Metrics on States from Actions of Compact Groups

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Abstract. Let a compact Lie group act ergodically on a unital C^*-algebra A. We consider several ways of using this structure to define metrics on the state space of A. These ways involve length functions, norms on the Lie algebra, and Dirac operators. The main thrust is to verify that the corresponding metric topologies on the state space agree with the weak-$*$ topology.

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Connes [C1, C2, C3] has shown us that Riemannian metrics on non-commutative spaces (C^*-algebras) can be specified by generalized Dirac operators. Although in this setting there is no underlying manifold on which one then obtains an ordinary metric, Connes has shown that one does obtain in a simple way an ordinary metric on the state space of the C^*-algebra, generalizing the Monge-Kantorovich metric on probability measures [Ra] (called the "Hutchinson metric" in the theory of fractals [Ba]).

But an aspect of this matter which has not received much attention so far [P] is the question of when the metric topology (that is, the topology from the metric coming from a Dirac operator) agrees with the underlying weak-$*$ topology on the state space. Note that for locally compact spaces their topology agrees with the weak-$*$ topology coming from viewing points as linear functionals (by evaluation) on the algebra of continuous functions vanishing at infinity.

In this paper we will consider metrics arising from actions of compact groups on C^*-algebras. For simplicity of exposition we will only deal with "compact" non-commutative spaces, that is, we will always assume that our C^*-algebras have an identity element. We will explain later what we mean by Dirac operators in this setting (section 4). In terms of this, a brief version of our main theorem is:

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THEOREM 4.2. *Let α be an ergodic action of a compact Lie group G on a unital C^*-algebra A, and let D be a corresponding Dirac operator. Then the metric topology on the state space of A defined by the metric from D agrees with the weak-$*$ topology.*

An important case to which this theorem applies consists of the non-commutative tori [Rf], since they carry ergodic actions of ordinary tori [OPT]. The metric geometry of non-commutative tori has recently become of interest in connection with string theory [CDS, RS, S].

We begin by showing in the first section of this paper that the mechanism for defining a metric on states can be formulated in a very rudimentary Banach space setting (with no algebras, groups, or Dirac operators). In this setting the discussion of agreement between the metric topology and the weak-$*$ topology takes a particularly simple form.

Then in the second section we will see how length functions on a compact group directly give (without Dirac operators) metrics on the state spaces of C^*-algebras on which the group acts ergodically. We then prove the analogue in this setting of the main theorem stated above.

In the third section we consider compact Lie groups, and show how norms on the Lie algebra directly give metrics on the state space. We again prove the corresponding analogue of our main theorem.

Finally, in section 4 we use the results of the previous sections to prove our main theorem, stated above, for the metrics which come from Dirac operators.

It is natural to ask about actions of non-compact groups. Examination of [Wv4] suggests that there may be very interesting phenomena there. The considerations of the present paper also make one wonder whether there is an appropriate analogue of length functions for compact quantum groups which might determine a metric on the state spaces of C^*- algebras on which a quantum group acts ergodically [Bo, Wn]. This would be especially interesting since for non-commutative compact groups there is only a sparse collection of known examples of ergodic actions [Ws], whereas in [Wn] a rich collection of ergodic actions of compact quantum groups is constructed. Closely related is the setting of ergodic coactions of discrete groups [N, Q]. But I have not explored any of these possibilities.

I developed a substantial part of the material discussed in the present paper during a visit of several weeks in the Spring of 1995 at the Fields Institute. I am appreciative of the hospitality of the Fields Institute, and of George Elliott's leadership there. But it took trying to present this material in a course which I was teaching this Spring, as well as benefit from [P, Wv1, Wv2, Wv3, Wv4], for me to find the simple development given here.

1. METRICS ON STATES

Let A be a unital C^*-algebra. Connes has shown [C1, C2, C3] that an appropriate way to specify a Riemannian metric in this non-commutative situation is by means of a spectral triple. This consists of a representation of A on a Hilbert space $\mathcal{H}$, together with an unbounded self-adjoint operator D on $\mathcal{H}$ (the generalized Dirac operator), satisfying certain conditions. The set $\mathcal{L}(A)$ of Lipschitz elements of A consists of those $a \in A$ such that the commutator $[D, a]$ is a bounded operator. It is required that $\mathcal{L}(A)$ be dense in A. The Lipschitz semi-norm, L, is defined on $\mathcal{L}(A)$ just by the operator norm $L(a) = \|[D,a]\|$.

Given states μ and ν of A, Connes defines the distance between them, $\rho(\mu,\nu)$, by

$$\rho(\mu,\nu) = \sup\{|\mu(a) - \nu(a)| : a \in \mathcal{L}(A),\ L(a) \ \leq\ 1\}\ . \tag{1.1}$$

(In the absence of further hypotheses it can easily happen that $\rho(\mu,\nu) = +\infty$. For one interesting situation where this sometimes happens see the end of the discussion of the second example following axiom 4' of [C3].)

The semi-norm L is an example of a general Lipschitz semi-norm, that is [BC, Cu, P, Wv1, Wv2], a semi-norm L on a dense subalgebra $\mathcal{L}$ of A satisfying the Leibniz property:

$$L(ab) \leq L(a)\|b\| + \|a\|L(b)\ . \tag{1.2}$$

Lipschitz norms carry some information about differentiable structure [BC, Cu], but not nearly as much as do spectral triples. But it is clear that just in terms of a given Lipschitz norm one can still define a metric on states by formula (1.1).

