Motivic Equivalence of Quadratic Forms

Oleg T. Izhboldin[1]

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Abstract. Let X_ϕ and X_ψ be projective quadrics corresponding to quadratic forms ϕ and ψ over a field F. If X_ϕ is isomorphic to X_ψ in the category of Chow motives, we say that ϕ and ψ are motivic isomorphic and write $\phi \overset{m}{\sim} \psi$. We show that in the case of odd-dimensional forms the condition $\phi \overset{m}{\sim} \psi$ is equivalent to the similarity of ϕ and ψ. After this, we discuss the case of even-dimensional forms. In particular, we construct examples of generalized Albert forms q_1 and q_2 such that $q_1 \overset{m}{\sim} q_2$ and $q_1 \not\sim q_2$.

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Let F be a field of characteristic $\neq 2$ and ϕ be a quadratic form of dimension ≥ 3 over F. By X_ϕ we denote the projective variety given by the equation $\phi = 0$. It is well known that the variety X_ϕ determines the form ϕ uniquely up to similarity. More precisely, the condition $X_\phi \simeq X_\psi$ holds if and only if $\phi \simeq k\psi$ for a suitable element $k \in F^*$. Now, let $\mathcal{M} : \mathcal{V}_F \to \mathcal{C}$ be an arbitrary functor from the category $\mathcal{V}_F$ of smooth projective F-varieties to a category $\mathcal{C}$. Is it possible to say anything specific about ϕ and ψ if we know that $\mathcal{M}(X_\phi) \simeq \mathcal{M}(X_\psi)$? Clearly, the answer depends on the category $\mathcal{C}$ and the functor $\mathcal{M}$. In the present paper, we mainly consider the example of the category $\mathcal{C} = \mathcal{MV}_F$ of Chow motives. In this particular case, we set $\mathcal{M}(X) = M(X)$, where $M(X)$ denotes the motive of X in the category of Chow motives. If $M(X_\phi) \simeq M(X_\psi)$, we say that ϕ is motivic equivalent to ψ (and we write $\phi \overset{m}{\sim} \psi$).

Recently, Alexander Vishik has proved that $\phi \overset{m}{\sim} \psi$ iff $\dim\phi = \dim\psi$ and $i_W(\phi_L) = i_W(\psi_L)$ for all extensions L/F (see [27]). His proof uses deep results concerning the Voevodsky motivic category. In [10], Nikita Karpenko found a new, more elementary, proof that, in contrast to Vishik's proof, deals only with Chow motives. In §2, we give an elementary proof of Vishik's theorem in the case of odd-dimensional forms. In fact, we prove a more precise result. Namely, we show that, in the case of odd-dimensional forms, the condition $\phi \overset{m}{\sim} \psi$ is equivalent to the similarity of the forms ϕ and ψ (here we do not use any results of the paper of Vishik). In other words, we prove that the condition $M(X_\phi) \simeq M(X_\psi)$ is equivalent to the condition

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$X_\phi \simeq X_\psi$ for the odd-dimensional quadrics X_ϕ and X_ψ. In the proof we use some results of §1 concerning low dimensional forms belonging to $W(F(\phi)/F)$.

In §3, we show that the condition $\phi \stackrel{m}{\sim} \psi$ is equivalent to the condition $\phi \sim \psi$ for all forms of dimension ≤ 7. Besides, we discuss the case of even-dimensional forms of dimension ≥ 8. This case is much more complicated. For instance, for all $n \geq 3$, there exists an example of anisotropic 2^n-dimensional forms ϕ and ψ such that $\phi \stackrel{m}{\sim} \psi$ but $\phi \not\sim \psi$. In §4, for any n and m such that $0 \leq m \leq n-3$, we construct generalized Albert forms q_1 and q_2 such that $\dim(q_1)_{an} = \dim(q_2)_{an} = 2(2^n - 2^m)$, $q_1 \stackrel{m}{\sim} q_2$ but $q_1 \not\sim q_2$. This example gives a negative answer to a question stated by T. Y. Lam [18].

Some words about terminology and notation. Mainly we use the same terminology and notation as in the book of T. Y. Lam [17], W. Scharlau [23], and the fundamental papers of M. Knebusch [11, 12]. However, there exist several differences. We use the notation $\langle\langle a_1, \dots, a_n \rangle\rangle$ for the Pfister form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ (in [17] and [23], $\langle\langle a_1, \dots, a_n \rangle\rangle = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$). We write $\phi \sim \psi$ if there exists an element $k \in F$ such that $k\phi \simeq \psi$ (i.e., if ϕ is similar to ψ). We say that ϕ and ψ are half-neighbors if $\dim \phi = \dim \psi$ and there exist $s, r \in F$ such that $\pi = s\phi \perp r\psi$ is a Pfister form (see, e.g., [6]). In this case, we will write $\phi \stackrel{hn}{\sim} \psi$ and we say that ϕ and ψ are half-neighbors of π. Our definition differs from the original definition of Knebusch [12]. However, we prefer to use the new definition since we want to regard any pair ϕ, ψ of 2^n-dimensional similar forms as half-neighbors. We denote by $P_n(F)$ the set of all n-fold Pfister forms. The set of all forms similar to n-fold Pfister forms is denoted by $GP_n(F)$. We also use the notation $P_*(F) = \cup_n P_n(F)$ and $GP_*(F) = \cup_n GP_n(F)$.

