ldots \cup Y_N$ and write $a_{i_0,\ldots,i_m} \in H^*(Y_{i_0,\ldots,i_m}, \mathbf{Q})$. The E_1-terms of the Steenbrink spectral sequence can be arrayed in a triangular diagram (compare [7], (2.3.8.1)) where each $\bullet$ denotes some $H^*(Y^{(m)}, \mathbf{Q}(n))$.

$$
\begin{array}{ccccccc}
\bullet & & & & & & \\
\uparrow & \nwarrow & & & & & \\
\bullet & \leftarrow & \bullet & & & & \\
\uparrow & \nwarrow & \uparrow & \nwarrow & & & \\
\bullet & \leftarrow & \bullet & \leftarrow & \bullet & & \\
\vdots & \diagup_{\mathrm{wt}0} & \vdots & \diagup_{\mathrm{wt}1} & \vdots & \ddots & \\
\bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet
\end{array}
\tag{7.2}
$$

Here the horizontal arrows are Gysin maps and the vertical arrows are restriction maps. The diagonal arrows are (upto twist) the maps N which, on the level of E_1 are either the identity or 0. The Steenbrink E_1-terms, i.e. the $H^*(Y, \mathrm{gr}_r^L \mathbf{R}\Psi(\mathbf{Q}))$, are direct sums of terms on a NE-SW diagonal, with weight r meeting the "x-axis" at $x = r$. The complex $H^*(Y^{(\bullet)})$ is embedded as the left hand column, and the resulting multiplication on it is the usual (associative) product

$$
a_{i_0,\ldots,i_m} \otimes b_{j_0,\ldots,j_n} \mapsto \begin{cases} 0 & i_m \neq j_0 \\ (a \cdot b)_{i_0,\ldots,i_m,j_1,\ldots,n_n} & i_m = j_0 \end{cases} \tag{7.3}
$$

The bottom row is a quotient complex calculating the homology of the closed fiber $H_*(Y)$ (with appropriate twist). Our multiplication induces an action of the left hand column on the bottom row, which we will show induces the cap product ([14], p. 254)

$$
H^q(Y) \otimes H_n(Y) \to H_{n-q}(Y). \tag{7.4}
$$

This module structure, unifying and extending the classical cocycle calculations for cup and cap product, is of independent interest. Quite possibly it can be extended to a product on the whole E_1-complex, but the daunting sign calculations involved have prevented us from working it out.
We will apply this construction to calculate the product

$$
[N^i] \cdot : \mathbf{H}^*(T, gr_r^L A^\bullet_{Z,\mathbf{Q}}) \to \mathbf{H}^{*+2d}(T, gr_{r-2i}^L A^\bullet_{Z,\mathbf{Q}}(d-i)) \tag{7.5}
$$

from (2.1).
We return to the situation in section 2. In particular, $Z \to X \times_S X$ is a resolution, and $T \subset Z$ is the special fiber, which we assume is a normal crossings divisor. We write $E_1(Z)$ for the Steenbrink spectral sequence associated to the degeneration Z/S.

LEMMA 7.1. *There exists a class $[N^i]$ in $E_1(Z)$ satisfying*

1. $d_1[N^i] = 0$, *and the induced class in* E_2 *is the* i*-th power of the monodromy operator*

$$N^i \in gr^L_{2(d-i)} H^{2d}(\tilde{X}^* \times \tilde{X}^*, \mathbf{Q}(d-i))$$

2. $N([N^i]) = 0$, *i.e. in the diagram* (7.2), $[N^i]$ *lies in the left hand vertical column.*

Proof. We see from proposition (2.1) that the class of N^i is killed by N in $E_2(Z)$. Let M denote the map on E_1 which is inverse to N insofar as possible, i.e. M maps down and to the right in diagram (7.2). M is zero on the bottom line. Let $x \in E_1$ represent N^i in E_2. Then $Nx = d_1 y$. (Here $d_1 = d' + d''$ is the total differential.) Since N commutes with d' and d'', and Nx has no term on the bottom row, it follows that $[N^i] := x - d_1 M y$ is supported on the left hand column, i.e. killed by N. □

