Classical Motivic Polylogarithm According to Beilinson and Deligne

Annette Huber, Jörg Wildeshaus [1]

Received: February 11, 1998
Revised: September 7, 1998

Communicated by Peter Schneider

Abstract. The main purpose of this paper is the construction in motivic cohomology of the *cyclotomic*, or *classical polylogarithm* on the projective line minus three points, and the identification of its image under the regulator to absolute (Deligne or l-adic) cohomology. By specialization to roots of unity, one obtains a compatibility statement on *cyclotomic elements* in motivic and absolute cohomology of abelian number fields. As shown in [BlK], this compatibility completes the proof of the Tamagawa number conjecture on special values of the Riemann zeta function.

The main constructions and ideas are contained in Beilinson's and Deligne's unpublished preprint "Motivic Polylogarithm and Zagier Conjecture" ([BD1]). We work out the details of the proof, setting up the foundational material which was missing from the original source: the paper contains an appendix on absolute Hodge cohomology with coefficients, and its interpretation in terms of Saito's Hodge modules. The second appendix treats K-theory and regulators for simplicial schemes.

1991 Mathematics Subject Classification: Primary 19F27; Secondary 11R18, 11R34, 11R42, 14D07, 14F99.

Keywords: Polylogarithm, motivic and absolute cohomology, regulators, cyclotomic elements.

Introduction

The aim of this work is to present the construction of the class of the cyclotomic, or classical polylogarithm in motivic cohomology. It maps to the elements in Deligne and l–adic cohomology defined and studied in Beilinson's "Polylogarithm and cyclotomic elements" ([B4]). The latter elements can be seen as being represented by a pro–variation of Hodge structure, or a pro–l–adic sheaf on the projective line minus three points.

[1] See correction on page 297 of this volume.

Our main interest lies in the specialization of these sheaves to roots of unity: they represent the "cyclotomic" one–extensions of Tate twists already studied by Soulé ([Sou5]), Deligne ([D5]) and Beilinson ([B2]).

Let us be more precise: denote by μ_d^0 the set of primitive d–th roots of unity in $\mathbb{Q}(\mu_d) = \mathbb{Q}[T]/\Phi_d(T)$, $d \geq 2$. We get an alternative proof of the following theorem of Beilinson's:

Corollary 9.6. Assume $n \geq 0$, and denote by $r_{\mathcal{D}}$ the regulator map

$$H^1_{\mathcal{M}}(\operatorname{Spec}\mathbb{Q}(\mu_d), \mathbb{Q}(n+1)) \longrightarrow \bigoplus_{\sigma:\mathbb{Q}(\mu_d)\hookrightarrow\mathbb{C}} \mathbb{C}/(2\pi i)^{n+1}\mathbb{R}\ .$$

There is a map of sets

$$\epsilon_{n+1} : \mu_d^0 \longrightarrow H^1_{\mathcal{M}}(\operatorname{Spec}\mathbb{Q}(\mu_d), \mathbb{Q}(n+1))$$

such that

$$r_{\mathcal{D}} \circ \epsilon_{n+1} : \mu_d^0 \longrightarrow \bigoplus_{\sigma:\mathbb{Q}(\mu_d)\hookrightarrow\mathbb{C}} \mathbb{C}/(2\pi i)^{n+1}\mathbb{R}$$

maps a root of unity ω to $(-Li_{n+1}(\sigma\omega))_\sigma = \left(-\sum_{k\geq 1}\frac{\sigma\omega^k}{k^{n+1}}\right)_\sigma$.

Now fix a d–th primitive root of unity ζ in $\overline{\mathbb{Q}}$. This choice allows to identify continuous étale cohomology $H^1_{cont}(\operatorname{Spec}\mathbb{Q}(\mu_d), \mathbb{Q}_l(n+1))$ with a $\mathbb{Q}_l$–subspace of

$$\left(\varprojlim_{r\geq 1}\left(\mathbb{Q}(\mu_{l^\infty},\zeta)^*/(\mathbb{Q}(\mu_{l^\infty},\zeta)^*)^{l^r} \otimes \mu_{l^r}^{\otimes n}\right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\right)^{\operatorname{Gal}(\mathbb{Q}(\mu_{l^\infty},\zeta)/\mathbb{Q}(\zeta))} .$$

Note that there is a distinguished root of unity T in $\mathbb{Q}(\mu_d)$. As was observed already in [B4], the study of the cyclotomic polylogarithm gives a proof of [BlK], Conjecture 6.2 (cf. [Sou5], Théorème 1 for the case $n = 1$; [Gr], Théorème IV.2.4 for the local version if $(l, d) = 1$):

Corollary 9.7. Let ϵ_{n+1} be the map constructed in 9.6. Under the above inclusion, the l–adic regulator

$$r_l : H^1_{\mathcal{M}}(\operatorname{Spec}\mathbb{Q}(\mu_d), \mathbb{Q}(n+1)) \longrightarrow H^1_{cont}(\operatorname{Spec}\mathbb{Q}(\mu_d), \mathbb{Q}_l(n+1))$$

maps $\epsilon_{n+1}(T^b)$ to

$$\frac{1}{d^n}\cdot\frac{1}{n!}\cdot\left(\sum_{\alpha^{l^r}=\zeta^b}[1-\alpha]\otimes(\alpha^d)^{\otimes n}\right)_r .$$

This result implies in particular that Soulé's cyclotomic elements in the group $K_{2n+1}(F)\otimes_{\mathbb{Z}}\mathbb{Z}_l$ (for an abelian number field F and a prime l) are induced by elements in K–theory itself (Corollary 9.8). Furthermore, the case $d = 2$ of 9.7 forms a central ingredient of the proof of the Tamagawa number conjecture modulo powers of 2 for odd Tate twists $\mathbb{Q}(n)$, $n \geq 2$ ([BlK], § 6). Finally, as shown in [KNF], Theorem 6.4, the general case of 9.7 implies the modified version of the Lichtenbaum conjecture for abelian number fields.

The main ideas necessary for both the construction of the motivic polylogarithm and the identification of the realization classes, together with a sketch of proof, are contained in the unpublished preprint "Motivic Polylogarithm and Zagier Conjecture" ([BD1]) and its predecessors [B4], [BD1p]. Our aim in this paper is to work out the details of the proofs. To do this we have to set up a lot of foundational material, which was missing from the original sources: K-theory of simplicial schemes, regulators to absolute (Hodge and l-adic) cohomology of simplicial schemes, and an interpretation of the latter as Ext groups of Hodge modules and l-adic sheaves respectively. This material is contained in the two appendices which we regard as our main contribution to the subject. We hope they prove to be useful in other contexts than that treated in the main text.

Other parts of [BD1] deal with (the weak version of) the Zagier conjecture. We do not treat this since a complete proof has been given by de Jeu ([Jeu]), although by somewhat different means from those used in [BD1].

We see two main groups of papers related to polylogarithms:

The first deals with mixed sheaves, i.e., variations of Hodge structure or l–adic mixed lisse sheaves. Maybe the nucleus of these papers is Deligne's observation that the analytic and topological properties of the dilogarithm Li_2, viewed as a multivalued holomorphic function on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$, can be coded by saying that Li_2 is an entry of the period matrix of a certain rank three variation of $\mathbb{Q}$–Tate–Hodge structure on $\mathbb{P}^1_{\mathbb{C}}\backslash\{0,1,\infty\}$.

We refer to [Rm], section 7.6 for a nice survey of the construction of a pro–variation on $\mathbb{P}^1_{\mathbb{C}}\backslash\{0,1,\infty\}$ containing all Li_k. The étale analogue is constructed in Beilinson's "Polylogarithm and Cyclotomic elements" ([B4]), where he defined pro–objects in the categories of l–adic sheaves on $\mathbb{P}^1\backslash\{0,1,\infty\}$. In both settings, the fibres at roots of unity different from 1 coincide with the cyclotomic extensions mentioned above.

The hope and indeed, the motivation underlying these papers is that once a satisfactory formalism of motivic sheaves is developed, the definition of polylogarithms should basically carry over. We would thus obtain polylogarithmic classes in Ext groups of motives, these groups being supposedly closely connected to K–theory, of which everything already defined on the level of realizations would turn out to be the respective regulator.

Nowhere is this hope documented more manifestly than in Beilinson's and Deligne's "Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs" ([BD2]): if there is such a motivic formalism, then the weak version of Zagier's conjecture necessarily holds: not only the values at roots of unity of higher logarithms, but also appropriate linear combinations of *arbitrary* values must lie in the image of the regulator.

For the time being, and in each case separately, honest work is needed to perform the K–theoretic constructions, and calculate their images under the regulators.

The second class of papers is concerned with precisely that task. In analogy with the above, one should first mention Bloch's "Application of the dilogarithm function in algebraic K–theory and algebraic geometry" ([Bl]).

Beilinson's "Higher regulators and values of L–functions" ([B2]) provided the K–theoretic construction of cyclotomic elements, together with the computation of their images in Deligne cohomology (loc. cit., Theorem 7.1.5, [Neu], [E]).

As for Zagier's conjecture, we mention Goncharov's "Polylogarithms and Motivic Galois Groups" ([Go]), where Zagier's conjecture, including the surjectivity statement is proved for K_5 of a number field, and de Jeu's "Zagier's Conjecture and Wedge Complexes in Algebraic K–theory" ([Jeu]), which contains the proof of the weak version of Zagier's conjecture, independently of motivic considerations, for K_{2n-1} of a number field, and arbitrary $n \geq 2$.

Typically, the objects of interest in this class of papers are complexes, cocycles, and symbols, i.e., objects which do not constantly afford a geometric, or sheaf–theoretic interpretation. It is by no means easy to see, say, how a concrete element in some Deligne cohomology group can be interpreted as an extension of variations of $\mathbb{R}$–Hodge structure. These and similar difficulties present themselves to the reader willing to translate from one class to the other.

The authors like to think of the present article as an attempt to bridge the gap between the two disciplines.

In a sense, the coarse structure of the article follows the above scheme: sections 1–6 are entirely sheaf–theoretic. Anything we say there is therefore a priori restricted to the level of realizations, i.e., non–motivic. In sections 7–9, K–theory enters. The appendices provide the foundations necessary to connect the two points of view.

Given that quite a lot has been said about the l–adic and Hodge theoretic incarnations of the classical polylogarithm ([B4], [BD2], [WiIV]), the reader may wonder why sheaf theoretic considerations still take up one third of this work.

Indeed, the construction of the motivic polylog could be achieved much more easily if a satisfactory formalism of mixed motivic sheaves were available. The necessity to replace a simple geometric situation by a rather complicated one, in order to replace complicated coefficients like $\mathcal{L}og$ by Tate twists, should be seen as the main source of difficulty in any attempt to the construction of motivic versions of polylogarithms.

We now turn to the description of the finer structure of the main text (sections 1–9):

In section 1, we normalize the sheaf theoretic notations used throughout the whole article.

Section 2 gives a quick axiomatic description of the logarithmic sheaf $\mathcal{L}og$, and the (small) polylogarithmic extension *pol*. The universal property (2.1) is needed only to connect the general definition of the logarithmic sheaf as a solution of a representability problem to the somewhat ad hoc, but much more geometric definition of section 4. A reader prepared to accept the results on the shape of the Hodge theoretic and l–adic incarnation of the polylogarithm (2.5, 2.6) may therefore take the constructions in sections 4 and 6 as a definition of both $\mathcal{L}og$ and *pol*, and view section 2 as an extended introduction providing background material.

In section 3, we establish the geometric situation used thereafter. As section 1, it is mainly intended for easier reference.

In section 4, we construct a pro-unipotent sheaf $\mathcal{G}$ on $\mathbb{U} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ as projective limit of relative cohomology objects of powers of $\mathbb{G}_m$ over $\mathbb{U}$ relative to certain singular subschemes. The transition maps are given by the boundary maps in the *relative residue sequence* (4.9). The universal property 2.1 then allows to identify $\mathcal{G}$ with the restriction of $\mathcal{L}og$ to $\mathbb{U}$ (4.11).

Section 5 contains a geometric proof of the *splitting principle* (5.2): the fibres of $\mathcal{L}og$ at roots of unity have split weight filtration. Since we need a proof which translates easily to the motivic situation, we return to Beilinson's original approach to the splitting principle ([B4], 4.2) which consists of an analysis of the action of the multiplication by natural numbers on our absolute cohomology groups.

The main objective of section 6 is the description of *pol* in terms of geometric data. The Leray spectral sequence suggests that one–extensions of $\mathbb{Q}(0)$ by $\mathcal{L}og$ should be described as elements of the projective limit of cohomology groups with Tate coefficients of powers of $\mathbb{G}_m$ relative to certain subschemes. The main result 6.6 allows to identify *pol* under this correspondence.

In Section 7 our main tool, the residue sequence is constructed in the setting of motivic cohomology (Proposition 7.2 and Lemma 7.3). The arguments are very much parallel to those used for absolute cohomology of realizations in section 4. However, we have to replace the singular schemes by explicit simplicial schemes with regular components. This is where the material of Appendix B enters.

Section 8 is the K–theoretic analogue of section 6. We consider a certain projective system of motivic cohomology groups. In order to identify its projective limit (Corollary 8.8) we use bijectivity or at least controlled injectivity of the regulator to Deligne cohomology, and the results of section 6. We are then able to define the universal motivic polylog (8.9).

In the final section 9 the motivic version of the splitting principle is shown (9.3). Again we strongly use the known behaviour of the regulator to show that the action of multiplication by natural numbers splits into eigenspaces. Applied to the universal motivic polylogarithm this induces the cyclotomic elements in motivic cohomology. In the light of section 5 it is clear from their very construction that they induce the right elements not only in Deligne but also in continuous étale cohomology. We conclude by drawing the corollaries which are the main results announced at the beginning (9.6–9.9).

The Appendices can be read independently of the main text and of each other. They are meant to be used as a reference, but a careful reader might actually want to read them first. We refer to the respective introductions for an account of their content.

The reader might find it useful to consult [HW] for an overview of the strategy of the proof of the main results.

Acknowledgments: It is a pleasure for us to thank both Beilinson and Deligne for not only letting us work with their ideas, but explicitly encouraging us to bring mathematics into a state in which the constructions of [BD1] can be performed.

We thank C. Deninger for suggesting to us that the methods developed and results obtained in our respective PhD theses ([H1], [Wi]) might form a sound basis of a successful treatment of this theme.

We are grateful to G. Banaszak, H. Esnault, W. Gajda, T. Geisser, U. Jannsen, M. Joachim, K. Künnemann, F. Lecomte, A.J. Scholl, and J. Schürmann for useful discussions. We thank R. de Jeu for his remarks on Appendix B, and C. Soulé for making [GSo1] available and explaining some critical points. We are very grateful to the referee for her or his helpful and detailed comments on Appendix B.

Part of this work was written up during a stay at the Fields Institute at Toronto in October 1995. We wish to thank M. Kolster and V. Snaith for the invitation.

Finally, we are obliged to Mrs. Gabi Weckermann for LaTeXing part of this paper.

Contents

1 Mixed sheaves

We start by defining the sheaf categories which will be relevant for us. For our purposes, it will be necessary to work in the settings of mixed l–adic perverse sheaves ([H2]), and of algebraic mixed Hodge modules over $\mathbb{R}$ (A.2). Since the procedures are entirely analogous, we introduce, for economical reasons, the following rules: whenever an area of paper is divided by a vertical bar

$$\Bigg|$$

the text on the left of it will concern the Hodge theoretic setting, while the text on the right will deal with the l–adic setting. Of course, we hope that before long, there will be a satisfactory formalism of mixed motivic sheaves providing a third setting to which our constructions can be applied. We let

$A := \mathbb{R}$, $F := \mathbb{Q}$,	$l :=$ a fixed prime number , $A := \mathbb{Z}\left[\frac{1}{l}\right]$, $F := \mathbb{Q}_l$

and set $B := \mathrm{Spec}(A)$.

For any reduced, separated and flat scheme X of finite type over B, we let

$X_{\mathrm{top}} := X(\mathbb{C})$ as a topol. space, $\mathrm{Sh}(X_{\mathrm{top}}) := \mathrm{Perv}(X_{\mathrm{top}}, \mathbb{Q})$,	$X_{\mathrm{top}} := X \otimes_A \overline{\mathbb{Q}}$, $\mathrm{Sh}(X_{\mathrm{top}}) := \mathrm{Perv}(X_{\mathrm{top}}, \mathbb{Q}_l)$

the latter categories denoting the respective categories of perverse sheaves on X_{top} ([BBD], 2.2).

Next we define the category $\mathrm{Sh}(X)$: in the l–adic setting, we fix a pair $(\mathbf{S}, L)$ consisting of a horizontal stratification $\mathbf{S}$ of X ([H2], § 2) and a collection $L = \{L(S) \mid S \in \mathbf{S}\}$, where each $L(S)$ is a set of irreducible lisse l–adic sheaves on S. For all $S \in \mathbf{S}$ and $F \in L(S)$, we require that for the inclusion $j : S \hookrightarrow X$, all higher direct images $R^n j_* F$ are *$(\mathbf{S}, L)$–constructible*, i.e., have lisse restrictions to all $S \in \mathbf{S}$, which are extensions of objects of $L(S)$. We assume that all $F \in L(S)$ are pure.

We can make this more explicit: in our computations X will always be a locally closed subscheme of some $\mathbb{A}^n$; the stratification is by the number of vanishing coordinates in $\mathbb{A}^n$; $L(S)$ is the set of all Tate sheaves on S.

Following [H2], § 3, we define $D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ as the full subcategory of $D^b_c(X, \mathbb{Q}_l)$ of complexes with $(\mathbf{S}, L)$–constructible cohomology objects. Note that all objects will be mixed. By [H2], § 3, $D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ admits a perverse t–structure, whose heart we denote by $\mathrm{Perv}_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$.

$\mathrm{Sh}(X) := \mathrm{MHM}_{\mathbb{Q}}(X/\mathbb{R})$ (see A.2.4)	$\mathrm{Sh}(X) := \mathrm{Perv}_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$.

Because of the horizontality requirement in the l–adic situation we have the full formalism of Grothendieck's functors only on the direct limit $D^b_m(\mathfrak{U}_X, \mathbb{Q}_l)$ of the $D^b_{(\mathbf{S},L)}(X_U, \mathbb{Q}_l)$, for U open in B, and $(\mathbf{S}, L)$ as above (see [H2], § 2). However, for a fixed morphism

$$\pi : X \longrightarrow Y \ ,$$

we have a notion of e.g. π_*–admissibility for a pair $(\mathbf{S}, L)$: this is the case if

$$D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l) \hookrightarrow D^b_m(\mathfrak{U}_X, \mathbb{Q}_l) \xrightarrow{\pi_*} D^b_m(\mathfrak{U}_Y, \mathbb{Q}_l)$$

factors through some $D^b_{(\mathbf{T},K)}(Y, \mathbb{Q}_l)$. Our computations will show, at least a posteriori, that for our choice of $(\mathbf{S}, L)$ all functors which appear are admissible. We will not stress these technical problems and even suppress $(\mathbf{S}, L)$ from our notation.

As in [BBD], we denote by $\pi_*, \pi^*, \mathrm{Hom}$ etc. the respective functors on the categories

$$D^b \operatorname{Sh}(X) := D^b \operatorname{MHM}_{\mathbb{Q}}(X/\mathbb{R}) \,, \qquad \Big| \qquad D^b \operatorname{Sh}(X) := D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l) \,,$$

and $\mathcal{H}^q$ for the (perverse) cohomology functors.

We refer to objects of $\operatorname{Sh}(X)$ as sheaves, and to objects of $\operatorname{Sh}(X_{\mathrm{top}})$ as topological sheaves. Let us denote by

$$\mathbb{V} \mapsto \mathbb{V}_{\mathrm{top}}$$

the forgetful functor from $\operatorname{Sh}(X)$ to $\operatorname{Sh}(X_{\mathrm{top}})$. If we use the symbol $W.$, it will always refer to the weight filtration.

If X is smooth, we let

$\operatorname{Sh}^s(X) := \operatorname{Var}_{\mathbb{Q}}(X/\mathbb{R}) \subset \operatorname{Sh}(X)$ (see A.2.1) , $\operatorname{Sh}^s(X_{\mathrm{top}}) :=$ the category of $\mathbb{Q}$–local systems on X_{top}.	$\operatorname{Sh}^s(X) := \operatorname{Et}^{l,m}_{\mathbb{Q}_l}(X) \subset \operatorname{Sh}(X)$, the category of lisse mixed $\mathbb{Q}_l$–sheaves on X, $\operatorname{Sh}^s(X_{\mathrm{top}}) :=$ the category of lisse $\mathbb{Q}_l$–sheaves on X_{top}.

We refer to objects of $\operatorname{Sh}^s(X)$ as smooth sheaves, and to objects of $\operatorname{Sh}^s(X_{\mathrm{top}})$ as smooth topological sheaves. Denote by $U\operatorname{Sh}^s(X)$ the category of unipotent objects of $\operatorname{Sh}^s(X)$, i.e., those smooth sheaves admitting a filtration whose graded parts are pullbacks of smooth sheaves of $\operatorname{Sh}^s(B)$ via the structure morphism. Similarly, one defines $U\operatorname{Sh}^s(X_{\mathrm{top}})$.

REMARK: Note that in the l–adic situation, the existence of a weight filtration, i.e., an ascending filtration $W.$ by subsheaves indexed by the integers, such that Gr^W_m is of weight m, is not incorporated in the definition of Sh^s – compare the warnings in [H2], § 3. In the Hodge theoretic setting, the existence of a weight filtration is part of the data.

REMARK: We have to deal with a shift of the index when viewing e.g. a variation as a Hodge module, which occurs either in the normalization of the embedding

$$\operatorname{Var}_{\mathbb{Q}}(X/\mathbb{R}) \longrightarrow D^b \operatorname{MHM}_{\mathbb{Q}}(X/\mathbb{R})$$

or in the numbering of cohomology objects of functors induced by morphisms between schemes of different dimension. In order to conform with the conventions laid down in appendix A and [Wil], chapter 4, we chose the second possibility: a variation *is* a Hodge module, not just a shift of one such. Similarly, a lisse mixed $\mathbb{Q}_l$–sheaf *is* a perverse mixed sheaf. Therefore, if X is of pure relative dimension d over B, then the embedding

$$\operatorname{Et}^{l,m}_{\mathbb{Q}_l}(X) \longrightarrow D^b_m(\mathfrak{U}_X, \mathbb{Q}_l)$$

associates to $\mathbb{V}$ the complex concentrated in degree $-d$, whose only non–trivial cohomology object is $\mathbb{V}$.

As a consequence, the numbering of cohomology objects of the direct image (say) will differ from what the reader might be used to: e.g., the cohomology of a curve is concentrated in degrees -1, 0, and 1 instead of 0, 1, and 2. Similarly, one has to distinguish between the "naive" pullback $(\pi^s)^*$ of a smooth sheaf and the pullback π^* on the level of $D^b\,\mathrm{Sh}(X)$: $(\pi^s)^*$ lands in the category of smooth sheaves, while π^* of a smooth sheaf yields only a smooth sheaf up to a shift.

In the special situation of pullbacks, we allow ourselves one notational inconsistency: if there is no danger of confusion (e.g. in Theorem 2.1), we use the notation π^* also for the naive pullback of smooth sheaves. Similar remarks apply for smooth topological sheaves.

For a scheme $a : X \to B$, we define

$$F(n)_X := a^* F(n) \in D^b\,\mathrm{Sh}(X)\ ,$$

where $F(n)$ is the usual Tate twist on B.

