A note on the Global Langlands Conjecture

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Abstract. The theory of base change is used to give some new examples of the Global Langlands Conjecture. The Galois representations involved have solvable image and are not monomial, although some multiple of them in the Grothendieck group is monomial. Thus, it gives nothing new about Artin's Conjecture itself. An application is given to a question which arises in studying multiplicities of cuspidal representations of SL_n. We explain how the (conjectural) adjoint lifting can prove GLC for a family of representations containing the tetrahedral 2-dimensional ones.

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1 Introduction

The Global Langlands Conjecture asserts that for any n-dimensional irreducible representation σ of the absolute Galois group (or more generally, the Weil group) of a number field F, there corresponds a cuspidal representation π of $GL_n(\mathbb{A}_F)$ with matching Langlands parameters almost everywhere. Specifically, if $\pi_v = \mathrm{Ind}_B^G(|\cdot|^{s_1}, \ldots, |\cdot|^{s_n})$ then the condition is that

$$\sigma(Fr_v) \sim \begin{pmatrix} q_v^{-s_1} & & \\ & \cdots & \\ & & q_v^{-s_n} \end{pmatrix}.$$

Such a cuspidal representation π is unique by virtue of strong multiplicity one. This conjecture implies that the (partial) L-functions of σ and π are the same, and hence Artin's Conjecture for σ is true. The case of $n = 1$ is essentially global class field theory. In ([L]) Langlands observed that the theory of base change, initiated by Saito and Shintani, can lead to the GLC for some irreducible 2-dimensional Galois representations with solvable image. This was extended to all such representations in [Tu]. Later, Arthur and Clozel ([AC]) proved cyclic base change for GL_n. Thus, the GLC is preserved under induction from a cyclic extension. In particular, it holds for representations of the type

$$\mathrm{Ind}_{W_E}^{W_F} \theta, \ F \subset_{sc} E, \theta \text{ a Hecke character of } E, \qquad (*)$$

where $F \subset_{sc} E$ means that the extension is obtained by a series of cyclic extensions. It is then a consequence of results of Jacquet and Shalika ([JS]) to deduce GLC for irreducible representations which are linear combinations *over* $\mathbb{Z}$ in the Grothendieck group of representations of the above type. Unfortunately, by a result of Dade ([Da]), such representations are themselves of type $(*)$. The purpose of this note is to indicate other cases for which the GLC can be proved, using base change. In all our cases the representation is a linear combination *over* $\mathbb{Q}$ of representations of type $(*)$. Again, by modifying the result of Dade, some multiple of such a representation is monomial, so that Artin's Conjecture is automatically satisfied for it. However, it is not necessarily monomial itself. The prototype of the new examples is an irreducible n-dimensional representation σ, factoring through a group G, whose image in $PGL_n(\mathbb{C})$ has order n^2 (which is the least possible). Equivalently, $n\sigma = \mathrm{Ind}_Z^G \zeta$ in the Grothendieck group where Z is the center of G and ζ is the central character of σ. We call such representations *minimal*. If G is solvable then we know that there exists an automorphic representation Π of $GL_{n^2}(F)$ induced from cuspidals corresponding to $\mathrm{Ind}_Z^G \zeta$. If we can show that Π is "isotypic", i.e. it is induced from $\pi \otimes \cdots \otimes \pi$ for π on $GL_n(F)$, then $\pi \leftrightarrow \sigma$ and GLC is valid for σ. This is not immediate; if $\Pi = \boxplus m_i \pi_i$ is the "decomposition" of Π then a standard argument of L-functions gives us that $\sum {m_i}^2 = n^2$. However, this is not enough to conclude that there is only one summand. On the other hand, there are lucky situations where it is not difficult to prove the "purity" of Π, even though σ may not be itself monomial. The main point is that base change and automorphic induction can *sometime* be used to provide analogues in the automorphic side of classical results from representation theory of finite groups, such as Frobenius reciprocity and Clifford theory. Even if we limit ourselves to the solvable case this does not always work, because we only have the constructions for normal subgroups. The following is a typical case of our main result.

Theorem 1. *Let σ be a minimal representation factoring through G. Suppose that either*

1. *G/Z has a composition series whose quotients have pairwise coprime orders (Theorem 4), or,*
2. *$m\sigma = \mathrm{Ind}_N^G \tau$ in $R(G)$ where τ is minimal, N is normal in G, $m^2 = |G/N|$, and both N and G/N are nilpotent (Theorem 5).*

Then σ is automorphic.

As an application, we address a question which was posed in [La]. The problem is pertaining to calculating multiplicities of representations of $SL(n)$ in the cuspidal spectrum. There is a heuristic analogue of the multiplicity formula of Labesse-Langlands for L-packets coming from representations of the Weil group. One would like to compare the two formulas. Our result here is that the comparison is valid for L-packets coming from irreducible representations of W_F induced from a Hecke character on E where $F \subset E$ is nilpotent.

In the last section, we focus on the simplest representations for which Artin's Conjecture is not known. These are higher dimensional analogues of the tetrahedral 2-dimensional representations. In more detail, for any prime power q we consider q-dimensional irreducible representations σ whose image in $PGL_q(\mathbb{C})$ is isomorphic to $\mathbb{F}_q \oplus \mathbb{F}_q \rtimes B$ where $B \le SL_2(\mathbb{F}_q)$ is solvable and $q \not| |B|$. The argument in [L] that proves GLC in the case $q = 2$ carries over to the general case, provided that we know that the adjoint lifting from automorphic representations on GL_q to GL_{q^2-1} exists. While this is far from being proved, even for $q = 3$, the example is given as an illustration of a "simple" case to stare at, while working on Artin's Conjecture.