Here is some notation. The special fiber will be $Y = \bigcup Y_i$, with $0 \le i \le N$. Write $H^*(Y)$ for cohomology in some fixed constant ring like $\mathbf{Z}$ or $\mathbf{C}$.

$$I = \{i_0, \dots, i_m\}; \quad J = \{j_0, \dots, j_n\} \quad \text{(strictly ordered)}; \quad Y_I = \bigcap_{i_k \in I} Y_{i_k}$$

We will say the pair I, J is *admissible* if

$$\exists p \text{ such that } i_m = \max(I) = j_p \text{ and } \{j_0, \dots, j_p\} \subset I.$$

In this case, write $j_0 = i_{b_0}, \dots, j_{p-1} = i_{b_{p-1}}$. Define

$$a(I, J) := b_0 + \ldots + b_{p-1} + mp.$$

With I, J admissible as above, write

$$J' = \{j_0, \dots, j_p\}; \; J'' = \{j_p, \dots, j_n\}; \; J = J' \cup J''; \; J' \cap J'' = \{j_p\} = \{i_m\}$$

Write

$$I' = J'; \; I'' = (I - J') \cup \{i_m\}; \; I = I' \cup I''; \; \{i_m\} = I' \cap I''$$

Let $K = I'' \cup J''$, and define

$$\theta(I, J) : H^\alpha(Y_I) \otimes H^\beta(Y_J) \to H^{\alpha+\beta+2p}(Y_K) \tag{7.6}$$

$$\theta(I, J)(x \otimes y) := (-1)^{a(I,J)} g_{j_0} \circ \cdots g_{j_{p-1}}(x \cdot y). \tag{7.7}$$

Here $x \cdot y \in H^{\alpha+\beta}(Y_{I \cup J})$, the g_j are Gysin maps, and

$$g_{j_0} \circ \cdots g_{j_{p-1}} : H^*(Y_{I \cup J}) \to H^{*+2p}(Y_{I'' \cup J''}).$$

If the pair I, J is not admissible, define $\theta(I, J) = 0$. Define for I as above and $0 \le k \le N$

$$\sigma(I, k) := \#\{i \in I \mid i < k\}$$

For $k \notin I$ we have the restriction $\mathrm{rest}_k : H^*(Y_I) \to H^*(Y_{I \cup \{k\}})$. Define

$$d' := \sum_{k \notin I} (-1)^{\sigma(I,k)} \mathrm{rest}_k : H^*(Y_I) \to \bigoplus_{k \notin I} H^*(Y_{I \cup \{k\}})$$

Similarly, for $k \in I$ we have the Gysin $g_k : H^*(Y_I) \to H^{*+2}(Y_{I-\{k\}})$. We define

$$d'' = \sum_{k \in I} (-1)^{\sigma(I,k)} g_k : H^*(Y_I) \to \bigoplus_{k \in I} H^{*+2}(Y_{I-\{k\}}).$$

THEOREM 7.2. *With notation as above (I, J not necessarily admissible) the following diagram is commutative:*

$$\begin{array}{ccc} H^*(Y_I) \otimes H^*(Y_J) & \xrightarrow{\theta(I,J)} & H^*(Y_K) \\ \Big\downarrow{\scriptstyle d'\otimes 1 + (-1)^m 1 \otimes (d'+d'')} & & \Big\downarrow{\scriptstyle d'+d''} \\ \bigoplus_{\tilde{I},\tilde{J}} H^*(Y_{\tilde{I}}) \otimes H^*(Y_{\tilde{J}}) & \xrightarrow{\theta(\tilde{I},\tilde{J})} & \bigoplus_{\tilde{K}=\tilde{I}'' \cup \tilde{J}''} H^*(Y_{\tilde{K}}) \end{array}$$

REMARK 7.3.