If X is smooth, we also have the naive Tate twist

$$F(n) \in \mathrm{Sh}^s(X) \subset \mathrm{Sh}(X)$$

on X. If X is of pure dimension d, then we have the equality

$$F(n) = F(n)_X[d]\ .$$

In order to keep our notation transparent, we have the following

Definition 1.1. *For any morphism $\pi : X \longrightarrow S$ of reduced, separated and flat B–schemes we let*

$$\mathcal{R}_S(X,\ \cdot\ ) := \pi_* : D^b\,\mathrm{Sh}(X) \longrightarrow D^b\,\mathrm{Sh}(S)\ ,$$
$$\mathcal{H}^i_S(X,\ \cdot\ ) := \mathcal{H}^i\pi_* : D^b\,\mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(S)\ .$$

Definition 1.2. *For a closed reduced subscheme Z of a separated, reduced, flat B–scheme X of finite type, with complement $j : U \hookrightarrow X$, and an object $M^\cdot$ of $D^b\,\mathrm{Sh}(X)$, define*

a)
$$R\Gamma_{\mathrm{abs}}(X, M^\cdot) := R\,\mathrm{Hom}_{D^b\,\mathrm{Sh}(X)}(F(0)_X, M^\cdot)\ ,$$
$$H^i_{\mathrm{abs}}(X, M^\cdot) := H^i R\Gamma_{\mathrm{abs}}(X, M^\cdot)\ ,$$

the absolute complex and absolute cohomology groups of X with coefficients in $M^\cdot$.

b)
$$R\Gamma_{\mathrm{abs}}(X, n) := R\Gamma_{\mathrm{abs}}(X, F(n)_X)\ ,$$
$$H^i_{\mathrm{abs}}(X, n) := H^i_{\mathrm{abs}}(X, F(n)_X)\ .$$

c)
$$R\Gamma_{\mathrm{abs}}(X \textit{ rel } Z, n) := R\Gamma_{\mathrm{abs}}(X, j_! F(n)_U)\ ,$$
$$H^i_{\mathrm{abs}}(X \textit{ rel } Z, n) := H^i_{\mathrm{abs}}(X, j_! F(n)_U)\ ,$$

the relative absolute complex and relative absolute cohomology with Tate coefficients.

In the Hodge setting, absolute cohomology with Tate coefficients coincides with Beilinson's absolute Hodge cohomology over $\mathbb{R}$ (Theorem A.2.7). In the l-adic setting, it yields continuous étale cohomology (see the remark following Definition B.4.2).

REMARK: If X is a scheme over S, then we have the formulae

$$\begin{aligned} R\Gamma_{\text{abs}}(X, \,\cdot\,) &= R\Gamma_{\text{abs}}\left(S, \mathcal{R}_S(X, \,\cdot\,)\right) \ , \\ H^i_{\text{abs}}(X, \,\cdot\,) &= H^i_{\text{abs}}(S, \mathcal{R}_S(X, \,\cdot\,)) \ . \end{aligned}$$

2 THE LOGARITHMIC SHEAF, AND THE POLYLOGARITHMIC EXTENSION

We aim at a sheaf theoretic description of the (small) classical polylogarithm on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The first step is an axiomatic definition of the *logarithmic pro–sheaf.* We need the following result:

THEOREM 2.1. *Let X be the complement in a smooth, proper B–scheme of an NC–divisor relative to B ([SGA1], Exp. XIII, 2.1), all of whose irreducible components are smooth over B. Let $x \in X(B)$, and write $a : X \to B$. The functor*

$$x^* : U\mathrm{Sh}^s(X) \longrightarrow \mathrm{Sh}^s(B)$$

is representable in the following sense:

a) *There is a pro–object*

$$\mathcal{G}en_x \in \textit{pro–}U\mathrm{Sh}^s(X) \ ,$$

the generic pro–unipotent sheaf with basepoint x on X, which has a weight filtration satisfying

$$\mathcal{G}en_x / W_{-n}\mathcal{G}en_x \in U\mathrm{Sh}^s(X) \quad \textit{for all } n \ .$$

Note that this implies that the direct system

$$\left(R^0 a_* \underline{\mathrm{Hom}}(\mathcal{G}en_x / W_{-n}\mathcal{G}en_x, \mathbb{V})\right)_{n \in \mathbb{N}}$$

of smooth sheaves on B becomes constant for any $\mathbb{V} \in U\mathrm{Sh}^s(X)$.

This constant value is denoted by

$$R^0 a_* \underline{\mathrm{Hom}}(\mathcal{G}en_x, \mathbb{V}) \ .$$

b) *There is a section*

$$1 \in \Gamma\left(B, x^*\mathcal{G}en_x\right) \ .$$

c) *The natural transformation of functors from $U\mathrm{Sh}^s(X)$ to $\mathrm{Sh}^s(B)$*

$$\begin{array}{rcl} ev : R^0 a_* \underline{\mathrm{Hom}}(\mathcal{G}en_x, \,\_\,) & \longrightarrow & x^* \ , \\ \varphi & \longmapsto & (x^*\varphi)(1) \end{array}$$

is an isomorphism. Similarly for the transformation of functors from $U\mathrm{Sh}^s(X_{\mathrm{top}})$ *to* $\mathrm{Sh}^s(B_{\mathrm{top}})$

$$\begin{aligned} ev : R^0 a_* \underline{\mathrm{Hom}}\,((\mathcal{G}en_x)_{\mathrm{top}}, \_) &\longrightarrow x^* \;, \\ \varphi &\longmapsto (x^*\varphi)(1) \;. \end{aligned}$$

Consequently, the pairs $(\mathcal{G}en_x, 1)$ *and* $((\mathcal{G}en_x)_{\mathrm{top}}, 1)$ *are unique up to unique isomorphism.*

d) *The natural transformations of functors*

$$\begin{aligned} \mathrm{Hom}_{U\mathrm{Sh}^s(X)}(\mathcal{G}en_x, \_) &\longrightarrow \mathrm{Hom}_{\mathrm{Sh}^s(B)}(F(0), x^*\_) \quad \text{and} \\ \mathrm{Hom}_{U\mathrm{Sh}^s(X_{\mathrm{top}})}((\mathcal{G}en_x)_{\mathrm{top}}, \_) &\longrightarrow \Gamma(B_{\mathrm{top}}, x^*\_) \end{aligned}$$

from $U\mathrm{Sh}^s(X)$ *and* $U\mathrm{Sh}^s(X_{\mathrm{top}})$ *respectively are isomorphisms.*

Proof. For a)–c), we refer to

[WiI], Remark d) after Theorem 3.6,	[WiI], Theorem 3.5.i),

and loc. cit., Theorem 3.5.ii). Apply the functors $\mathrm{Hom}_{\mathrm{Sh}^s(B)}(F(0), \_)$ and $\Gamma(B_{\mathrm{top}}, \_)$ to the result in c) in order to obtain d). □

REMARK: In the Hodge setting and for the constant base B, Theorem 2.1 is equivalent to the classification theorem for admissible unipotent variations of Hodge structure ([HZ], Theorem 1.6). In this case, $\mathcal{G}en_x$ is the *canonical variation with base point* x of loc. cit., section 1.

Now let

$$\begin{gathered} \mathbb{G}_m := \mathbb{G}_{m,B} \;, \quad \mathbb{U} := \mathbb{P}^1_B \setminus \{0,1,\infty\}_B \;, \\ j : \mathbb{U} \hookrightarrow \mathbb{G}_m \;, \\ p : \mathbb{G}_m \longrightarrow B \;, \quad \widetilde{p} := p \circ j : \mathbb{U} \longrightarrow B \;. \end{gathered}$$

We may form the generic pro–unipotent sheaf with basepoint 1 on $\mathbb{G}_m$.

DEFINITION 2.2. $\mathcal{L}og := \mathcal{G}en_1 \in$ *pro–*$U\mathrm{Sh}^s(\mathbb{G}_m)$ *is called the logarithmic pro–sheaf.*

As we shall see below, there is an isomorphism

$$\kappa : \mathrm{Gr}^W_{\cdot}\,\mathcal{L}og \xrightarrow{\sim} \prod_{k \geq 0} F(k) \;.$$

Assuming this for the moment, we now describe the higher direct images $\mathcal{H}^{\cdot}_B(\mathbb{U}, j^*\mathcal{L}og(1))$:

THEOREM 2.3. a) $\mathcal{H}^q_B(\mathbb{U}, j^*\mathcal{L}og(1)) = 0$ *for* $q \neq 0$.

b) $\mathcal{H}^0_B(\mathbb{U}, j^*\mathcal{L}og(1))$ *has a weight filtration, and* $W_{-1}\left(\mathcal{H}^0_B(\mathbb{U}, j^*\mathcal{L}og(1))\right)$ *is split. More precisely, any isomorphism* κ *as above induces an isomorphism*

$$W_{-1}\left(\mathcal{H}^0_B(\mathbb{U}, j^*\mathcal{L}og(1))\right) \xrightarrow{\sim} \prod_{k \geq 1} F(k) \;.$$

Remark: By these statements on the higher direct images of the pro–sheaf $j^*\mathcal{L}og(1)$, we mean the following:

a) For $q \neq 0$, the projective system

$$\mathcal{H}^q_B\left(\mathbb{U}, j^*(\mathcal{L}og/W_{-n}\mathcal{L}og)(1)\right)_{n\geq 1}$$

is ML–zero.

b) κ induces a morphism of projective systems

$$\mathcal{H}^0_B\left(\mathbb{U}, j^*(\mathcal{L}og/W_{-2m}\mathcal{L}og)(1)\right)_{m\geq 1} \longrightarrow \left(\prod_{k=0}^{m} F(k)\right)_{m\geq 1}$$

of sheaves with a weight filtration, such that the weight ≤ -1–parts of the projective systems of kernels and co-kernels are ML–zero.

Proof. One uses the exact triangle

$$\begin{tikzcd} 1_* 1^! \arrow[rr] & & \mathrm{id}_{\mathbb{G}_m} \arrow[dl] \\ & j_* j^* \arrow[ul, "{[1]}"] & \end{tikzcd}$$

or rather, $\mathcal{H}^{\cdot}_B(\mathbb{G}_m, \_)$ of it, and the fact that $\mathcal{H}^{\cdot}_B(\mathbb{G}_m, \mathcal{L}og)$ is easily computable. For the details, see [WiIII], Theorem 1.3. Or use 4.11 and 6.2, whose proof is independent of 2.3. □

A fixed choice of

$$\kappa : \mathrm{Gr}^W_{\cdot} \mathcal{L}og \xrightarrow{\sim} \prod_{k\geq 0} F(k)$$

induces in particular an isomorphism of $\mathrm{Gr}^W_{-2}\mathcal{L}og$ and $F(1)$. The theorem then enables one to define the *small polylogarithmic extension* as the extension

$$pol \in \mathrm{Ext}^1_{U\mathrm{Sh}^s(\mathbb{U})}\left(\mathrm{Gr}^W_{-2}\mathcal{L}og\,|_{\mathbb{U}}, \mathcal{L}og(1)\,|_{\mathbb{U}}\right)$$

mapping to the natural inclusion $F(1) \hookrightarrow \prod_{k\geq 1} F(k)$ under the isomorphism

$$\begin{aligned} \mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}(F(1), \mathcal{L}og(1)\,|_{\mathbb{U}}) &= \mathrm{Hom}_{D^b\,\mathrm{Sh}(\mathbb{U})}(F(1)_{\mathbb{U}}, \mathcal{L}og(1)\,|_{\mathbb{U}}) \\ &= \mathrm{Hom}_{D^b\,\mathrm{Sh}(\mathbb{U})}(\widetilde{p}^*F(1), j^*\mathcal{L}og(1)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sh}(B)}\left(F(1), \prod_{k\geq 1} F(k)\right) \end{aligned}$$

induced by the projective limit of the edge homomorphisms in the Leray spectral sequence for $\widetilde{p}$, and the isomorphism of 2.3.b). Note that the definition of *pol* is independent of the choice of κ. For the details, we refer to [WiIII], Theorem 1.5 – as there, we define

$$\mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}(F(1), \mathcal{L}og(1)\,|_{\mathbb{U}}) := \lim_{\substack{\longleftarrow \\ n}} \mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}(F(1), (\mathcal{L}og/W_{-n}\mathcal{L}og)(1)\,|_{\mathbb{U}}) \ .$$

A description of $\mathcal{L}og$ and pol, in both incarnations, was given by Beilinson and Deligne; see [B4], 2.1, 3.1 and [BD1], § 1 for the Hodge version and [B4], 3.3 for the l–adic setting. The reader may find it useful to also consult [WiIV], chapters 3 and 4, setting $N = 1$ in the notation of loc. cit.

We recall the "values" of pol at spectra of cyclotomic fields: let $d \geq 2$, and $C := \operatorname{Spec}(R)$, where $R := A\left[\frac{1}{d}, T\right] / \Phi_d(T)$, where $\Phi_d(T)$ is the d-th cyclotomic polynomial.

C is canonically a closed, reduced subscheme of $\mathbb{G}_m \otimes_A A\left[\frac{1}{d}\right]$. For any integer b prime to d, there is an embedding

$$\begin{array}{rcl} i_b : C & \xrightarrow{\sim} & C \hookrightarrow \mathbb{G}_m \otimes_A A\left[\dfrac{1}{d}\right] , \\ \zeta & \longmapsto & \zeta^b . \end{array}$$

Since d is invertible on C, the image of i_b is actually contained in $\mathbb{U}$, and hence we may form the pullback of pol via i_b,

$$pol_b \in \operatorname{Ext}^1_{\mathrm{Sh}^s(C)} \left(F(1), \mathcal{L}og_b(1)\right) ,$$

where $\mathcal{L}og_b$ denotes the pullback of $\mathcal{L}og$.

Now we have the following

Theorem 2.4 (Splitting Principle). *$\mathcal{L}og_b$ splits (uniquely) into a direct product*

$$\mathcal{L}og_b = \prod_{k \geq 0} \operatorname{Gr}^W_{-2k} \left(\mathcal{L}og_b\right) ,$$

and $\operatorname{Gr}^W_{-2k}\left(\mathcal{L}og_b\right)$ is isomorphic to $F(k)$ for any $k \geq 0$.

Proof. [B4], 4, or [BD1], 3.6, or [WiIV], Lemma 3.10. Or use 4.11 and 5.2, whose proof is independent of 2.4. ☐

In order to identify pol_b with an element of

$$\prod_{k \geq 1} \operatorname{Ext}^1_{\mathrm{Sh}^s(C)} \left(F(1), F(k)\right) ,$$

we need to fix an isomorphism

$$\kappa_b : \operatorname{Gr}^W_{\cdot} \mathcal{L}og_b \xrightarrow{\sim} \prod_{k \geq 0} F(k) .$$

By definition, κ_b is the pullback via i_b of the isomorphism

$$\kappa : \operatorname{Gr}^W_{\cdot} \mathcal{L}og \xrightarrow{\sim} \prod_{k \geq 0} F(k)$$

of pro–sheaves on $\mathbb{G}_m$ of [WiIV], chapters 3 and 4, which we briefly describe now:

By 2.1.d), there is a canonical projection

$$\varepsilon : \mathcal{L}og \longrightarrow F(0) .$$

Furthermore, there is a canonical isomorphism

$$\gamma : \mathrm{Gr}^W_{-2} \mathcal{L}og \xrightarrow{\sim} p^* \mathcal{H}^0_B (\mathbb{G}_m, F(0))^\vee$$

given by the fact that both sides are equal to p^* of the mixed structure on the (abelianized) fundamental group $\pi_1(\mathbb{G}_{m,\mathrm{top}}, 1)$ (see [WiI], chapter 2).

Observe that there is an isomorphism

$$\mathrm{res} : \mathcal{H}^0_B (\mathbb{G}_m, F(0)) \xrightarrow{\sim} F(1)$$

given by the map "residue at 0".

Finally, both $\mathrm{Gr}^W_\cdot \mathcal{L}og$ and $\prod_{k\geq 0} F(k)$ carry a canonical multiplicative structure: for $\mathrm{Gr}^W_\cdot \mathcal{L}og$, this is a formal consequence of

[WiI], Corollary 3.4.ii)	[WiI], Corollary 3.2.ii)

(see Remark b) at the end of chapter 3 of loc. cit.).

Our isomorphism

$$\kappa : \mathrm{Gr}^W_\cdot \mathcal{L}og \xrightarrow{\sim} \prod_{k\geq 0} F(k)$$

is the unique isomorphism compatible with ε, $(\mathrm{res})^\vee \circ \gamma$, and the multiplicative structure of both sides.

Using the framing of $\mathcal{L}og_b$ given by κ_b, we may identify pol_b with an element of

$$\prod_{k\geq 1} \mathrm{Ext}^1_{\mathrm{Sh}^s(C)} (F(1), F(k)) \ ,$$

or, after twisting and forgetting the component "$k = 0$", as an element of

$$\prod_{k\geq 1} \mathrm{Ext}^1_{\mathrm{Sh}^s(C)} (F(0), F(k)) \ .$$

Note that in the Hodge setting we do not lose any information by forgetting the component "$k = 0$" as there are no non–trivial extensions in $\mathrm{Sh}^s(C)$ of $F(0)$ by itself. This latter statement fails to hold in the l–adic context. It is however true that the zero–component of pol_b is trivial. One way to see this is via [WiIII], Corollary 2.2, where it is proved that there is in fact a mixed realization pol_b of which the above extensions are merely the Hodge and l–adic components. In the category of mixed realizations, there is a good concept of polarization, which ensures that there are no non–trivial extensions of pure realizations of the same weight. Alternatively, one uses Theorem 9.5, where it is proved that our pol_b lie in the image of the respective regulators. The claim then follows from the vanishing of $H^1_{\mathcal{M}}(C, 0)$.

THEOREM 2.5 (BEILINSON). *Under the isomorphism of A.2.12, we have in the Hodge setting:*

$$pol_b = \left((-1)^k \,\mathrm{Li}_k(\omega^b)\right)_{\omega,k} \in \prod_{k\geq 1} \left(\bigoplus_{\omega\in C(\mathbb{C})} \mathbb{C}/(2\pi i)^k \mathbb{Q} \right)^+ ,$$

where $\mathrm{Li}_k(z) := \sum_{n\geq 1} \frac{z^n}{n^k}$ *for* $|z| \leq 1$ *and* $z \neq 1$.

Proof. [B4], 4.1, or [BD1], 3.6.3.i), or [WiIV], Theorem 3.11. □

Note that one may identify $C(\mathbb{C})$ with $\{\sigma : \mathbb{Q}(\mu_d) \hookrightarrow \mathbb{C}\}$ by associating to ω the unique embedding mapping $T \in \mathbb{Q}(\mu_d) = \mathbb{Q}[T]/\Phi_d(T)$ to ω.

In the l–adic situation, choose a geometric point $\zeta \in C(\overline{\mathbb{Q}})$. It allows to identify C and

$$\operatorname{Spec}\left(\mathbb{Z}\left[\zeta, \frac{1}{ld}\right]\right) \ ,$$

and, furthermore, the category of continuous $\mathbb{Q}_l$–modules under the Galois group of $\mathbb{Q}(\zeta)$ that are mixed and unramified outside ld, and the category $\mathrm{Sh}^s(C) = \mathrm{Et}^{l,m}_{\mathbb{Q}_l}(C)$.

Given this, we think of $\mathrm{Ext}^1_{\mathrm{Sh}^s(C)}(\mathbb{Q}_l(0), \mathbb{Q}_l(k))$ as sitting inside

$$H^1_{cont}(\mathbb{Q}(\zeta), \mathbb{Q}_l(k)) \ .$$

Together with the natural map of Lemma B.4.9 we thus have an inclusion of $\mathrm{Ext}^1_{\mathrm{Sh}^s(C)}(\mathbb{Q}_l(0), \mathbb{Q}_l(k))$ into

$$\left(\varprojlim_{r\geq 1}\left(\mathbb{Q}(\mu_{l^\infty}, \zeta)^* / (\mathbb{Q}(\mu_{l^\infty}, \zeta)^*)^{l^r} \otimes \mu_{l^r}^{\otimes(k-1)}\right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\right)^{\mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty},\zeta)/\mathbb{Q}(\zeta))} .$$

Theorem 2.6 (Beilinson). *Under the above inclusion, we have in the l–adic setting:*

$$pol_b = \left((-1)^{k-1} \cdot \frac{1}{d^{k-1}} \cdot \frac{1}{(k-1)!} \cdot \sum_{\alpha^{l^r}=\zeta^b} ([1-\alpha] \otimes (\alpha^d)^{\otimes(k-1)})\right)_{r,k\geq 1} .$$

Proof. [B4], 4.1, or [BD1], 3.6.3.ii), or [WiIV], Theorem 4.5. □

Remarks: a) Using the defining property of *pol*, one can show (see [B4], 2.12 or [BD1], proof of 3.1.1) that it coincides with a specific subquotient of the generic pro-unipotent sheaf on $\mathbb{U}$. The specializations to spectra of cyclotomic fields of this subquotient were already studied in [D5], section 16. In particular, Theorems 2.5 and 2.6 are equivalent to the Hodge and l-adic versions of [D5], Théorème 16.24.
b) One of the main results of this work will be (Theorem 9.5) that the elements in 2.5 and 2.6, for fixed b and d, are the respective regulators of one and the same element in motivic cohomology. This implies that Soulé's construction of cyclotomic elements in the K–theory with $\mathbb{Z}_l$–coefficients of an abelian number field ([Sou2], Lemma 1, [Sou5]) actually factors over the image of K–theory proper (Corollary 9.8). As shown in [BlK], § 6, Theorem 9.5 also implies that the Tamagawa number conjecture modulo powers of 2 is also true for odd Tate twists (see our Corollary 9.9). Finally, 9.5 is used in [KNF], Theorem 6.4 to prove the modified version of the Lichtenbaum conjecture for abelian number fields.
c) There are relative versions of 2.1 and 2.3 for schemes over a base scheme S smooth over B. They allow to directly define the small polylogarithmic extension pol_S on $\mathbb{U} \times_B S$, which however turns out to be the base change to S of *pol*.

Remark: In our definition of *pol*, we chose not to follow [BD1], 3.1. The approach via the universal property of $\mathcal{L}og$ and the computation of its cohomology rather imitates that of Beilinson and Levin in the elliptic case ([BL], 1.2, 1.3). In fact, one of the predecessors of loc. cit. contains a unified definition of $\mathcal{L}og$ and *pol* for relative curves of arbitrary genus ([BLp], 1).

3 The Geometric Set-Up

For easier reference, we assemble the notation used in the next sections.
As before, we let

$A := \mathbb{R}$,	$l :=$ a fixed prime number, $A := \mathbb{Z}\left[\frac{1}{l}\right]$,

$$B := \operatorname{Spec}(A) \, ,$$
$$\mathbb{G}_m := \mathbb{G}_{m,B} \ , \ \mathbb{U} := \mathbb{P}^1_B \setminus \{0,1,\infty\}_B \, .$$

Furthermore, we let $\underline{S}$ denote a smooth separated scheme over B of pure relative dimension $d(\underline{S})$,

$$\underline{\alpha}, \underline{\beta} \in \mathbb{G}_m(\underline{S}) \, ,$$

$S \subset \underline{S}$ the open subscheme of $\underline{S}$ where $\underline{\alpha}$ and $\underline{\beta}$ are disjoint. We assume S to be dense in $\underline{S}$.

$$j : S \hookrightarrow \underline{S} \, ,$$
$$i : \underline{S} \setminus S \hookrightarrow \underline{S} \, ,$$

where $\underline{S} \setminus S$ is equipped with the reduced scheme structure.

$$\underline{Z} := \underline{\alpha}(\underline{S}) \cup \underline{\beta}(\underline{S})$$

with the reduced scheme structure,

$$\underline{V} := \mathbb{G}_{m,\underline{S}} \setminus \underline{Z} \, .$$

For $n \geq 0$, define

$$\underline{p}^n : \mathbb{G}^n_{m,\underline{S}} \to \underline{S} \, ,$$
$$\underline{v}^n : \underline{V}^n \hookrightarrow \mathbb{G}^n_{m,\underline{S}} \, ,$$
$$\underline{z}^{(n)} : \underline{Z}^{(n)} := \mathbb{G}^n_{m,\underline{S}} \setminus \underline{V}^n \hookrightarrow \mathbb{G}^n_{m,\underline{S}} \, ,$$

where $\underline{Z}^{(n)}$ carries the reduced scheme structure. (So $\underline{p}^0 = \underline{v}^0 = \mathrm{id}_{\underline{S}}$, and $\underline{Z}^{(0)} = \emptyset$.)