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1. LOW DIMENSIONAL FORMS IN $W(F(\phi)/F)$

In this section, we give slight generalizations of some results of M. Knebusch. In fact, we modify some proofs of [12] by using Hoffmann's theorem [5] [2]. We recall that Hoffmann's theorem asserts that for a pair of anisotropic quadratic forms ϕ and ψ satisfying the condition $\dim \phi \leq 2^n < \dim \psi$, the form ϕ remains anisotropic over $F(\psi)$.

PROPOSITION 1.1. *Let ϕ and ψ be anisotropic quadratic forms over F such that $\dim \phi \geq \dim \psi$. Suppose that the form $\pi \stackrel{\text{Def}}{=} \phi \perp \psi$ belongs to the group $W(F(\phi)/F)$. Then*

(1) *if π is isotropic, then π is hyperbolic,*
(2) *if π is anisotropic, then π is similar to a Pfister form.*

Proof. (1) Assume that π is isotropic but not hyperbolic. This means that $0 < \dim \pi_{an} < \dim \pi$. In the Witt ring $W(F)$, we have $\pi - \phi = \psi$. Therefore,

$$\dim(\pi_{an} \perp -\phi)_{an} = \dim \psi \leq \dim \phi < \dim \pi_{an} + \dim \phi = \dim(\pi_{an} \perp -\phi).$$

[2] see also [6, Prop. 2.4] and [3, Th. 1.6]

Consequently, the form $\pi_{an} \perp -\phi$ is isotropic. Hence the set $D_F(\pi_{an}) \cap D_F(\phi)$ is nonempty.

Since $\pi_{F(\phi)}$ is hyperbolic, it follows that $((\pi)_{an})_{F(\phi)}$ is also hyperbolic. Since the set $D_F(\pi_{an}) \cap D_F(\phi)$ is nonempty, the Cassels–Pfister subform theorem implies that $\phi \subset \pi_{an}$. Therefore,

$$\dim(\pi_{an} \perp -\phi)_{an} = \dim \pi_{an} - \dim \phi < \dim \pi - \dim \phi = \dim \psi.$$

This contradicts to the relation $\dim(\pi_{an} \perp -\phi)_{an} = \dim \psi$ proved above.

(2) Assume that π is not isotropic. To prove that π is similar to a Pfister form, it suffices to prove that $\pi_{F(\pi)}$ is hyperbolic (see [12]).

Let $\tilde{F} = F(\pi)$, $\tilde{\pi} = \pi_{\tilde{F}}$, $\tilde{\phi} = \phi_{\tilde{F}}$, and $\tilde{\psi} = \psi_{\tilde{F}}$. Since $\dim \psi \leq \frac{1}{2} \dim \pi$, Hoffmann's theorem implies that the form $\tilde{\psi} = \psi_{F(\pi)}$ is anisotropic. If we assume that $\tilde{\phi}$ is anisotropic, then we can apply item (1) of Proposition 1.1 to the $\tilde{F}$-forms $\tilde{\phi}$, $\tilde{\psi}$, and $\tilde{\pi}$. Then we conclude that $\tilde{\pi}$ is hyperbolic. Now, we assume that $\tilde{\phi} = \phi_{F(\pi)}$ is isotropic. Since $\pi_{F(\phi)}$ is hyperbolic and $\phi_{F(\pi)}$ is isotropic, it follows that $\pi_{F(\pi)}$ is hyperbolic. Thus, the form $\pi_{F(\pi)}$ is hyperbolic in any case and the proposition is proved. □

Corollary 1.2. (Fitzgerald, [3, Th. 1.6]). *Let ϕ be an F-form, and let $\pi \in W(F(\phi)/F)$ be an anisotropic nonzero form of dimension $\leq 2\dim\phi$. Then $\pi \in GP_*(F)$ and one of the following conditions holds:*

- *ϕ is a Pfister neighbor of π,*
- *ϕ is a half-neighbor of π,*

Proof. Since π is anisotropic and $\pi_{F(\phi)}$ is hyperbolic, the form ϕ is similar to a subform of π. Multiplying ϕ by a scalar, we may assume that $\phi \subset \pi$. Let ψ be the complement of ϕ in π. Then all hypotheses of Proposition 1.1 hold. Since π is anisotropic, Proposition 1.1 implies $\pi \in GP_*(F)$. The rest of the proof is an immediate consequence of the definitions of Pfister neighbors and half-neighbors, and the Cassels-Pfister subform theorem. □

Corollary 1.3. (cf. [12, Th. 8.9]). *Let ϕ and η be anisotropic forms such that $\dim\phi \geq \dim\eta$ and $(\phi_{F(\phi)})_{an} \simeq (\eta_{F(\phi)})_{an}$. Then either $\phi \simeq \eta$ or $\phi \perp -\eta \in GP_*(F)$.*

Proof. Let $\psi = -\eta$ and $\pi = \phi \perp -\eta = \phi \perp \psi$. All the hypotheses of Proposition 1.1 hold. In the case where π is isotropic, Proposition 1.1 implies that π is hyperbolic. Then $\phi = \eta$ in the Witt ring. Since ϕ and η are anisotropic, we have $\phi \simeq \eta$. If π is anisotropic, Proposition 1.1 implies that $\phi \perp -\eta = \pi \in GP_*(F)$. □

2. Motivic equivalence of odd-dimensional forms

Definition 2.1. To any field F, let be assigned an equivalence relation $\overset{*}{\sim}_F$ on the set of all quadratic forms over F such that the following conditions hold:

(i) If ϕ and ψ are forms over F such that $\phi \sim \psi$, then $\phi \overset{*}{\sim}_F \psi$.