However, for the purpose of understanding the relationship between the metric topology and the weak-$*$ topology, we do not need the Leibniz property (1.2), nor even that A be an algebra. The natural setting for these considerations seems to be the following very rudimentary one. The data is:

(1.3a) A normed space A, with norm $\| \ \|$, over either $\mathbb{C}$ or $\mathbb{R}$.

(1.3b) A subspace $\mathcal{L}$ of A, not necessarily closed.

(1.3c) A semi-norm L on $\mathcal{L}$.

(1.3d) A continuous (for $\| \ \|$) linear functional, η, on $\mathcal{K} = \{a \in \mathcal{L} : L(a) = 0\}$ with $\|\eta\| = 1$. (Thus, in particular, we require $\mathcal{K} \neq \{0\}$.)

Let A' denote the Banach-space dual of A, and set

$$S = \{\mu \in A' : \mu = \eta \text{ on } \mathcal{K},\ \text{and } \|\mu\| = 1\}\ .$$

Thus S is a norm-closed, bounded, convex subset of A', and so is weak-$*$ compact. In general S can be quite small; when A is a Hilbert space S will contain only one element. But in the applications we have in mind A will be a unital C^*-algebra, $\mathcal{K}$ will be the one-dimensional subspace spanned by the identity element, and η will be the functional on $\mathcal{K}$ taking value 1 on the identity element. Thus S will be the full state-space of A. (That $\mathcal{K}$ will consist only of the scalar multiples of the identity element in our examples will follow from our ergodicity hypothesis. We treat the case of general $\mathcal{K}$ here because this clarifies slightly some issues, and it might possibly be of eventual use, for example in non-ergodic situations.)

We do not assume that $\mathcal{L}$ is dense in A. But to avoid trivialities we do make one more assumption about our set-up, namely:

$$\text{(1.3e)} \qquad \mathcal{L} \quad \text{separates the points of} \quad S.$$

This means that given $\mu, \nu \in S$ there is an $a \in \mathcal{L}$ such that $\mu(a) \neq \nu(a)$. (Note that for $\mu \in S$ there exists $a \in \mathcal{L}$ with $\mu(a) \neq 0$, since we can just take an $a \in \mathcal{K}$ such that $\eta(a) \neq 0$.)

With notation as above, let $\tilde{\mathcal{L}} = \mathcal{L}/\mathcal{K}$. Then L drops to an actual norm on $\tilde{\mathcal{L}}$, which we denote by $\tilde{L}$. But on $\tilde{\mathcal{L}}$ we also have the quotient norm from $\| \ \|$ on $\mathcal{L}$, which we denote by $\| \ \|^{\sim}$. The image in $\tilde{\mathcal{L}}$ of $a \in \mathcal{L}$ will be denoted by $\tilde{a}$.

We remark that when $\mathcal{L}$ is a unital algebra (perhaps dense in a C^*-algebra), and when $\mathcal{K}$ is the span of the identity element, then the space of universal 1-forms Ω^1 over $\mathcal{L}$ is commonly identified [BC, Br, C2, Cu] with $\mathcal{L} \otimes \tilde{\mathcal{L}}$, and the differential $d : \mathcal{L} \to \Omega^1$ is given by $da = 1 \otimes \tilde{a}$. Thus in this setting our $\tilde{L}$ is a norm on the space of universal 1-coboundaries of $\mathcal{L}$. The definition of L which we will use in the examples of section 3 is also closely related to this view.

On S we can still define a metric, ρ, by formula (1.1), with $\mathcal{L}(A)$ replaced by $\mathcal{L}$. The symmetry of ρ is evident, and the triangle inequality is easily verified. Since we assume that $\mathcal{L}$ separates the points of S, so will ρ. But ρ can still take the value $+\infty$. We will refer to the topology on S defined by ρ as the "ρ-topology", or the "metric topology" when ρ is understood.

It will often be convenient to consider elements of A as (weak-$*$ continuous) functions on S. At times this will be done tacitly, but when it is useful to do this explicitly we will write $\hat{a}$ for the corresponding function, so that $\hat{a}(\mu) = \mu(a)$ for $\mu \in S$.

Without further hypotheses we have the following fact. It is closely related to proposition 3.1a of [P], where metrics are defined in terms of linear operators from an algebra into a Banach space.

1.4 PROPOSITION. *The ρ-topology on S is finer than the weak-$*$ topology.*

Proof. Let $\{\mu_k\}$ be a sequence in S which converges to $\mu \in S$ for the metric ρ. Then it is clear from the definition of ρ that $\{\mu_k(a)\}$ converges to $\mu(a)$ for any $a \in \mathcal{L}$ with $L(a) \leq 1$, and hence for all $a \in \mathcal{L}$.