The base change of the above objects and morphisms to S is denoted by the same letters not underlined:

$$\begin{aligned}
&\alpha, \beta : S \to \mathbb{G}_{m,S} \ , \\
&Z := \alpha(S) \amalg \beta(S) \ , \\
&V := \mathbb{G}_{m,S} \setminus Z \ , \\
&p^n : \mathbb{G}^n_{m,S} \to S \ , \\
&v^n : V^n \hookrightarrow \mathbb{G}^n_{m,S} \ , \\
&z^{(n)} : Z^{(n)} \hookrightarrow \mathbb{G}^n_{m,S} \ .
\end{aligned}$$

Also, we define partial compactifications of p^n:

$$\begin{aligned}
&g^n : \mathbb{G}^n_{m,S} \hookrightarrow \mathbb{A}^n_S \ , \\
&h^{(n)} : H^{(n)} := \mathbb{A}^n_S \setminus \mathbb{G}^n_{m,S} \hookrightarrow \mathbb{A}^n_S \ ,
\end{aligned}$$

where again $H^{(n)}$ has the reduced structure,

$$\begin{aligned}
&\overline{p}^n : \mathbb{A}^n_S \to S \ , \\
&\overline{V} := \mathbb{A}^1_S \setminus Z \ , \\
&\overline{v}^n : \overline{V}^n \hookrightarrow \mathbb{A}^n_S \ , \\
&\overline{z}^{(n)} : \overline{Z}^{(n)} := \mathbb{A}^n_S \setminus \overline{V}^n \hookrightarrow \mathbb{A}^n_S \ ,
\end{aligned}$$

where $\overline{Z}^{(n)}$ is equipped with the reduced structure. (So $\overline{Z}^{(1)} = Z^{(1)} = Z$.)

Remarks: a) The underlined objects should remind the reader that the partial compactification comes from the compactification j of the base S. The overlined objects refer to compactification upstairs, induced from g^n.
b) For fixed n, we have a natural action of the symmetric group $\mathfrak{S}_n$ on our geometric situation.

For the purposes of K-theory in section 7 we will have to replace the singular scheme $Z^{(n)}$ by some smooth simplicial scheme. Put

$$Z_0^{(n)} = Z \times_S \mathbb{G}^{n-1}_{m,S} \amalg \mathbb{G}_{m,S} \times Z \times_S \mathbb{G}^{n-2}_{m,S} \amalg \ldots \amalg \mathbb{G}^{n-1}_{m,S} \times_S Z$$

Note that $Z_0^{(n)}$ is a proper covering of $Z^{(n)}$. This is the easiest case of a morphism of schemes with cohomological descent, meaning that for any reasonable cohomology theory the cohomology of $Z^{(n)}$ will agree with the cohomology of the smooth simplicial scheme

$$Z_\cdot^{(n)} = \operatorname{cosk}_0(Z_0^{(n)}/\mathbb{G}^n_{m,S}) \ ,$$

i.e.,

$$Z_k^{(n)} = Z_0^{(n)} \times_{\mathbb{G}^n_{m,S}} \cdots \times_{\mathbb{G}^n_{m,S}} Z_0^{(n)} \quad (k+1\text{-fold product}).$$

Put $Z_{\cdot}^{(0)} = \star$ (corresponding to the empty scheme). We will also use the simplicial scheme $\overline{Z}_{\cdot}^{(n)}$ which is attached to $\overline{Z}^{(n)}$ sitting in $\mathbb{A}_S^n$ in the same way. Finally let

$$\begin{aligned} \mathbb{G}_{m,S}^{\vee n} &= \operatorname{Cone}(Z_{\cdot}^{(n)} \longrightarrow \mathbb{G}_{m,S}^n) \\ \mathbb{A}_S^{\vee n} &= \operatorname{Cone}(\overline{Z}_{\cdot}^{(n)} \longrightarrow \mathbb{A}_S^n) \end{aligned}$$

where the cone is taken in the category of pointed simplicial sheaves on the big Zariski site (cf. the discussion in appendix B.1).

4 Geometric Origin of the Logarithmic Sheaf

In section 2, we defined a pro–sheaf

$$\mathcal{L}og \in \text{pro}-U\,\mathrm{Sh}^s(\mathbb{G}_m)$$

and an element

$$\begin{aligned} pol &\in \operatorname{Ext}^1_{\mathrm{Sh}(\mathbb{U})}(F(1), \mathcal{L}og(1)\,|_{\mathbb{U}}) \\ &= \varprojlim_n \operatorname{Ext}^1_{\mathrm{Sh}(\mathbb{U})}(F(1), (\mathcal{L}og/W_{-n}\mathcal{L}og)(1)\,|_{\mathbb{U}}) \,. \end{aligned}$$

The aim of this section is to identify $\mathcal{L}og\,|_{\mathbb{U}}$, or rather, its Noetherian quotients, as relative cohomology objects with coefficients in Tate twists of certain schemes over $\mathbb{U}$ (Theorem 4.11).

Recall that according to our conventions, we have

$$F(0) = F(0)_{\mathbb{U}}[1] \quad ,$$

and hence we may view *pol* as an element of

$$\operatorname{Hom}_{D^b\,\mathrm{Sh}(\mathbb{U})}(F(0)_{\mathbb{U}}, \mathcal{L}og\,|_{\mathbb{U}}) = H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{L}og\,|_{\mathbb{U}}) \,,$$

where we have used the notation introduced in Definition 1.2.

For the schemes of section 3, we have the following

Definition 4.1. *For $n \geq 0$,*

$$\mathcal{G}^{(n)} := \mathcal{H}_S^0 \left(\mathbb{G}_{m,S}^n, v_!^n F(n)\right)^{sgn} = \mathcal{H}_S^{n+d(\underline{S})} \left(\mathbb{G}_{m,S}^n, v_!^n F(n)_{V^n}\right)^{sgn} \quad ,$$

where the superscript sgn refers to the sign–eigenspace under the natural action of the symmetric group $\mathfrak{S}_n$ on $\mathbb{G}_{m,S}^n$ and V^n.
Observe in particular that $\mathcal{G}^{(0)} = F(0)$.

The following is an immediate consequence of the Künneth formula:

Lemma 4.2. *There is a canonical isomorphism*

$$\mathcal{G}^{(n)} \xrightarrow{\sim} \operatorname{Sym}^n \mathcal{G}^{(1)} \,.$$

We want to compute $\mathcal{G}^{(n)}$, and simultaneously construct, for each $n \geq 1$, a projection

$$\mathcal{G}^{(n)} \twoheadrightarrow \mathcal{G}^{(n-1)}$$

via the "residue at 0", whose projective limit over n we shall then identify, for special α and β, and $S = \mathbb{U}$, with the restriction $\mathcal{L}og\,|_{\mathbb{U}}$ of the logarithmic pro–sheaf to $\mathbb{U}$.

Let $H^{(n)}_{\text{sing}}$ be the singular part of $H^{(n)}$ and $H^{(n)}_{\text{reg}} := H^{(n)} \setminus H^{(n)}_{\text{sing}}$ the smooth part. For any subscheme of $\mathbb{A}^n_S$, the subscript reg will mean the complement of $H^{(n)}_{\text{sing}}$. We work with the following geometric arrangement:

$$\begin{tikzcd}
\overline{V}^{\,n}_{\text{reg}} \cap H^{(n)}_{\text{reg}} \arrow[r] \arrow[d, "\overline{v}^{\,n}_{H,\text{reg}}"'] & \overline{V}^{\,n}_{\text{reg}} \arrow[d, "\overline{v}^{\,n}_{\text{reg}}"'] & V^n \arrow[l] \arrow[d, "v^n"'] \\
H^{(n)}_{\text{reg}} \arrow[r, "h^{(n)}_{\text{reg}}"] & \mathbb{A}^n_{S,\text{reg}} & \mathbb{G}^n_{m,S} \arrow[l, "g^n_{\text{reg}}"']
\end{tikzcd}$$

Both squares are cartesian. All maps are either open or closed immersions, and each line gives in fact a smooth pair of S–schemes.

Lemma 4.3. *For any complex $M \in D^b\,\mathrm{Sh}(\mathbb{A}^n_{S,\text{reg}})$ such that $(\overline{v}^{\,n}_{\text{reg}})^* M$ is a shift of a smooth sheaf on $\overline{V}^{\,n}_{\text{reg}}$, there is an exact triangle*

$$(*) \qquad \begin{tikzcd}
(h^{(n)}_{\text{reg}})_* \, (\overline{v}^{\,n}_{H,\text{reg}})_! \left(\overline{v}^{\,n}_{H,\text{reg}} \circ h^{(n)}_{\text{reg}}\right)^* M(-1)[-2] \arrow[rr] & & (\overline{v}^{\,n}_{\text{reg}})_! \, (\overline{v}^{\,n}_{\text{reg}})^* M \arrow[dl] \\
& (g^n_{\text{reg}})_* \; v^n_! \left(v^n \circ g^n_{\text{reg}}\right)^* M \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

Proof. This is $(\overline{v}^{\,n}_{\text{reg}})_!$ applied to the exact triangle obtained from purity for the closed immersion

$$\overline{V}^{\,n}_{\text{reg}} \cap H^{(n)}_{\text{reg}} \longrightarrow \overline{V}^{\,n}_{\text{reg}}$$

of smooth schemes. ◻

We apply this lemma to $M = F(n)_{\mathbb{A}^n_{S,\text{reg}}}$, and evaluate the cohomological functors $H^i_{\text{abs}}(\mathbb{A}^n_{S,\text{reg}}, \,\cdot\,)^{\text{sgn}}$ on the triangle $(*)$. Following 1.2.c), we write everything as relative cohomology with Tate coefficients:

$$\begin{aligned}
\ldots \longrightarrow & H^i_{\text{abs}}(\mathbb{A}^n_{S,\text{reg}} \text{ rel } \overline{Z}^{(n)}_{\text{reg}}, n)^{\text{sgn}} \longrightarrow H^i_{\text{abs}}(\mathbb{G}^n_{m,S} \text{ rel } Z^{(n)}, n)^{\text{sgn}} \\
& \qquad \longrightarrow H^{i-1}_{\text{abs}}(H^{(n)}_{\text{reg}} \text{ rel } (\overline{Z}^{\,(n)} \cap H^{(n)}_{\text{reg}}), n-1)^{\text{sgn}} \\
\longrightarrow & H^{i+1}_{\text{abs}}(\mathbb{A}^n_{S,\text{reg}} \text{ rel } \overline{Z}^{(n)}_{\text{reg}}, n)^{\text{sgn}} \longrightarrow \ldots
\end{aligned}$$

We refer to this as the *absolute residue sequence.*

Application of the cohomological functors $\mathcal{H}^i_S(\mathbb{A}^n_{S,\text{reg}}, \,\cdot\,)^{\text{sgn}}$ to the same exact triangle yields a long exact sequence of sheaves on S that we call the *relative residue* *sequence*:

$$\begin{aligned}\ldots \longrightarrow \mathcal{H}^i_S\left(\mathbb{A}^n_{S,\mathrm{reg}}, (\overline{v}^n_{\mathrm{reg}})_! F(n)_{\overline{V}^n_{\mathrm{reg}}}\right)^{\mathrm{sgn}} &\longrightarrow \mathcal{H}^i_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)_{V^n}\right)^{\mathrm{sgn}} \\ &\longrightarrow \mathcal{H}^{i-1}_S\left(H^{(n)}_{\mathrm{reg}}, (\overline{v}^n_{H,\mathrm{reg}})_! F(n-1)_{\overline{V}^n_{\mathrm{reg}}\cap H^{(n)}_{\mathrm{reg}}}\right)^{\mathrm{sgn}} \\ \longrightarrow \mathcal{H}^{i+1}_S\left(\mathbb{A}^n_{S,\mathrm{reg}}, (\overline{v}^n_{\mathrm{reg}})_! F(n)_{\overline{V}^n_{\mathrm{reg}}}\right)^{\mathrm{sgn}} &\longrightarrow \ldots\end{aligned}$$

Note that $\mathcal{G}^{(n)} = \mathcal{H}^{n+d(\underline{S})}_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)_{V^n}\right)^{\mathrm{sgn}}$ occurs in this sequence.

We are now going to further analyse, and reshape these sequences. The final form will be achieved in Proposition 4.8 and Theorem 4.9.

First, we need to identify the terms

$$H^{i-1}_{\mathrm{abs}}(H^{(n)}_{\mathrm{reg}} \text{ rel } (\overline{Z}^{(n)} \cap H^{(n)}_{\mathrm{reg}}), n-1)^{\mathrm{sgn}}\ , \quad n \geq 1\ ,$$
$$\mathcal{H}^{i-1}_S\left(H^{(n)}_{\mathrm{reg}}, (\overline{v}^n_{H,\mathrm{reg}})_! F(n-1)_{\overline{V}^n_{\mathrm{reg}}\cap H^{(n)}_{\mathrm{reg}}}\right)^{\mathrm{sgn}}\ , \quad n \geq 1\ .$$

The complement of $\overline{Z}^{(n)} \cap H^{(n)}_{\mathrm{reg}}$ in $H^{(n)}_{\mathrm{reg}}$ is given by

$$\overline{v}^n_{H,\mathrm{reg}} : \overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}} \to H^{(n)}_{\mathrm{reg}}\ .$$

Since $\overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}} = \coprod_{k=1}^n V^{n-1}$ under the identification

$$H^{(n)}_{\mathrm{reg}} = \coprod_{k=1}^n \mathbb{G}^{n-1}_{m,S}\ ,$$

and these components are permuted transitively by $\mathfrak{S}_n$, we conclude

Lemma 4.4.

a) $$(\overline{v}^n_{H,\mathrm{reg}})_! F(n-1)_{\overline{V}^n_{\mathrm{reg}}\cap H^{(n)}_{\mathrm{reg}}} = \left(\coprod_{k=1}^n v^{n-1}\right)_! F(n-1)_{\coprod_{k=1}^n V^{n-1}}\ .$$

b) $$H^{i-1}_{\mathrm{abs}}(H^{(n)}_{\mathrm{reg}} \textit{ rel } (\overline{Z}^{(n)} \cap H^{(n)}_{\mathrm{reg}}), n-1) = \bigoplus_{k=1}^n H^{i-1}_{\mathrm{abs}}(\mathbb{G}^{n-1}_{m,S} \textit{ rel } Z^{(n-1)}, n-1)\ ,$$

and hence the sign–eigenspace $H^{i-1}_{\mathrm{abs}}(H^{(n)}_{\mathrm{reg}} \textit{ rel } (\overline{Z}^{(n)} \cap H^{(n)}_{\mathrm{reg}}), n-1)^{sgn}$ *is isomorphic to*

$$H^{i-1}_{\mathrm{abs}}(\mathbb{G}^{n-1}_{m,S} \textit{ rel } Z^{(n-1)}, n-1)^{sgn}\ ,$$

where the last sgn refers to the action of $\mathfrak{S}_{n-1}$. *The isomorphism is given by projection onto the components unequal to* k, *for some* choice $k \in \{1, \ldots, n\}$. *It is independent of the choice of* k.

c) $$\mathcal{R}_S\left(H^{(n)}_{\mathrm{reg}}, (\overline{v}^n_{H,\mathrm{reg}})_! F(n-1)_{\overline{V}^n_{\mathrm{reg}}\cap H^{(n)}_{\mathrm{reg}}}\right) = \bigoplus_{k=1}^n \mathcal{R}_S\left(\mathbb{G}^{n-1}_{m,S}, v^{n-1}_! F(n-1)_{V^{n-1}}\right)\ .$$

As in b), the sign–eigenspace $\mathcal{H}_S^{i-1}\left(H_{\mathrm{reg}}^{(n)}, (\overline{v}_{H,\mathrm{reg}}^n)_! F(n-1)_{\overline{V}_{\mathrm{reg}}^n \cap H_{\mathrm{reg}}^{(n)}}\right)^{sgn}$ *is canonically isomorphic to*

$$\mathcal{H}_S^{i-1}\left(\mathbb{G}_{m,S}^{n-1}, v_!^{n-1} F(n-1)_{V^{n-1}}\right)^{sgn} .$$

For $i = n + d(\underline{S})$*, the latter equals* $\mathcal{G}^{(n-1)}$*.*

Proof. The only point that remains to be shown is the independence of the isomorphisms in b) and c) of the choice of k. Recall the identity

$$R\Gamma_{\mathrm{abs}}\left(\mathbb{G}_{m,S}^n \text{ rel } Z^{(n)}, n\right)^{\mathrm{sgn}} = R\Gamma_{\mathrm{abs}}\left(S, \mathcal{R}_S\left(\mathbb{G}_{m,S}^n, v_!^n F(n)_{V^n}\right)\right)^{\mathrm{sgn}} .$$

We are going to prove in 4.6.d) that $\mathcal{H}_S^q(\mathbb{G}_{m,S}^n, v_!^n F(n)_{V^n})^{\mathrm{sgn}} = 0$ for $q \neq n + d(\underline{S})$. So the associated spectral sequence degenerates, and shows that the independence of the map in b) follows from that of the map in c).
For c), we only need to consider $\mathcal{G}^{(n)} = \mathcal{H}_S^{n+d(\underline{S})}(\mathbb{G}_{m,S}^n, v_!^n F(n)_{V^n})^{\mathrm{sgn}}$. There, our claim follows from Lemma 4.2, and the graded–compatibility of the cup product with boundary morphisms ([GH], Proposition 2.2 and Corollary 2.3). ☐

Remark: The arguments of this section would become simpler if we could use an object $\mathcal{R}_S^{\mathrm{sgn}}$ in c). However, we do not know whether it is possible to make a decomposition into eigenspaces in our triangulated categories.

By the identification of the lemma, the residue sequences define canonical *residue maps*

$$\mathrm{res} : H_{\mathrm{abs}}^i(\mathbb{G}_{m,S}^n \text{ rel } Z^{(n)}, n)^{\mathrm{sgn}} \longrightarrow H_{\mathrm{abs}}^{i-1}(\mathbb{G}_{m,S}^{n-1} \text{ rel } Z^{(n-1)}, n-1)^{\mathrm{sgn}} ,$$
$$\mathrm{res} : \mathcal{H}_S^i\left(\mathbb{G}_{m,S}^n, v_!^n F(n)_{V^n}\right)^{\mathrm{sgn}} \longrightarrow \mathcal{H}_S^{i-1}\left(\mathbb{G}_{m,S}^{n-1}, v_!^{n-1} F(n-1)_{V^{n-1}}\right)^{\mathrm{sgn}}$$

fitting into the relative and absolute residue sequences. In particular, observe that we have a residue map

$$\mathrm{res} : \mathcal{G}^{(n)} \longrightarrow \mathcal{G}^{(n-1)} .$$

Now we concern ourselves with the identification of the remaining terms

$$H_{\mathrm{abs}}^i(\mathbb{A}_{S,\mathrm{reg}}^n \text{ rel } \overline{Z}_{\mathrm{reg}}^{(n)}, n)^{\mathrm{sgn}} , n \geq 0 ,$$
$$\mathcal{H}_S^i\left(\mathbb{A}_{S,\mathrm{reg}}^n, (\overline{v}_{\mathrm{reg}}^n)_! F(n)_{\overline{V}_{\mathrm{reg}}^n}\right)^{\mathrm{sgn}} , n \geq 0$$

of the residue sequences.

We use the following filtration of $\mathbb{A}_S^n$ by open subschemes:

$$F_k \mathbb{A}_S^n := \{(x_1, \ldots, x_n) \in \mathbb{A}_S^n \,|\, \text{at most } k \text{ coordinates vanish}\} .$$

So we have $F_n \mathbb{A}_S^n = \mathbb{A}_S^n$ and $F_0 \mathbb{A}_S^n = \mathbb{G}_{m,S}^n$.

The "graded pieces" of this filtration are

$$\begin{aligned} G_k \mathbb{A}_S^n &:= F_k \mathbb{A}_S^n \setminus F_{k-1} \mathbb{A}_S^n \\ &= \{(x_1, \ldots, x_n) \in \mathbb{A}_S^n \,|\, \text{precisely } k \text{ coordinates vanish}\} . \end{aligned}$$

$G_k\mathbb{A}^n_S$ is equipped with the reduced scheme structure. Note that it splits into several disjoint pieces. For $k \geq 2$ and any such piece, there is a transposition of $\mathfrak{S}_n$ acting trivially. By using triangles similar to $(*)$ for the inclusions

$$G_k\mathbb{A}^n_S \hookrightarrow F_k\mathbb{A}^n_S \hookleftarrow F_{k-1}\mathbb{A}^n_S \ ,$$

we conclude inductively that the sign–eigenpart of the cohomology of $H^{(n)}_{\text{sing}}$ is trivial:

Lemma 4.5. *The adjunction morphism induces isomorphisms*

$$H^i_{\text{abs}}(\mathbb{A}^n_S \ rel\ \overline{Z}^{(n)}, n)^{sgn} \xrightarrow{\sim} H^i_{\text{abs}}(\mathbb{A}^n_{S,\text{reg}} \ rel\ \overline{Z}^{(n)}_{\text{reg}}, n)^{sgn} ,$$
$$\mathcal{H}^i_S\,(\mathbb{A}^n_S, \overline{v}_!^n F(n)_{\overline{V}^n})^{sgn} \xrightarrow{\sim} \mathcal{H}^i_S\left(\mathbb{A}^n_{S,\text{reg}}, (\overline{v}^n_{\text{reg}})_! F(n)_{\overline{V}^n_{\text{reg}}}\right)^{sgn} .$$

By 4.4.b) and 4.5, the absolute residue sequence takes the form

$$\begin{aligned}\cdots \to & H^i_{\text{abs}}(\mathbb{A}^n_S \text{ rel } \overline{Z}^{(n)}, n)^{\text{sgn}} \to H^i_{\text{abs}}(\mathbb{G}^n_{m,S} \text{ rel } Z^{(n)}, n)^{\text{sgn}} \\ & \xrightarrow{\text{res}} H^{i-1}_{\text{abs}}(\mathbb{G}^{n-1}_{m,S} \text{ rel } Z^{(n-1)}, n-1)^{\text{sgn}} \\ \to & H^{i+1}_{\text{abs}}(\mathbb{A}^n_S \text{ rel } \overline{Z}^{(n)}, n)^{\text{sgn}} \to \dots\end{aligned}$$

Similarly, the relative residue sequence looks as follows:

$$\begin{aligned}\dots \longrightarrow & \mathcal{H}^i_S\,(\mathbb{A}^n_S, \overline{v}_!^n F(n)_{\overline{V}^n})^{\text{sgn}} \longrightarrow \mathcal{H}^i_S\left(\mathbb{G}^n_{m,S}, v_!^n F(n)_{V^n}\right)^{\text{sgn}} \\ & \xrightarrow{\text{res}} \mathcal{H}^{i-1}_S\left(\mathbb{G}^{n-1}_{m,S}, v_!^{n-1} F(n-1)_{V^{n-1}}\right)^{\text{sgn}} \\ \longrightarrow & \mathcal{H}^{i+1}_S\,(\mathbb{A}^n_S, \overline{v}_!^n F(n)_{\overline{V}^n})^{\text{sgn}} \longrightarrow \dots\end{aligned}$$

For the computation of the term

$$\mathcal{H}^i_S\,(\mathbb{A}^n_S, \overline{v}_!^n F(n)_{\overline{V}^n})^{\text{sgn}} \ ,$$

we use the Künneth formula:

Lemma 4.6. a) $\mathcal{R}_S\,(\mathbb{A}^n_S, \overline{v}_!^n F(n)) = \mathcal{H}^0_S\,(\mathbb{A}^n_S, \overline{v}_!^n F(n))\,[0]$, *and the Künneth formula gives an isomorphism*

$$\mathcal{H}^0_S\,(\mathbb{A}^n_S, \overline{v}_!^n F(n)) = \mathcal{H}^0_S\,(\mathbb{A}^n_S, \overline{v}_!^n F(n))^{sgn} \xrightarrow{\sim} \operatorname{Sym}^n \mathcal{H}^0_S\left(\mathbb{A}^1_S, \overline{v}_!^1 F(1)\right) .$$

b) *The choice of an ordering of the sections α and β gives an isomorphism*

$$\mathcal{R}_S\left(\mathbb{A}^1_S, \overline{v}_!^1 F(1)\right) = \mathcal{H}^0_S\left(\mathbb{A}^1_S \overline{v}_!^1 F(1)\right)[0] \xrightarrow{\sim} F(1)[0] .$$

Up to sign, it is canonical.
c) *The isomorphisms of a) and b) induce an isomorphism*

$$\mathcal{H}^{n+d(\underline{S})}_S\,(\mathbb{A}^n_S, \overline{v}_!^n F(n)_{\overline{V}^n}) = \mathcal{H}^0_S\,(\mathbb{A}^n_S, \overline{v}_!^n F(n)) \xrightarrow{\sim} F(n) .$$

It depends on the choice made in b) only up to the sign $(-1)^n$. The group $\mathfrak{S}_n$ acts on these objects via the sign character.
d) *For $i \neq 0$, we have*

$$\mathcal{H}^{i+n+d(\underline{S})}_S\left(\mathbb{G}^n_{m,S}, v_!^n F(n)_{V^n}\right)^{sgn} = \mathcal{H}^i_S\left(\mathbb{G}^n_{m,S}, v_!^n F(n)\right)^{sgn} = 0 .$$

Proof. For b), consider the long exact cohomology sequence associated to the triangle

$$(**)\qquad \begin{array}{ccc} \overline{v}_!^1 F(1) & \longrightarrow & F(1) \\ {}_{[1]}\nwarrow & & \swarrow \\ & \overline{z}_*^{(1)} F(1) & \end{array}$$

We have

$$\mathcal{H}^i_S\left(\mathbb{A}^1_S, F(1)\right) = \begin{cases} F(1)\,, & i=-1 \\ 0\,, & i \neq -1 \end{cases}$$

and

$$\mathcal{H}^i_S\left(\mathbb{A}^1_S, \overline{z}_*^{(1)} F(1)\right) = \begin{cases} \bigoplus_{\alpha,\beta} F(1)\,, & i=0 \\ 0\,, & i \neq 0 \end{cases}.$$

The long exact cohomology sequence thus reads

$$0 \to \mathcal{H}^{-1}_S\left(\mathbb{A}^1_S, \overline{v}_!^1 F(1)\right) \to F(1) \stackrel{\Delta}{\to} \bigoplus_{\alpha,\beta} F(1) \to \mathcal{H}^0_S\left(\mathbb{A}^1_S, \overline{v}_!^1 F(1)\right) \to 0\,.$$

If we let $\{\alpha, \beta\} = \{s_1, s_2\}$, then we identify the cokernel of

$$\Delta : F(1) \longrightarrow \bigoplus_{\alpha,\beta} F(1) = \bigoplus_{i=1}^{2} F(1)$$

with $F(1)$ by mapping $(f_{s_1}, f_{s_2}) \in \bigoplus_{i=1}^2 F(1)$ to $f_{s_2} - f_{s_1}$.
a) follows from b) since $\bigotimes^n F(1) = \mathrm{Sym}^n F(1)$.
c) is a consequence of a) and b).
d) follows from a) and the relative residue sequence by induction on n. $\square$

On the level of absolute cohomology, the isomorphism of 4.6.c) induces an isomorphism

$$H^{i+n}_{\mathrm{abs}}\left(\mathbb{A}^n_S \text{ rel } \overline{Z}^{(n)}, n\right) = H^{i+n}_{\mathrm{abs}}\left(\mathbb{A}^n_S \text{ rel } \overline{Z}^{(n)}, n\right)^{\mathrm{sgn}} \stackrel{\sim}{\longrightarrow} H^i_{\mathrm{abs}}(S, n)\,.$$

This gives the final shape of the absolute residue sequence:

$$\begin{aligned} \cdots \longrightarrow H^i_{\mathrm{abs}}(S,n) \stackrel{\delta}{\longrightarrow} & H^{i+n}_{\mathrm{abs}}(\mathbb{G}^n_{m,S} \text{ rel } Z^{(n)}, n)^{\mathrm{sgn}} \\ & \stackrel{\mathrm{res}}{\longrightarrow} H^{i+n-1}_{\mathrm{abs}}(\mathbb{G}^{n-1}_{m,S} \text{ rel } Z^{(n-1)}, n-1)^{\mathrm{sgn}} \\ \longrightarrow H^{i+1}_{\mathrm{abs}}(S,n) \stackrel{\delta}{\longrightarrow} \cdots & \end{aligned}$$

By 4.6.d), the relative residue sequence collapses into the short exact sequence of sheaves on S:

$$0 \longrightarrow F(n) \longrightarrow \mathcal{G}^{(n)} \stackrel{\mathrm{res}}{\longrightarrow} \mathcal{G}^{(n-1)} \longrightarrow 0\,.$$

In order to identify the long exact absolute cohomology sequence associated to this sequence with the absolute residue sequence, we need the following:

LEMMA 4.7. *Let $K \in D^b \operatorname{Sh}(X)$ be a complex of sheaves on a separated, reduced and flat B–scheme X. Suppose there is an action of a finite group G on K. Let χ be the character of an absolutely irreducible representation of G over F. For any object $\mathbb{V}$ with a G–action of an F-linear abelian category, denote by $\mathbb{V}(\chi)$ the χ–isotypical component of $\mathbb{V}$, i.e., the image under the projector*

$$e_\chi := \frac{1}{\#G} \sum_{g \in G} \chi(g^{-1}) \cdot g \ .$$

Suppose that $(\mathcal{H}^i K)(\chi)$ vanishes for all $i \neq 0$. Then

$$\operatorname{Hom}_{D^b}(F, K[i])(\chi) = \operatorname{Hom}_{D^b}(F, (\mathcal{H}^0 K)(\chi)[i])$$

Proof. By applying e_χ and $1 - e_\chi$, one checks the statement for a complex of the special form $K \cong \mathcal{H}^0 K$. For the general case, consider the spectral sequence for $\operatorname{Hom}_{D^b}(F, \ \cdot \ [i])$ induced by the truncation functors $\tau_{\leq n}$. It degenerates after applying e_χ. □

Now that we know that formation of absolute cohomology commutes with formation of sign eigenspaces, we have:

PROPOSITION 4.8. *The absolute residue sequence is the long exact sequence in absolute cohomology attached to the short exact sequence*

$$0 \longrightarrow F(n) \longrightarrow \mathcal{G}^{(n)} \xrightarrow{\text{res}} \mathcal{G}^{(n-1)} \longrightarrow 0 \, .$$

We conclude the computational part of this section by collecting our results:

THEOREM 4.9. a) *For $n \geq 0$, we have*

$$\mathcal{H}^0_S \left(\mathbb{G}^n_{m,S}, v^n_! F(n) \right)^{sgn} = \mathcal{G}^{(n)} \ ,$$

and $\mathcal{H}^i_S \left(\mathbb{G}^n_{m,S}, v^n_! F(n) \right)^{sgn} = 0$ for $i \neq 0$.
b) *The residue at 0, i.e., the boundary map of $(*)$, gives an epimorphism*

$$\text{res} : \mathcal{G}^{(n)} \twoheadrightarrow \mathcal{G}^{(n-1)}$$

for $n \geq 1$.
c) *The Künneth formula gives an isomorphism*

$$\mathcal{H}^0_S \left(\mathbb{A}^n_S, \overline{v}_!{}^n F(n) \right) = \mathcal{H}^0_S \left(\mathbb{A}^n_S, \overline{v}_!{}^n F(n) \right)^{sgn} \xrightarrow{\sim} \ker(\text{res})$$

for $n \geq 1$. A choice of an ordering of the sections α and β induces an isomorphism

$$F(n) \xrightarrow{\sim} \ker(\text{res}) \ ,$$

which depends on this choice only up to the sign $(-1)^n$.
d) *Let $\mathcal{G}^{(n)} \xrightarrow{\sim} \operatorname{Sym}^n \mathcal{G}^{(1)}$ be the canonical isomorphism of 4.2, and*

$$\operatorname{Sym}^n F(0) \xrightarrow{\sim} F(0) \ ,$$
$$\operatorname{Sym}^n F(1) \xrightarrow{\sim} F(n)$$

the isomorphisms given by multiplication. Then the diagrams

$$\begin{array}{ccc} \mathcal{G}^{(n)} & \longrightarrow & F(0) \\ \downarrow\wr & & \uparrow\wr \\ \mathrm{Sym}^n\mathcal{G}^{(1)} & \longrightarrow & \mathrm{Sym}^n F(0) \end{array}$$

and

$$\begin{array}{ccc} F(n) & \longrightarrow & \mathcal{G}^{(n)} \\ \uparrow\wr & & \downarrow\wr \\ \mathrm{Sym}^n F(1) & \longrightarrow & \mathrm{Sym}^n\mathcal{G}^{(1)} \end{array}$$

commute. Here, the horizontal maps are given by the successive residue maps, and by c) respectively.
e) *Let* $W_{-2n-1}\,\mathcal{G}^{(n)} := 0$,

$$W_{-2k}\,\mathcal{G}^{(n)} := W_{-2k+1}\,\mathcal{G}^{(n)} := \ker(\mathcal{G}^{(n)} \to \mathcal{G}^{(k-1)}) \quad for \quad 1 \le k \le n\,,$$

and $W_0\,\mathcal{G}^{(n)} := \mathcal{G}^{(n)}$. *The choice in c) induces isomorphisms*

$$\mathrm{Gr}^W_\cdot\,\mathcal{G}^{(n)} \xrightarrow{\sim} \bigoplus_{i=0}^{n} F(i)\;,$$

which by their construction fit into commutative diagrams

$$\begin{array}{ccc} \mathrm{Gr}^W_\cdot\,\mathcal{G}^{(n)} & \xrightarrow{\sim} & \bigoplus_{i=0}^{n} F(i) \\ \mathrm{Gr}^W_\cdot\,\mathrm{res}\downarrow & & \downarrow\mathrm{can} \\ \mathrm{Gr}^W_\cdot\,\mathcal{G}^{(n-1)} & \xrightarrow{\sim} & \bigoplus_{i=0}^{n-1} F(i) \end{array}$$

The filtration $W_\cdot$ *is therefore the weight filtration of* $\mathcal{G}^{(n)}$.

Proof. a), b) and c) follow from the previous results. The commutativity of the first diagram in d) follows from the definition of the residue map. For the second diagram, we use the fact that the Künneth formula of 4.2 is compatible with the Künneth formula of the proof of 4.6.a). For e), apply induction on n. □

Recall that S is the open subscheme of $\underline{S}$ where the sections $\underline{\alpha}$ and $\underline{\beta}$ of $\mathbb{G}_{m,\underline{S}}$ are disjoint. For special S, α and β, the following is the main step towards the identification of the projective limit of the $\mathcal{G}^{(n)}$ with the restriction $\mathcal{L}og\,|_{\mathbb{U}}$ of the logarithmic sheaf:

LEMMA 4.10. a) *There is a unique smooth sheaf* $\underline{\mathcal{G}}^{(n)}$ *on* $\underline{S}$ *extending* $\mathcal{G}^{(n)}$. *It has a weight filtration.*
b) *There is a canonical isomorphism*

$$\underline{\mathcal{G}}^{(n)} \xrightarrow{\sim} \mathrm{Sym}^n\underline{\mathcal{G}}^{(1)}\;,$$

and a unique isomorphism

$$\eta^{(n)} : \mathrm{Gr}^W_\cdot\,\underline{\mathcal{G}}^{(n)} \xrightarrow{\sim} \bigoplus_{i=0}^{n} F(i)\;,$$

which is compatible with the isomorphism of 4.9.e).
c) *The weight filtration of $i^*\underline{\mathcal{G}}^{(n)}$ is split: there is a canonical isomorphism*

$$i^*\underline{\mathcal{G}}^{(n)} \xrightarrow{\sim} \mathrm{Gr}^W_{\cdot}\, i^*\underline{\mathcal{G}}^{(n)} \underset{b)}{\xrightarrow{\sim}} \bigoplus_{i=0}^{n} F(i) \; .$$

Here, i denotes the inclusion of $\underline{S} \setminus S$ into $\underline{S}$.
d) *There is an exact sequence*

$$0 \longrightarrow i_*F(1) \longrightarrow \mathcal{H}^0_{\underline{S}}\left(\mathbb{G}_{m,\underline{S}}, \underline{v}^1_! F(1)\right) \longrightarrow \underline{\mathcal{G}}^{(1)} \longrightarrow 0$$

of sheaves on $\underline{S}$.

Proof. If there is any smooth sheaf as in a), then it will automatically be unique, and hence b) follows from a), and 4.9.d), e). Also, it will suffice, because of 4.9.d), to show the lemma for the case $n = 1$.
There we have the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
 & & i_*F(1) \arrow[d] & & & \\
\mathcal{H}^{-1} \arrow[r] & F(1) \arrow[r, "\Delta"] & K \arrow[r] \arrow[d] & \mathcal{H}^0 \arrow[r] & F(0) \arrow[r] & 0 \\
 & & \bigoplus_{\alpha,\beta} F(1) \arrow[d] & & & \\
 & & 0 & & &
\end{tikzcd}$$

where

$$K = \mathcal{H}^{-1}\,\mathrm{Cone}(\delta : \bigoplus_{\alpha,\beta} F(1)_{\underline{S}}[d(\underline{S})] \to i_*F(1)_{\underline{S}\setminus S}[d(\underline{S})])$$

with $\delta(v_1, v_2) := v_1 - v_2$ (in terms of constructible sheaves this is just $\mathrm{Ker}\,\delta$ shifted in the appropriate degree to define a perverse sheaf). The horizontal sequence is, as in the proof of 4.6.b), the long exact cohomology sequence on $\underline{S}$ associated to the short exact sequence on $\mathbb{G}_{m,\underline{S}}$

$$(**) \qquad 0 \to \underline{z}^{(1)}_* F(1) \to \underline{v}^1_! F(1) \to F(1) \to 0 \; ,$$

where we have set

$$\mathcal{H}^i := \mathcal{H}^i_{\underline{S}}\left(\mathbb{G}_{m,\underline{S}}, \underline{v}^1_! F(1)\right) \; .$$

We thus get the equality

$$\mathcal{R}_{\underline{S}}\left(\mathbb{G}_{m,\underline{S}}, \underline{v}^1_! F(1)\right) = \mathcal{H}^0_{\underline{S}}\left(\mathbb{G}_{m,\underline{S}}, \underline{v}^1_! F(1)\right)[0] \; ,$$

and an exact sequence of sheaves on $\underline{S}$

$$0 \to K/\Delta\,(F(1)) \to \mathcal{H}^0_{\underline{S}}\left(\mathbb{G}_{m,\underline{S}}, \underline{v}^1_! F(1)\right) \to F(0) \to 0\ ,$$

whose restriction to S is isomorphic, via the choice of an ordering of α and β, to

$$0 \to F(1) \to \mathcal{G}^{(1)} \to F(0) \to 0\ .$$

Push out of the above via the morphism

$$K/\Delta(F(1)) \to \left(\bigoplus_{\alpha,\beta} F(1)\right) / \Delta(F(1))\ ,$$

whose kernel is $i_*F(1)$ (recall again that we use perverse indices), gives the desired extension $\underline{\mathcal{G}}^{(1)}$. By construction b) and d) hold. Applying i^* to the pushout diagram and taking cohomology, we see that the sheaf $i^*\underline{\mathcal{G}}^{(1)}[-1]$ is the pushout of $F(0)$ via

$$0 \hookrightarrow F(1)\ ,$$

and we get c). □

We now specialize our geometric situation: we let

$$\begin{aligned} \underline{S} &:= \mathbb{G}_{m,B}\ , \\ \underline{\alpha} &:= 1 : \mathbb{G}_{m,B} \twoheadrightarrow B \hookrightarrow \mathbb{G}_{m,B}\ , \\ \underline{\beta} &:= \mathrm{id} : \mathbb{G}_{m,B} \to \mathbb{G}_{m,B}\ . \end{aligned}$$

So we have $S = \mathbb{U}$ and $\underline{S} \setminus S = 1_B$, the closed subscheme of $\mathbb{G}_{m,B}$ given by the immersion 1 of B into $\mathbb{G}_{m,B}$.

After having made precise which choice of normalization we have and in how far it affects our identifications, we now fix it: we let

$$s_1 := \alpha = 1 \text{ and } s_2 := \beta = \mathrm{id} \text{ in 4.9.c).}$$

We thus get a projective system $(\underline{\mathcal{G}}^{(n)})_{n\geq 0}$ of smooth Tate sheaves on $\mathbb{G}_{m,B}$ with

$$\underline{\mathcal{G}}^{(n)}\,|_{1_B} = \bigoplus_{i=0}^{n} F(i)\ .$$

By the universal property of $\mathcal{L}og$ (Theorem 2.1.d)), there is a unique morphism

$$\varphi : \mathcal{L}og \to \underline{\mathcal{G}} := \varprojlim_n \underline{\mathcal{G}}^{(n)}$$

such that $\varphi\,|_{1(B)}$ sends $1 \in \Gamma(B, \mathcal{L}og\,|_{1_B})$ to

$$1 : F(0) \hookrightarrow \prod_{i=0}^{\infty} F(i) = \underline{\mathcal{G}}\,|_{1(B)}\ .$$

Theorem 4.11. *φ is an isomorphism.*

Proof. The claim can be shown on the level of the underlying topological sheaves. The l–adic statement follows from the statement for the topological spaces of $\mathbb{C}$–valued points by comparison – recall that we are dealing with locally constant sheaves.
Over $\mathbb{C}$, the fibre at 1 of the pro–local system $\mathcal{L}og_{\mathrm{top}}$ equals the completion of the group ring $\mathbb{Q}[\pi_1]$ of $\pi_1 := \pi_1(\mathbb{G}_m(\mathbb{C}), 1) \cong \mathbb{Z}$ with respect to the augmentation ideal a. The representation of π_1 is given by multiplication; compare the general construction in [WiI], 2.5–2.7. In particular, we have

$$\mathcal{L}og_{\mathrm{top}} = \varprojlim_n \mathrm{Sym}^n(\mathcal{L}og_{\mathrm{top},\geq -2}) \, ,$$

where $\mathcal{L}og_{\mathrm{top},\geq -2} := \mathcal{L}og_{\mathrm{top}}/a^2$ is of dimension two. Now in the category of unipotent local systems on $\mathbb{G}_m(\mathbb{C})$, the pro–sheaf $\mathcal{L}og_{\mathrm{top}}$ has the universal property of Theorem 2.1.d).
We apply this universal property to $\underline{\mathcal{G}}_{\mathrm{top},\geq -2} := \underline{\mathcal{G}}^{(1)}_{\mathrm{top}}$. The resulting map factors over φ_{top}. Since $\underline{\mathcal{G}}_{\mathrm{top},\geq -2}$ is two-dimensional, the representation of $\mathbb{Q}[\pi_1]$ is necessarily trivial on a^2, and we get a morphism of local systems

$$\varphi_{\mathrm{top},\geq -2} : \mathcal{L}og_{\mathrm{top},\geq -2} \longrightarrow \underline{\mathcal{G}}_{\mathrm{top},\geq -2}$$

giving rise to a morphism

$$\varprojlim_n \mathrm{Sym}^n(\varphi_{\mathrm{top},\geq -2}) : \mathcal{L}og_{\mathrm{top}} \longrightarrow \underline{\mathcal{G}}_{\mathrm{top}} \, .$$

Again because of the universal property of $\mathcal{L}og_{\mathrm{top}}$, this morphism is identical to φ_{top}. It therefore suffices to show that $\varphi_{\mathrm{top},\geq -2}$ is bijective, which amounts to saying that the coinvariants of $\underline{\mathcal{G}}_{\mathrm{top},\geq -2}$ under the action of π_1 are one–dimensional. But taking coinvariants under π_1 of a unipotent variation $\mathbb{V}$ amounts to computing singular cohomology

$$H^1(\mathbb{G}_m(\mathbb{C}), \mathbb{V}) = \mathcal{H}^0_{\mathrm{Spec}(\mathbb{R})}(\mathbb{G}_{m,\mathbb{R}}, \mathbb{V}) \, .$$

Firstly, we claim that

$$\mathcal{H}^i_{\mathrm{Spec}(\mathbb{R})}\left(\mathbb{G}_{m,\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}, \underline{v}^1_! F(1)\right) = \begin{cases} F(-1)\, , & i = 0 \\ 0\, , & i \neq 0 \end{cases} \quad :$$

e.g., identify the left hand side with

$$\begin{aligned} & H^{i+2}(\mathbb{G}_m(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C}), \Delta(\mathbb{G}_m(\mathbb{C})) \cup (\{1\} \times \mathbb{G}_m(\mathbb{C})), F(1)) \\ \cong\ & H^{i+2}(\mathbb{G}_m(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C}), (\mathbb{G}_m(\mathbb{C}) \times \{1\}) \cup (\{1\} \times \mathbb{G}_m(\mathbb{C})), F(1)) \ , \end{aligned}$$

and apply the Künneth formula. From the proof of 4.10, we recall – remember that we have $\underline{S} = \mathbb{G}_m$:

$$\mathcal{R}_{\mathbb{G}_{m,\mathbb{R}}}\left(\mathbb{G}_{m,\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}, \underline{v}^1_! F(1)\right) = \mathcal{H}^0_{\mathbb{G}_{m,\mathbb{R}}}\left(\mathbb{G}_{m,\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}, \underline{v}^1_! F(1)\right)[0] \, ,$$

from which we conclude:

$$\mathcal{H}^i_{\mathrm{Spec}(\mathbb{R})}\left(\mathbb{G}_{m,\mathbb{R}}, \mathcal{H}^0_{\mathbb{G}_{m,\mathbb{R}}}\left(\mathbb{G}_{m,\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}, \underline{v}^1_! F(1)\right)\right) = \begin{cases} F(-1)\, , & i = 0 \\ 0\, , & i \neq 0 \end{cases} \quad .$$

The long exact sequence obtained by applying $\mathcal{R}_{\mathrm{Spec}(\mathbb{R})}(\mathbb{G}_{m,\mathbb{R}}, \_)$ to the exact sequence of 4.10.d)

$$0 \longrightarrow 1_* F(1) \longrightarrow \mathcal{H}^0_{\mathbb{G}_{m,\mathbb{R}}}\left(\mathbb{G}_{m,\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}, \underline{v}^1_! F(1)\right) \longrightarrow \underline{\mathcal{G}}^{(1)} \longrightarrow 0$$

then shows that

$$\mathcal{H}^0_{\mathrm{Spec}(\mathbb{R})}(\mathbb{G}_{m,\mathbb{R}}, \underline{\mathcal{G}}^{(1)}) = F(-1) \; .$$

$\square$

Remark: The geometric situation used in this section is identical to the one of [BD1], 4.1–4.3 (see in particular loc. cit., 4.1.9). The comparison statement of our Proposition 4.8 is implicit in loc. cit., 4.3.3. We mention that basically the same geometric arrangement was used in [Jeu]. More precisely, writing down the iterated cone construction of loc. cit., one arrives at a simplicial object which is homotopy equivalent to Beilinson's and Deligne's construction used here.

5 The Splitting Principle Revisited

In order to be able to translate easily to the motivic context, we recall Beilinson's original proof ([B4], 4) of the splitting of the logarithmic pro-sheaf over spectra of cyclotomic fields (Theorem 2.4).