1.1 Notations and preliminaries

Throughout this paper $\mathbb{A} = \mathbb{A}_F$ (resp. $\mathbb{I}_F$) will denote the ring of adeles (resp. group of ideles) over a number field F, W_F is the Weil group of F. For a cyclic extension $F \subset E$ we let $\omega_{E/F}$ be a character of $\mathbb{I}_F$ with kernel $F^* \operatorname{Nm}_F^E \mathbb{I}_E$. An extension $F \subset E$ is called subcyclic (written $F \subset_{sc} E$) if there exists a sequence of cyclic extensions $F = F_0 \subset F_1 \subset \cdots \subset F_r = E$. For any extension $F \subset E$ we let $\Gamma = \Gamma_{E/F}$ be the homogeneous space (or the group in the normal case) $\operatorname{Gal}(\bar{F}/F)/\operatorname{Gal}(\bar{F}/E)$.

For any group G, let $\operatorname{Irr}_n(G)$ be the set of equivalence classes of n-dimensional irreducible representations of G. Let also $\operatorname{Irr}(G) = \bigcup_n \operatorname{Irr}_n(G)$. We write $R(G)$ for the Grothendieck group of the category of finite dimensional representations of G. Let $c : R(G) \times R(G) \to \mathbb{Z}$ be the canonical pairing. The determinant character of $\sigma \in \operatorname{Irr}(G)$ will be denoted by χ_σ. We will often encounter the follow situation. Suppose that H is a normal subgroup of G of finite index and $\sigma \in \operatorname{Irr}(H)$. We let $(G/H)(\sigma) = \{g \in G/H : \sigma^g \simeq \sigma\}$. Suppose that $G/H(\sigma) = G/H$. Choose a transversal $\{g_x\}_{x \in G/H}$ and let $A_x : V_\sigma \to V_\sigma$ be intertwining operators between (σ, V_σ) and (σ^{g_x}, V_σ). The cocycle given by

$$\alpha_\sigma(x, y) = A_x A_y A_{xy}^{-1} \sigma(g_x g_y g_{xy}^{-1})^{-1} \in \mathbb{C}^* \tag{1}$$

defines an element in the Schur multipliers of G/H, which depends only on σ. It is the obstruction to extending σ to G. We have $\operatorname{End}_G[\operatorname{Ind}_H^G \sigma] \simeq \mathbb{C}[G/H, \alpha_\sigma]$ where the latter is the twisted group algebra of G/H. In particular, $\operatorname{Ind}_H^G \sigma$ is isotypic if and only if $\mathbb{C}[G/H, \alpha_\sigma]$ is simple. For the central character we have

$$\alpha_{\chi_\sigma} = \alpha_\sigma^m \tag{2}$$

where $m = \deg \sigma$.

We will deal with automorphic representations π of GL_n which are induced from cuspidals, i.e. there exists a parabolic P of type $(n_1, \ldots, n_r)$ and a cuspidal representation $\otimes \pi_i$ of $M_P = GL_{n_1}(F) \times \cdots \times GL_{n_r}(F)$ such that $\pi = \boxplus \pi_i = Ind_{P(\mathbb{A})}^{G(\mathbb{A})} \otimes \pi_i$. By the results of Jacquet-Shalika ([JS]) the π_i's are uniquely determined and we call them the *components* of π. We will denote by $\operatorname{Cusp}_n(F)$ the set of cuspidal representations of $GL_n(F)$. Let also $\operatorname{Cusp}(F) = \bigcup \operatorname{Cusp}_n(F)$. Let $R_{cusp}(F)$ be the semigroup of automorphic representations induced from cuspidal representations of $GL_n(F)$. If $\pi = \boxplus \pi_i$, $\tau = \boxplus \tau_j$ are the decompositions of $\pi, \tau \in R_{cusp}(F)$, we let

$$c(\pi, \tau) = \#\{(i, j) : \pi_i \simeq \tau_j\}.$$

Then $c(\pi, \tau)$ is the order of the pole at $s = 1$ of the partial Jacquet-Shalika L-function $L^S(\pi \otimes \tau^\vee, s)$ where $\tau^\vee$ is the contragredient of τ ([JS]). We call π, τ *disjoint* if $c(\pi, \tau) = 0$. Similarly to the notations above, if $\pi \in R_{cusp}(E)$ and $F \subset E$ is normal we let $\Gamma(\pi) = \{\gamma \in \Gamma_{E/F} : \pi^\gamma \simeq \pi\}$. Also, the central character of an automorphic representation π will be denoted by χ_π. If $\pi \in \mathrm{Cusp}_n(F)$ and $\sigma \in \mathrm{Irr}_n(W_F)$ have the same Langlands parameters almost everywhere we write $\pi \leftrightarrow \sigma$. Then of course $\chi_\pi = \chi_\sigma$, where we identify characters of W_F and $\mathbb{I}_F$. Let us call a Weil group representation *automorphic* if there exists an automorphic representation, necessarily unique, with matching Langlands parameters almost everywhere. We say that the corresponding automorphic representation is *of Galois type.* Finally we call an extension $F \subset E$ p-subnormal if it can be embedded in a normal extension of F of p-power order.