(ii) If ϕ and ψ are forms over F such that $\phi \overset{*}{\sim}_F \psi$, then, for any extension E/F, we have $\phi_E \overset{*}{\sim}_E \psi_E$.

(iii) If ϕ and ψ are forms over a field F such that $\phi \overset{*}{\sim}_F \psi$, then $\dim\phi = \dim\psi$ and $i_W(\phi) = i_W(\psi)$.

A collection of equivalence relations $\overset{*}{\sim}_F$ satisfying properties (i)–(iii) will be called *a good equivalence relation on quadratic forms* (*over all fields*).

Below we will drop the index F at $\overset{*}{\sim}_F$ and write simply $\overset{*}{\sim}$.

Definition 2.2. Let ϕ and ψ be F-forms. We say that the *quadratic form ϕ is equivalent to the quadratic form ψ in the sense of Vishik* if $\dim\phi = \dim\psi$ and for any field extension E/F we have $i_W(\phi_E) = i_W(\psi_E)$. In this case, we write $\phi \overset{v}{\sim} \psi$.

The following lemma is obvious.

Lemma 2.3. *The equivalence relation $\overset{v}{\sim}$ is a minimal good equivalence relation. More precisely,*

- *The equivalence relation $\overset{v}{\sim}$ is a good relation.*
- *For any good relation $\overset{*}{\sim}$, the condition $\phi \overset{*}{\sim} \psi$ implies $\phi \overset{v}{\sim} \psi$.* □

Example 2.4. *Let X be a smooth variety over F. By $M(X)$ we denote the motive of X in the category of Chow motives. Let us define the equivalence $\overset{m}{\sim}$ of quadratic forms ϕ and ψ as follows:*

$$\phi \overset{m}{\sim} \psi \quad \textit{if} \quad M(X_\phi) \simeq M(X_\psi).$$

Then $\overset{m}{\sim}$ is a good equivalence relation.

Proof. Clearly, conditions (i) and (ii) in Definition 2.1 are fulfilled. We need to verify only condition (iii). Let $X = X_\phi$, and let $\bar{F}$ denote the algebraic closure of F. By [9, Item (2.2) and Prop. 2.6] [3]

- $\dim\phi$ coincides with the largest integer m such that $\mathrm{CH}_{m-2}(X) \neq 0$,
- the integer $i_W(\phi)$ coincides with the largest integer m satisfying the conditions $m \leq \frac{1}{2}\dim\phi$ and $\mathrm{coker}(\mathrm{CH}_{m-1}(X) \to \mathrm{CH}_{m-1}(X_{\bar{F}})) = 0$.

Thus, it suffices to show that the groups $\mathrm{coker}(\mathrm{CH}^j(X) \to \mathrm{CH}^j(X_{\bar{F}}))$ and $\mathrm{CH}^j(X)$ depend only on the motive of X. This can easily be proved if we observe that the functor CH^j is representable in the category of Chow motives. Namely, $\mathrm{CH}^j(X) = \mathrm{Hom}_{\mathcal{MV}_F}(M(pt_F)(j), M(X))$, where $M(pt_F)$ is the motive of $pt_F = \mathrm{Spec}(F)$ and the object $M(pt_F)(j)$ is defined, e.g., in [24]. Thus, $\mathrm{CH}^j(X)$ depends only on the motive of X. Now, we consider the base change functor $\Phi : \mathcal{MV}_F \to \mathcal{MV}_{\bar{F}}$. Since the homomorphism $\mathrm{CH}^j(X) \to \mathrm{CH}^j(X_{\bar{F}})$ coincides with the homomorphism

$$\Phi : \underset{\mathcal{MV}_F}{\mathrm{Hom}}(M(pt_F)(j), M(X)) \to \underset{\mathcal{MV}_{\bar{F}}}{\mathrm{Hom}}(\Phi(M(pt_F)(j)), \Phi(M(X))),$$

it follows that the group $\mathrm{coker}(\mathrm{CH}^j(X) \to \mathrm{CH}^j(X_{\bar{F}}))$ also depends only on $M(X)$. □

Theorem 2.5. *Let $\overset{*}{\sim}$ be a good equivalence relation. Let ϕ and ψ be odd-dimensional quadratic forms over a field. Then the condition $\phi \overset{*}{\sim} \psi$ is equivalent to the condition $\phi \sim \psi$.*

Proof. We start the proof with three lemmas

Lemma 2.6. *Let ϕ and ψ be odd-dimensional anisotropic forms of dimension ≥ 3 such that* $\dim\phi = \dim\psi$ *and* $(\phi_{F(\phi)})_{an} \simeq (\psi_{F(\phi)})_{an}$. *Then $\phi \simeq \psi$.*

Proof. If $\phi \not\simeq \psi$, Corollary 1.3 shows that $\phi \perp -\psi \in GP_*(F)$. Since $\dim\phi = \dim\psi$, we conclude that $\dim\psi$ is a power of 2. Since $\dim\psi \geq 3$, we see that $\dim\psi$ is even. We get a contradiction to the assumption of the lemma. □

[3] see also [22, Prop. 2] and [25].

The following lemma is obvious.