This says that $\hat{a}(\mu_k)$ converges to $\hat{a}(\mu)$ for all $a \in \mathcal{L}$. But $\hat{\mathcal{L}}$ is a linear space of weak-$*$ continuous functions on S which separates the points of S by assumption (and which contains the constant functions, since they come from any $a \in \mathcal{K}$ on which η is not 0). A simple compactness argument shows then that $\hat{\mathcal{L}}$ determines the weak-$*$ topology of S. Thus $\{\mu_k\}$ converges to μ in the weak-$*$ topology, as desired. $\square$

There will be some situations in which we want to obtain information about $(\mathcal{L}, L)$ from information about S. It is clear that to do this S must "see" all of $\mathcal{L}$. The convenient formulation of this for our purposes is as follows. Let $\| \ \|_\infty$ denote the supremum norm on functions on S. Let it also denote the corresponding semi-norm on $\mathcal{L}$ defined by $\|a\|_\infty = \|\hat{a}\|_\infty$. Clearly $\|\hat{a}\|_\infty \leq \|a\|$ for $a \in \mathcal{L}$.

1.5 CONDITION. The semi-norm $\| \|_\infty$ on $\mathcal{L}$ is a norm, and it is equivalent to the norm $\| \|$, so that there is a constant k with

$$\|a\| \le k\|\hat{a}\|_\infty \qquad \text{for } \ a \in \mathcal{L}.$$

This condition clearly holds when A is a C^*-algebra, $\mathcal{L}$ is dense in A, and S is the state space of A, so that we are dealing with the usual Kadison functional representation [KR]. But we remark that even in this case the constant k above cannot always be taken to be 1 (bottom of page 263 of [KR]). This suggests that in using formula (1.1) one might want to restrict to using just the self-adjoint elements of $\mathcal{L}$, since there the function representation is isometric. But more experience with examples is needed.

We return to the general case. If we are to have the ρ-topology on S agree with the weak-$*$ topology, then S must at least have finite ρ-diameter, that is, ρ must be bounded. The following proposition is closely related to theorem 6.2 of [P].

1.6 PROPOSITION. *Suppose there is a constant, r, such that*

$$\| \|^{\sim} \le r\tilde{L} \ . \tag{1.7}$$

Then ρ is bounded (by $2r$).

Conversely, suppose that Condition 1.5 holds. If ρ is bounded, (say by d), then there is a constant r such that (1.7) holds (namely $r = kd$ where k is as in 1.5).

Proof. Suppose that (1.7) holds. If $a \in \mathcal{L}$ and $L(a) \le 1$, then $\tilde{L}(\tilde{a}) \le 1$ and so $\|\tilde{a}\|^{\sim} \le r$. This means that, given $\varepsilon > 0$, there is a $b \in \mathcal{K}$ such that $\|a - b\| \le r + \varepsilon$. Then for any $\mu, \nu \in S$, we have, because μ and ν agree on $\mathcal{K}$,

$$|\mu(a) - \nu(a)| = |\mu(a - b) - \nu(a - b)| \le \|\mu - \nu\| \ \|a - b\| \le 2(r + \varepsilon) \ .$$

Since ε is arbitrarily small, it follows that $|\mu(a) - \nu(a)| \le 2r$. Consequently $\rho(\mu, \nu) \le 2r$.

Assume conversely that ρ is bounded by d. Fix $\nu \in S$, and choose $b \in \mathcal{K}$ such that $\eta(b) = 1$. Then for any $\mu \in S$ and any $a \in \mathcal{L}$ with $L(a) \le 1$ we have

$$d \ge \rho(\mu, \nu) \ge |\mu(a) - \nu(a)| = |\mu(a - \nu(a)b)| \ .$$

Suppose now that Condition 1.5 holds. We apply it to $a - \nu(a)b$. Thus, since S is compact, we can find μ such that

$$\|a - \nu(a)b\| \le k|\mu(a - \nu(a)b)| \ .$$

Consequently $\|a - \nu(a)b\| \le kd$, so that $\|\tilde{a}\|^{\sim} \le kd$. All this was under the assumption that $L(a) \le 1$. It follows that for general $a \in \mathcal{L}$ we have $\|\tilde{a}\|^{\sim} \le kd\tilde{L}(\tilde{a})$, as desired. $\square$

We now turn to the question of when the ρ-topology and the weak-$*$ topology on S agree. The following theorem is closely related to theorem 6.3 of [P].

1.8 THEOREM. *Let the data be as in* (1.3a–e)*, and let* $\mathcal{L}_1 = \{a \in \mathcal{L} : L(a) \leq 1\}$*. If the image of* $\mathcal{L}_1$ *in* $\mathcal{L}^{\sim}$ *is totally bounded for* $\| \|^{\sim}$*, then the* ρ*-topology on* S *agrees with the weak-*$*$ *topology.*

Conversely, if Condition 1.5 holds and if the ρ*-topology on* S *agrees with the weak-*$*$ *topology, then the image of* $\mathcal{L}_1$ *in* $\mathcal{L}^{\sim}$ *is totally bounded for* $\| \|^{\sim}$*.*

Proof. We begin with the converse, so that we see why the total-boundedness assumption is natural. If the ρ-topology gives the weak-$*$ t opology on S, then ρ must be bounded since S is compact. Thus by Proposition 1.6 there is a constant, r_o, such that $\| \|^{\sim} \leq r_o L^{\sim}$, since we assume here that Condition 1.5 holds. Choose $r > r_o$. Then $\|a\|^{\sim} < r$ if $a \in \mathcal{L}_1$. Consequently, if we let

$$\mathcal{B}_r = \{a \in \mathcal{L} : L(a) \leq 1 \text{ and } \|a\| \leq r\} \ ,$$

then the image of $\mathcal{B}_r$ in $\mathcal{L}^{\sim}$ is the same as the image of $\mathcal{L}_1$. Thus it suffices to show that $\mathcal{B}_r$ is totally bounded.

