Singularities, Double Points, Controlled Topology and Chain Duality

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Abstract. A manifold is a Poincaré duality space without singularities. McCrory obtained a homological criterion of a global nature for deciding if a polyhedral Poincaré duality space is a homology manifold, i.e. if the singularities are homologically inessential. A homeomorphism of manifolds is a degree 1 map without double points. In this paper combinatorially controlled topology and chain complex methods are used to provide a homological criterion of a global nature for deciding if a degree 1 map of polyhedral homology manifolds has acyclic point inverses, i.e. if the double points are homologically inessential.

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Introduction

A *chain duality* on an additive category $\mathbb{A}$ is an involution on the derived category of finite chain complexes in $\mathbb{A}$ and chain homotopy classes of chain maps. The precise definition will be recalled in §1. Chain duality was introduced in Ranicki [29] in order to construct the algebraic surgery exact sequence of a space X

$$\cdots \to H_n(X;\mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X;\mathbb{L}_\bullet) \to \dots$$

with $L_*(\mathbb{Z}[\pi_1(X)])$ the surgery obstruction groups of Wall [43], and A the assembly map. Here, $\mathbb{L}_\bullet$ is the 1-connective simply-connected algebraic surgery spectrum of $\mathbb{Z}$, and the generalized homology groups are the (1-connective) L-theory of the X-controlled $\mathbb{Z}$-module category $\mathbb{A}(\mathbb{Z},X)$ of Ranicki and Weiss [34]

$$H_*(X;\mathbb{L}_\bullet) \;=\; L_*(\mathbb{A}(\mathbb{Z},X)) \ .$$

The algebraic surgery exact sequence was used in [29, Chapter 17] to give algebraic formulations of the obstructions to the two basic questions of Browder-Novikov-Sullivan-Wall surgery theory :

A1. Is an n-dimensional Poincaré duality space X homotopy equivalent to an n-dimensional manifold?

A2. Is a homotopy equivalence $f : M \to N$ of n-dimensional manifolds homotopic to a homeomorphism?

The following are the basic questions of Chapman-Ferry-Quinn controlled topology :

B1. How close is an n-dimensional controlled Poincaré duality space X to being an n-dimensional manifold?

B2. How close is a controlled homotopy equivalence $f : M \to N$ of n-dimensional manifolds to being a homeomorphism?

Here is a very crude approximation to controlled topology. Given a topological space X define an *X-controlled space* to be a space M equipped with a map $p_M : M \to X$. A *map* of X-controlled spaces $f : M \to N$ is a map of the underlying spaces such that there is defined a commutative diagram

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "p_M"'] & & N \arrow[dl, "p_N"] \\ & X & \end{tikzcd}$$

The map f is an *X-controlled homology equivalence* if the restrictions

$$f| : p_M^{-1}(x) \to p_N^{-1}(x) \quad (x \in X)$$

induce isomorphisms

$$(f|)_* : H_*(p_M^{-1}(x)) \cong H_*(p_N^{-1}(x)) \ .$$

An *n-dimensional X-controlled Poincaré space* is an X-controlled space N with Lefschetz duality isomorphisms

$$H^{n-*}(N, N\backslash p_N^{-1}(x)) \cong H_*(p_N^{-1}(x)) \quad (x \in X) \ .$$

There are two extreme cases :

- If $X = \{\text{pt.}\}$ then :
  - an X-controlled homology equivalence $f : M \to N$ of X-controlled spaces is just a homology equivalence, with

$$f_* : H_*(M) \cong H_*(N) \ ,$$

  - an n-dimensional X-controlled Poincaré space N is just an n-dimensional Poincaré space, with

$$H^{n-*}(N) \cong H_*(N) \ .$$

- If $p_N = 1 : N \to N = X$ then :
  - an N-controlled homology equivalence $f : M \to N$ of N-controlled spaces is just a map with acyclic point inverses, with

$$(f|)_* : H_*(f^{-1}(x)) \cong H_*(\{x\}) \ \ (x \in N) \ ,$$

  - an n-dimensional N-controlled Poincaré space N is just an n-dimensional homology manifold, with

$$H^{n-*}(N, N\backslash\{x\}) \cong H_*(\{x\}) \ \ (x \in N) \ .$$

In a more sophisticated exposition of controlled topology X would be a metric space, and the condition $p_M = p_N f$ in the definition of an X-controlled map would be weakened to

$$d(p_M(x), p_N f(x)) < \epsilon \ \ (x \in M)$$

for some $\epsilon > 0$. In principle, Quinn [24] characterized ANR homology manifolds X as metrically X-controlled Poincaré duality spaces. (See Ranicki and Yamasaki [37] for a preliminary account of the metrically controlled L-theory required for the details of the characterization).

The original development of controlled topology for metric spaces involved quite complicated controlled algebra, starting with Connell and Hollingsworth [5]. However, these questions will only be considered here in the combinatorial context of compact polyhedra, homology manifolds and PL maps, for which the controlled algebra is much easier :

C1. Is a polyhedral n-dimensional Poincaré duality space X an n-dimensional homology manifold?

C2. Does a degree 1 PL map $f : M \to N$ of polyhedral n-dimensional homology manifolds have acyclic point inverses?

McCrory [17] obtained a homological obstruction for C1 (under slightly different hypotheses), which was interpreted in Ranicki [29, 8.5] in terms of the chain duality on the X-controlled $\mathbb{Z}$-module category $\mathbb{A}(\mathbb{Z}, X)$. The obstruction is the image in $H^n(X \times X \backslash \Delta_X)$ of the Poincaré dual in $H^n(X \times X)$ of the diagonal class $\Delta_*[X] \in H_n(X \times X)$. The obstruction vanishes if and only if X is an n-dimensional homology manifold, if and only if the $\mathbb{Z}$-module Poincaré duality chain equivalence

$$[X] \cap - : \Delta(X)^{n-*} \to \Delta(X')$$

is an X-controlled chain equivalence.

The main results of this paper are the following homological obstructions for C1 and C2.

Theorem A. An n-dimensional polyhedral Poincaré complex X is an n-dimensional homology manifold if and only if there is defined a Lefschetz duality isomorphism

$$H^n(X \times X, \Delta_X) \cong H_n(X \times X \backslash \Delta_X) ,$$

with

$$\Delta_X \ = \ \{(x,x) \in X \times X \,|\, x \in X\}$$

the diagonal of X.

Theorem B. A simplicial map $f : M \to N$ of n-dimensional polyhedral homology manifolds has acyclic point inverses if and only if it has degree 1

$$f_*[M] \ = \ [N] \in H_n(N)$$

and

$$H_n((f \times f)^{-1}\Delta_N, \Delta_M) \ = \ 0 ,$$

with

$$(f \times f)^{-1}\Delta_N \ = \ \{(x,y) \in M \times M \,|\, f(x) = f(y) \in N\}$$

the double point set of f.

Theorems A, B are proved in §§6,7 respectively, appearing as Theorem 6.13 and Corollary 7.5.

Here are the contents of the rest of the paper.

In §8 the obstructions of Theorems A, B are interpreted using bundles, specifically the Spivak normal bundle of a Poincaré complex and the tangent topological block bundle of a homology manifold.

In §9 the obstructions of Theorems A, B are related to the 'total surgery obstruction' $s(X) \in S_n(X)$ of Ranicki [29] for the existence of a topological manifold in the homotopy type of a Poincaré space.

In §10 chain duality is used to develop a combinatorial version of the controlled surgery theory.

In §11 some standard results on intersections and self-intersections of manifolds are interpreted in terms of the chain duality.

In §12 (resp. §13) the controlled topology point of view on Whitehead torsion (resp. fibrations) is adapted to the combinatorially controlled chain homotopy theory.

In §14 some standard results in high-dimensional knot theory are interpreted in terms of the chain duality.

In this paper only *oriented* polyhedral Poincaré complexes and homology manifolds will be considered, and orientation-preserving PL maps between them.

A preliminary version of some of the material in this paper appeared in Ranicki [32].

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1. CHAIN DUALITY

Let $\mathbb{A}$ be an additive category, and let $\mathbb{B}(\mathbb{A})$ be the additive category of finite chain complexes in $\mathbb{A}$ and chain maps. A contravariant additive functor $T : \mathbb{A} \to \mathbb{B}(\mathbb{A})$ extends to $T : \mathbb{B}(\mathbb{A}) \to \mathbb{B}(\mathbb{A})$ by defining $T(C)$ for a chain complex C to be the total of a double complex, with

$$T(C)_n = \sum_{p+q=n} T(C_{-p})_q \ .$$

Definition 1.1 (Ranicki [29, 1.1])
A *chain duality* (T, e) on $\mathbb{A}$ is a contravariant additive functor $T : \mathbb{A} \to \mathbb{B}(\mathbb{A})$, together with a natural transformation $e : T^2 \to 1$ such that for each object A in $\mathbb{A}$:

- $e(T(A)) \,.\, T(e(A)) \;=\; 1 : T(A) \;\to\; T(A)\,,$
- $e(A) \colon T^2(A) \to A$ is a chain equivalence.

Chain duality has the following properties:

- The dual of an object A is a chain complex $T(A)$.
- The dual of a chain complex C is a chain complex $T(C)$.

Example 1.2 (i) An *involution* (T, e) on an additive category $\mathbb{A}$ is a chain duality such that $T(A)$ is a 0-dimensional chain complex (= object) for each object A in $\mathbb{A}$, with $e(A) : T^2(A) \to A$ an isomorphism.
(ii) An involution $R \to R; r \mapsto \overline{r}$ on a ring R determines the involution (T, e) on the additive category $\mathbb{A}(R)$ of f.g. free left R-modules with :

- $T(A) = \mathrm{Hom}_R(A, R)$
- $R \times T(A) \to T(A)\ ;\ (r, f) \mapsto (x \mapsto f(x)\overline{r})$
- $e(A)^{-1} : A \to T^2(A)\ ;\ x \mapsto (f \mapsto \overline{f(x)})\ .$

2. Simplicially controlled algebra

Let X be a simplicial complex, and let R be a commutative ring.

Definition 2.1 (Ranicki and Weiss [34])
(i) An (R, X)-*module* is a finitely generated free R-module A with direct sum decomposition

$$A\ =\ \sum_{\sigma \in X} A(\sigma)\ ,$$

such that each $A(\sigma)$ is a f.g. free R-module.
(ii) An (R, X)-module *morphism* $f : A \to B$ is an R-module morphism such that for each $\sigma \in X$

$$f(A(\sigma)) \subseteq \sum_{\tau \geq \sigma} B(\tau)\ .$$

Write the components of f as $f(\tau, \sigma) : A(\sigma) \to B(\tau)$.

Let $\mathbb{A}(R)$ be the additive category of f.g. free R-modules, and let $\mathbb{A}(R, X)$ be the additive category of (R, X)-modules. Regard the simplicial complex X as the category with objects the simplexes $\sigma \in X$, and morphisms the face inclusions $\sigma \leq \tau$. An (R, X)-module $A = \sum\limits_{\sigma \in X} A(\sigma)$ determines a contravariant functor

$$[A] : X \to \mathbb{A}(R)\ ;\ \sigma \mapsto [A][\sigma] = \sum_{\tau \geq \sigma} A(\tau)\ .$$

The (R, X)-module category $\mathbb{A}(R, X)$ is thus a full subcategory of the category of contravariant functors $X \to \mathbb{A}(R)$.

Proposition 2.2 (Ranicki and Weiss [34, 2.9])
The following conditions on a chain map $f : C \to D$ of finite chain complexes in $\mathbb{A}(R, X)$ are equivalent :
(i) *f is a chain equivalence,*
(ii) *the R-module chain maps*

$$f(\sigma, \sigma) : C(\sigma)\ \to\ D(\sigma)\quad (\sigma \in X)$$

are chain equivalences,
(iii) *the* R*-module chain maps*

$$[f][\sigma] : [C][\sigma] \ \to \ [D][\sigma] \quad (\sigma \in X)$$

are chain equivalences.

3. SIMPLICIALLY CONTROLLED TOPOLOGY

The *barycentric subdivision* X' of a simplicial complex X is the simplicial complex with the same polyhedron

$$|X'| \ = \ |X|$$

and one n-simplex $\widehat{\sigma}_0\widehat{\sigma}_1 \dots \widehat{\sigma}_n$ for each sequence of simplexes in X

$$\sigma_0 < \sigma_1 < \dots < \sigma_n \ .$$

The *dual cell* of a simplex $\sigma \in X$ is the contractible subcomplex

$$D(\sigma, X) \ = \ \{\widehat{\sigma}_0\widehat{\sigma}_1 \dots \widehat{\sigma}_n \,|\, \sigma \leq \sigma_0\} \subseteq X' \ ,$$

with boundary

$$\partial D(\sigma, X) \ = \ \{\widehat{\sigma}_0\widehat{\sigma}_1 \dots \widehat{\sigma}_n \,|\, \sigma < \sigma_0\} \subseteq D(\sigma, X) \ .$$

Definition 3.1 (i) An X*-controlled simplicial complex* (M, p_M) is a finite simplicial complex M with a simplicial map $p_M : M \to X'$, the *control map.*
(ii) A *map* $f : (M, p_M) \to (N, p_N)$ of X-controlled simplicial complexes is a simplicial map $f : M \to N$ such that $p_M = p_N f : M \to X'$.

In practice, (M, p_M) will be abbreviated to M.

Definition 3.2 The (R, X)*-module chain complex* $\Delta(M; R)$ of an X-controlled simplicial complex M is the R-coefficient simplicial chain complex of M with

$$\Delta(M; R)(\sigma) \ = \ \Delta(p_M^{-1} D(\sigma, X), p_M^{-1} \partial D(\sigma, X); R) \ .$$

and

$$\begin{aligned} [\Delta(M; R)_r][\sigma] \ &= \ \sum_{\tau \geq \sigma} \Delta(M; R)(\tau)_r \\ &= \ \Delta(p_M^{-1} D(\sigma, X); R)_r \quad (r \in \mathbb{Z}, \sigma \in X) \ . \end{aligned}$$

A map of X-controlled simplicial complexes $f : M \to N$ induces an (R, X)-module chain map

$$f : \Delta(M; R) \to \Delta(N; R) \ .$$

Definition 3.3 A map of X-controlled simplicial complexes $f : M \to N$ is an *X-controlled R-homology equivalence* if the restrictions

$$f| : p_M^{-1}D(\sigma,X) \to p_N^{-1}D(\sigma,X) \quad (\sigma \in X)$$

induce isomorphisms in R-homology

$$(f|)_* : H_*(p_M^{-1}D(\sigma,X);R) \cong H_*(p_N^{-1}D(\sigma,X);R) \quad (\sigma \in X) \ .$$

Proposition 3.4 A map of X-controlled simplicial complexes $f : M \to N$ is an X-controlled R-homology equivalence if and only if the induced (R,X)-module chain map $f : \Delta(M;R) \to \Delta(N;R)$ is a chain equivalence.
Proof Immediate from 2.2. □

Proposition 3.5 (i) *If $X = \{\mathrm{pt.}\}$ an X-controlled map $f : M \to N$ is an X-controlled R-homology equivalence if and only if f induces R-homology isomorphisms*

$$f_* : H_*(M;R) \cong H_*(N;R) \ .$$

(ii) *If $X = N$ an X-controlled map $f : M \to N$ is an X-controlled R-homology equivalence if and only if f has R-acyclic point inverses*

$$H_*(f^{-1}(x);R) \cong H_*(\{x\};R) \quad (x \in |X|) \ .$$

Proof (i) Immediate from 3.4, since a chain map of finite free R-module chain complexes is a chain equivalence if and only if it induces isomorphisms in homology.
(ii) Immediate from 3.4, since every point $x \in |X|$ is in the interior $D(\sigma,X)\backslash\partial D(\sigma,X)$ of a unique dual cell $D(\sigma,X)$, and

$$H_*(\{x\};R) \cong H_*(D(\sigma,X);R) \quad , \quad H_*(f^{-1}(x);R) \cong H_*(f^{-1}D(\sigma,X);R) \ .$$

□

Here is another way in which (R,X)-module chain complexes arise:

Definition 3.6 (Ranicki [29, 4.2])
Let $\Delta^{-*}(X;R)$ be the (R,X)-module chain complex defined by

$$\begin{aligned} \Delta^{-*}(X;R) &= \mathrm{Hom}_R(\Delta(X;R),R)_{-*} \ , \\ \Delta^{-*}(X;R)_r(\sigma) &= \begin{cases} R & \text{if } r = -|\sigma| \\ 0 & \text{otherwise.} \end{cases} \quad (r \in \mathbb{Z}, \sigma \in X) \ . \end{aligned}$$

As an R-module chain complex $\Delta^{-*}(X;R)$ is just the R-coefficient simplicial cochain complex of X regraded to be a chain complex.

4. THE (R,X)-MODULE CHAIN DUALITY

Proposition 4.1 (Ranicki [29, 5.1])
The additive category $\mathbb{A}(R,X)$ of (R,X)-modules has a chain duality (T,e) with the dual of an (R,X)-module A the (R,X)-module chain complex

$$T(A) \ = \ \mathrm{Hom}_R(\mathrm{Hom}_{(R,X)}(\Delta^{-*}(X;R),A),R)$$

with

- $TA(\sigma) \ = \ [A][\sigma]^{|\sigma|-*}$,
- $T(A)_r(\sigma) \ = \ \begin{cases} \sum\limits_{\tau\geq\sigma} \mathrm{Hom}_R(A(\tau),R) & \text{if } r=-|\sigma| \\ 0 & \text{if } r\neq -|\sigma| \ . \end{cases}$

The chain duality is such that

$$T(C) \simeq_R \mathrm{Hom}_{(R,X)}(C,\Delta(X';R))^{-*} \simeq_R \mathrm{Hom}_R(C,R)^{-*}$$

for any finite (R,X)-module chain complex C.

Definition 4.2 Given an X-controlled simplicial complex M let

$$\Delta(M;R)^{-*} \ = \ T(\Delta(M;R))$$

be the (R,X)-module chain complex dual to $\Delta(M;R)$.

Note that there is defined an R-module chain equivalence

$$\Delta(M;R)^{-*} \simeq_R \mathrm{Hom}_R(\Delta(M;R),R)^{-*} \ ,$$

with $\mathrm{Hom}_R(\Delta(M;R),R)^{-*}$ the simplicial R-coefficient cochain complex of M regraded to be a chain complex, and note also that

$$\Delta(M;R)^{-*}(\sigma)_r \ = \ \mathrm{Hom}_R(\Delta(p_M^{-1}D(\sigma,X);R)_{-r+|\sigma|},R) \ \ (r\in\mathbb{Z},\sigma\in X) \ .$$

A map of X-controlled simplicial complexes $f : M \to N$ induces an (R,X)-module chain map

$$f^* : \Delta(N;R)^{-*} \to \Delta(M;R)^{-*} \ .$$

The (R,X)-module chain complex $\Delta^{-*}(X;R)$ of 3.6 and the (R,X)-module chain complex $\Delta(X;R)^{-*}$ of 4.2 (with $p_M = 1 : M \to M = X'$) are related by the (R,X)-module chain equivalence

$$\Delta^{-*}(X;R) \simeq_{(R,X)} \Delta(X;R)^{-*}$$

induced by the projections $\Delta(D(\sigma,X);R) \to R$.