LEMMA 2.7. *Let ϕ and ψ be odd-dimensional forms such that* $\dim\phi = \dim\psi$ *and* $\det\phi = \det\psi$. *Then the condition $\psi \sim \phi$ is equivalent to the condition $\phi \simeq \phi$.* $\square$

LEMMA 2.8. *Let ϕ and ψ be odd-dimensional forms such that* $\dim\phi_{an} = \dim\psi_{an} \geq 3$. *Suppose that $\phi_{F(\phi_{an})} \sim \psi_{F(\phi_{an})}$. Then $\phi \sim \psi$.*

Proof. Replacing first ϕ and ψ by ϕ_{an} and ψ_{an}, respectively, we may assume that ϕ and ψ are anisotropic. Replacing then ϕ by $\frac{1}{\det\phi}\phi$ and ψ by $\frac{1}{\det\psi}\psi$, we may assume that $\det\phi = 1 = \det\psi$. Since $\phi_{F(\phi)} \sim \psi_{F(\phi)}$, Lemma 2.7 implies that $\phi_{F(\phi)} \simeq \psi_{F(\phi)}$. By Lemma 2.6, we have $\phi \simeq \psi$. $\square$

Now, we return to the proof of Theorem 2.5. We use induction on $n = \dim\phi_{an} = \dim\psi_{an}$. The case where $n = 1$ is obvious. So we may assume that $n \geq 3$. Since $\phi \overset{*}{\sim} \psi$, we have $\phi_{F(\phi_{an})} \overset{*}{\sim} \psi_{F(\phi_{an})}$. By the induction assumption, we have $\phi_{F(\phi_{an})} \sim \psi_{F(\phi_{an})}$. Now, Lemma 2.8 implies that $\phi \sim \psi$. $\square$

COROLLARY 2.9. *Let ϕ and ψ be odd-dimensional quadratic forms over a field. Then*

$$\phi \overset{v}{\sim} \psi \quad \text{iff} \quad \phi \overset{m}{\sim} \psi \quad \text{iff} \quad \phi \sim \psi.$$

3. EVEN-DIMENSIONAL FORMS

In this section, we study the relation $\overset{m}{\sim}$ in the case of even-dimensional forms. If quadratic forms ϕ and ψ of dimension ≥ 2 satisfy the condition $\phi \overset{v}{\sim} \psi$, then $\phi_{F(\psi)}$ and $\psi_{F(\phi)}$ are isotropic (because $\phi_{F(\phi)}$ and $\psi_{F(\psi)}$ are isotropic).

PROPOSITION 3.1. *Let ϕ and ψ be quadratic forms of dimension < 8. Then*

$$\phi \overset{v}{\sim} \psi \quad \text{iff} \quad \phi \overset{m}{\sim} \psi \quad \text{iff} \quad \phi \sim \psi.$$

Proof. In view of Corollary 2.9, we may assume that $d = \dim\phi = \dim\psi$ is even. Thus, it suffices to consider the cases $d = 2$, 4, and 6. The implications $\phi \sim \psi \Rightarrow \phi \overset{m}{\sim} \psi \Rightarrow \phi \overset{v}{\sim} \psi$ are obvious. Therefore, we must verify only that $\phi \overset{v}{\sim} \psi$ implies $\phi \sim \psi$. Since $\phi \overset{v}{\sim} \psi$, the forms $\phi_{F(\psi)}$ and $\psi_{F(\phi)}$ are isotropic. In the case $d = 2$, this obviously means that $\phi \sim \psi$. If $d = 4$, then $\phi \sim \psi$ by Wadsworth's theorem [28]. Thus, we may assume that $d = 6$. We need the following assertion concerning the isotropy of 6-dimensional forms.

LEMMA 3.2. (see [4, 13, 16, 21]). *Let ϕ and ψ be anisotropic 6-dimensional forms such that $\phi_{F(\psi)}$ is isotropic. Then either $\phi \sim \psi$ or ψ is a 3-fold Pfister neighbor.* $\square$

In view of this lemma, we may assume that ψ is a Pfister neighbor of a 3-fold Pfister form π. Since $\psi_{F(\phi)}$ is isotropic, it follows that $\pi_{F(\phi)}$ is isotropic. Hence ϕ is a Pfister neighbor of π. Therefore, $\phi \sim (\pi - \langle\langle d_\pm\phi\rangle\rangle)_{an}$ and $\psi \sim (\pi \perp -\langle\langle d_\pm\psi\rangle\rangle)_{an}$. Thus, it suffices to verify that $d_\pm\phi = d_\pm\psi$. This is a consequence of the following chain of equivalent conditions

$$a = d_\pm\phi \Leftrightarrow i_W(\phi_{F(\sqrt{a})}) = 3 \Leftrightarrow i_W(\psi_{F(\sqrt{a})}) = 3 \Leftrightarrow a = d_\pm\psi$$

The proof is complete. $\square$

Now, we begin to study even-dimensional forms of dimension ≥ 8.

LEMMA 3.3. (see, e.g., [27]). *Let ϕ and ψ be half-neighbors. Then $\phi \overset{v}{\sim} \psi$.*

For the reader's convenience, we cite the proof (which, in fact, is trivial).