Let $a \in \mathcal{B}_r$ and let $\mu, \nu \in S$. Then

$$|\hat{a}(\mu) - \hat{a}(\nu)| = |\mu(a) - \nu(a)| \leq \rho(\mu, \nu) \ .$$

Thus $(\mathcal{B}_r)\,\hat{}$ can be viewed as a bounded family of functions on S which is equicontinuous for the weak-$*$ topology, since ρ gives the weak-$*$ topology of S. It follows from Ascoli's theorem [Ru] that $(\mathcal{B}_r)\,\hat{}$ is totally bounded for $\| \|_\infty$. By Condition 1.5 this means that $\mathcal{B}_r$ is totally bounded for $\| \|$ as a subset of A, as desired.

For the other direction we do not need Condition 1.5. We suppose now that the image of $\mathcal{L}_1$ in $\tilde{L}$ is totally bounded for $\| \|^{\sim}$. Let $\mu \in S$ and $\varepsilon > 0$ be given, and let $B(\mu, \varepsilon)$ be the ρ-ball of radius ε about μ in S. In view of Proposition 1.4 it suffices to show that $B(\mu, \varepsilon)$ contains a weak-$*$ neighborhood of μ. Now by the total boundedness of the image of $\mathcal{L}_1$ we can find $a_1, \ldots, a_n \in \mathcal{L}_1$ such that the $\| \|^{\sim}$-balls of radius $\varepsilon/3$ about the $\hat{a}_j$'s cover the image of $\mathcal{L}_1$. We now show that the weak-$*$ neighborhood

$$\mathcal{O} = \mathcal{O}(\mu, \{a_j\}, \varepsilon/3) = \{\nu \in S : |(\mu - \nu)(a_j)| < \varepsilon/3, \ 1 \leq j \leq n\}$$

is contained in $B(\mu, \varepsilon)$. Consider any $a \in \mathcal{L}_1$. There is a j and a $b \in \mathcal{K}$, depending on a, such that

$$\|a - a_j - b\| < \varepsilon/3 \ .$$

Hence for any $\nu \in \mathcal{O}$ we have

$$\begin{aligned} |\mu(a) - \nu(a)| \ &\leq \ |\mu(a) - \mu(a_j + b)| + |\mu(a_j + b) - \nu(a_j + b)| + |\nu(a_j + b) - \nu(a)| \\ &< \ \varepsilon/3 + |\mu(a_j) - \nu(a_j)| + \varepsilon/3 \ < \ \varepsilon \ . \end{aligned}$$

Thus $\rho(\mu, \nu) < \varepsilon$. Consequently $\mathcal{O} \subseteq B(\mu, \varepsilon)$ as desired. □

Examination of the proof of the above theorem suggests a reformulation which provides a convenient subdivision of the problem of showing for specific examples that the ρ-topology agrees with the weak-$*$ topology. We will use this reformulation in the next sections.

1.9 Theorem. *Let the data be as in* (1.3a–e). *Then the ρ-topology on S will agree with the weak-$*$ topology if the following three hypotheses are satisfied:*

i) *Condition 1.5 holds.*
ii) *ρ is bounded.*
iii) *The set $\mathcal{B}_1 = \{a \in \mathcal{L} : L(a) \le 1$ and $\|a\| \le 1\}$ is totally bounded in A for $\|\ \|$.*

Conversely, if Condition 1.5 holds and if the ρ-topology agrees with the weak-$*$ topology, then the above three conditions are satisfied.

Proof. If conditions i) and ii) are satisfied, then, just as in the first part of the proof of Theorem 1.8, there is a constant r such that the image of $\mathcal{B}_r$ in $\tilde{\mathcal{L}}$ contains the image of $\mathcal{L}_1$. But $\mathcal{B}_r \subseteq r\mathcal{B}_1$. Thus if $\mathcal{B}_1$ is totally bounded then so is $\mathcal{B}_r$, as is then the image of $\mathcal{L}_1$. Then we can apply Theorem 1.8 to conclude that the ρ-topology agrees with the weak-$*$ topology.

Conversely, if the ρ-topology and the weak-$*$ topology agree, then condition ii) holds by Proposition 1.6. But by the first part of the proof of Theorem 1.8 there is then a constant r such that $\mathcal{B}_r$ is totally bounded. By scaling we see that $\mathcal{B}_1$ is also. ☐

We remark that if we take any 1-dimensional subspace $\mathcal{K}$ of an infinite-dimensional normed space A, set $\mathcal{L} = A$, and let L be the pull-back to A of $\|\ \|^{\sim}$on $A/\mathcal{K}$, we obtain an example where ρ is bounded but the image of $\mathcal{L}_1$ in $\mathcal{L}^{\sim}$ is not totally bounded, nor is $\mathcal{B}_1$ totally bounded in A.