2 Base Change

In this section we recollect some facts about base change and automorphic induction. Let $F \subset E$ be an extension of degree m and $GL_n(E) = RES_{E/F}GL_n$. Recall that the L-group of $GL_n(E)$ is isomorphic to $GL_n(\mathbb{C})^{\Gamma_{E/F}} \rtimes \mathrm{Gal}(\bar{F}/F)$ where $\mathrm{Gal}(\bar{F}/F)$ acts through its action on Γ. There are two "dual" homomorphisms

$$\begin{aligned} bc &: {}^L GL_n(F) \longrightarrow {}^L GL_n(E) \\ ai &: {}^L GL_n(E) \longrightarrow {}^L GL_{nm}(F) \end{aligned}$$

of L-groups in the theory of base change, corresponding to restriction and induction of representations of the Weil groups. They are defined by

$$\begin{aligned} bc(g,\sigma) &= ((g,\ldots,g),\sigma) \\ ai((g_1,\ldots,g_m),\sigma) &= (\mathrm{diag}(g_1,\ldots,g_m)R_\sigma,\sigma) \end{aligned}$$

where R_σ is the permutation matrix on the $n \times n$ blocks corresponding to σ. Langlands functoriality predicts the existence of liftings BC_F^E and AI_E^F of automorphic representations compatible with these homomorphisms.

Theorem 2 ([AC]). *Let $F \subset_{sc} E$. Then BC_F^E and AI_E^F exist and define additive morphisms*

$$\begin{aligned} \mathrm{BC}_F^E &: R_{cusp}(F) \longrightarrow R_{cusp}(E) \\ \mathrm{AI}_E^F &: R_{cusp}(E) \longrightarrow R_{cusp}(F). \end{aligned}$$

Furthermore, if $F \subset E$ is cyclic and $\rho_1, \rho_2 \in \mathrm{Cusp}(F)$, then $\mathrm{BC}(\rho_1) \simeq \mathrm{BC}(\rho_2)$ if and only if $\rho_2 \simeq \rho_1 \otimes \omega_{E/F}^i$ for some i. Similarly, if $\pi_1, \pi_2 \in \mathrm{Cusp}(E)$, then $\mathrm{AI}(\pi_1) \simeq \mathrm{AI}(\pi_2)$ if and only if $\pi_2 \simeq \pi_1^\gamma$ for some $\gamma \in \Gamma_{E/F}$.

These maps enjoy the following properties which are analogous to those of restriction and induction. Let $F \subset_{sc} E, K$, $\pi \in R_{cusp}(E)$, and $\rho \in R_{cusp}(F)$. Then:

$$\mathrm{AI}_E^F \pi = \mathrm{AI}_L^F \mathrm{AI}_E^L \pi \text{ for } F \subset_{sc} L \subset_{sc} E \text{ and similarly for } \mathrm{BC}. \tag{3}$$

$$(\mathrm{BC}_F^E \rho)^g = \mathrm{BC}_{F^g}^{E^g} \rho^g \text{ for } g \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \text{ and similarly for } \mathrm{AI}. \tag{4}$$

$$\mathrm{BC}_F^E(\mathrm{AI}_K^F(\pi)) = \boxplus_{\gamma \in W_K \backslash W_F / W_E} \mathrm{AI}_{E \cdot K^\gamma}^E (\mathrm{BC}_K^{E^{\gamma^{-1}} \cdot K} \pi)^\gamma. \tag{5}$$

These properties are mentioned, at least implicitly, in [AC], and follow easily from an unramified computation. Furthermore, the L-function identity

$$L^S(\mathrm{AI}_E^F \pi \otimes \rho^\vee, s) = L^S(\pi \otimes \mathrm{BC}_F^E \rho^\vee, s)$$

gives the following form of Frobenius reciprocity:

$$c(\mathrm{AI}_E^F \pi, \rho) = c(\pi, \mathrm{BC}_F^E \rho).$$

Finally, the same argument as in the group case proves that if $\pi \in \mathrm{Cusp}(E)$ then $\mathrm{AI}_E^F(\pi)$ is cuspidal if and only if $\mathrm{BC}_{E^\gamma}^{E\cdot E^\gamma}(\pi^\gamma)$ and $\mathrm{BC}_E^{E\cdot E^\gamma}(\pi)$ are disjoint for any $\gamma \in \Gamma_{\bar{E}/F} - \Gamma_{\bar{E}/E}$, where $\bar{E}$ is the normal closure of $F \subset E$.

Let $F \subset E$ be Galois, $\sigma \in \mathrm{Irr}(W_E)$ and $F \subset K \subset E$ be defined by $\Gamma(\sigma)$. Recall that by Clifford's Theory, the induction gives a bijection between the subsets of $\mathrm{Irr}(W_K)$ and $\mathrm{Irr}(W_F)$ of those representations whose restriction to W_E contains σ in their decomposition. The same will be true in the automorphic setup, if we assume that $F \subset E$ is solvable and $F \subset K$ is sub-normal.

We also need the following fact from group theory. Let $H < G$ be a normal subgroup with G/H nilpotent. Let λ be a character of H. Then any irreducible constituent of $\mathrm{Ind}_H^G \lambda$ is induced from a character on some subgroup $H < K < G$.

3 The main results

The heart of the matter is the following simple Lemma.

Lemma 1. *Let H be a normal subgroup of G with $[G:H] = p^k$, p prime, and let $\sigma \in \mathrm{Irr}_m(H)$ where $p \not| m$. Suppose that $\sigma^g \simeq \sigma$ for any $g \in G$. Then*

1. *There exists a subgroup $H < K < G$ with $[K:H]^2 \geq [G:H]$ such that σ extends to K.*
2. *The following conditions are equivalent*
   (a) *$\mathrm{Ind}_H^G \sigma$ is isotypic.*
   (b) *$\mathrm{Ind}_H^G \chi_\sigma$ is isotypic.*
   (c) *If $H < K < G$ and σ extends to K then $[K:H]^2 \leq [G:H]$.*
   (d) *If $H < K < G$ and χ_σ extends to a character of K then $[K:H]^2 \leq [G:H]$.*
3. *Under these conditions, if Σ is an extension of σ to K with $[K:H]^2 = [G:H]$ then $\mathrm{Ind}_K^G \Sigma$ and $\mathrm{Ind}_K^G \chi_\Sigma$ are irreducible and*

$$\mathrm{Ind}_H^G \sigma = [K:H]\,\mathrm{Ind}_K^G \Sigma, \quad \mathrm{Ind}_H^G \chi_\sigma = [K:H]\,\mathrm{Ind}_K^G \chi_\Sigma.$$