First, we return to the general situation considered at the beginning of section 4. For $N \geq 1$, we have the morphism of S-schemes

$$\begin{array}{rcl} \phi : \mathbb{G}_{m,S} & \longrightarrow & \mathbb{G}_{m,S} \; , \\ x & \longmapsto & x^N \; , \end{array}$$

and for each $n \geq 0$, the induced morphism

$$\phi^n : \mathbb{G}^n_{m,S} \longrightarrow \mathbb{G}^n_{m,S} \; .$$

We work under the additional assumption

$$\text{(A)} \qquad \phi \circ \alpha = \alpha \; , \quad \phi \circ \beta = \beta \; .$$

If this is the case, we have $(\phi^n)^{-1}(V^n) \subset V^n$, and hence get a morphism

$$(\phi^n)^* v^n_! F(n) \longrightarrow v^n_! F(n) \, ,$$

and hence a morphism

$$(\phi^n)^\sharp : v^n_! F(n) \longrightarrow \phi^n_* v_! F(n) \, ,$$

which after application of p^n_* and projection onto the sign-eigenpart induces

$$(\phi^n)^\sharp : \mathcal{G}^{(n)} \longrightarrow \mathcal{G}^{(n)} \; .$$

We need to understand the action of $(\phi^n)^\sharp$ on $\mathcal{G}^{(n)}$, and on absolute cohomology. First, we establish in how far $(\phi^n)^\sharp$ is compatible with the residue at 0:

Lemma 5.1. a) *Under any isomorphism*

$$\mathrm{Gr}_{\cdot}^{W}\,\mathcal{G}^{(n)} \xrightarrow{\sim} \bigoplus_{i=0}^{n} F(i)\ ,$$

the map $\mathrm{Gr}_{\cdot}^{W}(\phi^n)^\sharp$ *is multiplication by* N^{n-i} *on* $F(i)$.
b) *For any* $n \geq 1$, *the diagram*

$$\begin{tikzcd} \mathcal{G}^{(n)} \arrow[r, "(\phi^n)^\sharp"] \arrow[d, "\mathrm{res}_n"'] & \mathcal{G}^{(n)} \arrow[d, "\mathrm{res}_n"] \\ \mathcal{G}^{(n-1)} \arrow[r, "N\cdot(\phi^{n-1})^\sharp"] & \mathcal{G}^{(n-1)} \end{tikzcd}$$

commutes.

Proof. Since the morphisms in b) are strict with respect to the weight filtration, it suffices to check that

$$\mathrm{Gr}_{\cdot}^{W}(\mathrm{res}_n)\circ \mathrm{Gr}_{\cdot}^{W}(\phi^n)^\sharp = N\cdot \mathrm{Gr}_{\cdot}^{W}(\phi^{n-1})^\sharp \circ \mathrm{Gr}_{\cdot}^{W}(\mathrm{res}_n)\ .$$

But if we choose the isomorphism of 4.9.e), then $\mathrm{Gr}_{\cdot}^{W}(\mathrm{res}_n)$ is simply the canonical projection

$$\bigoplus_{i=0}^{n} F(i) \twoheadrightarrow \bigoplus_{i=0}^{n-1} F(i)\ ,$$

and therefore b) follows from a). For a), we note first that it suffices to show the statement for one choice of isomorphism

$$\mathrm{Gr}_{\cdot}^{W}\,\mathcal{G}^{(n)} \xrightarrow{\sim} \bigoplus_{i=0}^{n} F(i)\ .$$

This time, we use the isomorphism on graded objects induced by 4.2, thereby reducing ourselves to the case $n = 1$. There, we consider the long exact cohomology sequence associated to the exact sequence

$$0 \to z_*^{(1)} F(1) \to v_!^1 F(1) \to F(1) \to 0\ ,$$

and the cohomological functors $\mathcal{H}^i_S(\mathbb{G}_{m,S}\,,\ \cdot\ )$. We know the cohomology of $\mathbb{G}_m$:

$$\mathcal{H}^i_S\left(\mathbb{G}_{m,S}\,, F(1)\right) = \begin{cases} F(1) & , \quad i=-1 \\ F(0) & , \quad i=0 \\ 0 & , \quad i \notin \{-1,0\} \end{cases} .$$

Of course, we know the cohomology of two points:

$$\mathcal{H}^i_S\left(\mathbb{G}_{m,S}\,, z_*^{(1)} F(1)\right) = \begin{cases} \bigoplus_{\alpha,\beta} F(1) & , \quad i=0 \\ 0 & , \quad i\neq 0 \end{cases} .$$

We get an exact sequence

$$0 \to F(1) \stackrel{\Delta}{\to} \bigoplus_{\alpha,\beta} F(1) \to \mathcal{G}^{(1)} = \mathcal{H}^0_S\left(\mathbb{G}_{m,S}, v^1_! F(1)\right) \to F(0) \to 0 \ .$$

and because of assumption (A), it carries an action of $(\phi^n)^\sharp$. But this action can be identified on $\mathcal{H}^i_S(\mathbb{G}_{m,S}, F(1))$ and $\mathcal{H}^i_S\left(\mathbb{G}_{m,S}, z^{(1)}_* F(1)\right)$: it is trivial on the $F(1)$, and multiplication by N on $F(0)$. $\square$

Certainly (A) is only satisfied in very special situations, namely if α and β are supported in the schemes of $(N-1)$-torsion of $\mathbb{G}_{m,S}$.

Let again $d \geq 2$, $C := \operatorname{Spec}(R)$, where $R := A[\frac{1}{d}, T]/\Phi_d(T)$ as in section 2. For b prime to d, consider

$$\begin{array}{rcl} i_b : C & \stackrel{\sim}{\longrightarrow} & C \hookrightarrow \mathbb{G}_m \ , \\ \zeta & \longmapsto & \zeta^b \ . \end{array}$$

The pullback $\mathcal{L}og_b$ of the pro-sheaf $\mathcal{L}og|_{\mathbb{U}}$ on $\mathbb{U}$ via i_b is identical to the projective limit of the sheaves $\mathcal{G}^{(n)}_b$ obtained by setting

$$\begin{aligned} &\underline{S} := C \ , \\ &\underline{\alpha} := 1 : C \to B \hookrightarrow \mathbb{G}_m \ , \\ &\underline{\beta} := i_b \ . \end{aligned}$$

Since (A) is satisfied with $N = d+1$, we may apply 5.1, and conclude:

COROLLARY 5.2. *$\mathcal{G}^{(n)}_b$ splits into a direct sum*

$$\mathcal{G}^{(n)}_b = \bigoplus_{i=0}^n \operatorname{Gr}^W_{-2i} \mathcal{G}^{(n)}_b \ .$$

Therefore, there is a unique isomorphism

$$\eta^{(n)}_b : \mathcal{G}^{(n)}_b \stackrel{\sim}{\longrightarrow} \bigoplus_{i=0}^n F(i) \ ,$$

which is compatible with the isomorphism $\eta^{(n)}$ of 4.10.b).

Proof. $F(i) \subset \mathcal{G}^{(n)}_b$ is the eigenspace of $(d+1)^{n-i}$ under the morphism $(\phi^n)^\sharp$. $\square$

We conclude with the implications of 5.1 and 5.2 for absolute cohomology with coefficients. For this, recall the absolute residue sequence for $n \geq 1$

$$\ldots \to H^{\cdot}_{\text{abs}}(C, n) \to H^{\cdot+n}_{\text{abs}}(\mathbb{G}^{\vee n}_{m,C}, n)^{\text{sgn}} \stackrel{\text{res}}{\to} H^{\cdot+n-1}_{\text{abs}}(\mathbb{G}^{\vee n-1}_{m,C}, n-1)^{\text{sgn}} \to \ldots$$

introduced after 4.6, where we have set

$$H^{\cdot+n}_{\text{abs}}(\mathbb{G}^{\vee n}_{m,C}, n)^{\text{sgn}} := H^{\cdot+n}_{\text{abs}}(\mathbb{G}^n_{m,C} \text{ rel } Z^{(n)}, n)^{\text{sgn}} \ ,$$

thus saving enough space to get the above sequence into a single line.

COROLLARY 5.3. a) *For $n \geq 1$, the absolute residue sequence splits into short exact sequences*

$$0 \to H^{\cdot}_{\mathrm{abs}}(C,n) \to H^{\cdot+n}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{sgn} \to H^{\cdot+n-1}_{\mathrm{abs}}(\mathbb{G}^{\vee n-1}_{m,C},n-1)^{sgn} \to 0\ .$$

b) *For $N = d+1$, the map $(\phi^n)^*$ acts on the short exact sequences of* a)*: there is a commutative diagram*

$$\begin{array}{ccccc} H^{\cdot}_{\mathrm{abs}}(C,n) & \to & H^{\cdot+n}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{sgn} & \to & H^{\cdot+n-1}_{\mathrm{abs}}(\mathbb{G}^{\vee n-1}_{m,C},n-1)^{sgn} \\ \mathrm{id}\downarrow & & (\phi^n)^*\downarrow & & (d+1)\cdot(\phi^{n-1})^*\downarrow \\ H^{\cdot}_{\mathrm{abs}}(C,n) & \to & H^{\cdot+n}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{sgn} & \to & H^{\cdot+n-1}_{\mathrm{abs}}(\mathbb{G}^{\vee n-1}_{m,C},n-1)^{sgn} \end{array}$$

Proof. By 4.8, the absolute residue sequence is the absolute cohomology sequence for the exact sequence of sheaves on C

$$0 \to F(n) \to \mathcal{G}^{(n)}_b \stackrel{\mathrm{res}_b}{\to} \mathcal{G}^{(n-1)}_b \to 0\ .$$

Therefore, a) follows from 5.2, while b) follows from 5.1.b) and the fact that under the identification of 4.9.a)

$$H^{\cdot}_{\mathrm{abs}}(C,\mathcal{G}^{(n)}) \xrightarrow{\sim} H^{\cdot+n}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{\mathrm{sgn}}\ ,$$

the map induced by

$$(\phi^n)^\sharp : \mathcal{G}^{(n)}_b \to \mathcal{G}^{(n)}_b$$

is the map $(\phi^n)^*$ of the absolute cohomology groups. $\square$

It follows that the eigenvalues of $(\phi^n)^*$ on $H^{n+1}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{\mathrm{sgn}}$ are $1, d+1, \ldots, (d+1)^n$. The eigenspace decomposition yields

$$\eta^{(n)}_b : H^{n+1}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{\mathrm{sgn}} = H^{n+1}_{\mathrm{abs}}(\mathbb{G}^{n}_{m,C} \text{ rel } Z^{(n)},n)^{\mathrm{sgn}} \xrightarrow{\sim} \bigoplus_{i=0}^{n} H^1_{\mathrm{abs}}(C,i)\ ,$$

which in sheaf theoretic terms corresponds to the decomposition

$$\eta^{(n)}_b : \mathrm{Ext}^1_{\mathrm{Sh}(C)}(F(0),\mathcal{G}^{(n)}_b) \xrightarrow{\sim} \bigoplus_{i=0}^{n} \mathrm{Ext}^1_{\mathrm{Sh}(C)}(F(0),F(i))$$

given by Corollary 5.2.

The pullback pol_b of the small polylogarithmic extension pol on $\mathbb{U}$ is an element of

$$\begin{aligned} \varprojlim_{n\geq 1} \mathrm{Ext}^1_{\mathrm{Sh}(C)}(F(0),\mathcal{G}^{(n)}_b) &= \varprojlim_{n\geq 1} H^{n+1}_{\mathrm{abs}}(\mathbb{G}^{n}_{m,C} \text{ rel } Z^{(n)},n)^{\mathrm{sgn}} \\ &= \varprojlim_{n\geq 1} H^{n+1}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{\mathrm{sgn}}\ . \end{aligned}$$

We have shown that, using the eigenspace decomposition for the action of the $(\phi^n)^\sharp$, these groups are isomorphic to

$$\prod_{k\geq 0} \mathrm{Ext}^1_{\mathrm{Sh}(C)}(F(0), F(k)) = \prod_{k\geq 0} H^1_{\mathrm{abs}}(C, k) \ .$$

2.5 and 2.6 describe pol_b as an element in this group.

Actually, in order to relate the above decomposition to the one used for 2.5 and 2.6, we shall need to compare the isomorphism

$$\eta := \varprojlim_{n\geq 1} \eta^{(n)} : \mathrm{Gr}^W_\cdot \underline{\mathcal{G}} \xrightarrow{\sim} \prod_{k\geq 0} F(k)$$

of 4.10.b) to the isomorphism

$$\kappa : \mathrm{Gr}^W_\cdot \underline{\mathcal{G}} = \mathrm{Gr}^W_\cdot \mathcal{L}og \xrightarrow{\sim} \prod_{k\geq 0} F(k)$$

of section 2.

A priori, we know that the isomorphisms

$$\eta_{-2k} \ , \ \kappa_{-2k} : \mathrm{Gr}^W_{-2k} \underline{\mathcal{G}} \xrightarrow{\sim} F(k)$$

satisfy an identity of the type

$$\eta_{-2k} = q_{-2k} \cdot \kappa_{-2k} \ ,$$

for a constant $q_{-2k} \in F^*$.

We remark that in order to prove the main results announced in the introduction, all one needs to know is that q_{-2k} is a rational number, which is independent of whether we work in the Hodge or the l–adic setting.

In order to exhibit the precise relation of the motivic analogue of *pol* (see section 8) to the cyclotomic elements in K–theory (see Corollary 9.6.b)), we need to identify q_{-2k}.

Proposition 5.4. *We have the equality*

$$\eta_{-2k} = k! \cdot \kappa_{-2k} \ .$$

Proof. Because of the compatibility of κ_0 with the canonical projection

$$\varepsilon : \underline{\mathcal{G}} \longrightarrow F(0) \ ,$$

we have $\eta_0 = \kappa_0$. In order to show $\eta_{-2} = \kappa_{-2}$ we compare the classes of $\underline{\mathcal{G}}^{(1)}$ in

$$\mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{G}_m)}(F(0), F(1))$$

induced by η_{-2} and κ_{-2} respectively. Let

$K := \mathbb{C}$,	$K := \mathbb{Q}$,

and choose any K–valued point t of $\mathbb{U}$. Of course, the value of q_{-2} can still be detected from the extensions of

mixed $\mathbb{Q}$–Hodge structures	Galois modules

given by the pullback $t^*\underline{\mathcal{G}}^{(1)}$ of $\underline{\mathcal{G}}^{(1)}$ via t. In both settings, there is a natural morphism of $K^* \otimes_{\mathbb{Z}} F$ into the respective $\mathrm{Ext}^1\left(F(0), F(1)\right)$ (see e.g.

[WiIV], Theorem 3.7).	[WiIV], Theorem 4.6).

By

[WiIV], Proposition 3.13.a),	[WiIV], Proposition 4.7.a),

the class of $t^*\underline{\mathcal{G}}^{(1)}$, calculated in the framing given by κ_{-2}, equals the image of $t \in K^*$ under this morphism. By [Sch], 2.7, the same holds for the framing given by η_{-2} – note that here it is vital to choose the ordering of the sections $\underline{\alpha}$ and $\underline{\beta}$ in the way we did before 4.11. For $k \geq 2$, let

$$\varphi_0^{(k)} : \underline{\mathcal{G}}^{(k)} \xrightarrow{\sim} \mathrm{Sym}^k\, \underline{\mathcal{G}}^{(1)}$$

be the isomorphism of 4.10.b). By 4.9.d), the diagram

$$\begin{array}{ccc} \underline{\mathcal{G}}^{(k)} & \longrightarrow & F(0) \\ \varphi_0^{(k)} \downarrow \wr & & \uparrow \wr \\ \mathrm{Sym}^k \underline{\mathcal{G}}^{(1)} & \longrightarrow & \mathrm{Sym}^k F(0) \end{array}$$

commutes. By [WiIV], Theorem 3.12.a), the commutativity of this diagram characterizes $\varphi_0^{(k)}$ uniquely. From loc. cit., Theorem 3.12.b) and c), we know that the diagram

$$\begin{array}{ccc} F(k) & \xrightarrow{\frac{1}{k!}\cdot\kappa^{-1}} & \underline{\mathcal{G}}^{(k)} \\ \uparrow \wr & & \wr \downarrow \varphi_0^{(k)} \\ \mathrm{Sym}^k F(1) & \xrightarrow{\mathrm{Sym}^k\, \kappa^{-1}} & \mathrm{Sym}^k \underline{\mathcal{G}}^{(1)} \end{array}$$

commutes. So our identity

$$\eta_{-2k} = k! \cdot \kappa_{-2k}$$

follows from 4.9.d). $\square$

6 Polylogs in Absolute Cohomology Theories

In section 4, we showed that the logarithmic pro–sheaf is the projective limit of relative cohomology objects with coefficients in Tate twists of certain schemes over $\mathbb{U}$. The Leray spectral sequence suggests that is should be possible to recover *pol* as a projective limit of elements in absolute cohomology with Tate coefficients of these schemes, and indeed this is what we do in Theorem 6.6. That the coefficients are Tate is of course the central point: it allows us, in section 7, to imitate the construction of this section, and thus to define a motivic version of *pol*. This detour is necessary because we know, up to date, of no satisfactory formalism of mixed motivic sheaves, whose absolute cohomology with Tate coefficients would give back motivic cohomology defined via K-theory.

We return to the geometric situation set up before 4.11, and start by computing the higher direct images of the restriction of $\mathcal{L}og$ to $\mathbb{U}$:

LEMMA 6.1. a) *The inclusion $F(1) \hookrightarrow \underline{\mathcal{G}}^{(1)}$ and the projection $\underline{\mathcal{G}}^{(1)} \twoheadrightarrow F(0)$ induce natural isomorphisms*

$$\begin{aligned} F(1)_B &\xrightarrow{\sim} \mathcal{H}_B^{-1}\left(\mathbb{G}_m, \underline{\mathcal{G}}^{(1)}\right) , \\ \mathcal{H}_B^0\left(\mathbb{G}_m, \underline{\mathcal{G}}^{(1)}\right) &\xrightarrow{\sim} \mathcal{H}_B^0\left(\mathbb{G}_m, F(0)\right) , \end{aligned}$$

and the latter group is isomorphic to $F(-1)_B$ via the map "residue at 0".
b) *The inclusion $F(n) \hookrightarrow \underline{\mathcal{G}}^{(n)}$ and the projection $\underline{\mathcal{G}}^{(n)} \twoheadrightarrow F(0)$ induce natural identifications*

$$\mathcal{H}_B^i\left(\mathbb{G}_m, \underline{\mathcal{G}}^{(n)}\right) = \begin{cases} F(n)_B , & i=-1 \\ F(-1)_B , & i=0 \\ 0 , & i \notin \{-1,0\} \end{cases} .$$

Proof. The statements need only be checked on the level of local systems. Part a) is shown in the proof of 4.11. From there, we also recall that we have to compute the invariants and coinvariants under the action of the group $\pi_1 := \pi_1(\mathbb{G}_m(\mathbb{C}), 1)$, or equivalently, of a generator of π_1. Using 4.10.b), we may deduce b) from a). □

COROLLARY 6.2.

$$\mathcal{H}_B^i\left(\mathbb{U}, \mathcal{G}^{(n)}\right) = \begin{cases} F(n)_B & , \ i=-1 \\ 0 & , \ i \notin \{-1,0\} \end{cases} .$$

For $i = 0$, the sheaf $\mathcal{H}_B^0\left(\mathbb{U}, \mathcal{G}^{(n)}\right)$ is the direct sum of $\bigoplus_{k=1}^{n} F(k-1)_B$ and an object which is an extension of $F(-1)_B$ by itself.

Proof. By [WiI], Theorem 4.3, there is a weight filtration on $\mathcal{H}_B^i\left(\mathbb{U}, \mathcal{G}^{(n)}\right)$. Now use the exact triangle

$$\begin{tikzcd} 1_*1^! \arrow[rr] & & \mathrm{id}_{\mathbb{G}_m,B} \arrow[dl] \\ & j_*j^* \arrow[ul, "{[1]}"] & \end{tikzcd}$$

purity, and 4.10.c). □

REMARK: In the setting of Hodge modules, where a concept of polarization is available, any extension of pure objects of the same weight is necessarily split.

The map $\mathcal{H}_B^0\left(\mathbb{U}, \mathcal{G}^{(n)}\right) \to F(0)$ of the corollary yields in particular a map "residue at 1_B", for $n \geq 1$,

$$\mathrm{res} : H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) = H^0_{\mathrm{abs}}\left(B, \mathcal{R}_B(\mathbb{U}, \mathcal{G}^{(n)})\right) \to H^0_{\mathrm{abs}}(B, 0) .$$

Definition 6.3. *Let $n \geq 1$. The map*

$$\mathrm{res}: H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) = H^{n+1}_{\mathrm{abs}}(\mathbb{G}^n_{m,\mathbb{U}} \text{ } rel \text{ } Z^{(n)}, n)^{sgn} \to H^0_{\mathrm{abs}}(B, 0)$$

is called the total residue map.

For later reference, we note

Corollary 6.4. $H^1_{\mathrm{abs}}(\mathbb{G}^1_{m,\mathbb{U}} \text{ } rel \text{ } Z^{(1)}, 1) = 0$.

Proof. We have

$$H^1_{\mathrm{abs}}(\mathbb{G}^1_{m,\mathbb{U}} \text{ rel } Z^{(1)}, 1) = H^{-1}_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(1)}) \text{ ,}$$

which because of 6.2 equals $H^0_{\mathrm{abs}}(B, F(1)) = 0$. □

Next we have

Lemma 6.5. i) *The transition morphism*

$$\mathrm{res}: \mathcal{G}^{(n)} \twoheadrightarrow \mathcal{G}^{(n-1)}$$

satisfies

$$\begin{aligned} &\mathcal{H}^{-1}_B(\mathbb{U}, \mathrm{res}) = 0 : F(n)_B \to F(n-1)_B \text{ ,} \\ &\mathcal{H}^0_B(\mathbb{U}, \mathrm{res}) : \mathcal{H}^0_B(\mathbb{U}, \mathcal{G}^{(n)}) \to \mathcal{H}^0_B(\mathbb{U}, \mathcal{G}^{(n-1)}) \end{aligned}$$

is surjective with kernel $F(n-1)_B$.

In particular, the total residue for $n \geq 2$ factors over the total residue for $n-1$: there is a commutative diagram

$$\begin{array}{ccc} H^0_{\mathrm{abs}}(B,0) & \stackrel{\mathrm{res}}{\longleftarrow} & H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) \\ & \mathrm{res}\nwarrow & \downarrow \mathrm{res} \\ & & H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n-1)}) \end{array}$$

ii) *The Leray spectral sequences, for $n \geq 0$, give exact sequences*

$$0 \longrightarrow H^1_{\mathrm{abs}}(B, n) \xrightarrow{\delta} H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) \xrightarrow{\mathrm{res}} H^0_{\mathrm{abs}}(B, 0) \longrightarrow 0 \text{ .}$$

The map

$$\delta : H^1_{\mathrm{abs}}(B, n) \to H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)})$$

is the composition of $H^1_{\mathrm{abs}}(B, n) \to H^1_{\mathrm{abs}}(\mathbb{U}, n) = H^0_{\mathrm{abs}}(\mathbb{U}, F(n))$ and the map induced by the inclusion of $F(n)$ into $\mathcal{G}^{(n)}$, in other words, the same noted map of the residue sequence.