In the next sections we will find very useful the following:

1.10 Comparison Lemma. *Let the data be as in* (1.3a–e). *Suppose we have a subspace $\mathcal{M}$ of $\mathcal{L}$ which contains $\mathcal{K}$ and separates the points of S, and a semi-norm M on $\mathcal{M}$ which takes value* 0 *exactly on $\mathcal{K}$. Let ρ_L and ρ_M denote the corresponding metrics on S (possibly taking value $+\infty$). Assume that*

$$M \ \ge\ L \quad on \quad \mathcal{M},$$

in the sense that $M(a) \ \ge\ L(a)$ for all $a \in \mathcal{M}$. Then

$$\rho_M \ \le\ \rho_L\ ,$$

in the sense that $\rho_M(\mu,\nu) \le \rho_L(\mu,\nu)$ for all $\mu,\nu \in S$. Thus

i) *If ρ_L is finite then so is ρ_M.*
ii) *If ρ_L is bounded then so is ρ_M.*
iii) *If the ρ_L-topology on S agrees with the weak-$*$ topology then so does the ρ_M-topology.*

Proof. If $a \in \mathcal{M}$ and $M(a) \le 1$ then $L(a) \le 1$. Thus the supremum defining ρ_M is taken over a smaller set than that for ρ_L, and so $\rho_M \le \rho_L$. Conclusions i) and ii) are then obvious. Conclusion iii) follows from the fact that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism. ☐

For later use we record the following easily verified fact.

1.11 Proposition. *Let data be as above. Let t be a strictly positive real number. Set $M = tL$ on $\mathcal{L}$. Then $\rho_M = t^{-1}\rho_L$. Thus properties for ρ_L of finiteness, boundedness, and agreement of the ρ_L-topology with the weak-$*$ topology carry over to ρ_M.*

2. Metrics from actions and length functions

Let G be a compact group (with identity element denoted by e). We normalize Haar measure to give G mass 1. We recall that a length function on a group G is a continuous non-negative real-valued function, ℓ, on G such that

$$\ell(xy) \leq \ell(x) + \ell(y) \quad \text{for} \quad x, y \in G, \tag{2.1a}$$

$$\ell(x^{-1}) = \ell(x), \tag{2.1b}$$

$$\ell(x) = 0 \quad \text{exactly if} \quad x = e. \tag{2.1c}$$

Length functions arise in a number of ways. For example, if π is a faithful unitary representation of G on a finite-dimensional Hilbert space, then we can set $\ell(x) = \|\pi_x - \pi_e\|$. We will see another way in the next section. We will assume for the rest of this section that a length function has been chosen for G.

Let A be a unital C^*-algebra, and let α be an action (strongly continuous) of G by automorphisms of A. We let $\mathcal{L}$ denote the set of Lipschitz elements of A for α (and ℓ), with corresponding Lipschitz semi-norm L. That is [Ro1, Ro2], for $a \in A$ we set

$$L(a) = \sup\{\|\alpha_x(a) - a\|/\ell(x) : x \neq e\} \ ,$$

which may have value $+\infty$, and we set

$$\mathcal{L} = \{a \in A : L(a) < \infty\} \ .$$

It is easily verified that $\mathcal{L}$ is a $*$-subalgebra of A, and that L satisfies the Leibniz property 1.2. (More generally, for $0 < r < 1$ we could define L^r by

$$L^r(a) = \sup\{\|\alpha_x(a) - a\|/(\ell(x))^r : x \neq e\}$$

along the lines considered in [Ro1, Ro2]. For actions on the non-commutative torus this has been studied in [Wv2], but we will not pursue this here.)

It is not so clear whether $\mathcal{L}$ is carried into itself by α, but we do not need this fact here. (For Lie groups see theorem 4.1 of [Ro1] or the comments after theorem 6.1 of [Ro2].) Let us consider, however, the α-invariance of L. We find that

$$\begin{aligned} L(\alpha_z(a)) &= \sup\{\|\alpha_z(\alpha_{z^{-1}xz}(a) - a)\|/\ell(x) : x \neq e\} \\ &= \sup\{\|\alpha_x(a) - a\|/\ell(zxz^{-1}) : x \neq e\}. \end{aligned}$$

Thus if $\ell(zxz^{-1}) = \ell(x)$ for all $x, z \in G$, then L is α-invariant, and $\mathcal{L}$ is carried into itself by α. The metric ρ on S defined by L will then be α-invariant for the evident action on S. But we will not discuss this matter further here.

2.2 Proposition. *The $*$-algebra $\mathcal{L}$ is dense in A.*

Proof. For $f \in L^1(G)$ we define α_f as usual by $\alpha_f(a) = \int f(x)\alpha_x(a)\,dx$. It is standard [BR] that as f runs through an "approximate delta-function", $\alpha_f(a)$ converges to a. Thus the set of elements of form $\alpha_f(a)$ is dense in A. Let λ denote the action of G by left translation of functions on G. A quick standard calculation shows that $\alpha_x(\alpha_f(a)) = \alpha_{\lambda_x(f)}(a)$. Thus

$$\|\alpha_x(\alpha_f(a)) - \alpha_f(a)\| = \|\alpha_{(\lambda_x f - f)}(a)\| \leq \|\lambda_x f - f\|_1 \|a\|,$$

where $\| \ \|_1$ denotes the usual L^1-norm. Thus we see that $\alpha_f(a) \in \mathcal{L}$ if $f \in Lip^1_\lambda$, the space of Lipschitz functions in $L^1(G)$ for λ (and ℓ).