5. Products

Definition 5.1 The *product* of X-controlled simplicial complexes M, N is the pullback X-controlled simplicial complex

$$M \times_X N \;=\; \{(x,y) \in M \times N \,|\, p_M(x) = p_N(y) \in X\}$$

with control map

$$M \times_X N \to X \;;\; (x,y) \mapsto p_M(x) = p_N(y) \;.$$

(Strictly speaking, this only defines a polyhedron $M \times_X N$).

Definition 5.2 The *product* of (R,X)-modules A, B is the (R,X)-module

$$A \otimes_{(R,X)} B \;=\; \sum_{\lambda,\mu \in X, \lambda\cap\mu \neq \emptyset} A(\lambda) \otimes_R B(\mu) \subseteq A \otimes_R B$$

with

$$(A \otimes_{(R,X)} B)(\sigma) \;=\; \sum_{\lambda,\mu \in X, \lambda\cap\mu = \sigma} A(\lambda) \otimes_R B(\mu) \quad (\sigma \in X) \;.$$

Recall the following properties of the products in 5.1,5.2 from Ranicki [29, Chapter 7]. (The product $A \otimes_{(R,X)} B$ was denoted by $A \boxtimes_R B$ in [29, 7.1]).

Proposition 5.3 (i) *For any (R,X)-module chain complexes C, D*

- $C \otimes_{(R,X)} \Delta(X';R) \simeq_{(R,X)} C$,
- $TC \otimes_{(R,X)} D \simeq_R \mathrm{Hom}_{(R,X)}(C,D)$.

(ii) *For any X-controlled simplicial complexes M, N*

- $\Delta(M;R) \otimes_{(R,X)} \Delta(N;R) \simeq_{(R,X)} \Delta(M \times_X N;R)$,
- $\Delta(M;R)^{-*} \otimes_{(R,X)} \Delta(N;R)^{-*}$

  $\simeq_R \;\; Hom_R(\Delta(M \times N, M \times N \backslash M \times_X N;R),R)_{-*}$,

(iii) *The Alexander-Whitney diagonal chain approximation of the barycentric subdivision X' of X is an R-module chain map*

$$\Delta : \Delta(X';R) \to \Delta(X';R) \otimes_R \Delta(X';R) \;;\; (\widehat{x}_0 \dots \widehat{x}_n) \mapsto \sum_{i=0}^{n} (\widehat{x}_0 \dots \widehat{x}_i) \otimes (\widehat{x}_i \dots \widehat{x}_n)$$

which is the composite of an (R,X)-module chain equivalence

$$\Delta(X';R) \simeq_{(R,X)} \Delta(X';R) \otimes_{(R,X)} \Delta(X';R)$$

and the inclusion

$$\Delta(X';R) \otimes_{(R,X)} \Delta(X';R) \subseteq \Delta(X';R) \otimes_R \Delta(X';R) \;.$$

(iv) *The homology classes* $[X] \in H_n(X;R)$ *are in one-one correspondence with the chain homotopy classes of* (R,X)*-module chain maps*

$$[X] \cap - : \Delta(X;R)^{n-*} \ \to \ \Delta(X';R) \ ,$$

with

$$\begin{aligned} H_0(\mathrm{Hom}_{(R,X)}(\Delta(X;R)^{n-*}, \Delta(X';R))) \ &= \ H_n(\Delta(X';R) \otimes_{(R,X)} \Delta(X';R)) \\ &= \ H_n(X;R) \ . \end{aligned}$$

Remark 5.4 An X-controlled simplicial complex M is an example of a CW complex with a block system κ in the sense of Ranicki and Yamasaki [35]. The product $\Delta(M) \otimes_{(\mathbb{Z},X)} \Delta(M)$ is chain equivalent to the chain complex $D^\kappa(\Delta(M))$ of [35].

6. Homology manifolds and Poincaré complexes

Definition 6.1 An *n-dimensional R-homology manifold* is a finite simplicial complex M such that

$$H_*(M, M \backslash \widehat{\sigma}; R) \ = \ \begin{cases} R & \text{if } * = n \\ 0 & \text{otherwise} \end{cases} \quad (\sigma \in M) \ .$$

Definition 6.2 An *n-dimensional R-homology Poincaré complex* is a finite simplicial complex M with a homology class $[M] \in H_n(M;R)$ such that the cap products are R-module isomorphisms

$$[M] \cap - : H^{n-*}(M;R) \cong H_*(M;R) \ .$$

Similarly for an *n-dimensional R-homology Poincaré pair* $(M, \partial M)$, with $[M] \in H_n(M, \partial M; R)$ and

$$[M] \cap - : H^{n-*}(M, \partial M; R) \cong H_*(M;R) \ .$$

Proposition 6.3 A finite simplicial complex M is an n-dimensional R-homology manifold with fundamental class $[M] \in H_n(M;R)$ if and only if each $(D(\sigma,M), \partial D(\sigma,M))$ $(\sigma \in M)$ is an $(n-|\sigma|)$-dimensional R-homology Poincaré pair

$$H^{n-|\sigma|-*}(D(\sigma,M), \partial D(\sigma,M); R) \cong H_*(D(\sigma,M);R)$$

with fundamental class $[D(\sigma,M), \partial D(\sigma,M)] \in H_{n-|\sigma|}(D(\sigma,M), \partial D(\sigma,M); R)$ the image of $[M]$ under the composition of $|\sigma|$ codimension 1 boundary maps.

A $\mathbb{Z}$-homology manifold will just be called a homology manifold, and similarly for Poincaré complexes and pairs.

Definition 6.4 An *n-dimensional X-controlled R-homology Poincaré complex* M is an X-controlled simplicial complex with a homology class $[M] \in H_n(M;R)$ such that the cap product

$$[M] \cap - : \Delta(M;R)^{n-*} \to \Delta(M;R)$$

is an (R,X)-module chain equivalence.

Remark 6.5 An X-controlled simplicial complex M is an n-dimensional X-controlled R-homology Poincaré complex if and only if each

$$p_M^{-1}(D(\sigma,X),\partial D(\sigma,X)) \subseteq M \quad (\sigma \in X)$$

is an $(n-|\sigma|)$-dimensional R-homology Poincaré pair. In terms of the polyhedra $|M|$, $|X|$ this condition can be expressed as follows: for every $x \in |X|$ the inverse image $p_M^{-1}(x) \subseteq |M|$ has a closed regular neighbourhood $(U,\partial U)$ which is an n-dimensional R-homology Poincaré pair.

By analogy with 3.5:

Proposition 6.6 (i) *If $X = \{\text{pt.}\}$ an n-dimensional X-controlled R-homology Poincaré complex M is the same as an n-dimensional R-homology Poincaré complex.*
(ii) *If $X = M$ an n-dimensional X-controlled R-homology Poincaré complex M is the same as an n-dimensional R-homology manifold.*

Theorem 6.7 (Poincaré duality) *An n-dimensional R-homology manifold M is an n-dimensional X-controlled R-homology Poincaré complex, with an (R,X)-module chain equivalence*

$$\Delta(M;R)^{n-*} \simeq \Delta(M;R)$$

with respect to any control map $p_M : M \to X'$.
Proof An (R,M)-module chain equivalence

$$[M] \cap - : \Delta(M;R)^{n-*} \to \Delta(M;R)$$

can be regarded as an (R,X)-module chain equivalence, for any control map $p_M : M \to X'$. □

Corollary 6.8 (Poincaré-Lefschetz duality) *An n-dimensional R-homology manifold with boundary $(M,\partial M)$ is an n-dimensional X-controlled R-homology Poincaré pair, with an (R,X)-module chain equivalence*

$$\Delta(M;R)^{n-*} \simeq \Delta(M,\partial M;R)$$

with respect to any control map $p_M : M \to X'$.

Corollary 6.9 (Lefschetz duality) *If M is an n-dimensional R-homology manifold and $L \subseteq M$ is any subcomplex, there is defined an (R,X)-module chain equivalence*

$$\Delta(M,M\backslash L;R)^{n-*} \simeq \Delta(L;R)$$

with respect to any control map $p_M : M \to X'$. Similarly for an (R,X)-module chain equivalence

$$\Delta(M,L;R)^{n-*} \simeq \Delta(M\backslash L;R) .$$

Proof Let $(U,\partial U)$ be a closed regular neighbourhood of L in M, an n-dimensional R-homology manifold with boundary such that the inclusion $L \subset U$ is a homotopy equivalence. There are defined (R,X)-module chain equivalences

$$\begin{aligned}\Delta(M,M\backslash L;R)^{n-*} &\simeq \Delta(M,\mathrm{cl.}(M\backslash U);R)^{n-*} \text{ (homotopy invariance)}\\ &\simeq \Delta(U,\partial U;R)^{n-*} \text{ (excision)}\\ &\simeq \Delta(U;R) \text{ (Poincaré-Lefschetz duality)}\\ &\simeq \Delta(L;R) \text{ (homotopy invariance)} .\end{aligned}$$

□

Definition 6.10 Let M be an X-controlled simplicial complex, with a homology class $[M] \in H_n(M;R)$. The *X-controlled peripheral chain complex* of M is the algebraic mapping cone

$$C \ = \ \mathcal{C}([M]\cap - : \Delta(M;R)^{n-*} \to \Delta(M';R))_{*+1}$$

(with a dimension shift), a finite chain complex in $\mathbb{A}(R,X)$.

Proposition 6.11 The following conditions on an X-controlled simplicial complex M with a homology class $[M] \in H_n(M;R)$ and peripheral chain complex C are equivalent:
(i) *M is an n-dimensional X-controlled R-homology Poincaré complex,*
(ii) *C is chain contractible in $\mathbb{A}(R,X)$,*
(iii) *$H_{n-1}(C \otimes_{(R,X)} C) = 0$,*
(iv) *each $p^{-1}(D(\sigma,X),\partial D(\sigma,X))$ ($\sigma \in X$) is an $(n-|\sigma|)$-dimensional R-homology Poincaré pair.*

Proof (i) $\Longleftrightarrow$ (ii) The chain map $[M]\cap - : \Delta(M;R)^{n-*} \to \Delta(M';R)$ is a chain equivalence in $\mathbb{A}(R,X)$ if and only if the algebraic mapping cone is chain contractible in $\mathbb{A}(R,X)$.
(ii) $\Longleftrightarrow$ (iii) The (R,X)-module chain map

$$\alpha \ = \ [M]\cap - : \Delta(M;R)^{n-*} \to \Delta(M';R)$$

is chain homotopic to its chain dual, with a chain homotopy

$$\beta : \alpha \simeq T\alpha : \Delta(M;R)^{n-*} \to \Delta(M';R) .$$

Define a chain equivalence in $\mathbb{A}(R,X)$

$$\phi_X : C^{n-1-*} \to C \ = \ \mathcal{C}(\alpha)_{*+1}$$

by

$$\phi_X \ = \ \begin{pmatrix} \beta & 1 \\ 1 & 0 \end{pmatrix} :$$
$$C^{n-1-r} = \Delta(M;R)^{n-r} \oplus \Delta(M';R)_{r+1} \to C_r = \Delta(M';R)_{r+1} \oplus \Delta(M;R)^{n-r} .$$

(See §9 for a more detailed discussion of the quadratic Poincaré structure on C). The abelian group

$$\begin{aligned} H_{n-1}(C \otimes_{(R,X)} C) &= H_0(\mathrm{Hom}_{(R,X)}(C^{n-1-*}, C)) \\ &= H_0(\mathrm{Hom}_{(R,X)}(C, C)) \end{aligned}$$

consists of the chain homotopy classes of chain maps $C \to C$. This group is 0 if and only if C is chain contractible.
(ii) $\Longleftrightarrow$ (iv) By 2.2 C is chain contractible if and only if each component R-module chain complexes $C(\sigma)$ $(\sigma \in X)$ is chain contractible. Now

$$\begin{aligned} C(\sigma) \simeq_R \mathcal{C}([p^{-1}D(\sigma,X)] \cap - :& \\ &\Delta(p^{-1}(D(\sigma,X), \partial D(\sigma,X)); R)^{n-|\sigma|-*} \to \Delta(p^{-1}D(\sigma,X); R))_{*+1} \ , \end{aligned}$$

so that $C(\sigma) \simeq_R 0$ if and only if $p^{-1}(D(\sigma,X), \partial D(\sigma,X))$ $(\sigma \in X)$ is an $(n-|\sigma|)$-dimensional R-homology Poincaré pair. □

Example 6.12 Let $X = \{\mathrm{pt.}\}$. The following conditions on a simplicial complex M with a homology class $[M] \in H_n(M;R)$ and peripheral R-module chain complex C are equivalent:
(i) M is an n-dimensional R-homology Poincaré complex with fundamental class $[M]$,
(ii) $H_*(C) = 0$,
(iii) $H_{n-1}(C \otimes_R C) = 0$.

In the following result $X = M$.

Theorem 6.13 The following conditions on an n-dimensional R-homology Poincaré complex X are equivalent:
(i) *X is an n-dimensional R-homology manifold,*
(ii) *the peripheral chain complex*

$$C \ = \ \mathcal{C}([X] \cap - : \Delta(X;R)^{n-*} \to \Delta(X';R))_{*+1}$$

is (R,X)-module chain contractible,
(iii) $H_{n-1}(C \otimes_{(R,X)} C) = 0$,
(iv) *the cohomology class $V \in H^n(X \times X; R)$ Poincaré dual to the homology class $\Delta_*[X] \in H_n(X \times X; R)$ has image $0 \in H^n(X \times X \backslash \Delta_X; R)$,*
(v) *the fundamental class $[X] \in H_n(X;R)$ is such that*

$$[X] \in \mathrm{im}(H^n(X \times X, X \times X \backslash \Delta_X; R) \to H_n(X;R)) \ ,$$

(vi) *a particular R-module morphism*

$$H^n(X \times X \backslash \Delta_X; R) \to H_n(X \times X, \Delta_X; R)$$

(specified in the proof) is an isomorphism, namely the Lefschetz duality isomorphism.

Proof (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) This is a special case of 6.11.
(i) $\Longleftrightarrow$ (iv) There is defined an exact sequence

$$H^n(X\times X, X\times X\backslash\Delta_X;R)\to H^n(X\times X;R)\to H^n(X\times X\backslash\Delta_X;R)\ .$$

Thus V has image $0\in H^n(X\times X\backslash\Delta_X;R)$ if and only if there exists an element

$$U\in H^n(X\times X, X\times X\backslash\Delta_X;R)$$

with image V. Now U is a chain homotopy class of (R,X)-module chain maps $\Delta(X';R)\to\Delta(X;R)^{n-*}$, since

$$\begin{aligned} H^n(X\times X, X\times X\backslash\Delta_X;R) &= H_n(\Delta(X;R)^{-*}\otimes_{(R,X)}\Delta(X;R)^{-*}) \\ &= H_0(\mathrm{Hom}_{(R,X)}(\Delta(X';R),\Delta(X;R)^{n-*}))\ . \end{aligned}$$

U is a chain homotopy inverse of

$$\phi=[X]\cap - : \Delta(X;R)^{n-*}\to\Delta(X';R)$$

with

$$\begin{aligned} \phi U &= 1\in H_0(\mathrm{Hom}_{(R,X)}(\Delta(X';R),\Delta(X';R))) = H^0(X;R)\ , \\ \phi &= T\phi\in H_0(\mathrm{Hom}_{(R,X)}(\Delta(X;R)^{n-*},\Delta(X';R)))\ , \\ (TU)\phi &= (TU)(T\phi) = T(\phi U) = 1 \\ &\quad \in H_0(\mathrm{Hom}_{(R,X)}(\Delta(X';R)^{n-*},\Delta(X;R)^{n-*}))\ . \end{aligned}$$

(iv) $\Longleftrightarrow$ (v) $\Longleftrightarrow$ (vi) Immediate from the commutative braid of exact sequences

$$\begin{tikzcd}[column sep=tiny]
H^n(X\times X, X\times X\backslash\Delta_X;R) \arrow[dr] \arrow[rr, bend left] & & H^n(X\times X;R) \arrow[dr] \arrow[rr, bend left] & & H_n(X\times X,\Delta_X;R) \\
& H_n(X;R) \arrow[ur, "\Delta_*"] \arrow[dr] & & H^n(X\times X\backslash\Delta_X;R) \arrow[ur] \arrow[dr] & \\
H_{n+1}(X\times X,\Delta_X;R) \arrow[ur, "0"] \arrow[rr, bend right, "0"'] & & H_{n-1}(C\otimes_{(R,X)}C) \arrow[ur] \arrow[rr, bend right] & & H^{n+1}(X\times X, X\times X\backslash\Delta_X;R)
\end{tikzcd}$$

on noting that $X\times X$ is a $2n$-dimensional R-homology Poincaré complex with isomorphisms

$$[X\times X]\cap - : H^n(X\times X;R)\cong H_n(X\times X;R)$$

and that the diagonal map

$$\Delta : X\to X\times X\ ;\ x\mapsto(x,x)$$

is split by the projection

$$p : X \times X \to X \ ; \ (x, y) \mapsto x \ ,$$

so that

$$H_*(X \times X; R) \ = \ H_*(X; R) \oplus H_*(X \times X, \Delta_X; R) \ .$$

The classes

$$V \in H^n(X \times X, X \times X \backslash \Delta_X; R) \quad , \quad \phi_X \in H_{n-1}(C \otimes_{(R,X)} C)$$

(with ϕ_X as in the proof of 6.11) are both images of the fundamental class $[X] \in H_n(X; R)$, so that they have the same image in $H^n(X \times X \backslash \Delta_X; R)$. $\square$

Remark 6.14 The equivalence (i) $\Longleftrightarrow$ (iv) in 6.13 in the case $R = \mathbb{Z}$ is a slight generalization of the corresponding results of McCrory [17, Theorem 1] and Ranicki [29, 8.5] for n-circuits and n-dimensional pseudomanifolds respectively.

Remark 6.15 A Poincaré complex X is a homology manifold precisely when the dihomology spectral sequence of Zeeman [45] collapses. See McCrory [18] for a geometric interpretation in terms of moving cocycles in $X \times X$ off the diagonal.

There is also a version of 6.13 for Poincaré pairs with manifold boundary. Here is a special case:

Proposition 6.16 An n-dimensional R-homology Poincaré pair $(X, \partial X)$ with R-homology manifold boundary is an n-dimensional R-homology manifold with boundary if and only if the cohomology class $V \in H^n(X \times X, X \times \partial X; R)$ Poincaré-Lefschetz dual to the homology class $\Delta_*[X] \in H_n(X \times X, \partial X \times X; R)$ (with $[X] \in H_n(X, \partial X; R)$) is the image of a class

$$U \in H^n(X \times X, X \times \partial X \cup X \times X \backslash \Delta_X; R) \ .$$

Remark 6.17 In general, a singularity does not arise as a non-manifold point of a Poincaré complex, so 6.13 cannot be applied directly to obtain a homological invariant of the singularity. However, for an isolated singular point of a complex hypersurface it is possible to apply 6.16 to a related Poincaré pair with manifold boundary. Given a polynomial function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with an isolated critical point $z_0 \in V = f^{-1}(0)$ Milnor [20] relates the singularity of f at z_0 to the properties of the fibred knot

$$k : V \cap S_\epsilon \ = \ S^{2n-1} \subset S_\epsilon \ = \ S^{2n+1}$$

defined by intersecting V with

$$S_\epsilon \ = \ \{z \in \mathbb{C}^{n+1} \mid \|z - z_0\| = \epsilon\}$$

for a sufficiently small ϵ. (Only PL structures are considered here – the differentiable structure on $V \cap S_\epsilon$ could of course be exotic). In §14 below there will be associated to any fibred knot $k : S^{2n-1} \subset S^{2n+1}$ a $(2n+2)$-dimensional homology Poincaré pair $(X, \partial X)$ with manifold boundary, which is a homology manifold with boundary if k is unknotted; the obstruction to $(X, \partial X)$ being a homology manifold with boundary is related to homological invariants of k, and hence to the nature of the singularity.