The projective limit of the above sequences identifies

$$H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{L}og\,|_{\mathbb{U}}) := \varprojlim_n H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)})$$

and $H^0_{\mathrm{abs}}(B, 0)$.

iii) *There are unique splittings*

$$s_n : H^0_{\mathrm{abs}}(B,0) \hookrightarrow H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)})$$

of the sequences in ii), for any $n \geq 0$, such that for any $n \geq 1$ we have a commutative diagram

$$\begin{array}{ccc} H^0_{\mathrm{abs}}(B,0) & \xrightarrow{s_n} & H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) \\ & s_{n-1} \searrow & \downarrow \mathrm{res} \\ & & H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n-1)}) \end{array}$$

Proof. i) The first statement is clear. For the second, either go through the construction or observe that the direct image of the morphism $\mathbb{U}_{\mathrm{top}} \to B_{\mathrm{top}}$ has cohomological dimension one, hence $\mathcal{H}^0_B(\mathbb{U}, \,\cdot\,)$ is right exact on smooth sheaves.
ii) We have the Leray spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{abs}}\left(B, \mathcal{H}^q_B(\mathbb{U}, \mathcal{G}^{(n)})\right) \Rightarrow H^{p+q}_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) \ ,$$

whose low-term sequence reads

$$0 \to H^1_{\mathrm{abs}}(B,n) \to H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) \to H^0_{\mathrm{abs}}(B,0) \overset{d_2^{(n)}}{\to} H^2_{\mathrm{abs}}(B,n) \ .$$

By i), the Mittag–Leffler condition is satisfied for the projective system $(H^1_{\mathrm{abs}}(B,n))_{n\geq 0}$, and therefore,

$$H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{L}og|_{\mathbb{U}}) = \varprojlim_n \ker(d_2^{(n)}) = H^0_{\mathrm{abs}}(B,0)$$

since the projective system $(\mathrm{im}(d_2^{(n)}))_{n\geq 0} \subset (H^2_{\mathrm{abs}}(B,n))_{n\geq 0}$ is ML-zero.
But then any of the

$$H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) \to H^0_{\mathrm{abs}}(B,0)$$

must be surjective as well.
iii) Apply ii). □

Denote by $pol^{(n)}$ the image of the small polylogarithmic extension *pol* under

$$H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{L}og\,|_{\mathbb{U}}) \to H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) \ .$$

THEOREM 6.6. a) *Under the isomorphism*

$$H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{L}og\,|_{\mathbb{U}}) \xrightarrow{\sim} H^0_{\mathrm{abs}}(B,0)$$

of 6.5 ii), the small polylogarithmic extension pol is mapped to 1.
b) *For each $n \geq 0$, the map*

$$s_n : H^0_{\mathrm{abs}}(B,0) \to H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)})$$

maps 1 *to* $pol^{(n)}$.

Proof. This is the definition of *pol* and the s_n. □

Recall (4.9.a)) that we may identify

$$\begin{aligned} H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) &= H^0_{\mathrm{abs}}(\mathbb{G}^n_{m,\mathbb{U}}, v^n_! F(n))^{\mathrm{sgn}} \\ &= H^{n+1}_{\mathrm{abs}}(\mathbb{G}^n_{m,\mathbb{U}}, v^n_! F(n)_{V^n})^{\mathrm{sgn}} \\ &= H^{n+1}_{\mathrm{abs}}(\mathbb{G}^n_{m,\mathbb{U}} \text{ rel } Z^{(n)}, n)^{\mathrm{sgn}} . \end{aligned}$$

In section 8, we are going to prove a motivic analogue of 6.5.ii), and then *define pol* as the element in

$$\varprojlim_n H^{n+1}_{\mathcal{M}}(\mathbb{G}^n_{m,\mathbb{U}} \text{ rel } Z^{(n)}, n)^{\mathrm{sgn}}$$

mapping to 1 under the isomorphism to $H^0_{\mathcal{M}}(B, 0)$.

In order to prove a motivic version of 6.5.ii), we shall frequently use injectivity of the Beilinson regulator on certain motivic cohomology groups, and two technical results on $H^{\cdot}_{\mathrm{abs}}$, that will occupy the rest of this section.

While this may appear artificial at first sight, we remind the reader that in the motivic setting, we cannot make use of any sheaf theoretic means like Leray spectral sequences.

An important means will be the *localization sequence* associated to the geometric situation

$$\{0,1\}_B \hookrightarrow \mathbb{A}^1_B \hookleftarrow \mathbb{U} .$$

It is the result of the degeneration of the Leray spectral sequence and reads

$$\begin{aligned} \cdots \to & H^{\cdot}_{\mathrm{abs}}(\mathbb{A}^1_B, p) \to H^{\cdot}_{\mathrm{abs}}(\mathbb{U}, p) \to H^{\cdot-1}_{\mathrm{abs}}(\{0,1\}_B, p-1) \\ \to & H^{\cdot+1}_{\mathrm{abs}}(\mathbb{A}^1_B, p) \to \dots \end{aligned}$$

LEMMA 6.7. a) *The structure morphism is an isomorphism*

$$H^{\cdot}_{\mathrm{abs}}(B, p) \xrightarrow{\sim} H^{\cdot}_{\mathrm{abs}}(\mathbb{A}^1_B, p) .$$

b) *The boundary map is trivial, i.e., we have short exact sequences*

$$0 \to H^{\cdot}_{\mathrm{abs}}(B, p) \to H^{\cdot}_{\mathrm{abs}}(\mathbb{U}, p) \to \bigoplus_{i=0}^{1} H^{\cdot-1}_{\mathrm{abs}}(B, p-1) \to 0 .$$

Proof. For a), note that $\mathcal{R}_B\left(\mathbb{A}^1_B, F(p)_{\mathbb{A}^1_B}\right) = F(p)_B[0]$.
b) follows from the fact that there are B-valued points of $\mathbb{U}$. □

In particular, for $p = 1$, we have the exact sequence

$$0 \longrightarrow H^1_{\mathrm{abs}}(B, 1) \longrightarrow H^1_{\mathrm{abs}}(\mathbb{U}, 1) \xrightarrow{\partial} \bigoplus_{i=0}^{1} H^0_{\mathrm{abs}}(B, 0) \longrightarrow 0 .$$

The last map equals the map of Ext groups

$$\partial : \mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}(F(0), F(1)) \longrightarrow \mathrm{Hom}_{\mathrm{Sh}(B)}\left(F(0), \mathcal{H}^0_B(\mathbb{U}, F(1))\right)$$

obtained from the Leray spectral sequence; observe that the residues at 0_B and 1_B provide an isomorphism

$$\mathcal{H}^0_B\left(\mathbb{U}, F(1)\right) \xrightarrow{\sim} \bigoplus_{i=0}^{1} F(0) \ .$$

We have a natural map

$$\mathcal{O}(\mathbb{U})^* \to H^1_{\text{abs}}(\mathbb{U}, 1) \ .$$

Its composition with

$$\partial : H^1_{\text{abs}}(\mathbb{U}, 1) \longrightarrow \bigoplus_{i=0}^{1} H^0_{\text{abs}}(B, 0)$$

associates to a function on $\mathbb{U}$ its orders at 0 and 1 respectively.

We need to understand the composition

$$\begin{aligned} \text{res} \circ \partial : H^1_{\text{abs}}(\mathbb{U}, 1) &= \text{Ext}^1_{\text{Sh}(\mathbb{U})}\left(F(0), F(1)\right) \\ &\longrightarrow \text{Hom}_{\text{Sh}(B)}\left(F(0), \mathcal{H}^0_B\left(\mathbb{U}, \mathcal{G}^{(1)}\right)\right) \ . \end{aligned}$$

Observe that due to 6.2, the last group is equal to $H^0_{\text{abs}}(B, 0)$. Furthermore, we recall from the proof of 6.2 and the definition of res that the composition

$$\bigoplus_{i=0}^{1} F(0) = \mathcal{H}^0_B\left(\mathbb{U}, F(1)\right) \longrightarrow \mathcal{H}^0_B\left(\mathbb{U}, \mathcal{G}^{(1)}\right) \xrightarrow{\text{res}} F(0)$$

is given by projection onto the "1"–component of $\bigoplus_{i=0}^{1} F(0)$. We have thus proved:

LEMMA 6.8. *Consider the non–vanishing functions t and $1-t$ on $\mathbb{U}$. We have*

$$\text{res} \circ \partial(t) = 0 \ , \quad \text{res} \circ \partial(1-t) = 1 \ .$$

In particular, the map

$$\delta : H^1_{\text{abs}}(\mathbb{U}, 1) \longrightarrow H^0_{\text{abs}}(\mathbb{U}, \mathcal{G}^{(1)}) = H^2_{\text{abs}}\left(\mathbb{G}^1_{m,\mathbb{U}} \text{ rel } Z^{(1)}, 1\right)$$

does not map $1-t \in \mathcal{O}(\mathbb{U})^*$ to zero.

Proof. Observe that $\text{res} \circ \partial$ factorizes through δ. □

REMARK: The main technical result of this section, 6.5.ii) corresponds to [BD1], 3.1.6.ii). Observe that *pol* and the *polylogarithmic class* Π_ϕ of loc. cit. do not quite agree: in our notation,

$$\Pi_\phi \in H^0_{\text{abs}}\left(\mathbb{U}, \mathcal{L}og(1)|_{\mathbb{U}}\right) \ ,$$

while *pol* $\in H^0_{\text{abs}}(\mathbb{U}, \mathcal{L}og\,|_{\mathbb{U}})$. The connection is as follows: there is a canonical monomorphism

$$\iota : \mathcal{L}og(1) \longrightarrow \mathcal{L}og$$

(identifying $\mathcal{L}og(1)$ with $W_{-2}\mathcal{L}og$), and *pol* is the push out of Π_ϕ via ι. The present definition of the polylog seems more natural since it is an element of an $H^0_{\text{abs}}(B, 0)$-module of rank one, which is canonically trivialized. By contrast, $H^0_{\text{abs}}(\mathbb{U}, \mathcal{L}og(1)|_{\mathbb{U}})$ is of rank two.

7 Calculations in K-theory

The next step is to do the constructions of section 4 with K-groups, or more precisely, with relative K-cohomology as introduced in appendix B.2. For technical reasons we will have to use simplicial schemes to replace the singular schemes that appeared before. All constructions will be compatible with the regulator maps to absolute Hodge cohomology (appendix A and B.5.8) and to continuous étale cohomology (appendix B.4.6).

A priori these regulators have values in absolute cohomology groups for the same simplicial object (cf. B.4.2 and B.5.2). Using B.4.5 and B.5.7 these absolute cohomology groups are then identified with (relative) cohomology of singular schemes. This identification is made tacitly.

Let $B = \operatorname{Spec}(\mathbb{Z})$ and S a smooth affine B-scheme. We will work in the category of smooth S-schemes. K-cohomology is taken on the Zariski site over B.

Before returning to the geometric situation introduced in section 3, we have to check a technical lemma. Let us consider the following general construction: Let X be a smooth quasi-projective S-scheme and Y a closed subscheme of X which is itself also smooth over S. Put

$$Y_0^{(n)} = Y \times_S X^{n-1} \amalg X \times_S Y \times X^{n-2} \amalg \ldots \amalg X^{n-1} \times_S Y \ .$$

Note that $Y_0^{(n)}$ is a proper covering of the singular scheme

$$Y^{(n)} = X^n \smallsetminus (X \smallsetminus Y)^n \ .$$

This is the easiest case of a morphism of schemes with cohomological descent, meaning that for any reasonable cohomology theory the cohomology of $Y^{(n)}$ will agree with the cohomology of the smooth simplicial scheme

$$Y_.^{(n)} = \operatorname{cosk}_0(Y_0^{(n)}/X^n) \ ,$$

i.e.,

$$Y_k^{(n)} = Y_0^{(n)} \times_{X^n} \cdots \times_{X^n} Y_0^{(n)} \qquad (k+1\text{-fold product}).$$

For étale cohomology and absolute Hodge cohomology, the corresponding results are B.4.5 and B.5.6 respectively.

We will work in the setting of spaces, i.e., pointed simplicial sheaves of sets on the Zariski site of smooth B-schemes. We refer to appendix B.1 for details and terminology. We use the notation

$$X_.^{\vee n} = \operatorname{Cone}(Y_.^{(n)} \longrightarrow X^n)$$

for the space that computes relative cohomology for the closed embedding (cf. B.1.5).

The space $Y_.^{(n)}$ does not become degenerate above any simplicial degree. However, we have:

Lemma 7.1. a) *$Y_.^{(n)}$ is isomorphic in* $\operatorname{Ho} s\mathbf{T}$ *to a simplicial scheme which is degenerate above degree $n-1$.*

b) *In particular,.* $Y_\cdot^{(n)}$ *and* $X_\cdot^{\vee n}$ *are* K*-coherent.*

c) $X_\cdot^{\vee n}$ *is a space constructed from schemes in a finite diagram over* X^n *in the sense of B.2.13.*

d) *If* T *is another closed subscheme of* X *which is smooth over* S *and disjoint of* Y*, then the inclusions*

$$T^i \times_S X^{n-i} \longrightarrow X^n$$

are tor-independent of all morphisms in the diagram in c).

Proof. By definition

$$Y_0^{(n)} = Y_1 \amalg \cdots \amalg Y_n$$

where Y_i is the reduced closed subscheme of X^n of those points, whose i-th coordinate lies in Y. This induces a decomposition of $Y_k^{(n)}$ into disjoint subschemes of the form $Y_{i_1} \times_{X^n} \cdots \times_{X^n} Y_{i_k}$. Actually this subscheme is canonically isomorphic to

$$Y_{i_1} \cap \cdots \cap Y_{i_k} = \{(x_1, \ldots, x_n) \in X^n \mid x_{i_j} \in Y \text{ for } 1 \leq j \leq k\} \ .$$

We get the following more familiar form of the simplicial scheme

$$Y_k^{(n)} = \coprod_{I \in \{1,\ldots,n\}^k} \bigcap_{i \in I} Y_i \ .$$

Let $\Delta(n)$ be the simplicial set with

$$\Delta(n)_k = \{(i_0, \ldots, i_k) \mid 1 \leq i_0 \leq \cdots \leq \ldots i_k \leq n\} \ .$$

We define the simplicial scheme $Y_\cdot^{\Delta(n)}$ by

$$Y_k^{\Delta(n)} = \coprod_{I \in \Delta(n)_k} \bigcap_{i \in I} Y_i$$

It is degenerate above the simplicial degree $n-1$ and from our previous considerations we see that it is a natural subspace of $Y_\cdot^{(n)}$. We consider these simplicial schemes as spaces in the sense of appendix B.1 by adding a disjoint base point $\star$.
For a scheme U in the big Zariski site over B we consider the morphism of simplicial sets

$$Y_\cdot^{\Delta(n)}(U) \longrightarrow Y_\cdot^{(n)}(U) \ .$$

By the combinatorial Lemma B.6.2 it induces an isomorphism of homotopy sets. Hence the inclusion is a weak homotopy equivalence of spaces.
b) is an immediate consequence of a) and B.2.3.b). Recall that Y and X were assumed smooth over B. We already have seen that all components of $X^{\vee n}$ are disjoint unions of X^n-schemes of the form $Y_{i_1} \cap \cdots \cap Y_{i_k}$ and a disjoint base point. All morphisms between the scheme components are given by the natural closed immersions between

them. The condition on the tor-dimension required in B.2.13 follows because are schemes are regular. T, Y and X are all flat over S, hence the maps in the diagram

$$\begin{tikzcd} & X \times_S Y \arrow[d] \\ T \times_S X \arrow[r] & X \times_S X \end{tikzcd}$$

are easily seen to be tor-independent. The inclusions of T and Y into X are trivially tor-independent because this is a local condition. $\square$

Basically this lemma tells us that all conditions hold that are needed to apply the machinery of appendix B.2. We have a well-behaved relative motivic cohomology theory (cf. B.2.11).

Now we return to the geometric situation set up in section 3. We consider

$$\begin{tikzcd} Z_\cdot^{(n)} \arrow[r] \arrow[d] & \mathbb{G}_{m,S}^n \arrow[d] \\ \overline{Z}_\cdot^{(n)} \arrow[r] & \mathbb{A}_S^n \end{tikzcd}$$

where $Z = \overline{Z} = \alpha(S) \amalg \beta(S)$ with disjoint S-rational points α and β of $\mathbb{G}_{m,S}$. There is a simplicial operation of $\mathfrak{S}^n$ on the situation which induces an operation on relative K-cohomology and on motivic cohomology.

Proposition 7.2. *There is a natural residue map*

$$H^i_{\mathcal{M}}(\mathbb{G}^n_{m,S} \ rel\ Z_\cdot^{(n)}, j)^{sgn} \xrightarrow{\mathrm{res}_n} H^{i-1}_{\mathcal{M}}(\mathbb{G}^{n-1}_{m,S} \ rel\ Z_\cdot^{(n-1)}, j-1)^{sgn}$$

where sgn means the sign eigen-space under the operation of the respective symmetric group.

Moreover, there is a long exact sequence

$$\begin{aligned} \cdots \longrightarrow & H^{i-2}_{\mathcal{M}}(\mathbb{G}^{n-1}_{m,S} \ rel\ Z_\cdot^{(n-1)}, j-1)^{sgn} \longrightarrow H^i_{\mathcal{M}}(\mathbb{A}^n_S \ rel\ \overline{Z}_\cdot^{(n)}, j)^{sgn} \\ & \longrightarrow H^i_{\mathcal{M}}(\mathbb{G}^n_{m,S} \ rel\ Z_\cdot^{(n)}, j)^{sgn} \\ \longrightarrow & H^{i-1}_{\mathcal{M}}(\mathbb{G}^{n-1}_{m,S} \ rel\ Z_\cdot^{(n-1)}, j-1)^{sgn} \longrightarrow \cdots \ . \end{aligned}$$

Under the regulators, the long exact sequences are compatible with the ones in absolute cohomology (after 4.5).

Remark: Recall that $Z_\cdot^{(0)} = \star$ and hence $H^k_{\mathcal{M}}(\mathbb{G}^0_{m,S} \text{ rel } Z_\cdot^{(0)}, j) = H^k_{\mathcal{M}}(S, j)$ by definition.

Proof. We filter $\mathbb{A}^n_S$ by the open subschemes $F_k\mathbb{A}^n_S$ defined just before Lemma 4.5. In particular, $F_0\mathbb{A}^n_S = \mathbb{G}^n_{m,S}$. Again $G_k\mathbb{A}^n_S = F_k\mathbb{A}^n_S \smallsetminus F_{k-1}\mathbb{A}^n_S$. We use the notation $F_k\mathbb{A}_\cdot^{\vee n}$ and $G_k\mathbb{A}_\cdot^{\vee n}$ for the induced open respectively locally closed subspaces of $\mathbb{A}_\cdot^{\vee n}$. Note that the situation is still symmetric under permutation of coordinates. Hence there is a compatible operation of the symmetric group on the space constructed from schemes $F_k\mathbb{A}_\cdot^{\vee n}$.

The closed immersion $G_k\mathbb{A}^{\vee n}_{\cdot} \to F_k\mathbb{A}^{\vee n}_{\cdot}$ satisfies the first condition in (TC) in B.2.13. The maps we have to consider for the rest of (TC) are locally of the form considered in 7.1.d). Hence B.2.19 applies, i.e., we can use the localization sequences for motivic cohomology induced by the triples $F_{k-1}\mathbb{A}^{\vee n}_{\cdot} \to F_k\mathbb{A}^{\vee n}_{\cdot} \leftarrow G_k\mathbb{A}^{\vee n}_{\cdot}$. We get

$$\begin{aligned} \ldots \longrightarrow & H^i_{\mathcal{M}}(G_k\mathbb{A}^{\vee n}_{\cdot}, j) \longrightarrow H^{i+2}_{\mathcal{M}}(F_k\mathbb{A}^{\vee n}_{\cdot}, j+1) \longrightarrow H^{i+2}_{\mathcal{M}}(F_{k-1}\mathbb{A}^{\vee n}_{\cdot}, j+1) \\ \longrightarrow & H^{i+1}_{\mathcal{M}}(G_k\mathbb{A}^{\vee n}_{\cdot}, j) \longrightarrow \ldots \end{aligned}$$

The sequence remains exact when we take sign–eigenspaces. Now let us compute one of the groups involved.

$$H^i_{\mathcal{M}}(G_k\mathbb{A}^{\vee n}_{\cdot}, j) = \bigoplus_{\{1 \le a_1 < a_2 < \cdots < a_k \le n\}} H^i_{\mathcal{M}}(\mathbb{A}^{\vee n}_{\cdot} \times_{\mathbb{A}^n} \mathbb{G}_{m,S}(a_1, \ldots, a_k), j)$$

where

$$\mathbb{G}_{m,S}(a_1, \ldots, a_k) = \{(x_1, \ldots, x_n) \mid x_i = 0 \text{ if } i = a_j \text{ for some } j;\ x_i \neq 0 \text{ else }\}\ .$$

The decomposition corresponds to the decomposition of $G_k\mathbb{A}^n$ into its connected components. The notation $\mathbb{A}^{\vee n}_{\cdot} \times_{\mathbb{A}^n} \mathbb{G}_{m,S}(a_1, \ldots, a_k)$ means the open subspace lying over the locally closed scheme. Now consider the operation of the symmetric group. If $k > 1$, then there is for each component some transposition which acts trivially, namely one that interchanges two vanishing coordinates. Hence the sign–eigenspace vanishes altogether. For $k = 1$, the decomposition has the form

$$H^i_{\mathcal{M}}(G_1\mathbb{A}^{\vee n}_{\cdot}, j) = \bigoplus_{a=1,\cdots,n} H^i_{\mathcal{M}}(\mathbb{A}^{\vee n}_{\cdot} \times_{\mathbb{A}^n} (\mathbb{G}^{a-1}_{m,S} \times \{0\} \times \mathbb{G}^{n-a}_{m,S}), j)\ .$$

The operation of the symmetric group permutes the factors transitively. The stabilizer of one summand is the symmetric group $\mathfrak{S}^{n-1}$. We get

$$H^i_{\mathcal{M}}(G_1\mathbb{A}^{\vee n}_{\cdot}, j)^{\text{sgn}} \cong H^i_{\mathcal{M}}(\mathbb{G}^{\vee n-1}_{m,S}, j)^{\text{sgn}}$$

where the sign eigenspace on the right hand side is taken with respect to the smaller symmetric group $\mathfrak{S}^{n-1}$. We have a choice of isomorphism here and use the one that identifies $\mathbb{G}^{n-1}_{m,S}$ with $\mathbb{G}^{n-1}_{m,S} \times \{0\}$. Putting these results in the long exact sequences we get iteratively

$$H^i_{\mathcal{M}}(\mathbb{A}^n \text{ rel } \overline{Z}^{(n)}_{\cdot}, j)^{\text{sgn}} = H^i_{\mathcal{M}}(F_n\mathbb{A}^{\vee n}_{\cdot}, j)^{\text{sgn}} \xrightarrow{\cong} \cdots H^i_{\mathcal{M}}(F_1\mathbb{A}^{\vee n}_{\cdot}, j)^{\text{sgn}}\ .$$

So the above sequence, for $k = 1$, gives the desired residue sequence. We can do the same construction for absolute cohomology (Hodge or l-adic) considered as generalized cohomology theories. By B.4.6, B.5.8 and B.3.7, the long exact sequences for motivic cohomology will be compatible via the regulator with the ones in generalized cohomology. The next step is to pass from generalized cohomology to cohomology of abelian sheaves. By B.4.5 and B.5.7 this can be done. In fact we get precisely the residue sequence for absolute cohomology constructed in section 4. □

REMARK: a) By B.2.19, we have the same maps and long exact sequences for the K-cohomology groups themselves. However, note that there is a Riemann-Roch hidden in the compatibility of the localization sequence in K-cohomology and absolute cohomology.
b) We shall show injectivity of the Beilinson regulator on

$$H^{n+1}_{\mathcal{M}}(\mathbb{G}^n_{m,S} \text{ rel } Z^{(n)}_{\cdot}, n)^{\mathrm{sgn}}$$

in Proposition 8.7. Together with Lemma 4.4.b), it shows that the residue map on

$$H^{i}_{\mathcal{M}}(\mathbb{G}^n_{m,S} \text{ rel } Z^{(n)}_{\cdot}, j)^{\mathrm{sgn}}$$

does not depend on the choice of embedding of $\mathbb{G}^{n-1}_{m,S}$ in

$$\bigcup_{a=1,\cdots,n} \mathbb{G}^{a-1}_{m,S} \times \{0\} \times \mathbb{G}^{n-a}_{m,S}$$

of the above proof, if $(i,j) = (n+1,n)$. Since we are only interested in these special indices, we chose to exclude from the statement of 7.2 the dependence of res_n in the general case from the above choice.