Proof. Let α_σ be as in (1). The relation (2) together with the fact that $p \not| m$ implies that 2a $\Longleftrightarrow$ 2b and 2c $\Longleftrightarrow$ 2d. Decompose $\mathrm{Ind}_H^G \chi_\sigma$ as $\oplus_{i=1}^r m_i \lambda_i$ with $\lambda_i \in \mathrm{Irr}(G)$. Then $\sum m_i^2 = [G:H]$ and $\sum m_i \dim(\lambda_i) = [G:H]$. We also know (see end of §2) that for all i, $\lambda_i = \mathrm{Ind}_{K_i}^G \theta_i$ for some 1-dim character θ_i on $K_i > H$ extending χ_σ. For some i, $\dim(\lambda_i) \leq m_i \leq [G:H]^{1/2}$, and hence $[K_i:H]^2 \geq [G:H]$. Moreover, if $r > 1$ then we get a strict inequality. This proves the first part and that 2d implies 2b. Suppose that $r = 1$. Then

$$m_1 = \dim(\lambda_1) = [K_1:H] = [G:H]^{1/2}.$$

If λ were an extension of χ_σ to K with $[K:H] > [G:H]^{1/2}$ then $\mathrm{Ind}_K^G \lambda$ would be a subrepresentation of $\mathrm{Ind}_H^G \chi_\sigma$ of dimension $< [G:H]^{1/2}$ which is absurd. Finally, to prove the last statement, note that by the considerations above $\mathrm{Ind}_H^G \sigma = [K:H]\rho$ with ρ irreducible. It remains to observe that $\mathrm{Ind}_K^G \Sigma$ is a subrepresentation of $\mathrm{Ind}_H^G \sigma$ of dimension $m[K:H]$. □

The analogue of the following Lemma to the group case (valid for any Galois extension) is proved easily using Schur's lemma.

Lemma 2. *Let $F \subset E$ be a solvable extension, and let $\rho_1, \rho_2 \in \mathrm{Cusp}(F)$. Assume that $\mathrm{BC}_F^E \rho_1$ is cuspidal and $\mathrm{BC}_F^E \rho_2 \simeq \mathrm{BC}_F^E \rho_1$. Then $\rho_2 \simeq \rho_1 \otimes \omega$ for some character ω of W_F/W_E.*

Proof. We use induction on $[E:F]$, the cyclic case being covered by Theorem 2. Let $F \subset K$ be a cyclic extension in E. Using

$$\mathrm{BC}_K^E \mathrm{BC}_F^K \rho_1 \simeq \mathrm{BC}_K^E \mathrm{BC}_F^K \rho_2$$

and the induction hypothesis we get $\mathrm{BC}_F^K \rho_1 \simeq \mathrm{BC}_F^K \rho_2 \otimes \omega$ for some character ω of W_K/W_E. Conjugating by $\gamma \in \Gamma_{K/F}$ we also have $\mathrm{BC}_F^K \rho_1 \simeq \mathrm{BC}_F^K \rho_2 \otimes \omega^\gamma$, from which

$$\mathrm{BC}_F^K \rho_1 \otimes \omega^\gamma \omega^{-1} \simeq \mathrm{BC}_F^K \rho_1.$$

If $\omega^\gamma \neq \omega$ then $\mathrm{BC}_F^E \rho_1 = \mathrm{BC}_K^E \mathrm{BC}_F^K \rho_1$ is not cuspidal, contradicting our assumption. Thus ω is $\Gamma_{K/F}$-invariant. Hence, we can extend ω to a character μ of W_F/W_E. We have $\mathrm{BC}_F^K \rho_1 \simeq \mathrm{BC}_F^K(\rho_2 \otimes \mu)$ and we can appeal to the cyclic case. □

Let us now give a simple descent criterion for base change, whose analogue in the group case is well-known.

Proposition 1. *Let $F \subset E$ be a solvable extension and let $\pi \in \mathrm{Cusp}_n(E)$ with $([E:F], n) = 1$. Suppose that $\pi^\gamma \simeq \pi$ for any $\gamma \in \Gamma_{E/F}$ and that there exists a character χ of W_F which extends χ_π. Then there exists a unique $\rho \in \mathrm{Cusp}_n(F)$ such that $\mathrm{BC}_F^E(\rho) \simeq \pi$ and $\chi_\rho = \chi$.*

Proof. The uniqueness part follows immediately from the Lemma above. To prove the existence we proceed by induction. Let $F \subset K$ be a cyclic extension contained in E. Using the induction hypothesis for $K \subset E$ we extend π to K as ρ' with $\chi_{\rho'} = \chi \circ \mathrm{Nm}_F^K$. Now, for any $\gamma \in \Gamma_{K/F}$

$$\mathrm{BC}_K^F {\rho'}^\gamma \simeq \pi^\gamma \simeq \pi \simeq \mathrm{BC}_K^F \rho'.$$

By uniqueness, ${\rho'}^\gamma \simeq \rho'$ and we can use the descent criterion for cyclic extensions (Theorem 2). After a possible twist by a character, we get the required central character. □

Theorem 3. *Let E/F be a p-extension of number fields and suppose that GLC holds for $\sigma \in \mathrm{Irr}_m(W_E)$ with $p \not| m$. Assume that $\mathrm{Ind}_{W_E}^{W_F} \sigma = n \cdot \tau$ with $\tau \in \mathrm{Irr}(W_F)$. Then τ is automorphic.*