Proof. The condition $\phi \overset{hn}{\sim} \psi$ means that $\dim\phi = \dim\psi$, and there exist $s, r \in F^*$ such that $s\phi \perp r\psi = \pi \in P_*(F)$. Let L/F be a field extension. If both ϕ_L and ψ_L are anisotropic, then $i_W(\phi_L) = 0 = i_W(\psi_L)$. If at least one of the forms ϕ_L or ψ_L is isotropic, then π_L is also isotropic. Taking into account the condition $\pi \in P_*(F)$, we conclude that π_L is hyperbolic. Therefore, $s\phi_L = -r\psi_L$ in the Witt ring. Since $\dim\phi = \dim\psi$, we have $s\phi_L \simeq -r\psi_L$. Hence $i_W(\phi_L) = i_W(\psi_L)$. □

The following lemma shows that there exist examples of nonsimilar half-neighbors.

LEMMA 3.4. (see [6], [8]). *For any $n \geq 3$, there exists a field F and 2^n-dimensional half-neighbors ϕ and ψ such that $\phi \not\sim \psi$.*

As a consequence of this result, we see that, for any $n \geq 3$, there exists a pair of 2^n dimensional forms ϕ and ψ such that $\phi \overset{v}{\sim} \psi$ and $\phi \not\sim \psi$. In particular, Proposition 3.1 cannot always be generalized for 8-dimensional forms.

Nevertheless, for 8-dimensional forms with trivial determinant, we have the following

PROPOSITION 3.5. *Let ϕ and ψ be 8-dimensional forms with trivial determinant. Then the following conditions are equivalent:*

(1) $\phi \overset{v}{\sim} \psi$;
(2) $\phi_{F(\psi)}$ *and* $\psi_{F(\phi)}$ *are isotropic;*
(3) ϕ *and* ψ *are half-neighbors.*

Proof. The implications (3)⇒(1)⇒(2) are obvious. The implication (2)⇒(3) follows immediately from the results of A. Laghribi [16], [15], [14]. □

4. GENERALIZED ALBERT FORMS

In this section, we construct examples of nonsimilar $\overset{v}{\sim}$-equivalent forms based on the so-called generalized Albert forms.

DEFINITION 4.1. A generalized Albert form (or n-Albert form) is a form of type $q = \pi' \perp -\tau'$, where π' and τ' are pure parts of n-fold Pfister forms π and τ.

REMARK 4.2.
- Any n-Albert form has dimension $2(2^n - 1)$.
- Suppose that q is an n-Albert form. By [2, Proof of Prop. 4.4], the anisotropic part q_{an} looks like $q_{an} = \langle\langle a_1, \ldots, a_m\rangle\rangle\, q'$, where q' is an anisotropic $(n-m)$-Albert form. In particular, $\dim q_{an}$ has dimension $2^m \cdot 2(2^{n-m}-1) = 2(2^n - 2^m)$, where $0 \leq m \leq n$. We say that m is the linkage number of the n-Albert from q.
- Every 1-Albert form has the form $q = \langle\langle a\rangle\rangle' \perp -\langle\langle b\rangle\rangle = \langle -a, b\rangle$. Hence any 2-dimensional form is a 1-Albert form.
- Every 2-Albert form has the form
$$q = \langle\langle a_1, a_2\rangle\rangle' \perp -\langle\langle b_1, b_2\rangle\rangle' = \langle -a_1, -a_2, a_1a_2, b_1, b_2, -b_1b_2\rangle\,.$$
Thus, a 2-Albert form is the "classical" 6-dimensional Albert form.

Our interest in n-Albert forms is motivated by the following observation of A. Vishik (see [27]): *if q_1 and q_2 are n-Albert forms such that $q_1 \equiv q_2 \pmod{I^{n+1}(F)}$, then $q_1 \overset{v}{\sim} q_2$.*

The following question is due to Lam [18, Item (6.6), Page 28].

Question 4.3. *Let q_1 and q_2 be n-Albert forms such that $q_1 \equiv q_2 \pmod{I^{n+1}(F)}$. Is it always true that $q_1 \sim q_2$?*

The answer to this question is obviously positive in the case $n = 1$. In the case $n = 2$, the answer is also positive. This is a version of a Jacobson's theorem (see, e.g., [19, Prop. 2.4]). In this section, we construct a counterexample to this question for any $n \geq 3$.

Theorem 4.4. *There exists a field F and anisotropic 3-Albert forms q_1 and q_2 over F such that $q_1 \equiv q_2 \pmod{I^4(F)}$ and $q_1 \not\sim q_2$. In particular, the answer to Question 4.3 is negative in the case $n = 3$.*

Proof. We need the following theorem of Hoffmann.

Theorem 4.5. (see [6, Th. 4.3]). *There exists a field k and anisotropic 8-dimensional quadratic forms over k,*

$$\phi_1 = s_1 \langle\langle a_1, b_1 \rangle\rangle \perp -k_1 \langle\langle c_1, d_1 \rangle\rangle ,$$
$$\phi_2 = s_2 \langle\langle a_2, b_2 \rangle\rangle \perp -k_2 \langle\langle c_2, d_2 \rangle\rangle$$

such that $\phi_1 \equiv \phi_2 \pmod{I^4(k)}$, $\operatorname{ind} C(\phi_1) = \operatorname{ind} C(\phi_2) = 4$ and $\phi_1 \not\sim \phi_2$. □

Remark 4.6. In fact, the formulation of Theorem 4.3 in [6] differs from the one presented above. In his theorem, Hoffmann has constructed a pair $\phi, \psi \in I^2(k)$ of 8-dimension quadratic forms such that $\phi \not\sim \psi$ and $\phi \overset{hn}{\sim} \psi$. Clearly, changing ψ by a scalar, we may always assume that $\phi \equiv \psi \pmod{I^4(k)}$. To obtain Theorem 4.5, it suffices to show that we may always take ϕ and ψ in the form of direct sums of forms belonging to $GP_2(k)$. In the proof of [6, Theorem 4.3] it is so for the form ϕ (the explicit formula for ϕ in [6] shows that ϕ contains a subform $a\langle 1, x, y, xy\rangle$). The required statement concerning ψ is obvious since $i_W(\psi_{k(\sqrt{-x})}) = i_W(\phi_{k(\sqrt{-x})}) \geq 2$.