7. Degree 1 maps and homology equivalences

This section investigates the extent to which a degree 1 map of n-dimensional homology manifolds has acyclic point inverses. It is shown that this is the case if and only if the n-dimensional homology of the double point set relative to the diagonal is zero.

Definition 7.1 The *double point set* of a map $f : M \to N$ is the pullback (5.1)

$$\begin{aligned} M \times_N M &= (f \times f)^{-1}(\Delta_N) \\ &= \{(x,y) \in M \times M \,|\, f(x) = f(y) \in N\} \ . \end{aligned}$$

If f is a simplicial map then $M \times_N M$ is an N-controlled simplicial complex.

Given a map $f : M \to N$ define the maps

$$\begin{aligned} & i : M \to M \times_N M \ ; \ x \mapsto (x,x) \ , \\ & j : M \times_N M \to N \ ; \ (x,y) \mapsto f(x) = f(y) \ , \\ & k : M \times_N M \to M \ ; \ (x,y) \mapsto x \ . \end{aligned}$$

There is defined a commutative diagram

$$\begin{tikzcd} & M \times M \arrow[r, "f \times f"] & N \times N \\ & M \times_N M \arrow[u] \arrow[r, "j"] & N \arrow[u, "\Delta_N"'] \\ M \arrow[uur, "\Delta_M"] \arrow[ur, "i"] \arrow[urr, "f"'] & & \end{tikzcd}$$

It follows from $ki = 1 : M \to M$ that

$$H_*(M \times_N M) \ = \ H_*(M) \oplus H_*(M \times_N M, \Delta_M) \ .$$

Definition 7.2 Let $f : M \to N$ be a map of X-controlled R-homology Poincaré complexes, with $\dim(M) = m$, $\dim(N) = n$.
(i) The *Umkehr* of f is the (R,X)-module chain map

$$f^! : \Delta(N;R) \simeq \Delta(N;R)^{n-*} \xrightarrow{\ f^*\ } \Delta(M;R)^{n-*} \simeq \Delta(M;R)_{*+m-n} \ .$$

(ii) f has *degree 1* if $m = n$ and

$$f_*[M] \ = \ [N] \in H_n(N;R) \ .$$

Proposition 7.3 (i) *If $f : M \to N$ is a degree 1 map of n-dimensional X-controlled R-homology Poincaré complexes the Umkehr (R,X)-module chain map $f^! : \Delta(N;R) \to \Delta(M;R)$ is such that*

$$ff^! \simeq 1 : \Delta(N;R) \to \Delta(N;R)$$

and there exists an (R,X)-module chain equivalence

$$\Delta(M;R) \simeq_{(R,X)} \Delta(N;R) \oplus \Delta(f^!) \ .$$

(ii) *If $f : M \to N$ is a degree 1 map of n-dimensional R-homology manifolds then*

$$H_n(\Delta(f^!) \otimes_{(R,N)} \Delta(f^!)) \ = \ H_n(M \times_N M, \Delta_M; R) \ .$$

Proof (i) Immediate from $f_*[M] = [N] \in H_n(N;R)$ and the naturality properties of the cap product.
(ii) Apply $\Delta(M) \otimes_{(\mathbb{Z},N)} -$ to the $(\mathbb{Z},N)$-module chain equivalence given by (i)

$$\Delta(M) \simeq_{(\mathbb{Z},N)} \Delta(N) \oplus \Delta(f^!) \ ,$$

to obtain

$$\begin{aligned}
&\Delta(M) \otimes_{(\mathbb{Z},N)} \Delta(M) \\
&\quad \simeq_{(\mathbb{Z},N)} (\Delta(M) \otimes_{(\mathbb{Z},N)} \Delta(N)) \oplus (\Delta(M) \otimes_{(\mathbb{Z},N)} \Delta(f^!)) \\
&\quad \simeq_{(\mathbb{Z},N)} (\Delta(M) \otimes_{(\mathbb{Z},N)} \Delta(N)) \oplus (\Delta(N) \otimes_{(\mathbb{Z},N)} \Delta(f^!)) \oplus (\Delta(f^!) \otimes_{(\mathbb{Z},N)} \Delta(f^!)) \\
&\quad \simeq_{(\mathbb{Z},N)} \Delta(M) \oplus \Delta(f^!) \oplus (\Delta(f^!) \otimes_{(\mathbb{Z},N)} \Delta(f^!)) \ .
\end{aligned}$$

Since $H_n(f^!) = 0$, it follows that

$$\begin{aligned}
H_n(M \times_N M) \ &= \ H_n(\Delta(M) \otimes_{(\mathbb{Z},N)} \Delta(M)) \\
&= \ H_n(M) \oplus H_n(f^!) \oplus H_n(\Delta(f^!) \otimes_{(\mathbb{Z},N)} \Delta(f^!)) \\
&= \ H_n(M) \oplus H_n(\Delta(f^!) \otimes_{(\mathbb{Z},N)} \Delta(f^!)) \ .
\end{aligned}$$

□

Theorem 7.4 The following conditions on a degree 1 map $f : M \to N$ of n-dimensional X-controlled R-homology Poincaré complexes are equivalent:

(i) *f is an X-controlled R-homology equivalence (3.3),*

(ii) *$f : \Delta(M;R) \to \Delta(N;R)$ is an (R,X)-module chain equivalence,*

(iii) *there exists an (R,X)-module chain homotopy*

$$f^! f \simeq 1 : \Delta(M;R) \to \Delta(M;R) \ ,$$

(iv) $\Delta_*[M] = (f^! \otimes f^!)\Delta_*[N] \in H_n(M \times_X M; R)$,

(v) $(f^! \otimes f^!)\Delta_*[N] = 0 \in H_n(M \times_X M, \Delta_M; R)$,

(vi) $(f \times f)_* : H_n(M \times_X M; R) \cong H_n(N \times_X N; R)$.

Proof (i) $\Longleftrightarrow$ (ii) This is a special case of 3.4.
(ii) $\Longleftrightarrow$ (iii) Immediate from 7.3.
(iii) $\Longleftrightarrow$ (iv) Immediate from the identifications

$$\begin{aligned} 1 \;=\; \Delta_*[M] \quad , \quad f^!f \;&=\; (f^! \otimes f^!)\Delta_*[N] \\ &\in H_0(\mathrm{Hom}_{(R,X)}(\Delta(M;R),\Delta(M;R))) \;=\; H_n(M \times_X M;R) \ . \end{aligned}$$

(iv) $\Longleftrightarrow$ (v) Immediate from the identity

$$\begin{aligned} (f^! \otimes f^!)\Delta_*[N] \;&=\; ([M],(f^! \otimes f^!)\Delta_*[N] - \Delta_*[M]) \\ &\in H_n(M \times_X M;R) \;=\; H_n(M;R) \oplus H_n(M \times_X M, \Delta_M;R) \ . \end{aligned}$$

(ii) $\Longrightarrow$ (vi) If $f : \Delta(M;R) \to \Delta(N;R)$ is an (R,X)-module chain equivalence then so is

$$f \otimes f : \Delta(M;R) \otimes_{(R,X)} \Delta(M;R) \to \Delta(N;R) \otimes_{(R,X)} \Delta(N;R) \ .$$

(vi) $\Longrightarrow$ (iv) It follows from $ff^! \simeq 1$ and

$$(f \otimes f)_*\Delta_*[M] \;=\; \Delta_*[N] \in H_n(N \times_X N;R)$$

that

$$\begin{aligned} &\Delta_*[M] - (f^! \otimes f^!)\Delta_*[N] \\ &\in \ker((f \times f)_* : H_n(M \times_X M;R) \to H_n(N \times_X N;R)) \;=\; \{0\} \ . \end{aligned}$$

$\square$

Corollary 7.5 The following conditions on a degree 1 map $f : M \to N$ of n-dimensional homology manifolds are equivalent:
(i) *f has acyclic point inverses,*
(ii) *$H_n(M \times_N M, \Delta_M) = 0$,*
(iii) *$H_n(\Delta(f^!) \otimes_{(\mathbb{Z},N)} \Delta(f^!)) = 0$.*
Proof (i) $\Longleftrightarrow$ (ii) Apply 7.3 with $R = \mathbb{Z}$, $X = N$, so that

$$\begin{aligned} &M \times_X M \;=\; M \times_N M \;=\; (f \times f)^{-1}\Delta_N \quad , \quad N \times_X N \;=\; N \ , \\ &H_n(M \times_X M) \;=\; H_n(M) \oplus H_n(M \times_N M, \Delta_M) \ . \end{aligned}$$

Since $f_* : H_n(M) \cong H_n(N)$, condition 7.4 (vi)

$$(f \times f)_* : H_n(M \times_N M) \cong H_n(N \times_N N)$$

for f to be a $(\mathbb{Z},N)$-homology equivalence is equivalent to

$$H_n(M \times_N M, \Delta_M) \;=\; 0 \ .$$

As in 3.5 (ii) a map f is a $(\mathbb{Z},N)$-homology equivalence if and only if it has acyclic point inverses.
(ii) $\Longleftrightarrow$ (iii) By 7.3 (ii) $H_n(\Delta(f^!) \otimes_{(\mathbb{Z},N)} \Delta(f^!)) = H_n(M \times_N M, \Delta_M)$. $\square$

Remark 7.6 (i) A map $f : M \to N$ is injective if and only if

$$M \times_N M \;=\; \Delta_M \;.$$

The condition of 7.5 (ii) is automatically satisfied for injective f.
(ii) A degree 1 map $f : M \to N$ of n-dimensional R-homology manifolds is surjective by the following argument, which does not require M, N to be polyhedra. If $x \in N \backslash f(M)$ then

$$H_n(M, M \backslash f^{-1}(x); R) \;=\; 0 \quad , \quad H_n(N, N \backslash \{x\}; R) \;=\; R \;,$$

leading to a contradiction in the commutative diagram

$$\begin{array}{ccc} H_n(M;R) = R & \xrightarrow{\cong} & H_n(N;R) = R \\ \big\downarrow & & \cong \big\downarrow \\ H_n(M, M\backslash f^{-1}(x); R) = 0 & \xrightarrow{f_*} & H_n(N, N\backslash\{x\}; R) = R \end{array}$$

(assuming M, N are connected).

Corollary 7.7 (i) *A map $f : M \to N$ of n-dimensional R-homology Poincaré complexes is an R-homology equivalence if and only if it is degree 1 and*

$$\Delta_*[M] \;=\; (f^! \otimes f^!)\Delta_*[N] \in H_n(M \times M; R) \;.$$

(ii) *A map $f : M \to N$ of n-dimensional R-homology manifolds has R-acyclic point inverses if and only if it is degree 1 and*

$$\Delta_*[M] \;=\; (f^! \otimes f^!)\Delta_*[N] \in H_n(M \times_N M; R) \;.$$

Proof (i) Apply 7.4 with $X = \{\text{pt.}\}$.
(ii) Apply 7.4 with $X = N$. □

Definition 7.8 Given a map $f : M \to N$ of R-homology manifolds with $\dim(M) = m$, $\dim(N) = n$ let the *Umkehr* of the map

$$j : M \times_N M \to N \;;\; (x, y) \mapsto f(x) = f(y)$$

be the (R, N)-module chain map

$$j^! : \Delta(N; R) \to \Delta(M \times_N M; R)_{*+2m-2n}$$

given by the composite

$$\begin{aligned} j^! : \Delta(N;R) &\simeq_{(R,N)} \Delta(N \times N, N \times N \backslash \Delta_N; R)^{2n-*} \\ &\xrightarrow{(f\times f)^*} \Delta(M \times M, M \times M \backslash M \times_N M; R)^{2n-*} \\ &\simeq_{(R,N)} \Delta(M \times_N M; R)_{*+2m-2n} \;. \end{aligned}$$

Proposition 7.9 The following conditions on a degree 1 map $f : M \to N$ of n-dimensional R-homology manifolds are equivalent:
(i) *f has R-acyclic point inverses,*
(ii) *there exists an (R, N)-module chain homotopy*

$$i_* f^! \simeq j^! : \Delta(N; R) \to \Delta(M \times_N M; R) ,$$

(iii) *there exists an (R, N)-module chain map $g : \Delta(N) \to \Delta(M)$ with an (R, N)-module chain homotopy*

$$i_* g \simeq j^! : \Delta(N; R) \to \Delta(M \times_N M; R) .$$

Proof (i) $\iff$ (ii) Identify

$$\begin{aligned} i_* f^! \;=\; \Delta_*[M] \;\;,\;\; j^! \;=\; (f^! \otimes f^!)\Delta_*[N] \\ \in H_0(\mathrm{Hom}_{(R,N)}(\Delta(N; R), \Delta(M \times_N M; R))) \;=\; H_n(M \times_N M; R) \end{aligned}$$

and apply the equivalence (i) $\iff$ (iv) of 7.4, with $X = N$.
(ii) $\Longrightarrow$ (iii) Take $g = f^!$.
(iii) $\Longrightarrow$ (i) It follows from the exact sequence

$$\begin{aligned} & H_0(\mathrm{Hom}_{(R,N)}(\Delta(N; R), \Delta(M; R))) \\ & \qquad \xrightarrow{i_*} H_0(\mathrm{Hom}_{(R,N)}(\Delta(N; R), \Delta(M \times_N M; R))) \\ & \qquad \longrightarrow H_0(\mathrm{Hom}_{(R,N)}(\Delta(N; R), \Delta(M \times_N M, \Delta_M; R))) \end{aligned}$$

that such a g exists if and only if the (R, N)-module chain homotopy class

$$j^! \in H_0(\mathrm{Hom}_{(R,N)}(\Delta(N; R), \Delta(M \times_N M; R)))$$

has 0 image in

$$H_0(\mathrm{Hom}_{(R,N)}(\Delta(N; R), \Delta(M \times_N M, \Delta_M; R))) \;=\; H_n(M \times_N M, \Delta_M; R) .$$

But this image is precisely the element $(f^! \otimes f^!)\Delta_*[N] \in H_n(M \times_N M, \Delta_M; R)$ of 7.4 (v) whose vanishing is (necessary and) sufficient for f to have R-acyclic point inverses. □

8. BUNDLES

The results of §§6,7 will now be interpreted from the bundle point of view, aftre a brief review of the various bundle theories involved.

Oriented spherical fibrations η over a space X

$$(D^k, S^{k-1}) \to (E(\eta), S(\eta)) \to X$$

are classified up to oriented fibre homotopy equivalence by the homotopy classes of maps $\eta : X \to BG(k)$ to a classifying space $BG(k)$. Every such fibration has a *Thom space*

$$T(\eta) \;=\; E(\eta)/S(\eta)$$

and a *Thom class*

$$U_\eta \in \widetilde{H}^k(T(\eta)) \ .$$

See Rourke and Sanderson [38] for the theory of (oriented) PL k-block bundles, with a classifying space $B\widetilde{SPL}(k)$. A codimension k embedding $M^n \subset N^{n+k}$ of PL manifolds has a normal PL k-block bundle $\nu_{M\subset N} : M \to BPL(k)$.

See Martin and Maunder [15] for the theory of homology cobordism bundles, with a classifying space $BSH(k)$ and forgetful maps

$$B\widetilde{SPL}(k) \to BSH(k) \quad , \quad BSH(k) \to BSG(k) \ .$$

A codimension k embedding $M^n \subset N^{n+k}$ of homology manifolds (i.e. a PL map which is an injection) has a normal homology cobordism S^{k-1}-bundle $\nu_{M\subset N} : M \to BSH(k)$.

See Rourke and Sanderson [39] for the theory of (oriented) topological k-block bundles, with a classifying space $B\widetilde{STOP}(k)$ and forgetful maps

$$B\widetilde{SPL}(k) \to B\widetilde{STOP}(k) \quad , \quad B\widetilde{STOP}(k) \to BSG(k) \ .$$

Galewski and Stern [7] proved that every homology cobordism S^{k-1}-bundle has a canonical lift to a topological k-block bundle, so that there is defined a commutative diagram of classifying spaces and forgetful maps

$$\begin{tikzcd}
B\widetilde{SPL}(k) \arrow[r] \arrow[d] & B\widetilde{STOP}(k) \arrow[d] \\
BSH(k) \arrow[r] \arrow[ur] & BSG(k) \ .
\end{tikzcd}$$

The diagonal embedding of an n-dimensional homology manifold M

$$\Delta : M \to M \times M \ ; \ x \mapsto (x,x)$$

has a normal homology cobordism S^{n-1}-bundle, the *tangent homology cobordism S^{n-1}-bundle* ([15, 5.3])

$$\tau_M \;=\; \nu_\Delta : M \to BSH(n) \ ,$$

and hence a *tangent topological n-block bundle* $\tau_M : M \to B\widetilde{STOP}(n)$. The Euler class of τ_M may be identified with the Euler characteristic of M, as follows.

The *Euler characteristic* of a finite simplicial complex X is

$$\chi(X) \;=\; \sum_{r=0}^{\infty}(-)^r \dim_{\mathbb{R}} H_r(X;\mathbb{R}) \in \mathbb{Z}\ .$$

Proposition 8.1 (i) *For a connected n-dimensional Poincaré complex X*

$$\chi(X) \;=\; \Delta^*(V) \in H^n(X) \;=\; \mathbb{Z}$$

with $V \in H^n(X \times X)$ the Poincaré dual of $\Delta_*[X] \in H_n(X \times X)$.
(ii) *The obstruction to a degree 1 map $f : M \to N$ of connected n-dimensional Poincaré complexes being a homology equivalence (7.7 (i))*

$$\Delta_*[M] - (f^! \otimes f^!)\Delta_*[N] \in H_n(M \times M)$$

has image $\chi(M) - \chi(N) \in \mathbb{Z}$ under the composite

$$H_n(M \times M) \cong H^n(M \times M) \xrightarrow{\Delta^*} H^n(M) \;=\; \mathbb{Z}\ .$$

Proof (i) As for smooth manifolds (Milnor and Stasheff [21, 11.13]).
(ii) Immediate from (i). □

It is well known that $\chi(M) = \chi(\tau_M)$ for a smooth manifold M ([21, 11.13]). More generally :

Proposition 8.2 The Euler characteristic of a connected n-dimensional homology manifold M is the Euler class of the tangent n-block bundle τ_M

$$\chi(M) \;=\; \chi(\tau_M) \in H^n(M) \;=\; \mathbb{Z}\ .$$

Proof The homology tangent bundle of M (Spanier [40, p.294]) is the homology fibration

$$(M, M\backslash\{*\}) \to (M \times M, M \times M\backslash\Delta_M) \to M$$

with

$$M \to M \times M\ ;\ x \mapsto (*, x)\ ,$$
$$M \times M \to M\ ;\ (x, y) \mapsto x\ .$$

The tangent topological n-block bundle of M

$$(D^n, S^{n-1}) \to (E(\tau_M), S(\tau_M)) \to M$$

is related to the homology tangent bundle by a homotopy pushout diagram

$$\begin{tikzcd} S(\tau_M) \arrow[r] \arrow[d] & M \times M\backslash\Delta_M \arrow[d] \\ E(\tau_M) \simeq M \arrow[r, "\Delta"] & M \times M\ . \end{tikzcd}$$

The Thom space, Thom class and Euler class of τ_M are such that

$$\begin{aligned} T(\tau_M) &= E(\tau_M)/S(\tau_M) = (M\times M)/(M\times M\backslash\Delta_M) , \\ U_M &\in \widetilde{H}^n(T(\tau_M)) = H^n(M\times M, M\times M\backslash\Delta_M) , \\ e(\tau_M) &= z^*(U_M)\in H^n(M) , \end{aligned}$$

with $z: M\to T(\tau_M)$ the zero section. Furthermore, there is defined a commutative diagram

$$\begin{tikzcd} H^n(M\times M, M\times M\backslash\Delta_M) \arrow[r, "i^*"] \arrow[d, "\cong"'] & H^n(M\times M) \arrow[d, "\Delta^*"] \\ \widetilde{H}^n(T(\tau_M)) \arrow[r, "z^*"] & H^n(M) \end{tikzcd}$$

with $i: M\times M\to (M\times M, M\times M\backslash\Delta_M)$ the natural map. As before, let $V\in H^n(M\times M)$ be the Poincaré dual of $\Delta_*[M]\in H_n(M\times M)$. The Thom class $U_M\in\widetilde{H}^n(T(\tau_M))$ has image

$$i^*(U_M) = V\in H^n(M\times M) ,$$

and

$$e(\tau_M) = z^*(U_M) = \Delta^*(i^*(U_M)) = \Delta^*(V) = \chi(M)\in H^n(M) = \mathbb{Z} .$$

□

Remark 8.3 Theorem 6.13 can be regarded as a converse of 8.2 :
A connected n-dimensional Poincaré complex X is an n-dimensional homology manifold if and only if the Poincaré dual $V\in H^n(X\times X)$ of $\Delta_*[X]\in H_n(X\times X)$ is the image of a Thom class $U\in\widetilde{H}^n(T(\tau_X))$, in which case

$$\chi(X) = e(\tau_X)\in H^n(X) = \mathbb{Z} .$$

McCrory [17] called such U a *geometric Thom class* for X.