Consequently it suffices to show that Lip^1_λ is dense in $L^1(G)$. We first note that it contains a non-trivial element, namely ℓ itself. For if $x, y \in G$, then

$$|(\lambda_x\ell)(y) - \ell(y)| = |\ell(x^{-1}y) - \ell(y)| \leq \ell(x),$$

where the inequality follows from 2.1a and 2.1b above. We momentarily switch attention to $C(G)$ with $\| \ \|_\infty$, and the action λ of G on it. Of course $\ell \in C(G)$. The above inequality then says that $\ell \in Lip^\infty_\lambda$, the space of Lipschitz functions in $C(G)$ for λ. But as mentioned earlier, Lip^∞_λ is easily seen to be a $*$-subalgebra of $C(G)$ for the pointwise product, and it contains the constant functions. Furthermore, a simple calculation shows that Lip^∞_λ is carried into itself by *right* translation. Since Lip^∞_λ contains ℓ, which separates e from any other point, it follows that Lip^∞_λ separates the points of G. Thus Lip^∞_λ is dense in $C(G)$ by the Stone-Weierstrass theorem. Since $\| \ \|_\infty$ dominates $\| \ \|_1$ for compact G, it follows that Lip^1_λ is dense in $L^1(G)$ as needed. $\square$

For simplicity of exposition we will deal only with the case in which we obtain metrics on the entire state space of the C^*-algebra A. For this purpose we want the subspace where L takes the value 0 to be one-dimensional. It is evident that L takes value 0 on exactly those elements of A which are α-invariant, and in particular on the scalar multiples of the identity element of A. Thus we need to assume that the action α is *ergodic*, in the sense that the only α-invariant elements are the scalar multiples of the identity.

The main theorem of this section is:

2.3 Theorem. *Let α be an ergodic action of a compact group G on a unital C^*-algebra A. Let ℓ be a length function on G, and define $\mathcal{L}$ and L as above. Let ρ be the corresponding metric on the state space S of A. Then the ρ-topology on S agrees with the weak-$*$ topology.*

Proof. Because $\mathcal{L}$ is dense by Proposition 2.2, it separates the points of S. Consequently the conditions 1.3a–e are fulfilled (for the evident η). Thus L indeed defines a metric, ρ, on S (perhaps taking value $+\infty$).

Since G is compact, we can average α over G to obtain a conditional expectation from A onto its fixed-point subalgebra. Because we assume that α is ergodic, this conditional expectation can be viewed as a state on A. By abuse of notation we will denote it again by η, since it extends the evident state η on the fixed-point algebra. Thus

$$\eta(a) = \int_G \alpha_x(a)\,dx$$

for $a \in A$, interpreted as a complex number when convenient.

We will follow the approach suggested by Theorem 1.9. Now hypothesis (i) of that theorem is satisfied in the present setting, as discussed right after Condition 1.5 above. We now check hypothesis (ii), that is:

2.4 Lemma. *ρ is bounded.*

Proof. Let $\mu \in S$. Then for any $a \in \mathcal{L}$ we have

$$|\mu(a) - \eta(a)| = |\int \mu(a)dx - \mu(\int \alpha_x(a)dx)| = |\int \mu(a - \alpha_x(a))dx| \leq L(a) \int_G \ell(x)dx \ .$$

It follows that $\rho(\mu, \eta) \leq \int \ell(x)dx$. Thus for any $\mu, \nu \in S$ we have

$$\rho(\mu, \nu) \leq 2 \int_G \ell(x)dx \ ,$$

which is finite since ℓ is bounded. □

We now begin the verification of hypothesis (iii) of Theorem 1.9. For this we need the unobvious fact [HLS, Bo] that because G is compact and α is ergodic, each irreducible representation of G occurs with at most finite multiplicity in A. (In [HLS] it is also shown that η is a trace, but we do not need this fact here.) The following lemma is undoubtedly well-known, but I do not know a reference for it.

2.5 Lemma. *Let α be a (strongly continuous) action of a compact group G on a Banach space A. Suppose that each irreducible representation of G occurs in A with at most finite multiplicity. Then for any $f \in L^1(G)$ the operator α_f defined by*

$$\alpha_f(a) = \int_G f(x)\alpha_x(a)dx$$

is compact.

Proof. If f is a coordinate function for an irreducible representation π of G, then it is not hard to see (ch. IX of [FD]) that α_f will have range in the π-isotypic component of A, which we are assuming is finite-dimensional. Thus α_f is of finite rank in this case. But by the Peter-Weyl theorem [FD] the linear span of the coordinate functions for all irreducible representations is dense in $L^1(G)$. So any α_f can be approximated by finite rank operators. □

Proof of Theorem 2.3. We show now that $\mathcal{B}_1$, as in (iii) of Theorem 1.9, is totally bounded. Let $\varepsilon > 0$ be given. Since $\ell(e) = 0$ and ℓ is continuous at e, we can find $f \in L^1(G)$ such that $f \geq 0$, $\int_G f(x)dx = 1$, and $\int_G f(x)\ell(x)dx < \varepsilon/2$. By the previous lemma α_f is compact. Since $\mathcal{B}_1$ is bounded, it follows that $\alpha_f(\mathcal{B}_1)$ is totally bounded. Thus it can be covered by a finite number of balls of radius $\varepsilon/2$. But for any $a \in \mathcal{B}_1$ we have

$$\begin{aligned} \|a - \alpha_f(a)\| = \|a \int f(x)dx - \int f(x)\alpha_x(a)dx\| &\leq \int f(x)\|a - \alpha_x(a)\|dx \\ &\leq L(a) \int f(x)\ell(x)dx \leq \varepsilon/2 \ . \end{aligned}$$