A priori the theorem does not suffice to determine the desired mapping

$$H^*(Y^\bullet) \otimes E_1 \to E_1 \quad a \otimes b \mapsto a * b$$

because a given $H^*(Y_K)$ occurs many times in the diagram (2.1) (at every point along a NW pointing diagonal). However, if we add the condition that the weights (SW-NE diagonals in (7.2)) should be added, the mapping is defined. It has the property that

$$a * Nb = N(a * b)$$

In particular, there is an induced action on E_1/NE_1 which we identify with the bottom row in (7.2). This simple complex calculates $H_*(Y)$, and the product coincides with the cap product. To see this, one notes that the product is correct for two elements in weight 0, and that if each $H^*(Y_I)$ is replaced by $\mathbf{Z}$, the acyclic model theorem ([14], p. 165) can be applied.

proof of theorem. The proof consists of many separate cases. In each case we will check the sign carefully (this is the delicate part) and omit checking that the maps coincide set-theoretically (which is straightforward).
case: $i_m \notin J$.
In this case, the pair I, J is not admissible, so $\theta(I,J) = 0$. We must show

$$\bigoplus_{\tilde{I},\tilde{J}} \theta(\tilde{I},\tilde{J}) \circ (d' \otimes 1 + (-1)^m 1 \otimes (d' + d'')) = 0. \tag{7.8}$$

We may ignore non-admissible $\tilde{I}, \tilde{J}$. The only way admissible $\tilde{I}, \tilde{J}$ can occur in this situation is if for some $p \geq 0$ we have $j_{p-1} < i_m < j_p$ and $\{j_0, \dots, j_{p-1}\} \subset I$. (If a subscript for j doesn't fall in $\{0, \dots, n\}$, ignore it, i.e. take $j_{-1} = -\infty$, $j_{n+1} = +\infty$.) Assume these conditions hold. Then the pair $I \cup \{j_p\}$, J is admissible and occurs in the image of $d' \otimes 1$. Also the pair I, $J \cup \{i_m\}$ is admissible and occurs in the image of $(-1)^m(1 \otimes d')$. We must show these two contributions cancel. Suppose $j_0 = i_{b_0}, \dots, j_{p-1} = i_{b_{p-1}}$. Then the sign condition we need to verify is

$$\sigma(I,j_p)+b_0+\cdots+b_{p-1}+p(m+1)\equiv \\ 1+m+\sigma(J,i_m)+b_0+\cdots+b_{p-1}+pm \mod (2)$$

This is correct because $\sigma(I,j_p)=m+1$ and $\sigma(J,i_m)=p$.
case: $i_m=j_p\in J,\ \{j_0,\dots,j_{p-1}\}\not\subset J$.
This is the other case where I,J is not admissible, so $\theta(I,J)=0$. To get admissible $\tilde{I},\tilde{J}$ we must have

$$\exists k,\ 0\le k\le p-1 \text{ such that } j_k\notin I,\ \{j_0,\dots,\hat{j}_k,\dots,j_{p-1}\}\subset I.$$

Assume this. Then the pairs $(I\cup\{j_k\},J)$ and $(I,J-\{k_k\})$ are admissible. The first occurs in $\theta(I\cup\{j_k\},J)\circ(d'\otimes 1)$ and the second in $(-1)^m\theta(I,J-\{j_k\})\circ 1\otimes d''$. The necessary sign condition for cancellation is

$$\sigma(I,j_k)+a(I\cup\{j_k\},J)\overset{?}{\equiv} m+1+k+a(I,J-\{j_k\}) \mod (2).$$

To check this sign condition write $j_r=i_{b_r}$ for $0\le r\le p-1,\ r\neq k$. Then

$$a(I,J-\{j_k\})=b_0+\cdots+b_{k-1}+b_{k+1}+\cdots+b_{p-1}+(p-1)m$$
$$a(I\cup\{j_k\},J)=b_0+\cdots+b_{k-1}+\sigma(I,j_k)+(b_{k+1}+1)+ \\ +\cdots+(b_{p-1}+1)+p(m+1).$$