Now we return to the proof of Theorem 4.4. Under the conditions of this theorem, we obviously have $(a_1, b_1) + (c_1, d_1) = c(\phi_1) = c(\phi_2) = (a_2, b_2) + (c_2, d_2)$. Hence there exists an Albert form ρ (of dimension 6) such that $c(\phi_1) = c(\phi_2) = c(\rho)$. Hence $\operatorname{ind} C(\rho) = \operatorname{ind} C(\phi_1) = 4$. By an Albert's theorem, ρ is anisotropic (see [1, Th. 3] or [26, Th. 3]). Since $(a_i, b_i) + (c_i, d_i) = c(\rho)$ for $i = 1, 2$, there exist r_1 and r_2 such that

$$\langle\langle a_1, b_1 \rangle\rangle' \perp - \langle\langle c_1, d_1 \rangle\rangle' \simeq r_1 \rho,$$
$$\langle\langle a_2, b_2 \rangle\rangle' \perp - \langle\langle c_2, d_2 \rangle\rangle' \simeq r_2 \rho.$$

In the Witt ring $W(k(t))$, we have

$$\begin{aligned} t\rho - \phi_i &= tr_i(\langle\langle a_i, b_i \rangle\rangle - \langle\langle c_i, d_i \rangle\rangle) - (s_i \langle\langle a_i, b_i \rangle\rangle - k_i \langle\langle c_i, d_i \rangle\rangle) \\ &= tr_i(\langle\langle a_i, b_i \rangle\rangle - tr_i s_i \langle\langle a_i, b_i \rangle\rangle) - tr_i(\langle\langle c_i, d_i \rangle\rangle - tr_i k_i \langle\langle c_i, d_i \rangle\rangle) \\ &= tr_i(\langle\langle a_i, b_i, tr_i s_i \rangle\rangle - \langle\langle c_i, d_i, tr_i k_i \rangle\rangle). \end{aligned}$$

We set $q_i = \langle\langle a_i, b_i, tr_i s_i \rangle\rangle' \perp - \langle\langle c_i, d_i, tr_i k_i \rangle\rangle'$ and $F = k(t)$. Since $t\rho - \phi_i = tr_i q_i$ in the Witt ring $W(F)$ and $\dim(t\rho \perp -\phi_i) = 6 + 8 = 14 = \dim q_i$, we have $t\rho \perp -\phi_i \simeq tr_i q_i$.

Since ρ and ϕ_i are anisotropic, q_i is also anisotropic by Springer's theorem (see [17, Ch. 6, Th. 1.4] or [23, Ch. 6, Cor. 2.6]).

Now, we need the following obvious assertion.

Lemma 4.7. (see, e.g., [6, Lemma 3.1]). *Let $\mu_1, \mu_2, \nu_1, \nu_2$ be anisotropic quadratic forms over k. Suppose that the form $\mu_1 \perp t\nu_1$ is similar to $\mu_2 \perp t\nu_2$ over the field of rational functions $k(t)$. Then*

- *either $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$,*
- *or $\mu_1 \sim \nu_2$ and $\nu_1 \sim \mu_2$.* □

Since $\phi_1 \not\sim \phi_2$ and $\dim \rho < \dim \phi_1 = \dim \phi_2$, Lemma 4.7 shows that $(t\rho \perp -\phi_1) \not\sim (t\rho \perp -\phi_2)$. Hence $q_1 \not\sim q_2$. On the other hand, the conditions $q_1, q_2 \in I^3(F)$ and $\phi_1 \equiv \phi_2 \pmod{I^4(F)}$ imply that

$$q_1 \equiv tr_1 q_1 \equiv (t\rho \perp -\phi_1) \equiv (t\rho \perp -\phi_2) \equiv tr_2 q_2 \equiv q_2 \pmod{I^4(F)}.$$

Thus, we have proved that q_1 and q_2 are anisotropic 3-Albert forms such that $q_1 \equiv q_2 \pmod{I^4(F)}$ and $q_1 \not\sim q_2$. The theorem is proved. □

Corollary 4.8. *For any $n \geq 3$, there exists a field E and n-Albert forms γ_1 and γ_2 over E such that $\gamma_1 \equiv \gamma_2 \pmod{I^{n+1}(E)}$ and $\gamma_1 \not\sim \gamma_2$. In other words, the answer to Question 4.3 is negative for any $n \geq 3$.*