Proposition 8.4 A degree 1 map $f: M\to N$ of n-dimensional R-homology manifolds has acyclic point inverses if and only if the Thom classes

$$U_M\in H^n(M\times M, M\times M\backslash\Delta_M;R) ,\ U_N\in H^n(N\times N, N\times N\backslash\Delta_N;R)$$

have the same image in $H^n(M\times M, M\times M\backslash M\times_N M;R)$

$$c^*(U_M) = (f\times f)^*(U_N)\in H^n(M\times M, M\times M\backslash M\times_N M;R) ,$$

with $c: (M\times M, M\times M\backslash M\times_N M)\to (M\times M, M\times M\backslash\Delta_M)$ the inclusion of pairs.

Proof This is just the cohomology version of 7.7 (ii), after Lefschetz duality (6.8) identifications

$$\begin{aligned} U_M &= [M] \in H^n(M \times M, M \times M \backslash \Delta_M; R) = H_n(M; R) , \\ U_N &= [N] \in H^n(N \times N, N \times N \backslash \Delta_N; R) = H_n(N; R) , \\ H^n(M \times M, M \times M \backslash M \times_N M; R) &= H_n(M \times_N M; R) , \end{aligned}$$

noting that $M \times M$ and $N \times N$ are $2n$-dimensional R-homology manifolds. □

Remark 8.5 Suppose that $f : M \to N$ is a degree 1 map of n-dimensional homology manifolds which is covered by a stable map

$$b : \tau_M \oplus \epsilon^\infty \to \tau_N \oplus \epsilon^\infty$$

of the tangent block bundles. (For example, if M, N have trivial tangent block bundles then any map $f : M \to N$ is covered by an unstable map $b : \tau_M \to \tau_N$). In general, the diagram

$$\begin{tikzcd} \widetilde{H}^n(T(\tau_N)) \arrow[rr, "T(b)^*"] \arrow[d, phantom, "\cong"] & & \widetilde{H}^n(T(\tau_M)) \arrow[d, phantom, "\cong"] \\ H^n(N \times N, N \times N \backslash \Delta_N) \arrow[dr, "(f \times f)^*"] & & H^n(M \times M, M \times M \backslash \Delta_M) \arrow[dl, "c^*"'] \\ & H^n(M \times M, M \times M \backslash M \times_N M) & \end{tikzcd}$$

is not commutative, with the obstruction in 8.4 non-zero:

$$\begin{aligned} c^*T(b)^*(U_N) - (f \times f)^*(U_N) &= c^*(U_M) - (f \times f)^*(U_N) \\ &\neq 0 \in H^n(M \times M, M \times M \backslash M \times_N M) . \end{aligned}$$

In §9 below this difference will be expressed in terms of an N-controlled refinement of the (symmetrization of the) quadratic structure used in Ranicki [27] to obtain a chain level expression for the Wall surgery obstruction.

Proposition 8.6 Let $f : M \to N$ be a degree 1 map of n-dimensional R-homology manifolds. If there exists an N-controlled map

$$a : (M \times M, M \times M \backslash \Delta_M) \to (N \times N, N \times N \backslash \Delta_N)$$

such that the diagram

$$\begin{tikzcd}[column sep=small]
 & (M\times M, M\times M\backslash M\times_N M) \arrow[dl, "c"'] \arrow[dr, "f\times f"] & \\
(M\times M, M\times M\backslash\Delta_M) \arrow[rr, "a"] & & (N\times N, N\times N\backslash\Delta_N)
\end{tikzcd}$$

is N-controlled homotopy commutative, then

$$(f\times f)^*(U_N) \;=\; c^*(U_M)\in H^n(M\times M, M\times M\backslash M\times_N M;R)$$

and f has acyclic point inverses. Moreover,

$$a^*(U_N) \;=\; U_M\in H^n(M\times M, M\times M\backslash\Delta_M;R)\ .$$

Proof Define the (R,N)-module chain map

$$\begin{aligned}g:\Delta(N;R)\simeq_{(R,N)}&\Delta(N\times N, N\times N\backslash\Delta_N;R)^{2n-*}\\ &\xrightarrow{a^*}\Delta(M\times M, M\times M\backslash\Delta_M;R)^{2n-*}\simeq_{(R,N)}\Delta(M;R)\end{aligned}$$

such that

$$g[N] \;=\; a^*(U_N)\in H_n(M) \;=\; H^n(M\times M, M\times M\backslash\Delta_M)\ .$$

The N-controlled homotopy of pairs

$$ac\simeq f\times f:(M\times M, M\times M\backslash M\times_N M)\to(N\times N, N\times N\backslash\Delta_N)$$

induces an (R,N)-module chain homotopy

$$\begin{aligned}ac\simeq f\times f:\Delta(M\times M, M\times M\backslash M\times_N M;R)&\simeq_{(R,N)}\Delta(M\times_N M;R)^{2n-*}\\ \to\Delta(N\times N, N\times N\backslash\Delta_N;R)&\simeq_{(R,N)}\Delta(N;R)^{2n-*}\ .\end{aligned}$$

The chain dual is an (R,N)-module chain homotopy

$$i_*g\simeq j^!:\Delta(N;R)\to\Delta(M\times_N M;R)\ ,$$

so that

$$i_*g[N] \;=\; j^![N] \;=\; [M\times_N M]\in H_n(M\times_N M;R)\ ,$$

with dual the identity

$$c^*a^*(U_N) \;=\; (f\times f)^*(U_N)\in H^n(M\times M, M\times M\backslash M\times_N M;R)\ ,$$

so that f has R-acyclic point inverses by 8.4, and

$$\begin{aligned} g \simeq f^{-1} \simeq f^! : \Delta(N;R) \to \Delta(M;R)\ , \\ g[N] \ = \ [M] \in H_n(M;R)\ , \\ a^*(U_N) \ = \ U_M \in H^n(M \times M, M \times M\backslash\Delta_M;R)\ . \end{aligned}$$

□

Remark 8.7 A degree 1 map $f : M \to N$ of n-dimensional homology manifolds which is covered by a map of the tangent n-block bundles $b : \tau_M \to \tau_N$ need not be covered by a map of homology tangent bundles a as in 8.6.

9. The total surgery obstruction

The total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ of Ranicki [29] is defined for a finite simplicial complex X satisfying n-dimensional Poincaré duality with respect to all coefficients – such Poincaré complexes are considered further below. For $n \geq 5$ the total surgery obstruction is $s(X) = 0$ if and only if the polyhedron $|X|$ is homotopy equivalent to a topological manifold (which need not be triangulable). On the other hand, an n-dimensional homology Poincaré complex X is a homology manifold if and only if an obstruction in $H^n(X \times X\backslash\Delta_X)$ (6.13) is 0. The obstruction of 6.13 will now be related to the total surgery obstruction and its $\mathbb{Z}$-homology analogue.

So far, only the homology $H_*(X;R)$ and cohomology $H^*(X;R)$ of a simplicial complex X with coefficients in a commutative ring R have been considered. For non-simply-connected X the homology $H_*(X;\Lambda)$ and cohomology $H^*(X;\Lambda)$ and with coefficients in an $R[\pi_1(X)]$-module Λ will also be considered.

Given a commutative ring R and a group π let the group ring $R[\pi]$ have the involution

$$R[\pi] \to R[\pi]\ ;\ a = \sum_{g\in\pi} n_g g \mapsto \overline{a} = \sum_{g\in\pi} n_g g^{-1} \quad (n_g \in R)\ .$$

Use the involution to convert every left $R[\pi]$-module M into a right $R[\pi]$-module M^t, with the same additive group and

$$M^t \times R[\pi] \to M^t\ ;\ (x,a) \mapsto \overline{a}.x\ .$$

Define an involution (1.2) on the additive category $\mathbb{A}(R[\pi])$ of f.g. free (left) $R[\pi]$-modules

$$* : \mathbb{A}(R[\pi]) \to \mathbb{A}(R[\pi])\ ;\ A \mapsto A^* \ = \ \mathrm{Hom}_{R[\pi]}(A, R[\pi])$$

with

$$R[\pi] \times A^* \to A^*\ ;\ (a,f) \mapsto (x \mapsto f(x).\overline{a})\ .$$

Definition 9.1 Given a connected simplicial complex X with universal cover $\widetilde{X}$ and an $R[\pi_1(X)]$-module Λ define the Λ-*coefficient homology* and *cohomology* R-modules of X to be

$$\begin{aligned} H_*(X;\Lambda) &= H_*(\Lambda^t \otimes_{R[\pi_1(X)]} \Delta(\widetilde{X};R)) , \\ H^*(X;\Lambda) &= H_*(\mathrm{Hom}_{R[\pi_1(X)]}(\Delta(\widetilde{X};R),\Lambda)) . \end{aligned}$$

The Λ-coefficient homology and cohomology R-modules are related by a cap product pairing

$$H_n(X;R) \otimes_R H^m(X;\Lambda) \to H_{n-m}(X;\Lambda); x \otimes y \mapsto x \cap y .$$

For $\Lambda = R[\pi_1(X)]$ the Λ-coefficient homology and cohomology groups are $R[\pi_1(X)]$-modules

$$\begin{aligned} H_*(X;R[\pi_1(X)]) &= H_*(\Delta(\widetilde{X};R)) = H_*(\widetilde{X};R) , \\ H^*(X;R[\pi_1(X)]) &= H_{-*}(\mathrm{Hom}_{R[\pi_1(X)]}(\Delta(\widetilde{X};R),R[\pi_1(X)])) \end{aligned}$$

Definition 9.2 An *n-dimensional universal R-homology Poincaré complex* is a finite connected simplicial complex X with a homology class $[X] \in H_n(X;R)$ such that the cap products are $R[\pi_1(X)]$-module isomorphisms

$$[X] \cap - : H^{n-*}(X;R[\pi_1(X)]) \cong H_*(X;R[\pi_1(X)]) .$$

A universal $\mathbb{Z}$-homology Poincaré complex will just be called a universal homology Poincaré complex.

Remark 9.3 (i) A universal homology Poincaré complex is just a Poincaré complex in the sense of Wall [42].
(ii) If X is a universal R-homology Poincaré complex with universal cover $\widetilde{X}$ then the $R[\pi_1(X)]$-module chain map

$$[X] \cap - : \Delta(\widetilde{X};R)^{n-*} = \mathrm{Hom}_{R[\pi_1(X)]}(\Delta(\widetilde{X};R),R[\pi_1(X)])_{*-n} \to \Delta(\widetilde{X};R)$$

is a chain equivalence, and there are defined Poincaré duality isomorphisms

$$[X] \cap - : H^{n-*}(X;\Lambda) \cong H_*(X;\Lambda)$$

for any $R[\pi_1(X)]$-module Λ.
(iii) A connected finite simplicial complex X with finite fundamental group $\pi_1(X)$ is an n-dimensional universal R-homology Poincaré complex if and only if the universal cover $\widetilde{X}$ is an n-dimensional R-homology Poincaré complex in the sense of 6.2.

Proposition 9.4 A connected n-dimensional R-homology manifold X is an n-dimensional universal R-homology Poincaré complex.

Proof The assembly functor of Ranicki and Weiss [34]

$$A : \mathbb{A}(R,X) \to \mathbb{A}(R[\pi_1(X)]) \ ; \ A \ = \ \sum_{\sigma\in X} A(\sigma) \mapsto A(\widetilde{X}) \ = \ \sum_{\widetilde{\sigma}\in\widetilde{X}} A(p\widetilde{\sigma})$$

is defined for any connected simplicial complex X, with $p : \widetilde{X} \to X$ the universal covering projection. The assembly is a natural transformation of additive categories with chain duality ([29, 9.11]), so that the assembly of the n-dimensional symmetric Poincaré complex $(\Delta(X';R),\Delta[X])$ in $\mathbb{A}(R,X)$ is the n-dimensional symmetric Poincaré complex $(\Delta(\widetilde{X}';R),\Delta[X])$ in $\mathbb{A}(R[\pi_1(X)])$. (This is just a formalization of the standard dual cell proof of Poincaré duality, e.g. Wall [43, Thm. 2.1]). □

In particular, a homology manifold is a universal homology Poincaré complex.

Definition 9.5 (Quinn [22])
(i) An n-dimensional *normal complex* (X,ν_X,ρ_X) is a finite simplicial complex X together with a *normal structure*

$$(\nu_X : X \to BSG(k) \ , \ \rho_X : S^{n+k} \to T(\nu_X)) \ \ (k \text{ large}) \ .$$

The homology class

$$[X] \ = \ U_{\nu_X} \cap h(\rho_X) \ = \ [X] \in H_n(X) \ \ (h = \text{Hurewicz})$$

is the *fundamental class* of X.
(ii) A *normal structure* on an n-dimensional homology Poincaré complex X is a normal structure (ν_X,ρ_X) realizing the fundamental class $[X] \in H_n(X)$.

Remark 9.6 (i) A finite simplicial complex X is an n-dimensional universal homology Poincaré complex if and only if a regular neighbourhood $(U,\partial U)$ of an embedding $X \subset S^{n+k}$ defines a fibration

$$(D^k,S^{k-1}) \to (U,\partial U) \to X$$

(Spivak [41], Wall [42], Ranicki [27]). A n-dimensional universal homology Poincaré complex X has a canonical class of Spivak normal structures $(\nu_X : X \to BSG(k), \rho_X : S^{n+k} \to T(\nu_X))$, namely those represented by such regular neighbourhoods $(U,\partial U)$ with

$$\rho_X : S^{n+k} \to S^{n+k}/\text{cl.}(S^{n+k}\backslash U) \ = \ U/\partial U \ = \ T(\nu_X) \ .$$

(ii) Browder [1] used Poincaré surgery on $\pi_1(X)$ to prove that every n-dimensional homology Poincaré complex X admits normal structures $(\nu_X : X \to BSG(k), \rho_X : S^{n+k} \to T(\nu_X))$, and that for any such structure $\nu_X \oplus \epsilon : X \to BSG(k+1)$ is the normal fibration of a Poincaré embedding $X \subset S^{n+k+1}$ with complement $T(\nu_X) \cup_{\rho_X} D^{n+k+1}$.

Definition 9.7 (Ranicki [29, 17.1])
The *peripheral quadratic complex* of an n-dimensional normal complex X is the $(n-1)$-dimensional quadratic Poincaré complex (C, ψ_X) in $\mathbb{A}(\mathbb{Z}, X)$ with C the X-controlled peripheral chain complex (6.10)

$$C = \mathcal{C}([X] \cap - : \Delta(X)^{n-*} \to \Delta(X'))_{*+1}$$

and

$$\psi_X \in Q^X_{n-1}(C) = H_{n-1}(W \otimes_{\mathbb{Z}[\Sigma_2]} (C \otimes_{(\mathbb{Z},X)} C))$$

the X-controlled quadratic class obtained by the boundary construction of [29, 2.6].

Note that the normal complex X is a universal homology Poincaré complex if and only if the peripheral chain complex C is $\mathbb{A}(\mathbb{Z}[\pi_1(X)])$-contractible.

Remark 9.8 The X-controlled quadratic class $\psi_X \in Q^X_{n-1}(C)$ in 9.7 has symmetrization

$$(1+T)\psi_X = \phi_X \in H_{n-1}(C \otimes_{(\mathbb{Z},X)} C)$$

the chain homotopy class of chain equivalences $\phi_X : C^{n-1-*} \to C$ (6.11). In fact, ψ_X is an X-controlled version of the quadratic class

$$\psi = \psi_F(U_{\nu_X}) \in Q_{n-1}(C) = H_{n-1}(W \otimes_{\mathbb{Z}[\Sigma_2]} (C \otimes_{\mathbb{Z}} C))$$

obtained by evaluating the spectral quadratic construction of Ranicki [28, 7.3]

$$\psi_F : \widetilde{H}^k(T(\nu_X)) \to Q_{n-1}(C)$$

on the Thom class $U_{\nu_X} \in \widetilde{H}^k(T(\nu_X))$. Here, $F : T(\nu_X)^* \to \Sigma^\infty X_+$ is a stable map inducing the chain map $[X] \cap - : \Delta(X)^{n-*} \to \Delta(X')$, with $T(\nu_X)^*$ the spectrum S-dual of the Thom space $T(\nu_X)$. If X is homology Poincaré then $T(\nu_X)^* = \Sigma^\infty X_+$. If X is R-homology Poincaré $\psi = 0 \in Q_{n-1}(C) = 0$, but in general $\psi_X \neq 0$.