Proof. By Clifford, $\mathrm{Ind}_{W_E}^{W_L} \sigma = n \cdot \tau'$ and $\mathrm{Ind}_{W_L}^{W_F} \tau' = \tau$, where L is the subfield corresponding to $\Gamma_{E/F}(\sigma)$. Let $\pi \leftrightarrow \sigma$. Suppose that we know that

$$\mathrm{AI}_E^L(\pi) = n\rho' \tag{6}$$

with $\rho' \in \mathrm{Cusp}(L)$. We can then conclude, by comparing the parameters, that $\rho' \leftrightarrow \tau'$ and $\mathrm{AI}_L^F \rho' \leftrightarrow \tau$. Let us prove (6). We can assume that $L = F$. By Lemma 1 we know that $\tau = \mathrm{Ind}_{W_K}^{W_F} \Sigma$ where $F \subset K \subset E$ and $\Sigma \in \mathrm{Irr}_m(W_K)$ extends σ. By the Proposition above, there exists $\Pi \in \mathrm{Cusp}_m(K)$ such that $\mathrm{BC}_K^E \Pi = \pi$ and $\chi_\Pi = \chi_\Sigma$. We claim that $\rho = \mathrm{AI}_K^F \Pi$ is cuspidal. If not, then the condition for cuspidality (§2), and the fact that $\mathrm{BC}_K^E \Pi$ is cuspidal imply that we have $\mathrm{BC}_{K^\gamma}^{K\cdot K^\gamma} \Pi^\gamma \simeq \mathrm{BC}_K^{K\cdot K^\gamma} \Pi$ for some $\gamma \in \Gamma_{E/F} - \Gamma_{E/K}$. In particular,

$$\chi_\Pi^\gamma\big|_{W_K \cap W_K^\gamma} = \chi_\Pi\big|_{W_K \cap W_K^\gamma}. \tag{7}$$

However, $\mathrm{Ind}_{W_K}^{W_F} \chi_\Sigma$ is irreducible. This contradicts (7), because $\chi_\Sigma = \chi_\Pi$.

Finally,

$$c(\rho, \mathrm{AI}_E^F \pi) = c(\mathrm{BC}_F^E(\mathrm{AI}_K^F \Pi), \pi) = \sum_{\gamma \in \Gamma_{E/F}/\Gamma_{E/K}} c(\mathrm{BC}_K^E(\Pi)^\gamma, \pi) = [K:F] = n$$

so that $\mathrm{AI}_E^F \pi = n\rho$ as required. □

Theorem 4. *Let $F \subset E$ be a normal extension of number fields with the property that there exists a sequence of distinct primes $p_i, i = 1, 2, \ldots r$ and a sequence of extensions $F = F_0 \subset F_1 \subset \cdots \subset F_r = E$ where $F_{i-1} \subset F_i$ is a p_i-extension. Let $\sigma \in \mathrm{Irr}_m(W_E)$ with $p_i \nmid m$ for any i, be such that $\mathrm{Ind}_{W_E}^{W_F} \sigma = n\tau$ for some n and $\tau \in \mathrm{Irr}(W_F)$. Then τ is automorphic.*

Proof. Using induction and the previous Theorem we have to show that the conditions of the Theorem hold for $F = F_1$. Suppose on the contrary that $\Sigma_1 = \mathrm{Ind}_{W_E}^{W_{F_1}} \sigma$ is not isotypic. First, we claim that the irreducible constituents of Σ_1 lie in the same orbit under $\Gamma_{F_1/F}$. Indeed, if τ_1 is an irreducible component of Σ_1, then by our condition $\mathrm{Ind}_{W_{F_1}}^{W_F} \tau_1$ is isotypic and the type does not depend on τ_1. In particular, for any irreducible component τ' of Σ_1 we have $c(\mathrm{Ind}_{W_{F_1}}^{W_F} \tau', \mathrm{Ind}_{W_{F_1}}^{W_F} \tau_1) > 0$. This implies that $\tau' = \tau_1^\gamma$ for some $\gamma \in \Gamma_{F_1/F}$. Next, note that for $\gamma \in \Gamma_{F_1/F}$, $c(\Sigma_1^\gamma, \Sigma_1) = 0$ if γ does not lie in the image $\bar\Gamma$ of $\Gamma_{E/F}(\sigma)$ under $\Gamma_{E/F} \to \Gamma_{F_1/F}$, and $\Sigma_1^\gamma \simeq \Sigma_1$ otherwise. Thus $\Sigma_1 = k(\oplus \tau_1^\gamma)$ for some k where the sum is over the orbit of τ_1 under $\bar\Gamma$. Since $p_1 \nmid \dim \Sigma_1$, this orbit is a singleton and we get $\Sigma_1 = k\tau_1$ as required. □

For the application we have in mind, we would also like to have the dual statement, which is proved in a similar way.

Theorem 4'. *Let $F \subset E$ be as before. Suppose that $\pi \leftrightarrow \sigma$ and $\mathrm{AI}_E^F \pi = n\rho$ with ρ cuspidal. Then $\mathrm{Ind}_{W_E}^{W_F} \sigma = n\tau$ with τ irreducible and $\rho \leftrightarrow \tau$.*

Proof. The reduction to the case where $F \subset E$ is a p-extension is as in Theorem 4, using only the formal properties of base change described in §2. Then again, by using 'Clifford Theory' in the automorphic side, we are reduced to the case where $\Gamma(\sigma) = \Gamma(\pi) = \Gamma$. Assume on the contrary that $\mathrm{Ind}_{W_E}^{W_F} \sigma$ is not isotypic. According to Lemma 1, we can extend σ to a representation $\Sigma \in \mathrm{Irr}_m(W_K)$ where $[K:F]^2 < [E:F]$. By Proposition 1 we have a cuspidal representation $\Pi \in \mathrm{Cusp}_m(K)$ with $\mathrm{BC}_K^E \Pi \simeq \pi$. But then,

$$c(\rho, \mathrm{AI}_K^F \Pi) = \frac{1}{n} c(\mathrm{AI}_E^F \pi, \mathrm{AI}_K^F \Pi) = \frac{1}{n} c(\pi, \mathrm{BC}_F^E \mathrm{AI}_K^F \Pi) \geq \frac{1}{n} c(\pi, \mathrm{BC}_K^E \Pi) > 0.$$