LEMMA 7.3. *Let $2j \geq k$. Then*

$$H^k_{\mathcal{M}}(\mathbb{A}^n_S \text{ rel } \overline{Z}^{(n)}_{\cdot}, j) \cong H^{k-n}_{\mathcal{M}}(S,j)$$

where the isomorphism is induced by a choice of ordering of the sections α and β. It is compatible with the identification in 4.6 under the regulator map. $\mathfrak{S}_n$ operates by sign on the left hand side.

REMARK: Here and in the sequel we put $H^i_{\mathcal{M}}(S,j) = 0$ if $j < 2i$. This makes sense as S is regular and the corresponding K–group vanishes (see B.2.3).

Proof. Fix j. We consider the skeletal spectral sequence B.2.12. We have

$$E_1^{p,q} = H^q_{\mathcal{M}}((\mathbb{A}^{\vee n}_S)_p, j) \ .$$

We will show that the only non-trivial E_2-terms are concentrated in one vertical line

$$E_2^{n,q} = H^q_{\mathcal{M}}(S,j) \ .$$

This means that the spectral sequence converges in the strongest possible way. This yields isomorphisms as stated. Before we can check this we need some preparation. If $X_{\cdot}$ is a space constructed from schemes, we denote by $Cp(X_{\cdot})$ the simplicial set of its connected components. $Cp(\overline{Z}^{(n)}_{\cdot})$ has the same singular cohomology as $Cp(\overline{Z}^{\Delta(n)}_{\cdot})$ (cf. proof of 7.1) which is the simplicial set attached to a CW-complex dual to the boundary of the n-dimensional hypercube (note that $\overline{Z}$ has two disjoint components). This means that $Cp(\overline{Z}^{\Delta(n)}_{\cdot})$ has a 1-vertex for every $(n-1)$-cell of the cube etc. In particular we see that it has the homotopy type of an $(n-1)$-sphere. $Cp(\mathbb{A}^n_S)$ is of course contractible. It follows that $Cp(\mathbb{A}^{\vee n}_{\cdot})$ has singular cohomology concentrated in degree n where it is one-dimensional.

Let us make this more explicit:
In order to compute the cohomology of a cosimplicial group it suffices to consider the sub-complex corresponding to nondegenerate simplices. $Cp(\overline{Z}_\cdot^{\Delta(n)})$ is completely degenerate from cosimplicial degree n on. In degree $n-1$, there is one nondegenerate simplex for each vertex of the hypercube. They are indexed by $\{\alpha,\beta\}^n$. Hence any element of $H^nCp(\mathbb{A}_\cdot^{\vee n}) = H^{n-1}(Cp(\overline{Z}_\cdot^{\Delta(n)}))$ is represented by an element of

$$K^{n-1} = \bigoplus_{\{\alpha,\beta\}^n} \mathbb{Q} \, .$$

Let g be a generator of the cohomology group. $Cp(\overline{Z}_\cdot^{(n)})$ does not become degenerate. The nondegenerate part in degree $n-1$ is given by one copy of $\{\alpha,\beta\}^n$ for each possible permutation of the numbers $0,\dots,n-1$. It is easy to see that $((-1)^{\mathrm{sgn}(\sigma)}g)_\sigma$ is in the kernel of the differential. It represents the generator of cohomology of $Cp(\overline{Z}_\cdot^{(n)})$. We see that $\mathfrak{S}_n$ operates by the sign of the permutation.

We choose the generator g of cohomology given by the tuple

$$(-1)^{s(i_1)+\dots+s(i_n)} \in \mathbb{Q}_{i_1\times\dots\times i_n}$$

where $i_k \in \{\alpha,\beta\}$ and $s(\alpha)=1$, $s(\beta)=0$. This choice of generator amounts to picking the ordering $\alpha<\beta$ and extending it by the Künneth-formula. Now let us analyze our E_1-term: For fixed q we have the complex attached to the cosimplicial abelian group $H^q_{\mathcal{M}}((\mathbb{A}_S^{\vee n})_p, j)_{p\in\mathbb{N}_0}$. All connected components of $\mathbb{A}_\cdot^{\vee n}$ are isomorphic to a copy of some power of $\mathbb{A}^1_S$. By the homotopy property of K-theory we have

$$H^q_{\mathcal{M}}((\mathbb{A}_S)_p^{\vee n}, j)_{p\in\mathbb{N}_0} = H^q_{\mathcal{M}}(S,j)\otimes_{\mathbb{Q}} C_\cdot^{\vee n}$$

where $C_\cdot^{\vee n}$ is the cosimplicial vector space computing singular cohomology of $Cp(\mathbb{A}_\cdot^{\vee n})$. By the previous considerations we already know its cohomology. It also follows that the operation of $\mathfrak{S}_n$ on our motivic cohomology is by the sign.
Now compare our isomorphism to the one constructed in the realization. We have the same spectral sequence there (attached to the weight filtration). The identification of the E_2-term also uses Künneth-formula and choice of an ordering of the sections. $\square$

Using this identification we obtain the *motivic residue sequence*:

$$\begin{aligned}\dots \longrightarrow & H^{k-n}_{\mathcal{M}}(S,j) \longrightarrow H^k_{\mathcal{M}}(\mathbb{G}_{m,S}^{\vee n}, j)^{\mathrm{sgn}} \longrightarrow H^{k-1}_{\mathcal{M}}(\mathbb{G}_{m,S}^{\vee n-1}, j-1)^{\mathrm{sgn}}\\ \longrightarrow & H^{k-n+1}_{\mathcal{M}}(S,j) \longrightarrow \dots\end{aligned}$$

for $2j\geq k$. By construction, we have the following:

Theorem 7.4. *Under the regulator, the motivic residue sequence maps to the absolute residue sequence of section 4.*

Note that the residue sequences for all indices k and n organize into a spectral sequence connecting the relative motivic cohomology of $\mathbb{A}_\cdot^{\vee ?}$ and the relative motivic cohomology of $\mathbb{G}_m^{\vee ?}$. In particular for each n there is the converging cohomological spectral sequence

$$E_1^{pq} = H^{p+q-n}_{\mathcal{M}}(S,p) \Rightarrow H^{p+q}_{\mathcal{M}}(\mathbb{G}_{m,S}^{\vee n}, n) = H^{p+q}_{\mathcal{M}}(\mathbb{G}_{m,S}^n \text{ rel } Z_\cdot^{(n)}, n)\, .$$

This is the motivic version of the weight spectral sequence in absolute cohomology. We refer to it as the *motivic residue spectral sequence*.

Remark: As in section 6, the residue sequence, or equivalently, the residue spectral sequence turns out to be the central technical tool in the construction of the motivic polylog (see Definition 8.9). The spectral sequence is identical to the one constructed in [BD1], 4.2.6. The definition and basic properties of motivic cohomology of simplicial schemes (B.1, B.2) allow to justify the construction.

At this point, we should stress that the proof of the innocent looking Theorem 7.4 requires the whole of the theory covered in the appendices.

8 Universal Motivic Polylogarithm

We now return to the special situation used in section 6. Let $B = \operatorname{Spec}(\mathbb{Z})$. We consider now the case $S = \mathbb{U}$. Let $\alpha = 1$, and β the diagonal section of $\mathbb{U} \times_B \mathbb{G}_{m,B}$.

First we compute the motivic cohomology of $\mathbb{U}$. We use the embedding of $\mathbb{U}$ into $\mathbb{A}^1$ to do so. The long exact localization sequence B.2.18 reads

$$\begin{aligned}\ldots \longrightarrow & H^{n-2}_{\mathcal{M}}(0(B) \amalg 1(B), j-1) \longrightarrow H^n_{\mathcal{M}}(\mathbb{A}^1_B, j) \longrightarrow H^n_{\mathcal{M}}(\mathbb{U}, j)\\ \longrightarrow & H^{n-1}_{\mathcal{M}}(0(B) \amalg 1(B), j-1) \longrightarrow \ldots\end{aligned}$$

By the homotopy property of K-theory we get

$$\begin{aligned}\ldots \longrightarrow & H^n_{\mathcal{M}}(B, j) \longrightarrow H^n_{\mathcal{M}}(\mathbb{U}, j) \longrightarrow H^{n-1}_{\mathcal{M}}(B, j-1) \oplus H^{n-1}_{\mathcal{M}}(B, j-1)\\ \longrightarrow & H^{n+1}_{\mathcal{M}}(B, j) \longrightarrow \ldots\end{aligned}$$

The Gysin map for the inclusion of a point in the affine line vanishes by [Q2] Thm 8 ii. Hence we are actually dealing with a system of short exact sequences. As all motivic cohomology groups of B vanish for $n > 1$ this sequence only gives non-trivial cohomology of $\mathbb{U}$ for $n = 0, 1, 2$.

Lemma 8.1. *For $B = \operatorname{Spec}(\mathbb{Z})$ we have*

$$\begin{aligned}
H^0_{\mathcal{M}}(\mathbb{U}, i) &= \begin{cases} \mathbb{Q} & \text{if } i = 0,\\ 0 & \text{else,}\end{cases}\\
H^1_{\mathcal{M}}(\mathbb{U}, j) &= \begin{cases} 0 & \text{for } j < 1,\\ \mathbb{Q} \oplus \mathbb{Q} & \text{for } j = 1,\\ H^1_{\mathcal{M}}(B, j) & \text{for } j > 1,\end{cases}\\
H^2_{\mathcal{M}}(\mathbb{U}, j) &= H^1_{\mathcal{M}}(B, j-1) \oplus H^1_{\mathcal{M}}(B, j-1)\ ,\\
H^n_{\mathcal{M}}(\mathbb{U}, j) &= 0 \text{ if } n > 2.
\end{aligned}$$

Proof. Clear from the above using B.2.20 □

By Borel's Theorem (B.5.9) the Beilinson regulator

$$H^i_{\mathcal{M}}(X, j) \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow H^i_{\mathfrak{H}^p}(X_{\mathbb{R}}/\mathbb{R}, j)$$

is injective for $X = \operatorname{Spec}(\mathbb{Z})$, even an isomorphism but in the one case $H^1_{\mathcal{M}}(B, 1)$ where the codimension is one. (We call Beilinson regulator what strictly speaking is its tensor product with $\mathbb{R}$.) This implies that it is also an isomorphism for $H^i_{\mathcal{M}}(\mathbb{U},k)$ with the exception of the indices $(1,1)$ and $(2,2)$ where the codimension is 1 resp. 2.

This means that many of the residue maps are actually isomorphisms. The following computations are carried out in the case $B = \mathrm{Spec}(\mathbb{Z})$. With a little more effort they generalize to the case of the ring of integers of a number field.

Consider the residue sequence for $n = j = 1$ and $S = \mathbb{U}$.

$$\begin{aligned} 0 = H^0_{\mathcal{M}}(\mathbb{U},1) \longrightarrow H^1_{\mathcal{M}}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1) \longrightarrow H^0_{\mathcal{M}}(\mathbb{U},0) \\ \longrightarrow H^1_{\mathcal{M}}(\mathbb{U},1) \xrightarrow{\delta} H^2_{\mathcal{M}}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1) \longrightarrow H^1_{\mathcal{M}}(\mathbb{U},0) = 0\ . \end{aligned}$$

The Beilinson regulator induces a map between the above sequence and the residue sequence in section 4. On $H^0_{\mathcal{M}}(\mathbb{U},0)\otimes\mathbb{R}$, the regulator is an isomorphism, and on $H^1_{\mathcal{M}}(\mathbb{U},1)\otimes\mathbb{R}$ it is injective of codimension one. By 6.4, the absolute Hodge cohomology group $H^1_{\mathfrak{H}^p}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}_{\mathbb{R}}}/\mathbb{R},1)$ vanishes. Hence the map from the first to the second line is injective and the regulator is injective of codimension one on $H^2_{\mathcal{M}}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1)$. Furthermore, this last group is one dimensional.

The image of δ under the Beilinson regulator is the map occurring in 6.8 for $n = 1$.

Definition 8.2. *Let s_1 be the composition of the maps*

$$\mathbb{Q} \quad = \quad H^0_{\mathcal{M}}(B,0) \quad \xrightarrow{i_1} \quad \bigoplus_{i=0,1} H^0_{\mathcal{M}}(B,0) \quad = \quad H^1_{\mathcal{M}}(\mathbb{U},1) \quad \xrightarrow{\delta} \quad H^2_{\mathcal{M}}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1)$$

where i_1 is the inclusion of the 1-summand and δ is the map of the residue sequence.

Lemma 8.3. *s_1 is an isomorphism.*

Proof. Because of dimension reasons we only have to check that δ does not vanish on the image of i_1. This follows from 6.8. □

Definition 8.4. *Let res_1 be the inverse of s_1. We define the total residue map*

$$\mathrm{res} : H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{sgn} \longrightarrow \mathbb{Q}\ .$$

by composition of the residue maps in our long exact sequence 7.2 with res_1.

We now have to check that the total residue map deserves its name. By definition and 6.5.i) it suffices to consider res_1.

Lemma 8.5. *The regulators map the motivic res_1 to res_1 in absolute cohomology.*

Proof. Let us consider the situation of 6.8. The morphism

$$\mathcal{O}(\mathbb{U})^* \longrightarrow H^1_{\mathfrak{H}^p}(\mathbb{U}_{\mathbb{R}}/\mathbb{R},1)$$

factors through $H^1_{\mathcal{M}}(\mathbb{U},1) = K_1(\mathbb{U})_{\mathbb{Q}}$. There is a commutative diagram

$$\begin{tikzcd} & H^1_{\mathfrak{H}^p}(\mathbb{U}_{\mathbb{R}}/\mathbb{R},1) \arrow[r] & \bigoplus_{i=0,1} H^0_{H_{\mathfrak{H}^p}}(B_{\mathbb{R}}/\mathbb{R},0) \\ \mathcal{O}(\mathbb{U})^* \arrow[r] & H^1_{\mathcal{M}}(\mathbb{U},1) \arrow[u] \arrow[r, "\cong"] & \bigoplus_{i=0,1} H^0_{\mathcal{M}}(B,0) \arrow[u, "\cong"'] \end{tikzcd},$$

hence the functions t and $1-t$ on $\mathbb{U}$ correspond to the canonical generators of the two summands. We consider the commutative diagram for absolute Hodge cohomology

$$\begin{array}{ccccc} H^1_{\mathfrak{H}^p}(\mathbb{U}_\mathbb{R}/\mathbb{R},1) & \xrightarrow{\delta_\mathbb{R}} & H^2_{\mathfrak{H}^p}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}_\mathbb{R}}/\mathbb{R},1) & \xrightarrow{\mathrm{res}_\mathbb{R}} & H^0_{\mathfrak{H}^p}(B_\mathbb{R}/\mathbb{R},0) \\ \uparrow & & \uparrow r & & \\ \bigoplus_{i=0,1} H^0_\mathcal{M}(B,0) & \xrightarrow{\delta} & H^2_\mathcal{M}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1) & & \end{array},$$

By 6.8 the composition from the bottom left to the top right corner is given by the projection to the 1-component tensored by $\mathbb{R}$. It follows that $(\mathrm{res}_R \circ r) \otimes \mathbb{R}$ is an isomorphism. In turn δ vanishes on the 0-component and is an isomorphism on the 1-component. But then by definition $\mathrm{res}_1 \circ \delta$ is also the projection to the 1-summand. As δ is surjective, this suffices. The same argument works in the étale situation. $\square$

LEMMA 8.6. *There is a short exact sequence*

$$0 \longrightarrow H^1_\mathcal{M}(B,2) \longrightarrow H^3_\mathcal{M}(\mathbb{G}^{\vee 2}_{m,\mathbb{U}},2)^{sgn} \xrightarrow{res} \mathbb{Q} \longrightarrow 0$$

and the Beilinson regulator is an isomorphism on the middle term.

Proof. This is nothing but the residue sequence using our computation of $H^2_\mathcal{M}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1)$. The zeroes on both sides come from vanishing cohomology groups. Comparison with the short exact sequence 6.5.ii) shows that the regulator is an isomorphism. $\square$

PROPOSITION 8.7. *There are short exact sequences*

$$0 \longrightarrow H^1_\mathcal{M}(B,n) \xrightarrow{\delta_n} H^{n+1}_\mathcal{M}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{sgn} \xrightarrow{\mathrm{res}} \mathbb{Q} \longrightarrow 0\,.$$

The Beilinson regulator is injective on all $H^{n+1}_\mathcal{M}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{sgn}$. It is even an isomorphism for $n>1$.

Proof. The $n=1$ and $n=2$ cases are the previous lemmas. By induction, one checks that all $H^n_\mathcal{M}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{\mathrm{sgn}}$ vanish for $n \geq 1$. Hence the residue sequence reads

$$0 \to H^1_\mathcal{M}(B,n) \xrightarrow{\delta_n} H^{n+1}_\mathcal{M}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{\mathrm{sgn}} \to H^n_\mathcal{M}(\mathbb{G}^{\vee n-1}_{m,\mathbb{U}},n-1)^{\mathrm{sgn}} \to H^2_\mathcal{M}(\mathbb{U},n)\,.$$

By the five lemma and inductive hypothesis we see that the regulator is an isomorphism on the middle term for n. We need the previous lemma to get started.
Now consider the sequences of the proposition. All maps are well-defined. It follows from 6.5.ii) that the sequence is exact. $\square$

COROLLARY 8.8. *There are canonical splittings $s_n : \mathbb{Q} \to H^{n+1}_\mathcal{M}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{sgn}$ such that the diagram*

$$\begin{array}{ccc} H^0_\mathcal{M}(B,0) & \xrightarrow{s_n} & H^{n+1}_\mathcal{M}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{sgn} \\ & s_{n-1}\searrow & \downarrow \mathrm{res} \\ & & H^n_\mathcal{M}(\mathbb{G}^{\vee n-1}_{m,\mathbb{U}},n-1)^{sgn} \end{array}$$

commutes. They are compatible with the ones in 6.5.iii). Furthermore, the group $\varprojlim H^{n+1}_\mathcal{M}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{sgn}$ is canonically isomorphic to $\mathbb{Q}$.

Proof. $\mathrm{Im}(res_n)$ is isomorphic to $\mathbb{Q}$ by the total residue on $H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,\mathbb{U}}, n)^{\mathrm{sgn}}$. This induces the same splitting as in 6.5. □

Definition 8.9. *For $n \in \mathbb{N}$ the system $pol_n = s_n(1)$ defines the universal motivic polylogarithm.*

By construction pol_n is mapped to the polylogarithmic system in absolute Hodge cohomology and continuous étale cohomology.

Remark: The main result of this section, 8.7 is identical to [BD1], 4.3.4. Although part of the argument involves only constructions within K-theory, the proof of 8.7 relies heavily on a detailed analysis of the behaviour of the regulator between the motivic and absolute residue sequences.

9 The Cyclotomic Case

Let $d \geq 2$. As before let $R = A[1/d, T]/\Phi_d(T)$ the ring of d-integers of the cyclotomic field of d-th roots of unity. Put $C = \operatorname{Spec} R$. Let ζ be a primitive d-th root of unity in $\overline{\mathbb{Q}}$, and b an integer prime to d. We work in the situation $S = C$, $\alpha = 1 \in \mathbb{G}_m(C)$, and $\beta = i_b \in \mathbb{G}_m(C)$ as in section 5.

Lemma 9.1. a) *For $n \geq 0$ we have*

$$H^n_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C}, n)^{sgn} = H^n_{\mathcal{M}}(\mathbb{G}^n_{m,C} \ rel\ Z^{(n)}_{\cdot}, n)^{sgn} = \mathbb{Q} \quad .$$

The Beilinson and the l-adic regulators are isomorphisms.

b) *For $n \geq 1$, the residue sequence induces short exact sequences*

$$0 \longrightarrow H^1_{\mathcal{M}}(C, n) \longrightarrow H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C}, n)^{sgn} \longrightarrow H^n_{\mathcal{M}}(\mathbb{G}^{\vee n-1}_{m,C}, n-1)^{sgn} \longrightarrow 0 \quad .$$

The l-adic regulator is injective on the group $H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C}, n)^{sgn}$ for $n \geq 1$.

Proof. For $n = 0$ we have $H^0_{\mathcal{M}}(\mathbb{G}^{\vee 0}_{m,C}, 0) = H^0_{\mathcal{M}}(C, 0)$, which is canonically isomorphic to $\mathbb{Q}$ by B.2.20. In particular both regulator are isomorphisms.
$H^1_{\mathcal{M}}(\mathbb{G}^{\vee 0}_{m,C}, 0)$ and its counterpart in absolute cohomology vanish.
Consider the following bit of the residue sequence for $n \geq 1$:

$$H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n+1}_{m,C}, n+1)^{\mathrm{sgn}} \to H^n_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C}, n)^{\mathrm{sgn}} \to H^1_{\mathcal{M}}(C, n+1)$$

The first map is injective since $H^0_{\mathcal{M}}(C, n+1) = 0$. The l-adic regulator is always injective on the last term by B.4.8. By inductive hypothesis it is an isomorphism on the middle term. By Cor. 5.3, the last map vanishes in absolute cohomology. This implies a) for $n+1$. In the next bit of the long exact sequence

$$H^1_{\mathcal{M}}(C, n) \longrightarrow H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C}, n)^{\mathrm{sgn}} \longrightarrow H^n_{\mathcal{M}}(\mathbb{G}^{\vee n-1}_{m,C}, n-1)^{\mathrm{sgn}} \longrightarrow (*) \quad ,$$

the first map is injective by a). For $n \geq 2$ we have $(*) = H^2_{\mathcal{M}}(C, n) = 0$, while for $n = 1$ the term

$$H^n_{\mathcal{M}}(\mathbb{G}^{\vee n-1}_{m,C}, n-1) = H^1_{\mathcal{M}}(C, 0)$$

vanishes. Hence in any case we end up with the short exact sequence in b). The regulator maps it to the short exact sequence 5.3. By induction and B.4.8 we can control the injectivity of the l–adic regulator. □

REMARK: The Beilinson regulator is not injective on $H^1_{\mathcal{M}}(C,1)$ because d is inverted in C.

Consider the morphism $\phi : \mathbb{G}_{m,C} \to \mathbb{G}_{m,C}$ that raises points to the $d+1$-th power. As in section 5 it induces a morphism of spaces $\phi^n : \mathbb{A}_C^{\vee n} \to \mathbb{A}_C^{\vee n}$. By contravariance it induces an operation on motivic cohomology.

LEMMA 9.2 ([BD1], REMARK (II) ON PAGE 78).
$(\phi^n)^*$ operates on the short exact sequence of the previous lemma as follows:

$$\begin{array}{ccccc} H^1_{\mathcal{M}}(C,n) & \longrightarrow & H^{n+1}_{\mathcal{M}}(\mathbb{G}_{m,C}^{\vee n},n)^{sgn} & \longrightarrow & H^n_{\mathcal{M}}(\mathbb{G}_{m,C}^{\vee n-1},n-1)^{sgn} \\ {\scriptstyle id}\downarrow & & {\scriptstyle (\phi^n)^*}\downarrow & & \downarrow{\scriptstyle (d+1)(\phi^{n-1})^*} . \\ H^1_{\mathcal{M}}(C,n) & \longrightarrow & H^{n+1}_{\mathcal{M}}(\mathbb{G}_{m,C}^{\vee n},n)^{sgn} & \longrightarrow & H^n_{\mathcal{M}}(\mathbb{G}_{m,C}^{\vee n-1},n-1)^{sgn} \end{array}$$

Proof. This description follows immediately from the injectivity of the l-adic regulator and Cor. 5.3.b). □

REMARK: The operation $(\phi^n)^*$ on $H^1_{\mathcal{M}}(C,n)$ is given by the operation on $H^{n+1}_{\mathcal{M}}(\mathbb{A}_C^{\vee n},n)$. It is easy to check that it is trivial by considering the operation on the starting terms of the degenerating skeletal spectral sequence. To understand the compatibility with the residue map in terms of K-theory is a lot harder. The factor $d+1$ is induced by a push-forward from a non-reduced scheme to its reduction. The theory in Appendix B is not even set up to handle such schemes.