Refer to Ranicki [29, p.148] for the *algebraic surgery exact sequence* of a simplicial complex X

$$\cdots \to H_n(X; \mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X; \mathbb{L}_\bullet) \to \dots ,$$

with A the assembly map. The generalized homology group

$$H_n(X; \mathbb{L}_\bullet) = L_n(\mathbb{A}(\mathbb{Z}, X))$$

is the cobordism group of 1-connective n-dimensional quadratic Poincaré complexes (C, ψ_X) in $\mathbb{A}(\mathbb{Z}, X)$, with C an n-dimensional chain complex in $\mathbb{A}(\mathbb{Z}, X)$ and

$$\psi_X \in Q^X_n(C) = H_n(W \otimes_{\mathbb{Z}[\Sigma_2]} (C \otimes_{(\mathbb{Z},X)} C))$$

such that

$$(1+T)\psi_X \in H_n(C \otimes_{(\mathbb{Z},X)} C) \;=\; H_0(\mathrm{Hom}_{(\mathbb{Z},X)}(C^{n-*},C))$$

is a chain homotopy class of $(\mathbb{Z},X)$-module chain equivalences $C^{n-*} \to C$. Here, W is a free $\mathbb{Z}[\Sigma_2]$-module resolution of $\mathbb{Z}$

$$W : \cdots \to \mathbb{Z}[\Sigma_2] \xrightarrow{1-T} \mathbb{Z}[\Sigma_2] \xrightarrow{1+T} \mathbb{Z}[\Sigma_2] \xrightarrow{1-T} \mathbb{Z}[\Sigma_2]$$

and the generator $T \in \Sigma_2$ acts on $C \otimes_{(\mathbb{Z},X)} C$ by signed transposition. The quadratic L-group

$$L_n(\mathbb{Z}[\pi_1(X)]) \;=\; L_n(\mathbb{A}(\mathbb{Z}[\pi_1(X)]))$$

is the cobordism group of n-dimensional quadratic Poincaré complexes (C,ψ) over the group ring $\mathbb{Z}[\pi_1(X)]$ with

$$\psi \in Q_n(C) \;=\; H_n(W \otimes_{\mathbb{Z}[\Sigma_2]} (C \otimes_{\mathbb{Z}[\pi_1(X)]} C)) \;.$$

The *structure group* $\mathbb{S}_n(X)$ is the cobordism group of 1/2-connective $\mathbb{A}(\mathbb{Z}[\pi_1(X)])$-contractible $(n-1)$-dimensional quadratic Poincaré complexes in $\mathbb{A}(\mathbb{Z},X)$.

Definition 9.9 (Ranicki [29, 17.4])
The *total surgery obstruction* of an n-dimensional universal homology Poincaré complex X is the cobordism class of the peripheral quadratic Poincaré complex in $\mathbb{A}(\mathbb{Z},X)$

$$s(X) \;=\; (C,\psi_X) \in \mathbb{S}_n(X) \;.$$

Proposition 9.10 Let X be an n-dimensional universal Poincaré complex, with peripheral complex (C,ψ_X).
(i) *The following conditions are equivalent :*

(a) *X is an n-dimensional homology manifold,*

(b) *C is $\mathbb{A}(\mathbb{Z},X)$-contractible,*

(c) *$(1+T)\psi_X = 0 \in H_{n-1}(C \otimes_{(\mathbb{Z},X)} C)$.*

(ii) *The total surgery obstruction is such that $s(X)=0$ if (and for $n \geq 5$ only if) the polyhedron $|X|$ is homotopy equivalent to an n-dimensional topological manifold. The image of the total surgery obstruction*

$$t(X) \;=\; [s(X)] \in H_{n-1}(X;\mathbb{L}_\bullet)$$

is such that $t(X)=0$ if and only if the Spivak normal fibration $\nu_X : X \to BSG$ admits a topological reduction $\widetilde{\nu}_X : X \to BSTOP$.

Proof (i) (a) $\Longleftrightarrow$ (b) The peripheral quadratic complex (C, ψ_X) is $\mathbb{A}(\mathbb{Z}, X)$-contractible if and only if the peripheral chain complex C is $\mathbb{A}(\mathbb{Z}, X)$-contractible, if and only if X is a homology manifold (6.11).
(b) $\Longleftrightarrow$ (c) The map

$$H_n(X) \to H_{n-1}(C \otimes_{(\mathbb{Z},X)} C)$$

in the braid used in the proof of Theorem 6.13 sends the fundamental class $[X] \in H_n(X)$ to the homology class

$$(1+T)\psi_X \in H_{n-1}(C \otimes_{(\mathbb{Z},X)} C) \ ,$$

and C is $\mathbb{A}(\mathbb{Z}, X)$-contractible if and only if $(1+T)\psi_X = 0$.
(ii) See [29, 17.4]. □

Remark 9.11 There is also an R-coefficient version, for any commutative ring R. The *R-coefficient peripheral complex* (C, ψ_X) of an n-dimensional universal R-homology Poincaré complex X is the $\mathbb{A}(R[\pi_1(X)])$-contractible $(n-1)$-dimensional quadratic Poincaré complex in $\mathbb{A}(R, X)$ with

$$C \ = \ \mathcal{C}([X] \cap - : \Delta(X; R)^{n-*} \to \Delta(X'; R))_{*+1}$$

The *R-coefficient total surgery obstruction* ([29, 26.1]) of X is the cobordism class

$$s(X; R) \ = \ (C, \psi_X) \in \mathbb{S}_n(X; R) \ ,$$

taking value in the R-coefficient structure group fitting into the R-coefficient algebraic surgery exact sequence

$$\cdots \to H_n(X; \mathbb{L}_\bullet) \xrightarrow{A} \Gamma_n(R[\pi_1(X)] \to R) \to \mathbb{S}_n(X; R) \to H_{n-1}(X; \mathbb{L}_\bullet) \to \dots$$

with Γ_* the R-homology surgery obstruction groups of Cappell and Shaneson [3]. The R-coefficient total surgery obstruction is such that $s(X; R) = 0$ if (and for $n \geq 5$ only if) the polyhedron $|X|$ is R-homology equivalent to an n-dimensional topological manifold (Ranicki [29, 26.1]). See §14 below for the application to knot theory, with $R = \mathbb{Z}$.

10. Combinatorially controlled surgery theory

This section develops the combinatorial version of the topological controlled surgery theory proposed by Quinn [23] and Ranicki and Yamasaki [37]. In principle, it is possible to construct the topological theory using the combinatorial version and the Čech nerves of open covers (cf. Quinn [25, 1.4]), but this will not be done here.

A degree 1 map $f : M \to N$ of n-dimensional homology manifolds has acyclic point inverses if and only if

$$\Delta_*[M] - (f^! \otimes f^!)\Delta_*[N] = 0 \in H_n(M \times_N M)$$

by 7.7 (ii). For a normal map $(f,b): M \to N$ this obstruction will now be related to the chain level surgery obstruction. The Wall surgery obstruction of (f,b) was expressed in Ranicki [27],[29] as the cobordism class of a kernel n-dimensional quadratic Poincaré complex in $\mathbb{A}(\mathbb{Z}[\pi_1(N)])$

$$\sigma_*(f,b) \ = \ (\Delta(f^!),\psi_b) \in L_n(\mathbb{Z}[\pi_1(N)]) \ .$$

The quadratic class ψ_b will be refined to an N-controlled version $\psi_{b,N}$, with symmetrization

$$(1+T)\psi_{b,N} \ = \ \Delta_*[M] - (f^! \otimes f^!)\Delta_*[N] \in H_n(M \times_N M) \ .$$

Galewski and Stern [7], [8, 1.7] proved that the Spivak normal fibration $\nu_M : M \to BSG$ of a homology manifold M has a canonical topological bundle reduction $\nu_M : M \to BSTOP$, namely the canonical topological bundle reduction of the normal homology cobordism bundle $\nu_M : M \to BSH$, and that in fact for $\dim(M) \geq 5$ there exists a polyhedral topological manifold M_{TOP} with a map $M_{TOP} \to M$ with contractible point inverses.

Definition 10.1 A *normal map* $(f,b): M \to N$ from an n-dimensional homology manifold M to an n-dimensional Poincaré complex N is a degree 1 map $f : M \to N$ with a map of (stable) topological bundles $b : \nu_M \to \eta$ over f.

The surgery obstruction $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(N)])$ of a normal map $(f,b): M \to N$ is defined by Maunder [16] following Wall [43]. The surgery obstruction is shown in [16] to be such that $\sigma_*(f,b) = 0$ if (and for $n \geq 5$ only if) (f,b) is normal bordant to a homotopy equivalence. The surgery obstruction can also be defined using the chain complex method of Ranicki [26], [27].

Definition 10.2 The *N-controlled quadratic structure* of a normal map $(f,b): M \to N$ of n-dimensional homology manifolds is the element

$$\psi_{b,N} \ = \ \psi_{F,N}[N] \in Q_n^N(\Delta(M)) \ = \ H_n(E\Sigma_2 \times_{\Sigma_2} (M \times_N M))$$

with $\psi_{F,N} : H_*(N) \to Q_*^N(\Delta(M))$ the N-controlled version of the quadratic construction of [27, Chapter 1]

$$\psi_F : H_*(N) \to Q_*(\Delta(M)) \ = \ H_*(E\Sigma_2 \times_{\Sigma_2} (M \times M)) \ .$$

Here, $b : \nu_M \to \eta$ is a stable bundle map over f from the stable normal bundle ν_M of M, η is a bundle over N, $E\Sigma_2$ is a contractible space with a free Σ_2-action, the generator $T \in \Sigma_2$ acts on $M \times_N M$ by transposition

$$T : M \times_N M \ \to \ M \times_N M \ ; \ (x,y) \ \mapsto \ (y,x)$$

and $F : \Sigma^\infty N_+ \to \Sigma^\infty M_+$ is a geometric Umkehr map (= the S-dual of $T(b) : \Sigma^\infty T(\nu_M) \to \Sigma^\infty T(\eta)$) inducing $f^!$ on the chain level.

As usual, write $W = \Delta(E\Sigma_2)$, so that

$$\begin{aligned} Q_n(\Delta(M)) &= H_n(W \otimes_{\mathbb{Z}[\Sigma_2]} (\Delta(M) \otimes_{\mathbb{Z}} \Delta(M))) \ , \\ Q_n^N(\Delta(M)) &= H_n(W \otimes_{\mathbb{Z}[\Sigma_2]} (\Delta(M) \otimes_{(\mathbb{Z},N)} \Delta(M))) \ . \end{aligned}$$

Remark 10.3 As defined in [27] the quadratic construction ψ_F only gives an element $\psi_b = \psi_F[N] \in Q_n(\Delta(M))$. There are two ways of checking that there is a lift of ψ_b to an N-controlled element $\psi_{b,N} \in Q_n^N(\Delta(M))$:

- Note that the natural chain level transformation in [27, Chapter 1]

  $$\psi_F : \Delta(N) \to W \otimes_{\mathbb{Z}[\Sigma_2]} (\Delta(M) \otimes_{\mathbb{Z}} \Delta(M))$$

  factors through

  $$\psi_{F,N} : \Delta(N) \to W \otimes_{\mathbb{Z}[\Sigma_2]} (\Delta(M) \otimes_{(\mathbb{Z},N)} \Delta(M))$$

  exactly as for the Alexander-Whitney diagonal chain approximation (5.3 (iii)), so that

  $$\psi_F : H_n(N) \xrightarrow{\psi_{F,N}} Q_n^N(\Delta(M)) \longrightarrow Q_n(\Delta(M)) \ .$$

- Note that (f,b) determines an algebraic normal map in $\mathbb{A}(\mathbb{Z},N)$ in the sense of [29, 2.16], with a corresponding quadratic class $\psi_{b,N}$.

An n-dimensional homology manifold M determines an n-dimensional symmetric Poincaré complex in $\mathbb{A}(\mathbb{Z},N)$

$$\sigma_N^*(M) = (\Delta(M), \Delta_*[M] \in Q_N^n(\Delta(M)))$$

for any simplicial map $M \to N$. Here, the Q-group is defined by

$$Q_N^n(\Delta(M)) = H_n(\mathrm{Hom}_{\mathbb{Z}[\Sigma_2]}(W, \Delta(M) \otimes_{(\mathbb{Z},N)} \Delta(M))) \ ,$$

and $\Delta_* : H_n(M) \to Q_N^n(\Delta(M))$ is induced by the Alexander-Whitney diagonal chain approximation. (Note that Δ_* is an isomorphism for $1 : M \to N = M$). The fundamental $\mathbb{L}^\bullet(\mathbb{Z})$-homology class of M (Ranicki [29, 16.16]) is the cobordism class

$$[M]_{\mathbb{L}} = \sigma_M^*(M) \in L^n(\mathbb{A}(\mathbb{Z},M)) = H_n(M; \mathbb{L}^\bullet(\mathbb{Z})) \ .$$

For a degree 1 map $f : M \to N$ the algebraic mapping cone of the Umkehr chain map $f^! : \Delta(N) \to \Delta(M)$ is a $(\mathbb{Z},N)$-module chain complex

$$\Delta(f^!) = \mathcal{C}(f^! : \Delta(N) \to \Delta(M)) \ .$$

Let $e : \Delta(M) \to \Delta(f^!)$ be the inclusion. The kernel n-dimensional symmetric Poincaré complex in $\mathbb{A}(\mathbb{Z},N)$

$$\sigma_N^*(f) = (\Delta(f^!), (e \otimes e)\Delta_*[M])$$

is such that up to homotopy equivalence

$$\sigma^*_N(M) \ = \ \sigma^*_N(N)\oplus\sigma^*_N(f) \ ,$$

with cobordism class the difference of the fundamental $\mathbb{L}^\bullet(\mathbb{Z})$-homology classes

$$\sigma^*_N(f) \ = \ f_*[M]_{\mathbb{L}}-[N]_{\mathbb{L}}\in H_n(N;\mathbb{L}^\bullet(\mathbb{Z})) \ .$$

Definition 10.4 (Ranicki [29, 18.3])
The *normal invariant* of a normal map $(f,b):M\to N$ of n-dimensional homology manifolds is the cobordism class

$$\begin{aligned}[f,b]_{\mathbb{L}} \ &= \ (\Delta(f^!),(e\otimes e)\psi_{b,N})\\ &\in L_n(\mathbb{A}(\mathbb{Z},N)) \ = \ H_n(N;\mathbb{L}_\bullet) \ = \ [N,G/TOP] \ .\end{aligned}$$

The normal invariant of 10.4 is a (mild) generalization of the traditional normal invariant in surgery theory, and has the following properties:

- $[f,b]_{\mathbb{L}}\in H_n(N;\mathbb{L}_\bullet)$ is a normal bordism invariant, such that $[f,b]_{\mathbb{L}}=0$ if f has acyclic point inverses.
- For a normal map of polyhedral topological manifolds $[f,b]_{\mathbb{L}}=0$ if (and for $n\geq 5$ only if) (f,b) is normal bordant to a homeomorphism.
- The assembly of $[f,b]_{\mathbb{L}}$ in the Wall surgery group is the surgery obstruction of (f,b)
$$A[f,b]_{\mathbb{L}} \ = \ \sigma_*(f,b)\in L_n(\mathbb{Z}[\pi_1(N)]) \ .$$
- The image of $\sigma_*(f,b)$ in the homology surgery Γ-group of Cappell and Shaneson [3]
$$A^H[f,b]_{\mathbb{L}} \ = \ \sigma^H_*(f,b)\in\Gamma_n(\mathbb{Z}[\pi_1(N)]\to\mathbb{Z})$$
is such that $\sigma^H_*(f,b)=0$ if (and for $n\geq 5$ only if) (f,b) is normal bordant to a homology equivalence.

For PL manifolds these are direct applications of the surgery obstruction theory of Wall [43]. In the general case, apply the extension of the theory to polyhedral homology manifolds due to Maunder [16], or else combine with the result of Galewski and Stern [7], [8, 1.7] that every polyhedral homology manifold can be resolved by a polyhedral topological manifold and the TOP version of Wall's theory.

Proposition 10.5 The N-controlled quadratic class $\psi_{b,N}$ of a normal map $(f,b):M\to N$ of n-dimensional homology manifolds determines a kernel n-dimensional quadratic Poincaré complex in $\mathbb{A}(\mathbb{Z},N)$

$$\sigma^N_*(f,b) \ = \ (\Delta(f^!),(e\otimes e)\psi_{b,N})$$

with cobordism class the normal invariant of (f,b)

$$[f,b]_{\mathbb{L}} \;=\; \sigma^N_*(f,b) \in L_n(\mathbb{A}(\mathbb{Z},N)) \;=\; H_n(N;\mathbb{L}_\bullet)\ .$$

The Poincaré duality chain equivalence of the symmetrization

$$(1+T)\sigma^N_*(f,b) \;=\; \sigma^*_N(f)$$

is such that up to chain homotopy

$$(1+T)(e\otimes e)\psi_{b,N} \;=\; (e\otimes e)\Delta_*[M] : \Delta(f^!)^{n-*} \to \Delta(f^!)\ ,$$

which is the obstruction to f having acyclic point inverses (7.7 (ii))

$$\begin{aligned}(1+T)(e\otimes e)\psi_{b,N} \;&=\; \Delta_*[M] - (f^!\otimes f^!)\Delta_*[N]\\ &\in H_n(\Delta(f^!)\otimes_{(\mathbb{Z},N)}\Delta(f^!)) \;=\; H_n(M\times_N M,\Delta_M)\ \ (7.3\ (\mathrm{ii}))\ .\end{aligned}$$

Proof The identity

$$(1+T)\psi_{b,N} \;=\; \Delta_*[M] - (f^!\otimes f^!)\Delta_*[N] \in H_n(M\times_N M)$$

is just the N-controlled analogue of the standard property of the quadratic construction ([27])

$$(1+T)\psi_b \;=\; \Delta_*[M] - (f^!\otimes f^!)\Delta_*[N] \in H_n(M\times M)\ .$$

□

Remark 10.6 The quadratic class $\psi_{b,N}\in Q^N_n(\Delta(M))$ can be defined for any degree 1 map $f:M\to N$ of n-dimensional universal Poincaré complexes with a map $b:\nu_M\to\nu_N$ of the Spivak normal fibrations, with all the properties of $\psi_{b,N}$ in 10.2 except that the n-dimensional quadratic complex $(\Delta(f^!),(e\otimes e)\psi_{b,N})$ in $\mathbb{A}(\mathbb{Z},N)$ will only be Poincaré in $\mathbb{A}(\mathbb{Z}[\pi_1(N)])$.

A homotopy equivalence $f:M\to N$ of n-dimensional homology manifolds can be regarded as a normal map $(f,b):M\to N$ with $b:\nu_M\to(f^{-1})^*\nu_M$.

Definition 10.7 (Ranicki [29, 18.3])
The *structure invariant* of a homotopy equivalence $f:M\to N$ of n-dimensional homology manifolds is the cobordism class

$$s(f) \;=\; (\Delta(f^!),\psi_{b,N}) \in \mathbb{S}_{n+1}(N)$$

with image the normal invariant $[f,b]_{\mathbb{L}}\in H_n(N;\mathbb{L}_\bullet)$.

Proposition 10.8 (Ranicki [29, 18.3])
The structure invariant of a homotopy equivalence $f:M\to N$ of polyhedral n-dimensional topological manifolds is such that $s(f)=0\in\mathbb{S}_n(N)$ if (and for $n\geq 5$ only if) f is h-cobordant to a homeomorphism.

In §13 below there will be obtained controlled versions of 10.7 and 10.8.

There is also a simple version of the structure invariant, which is defined for a simple homotopy equivalence $f : M \to N$ of n-dimensional homology manifolds, taking value in the relative group $\mathbb{S}^s_n(N)$ in the exact sequence

$$\cdots \to H_n(N;\mathbb{L}_\bullet) \xrightarrow{A} L^s_n(\mathbb{Z}[\pi_1(N)]) \to \mathbb{S}^s_n(N) \to H_{n-1}(N;\mathbb{L}_\bullet) \to \dots \; .$$

Remark 10.9 The simple homotopy theory version of surgery theory allows an application of the s-cobordism theorem, to obtain:
The simple structure invariant of a simple homotopy equivalence $f : M \to N$ of polyhedral n-dimensional topological manifolds is such that $s(f) = 0 \in \mathbb{S}^s_n(N)$ if (and for $n \geq 5$ only if) f is homotopic to a homeomorphism.