This contradicts the fact $\dim \mathrm{AI}_K^F \Pi = [K:F]m < \dim \rho$. ☐

Lemma 3. *Let $F \subset E$ be a nilpotent extension and let $\sigma \in \mathrm{Irr}(W_E)$. Then $\mathrm{Ind}_{W_E}^{W_F} \sigma$ is isotypic if and only if for any p, $\mathrm{Ind}_{W_E}^{W_{E_p}} \sigma$ is isotypic, where E_p is the field defined by the p-Sylow subgroup Γ_p of $\Gamma = \Gamma_{E/F}$. The analogous statement for cuspidal representations also holds.*

Proof. If $\mathrm{Ind}_{W_E}^{W_{E_p}} \sigma = n_p \tau_p$ with τ_p irreducible then $n_p^2 = |\Gamma_p(\sigma)|$. Thus $\mathrm{Ind}_{W_E}^{W_F} \sigma = \mathrm{Ind}_{W_{E_p}}^{W_F} \mathrm{Ind}_{W_E}^{W_{E_p}} \sigma$ is divisible by n_p in $R(W_F)$. Hence, it is divisible by n where $n^2 = |\Gamma(\sigma)|$. Thus, $\mathrm{Ind}_{W_E}^{W_F} \sigma$ is isotypic. The converse was proved in the proof of Theorem 4. The cuspidal side is similar, with $R_{cusp}(F)$ playing the role of $R(W_F)$. ☐

Theorem 5. *Let again $\sigma \in \mathrm{Irr}_m(W_E)$ satisfy GLC and suppose that there exists $F \subset K \subset E$ such that $\mathrm{Ind}_{W_E}^{W_K} \sigma = k\rho$ and $\mathrm{Ind}_{W_K}^{W_F} \rho = l\tau$ for $\rho \in \mathrm{Irr}(W_K)$, $\tau \in \mathrm{Irr}(W_F)$. Assume that $F \subset E$, $F \subset K$ are normal and both $\Gamma_{E/K}$ and $\Gamma_{K/F}$ are nilpotent. Also, assume that $(m, [E:F]) = 1$. Then τ is automorphic.*

Proof. By Theorem 4, we know that ρ is automorphic. Let ξ be the corresponding cuspidal representation, and $\Xi = \mathrm{AI}_K^F \xi$. We have to prove that Ξ is isotypic. By the previous Lemma it is enough to consider the case where $F \subset K$ is a p-extension. Let $K \subset E_{\bar{p}} \subset E$ be the subfield corresponding to the $\bar{p}$-Hall subgroup of $\Gamma_{E/K}$. Let $p_1, \dots, p_r$ be the other prime divisors of $[E:K]$. The sequence $F \subset E_{\bar{p}} \subset E_{\overline{\{p,p_1\}}} \subset \dots \subset E$ satisfies the conditions of Theorem 4. ☐

Again, we also have the dual statement.

Theorem 5'. *Let $F \subset K \subset E$ and m be as before, and let $\sigma \in \mathrm{Irr}_m(W_E)$. Suppose that $\pi \leftrightarrow \sigma$ and that both $\mathrm{AI}_E^F \pi$ and $\mathrm{AI}_E^K \pi$ are isotypic. Then $\mathrm{Ind}_{W_E}^{W_F} \sigma$ is isotypic. If μ is the cuspidal type of $\mathrm{AI}_E^F \pi$ and τ is the irreducible constituent of $\mathrm{Ind}_{W_E}^{W_F} \sigma$, then $\mu \leftrightarrow \tau$.*

Example. Recall that for any Abelian group A, the Schur multipliers $H^2(A, \mathbb{Q}/\mathbb{Z})$ can be canonically identified with the alternating bilinear forms on A with values in $Z = \mathbb{Q}/\mathbb{Z}$. As in [La], let $(\cdot,\cdot)$ be a non-degenerate form, and H be the corresponding Heisenberg group. That is, H sits in an exact sequence

$$0 \longrightarrow Z \longrightarrow H \longrightarrow A \longrightarrow 0$$

and the commutator pairing induces $(\cdot,\cdot)$ on A. Let σ be the Stone–von-Neumann representation with central character $\psi(z) = e^{2\pi i z}$. Let $\alpha : B \to \mathrm{Aut}(A, (\cdot,\cdot))$ be an action of the Abelian group B on A by symplectic automorphisms. Assume that $(|A|, |B|) = 1$. Then, we can lift α to an action on H, and σ extends to an irreducible representation of $H \rtimes B$. Suppose now that $[\cdot,\cdot]$ is a non-degenerate alternating form on B. Let $\gamma \in H^2(B, Z)$ be the corresponding cocycle, and G be the extension of H by B defined by γ. Then $\mathrm{Ind}_H^G \sigma = m\tau$ with τ irreducible and $m^2 = |B|$. Let τ' be the restriction of τ to a finite subgroup G' of G with $G = ZG'$. According to Theorem 4 any Galois representation which factors through τ' satisfies GLC. However, τ' is not monomial, unless there exist maximal isotropic subgroups A_1,B_1 of A and B respectively so that A_1 is invariant under B_1.