Proof. Let q_1, q_2 and F be as in Theorem 4.4. We write q_1 and q_2 in the form $q_1 = \pi_1' \perp -\tau_1'$, $q_2 = \pi_2' \perp -\tau_2'$ with $\pi_1, \pi_2, \tau_1, \tau_2 \in P_3(F)$ and put $E = F(x_1, \ldots, x_{n-3})$ and

$$\gamma_1 = (\pi_1 \langle\langle x_1, \ldots, x_{n-3}\rangle\rangle)' \perp -(\tau_1 \langle\langle x_1, \ldots, x_{n-3}\rangle\rangle)',$$
$$\gamma_2 = (\pi_2 \langle\langle x_1, \ldots, x_{n-3}\rangle\rangle)' \perp -(\tau_2 \langle\langle x_1, \ldots, x_{n-3}\rangle\rangle)'.$$

Obviously, $\gamma_i = q_i \langle\langle x_1, \ldots, x_{n-3}\rangle\rangle$ in the Witt ring $W(E)$. Since $q_1 \equiv q_2 \pmod{I^4(F)}$, we have $\gamma_1 \equiv \gamma_2 \pmod{I^{n+1}(E)}$. Since $q_1 \not\sim q_2$, we have $q_1 \langle\langle x_1, \ldots, x_{n-3}\rangle\rangle \not\sim q_2 \langle\langle x_1, \ldots, x_{n-3}\rangle\rangle$ (see, e.g., Lemma 4.7). Hence $\gamma_1 \not\sim \gamma_2$. □

We have constructed a pair of n-Albert forms γ_1 and γ_2 such that $\gamma_1 \overset{m}{\sim} \gamma_2$ and $\gamma_1 \not\sim \gamma_2$. Obviously, in our example, we have $\dim(\gamma_i)_{an} = 2^{n-3} \cdot 14 = 2^{n-3}(2^3 - 2) = 2(2^n - 2^{n-3})$. In other words, both n-Albert forms γ_1 and γ_2 are $(n-3)$-linked. We can generalize this example as follows.

Theorem 4.9. *For any $n \geq 3$ and m such that $0 \leq m \leq n-3$, there exists a field F and n-Albert forms q_1 and q_2 over F such that $q_1 \equiv q_2 \pmod{I^{n+1}(F)}$, $q_1 \not\sim q_2$, and $\dim(q_1)_{an} = \dim(q_2)_{an} = 2(2^n - 2^m)$.*

Here we only outline the proof of the theorem.

Step 1. It suffices to prove this theorem only in the case $m = 0$ (this means that q_1 and q_2 are anisotropic). After this, the general case can be obtained in the same way as Corollary 4.8.

Step 2. Consider a field E and n-Albert forms γ_1 and γ_2 as in Corollary 4.8. Since $\gamma_1 \equiv \gamma_2 \pmod{I^{n+1}(E)}$, there exist $\pi_1, \ldots, \pi_N \in P_{n+1}(E)$ for some integer N

such that $\gamma_1 - \gamma_2 = \sum_{i=1}^N \pi_i$. We consider the quadratic forms

$$\tilde{q}_1 = \langle\langle x_1, \dots, x_n\rangle\rangle' \perp -\langle\langle y_1, \dots, y_n\rangle\rangle',$$
$$\tilde{q}_2 = \langle\langle z_1, \dots, z_n\rangle\rangle' \perp -\langle\langle t_1, \dots, t_n\rangle\rangle',$$
$$\tau = \perp_{i=1}^N \langle\langle u_{i,1}, \dots, u_{i,n+1}\rangle\rangle.$$

over the field of rational functions

$$\tilde{E} = E(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n, t_1, \dots, t_n, u_{1,1}, \dots, u_{N,n+1}).$$

Obviously there exists a place $\tilde{s} : \tilde{E} \to E$ such that $\tilde{q}_1 \mapsto \gamma_1$, $\tilde{q}_2 \mapsto \gamma_2$, and $\langle\langle u_{i,1}, \dots, u_{i,n+1}\rangle\rangle \mapsto \pi_i$ for all $i = 1, \dots, N$. Since $\gamma_1 - \gamma_2 = \sum_{i=1}^N \pi_i$, the form $\tilde{s}_*(\tilde{q}_1 \perp -\tilde{q}_2 \perp -\tau)$ is hyperbolic.

Step 3. We define the field F as a "generic" extension $F/\tilde{E}$ such that $(\tilde{q}_1)_F - (\tilde{q}_2)_F = \tau_F$. More precisely, we set $F = \tilde{E}_h$, where $\tilde{E}_0, \tilde{E}_1, \dots, \tilde{E}_h$ is the generic splitting tower for the $\tilde{E}$-form $\tilde{q}_1 \perp -\tilde{q}_2 \perp -\tau$. We claim that the F-forms $q_1 \overset{\text{Def}}{=} (\tilde{q}_1)_F$ and $q_2 \overset{\text{Def}}{=} (\tilde{q}_1)_F$ satisfy the hypotheses of Theorem 4.9. Since $q_1 - q_2 = \tau_F$, we have $q_1 \equiv q_2 \pmod{I^{n+1}(F)}$. Thus, it suffices to verify that q_1 and q_2 are anisotropic and $q_1 \not\sim q_1$.

Step 4. Using properties of generic splitting fields (see [23, Ch. 4, Cor. 6.10] or [11, Th. 5.1]), we can extend $\tilde{s} : \tilde{E} \to E$ to a place $s : F \to E$. Obviously, $s_*(q_1) = \gamma_1$ and $s_*(q_2) = \gamma_2$. Therefore, the condition $\gamma_1 \not\sim \gamma_2$ implies $q_1 \not\sim q_2$.