Thus $\mathcal{B}_1$ itself can be covered by a finite number of balls of radius ε. □

3. Metrics from actions of Lie groups

We suppose now that G is a connected Lie group (compact). We let $\mathfrak{g}$ denote the Lie algebra of G. Fix a norm $\| \ \|$ on $\mathfrak{g}$. For any action α of G on a Banach space A we let A^1 denote the space of α-differentiable elements of A. Thus [BR] if $a \in A^1$ then for each $X \in \mathfrak{g}$ there is a $d_X a \in A$ such that

$$\lim_{t \to 0} (\alpha_{\exp(tX)}(a) - a)/t = d_X a \ ,$$

and $X \mapsto d_X a$ is a linear map from $\mathfrak{g}$ into A, which we denote by da. Since $\mathfrak{g}$ and A both have norms, the operator norm, $\|da\|$, of da is defined (and finite). A standard smoothing argument [BR] shows that A^1 is dense in A.

Suppose now that A is a C^*-algebra and that α is an action by automorphisms of A. We can set $\mathcal{L} = A^1$ and $L(a) = \|da\|$. It is easily verified that $\mathcal{L}$ is a $*$-subalgebra of A and that L satisfies the Leibniz property 1.2, though we do not need these facts here. Because G is connected, $L(a) = 0$ exactly if a is α-invariant.

3.1 Theorem. *Let G be a compact connected Lie group, and fix a norm on $\mathfrak{g}$. Let α be an ergodic action of G on a unital C^*-algebra A. Let $\mathcal{L} = A^1$ and $L(a) = \|da\|$, and let ρ denote the corresponding metric on the state space S. Then the ρ-topology on S agrees with the weak-$*$ topology.*

Proof. Choose an inner-product on $\mathfrak{g}$. Its corresponding norm is equivalent to the given norm, and so by the Comparison Lemma 1.10 it suffices to deal with the norm from the inner-p roduct. We can left-translate this inner-product over G to obtain a left-invariant Riemannian metric on G, and then a corresponding left-invariant ordinary metric on G. We let $\ell(x)$ denote the corresponding distance from x to e. Then ℓ is a continuous length function on G satisfying conditions 2.1 [G, Ro2].

Then the elements of $\mathcal{L} = A^1$ are Lipschitz for ℓ. This essentially just involves the following standard argument [G, Ro2], which we include for the reader's convenience. Let $a \in A^1$ and let c be a smooth path in G from e to a point $x \in G$. Then ϕ, defined by $\phi(t) = \alpha_{c(t)}(a)$, is differentiable, and so we have

$$\|\alpha_x(a) - a\| = \| \int \phi'(t) dt \| \leq \int \|\alpha_{c(t)}(d_{c'(t)} a)\| dt \leq \|da\| \int \|c'(t)\| dt \ .$$

But the last integral is just the length of c. Thus from the definition of the ordinary metric on G, with its length function ℓ, we obtain

$$\|\alpha_x(a) - a\| \leq \|da\| \ell(x) \ .$$

(Actually, the above argument works for any norm on $\mathfrak{g}$.) Then if we let $\mathcal{L}_0$ and L_0 be defined just in terms of ℓ as in the previous section, we see that $\mathcal{L} \subseteq \mathcal{L}_0$ and $L_0 \leq L$. Thus we are exactly in position to apply the Comparison Lemma 1.10 to obtain the desired conclusion. ☐

We remark that Weaver (theorem 24 of [Wv1]) in effect proved for this setting the total boundedness of $\mathcal{B}_1$ for the particular case of non-commutative 2-tori, by different methods.

4. METRICS FROM DIRAC OPERATORS

Suppose again that G is a compact connected Lie group, and that α is an ergodic action of G on a unital C^*-algebra A. Let $\mathfrak{g}$ denote the Lie algebra of G, and let $\mathfrak{g}'$ denote its vector-space dual. Fix any inner-product on $\mathfrak{g}'$. We will denote it by g, or by $\langle \ , \rangle_g$, to distinguish it from the Hilbert space inner-products which will arise.

With this data we can define a spectral triple [C1, C2, C3] for A. For simplicity of exposition we will not include gradings and real structure, and we will oversimplify our treatment of spinors, since the details are not essential for our purposes. But with more care they can be included. (See, e.g. [V, VB].) We proceed as follows. Let $C = \mathrm{Clif}(\mathfrak{g}', -g)$ be the complex Clifford C^*-algebra over $\mathfrak{g}'$ for $-g$. Thus each $\omega \in \mathfrak{g}'$ determines a skew-adjoint element of C such that

$$\omega^2 = -\langle \omega, \omega \rangle_g \ 1_C \ .$$

Depending on whether $\mathfrak{g}$ is even or odd dimensional, C will be a full matrix algebra, or the direct sum of two such. We let $\mathcal{S}$ be the Hilbert space of a finite-dimensional faithful representation of C (the "spinors").