4 An Application

As an application, we refer to a problem considered in [La]. Recall that the *global multiplicity* $\mathcal{M}(\mathcal{L})$ of an L-packet $\mathcal{L}$ of SL_n was defined to be the sum of multiplicities of L-packets which coincide with $\mathcal{L}$ almost everywhere. For its computation we considered two equivalence relations on cuspidal representations of GL_n:

1. $\tilde{\pi}\sim_s\tilde{\rho}$ if there exists a Hecke character ω of $\mathbb{I}_F/F^*$ such that $\tilde{\rho}\simeq\tilde{\pi}\otimes\omega$,
2. $\tilde{\pi}\sim_w\tilde{\rho}$ if for almost every place v there exists a character ω_v of F_v^* such that $\tilde{\rho}_v\simeq\tilde{\pi}_v\otimes\omega_v$.

Let $\mathcal{L}(\tilde{\pi})$ be the L-packet defined by a cuspidal representation $\tilde{\pi}$ of GL_n. By the multiplicity formula of Labesse and Langlands ([LL])

$$\mathcal{M}(\mathcal{L}(\tilde{\pi})) = |\{\tilde{\pi}' : \tilde{\pi}'\sim_w\tilde{\pi}\}/\sim_s|.$$

There are also analogous equivalence relations for projective representations of a group G. Let $\phi_i : G \to PGL_n(\mathbb{C})$, $i = 1,2$. Define

1. $\phi_1\sim_s\phi_2$ if there exists $x\in PGL_n(\mathbb{C})$ such that $\phi_1(g) = x^{-1}\phi_2(g)x$ for all $g\in G$,
2. $\phi_1\sim_w\phi_2$ if for any $g \in G$ there exists $x \in PGL_n(\mathbb{C})$ such that $\phi_1(g) = x^{-1}\phi_2(g)x$.

We can also define

$$\mathcal{M}(\phi) = |\{\phi' : \phi'\sim_w\phi\}/\sim_s|$$

for any $\phi : G\to PGL_n(\mathbb{C})$ (the latter is always finite, and in fact bounded in terms of n only). We denote by $\bar{\sigma}$ the projective representation obtained from an ordinary representation σ by the projection $GL_n(\mathbb{C}) \to PGL_n(\mathbb{C})$. The problem is to show that for automorphic representations σ of W_F with $\tilde{\pi}\leftrightarrow\sigma$ we have $\mathcal{M}(\mathcal{L}(\tilde{\pi})) = \mathcal{M}(\bar{\sigma})$. This was proved in the case where σ is induced from a character of an extension which is either Abelian or a sub-p-extension for some p. We can now show

Theorem 6. *Let $\sigma = \mathrm{Ind}_{W_K}^{W_F}\theta$ where θ is a Hecke character of K and $F\subset K$ is nilpotent. Let $\tilde{\pi}\leftrightarrow\sigma$. Then $\mathcal{M}(\mathcal{L}(\tilde{\pi})) = \mathcal{M}(\bar{\sigma})$.*

Proof. The statement in the Theorem is equivalent to the following two statements:

1. If $\tilde{\pi}'\sim_w\tilde{\pi}$ then $\tilde{\pi}'$ is of Galois type.
2. If $\overline{\sigma'}\sim_w\bar{\sigma}$ then σ' is automorphic.

Let us prove the second statement (the dual statement is proved similarly). As in the proof of Theorem 2 in [La] we have the following properties for σ':

1. $\sigma'\big|_{W_K} = d(\oplus_{g\in\Gamma_{K/F}(\rho)\backslash\Gamma_{K/F}}\rho^g)$ with $\rho\in\mathrm{Irr}_d(W_K)$.
2. $\mathrm{Ind}_{W_K}^{W_F}\rho\simeq d\sigma'$.
3. The kernel of $\bar{\rho}$ is W_E where E is an Abelian extension of order d^2 over K, normal over F.

4. $\rho\big|_{W_E} = d\zeta$ for a Hecke character ζ of E and $\mathrm{Ind}_{W_E}^{W_K} \zeta = d\rho$.

We can now use Theorem 5 to conclude the proof. $\square$

Unfortunately, the general case where $F \subset E$ is solvable lies beyond the limitations of the method described in this paper.

5 A generalized tetrahedral representation

We conclude by analyzing the simplest case of a Galois representation no multiple of which is monomial. Let $\mathbb{F}_q$ be the finite field with q elements and V be a 2-dimensional vector space over $\mathbb{F}_q$ with a non-degenerate $\mathbb{F}_q$-bilinear alternating form. Let H be the corresponding Heisenberg group. Let $\mathrm{Aut}_c(H)$ be the automorphisms of H which act trivially on the center. The exact sequence

$$0 \longrightarrow \mathrm{Inn}(H) \longrightarrow \mathrm{Aut}_c(H) \longrightarrow Sp(V) \longrightarrow 0 \tag{8}$$