This yields the necessary congruence.
For the rest of the proof we assume I,J is admissible. We examine the various terms in (7.8) and show they occur with the same signs in $(d'+d'')\circ\theta(I,J)$. We first consider terms coming from $d'\otimes 1$, so the target is labelled by $\tilde{I}=I\cup\{k\},\ \tilde{J}=J$.
case: $k<i_m=j_p$. In this case, since $j_p=\min J''$ and $k\notin I\supset J'$, we have $k\notin J$. The pair $\tilde{I}=I\cup\{k\},\ J$ is admissible with $\tilde{I}''=I''\cup\{k\}$ and the same decomposition $J=J'\cup J''$. Let $\tilde{K}=\tilde{I}''\cup J''=K\cup\{k\}$. Since $k<j_p=\min J''$, we have

$$\sigma(K,k)=\sigma(I'',k)=\sigma(I,k)-\sigma(J',k)$$

What we must show, therefore, is that

$$a(I,J)-a(\tilde{I},J)\equiv\sigma(J',k) \mod (2)$$

Write

$$\tilde{I}=\{\tilde{i}_0,\dots,\tilde{i}_{m+1}\};\ j_0=\tilde{i}_{\tilde{b}_0},\dots,j_{p-1}=\tilde{i}_{\tilde{b}_{p-1}};$$
$$a(\tilde{I},J)=\tilde{b}_0+\cdots+\tilde{b}_{p-1}+(m+1)p$$
$$I=\{i_0,\dots,i_m\};\ j_r=i_{b_r},0\le r\le p-1$$
$$a(I,J)=b_0+\cdots+b_{p-1}+mp$$

where

$$\tilde{b}_\ell=\begin{cases} b_\ell & i_{b_\ell}<k \\ b_\ell+1 & i_{b_\ell}>k\end{cases}$$

Thus

$$a(\tilde I, J) - a(I,J) = p - \#\{j \in J' - \{j_p\} \mid j > k\} = \\ \#\{j \in J' \mid j < k\} = \sigma(J', k).$$

This is the desired congruence.
We continue to consider the contribution of $d' \otimes 1$ with I, J admissible.
case: $k > i_m,\ k \neq j_{p+1}$.
In this case $I \cup \{k\}, J$ is not admissible so $\theta(\tilde I, J) = 0$.
case: $k = j_{p+1}$.
Here $\tilde I := I \cup \{k\}, \tilde J := J$ is admissible with

$$\tilde J' = \{j_0, \dots, j_{p+1}\} = J' \cup \{k\} = J' \cup \{j_{p+1}\}$$
$$\tilde J'' = \{j_{p+1}, \dots, j_n\} = J'' - \{j_p\};\ \tilde K = \tilde I'' \cup \tilde J'' = K - \{j_p\}$$

Note in this case $k > i_m$ so $\sigma(I,k) = m+1$. The claim is here that the diagram

$$\begin{array}{ccc} H^*(Y_I) \otimes H^*(Y_J) & \xrightarrow{\theta(I,J)} & H^*(Y_K) \\ \Big\downarrow{\scriptstyle (-1)^{m+1}\mathrm{rest}.\otimes 1} & & \Big\downarrow{\scriptstyle (-1)^{\sigma(K,j_p)}\mathrm{Gysin}_{j_p}} \\ H^*(Y_{\tilde I}) \otimes H^*(Y_J) & \xrightarrow{\theta(\tilde I,J)} & H^*(Y_{\tilde K}) \end{array}$$

commutes. Note that the right hand vertical arrow (with the sign) is part of $1 \otimes d''$. To verify the signs we need

$$a(I,J) + \sigma(K, j_p) \equiv m + 1 + a(\tilde I, J).$$

Since $K = I'' \cup J''$ and $k = \max(I'') = \min(J'')$ it is clear that

$$\sigma(K, j_p) = \#I'' - 1 = m - p.$$

Also $j_p = i_m$ so with the usual notation $j_r = i_{b_r}$ we get

$$a(\tilde I, J) = b_0 + \cdots + b_{p-1} + m + (m+1)(p+1).$$

Now the desired congruence becomes

$$b_0 + \cdots + b_{p-1} + pm + m - p \equiv b_0 + \cdots + b_{p-1} + m + (m+1)(p+1) + m + 1$$