Let A^∞ denote the space of smooth elements of A. (We could just as well use the A^1 of the previous section. We use A^∞ here for variety. It is still a dense $*$-subalgebra [BR].) Let $W = A^\infty \otimes \mathcal{S}$, viewed as a free right A^∞-module. From the Hilbert-space inner-product on $\mathcal{S}$ we obtain an A^∞-valued inner-product on W. Let η be as in the previous section, viewed as a faithful state on A. Combined with the A-valued inner product on W, it gives an ordinary inner-product on W. We will denote the Hilbert space completion by $L^2(W, \eta)$.

Now A^∞ and C have evident commuting left actions on W. These are easily seen to give $*$-representations of A and C on $L^2(W, \eta)$, which we denote by λ and c respectively.

We define the Dirac operator, D, on $L^2(W, \eta)$ in the usual way. Its domain will be W, and it is defined as the composition of operators

$$W \xrightarrow{d} \mathfrak{g}' \otimes W \xrightarrow{i} C \otimes W \xrightarrow{c} W \ .$$

Here d is the operator which takes $b \in A^\infty$ to $db \in \mathfrak{g}' \otimes A^\infty$, defined by $db(X) = d_X(b)$, which we then extend to W so that it takes $b \otimes s$ to $db \otimes s$. The operator i just comes from the canonical inclusion of $\mathfrak{g}'$ into C. The operator c just comes from applying the representation of C on $\mathcal{S}$, and so on W.

It is easily seen that D is a symmetric operator on $L^2(W, \eta)$. It will not be important for us to verify that D is essentially self-adjoint, and that its closure has compact resolvant.

Let $\{e_j\}$ denote an orthonormal basis for $\mathfrak{g}'$, and let $\{E_j\}$ denote the dual basis for $\mathfrak{g}$. Then in terms of these bases we have

$$D(b \otimes s) = \sum \alpha_{E_j}(b) \otimes c(e_j)s \ .$$

When we use this to compute $[D, \lambda_a]$ for $a \in A^\infty$, a straightforward calculation shows that we obtain

$$[D, \lambda_a](b \otimes s) = \sum (\alpha_{E_j}(a) \otimes c(e_j)))(b \otimes s) \ .$$

That is,

$$[D, \lambda_a] = \sum \alpha_{E_j}(a) \otimes e_j \ , \tag{4.1}$$

acting on $L^2(W, \eta)$ through the representations λ and c. It is clear from (4.1) that $[D, \lambda_a]$ is bounded for the operator norm from $L^2(W, \eta)$.

We can now set $\mathcal{L} = A^\infty$, and

$$L(a) = \|[D, \lambda_a]\| \ .$$

It is clear that $L(1_A) = 0$. To proceed further we compare L with the semi-norm of the last section. If we view $\mathfrak{g}'$ as contained in the C^*-algebra C, we have $e_j^2 = -1$ and $e_j^* = -e_j$ for each j. In particular, $\|e_j\| = 1$. From (4.1) it is then easy to see that there is a constant, K, such that

$$L(a) \leq K\|da\|$$

for all $a \in \mathcal{L}$, where $\|da\|$ is as in the previous section, for the inner-product dual to that on $\mathfrak{g}'$. However, what we need is an inequality in the reverse direction so that we will be able to apply the Comparison Lemma 1.10.

For this purpose, consider any element $t = \sum b_j \otimes e_j$ in $A \otimes C$, with the e_j as above. Let $f_j = ie_j$, so that $f_j^* = f_j$, $f_j^2 = 1$, and $f_j f_k = -f_k f_j$ for $j \neq k$. Let $p_j = (1 + f_j)/2$ and $q_j = 1 - p_j = (1 - f_j)/2$, both being self-adjoint projections. Then $p_j f_k = f_k q_j$ for $j \neq k$. Consequently $p_j f_k p_j = 0 = q_j f_k q_j$ for $j \neq k$. Thus

$$(1 \otimes p_j)t(1 \otimes p_j) = b_j \otimes p_j e_j p_j = b_j \otimes ip_j$$

and

$$(1 \otimes q_j)t(1 \otimes q_j) = -b_j \otimes iq_j \ .$$

Since at least one of p_j and q_j must be non-zero, it becomes clear that $\|t\| \geq \|b_j\|$ for each j. When we apply this to (4.1) we see that

$$L(a) \geq \|\alpha_{E_j}(a)\|$$

for each j. Consequently, for a suitable constant k we have

$$L(a) \geq k\|da\| \ ,$$

where again $\|da\|$ is as in the previous section. On applying Proposition 1.11, Theorem 3.1, and the Comparison Lemma 1.10, we obtain the proof of:

4.2 THEOREM. *Let α be an ergodic action of the compact connected Lie group G with Lie algebra $\mathfrak{g}$ on the unital C^*-algebra A. Pick any inner-product on the dual, $\mathfrak{g}'$, of $\mathfrak{g}$. Let D denote the corresponding Dirac operator, as defined above. Let $\mathcal{L} = A^\infty$, and let L be defined by*

$$L(a) = \|[D, a]\|$$

for $a \in A$. Let ρ be the corresponding metric on S. Then the ρ-topology on S agrees with the weak-$*$ topology.

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