Step 5. To prove that q_1 and q_2 are anisotropic, it suffices to construct a field extension $K/\tilde{E}$ with the same key property as F (i.e., $(\tilde{q}_1)_K - (\tilde{q}_2)_K = \tau_K$) and such that $(\tilde{q}_1)_K$ and $(\tilde{q}_2)_K$ are anisotropic. Since $F/\tilde{E}$ is a "generic" extension, we necessarily get that $q_1 = (\tilde{q}_1)_F$ and $q_2 = (\tilde{q}_2)_F$ are anisotropic. The following extension $K/\tilde{E}$ has the required properties:

$$K = \tilde{E}\Big(\sqrt{\frac{x_1}{z_1}}, \dots, \sqrt{\frac{x_n}{z_n}}, \sqrt{\frac{y_1}{t_1}}, \dots, \sqrt{\frac{y_n}{t_n}}, \sqrt{u_{1,1}}, \dots, \sqrt{u_{N,1}}\Big).$$

The "sketch" of the proof is complete. In fact, Steps 4 and 5 are the most difficult points. We refer the reader to the paper [7, Proof of Lemma 2.2], where similar arguments (as in Step 5) are presented with complete proofs.

Corollary 4.10. *For any m and n such that $0 \le m \le n-3$, there exists a field F and anisotropic $2(2^n - 2^m)$-dimensional forms q_1 and q_2 over F such that $q_1 \overset{v}{\sim} q_2$ and $q_1 \not\sim q_2$.*

5. Open questions

Obviously, Theorem 4.9 cannot be generalized to the cases $m = n-1$ and $m = n$ because in these cases the anisotropic parts of n-Albert forms either belong to $GP_n(F)$ or are zero. There is only one case, where we cannot say anything definite. Namely, $m = n-2$. For this reason, we propose the following modification of Lam's Question 4.3.

Conjecture 5.1. *Let q_1 and q_2 be Albert forms (i.e., 6-dimensional forms with trivial discriminants). Let $\phi_1 = \langle\langle a_1, \dots, a_k\rangle\rangle q_1$ and $\phi_2 = \langle\langle b_1, \dots, b_k\rangle\rangle q_2$. Suppose that $\phi_1 \equiv \phi_2 \pmod{I^{k+3}(F)}$. Then $\phi_1 \sim \phi_2$.*

We note that, in this conjecture, we always may assume that $a_i = b_i$ for $i = 1, \dots, k$. Indeed, putting $\pi = \langle\langle a_1, \dots, a_k \rangle\rangle$, we obtain $(\phi_2)_{F(\pi)} \equiv (\phi_1)_{F(\pi)} = 0 \pmod{I^{k+1}(F(\pi))}$. By the Arason–Pfister theorem, we conclude that ϕ_2 is hyperbolic over the field $F(\pi)$. Hence ϕ_2 has the form $\phi_2 = \pi q_2' = \langle\langle a_1, \dots, a_k \rangle\rangle\, q_2'$. Comparing dimensions, we get $\dim q_2' = 6$. Let us write q_2' in the form $q_2' = \langle c_1, \dots, c_6 \rangle$ and set $q_2'' = \langle c_1, \dots, c_5, c_6' \rangle$, where $c_6' = -c_1 \dots c_5$. We have $\pi \langle c_6, -c_6' \rangle = \pi q_2' - \pi q_2'' = \phi_2 - \pi q_2'' \in I^{k+2}(F) + I^k(F) \cdot I^2(F) = I^{k+2}(F)$. Since $\dim \pi \langle c_6, -c_6' \rangle = 2^k \cdot 2 < 2^{k+2}$, the Arason–Pfister theorem shows that $\pi \langle c_6, -c_6' \rangle$ is hyperbolic. Hence $\pi q_2' = \pi q_2''$. Therefore, $\phi_2 = \pi q_2'' = \langle\langle a_1, \dots, a_k \rangle\rangle\, q_2''$. Since q_2'' is an Albert form, we have proved, that the conjecture reduces to the case where $b_i = a_i$. □

Another question concerning the $\overset{v}{\sim}$-equivalence is motivated by the results of §3 and §4. First of all, in view of Lemma 3.4 and Corollary 4.10, we have the following assertion.

PROPOSITION 5.2. *Let d be an integer belonging to the set*

$$\{2^n | n \geq 3\} \cup \{2^i(2^j - 1) | i \geq 1, j \geq 3\}$$

Then there exist anisotropic d-dimensional quadratic forms ϕ and ψ over a suitable field such that $\phi \overset{v}{\sim} \psi$ and $\phi \not\sim \psi$. □

Here we state the following

PROBLEM 5.3. *Describe the set $\mathcal{VE}$ of all integers d for which there exist anisotropic d-dimensional quadratic forms ϕ and ψ over a suitable field such that $\phi \overset{v}{\sim} \psi$ and $\phi \not\sim \psi$.*

We know almost the full answer to this problem. The results of the previous sections imply that $\mathcal{VE} \subset \{8, 10, 12, \dots, 2i, \dots\}$. Besides, we can prove that any even integer ≥ 8 (except possibly 12) belongs to $\mathcal{VE}$.

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Oleg Izhboldin
Department of Mathematics and Mechanics
St.-Petersburg State University
Petrodvorets, 198904
Russia
oleg@mathematik.uni-bielefeld.de