splits if $2 \not| q$. In any case, it splits over any subgroup B of $Sp(V) \simeq SL_2(\mathbb{F}_q)$ with $q \not| |B|$. The classification of these subgroups B is well known (e.g. [Di]) and runs parallel to the case of $SL_2(\mathbb{C})$. Let $B \neq 1$ be a solvable group of this kind. Then, the image of B in $PSL_2(\mathbb{F}_q)$ is either cyclic, dihedral, A_4 or S_4. The Stone–von-Neumann representation of H extends to a q-dimensional irreducible representation of $G = H \rtimes B$. In fact, if q is odd, it extends to $H \rtimes SL_2(\mathbb{F}_q)$ and the restriction to $SL_2(\mathbb{F}_q)$ is the Weil representation. Suppose that $\sigma \in \mathrm{Irr}_q(W_F)$ factors through G. (By abuse of notation we will also regard $\sigma \in \mathrm{Irr}_q(G)$.) The image of $\bar{\sigma}$ is isomorphic to $V \rtimes B$. Let $F \subset E \subset K$ be the extensions corresponding to the inverse image of V and the kernel of $\bar{\sigma}$ respectively. We know that GLC holds for $\sigma\big|_{W_E}$; let $\pi_E \leftrightarrow \sigma\big|_E$. Clearly $\pi_E^\gamma \simeq \pi_E$ for any $\gamma \in \Gamma_{E/F} \simeq B$. By Proposition 1 there exists a unique $\pi \in \mathrm{Cusp}_q(F)$ so that $\mathrm{BC}_F^E \pi \simeq \pi_E$ and $\chi_\pi = \chi_\sigma$. What are the obstacles to proving that $\pi \leftrightarrow \sigma$? If v is a place in F which splits completely in E, then clearly $g(\pi_v) \sim \sigma(Fr_v)$. However, if v has relative degree d then we only know that

$$g(\pi_v)^d \sim \sigma(Fr_v)^d. \tag{9}$$

At this point we must assume some functoriality hypothesis, which looks inaccessible by today's methods.

Assumption 1. *There exists a lifting of automorphic representations corresponding to the adjoint representation* $\mathrm{Ad} : GL_q \longrightarrow GL_{q^2-1}$.

Granting the assumption, GLC for σ would follow from the following

Proposition 2. *Let* Π *be the adjoint lift of* π. *Then*

1. $\Pi \leftrightarrow \mathrm{Ad}(\sigma)$
2. $\mathrm{Ad}(\sigma(Fr_v)) \sim \mathrm{Ad}(g(\pi_v))$
3. $\bar{\sigma}(Fr_v) \sim \overline{g(\pi_v)}$
4. $\sigma(Fr_v) \sim g(\pi_v)$ *and thus* $\pi \leftrightarrow \sigma$.

Proof. 1. Note that

$$\mathrm{Ad}(\sigma)|_{W_E} \simeq \oplus_{1\neq\theta\in\widehat{W_E/W_K}}\theta.$$

Since B acts freely on $V-0$, there is a unique representation of W_F, namely $\mathrm{Ad}(\sigma)$, whose restriction to W_E is $\mathrm{Ad}(\sigma)|_{W_E}$. Moreover, $\mathrm{Ad}(\sigma) = \oplus_O \mathrm{Ind}_{W_E}^{W_F}\theta$ where O is a set of representatives of the B-orbits of non-trivial characters of W_E/W_K. Similarly, $\Pi' = \boxplus_O \mathrm{AI}_E^F\theta$ is the unique element of $R_{cusp}(F)$ satisfying $\mathrm{BC}_F^E\,\Pi' = \boxplus_{1\neq\theta\in\widehat{W_E/W_K}}\theta$. Clearly $\Pi' \leftrightarrow \mathrm{Ad}(\sigma)$. On the other hand, by functoriality, $\mathrm{BC}_F^E\,\Pi \leftrightarrow \mathrm{Ad}(\sigma)\big|_{W_E} \simeq \sum_{1\neq\theta\in\widehat{W_E/W_K}}\theta$ and thus $\Pi\simeq\Pi'$.
2. This follows immediately from 1.
3. We can assume that v does not split completely in E. Then, since B acts freely on $V-0$, $\bar\sigma(Fr_v)\sim\bar\sigma(g)$ for some $g\in B$. If q is odd, the Weil representation Θ decomposes as the sum of the two irreducible representations (uniquely determined up to conjugation by $GL_2(\mathbb{F}_q)$) of $SL_2(\mathbb{F}_q)$ of dimensions $\frac{q\pm1}{2}$. From the character table one sees that

$$\Theta\big|_T \simeq R_{reg}(T) + (-1)^{\epsilon(T)}\eta$$

where T is a torus of $SL_2(\mathbb{F}_q)$ (split or non-split), R_{reg} is the regular representation, $\epsilon(T)$ is 0 if T is split, and 1 otherwise, and finally η is the unique character of T of order 2. Since $\sigma\big|_B = \Theta\big|_B$ we conclude that

$$\bar\sigma(Fr_v) \sim \overline{\begin{pmatrix} 1 & & & \\ & \zeta & & \\ & & \cdots & \\ & & & \zeta^{q-1}\end{pmatrix}} \tag{10}$$

where ζ is a root of unity of order $|g|$. If q is even, (10) still holds (for a more general setup, see [Is]). It is now easy to see that $\bar\sigma(Fr_v)$ is the unique element in $PGL_q(\mathbb{C})$, up to conjugacy, which maps under the adjoint representation to the conjugacy class of the diagonal element consisting of all roots of unity of order $|g|$, each appearing $(q^2-1)/|g|$ times. Thus, 2 implies 3.
4. This follows from 3, (9), and the fact that $\chi_\sigma = \chi_\pi$. □

Remark. 1. The case $q=2$ is the classical dihedral case, proved by Langlands in [L], using the adjoint lifting for $GL(2)$ ([GJ]). The only difference in the argument above is that we use base change for GL_n with $n>2$ in step 1. Langlands avoids this (which was not known then) by using an L-function argument. This argument uses the equality

$$L^S(\Pi\otimes\Pi'^\vee, s) = L^S(\Pi'\otimes\Pi'^\vee, s) \tag{11}$$

to conclude that $\Pi\simeq\Pi'$ since Π,Π' are cuspidal in that case. However, if $q>2$, Π,Π' are not cuspidal, and the relation (11), which can be proved in the same way, is not sufficient to conclude that $\Pi\simeq\Pi'$.
2. In the case $q=3$ and $|B|=2$, we get a three-dimensional monomial representation. Thus, GLC follows from [JPS].
3. No other cases seem to be known.

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