Proposition 10.10 (i) *A map $f : M \to N$ of simplicial complexes with acyclic point inverses is simple, with $\tau(f) = 0 \in Wh(\pi_1(N))$.*
(ii) *A homotopy equivalence $f : M \to N$ of n-dimensional homology manifolds with acyclic point inverses is simple, with simple structure invariant $s(f) = 0 \in \mathbb{S}^s_n(N)$.*
(iii) *For $n \geq 5$ a homotopy equivalence $f : M \to N$ of n-dimensional polyhedral topological manifolds with acyclic point inverses is homotopic to a homeomorphism.*
Proof (i) As in the proof of 9.4 use the assembly functor of Ranicki and Weiss [34]

$$A : \mathbb{A}(\mathbb{Z},N) \to \mathbb{A}(\mathbb{Z}[\pi_1(N)]) \; ; \; A \;=\; \sum_{\sigma\in N} A(\sigma) \mapsto A(\widetilde{N}) \;=\; \sum_{\widetilde{\sigma}\in\widetilde{N}} A(p\widetilde{\sigma})$$

with $p : \widetilde{N} \to N$ the universal covering projection. A choice of basis for each of the f.g. free $\mathbb{Z}$-modules $A(\sigma)$ $(\sigma \in N)$ determines a basis for the assembly f.g. free $\mathbb{Z}[\pi_1(N)]$-module $A(\widetilde{N})$, uniquely up to multiplication by the group elements $g \in \pi_1(N)$. Thus if C is a based $(\mathbb{Z},N)$-module chain complex such that $C(\widetilde{N})$ is contractible there is a well-defined Whitehead torsion

$$\tau(C(\widetilde{N})) \in Wh(\pi_1(N)) \; .$$

For any simplicial map $f : M \to N$ there is defined a based $(\mathbb{Z},N)$-module chain complex

$$C \;=\; \mathcal{C}(f : \Delta(M) \to \Delta(N))$$

with

$$\begin{aligned} &C(\sigma) \;=\\ &\mathcal{C}(f| : \Delta(f^{-1}D(\sigma,N), f^{-1}\partial D(\sigma,N)) \to \Delta(D(\sigma,N),\partial D(\sigma,N))) \;\; (\sigma \in N) \; .\end{aligned}$$

The assembly of C is the based f.g. free $\mathbb{Z}[\pi_1(N)]$-module chain complex

$$C(\widetilde{N}) \;=\; \mathcal{C}(\widetilde{f} : \Delta(\widetilde{M}) \to \Delta(\widetilde{N}))$$

with $\widetilde{M} = f^*\widetilde{N}$ the pullback cover of M. If $\widetilde{f}$ is a $\mathbb{Z}[\pi_1(N)]$-module chain equivalence (e.g. if f is a homotopy equivalence) the torsion of f is defined by

$$\tau(f) \ = \ \tau(C(\widetilde{N})) \in Wh(\pi_1(N)) \ .$$

If f has acyclic point inverses each $C(\sigma)$ is contractible, and $\widetilde{f}$ is a $\mathbb{Z}[\pi_1(N)]$-module chain equivalence, with the torsion of f such that

$$\tau(f) \ = \ \tau(C(\widetilde{N})) \in \mathrm{im}(Wh(\{1\}) \to Wh(\pi_1(N))) \ = \ \{0\} \ ,$$

so that $\tau(f) = 0$. (This uses $Wh(\{1\}) = 0$, exactly as in the proof of the combinatorial invariance of Whitehead torsion in Milnor [19]).
(ii) The simple structure invariant $s(f)$ is the cobordism class of the simple $\mathbb{Z}[\pi_1(N)]$-contractible n-dimensional quadratic Poincaré complex $(\Delta(f^!), \psi_{b,N})$ in $\mathbb{A}(\mathbb{Z}, N)$ with

$$f^! \ = \ f^{-1} : \Delta(N) \to \Delta(M) \ .$$

By (i) $\Delta(f^!)$ is simple $(\mathbb{Z}, N)$-contractible, and so represents 0 in the simple structure group.
(iii) By (ii) f is a simple homotopy equivalence with zero simple structure invariant, so that 10.9 applies. □

Remark 10.11 Let $n \geq 5$.
(i) A map $f : M \to N$ of n-dimensional PL manifolds with acyclic point inverses is homotopic through maps with acyclic point inverses to a PL homeomorphism if and only if the Cohen-Sato-Sullivan obstruction

$$c^H(f) \in H^3(N; \theta_3^H)$$

is 0, with θ_3^H the Kervaire-Milnor cobordism group of oriented 3-dimensional PL homology spheres (Ranicki [31, pp.26–28]). The obstruction is 0 if f has contractible point inverses. The obstruction is the homotopy class of the map

$$c^H(f) : N \to H/PL \simeq K(\theta_3^H, 3)$$

classifying the difference between the PL reductions of the normal homology cobordism bundles of M and N. The combination of the Kirby-Siebenmann result

$$TOP/PL \simeq K(\mathbb{Z}_2, 3)$$

with the work of Galewski and Stern [7] shows that the various classifying spaces are related by a commutative braid of fibration sequences

$$\begin{tikzcd}
K(\theta_3^H,3) \arrow[dr, "\alpha"] \arrow[rr, bend left] & & BPL \arrow[dr] \arrow[rr, bend left] & & BTOP \arrow[dr, "\kappa"] \arrow[rr, bend left] & & K(\ker(\alpha),5) \\
& K(\mathbb{Z}_2,3) \arrow[ur] \arrow[dr, "\delta"] & & BH \arrow[ur] \arrow[dr, "c^H"] & & K(\mathbb{Z}_2,4) \arrow[ur, "\delta"] & \\
& & K(\ker(\alpha),4) \arrow[ur] \arrow[rr, bend right] & & K(\theta_3^H,4) \arrow[ur, "\alpha"] & &
\end{tikzcd}$$

with $\alpha : \theta_3^H \to \mathbb{Z}_2$ the Rochlin map ([31, p.26]).
(ii) A homeomorphism $f : M \to N$ of n-dimensional PL manifolds is homotopic to a PL homeomorphism if and only if the Casson-Sullivan obstruction

$$\kappa(f) \ = \ \alpha(c^H(f)) \in H^3(N; \mathbb{Z}_2)$$

is 0 ([31, p.14]). The obstruction is the homotopy class of the map

$$\kappa(f) : N \to TOP/PL \simeq K(\mathbb{Z}_2, 3)$$

classifying the difference between the PL reductions of the normal topological block bundles of M and N.
(iii) A homeomorphism $f : M \to N$ of n-dimensional PL manifolds is homotopic to a simplicial map with acyclic point inverses if and only if the Galewski-Matumoto-Stern obstruction

$$\delta\kappa(f) \in H^4(N; \ker(\alpha))$$

is 0 ([31, p.28]).
(iv) Galewski and Stern [8] proved that an n-dimensional topological manifold N is polyhedral (i.e. can be triangulated by a polyhedron) if and only if the element

$$\delta\kappa(\nu_N) \in H^5(N; \ker(\alpha))$$

is 0. In particular, this obstruction is 0 for the topological manifold $N = M_{TOP}$ resolving a (polyhedral) homology manifold M given by Galewski and Stern [7], so that M_{TOP} can be taken to be polyhedral.

11. Intersections and self-intersections

The chain complex methods of this paper will now be applied to obtained a combinatorially controlled homology version of the intersection theory of homology submanifolds.

Definition 11.1 Given maps of X-controlled R-homology Poincaré complexes

$$f_1 : M_1 \to N \quad , \quad f_2 : M_2 \to N$$

with

$$\dim(M_1) \ = \ m_1 \quad , \quad \dim(M_2) \ = \ m_2 \quad , \quad \dim(N) \ = \ n$$

define the *X-controlled intersection class*

$$[M_1 \times_X M_2] \in H_{m_1+m_2-n}(M_1 \times_X M_2; R)$$

to be the chain homotopy class of the (R,X)-module chain map

$$\begin{aligned}\Delta(M_1;R)^{m_1-*} \simeq \Delta(M_1;R) &\xrightarrow{f_1} \Delta(N;R) \simeq \Delta(N;R)^{n-*} \\ &\xrightarrow{(f_2)^*} \Delta(M_2;R)^{n-*} \simeq \Delta(M_2;R)_{*+m_2-n}\ ,\end{aligned}$$

using the identifications

$$\begin{aligned}&H_{m_1+m_2-n}(M_1 \times_X M_2;R) \\ &\qquad = H_{m_1+m_2-n}(\Delta(M_1;R) \otimes_{(R,X)} \Delta(M_2;R)) \\ &\qquad = H_0(\mathrm{Hom}_{(R,X)}(\Delta(M_1;R)^{m_1-*}, \Delta(M_2;R)_{*+m_2-n}))\ .\end{aligned}$$

In terms of the Umkehr (R,X)-module chain maps (7.1)

$$\begin{aligned}f_1^! : \Delta(N;R) \simeq \Delta(N;R)^{n-*} &\xrightarrow{(f_1)^*} \Delta(M_1;R)^{n-*} \simeq \Delta(M_1;R)_{*+m_1-n}\ , \\ f_2^! : \Delta(N;R) \simeq \Delta(N;R)^{n-*} &\xrightarrow{(f_2)^*} \Delta(M_2;R)^{n-*} \simeq \Delta(M_2;R)_{*+m_2-n}\end{aligned}$$

the X-controlled intersection class is given by the evaluation on the fundamental class $[N] \in H_n(N)$ of the composite

$$H_n(N;R) \xrightarrow{\Delta} H_n(N \times_X N;R) \xrightarrow{f_1^! \otimes f_2^!} H_{m_1+m_2-n}(M_1 \times_X M_2;R)\ ,$$

that is

$$[M_1 \times_N M_2]\ =\ (f_1^! \otimes f_2^!)\Delta[N] \in H_{m_1+m_2-n}(M_1 \times_X M_2;R)\ .$$

For the remainder of §11 $R = \mathbb{Z}$, $X = N$, i.e. only homology manifolds will be considered.

Definition 11.2 Embeddings of homology manifolds

$$f_1 : (M_1)^{m_1} \to N^n \quad , \quad f_2 : (M_2)^{m_2} \to N^n$$

are *transverse* if

- the intersection $M_1 \cap M_2$ is an $(m_1 + m_2 - n)$-dimensional homology manifold,
- the product embedding $f_1 \times f_2 : M_1 \times M_2 \to N \times N$ has a normal homology cobordism bundle

  $$\nu_{f_1 \times f_2}\ :\ M_1 \times M_2 \to BSH(2n - m_1 - m_2)$$

  whose restriction to $M_1 \cap M_2$ (viewed as a submanifold of $M_1 \times M_2$) is a normal homology cobordism bundle for $M_1 \cap M_2 \subset N$.

(Compare with the notion of homology manifold transversality considered by Galewski and Stern [7, Chapter 3].)

Proposition 11.3 The N-controlled intersection class of transversely intersecting embeddings of homology manifolds $f_1 : (M_1)^{m_1} \to N^n$, $f_2 : (M_2)^{m_2} \to N^n$ is the fundamental class of the (m_1+m_2-n)-dimensional homology submanifold

$$M_1 \times_N M_2 \ = \ M_1 \cap M_2 \subset N \ ,$$

that is

$$[M_1 \times_N M_2] \ = \ [M_1 \cap M_2] \in H_{m_1+m_2-n}(M_1 \times_N M_2) \ .$$

Proof The normal homology cobordism bundle

$$\nu = \nu_{M_1 \cap M_2 \subset N} : M_1 \cap M_2 \to BSH(2n - m_1 - m_2)$$

is such that there are defined isomorphisms

$$\begin{aligned} H_*(N, N \backslash (M_1 \cap M_2)) \ &\cong \ H_*(N, \mathrm{cl.}(N \backslash E(\nu))) \\ &\cong \ H_*(E(\nu), S(\nu)) \\ &\cong \ H_{*+m_1+m_2-2n}(M_1 \cap M_2) \ . \end{aligned}$$

The identity $[M_1 \times_N M_2] = [M_1 \cap M_2]$ follows from the evaluation of $[N] \in H_n(N)$ in the commutative diagram

$$\begin{tikzcd} H_n(N) \arrow[r, "\cong"] \arrow[d] & H_n(N \times_N N) \arrow[d, "f_1^! \otimes f_2^!"] \\ H_n(N, N \backslash (M_1 \cap M_2)) \arrow[r, "\cong"] & H_{m_1+m_2-n}(M_1 \times_N M_2) \ . \end{tikzcd}$$

□

Given a map $f : M \to N$ define the maps

$$\begin{aligned} & i : M \to M \times_N M \ ; \ x \mapsto (x, x) \ , \\ & j : M \times_N M \to N \ ; \ (x, y) \mapsto f(x) = f(y) \ , \\ & k : M \times_N N \to M \ ; \ (x, y) \mapsto x \end{aligned}$$

(as in §7) such that

$$ji \ = \ f : M \to N \quad , \quad ki \ = \ 1 : M \to M \ .$$

The induced maps

$$i_* : H_*(M) \to H_*(M \times_N M)$$

are split injections, with

$$H_*(M \times_N M) \ = \ H_*(M) \oplus H_*(M \times_N M, \Delta_M) \ .$$

If M, N are homology manifolds with $\dim(M) = m$, $\dim(N) = n$ the Umkehr $(\mathbb{Z}, N)$-module chain maps

$$f^! : \Delta(N) \to \Delta(M)_{*+m-n} \quad , \quad j^! : \Delta(N) \to \Delta(M \times_N M)_{*+2m-2n}$$

are defined as in 7.1,7.8.

Proposition 11.4 Let $f : M^m \to N^n$ be a map of homology manifolds.
(i) *The N-controlled intersection class of f with itself*

$$[M \times_N M] \;=\; j^![N] \in H_{2m-n}(M \times_N M)$$

is such that $[M \times_N M] = 0 \in H_{2m-n}(M \times_N M, \Delta_M)$ if and only if

$$[M \times_N M] \in \mathrm{im}(i_* : H_{2m-n}(M) \to H_{2m-n}(M \times_N M)) \ .$$

(ii) *If f is an embedding then*

$$[M \times_N M] \;=\; [M] \cap e(\nu_f) \in H_{2m-n}(M \times_N M) \;=\; H_{2m-n}(M) \ ,$$

with $e(\nu_f) \in H^{n-m}(M)$ the Euler class of the normal homology cobordism bundle $\nu_f : M \to BSH(n-m)$.
Proof (i) Immediate from the definition of $[M \times_N M]$, and the (split) exact sequence

$$0 \to H_{2m-n}(M) \xrightarrow{i_*} H_{2m-n}(M \times_N M) \to H_{2m-n}(M \times_N M, \Delta_M) \to 0 \ .$$

(ii) For an embedding f

$$\begin{aligned} i \;&=\; 1 : M \to M \times_N M \;=\; M \ , \\ j \;&=\; f : M \times_N M \;=\; M \to N \ . \end{aligned}$$

It follows from the commutative diagram

$$\begin{tikzcd}
H_n(N) \arrow[rrr, "\Delta_N"] \arrow[dr, "\cong"] \arrow[ddd, "j^!"'] & & & H_n(N\times N) \arrow[dl, "\cong"'] \arrow[ddd, "f^! \otimes f^!"] \\
& H^n(N\times N, N\times N\backslash\Delta_N) \arrow[r] \arrow[d, "(f\times f)^*"'] & H^n(N\times N) \arrow[d, "(f\times f)^*"] & \\
& H^n(M\times M, M\times M\backslash\Delta_M) \arrow[r] \arrow[dl, "\cong"'] & H^n(M\times M) \arrow[dr, "\cong"] & \\
H_{2m-n}(M) \arrow[rrr, "\Delta_M"] & & & H_{2m-n}(M\times M)
\end{tikzcd}$$

that

$$\Delta_M[M \times_N M] \;=\; \Delta_M j^![N] \;=\; (f^! \otimes f^!)\Delta_N[N] \in H_{2m-n}(M \times M) \ .$$

The Pontrjagin-Thom collapse map

$$F : N_+ \to N/\mathrm{cl.}(N\backslash E(\nu_f)) \;=\; E(\nu_f)/S(\nu_f) \;=\; T(\nu_f)$$

induces the Umkehr $\mathbb{Z}$-module chain map

$$F \;=\; f^! : \Delta(N) \;=\; \widetilde{\Delta}(N_+) \to \widetilde{\Delta}(T(\nu_f)) \simeq \Delta(M)_{*+m-n} \;.$$

It follows from the commutative diagram

$$\begin{tikzcd}
H_n(N) \arrow[r, "F = f^!"] \arrow[d, "\Delta_N"'] & \widetilde{H}_n(T(\nu_f)) \cong H_m(M) \arrow[d, "\Delta_{T(\nu_f)}"] \arrow[dr, "- \cap e(\nu_f)"] & \\
 & & H_{2m-n}(M) \arrow[dl, "\Delta_M"] \\
H_n(N\times N) \arrow[r, "f^! \otimes f^!"] & \widetilde{H}_n(T(\nu_f) \wedge T(\nu_f)) \cong H_{2m-n}(M \times M) &
\end{tikzcd}$$

that

$$\begin{aligned}
(f^! \otimes f^!)\Delta_N[N] \;&=\; \Delta_M(f^![N] \cap e(\nu_f)) \\
&=\; \Delta_M([M] \cap e(\nu_f)) \in H_{2m-n}(M \times M) \;.
\end{aligned}$$

Thus

$$\begin{aligned}
\Delta_M[M \times_N M] \;&=\; (f^! \otimes f^!)\Delta_N[N] \\
&=\; \Delta_M([M] \cap e(\nu_f)) \in H_{2m-n}(M \times M) \;.
\end{aligned}$$

Now $\Delta_M : H_{2m-n}(M) \to H_{2m-n}(M \times M)$ is a (split) injection, so that

$$[M \times_N M] \;=\; [M] \cap e(\nu_f) \in H_{2m-n}(M) \;.$$

□

Remark 11.5 (i) If $f : M^m \to N^n$ is a map of homology manifolds with an N-controlled map

$$a : (M \times M, M \times M\backslash\Delta_M) \to (N \times N, N \times N\backslash\Delta_N)$$

such that the diagram

$$\begin{tikzcd}
 & (M \times M, M \times M\backslash M \times_N M) \arrow[dl, "c"'] \arrow[dr, "f \times f"] & \\
(M \times M, M \times M\backslash\Delta_M) \arrow[rr, "a"] & & (N \times N, N \times N\backslash\Delta_N)
\end{tikzcd}$$

is N-controlled homotopy commutative then

$$[M \times_N M] \in \operatorname{im}(i_* : H_{2m-n}(M) \to H_{2m-n}(M \times_N M)) ,$$

where c is the inclusion. (For $m = n$ this is essentially the same as 8.6.) The property $ac \simeq f \times f$ is related to the necessary and sufficient condition obtained by Haefliger [10] for a map $f : M^m \to N^n$ of differentiable manifolds in the stable range $2n \geq 3(m+1)$ to be homotopic to an embedding, namely that $f \times f : M \times M \to N \times N$ be Σ_2-equivariantly homotopic to a map $h : M \times M \to N \times N$ with $h^{-1}(\Delta_N) = \Delta_M$, so that h defines a map of pairs

$$h : (M \times M, M \times M \backslash \Delta_M) \to (N \times N, N \times N \backslash \Delta_N) .$$

The action of Σ_2 is by transposition $(x, y) \mapsto (y, x)$. See 11.11 below for a more detailed discussion of the case $n = 2m$.
(ii) The identity of 11.4 (ii) for an embedding $f : M^m \to N^n$ can also be proved geometrically, whenever there exists an isotopic embedding $f' : M' = M \to N$ such that:

- $M, M' \subset N$ intersect transversely in a $(2m-n)$-dimensional homology submanifold $M \cap M' \subset N$,
- $[M \cap M'] \in H_{2m-n}(M)$ is Poincaré dual to $e(\nu_f) \in H^{n-m}(M)$,
- $[M \times_N M] = [M \cap M'] \in H_{2m-n}(M)$.