This is correct.
We now consider terms occurring in $(-1)^m(1 \otimes d')$ on the left of the diagram in the statement of the theorem. We assume given $k \notin J$.
case: $k > j_p$.
Note in this case $k \notin I$. Taking $\tilde J = J \cup \{k\},\ \tilde K = K \cup \{k\}$, I claim the diagram below is commutative:

$$\begin{array}{ccc} H^*(Y_I) \otimes H^*(Y_J) & \xrightarrow{\theta(I,J)} & H^*(Y_K) \\ \Big\downarrow{\scriptstyle (-1)^{m+\sigma(J,k)}1\otimes\mathrm{rest}} & & \Big\downarrow{\scriptstyle (-1)^{\sigma(K,k)}\mathrm{rest}} \\ H^*(Y_I) \otimes H^*(Y_{\tilde J}) & \xrightarrow{\theta(I,\tilde J)} & H^*(K_{\tilde K}) \end{array}$$

(In other words, the contribution in this case is to d' on the right.) Set

$$\tilde{J} = J' \cup \tilde{J}''; \ \tilde{J}'' = J'' \cup \{k\}; \ K = I'' \cup J''.$$

We have

$$\begin{gathered} a(I,J) = a(I,\tilde{J}) \\ \sigma(J,k) = \sigma(J'',k) + p + 1 \\ \sigma(K,k) = \sigma(J'',k) + \#I'' = \sigma(J'',k) + m + 1 - p \end{gathered}$$

It follows that

$$m + \sigma(J,k) + a(I,\tilde{J}) \equiv \sigma(K,k) + a(I,J) \mod (2)$$

which is the desired sign relation in this case.
case: $k < j_p, \ k \notin I$.
In this case, the pair $I, J \cup \{k\}$ is not admissible, so the contribution is zero.
case: $k < j_p, \ k \in I$.

In this case the pair $I, \tilde{J}$ is admissible with

$$\begin{gathered} \tilde{J} := J \cup \{k\} = \tilde{J}' \cup J''; \ \tilde{J}' = J' \cup \{k\} \\ I = \tilde{I} = \tilde{J}' \cup \tilde{I}''; \ \tilde{I}'' = I'' - \{k\}; \ \tilde{K} = K - \{k\} = \tilde{I}'' \cup J'' \end{gathered}$$

The term in question contributes to d'' on the right, and the diagram which commutes is:

$$\begin{array}{ccc} H^*(Y_I) \otimes H^*(Y_J) & \xrightarrow{\theta(I,J)} & H^*(Y_K) \\ \Big\downarrow {\scriptstyle (-1)^{m+\sigma(J,k)}\mathrm{rest}} & & \Big\downarrow {\scriptstyle (-1)^{\sigma(K,k)}\mathrm{Gysin}_k} \\ H^*(Y_I) \otimes H^*(Y_{\tilde{J}}) & \xrightarrow{\theta(I,\tilde{J})} & H^*(Y_{\tilde{K}}) \end{array}$$

The signs will be correct if

$$a(I,J) + \sigma(K,k) \equiv m + \sigma(J,k) + \theta(I,\tilde{J}) \mod (2)$$

Write $\tilde{J} = \{\tilde{j}_0, \dots, \tilde{j}_{m+1}\}$ and $\tilde{j}_r = i_{\tilde{b}_r}, \ r \leq p$. The desired congruence reads

$$b_0 + \cdots + b_{p-1} + mp + \sigma(K,k) \stackrel{?}{\equiv} m + \sigma(J,k) + \tilde{b}_0 + \cdots + \tilde{b}_p + (p+1)m$$