Applying 11.3, it follows that

$$\begin{aligned} [M \times_N M] &= [M \cap M'] \\ &= [M] \cap e(\nu_f) \in H_{2m-n}(M \times_N M) \ = \ H_{2m-n}(M) . \end{aligned}$$

(iii) Let $f : M \to X$ be a degree 1 map of n-dimensional manifolds, which is covered by a stable bundle map

$$b : \tau_M \oplus \epsilon^\infty \to \tau_X \oplus \epsilon^\infty .$$

The induced stable map of Thom spaces

$$T(b) : T(\tau_M \oplus \epsilon^\infty) = \Sigma^\infty T(\tau_M) \to T(\tau_X \oplus \epsilon^\infty) = \Sigma^\infty T(\tau_X)$$

sends the Thom class of τ_X to the Thom class of τ_M

$$T(b)^* : \widetilde{H}^n(T(\tau_X)) \to \widetilde{H}^n(T(\tau_M)) \ ; \ U_X \to U_M$$

The images of the Thom classes U_M, U_X under the maps

$$\begin{aligned} \text{inclusion}^* : \widetilde{H}^n(T(\tau_M)) &\cong H^n(M \times M, M \times M \backslash \Delta_M) \\ &\to H^n(M \times M, M \times M \backslash (f \times f)^{-1}\Delta_X) , \\ f \times f^* : H^n(X \times X, X \times X \backslash \Delta_X) &\to H^n(M \times M, M \times M \backslash (f \times f)^{-1}\Delta_X) \end{aligned}$$

are not the same (in general), since the diagram I in

$$\begin{tikzcd}
\widetilde{H}^n(T(\tau_M)) & & \widetilde{H}^n(T(\tau_X)) \arrow[ll, "T(b)^*"'] \\
\cong & & \cong \\
H^n(M\times M, M\times M\backslash\Delta_M) \arrow[dd] \arrow[dr] & I & H^n(X\times X, X\times X\backslash\Delta_X) \arrow[dl] \arrow[dd] \\
& H^n(M\times M, M\times M\backslash(f\times f)^{-1}\Delta_X) \arrow[dl] \arrow[dr] & \\
H^n(M\times M) & & H^n(X\times X) \arrow[ll, "(f\times f)^*"']
\end{tikzcd}$$

does not commute. However, it does commute in the unstable case $b : \tau_M \to \tau_X$, with a commutative diagram

$$\begin{tikzcd}
(M\times M, M\times M\backslash\Delta_M) \arrow[r, "b"] \arrow[d] & (X\times X, X\times X\backslash\Delta_X) \arrow[d] \\
M \arrow[r, "f"] & X
\end{tikzcd}$$

Definition 11.6 The *homotopy double point set* $P(f)$ of a map $f : M \to N$ is the homotopy pullback in the diagram

$$\begin{tikzcd}
P(f) \arrow[r] \arrow[d] & M \arrow[d, "f"] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

Thus $P(f)$ is the space of triples (x, y, ω) with $x, y \in M$ and $\omega : [0,1] \to N$ a path such that

$$\omega(0) \ = \ f(x) \ \ , \ \ \omega(1) \ = \ f(y) \in N \ .$$

The space $P(f)$ is a homotopy model for the actual double point set $M \times_N M$, and there is an evident inclusion

$$M \times_N M \to P(f) \ ; \ (x, y) \mapsto (x, y, \omega)$$

with $\omega(t) = f(x) = f(y) \in N$ $(0 \le t \le 1)$.

Proposition 11.7 If $f : M^m \to N^n$ is a map of homology manifolds the image of $[M \times_N M] \in H_{2m-n}(M \times_N M, \Delta_M)$ in $H_{2m-n}(P(f), \Delta_M)$ is a homotopy invariant of f, which is 0 if f is homotopic to an embedding.
Proof Immediate from 11.4. □

Remark 11.8 See Hatcher and Quinn [11] for the systematic use of homotopy pullbacks to define intersection invariants of submanifolds.

Next, consider an immersion of an m-dimensional homology manifold in an n-dimensional homology manifold

$$f : M^m \to N^n$$

with $m < n$. Let $\nu_f : M \to BSH(n-m)$ classify the normal homology cobordism bundle, so that there is defined a fibration

$$(D^{n-m}, S^{n-m-1}) \to (E(\nu_f), S(\nu_f)) \to M$$

and the Thom space is given by

$$T(\nu_f) \ = \ E(\nu_f)/S(\nu_f) \ .$$

For sufficiently large k there exists a map $g : M \to \mathrm{int}(D^k)$ such that

$$f \times g : M \to N \times D^k \ ; \ x \mapsto (f(x), g(x))$$

is an embedding with normal homology cobordism bundle

$$\nu_{f\times g} \ = \ \nu_f \oplus \epsilon^k : M \to BSH(n-m+k) \ .$$

The corresponding Pontrjagin-Thom collapse map

$$F : \Sigma^k N_+ \ = \ N \times D^k / N \times S^{k-1} \to T(\nu_f \oplus \epsilon^k) \ = \ \Sigma^k T(\nu_f)$$

induces the Umkehr $(\mathbb{Z}, N)$-module chain map

$$f^! : \Delta(N) \simeq \Delta(N)^{n-*} \xrightarrow{f^*} \Delta(M)^{n-*} \simeq \Delta(M)_{*+m-n} \simeq \widetilde{\Delta}(T(\nu_f)) \ .$$

Let

$$\nu_f \times_N \nu_f : M \times_N M \to BSH(2(n-m))$$

be the homology cobordism bundle defined by the restriction of the product

$$\nu_f \times \nu_f : M \times M \to BSH(2(n-m))$$

to $M \times_N M \subseteq M \times M$, with Thom space

$$\begin{aligned} T(\nu_f \times_N \nu_f) \ &= \ E(\nu_f \times_N \nu_f)/S(\nu_f \times_N \nu_f) \\ &= \ E(\nu_f) \times_N E(\nu_f)/(E(\nu_f) \times_N S(\nu_f) \cup S(\nu_f) \times_N E(\nu_f)) \ . \end{aligned}$$

Definition 11.9 The *N-controlled self-intersection class* of an immersion of homology manifolds $f : M^m \to N^n$ is the N-controlled version of the homology class of Ranicki [27, pp.279-282]

$$\begin{aligned} \mu_N(f) \ &= \ -\psi_{F,N}[N] \\ &\in \widetilde{H}_n(E\Sigma_2 \ltimes_{\Sigma_2} T(\nu_f \times_N \nu_f)) \\ &\qquad = \ H_{2m-n}(E\Sigma_2 \times_{\Sigma_2} (M \times_N M); \mathbb{Z}^{(-)^{n-m}}) \\ &\qquad = \ H_{2m-n}(W\mathbb{Z}^{(-)^{n-m}} \otimes_{\mathbb{Z}[\Sigma_2]} (\Delta(M) \otimes_{(\mathbb{Z},N)} \Delta(M))) \end{aligned}$$

with $\psi_{F,N}$ (9.2) the N-controlled version of the quadratic construction ψ_F of [27, Chapter 1] applied to a geometric Umkehr map $F : \Sigma^k N_+ \to \Sigma^k T(\nu_f)$ (k large) inducing $f^!$ on the chain level. Here, $\mathbb{Z}^{(-)^{n-m}}$ refers to $\mathbb{Z}$ twisted by the orientation character of the extended power homology cobordism bundle

$$e_2(\nu_f) : E\Sigma_2 \times_{\Sigma_2} (M \times_N M) \to BH(2(n-m))$$

with

$$\begin{aligned}
E(e_2(\nu_f)) &= E\Sigma_2 \times_{\Sigma_2} (E(\nu_f) \times_N E(\nu_f)) \ , \\
S(e_2(\nu_f)) &= E\Sigma_2 \times_{\Sigma_2} (E(\nu_f) \times_N S(\nu_f) \cup S(\nu_f) \times_N E(\nu_f)) \ , \\
T(e_2(\nu_f)) &= E(e_2(\nu_f))/S(e_2(\nu_f)) \ = \ E\Sigma_2 \ltimes_{\Sigma_2} T(\nu_f \times_N \nu_f) \ ,
\end{aligned}$$

and $W\mathbb{Z}^{(-)^{n-m}}$ is a free $\mathbb{Z}[\Sigma_2]$-resolution of $\mathbb{Z}^{(-)^{n-m}}$.

Proposition 11.10 (i) *The N-controlled self-intersection class has symmetrization*

$$\begin{aligned}
(1+T)\mu_N(f) &= [M \times_N M] - i_*(e(\nu_f) \cap [M]) \\
&\in \widetilde{H}_n(T(\nu_f \times_N \nu_f)) \ = \ H_{2m-n}(M \times_N M) \ ,
\end{aligned}$$

with

$$[M \times_N M] \ = \ (f^! \otimes f^!)\Delta_N[N] \in H_{2m-n}(M \times_N M) \ .$$

(ii) *The image of $\mu_N(f)$ in*

$$\begin{aligned}
&H_{2m-n}(E\Sigma_2 \times_{\Sigma_2} (M \times_N M), E\Sigma_2 \times_{\Sigma_2} \Delta_M; \mathbb{Z}^{(-)^{n-m}}) \\
&\qquad = \ H^{lf}_{2m-n}(E\Sigma_2 \times_{\Sigma_2} (M \times_N M \backslash \Delta_M); \mathbb{Z}^{(-)^{n-m}}) \\
&\qquad = \ H^{lf}_{2m-n}((M \times_N M \backslash \Delta_M)/\Sigma_2; \mathbb{Z}^{(-)^{n-m}})
\end{aligned}$$

is a $\mathbb{Z}^{(-)^{n-m}}$-twisted fundamental class for the stratified set of unordered double points[1]

$$(M \times_N M \backslash \Delta_M)/\Sigma_2 = \{(x,y) \in M \times M \,|\, x \neq y, f(x) = f(y)\}/\{(x,y) \sim (y,x)\} \ .$$

(iii) *If $f : M \to N$ is an embedding then it is possible to chose $k = 0$, $F : N_+ \to T(\nu_f)$ and $\mu_N(f) = 0$.*
(iv) *The image of $\mu_N(f)$ in $H^{lf}_{2m-n}((M \times M \backslash \Delta_M)/\Sigma_2; \mathbb{Z}^{(-)^{n-m}})$ is a regular homotopy invariant of f, which is 0 if f is regular homotopic to an embedding.*
Proof These are the N-controlled versions of standard properties of the self-intersection form μ of Chapter 5 of Wall [43]. □

Let $f : M^m \to N^n$ be a map of connected homology manifolds with $n = 2m$, such that $f_* : \pi_1(M) \to \pi_1(N)$ is trivial. Write $\pi_1(N) = \pi$, and let $g : N \to B\pi$

[1] The unordered double point set of an immersion of manifolds $f : M^m \to N^n$ is an open $(2m-n)$-dimensional manifold in the metastable range $3m < 2n$, when there are no triple points.

be the classifying map for the universal cover $\widetilde{N} = g^*E\pi$ of N. A choice of null-homotopy $gf \simeq \{*\} : M \to B\pi$ determines a homotopy equivalence $P(gf) \simeq \pi \times M \times M$, with $P(gf)$ the homotopy double point set (11.6), as well as a lift $\widetilde{f} : M \to \widetilde{N}$ of $f : M \to N$. The N-controlled intersection class (11.1) is an element

$$[M \times_N M] \in H_0(M \times_N M)$$

with image the intersection class of Wall [43, 5.2]

$$\lambda(f,f) \in H_0(P(gf)) \ = \ \mathbb{Z}[\pi] \ ,$$

which is a homotopy invariant of f. The following result was first obtained in the differentiable category.

Proposition 11.11 (Haefliger [10])
The reduced intersection class of a map $f : M^m \to N^{2m}$

$$\widetilde{\lambda}(f,f) \ = \ [\lambda(f,f)] \in H_0(P(gf), \Delta_M) \ = \ \mathbb{Z}[\pi]/\mathbb{Z}$$

is such that $\widetilde{\lambda}(f,f) = 0$ if (and for $m \geq 3$ only if) f is homotopic to an embedding.

Now assume that $f : M^m \to N^{2m}$ is an immersion, so that the double point set $M \times_N M$ is the disjoint union of Δ_M and a finite set $M \times_N M \backslash \Delta_M$. The N-controlled self-intersection class (11.9)

$$\mu_N(f) \in H_0(E\Sigma_2 \times_{\Sigma_2} (M \times_N M); \mathbb{Z}^{(-)^m})$$

has image the self-intersection form of [43, 5.2]

$$\begin{aligned}\mu(f) \ &= \sum_{(x,y)\in(M\times_N M\backslash\Delta_M)/\Sigma_2} w(x,y)g(x,y) \\ &\in H_0(E\Sigma_2 \times_{\Sigma_2} P(gf); \mathbb{Z}^{(-)^m}) \ = \ \mathbb{Z}[\pi]/\{a-(-)^m\overline{a}\}\end{aligned}$$

where

- $a \mapsto \overline{a}$ is the involution on the fundamental group ring $\mathbb{Z}[\pi]$ defined (as in §9) by
$$\mathbb{Z}[\pi] \to \mathbb{Z}[\pi] \ ; \ a \ = \ \sum_{g\in\pi} n_g g \mapsto \overline{a} \ = \ \sum_{g\in\pi} n_g g^{-1} \ ,$$
- $g(x,y) \in \pi$ is the fundamental group element determined for each non-trivial ordered double point $(x,y) \in M \times_N M \backslash \Delta_N$ by
$$\widetilde{f}(x) \ = \ g(x,y)\widetilde{f}(y) \in \widetilde{N} \ ,$$
- $w(x,y) = \pm 1$ according to the matching up of the orientations of M and N at $f(x) = f(y) \in N$.

The symmetrization of $\mu(f)$ is such that

$$\mu(f) + (-)^m\overline{\mu(f)} \;=\; \lambda(f,f) - \chi(\nu_f) \in \mathbb{Z}[\pi] ,$$

a special case of 11.10 (i), with $\chi(\nu_f) \in \mathbb{Z} \subseteq \mathbb{Z}[\pi]$.

Proposition 11.12 (Wall [43, 5.2])
The self-intersection form of an immersion $f : M^m \to N^{2m}$

$$\mu(f) \in \mathbb{Z}[\pi]/\{a - (-)^m\overline{a}\}$$

is a regular homotopy invariant such that $\mu(f) = 0$ if (and for $m \geq 3$) if f is regular homotopic to an embedding.

In fact, the reduced self-intersection form

$$\widetilde{\mu}(f) \in \mathbb{Z}[\pi]/(\mathbb{Z} + \{a - (-)^m\overline{a}\})$$

is a homotopy invariant of f. The condition $m \geq 3$ in 11.12 is necessary for the application of the Whitney trick to remove pairs of double points, with $\mu(f) = 0$ being just the algebraic condition for the double points to appear in potentially cancelling pairs. The result of 11.12 for an immersion $f : S^m \to N^{2m}$ is of course essential for even-dimensional surgery obstruction theory.

12. Whitehead torsion

It is a commonplace of controlled topology that the Whitehead torsion of an X-controlled homotopy equivalence of X-controlled complexes has zero image in $Wh(\pi_1(X))$. See for example the controlled K-theory proof in Ranicki and Yamasaki [36] of Chapman's theorem on the topological invariance of Whitehead torsion.

Proposition 12.1 If $f : M \to N$ is a homotopy equivalence of simplicial complexes which is also an X-controlled homology equivalence then the Whitehead torsion of f is such that

$$\tau(f) \in \ker((p_N)_* : Wh(\pi_1(N)) \to Wh(\pi_1(X))) .$$

Proof Work as in 9.10 (i): the algebraic mapping cone of the $(\mathbb{Z},X)$-module chain equivalence $f : \Delta(M) \to \Delta(N)$

$$C \;=\; \mathcal{C}(f : \Delta(M) \to \Delta(N))$$

is a based contractible finite chain complex in $\mathbb{A}(\mathbb{Z},X)$, with assembly the based contractible finite chain complex in $\mathbb{A}(\mathbb{Z}[\pi_1(X)])$

$$C(\widetilde{X}) \;=\; \mathcal{C}(\widetilde{f} : \Delta(\widetilde{M}) \to \Delta(\widetilde{N}))$$

with 0 torsion in $Wh(\pi_1(X))$. The image of $\tau(f) \in Wh(\pi_1(N))$ in $Wh(\pi_1(X))$ is thus

$$(p_N)_*\tau(f) \ = \ \tau(C(\widetilde{X})) \ = \ 0 \in Wh(\pi_1(X)) \ .$$

□

Definition 12.2 An *X-controlled h-cobordism* $(W;M,N)$ of homology manifolds is an h-cobordism together with a simplicial map $p_W : W \to X'$ such that the inclusions $M \to W$, $N \to W$ are X-controlled homology equivalences.

Proposition 12.3 The Whitehead torsion of an X-controlled h-cobordism $(W;M,N)$ of homology manifolds is such that

$$\tau(W;M,N) \in \ker((p_W)_* : Wh(\pi_1(W)) \to Wh(\pi_1(X))) \ .$$

Proof By definition

$$\tau(W;M,N) \ = \ \tau(M \to W) \in Wh(\pi_1(W)) \ .$$

Apply 12.2 to the X-controlled homotopy equivalence $M \to W$. □

Corollary 12.4 If $\pi_1(W) \cong \pi_1(X)$ an N-controlled h-cobordism $(W;M,N)$ of homology manifolds is an s-cobordism, with

$$\tau(W;M,N) \ = \ 0 \in Wh(\pi_1(W)) \ .$$

Proof Immediate from 12.3, since in this case $p_W : W \to X = N$ is a homotopy equivalence. □

In principle, it would be possible to investigate X-controlled versions of the classical h- and s-cobordism theorems of high-dimensional manifold theory, taking the controlled h-cobordism theorem of Quinn [23] as a model.

13. HOMOLOGY FIBRATIONS

It is a theme of controlled topology that if $F \to E \to B$ is a fibre bundle of manifolds and $f : M \to E$ is a homotopy equivalence of manifolds then M is the total space of a fibre bundle $F \to M \to B$ if and only if f is a B-controlled homotopy equivalence. For example, see Chapman [4]. (All niceties to do with fibre bundles, block bundles, approximate fibrations etc. are being ignored here!). An analogous result will now be obtained in the combinatorial context of this paper.