We have

$$\tilde{b}_\ell = \begin{cases} b_\ell & \ell < \sigma(J',k) \\ \sigma(I,k) & \ell = \sigma(J',k) \\ b_{\ell-1} & \ell > \sigma(J',k) \end{cases}$$

The condition becomes

$$\sigma(K,k) \stackrel{?}{\equiv} \sigma(J,k) + \sigma(I,k) = \sigma(J',k) + \sigma(J',k) + \sigma(I'',k),$$

which is true.
Finally we consider terms coming from $(-1)^m(1 \otimes d'')$ in the lefthand vertical

arrow in the diagram of the theorem. In what follows $j \in J$.
case: $j \in J''$, $j \neq j_p$. Define

$$\tilde{J} = J - \{j\};\ K = I'' \cup J'';\ \tilde{K} = K - \{j\} = I'' \cup \tilde{J}''.$$

The diagram which commutes is:

$$\begin{array}{ccc} H^*(Y_I) \otimes H^*(Y_J) & \xrightarrow{\theta(I,J)} & H^*(Y_K) \\ \Big\downarrow{\scriptstyle 1\otimes(-1)^{m+\sigma(J,j)}\mathrm{Gysin}_j} & & \Big\downarrow{\scriptstyle (-1)^{\sigma(K,j)}\mathrm{Gysin}_j} \\ H^*(Y_I) \otimes H^*(Y_{\tilde{J}}) & \xrightarrow{\theta(I,\tilde{J})} & H^*(Y_{\tilde{K}}) \end{array}$$

The sign condition to be checked is

$$m + \sigma(J,j) + a(I,\tilde{J}) \stackrel{?}{\equiv} a(I,J) + \sigma(K,j) \mod (2).$$

Our conditions imply $j > j_p$ so $a(I,J) = a(I,\tilde{J})$. Also,

$$\#I'' + \#J' = m + 2 \equiv m \mod (2),$$

so

$$\begin{aligned} \sigma(K,j) &= \#I'' + \sigma(J'',j) - 1 \equiv m + \#J' + \sigma(J'',j) - 1 \\ \sigma(J,j) &= \sigma(J',j) + \sigma(J'',j) - 1 = \#J' - 1 + \sigma(J'',j). \end{aligned}$$

This is the desired condition.
case: $j = j_p$.
In this case, $I, J - \{j\}$ is not admissible, so we get no contribution.
case: $j \in J,\ j < j_p$.
In this case, $j \in J',\ j \neq j_p$. Set

$$\begin{aligned} \tilde{J} &= J - \{j\};\ \tilde{J}' = J' - \{j\}; \tilde{J}'' = J'' \\ \tilde{I} &= I;\ \tilde{I}'' = I'' \cup \{j\};\ I = \tilde{I} = \tilde{J}' \cup \tilde{I}'' \\ K &= I'' \cup J'';\ \tilde{K} = \tilde{I}'' \cup \tilde{J}'' = K - \{j\}. \end{aligned}$$

The sign condition to show we gat a contribution to d'' on the right is

$$a(I,J) + \sigma(K,j) \stackrel{?}{\equiv} m + \sigma(J,j) + a(\tilde{I},\tilde{J}) \mod (2).$$

Writing $j = j_\ell = i_{b_\ell}$ the condition becomes

$$\begin{aligned} b_0 + \cdots + b_{p-1} + mp + \sigma(K,j) &\stackrel{?}{\equiv} \\ & b_0 + \cdots + \hat{b}_\ell + b_{\ell+1} + \cdots + b_{p-1} + m(p-1) + m + \sigma(J,j) \end{aligned}$$

This is true because

$$\begin{aligned} b_\ell = \sigma(I,j) &= \sigma(I'',j) + \sigma(J',j) \\ \sigma(K,j) = \sigma(I'',j); &\quad \sigma(J,j) = \sigma(J',j). \end{aligned}$$

The proof is completed by checking that all the terms on the right in the theorem (i.e. in $d' + d''$) are accounted for precisely once in the above enumeration of cases. $\square$

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