Definition 13.1 A *B-controlled R-homology fibration* E is a B-controlled simplicial complex E such that the inclusions

$$p_E^{-1}D(\tau,B) \to p_E^{-1}D(\sigma,B) \quad (\sigma \leq \tau \in B)$$

are R-homology equivalences, i.e. induce isomorphisms

$$H_*(p_E^{-1}D(\tau,B);R) \cong H_*(p_E^{-1}D(\sigma,B);R) \ .$$

The R-module chain homotopy type of $\Delta(p_E^{-1}D(\sigma,B);R)$ is the *chain fibre* of E. (It is assumed here that B is connected, so that the chain fibre is well-defined.)

Remark 13.2 An (R,B)-module chain complex C is *homogeneous* if the inclusions define R-module chain equivalences

$$[C][\sigma] \xrightarrow{\simeq} [C][\tau] \quad (\tau \leq \sigma \in B)$$

(Ranicki and Weiss [34, 4.5], Ranicki [29, p.110]). A B-controlled simplicial complex E is a B-controlled R-homology fibration if and only if the (R,B)-module chain complex $\Delta(E;R)$ is homogeneous.

Example 13.3 Let E be a B-controlled simplicial complex.
(i) The control map $p_E : E \to B'$ has R-acyclic point inverses if and only if E is a B-controlled R-homology fibration with R-acyclic chain fibre.
(ii) The control map $p_E : E \to B'$ is a quasifibration in the sense of Dold and Thom [6] with fibre $F = p_E^{-1}(*)$ if and only if the inclusions

$$p_E^{-1}D(\tau,B) \to p_E^{-1}D(\sigma,B) \quad (\sigma \leq \tau \in B)$$

are homotopy equivalences, in which case E is a B-controlled R-homology fibration with chain fibre $\Delta(F;R)$.

Definition 13.4 A *d-dimensional B-controlled R-homology Poincaré fibration E* is a B-controlled R-homology fibration such that each $p_E^{-1}D(\sigma,B)$ $(\sigma \in B)$ is a d-dimensional R-homology Poincaré complex, with each inclusion

$$p_E^{-1}D(\tau,B) \to p_E^{-1}D(\sigma,B) \quad (\sigma \leq \tau \in B)$$

an R-homology equivalence such that the induced isomorphism

$$H_d(p_E^{-1}D(\tau,B);R) \cong H_d(p_E^{-1}D(\sigma,B);R)$$

sends $[p_E^{-1}D(\tau,B)]$ to $[p_E^{-1}D(\sigma,B)]$.

The chain fibre C of a d-dimensional B-controlled R-homology Poincaré fibration E is a d-dimensional symmetric Poincaré complex over R. (See Ranicki [26] for the theory of symmetric Poincaré complexes.)

Proposition 13.5 Let B be an n-dimensional R-homology manifold B, and let E be a d-dimensional B-controlled R-homology Poincaré fibration, with chain fibre C.
(i) E is an $(n+d)$-dimensional B-controlled R-homology Poincaré complex.
(ii) $E \times_B E$ is an $(n+2d)$-dimensional B-controlled R-homology Poincaré fibration with chain fibre the $2d$-dimensional symmetric Poincaré complex $C \otimes_R C$ over R. In particular, $E \times_B E$ is an $(n+2d)$-dimensional B-controlled R-homology Poincaré complex.

Proof (i) Use the algebraic Poincaré cycle theory of Ranicki [29], involving the symmetric L-spectrum $\mathbb{L}^\bullet(R)$ with homotopy groups the symmetric L-groups of R

$$\pi_*(\mathbb{L}^\bullet(R)) \ = \ L^*(R) \ .$$

The $\mathbb{L}^\bullet(R)$-homology group $H_m(B;\mathbb{L}^\bullet(R))$ is the cobordism group of m-dimensional symmetric Poincaré cycles in $\mathbb{A}(R,B)$, and the cap product

$$\cap : H_n(B;\mathbb{L}^\bullet(R)) \otimes H^{-d}(B;\mathbb{L}^\bullet(R)) \to H_{n+d}(B;\mathbb{L}^\bullet(R))$$

is defined using the ring spectrum structure of $\mathbb{L}^\bullet(R)$. The R-coefficient homology class

$$[E] \ = \ [B]\otimes[F] \in H_{n+d}(E;R) \ = \ H_n(B;R)\otimes_R H_d(F;R)$$

determines an $(n+d)$-dimensional symmetric cycle $[E]_{\mathbb{L}} = (\Delta(E;R),\Delta[E])$ in $\mathbb{A}(R,B)$ which is Poincaré if and only if E is an $(n+d)$-dimensional B-controlled R-homology Poincaré complex, in which case $[E]_{\mathbb{L}} \in H_{n+d}(E;\mathbb{L}^\bullet(R))$ is a fundamental $\mathbb{L}^\bullet(R)$-homology class. The cap product (on the algebraic Poincaré cycle level) of the fundamental $\mathbb{L}^\bullet(R)$-homology class of [29, 16.16]

$$[B]_{\mathbb{L}} \in H_n(B;\mathbb{L}^\bullet(R))$$

and the $\mathbb{L}^\bullet(R)$-cohomology class

$$[C,p_E]_{\mathbb{L}} \in H^{-d}(B;\mathbb{L}^\bullet(R))$$

of Lück and Ranicki [14, Appendix] identifies

$$[E]_{\mathbb{L}} \ = \ [B]_{\mathbb{L}} \cap [C,p_E]_{\mathbb{L}} \in H_{n+d}(B;\mathbb{L}^\bullet(R)) \ ,$$

so that $[E]_{\mathbb{L}}$ is a Poincaré cycle, as required.
(ii) For any B-controlled R-homology fibration E with chain fibre C the product $E\times_B E$ is a B-controlled R-homology fibration with chain fibre $C\otimes_R C$. Thus if E is a d-dimensional B-controlled R-homology Poincaré fibration then $E\times_B E$ is a $2d$-dimensional B-controlled R-homology Poincaré fibration, and (i) applies.
□

Remark 13.6 The result of 13.5 (i) is a combinatorial version of the result of Buoncristiano, Rourke and Sanderson [2, p.21] that the total space of a mock bundle is a manifold, and of the result of Gottlieb [9] (announced by Quinn [22]) that the total space of a fibration $F \to E \to B$ with base B an n-dimensional Poincaré complex and fibre F a d-dimensional Poincaré complex is an $(n+d)$-dimensional Poincaré complex E.

Remark 13.7 Let E be a $(d{+}1)$-dimensional homology manifold with a simplicial map $p_E : E \to S^1$ such that the induced infinite cyclic cover of E

$$\overline{E} \ = \ (p_E)^*\mathbb{R}$$

is finitely dominated. Let $\zeta : \overline{E} \to \overline{E}$ be a generating covering translation, with mapping torus

$$T(\zeta) \;=\; \overline{E} \times [0,1]/\{(x,0) = (\zeta(x),1) \,|\, x \in \overline{E}\}\ .$$

The fibering obstruction of E

$$\Phi(E) \;=\; \tau(T(\zeta) \to E) \in Wh(\pi_1(E))$$

is such that $\Phi(E) = 0$ if (and for $d \geq 5$ only if) $p_E : E \to S^1$ is homotopic to the projection of a d-dimensional S^1-controlled homology Poincaré fibration. For an actual manifold E this is the original fibering obstruction of Farrell and Siebenmann, and the S^1-controlled homology Poincaré fibration can be taken to be an actual fibre bundle over S^1. See Ranicki [30],[33] and Hughes and Ranicki [12] for more recent accounts of the fibering obstruction over S^1.

Theorem 13.8 Let B be an n-dimensional R-homology manifold, and let E be a d-dimensional B-controlled R-homology Poincaré fibration with chain fibre C, so that E is an $(n+d)$-dimensional B-controlled R-homology Poincaré complex (13.5 (i)). If M is an $(n+d)$-dimensional B-controlled R-homology Poincaré complex and $f : M \to E$ is a degree 1 B-controlled map, the following conditions are equivalent:

(i) *M is a B-controlled R-homology fibration with chain fibre C,*

(ii) *f is a B-controlled R-homology equivalence,*

(iii) $(f \times f)_* : H_{n+d}(M \times_B M; R) \cong H_{n+d}(E \times_B E; R)$.

Proof (i) $\Longleftrightarrow$ (ii) A map $f : M \to E$ of B-controlled simplicial complexes is a B-controlled R-homology equivalence if and only if the restrictions

$$f| : p_M^{-1}D(\sigma, B) \to p_E^{-1}D(\sigma, B) \quad (\sigma \in B)$$

are R-homology equivalences.
(ii) $\Longleftrightarrow$ (iii) This is a special case of 7.3. □

Remark 13.9 Corollary 7.5 is the special case of 13.8 with $R = \mathbb{Z}$, $B = E$, $C = R$, $d = 0$ (cf. 13.3 (i)).

14. Knot theory

The results of §§7,13 are now illustrated by showing how they apply to high-dimensional knot theory. No actual new results are obtained in knot theory, however; known results are restated in terms of the chain theory developed in this paper.

The algebraic theory of surgery was used in Ranicki [28, 7.8], [33] to obtain a chain complex treatment of the algebraic invariants of high-dimensional knot theory, using the following construction. Let $k : S^n \subset S^{n+2}$ ($n \geq 1$) be a locally flat n-knot, with closed regular neighbourhood

$$(U, \partial U) \;=\; (S^n \times D^2, S^n \times S^1) \subset S^{n+2} \ .$$

The knot complement

$$(T, \partial T) \;=\; (\mathrm{cl.}(S^{n+2} \backslash U), \partial U)$$

is an $(n+2)$-dimensional manifold with boundary, such that the generator $1 \in H^1(T) = H_n(U) = \mathbb{Z}$ is realized by a normal map

$$(f, b) : (T, \partial T) \to (D^{n+1} \times S^1, S^n \times S^1)$$

with $f : T \to D^{n+1} \times S^1$ a $\mathbb{Z}$-homology equivalence, and $f| = 1 : \partial T \to S^n \times S^1$. Define an $(n+3)$-dimensional $\mathbb{Z}$-homology Poincaré pair $(X, \partial X)$ with X the mapping cylinder of f, and the boundary $\partial X = T \cup_\partial D^{n+1} \times S^1$ an $(n+2)$-dimensional manifold. The peripheral complex of $(X, \partial X)$ is a $\mathbb{Z}$-contractible $(n+2)$-dimensional quadratic Poincaré complex (C, ψ_X) in $\mathbb{A}(\mathbb{Z}, X)$, with

$$C \;=\; \mathcal{C}([X] \cap - : \Delta(X, \partial X)^{n+3-*} \to \Delta(X))_{*+1} \ .$$

The cobordism class

$$s_\partial(X; \mathbb{Z}) \;=\; (C, \psi_X) \in \mathbb{S}_{n+3}(X; \mathbb{Z})$$

is the rel ∂ total homology surgery obstruction (9.11), such that $s_\partial(X; \mathbb{Z}) = 0$ if (and for $n \geq 5$ only if) $(X, \partial X)$ is homology equivalent rel ∂ to an $(n+2)$-dimensional topological manifold with boundary. The projection $X \to S^1$ is a homotopy equivalence, so that

$$\begin{aligned} \mathbb{S}_{n+3}(X; \mathbb{Z}) \;&=\; \mathbb{S}_{n+3}(S^1; \mathbb{Z}) \\ &=\; \Gamma_{n+3}\left(\begin{array}{ccc} \mathbb{Z}[z, z^{-1}] & \longrightarrow & \mathbb{Z}[z, z^{-1}] \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z}[z, z^{-1}] & \longrightarrow & \mathbb{Z} \end{array} \right) \end{aligned}$$

The induced functor $\mathbb{A}(\mathbb{Z}, X) \to \mathbb{A}(\mathbb{Z}, S^1)$ sends the peripheral complex (C, ψ_X) to the kernel $\mathbb{Z}$-contractible $(n+2)$-dimensional quadratic Poincaré complex of (f, b) in $\mathbb{A}(\mathbb{Z}, S^1)$

$$\sigma_*^{S^1}(f, b) \;=\; (\Delta(f^! : \Delta(D^{n+1} \times S^1) \to \Delta(T)), \psi_b) \ .$$

The assembly functor $A : \mathbb{A}(\mathbb{Z}, S^1) \to \mathbb{A}(\mathbb{Z}[z, z^{-1}])$ sends $\sigma_*^{S^1}(f, b)$ to the $\mathbb{Z}$-contractible $(n+2)$-dimensional quadratic Poincaré complex in $\mathbb{A}(\mathbb{Z}[z, z^{-1}])$

$$A\sigma_*^{S^1}(f, b) \;=\; (\Delta(f^! : \Delta(D^{n+1} \times \mathbb{R}) \to \Delta(\overline{T})), A\psi_b) \ ,$$

with $\overline{T} = f^*(D^{n+1} \times \mathbb{R})$ the canonical infinite cyclic cover of T. The total homology surgery obstruction

$$s_\partial(X;\mathbb{Z}) = A\sigma_*^{S^1}(f,b) \in \mathbb{S}_{n+3}(S^1;\mathbb{Z})$$

is a cobordism invariant of k. For $n \geq 3$ it is in fact the cobordism class of k, with $\mathbb{S}_{n+3}(S^1;\mathbb{Z}) = C_n$ the n-dimensional knot cobordism group (Ranicki [28, p.836]).

The chain homotopy type of $\sigma_*^{S^1}(f,b)$ in $\mathbb{A}(\mathbb{Z},S^1)$ is not an isotopy invariant of the n-knot k, since it depends on the choice of the map $f: T \to D^{n+1} \times S^1$ within its homotopy class. Working as in the proof of 7.3 (ii) it follows from the $(\mathbb{Z},S^1)$-module chain equivalences

$$\begin{aligned} &\Delta(T) \simeq_{(\mathbb{Z},S^1)} \Delta(f^!) \oplus \Delta(S^1) \ , \\ &\Delta(f^!)^{n+2-*} \simeq_{(\mathbb{Z},S^1)} \Delta(f^!) \end{aligned}$$

that there is defined a $\mathbb{Z}$-module chain equivalence

$$\Delta(T) \otimes_{(\mathbb{Z},S^1)} \Delta(T) \simeq_{\mathbb{Z}} (\Delta(f^!) \otimes_{(\mathbb{Z},S^1)} \Delta(f^!)) \oplus \Delta(f^!) \oplus \Delta(f^!) \oplus \Delta(S^1) \ ,$$

and that

$$\begin{aligned} H_{n+2}(T \times_{S^1} T) &= H_{n+2}(\Delta(T) \otimes_{(\mathbb{Z},S^1)} \Delta(T)) \\ &= H_{n+2}(\Delta(f^!) \otimes_{(\mathbb{Z},S^1)} \Delta(f^!)) \\ &= H_0(\mathrm{Hom}_{(\mathbb{Z},S^1)}(\Delta(f^!),\Delta(f^!))) \ . \end{aligned}$$

The following conditions are equivalent:

(a) $H_{n+2}(T \times_{S^1} T) = 0$,

(b) $\sigma_*^{S^1}(f,b)$ is chain equivalent to 0 in $\mathbb{A}(\mathbb{Z},S^1)$,

(c) $f: T \to D^{n+1} \times S^1$ is an S^1-controlled homology equivalence.

In view of 13.8 it is possible to choose f to satisfy these conditions if and only if T is an S^1-controlled homology fibration – see further below for fibred knots.

The chain homotopy type of $A\sigma_*^{S^1}(f,b)$ in $\mathbb{A}(\mathbb{Z}[z,z^{-1}])$ is an isotopy invariant of k, since it only depends on the homotopy class of $f: T \to D^{n+1} \times S^1$. Let $\zeta : \overline{T} \to \overline{T}$ be a generating covering translation of the infinite cyclic cover $\overline{T}$ of T. The quotient of $\overline{T} \times \overline{T}$ by the diagonal $\mathbb{Z}$-action

$$\overline{T} \times_{\mathbb{Z}} \overline{T} = (\overline{T} \times \overline{T})/\{(x,y) \simeq (\zeta x, \zeta y)\}$$

is such that

$$\begin{aligned} H_{n+2}(\overline{T} \times_{\mathbb{Z}} \overline{T}) &= H_{n+2}(\Delta(\overline{T}) \otimes_{\mathbb{Z}[z,z^{-1}]} \Delta(\overline{T})) \\ &= H_{n+2}(A\Delta(f^!) \otimes_{\mathbb{Z}[z,z^{-1}]} A\Delta(f^!)) \end{aligned}$$

The following conditions are equivalent:

(d) $H_{n+2}(\overline{T}\times_{\mathbb{Z}}\overline{T})=0$,

(e) $A\sigma^{S^1}_*(f,b)$ is chain equivalent to 0 in $\mathbb{A}(\mathbb{Z}[z,z^{-1}])$,

(f) $f:T\to D^{n+1}\times S^1$ is homotopic to an S^1-controlled homology equivalence.

See Ranicki [28, 7.8] for the relationship between $A\sigma^{S^1}_*(f,b)$, the Seifert form, the Alexander polynomials and the Blanchfield pairing of k. If k is simple (i.e. $H_r(\overline{T})=0$ for $1\leq r\leq (n-1)/2$) and $n\geq 3$ the chain homotopy type of $A\sigma^{S^1}_*(f,b)$ is the complete isotopy invariant, by the classification results of Trotter, Levine and Kearton, and the conditions (d),(e),(f) are equivalent to k being unknotted, i.e. isotopic to the trivial n-knot $k_0:S^n\subset S^{n+2}$.

Now suppose that $k:S^n\subset S^{n+2}$ is a fibred n-knot, i.e. that the knot complement T fibres over S^1 (cf. Remark 13.7 above). For example, the link of an isolated singular point of a complex hypersurface $f^{-1}(0)\subset\mathbb{C}^m$ $(f:\mathbb{C}^m\to\mathbb{C})$ is a fibred $(2m-3)$-knot

$$S^{2m-3} \;=\; S^{2m-1}\cap f^{-1}(0)\subset S^{2m-1}\subset\mathbb{C}^m \ ,$$

by Milnor [20] (cf. Remark 6.17 above). Let $F^{n+1}\subset S^{n+2}$ be a Seifert surface for k, with $\partial F=k(S^n)$, and let $h:F\to F$ be the monodromy. The knot complement

$$(T,\partial T) \;=\; (T(h),S^n\times S^1)$$

is the total space of a fibre bundle

$$(F^{n+1},S^n)\to(T,\partial T)\to S^1 \ ,$$

and $f:T\to D^{n+1}\times S^1$ may be chosen to be a map of fibre bundles over S^1. The infinite cyclic cover of T is such that

$$\zeta:\overline{T} \;=\; F\times\mathbb{R}\to\overline{T}\ ;\ (x,t)\to(h(x),t+1)$$

and

$$T\times_{S^1}T \;=\; T(h\times h:F\times F\to F\times F)$$

is homotopy equivalent to

$$\overline{T}\times_{\mathbb{Z}}\overline{T} \;=\; T(h\times h)\times\mathbb{R}\ .$$

Thus

$$H_*(T\times_{S^1}T) \;=\; H_*(\overline{T}\times_{\mathbb{Z}}\overline{T})$$

and in the fibred case

$$\text{(a)}\iff\text{(b)}\iff\text{(c)}\iff\text{(d)}\iff\text{(e)}\iff\text{(f)}\ .$$

15. Other categories

Weiss [44] constructed a chain duality on the additive category of X-controlled $\mathbb{Z}$-modules, for any Δ-set X. Hutt [13] constructed a chain duality on the additive category of sheaves of $\mathbb{Z}$-modules over any space X. In principle, all the results in this paper can therefore be generalized to these categories.

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