As in the case of absolute cohomology it follows that the eigenvalues of $(\phi^n)^*$ on $H^{n+1}_{\mathcal{M}}(\mathbb{G}_{m,C} \text{ rel } Z^{(n)},n)^{\text{sgn}}$ are $1, d+1, \dots, (d+1)^{n-1}$.

LEMMA 9.3. *The eigenspace decomposition yields a splitting*

$$\eta_b^{(n)} : H^{n+1}_{\mathcal{M}}(\mathbb{G}_{m,C}^{\vee n},n)^{sgn} \xrightarrow{\sim} \bigoplus_{1\le i\le n} H^1_{\mathcal{M}}(C,i) ,$$

which is compatible with the splitting $\eta_b^{(n)}$ after Cor. 5.3. There is a canonical isomorphism

$$\eta_b : \varprojlim H^{n+1}_{\mathcal{M}}(\mathbb{G}_{m,C}^{\vee n},n)^{sgn} \xrightarrow{\sim} \prod_{i\ge 1} H^1_{\mathcal{M}}(C,i) .$$

Proof. The first assertion is clear by construction. The second follows because the eigenspace decomposition is compatible with the residue map. □

DEFINITION 9.4. *Let $i_b : C \to \mathbb{U}$ be as before. Let pol_b be the pullback of the universal polylogarithm system pol defined in 8.9 to the inverse limit $\varprojlim H^{n+1}_{\mathcal{M}}(\mathbb{G}_{m,C}^{\vee n},n)^{sgn} = \varprojlim H^{n+1}_{\mathcal{M}}(\mathbb{G}_{m,C}^n \text{ rel } Z^{(n)}_{\cdot},n)^{sgn}$. Via the isomorphism η_b of 9.3, we have constructed an element in $\prod_{i\ge 1} H^1_{\mathcal{M}}(C,i)$.*

THEOREM 9.5. *Under the regulators, the element*

$$pol_b \in \varprojlim H^{n+1}_{\mathcal{M}}(\mathbb{G}_{m,C}^{\vee n},n)^{sgn} = \prod_{i\ge 1} H^1_{\mathcal{M}}(C,i)$$

is mapped to the elements

$$pol_b \in \varprojlim H^{n+1}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C}, n)^{sgn} = \prod_{i \geq 1} H^1_{\mathrm{abs}}(C, i)$$

constructed at the end of section 5.

Proof. This follows from the construction. □

We list the consequences of this result: denote by μ_d^0 the set of primitive d–th roots of unity in $\mathbb{Q}(\mu_d)$.

Firstly, the description of the regulator to absolute Hodge cohomology yields an alternative proof of the following:

COROLLARY 9.6. *Assume* $n \geq 0$.

a) ([B2], 7.1.5, [Neu], II.1.1, [E], 3.9.)
There is a map of sets

$$\begin{aligned} \epsilon_{n+1} : \mu_d^0 \longrightarrow\ & H^1_{\mathcal{M}}(C, n+1) \\ & (= H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{Q}(\mu_d), n+1) \text{ for } n \geq 1) \end{aligned}$$

such that

$$\begin{aligned} r_{\mathcal{D}} \circ \epsilon_{n+1} : \mu_d^0 \longrightarrow\ & H^1_{\mathfrak{H}^p}(\operatorname{Spec} \mathbb{Q}(\mu_d)_{\mathbb{R}}/\mathbb{R}, n+1) \\ & \xrightarrow[A.2.12]{\sim} \left(\bigoplus_{\sigma : \mathbb{Q}(\mu_d) \hookrightarrow \mathbb{C}} \mathbb{C}/(2\pi i)^{n+1} \mathbb{R} \right)^+ \end{aligned}$$

maps a root of unity ω *to* $\left(-Li_{n+1}(\sigma\omega)\right)_\sigma$. *For* $n \geq 1$, *this property characterizes the map* ϵ_{n+1} *uniquely.*
b) For a root of unity $T^b \in \mathbb{Q}(\mu_d) = \mathbb{Q}[T]/\Phi_d(T)$, *the element*

$$\epsilon_{n+1}(T^b) \in H^1_{\mathcal{M}}(C, n+1)$$

is given by

$$\epsilon_{n+1}(T^b) := (-1)^n \cdot \frac{1}{(n+1)!} \cdot \left(\ (n+1)\text{–component of } pol_b \right) \ .$$

Proof. Note that a) really *is* Beilinson's formulation of the result: his normalization of the isomorphism

$$H^1_{\mathfrak{H}^p}(\operatorname{Spec} \mathbb{Q}(\mu_d)_{\mathbb{R}}/\mathbb{R}, n+1) \xrightarrow{\sim} \left(\bigoplus_{\sigma} \mathbb{C}/(2\pi i)^{n+1} \mathbb{R} \right)^+$$

differs from ours by the factor -1. The unicity assertion is a direct consequence of the injectivity of the regulator. So our claim follows from 2.5, and from 5.4. □

In [B2], the above compatibility statement is used to prove Gross's conjecture about special values of Dirichlet L-functions. An alternative proof of this conjecture, using an entirely different geometric construction, is given in section 3 of [Den].

Recall that the l–adic regulator r_l factorizes as follows:

$$\begin{aligned} K_{2n+1}(C) \otimes_{\mathbb{Z}} \mathbb{Q} = H^1_{\mathcal{M}}(C, n+1) &\hookrightarrow H^1_{\mathcal{M}}(C_{(l)}, n+1) \\ &\hookrightarrow H^1_{cont}(C_{(l)}, n+1) \\ &\hookrightarrow H^1_{cont}(\operatorname{Spec} \mathbb{Q}(\mu_d), n+1) \ , \end{aligned}$$

where we let $C_{(l)} := C \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{l}]$.

For the rest of this section, we fix $\zeta \in C(\overline{\mathbb{Q}})$. As was observed already in [B4], the study of the cyclotomic polylog yields a proof of the following result:

Corollary 9.7. *Assume $n \geq 0$.*
a) ([Sou5], Théorème 1 for the case $n = 1$; [Gr], Théorème IV.2.4 for the local version if $(l, d) = 1$.)
Let d and ϵ_{n+1} be as in 9.6. Let l be a prime. Under the embedding of 2.6, the l–adic regulator

$$r_l : H^1_{\mathcal{M}}(C, n+1) \longrightarrow H^1_{cont}(\operatorname{Spec} \mathbb{Q}(\mu_d), n+1)$$

maps $\epsilon_{n+1}(T^b)$ to

$$\frac{1}{d^n} \cdot \frac{1}{n!} \cdot \left(\sum_{\alpha^{l^r} = \zeta^b} [1 - \alpha] \otimes (\alpha^d)^{\otimes n} \right)_r \ .$$

b) Conjecture 6.2 of [BlK] holds.

Proof. a) is 2.6 and 5.4. As for b), it remains to check the comparison statement of [BlK], Conjecture 6.2 for the root of unity 1. For this, observe the relations

$$\begin{aligned} c_{n+1}(1) &= \frac{2^n}{1 - 2^n} c_{n+1}(-1) \ , \\ c_{n+1,2}(1) &= \frac{2^n}{1 - 2^n} c_{n+1,2}(-1) \end{aligned}$$

in the notation of loc. cit., if $n \geq 1$ ([D5], Proposition 3.13.1.i)). □

Soulé has constructed maps

$$\varphi_l : \mu_d^0 \to K_{2n+1}(C_{(l)}) \otimes_{\mathbb{Z}} \mathbb{Z}_l$$

for any prime l (see end of Appendix B.4 for more details).

The l–adic regulator

$$r_l : K_{2n+1}(C_{(l)}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to H^1_{cont}(\operatorname{Spec} \mathbb{Q}(\mu_d), n+1) \qquad \text{(Prop. B.4.10)}$$

takes $\varphi_l(T^b)$ to the cyclotomic element in continuous Galois cohomology

$$\left(\sum_{\alpha^{l^r} = \zeta^b} [1 - \alpha] \otimes (\alpha^d)^{\otimes n} \right)_r$$

defined by Soulé and Deligne (cf. [Sou2], page 384, [D5], 3.1, 3.3).

Corollary 9.8. *For each d and n, there is a unique map*

$$\varphi : \mu_d^0 \to K_{2n+1}(\operatorname{Spec} \mathbb{Q}(\mu_d))$$

such that for each prime number l, the map

$$\begin{aligned} \varphi_l : \mu_d^0 &\to K_{2n+1}(C_{(l)}) \otimes_{\mathbb{Z}} \mathbb{Z}_l \\ &\hookrightarrow K_{2n+1}(\operatorname{Spec} \mathbb{Q}(\mu_d)) \otimes_{\mathbb{Z}} \mathbb{Z}_l \end{aligned}$$

equals the composition of φ and the natural map

$$K_{2n+1}(\operatorname{Spec} \mathbb{Q}(\mu_d)) \to K_{2n+1}(\operatorname{Spec} \mathbb{Q}(\mu_d)) \otimes_{\mathbb{Z}} \mathbb{Z}_l \,.$$

Furthermore, the map $\varphi \otimes_{\mathbb{Z}} \mathbb{Q}$ agrees with

$$\epsilon'_{n+1} : \mu_d^0 \to H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{Q}(\mu_d), n+1)$$

given by $d^n \cdot n! \cdot \epsilon_{n+1}$.

Proof. The uniqueness assertion is a formal consequence of the finite generation of $K_{2n+1}(\operatorname{Spec} \mathbb{Q}(\mu_d))$: to give an element in a finitely generated abelian group M is the same as giving elements in $M \otimes_{\mathbb{Z}} \mathbb{Q}$ and all $M \otimes_{\mathbb{Z}} \mathbb{Z}_l$, which coincide in $M \otimes_{\mathbb{Z}} \mathbb{Q}_l$. By 9.7, the maps $r_l \circ \varphi_l$ and $r_l \circ \epsilon'_{n+1}$ agree for all l. From Theorem B.4.8, we conclude that φ_l and ϵ'_{n+1} agree as maps to $K_{2n+1} \otimes_{\mathbb{Z}} \mathbb{Q}_l$. ∎

As shown by Bloch and Kato, Corollary 9.7 implies the validity of the following also for even n:

Corollary 9.9. *Let $n \geq 1$.*
Then the Tamagawa number conjecture ([BlK], Conjecture 5.15) is true modulo a power of 2 for the motif $\mathbb{Q}(n+1)$.

Proof. [BlK], Theorem 6.1.i) gives the complete proof for odd n, which is independent of anything said in the present article. In loc.cit., Theorem 6.1.ii), it is shown that the conjecture holds for even n if [BlK], 6.2 holds. But the latter is the content of 9.7. ∎

Finally, the compatibility statement of 9.7 forms a central ingredient in the proof of the modified version of the Lichtenbaum conjecture for abelian number fields ([KNF], Theorem 6.4).

A Absolute Hodge Cohomology with Coefficients

The aim of this appendix is to provide a natural interpretation of absolute Hodge cohomology as extension groups in the category of algebraic Hodge modules over $\mathbb{R}$ (A.2.7). That such a sheaf–theoretic interpretation should be possible was already anticipated by Beilinson ([B1], 0.3), long before Hodge modules were defined.

The appendix is divided into two subsections. The first (A.1) starts with a summary of those parts of Saito's theory relevant to us. The central result is A.1.8, where we prove that for a smooth scheme $a : U \to \operatorname{Spec}(\mathbb{C})$, the polarizable Hodge complex $\underline{R\Gamma}(U, F)$ of [D3], (8.1.12) and [B1], § 4 is a representative for $a_* F(0)_U$, the object in the derived category of polarizable F–Hodge structures defined via Saito's formalism ([S2], 4.3). As a consequence, we are able (A.1.10) to identify absolute Hodge cohomology of a smooth scheme U over $\mathbb{C}$, as defined in [B1], § 5: it equals the Ext groups of Tate twists in the category of algebraic Hodge modules on U. The compatibility between the approaches of Deligne–Beilinson and of Saito will come as no surprise to the experts (see e.g. [S3], (2.8)). However, we were unable to find a quotable reference.

In A.2, we turn to the variant of the theory we really need: algebraic Hodge modules over $\mathbb{R}$. These live on the complexification of separated, reduced schemes of finite type over $\mathbb{R}$, and are basically the objects fixed by the natural involution on the category of mixed Hodge modules given by complex conjugation. The comparison statement for absolute Hodge cohomology over $\mathbb{R}$ (Theorem A.2.7) then follows formally from A.1.10.

A.1 Algebraic Mixed Hodge Modules

In [S2], § 4, the category $\mathrm{MHM}_A(X)$ of *algebraic mixed A–Hodge modules* is defined, where A is a field contained in $\mathbb{R}$, and X a separated reduced scheme of finite type over $\mathbb{C}$.

Saito's construction admits the full formalism of Grothendieck's functors $\pi_!$, $\pi^!$, π^*, π_*, $\underline{\mathrm{Hom}}$, $\otimes$, $\mathbb{D}$ on the level of bounded derived categories $D^b \mathrm{MHM}_A$ ([S2], 4.3, 4.4) and a forgetful functor

$$\mathrm{rat} : \mathrm{MHM}_A(X) \longrightarrow \mathrm{Perv}_A(\overline{X})$$

to the category of perverse sheaves on the topological space $\overline{X}$ underlying $X(\mathbb{C})$, which have algebraic stratifications such that the restrictions of their cohomology sheaves to the strata are local systems. By the definition of MHM_A, which we shall partly sketch in a moment, rat is faithful and exact. The functor rat on the level of derived categories is compatible with Grothendieck's functors ([S2], 4.3, 4.4).

For smooth X, one constructs $\mathrm{MHM}_A(X)$ as an abelian subcategory ([S1], Proposition 5.1.14) of the category $\mathrm{MF}_h\mathrm{W}(\mathcal{D}_X, A)$, whose objects are

$$((M, F^{\cdot}, W_{\cdot}), (K, W_{\cdot}), \alpha) \ ,$$

where $(M, F^{\cdot})$ is an object of the category $\mathrm{MF}_h(\mathcal{D}_X)$, i.e., a regular holonomic algebraic $\mathcal{D}_X$–module M together with a good filtration $F^{\cdot}$, and $K \in \mathrm{Perv}_A(\overline{X})$. $W_{\cdot}$ is a locally finite ascending filtration, and α is an isomorphism

$$DR(M) \xrightarrow{\sim} K \otimes_A \mathbb{C}$$

respecting $W_{\cdot}$. Here, DR denotes the de Rham functor from the category of $\mathcal{D}_X$–modules to the category of perverse sheaves.

We note that by definition, the weight graded objects of all algebraic Hodge modules satisfy a certain polarizability condition (see [S1], 5.2.10).

Call an algebraic Hodge module on a smooth variety *smooth* if the underlying perverse sheaf is a local system up to a shift.

Theorem A.1.1 (Saito). *Let X be smooth and separated. Then there is an equivalence*

$$\mathrm{Var}_A(X) \xrightarrow{\sim} \mathrm{MHM}_A(X)^s$$

between the category of admissible variations of mixed A–Hodge structure ([Ks]) and the category of smooth algebraic A–Hodge modules on X.

Proof. This is the remark following [S2], Theorem 3.27. ☐

In particular, we see that $\mathrm{MHM}_A(\mathrm{Spec}(\mathbb{C}))$ is the category MHS_A of polarizable mixed A–Hodge structures.

If $\mathbb{V}$ is a variation on X with underlying local system $\mathrm{For}(\mathbb{V})$, then the perverse sheaf underlying the Hodge module $\mathbb{V}$ under the correspondence of A.1.1 is

$$\mathrm{For}(\mathbb{V})[d]$$

if X is of pure dimension d.

It turns out that the definition of Tate twists in $\mathrm{MHM}_A(X)$ is compatible with the above equivalence only up to shift:

Definition A.1.2 ([S2], (4.5.5)). *Let $n \in \mathbb{Z}$, and $A(n) \in \mathrm{MHS}_A$ the usual Tate twist. For a separated reduced scheme $a : X \to \mathrm{Spec}(\mathbb{C})$, define*

$$A(n)_X := a^* A(n) \in D^b\,\mathrm{MHM}_A(X)\ .$$

If X is smooth and of pure dimension d, then $A(n)_X[d]$ is the variation of Hodge structure, which one denotes $A(n)$.

For arbitrary X, the complex $A(n)_X$ will not even be the shift of a Hodge module, but a proper element of $D^b\,\mathrm{MHM}_A(X)$, whose cohomology objects $\mathcal{H}^p A(n)_X$ are a priori trivial only for $p > \dim X$ ([S2], (4.5.6)).

We note again that we follow Saito's convention and write e.g. π_* for the functor on derived categories

$$D^b\,\mathrm{MHM}_A(X) \longrightarrow D^b\,\mathrm{MHM}_A(Y)$$

induced by a morphism $\pi : X \to Y$.

In order to compare the Hodge structures on Betti cohomology given by Saito's and Deligne's constructions, we need to go into the details of [S2]:

Theorem A.1.3 (Saito). *Let $j : U \hookrightarrow X$ be an open immersion of smooth separated schemes over $\mathbb{C}$, with $Y := X \setminus U$ a divisor with normal crossings. If X is of pure dimension d, then*

$$j_* A(0)_U[d] = \mathcal{H}^d j_* A(0)_U \in \mathrm{MHM}_A(X) \subset \mathrm{MF}_h\mathrm{W}(\mathcal{D}_X, A)$$

equals the object

$$(w_X(*Y), (j_{\mathrm{top}})_* A_U[d], \alpha)\ ,$$

where $w_X(*Y)$ denotes the $\mathcal{D}_X$–module $\Omega^d_X(\log Y)$, and $(j_{\mathrm{top}})_*$ the direct image for the derived category of perverse sheaves.

The de Rham complex with logarithmic singularities is quasi-isomorphic to $w_X(*Y) \overset{L}{\otimes}_{\mathcal{D}_X} \mathcal{O}_X[-d] = DR(w_X(*Y))[-d]$, *hence*

$$DR(w_X(*Y)) = \Omega^{\cdot}_X(\log Y)[d]$$

(compare [Bo3], VIII, 13.1), and

$$\alpha : \Omega^{\cdot}_X(\log Y)[d] \xrightarrow{\sim} (j_{\mathrm{top}})_* \mathbb{C}[d]$$

is the usual quasi-isomorphism

$$\Omega^{\cdot}_X(\log Y) \xrightarrow{\sim} (j_{\mathrm{top}})_* \Omega^{\cdot}_U \xleftarrow{\sim} (j_{\mathrm{top}})_* \mathbb{C}$$

(compare [D2], 3.1), shifted by d.

The Hodge filtration $F^{\cdot}$ *on* $w_X(*Y)$ *is induced from the stupid filtration, while the weight filtrations* $W_{\cdot}$ *on* $w_X(*Y)$ *and* $(j_{\mathrm{top}})_*\mathbb{C}[d]$ *are those induced from the canonical filtration on* $(j_{\mathrm{top}})_*\Omega_U$, *shifted by d.*

Proof. The equation $j_*A(0)_U[d] = \mathcal{H}^d j_* A(0)_U$ follows from the faithfulness of rat and the fact that the corresponding statement for $(j_{\mathrm{top}})_*$ is true since j is affine. In our geometric situation, the explicit construction of j_* of *any* admissible variation of A–Hodge structure is carried out in the proof of [S2], Theorem 3.27. For $A(0)_U$, it specializes to our claim. □

In [B1], 3.9, Beilinson extends Deligne's notion of Hodge complexes ([D3], 8.1) to the polarizable situation:

DEFINITION A.1.4 (BEILINSON). *A mixed A–Hodge complex*

$$K = ((K_{\mathbb{C}}, F^{\cdot}, W_{\cdot}), (K, W_{\cdot}), \alpha)$$

is called polarizable if the cohomology objects of the weight n Hodge complexes $\mathrm{Gr}^W_n(K)$ *are polarizable A–Hodge structures.*

REMARK: The weight filtration $W_{\cdot}$ of a mixed Hodge complex K induces mixed Hodge structures on its cohomology. Observe however that $\mathrm{Gr}^W_n(H^iK)$ is of weight $n+i$.

As in the non–polarizable situation, Beilinson proves:

THEOREM A.1.5 ([B1], LEMMA 3.11). *There is an equivalence of categories between* $D^b\,\mathrm{MHS}_A$ *and the derived category of polarizable A–Hodge complexes.*

Let X be smooth and separated over $\mathbb{C}$. Forgetting part of the structure of a Hodge module yields a functor

$$\mathrm{For} : C^b\,\mathrm{MHM}_A(X) \longrightarrow T(X)\ .$$

Here, $T(X)$ is the category of triples

$$M^{\cdot} = ((M^{\cdot}, F^{\cdot\cdot}, W^{\cdot}_{\cdot}), (K^{\cdot}, W^{\cdot}_{\cdot}), \alpha^{\cdot})\ ,$$

where $(M^\cdot, F^{\cdot\cdot}, W_\cdot^\cdot)$ is a class in the filtered derived category $D^bW(\mathrm{MF}_h(\mathcal{D}_X))$ of $\mathrm{MF}_h(\mathcal{D}_X)$, and $(K^\cdot, W_\cdot^\cdot)$ a class in the filtered derived category of sheaves of A–vector spaces on $X(\mathbb{C})$, denoted by $D^bW(X(\mathbb{C}), A)$. Furthermore, the map $\alpha^\cdot$ is an isomorphism

$$DR(M^\cdot) \xrightarrow{\sim} K^\cdot \otimes_A \mathbb{C}$$

respecting $W_\cdot^\cdot$.

Recall that in order to obtain a class in $D^bW(X(\mathbb{C}), A)$ from a complex of perverse sheaves, one applies the realization functor of [BBD], 3.1.9.

The global section functor Γ can be derived on $D^bW(X(\mathbb{C}), A)$. By [S1], 2.3, we have a functor $R\Gamma$ on $D^bW(\mathrm{MF}_h(\mathcal{D}_X))$ if X is proper, and the two constructions are compatible with the comparison isomorphism $\alpha^\cdot$ of any object in $T(X)$ ([S1], 2.3.7). We end up with an object

$$R\Gamma M^\cdot = (R\Gamma(M^\cdot, F^{\cdot\cdot}, W_\cdot^\cdot), R\Gamma(K^\cdot, W_\cdot^\cdot), R\Gamma\alpha^\cdot)$$

of $T(\mathrm{Spec}(\mathbb{C}))$. The functor

$$\underline{R\Gamma} := R\Gamma \circ \mathrm{For} : C^b\,\mathrm{MHM}_A(X) \longrightarrow T(\mathrm{Spec}(\mathbb{C}))$$

factorizes through $D^b\,\mathrm{MHM}_A(X)$.

Our second comparison result is the following:

Theorem A.1.6. *Let $a : X \to \mathrm{Spec}(\mathbb{C})$ be smooth and proper, and $M^\cdot$ an object of $D^b\,\mathrm{MHM}_A(X)$. Write*

$$\text{For } M^\cdot = ((M^\cdot, F^{\cdot\cdot}, W_\cdot^\cdot), (K^\cdot, W_\cdot^\cdot), \alpha^\cdot) \in T(X)\ .$$

a)

$$\underline{R\Gamma} M^\cdot = (R\Gamma(M^\cdot, F^{\cdot\cdot}, W_\cdot^\cdot), R\Gamma(K^\cdot, W_\cdot^\cdot), R\Gamma\alpha^\cdot)$$

is a mixed polarizable A–Hodge complex.

b) *The class of $\underline{R\Gamma} M^\cdot$ in the derived category of polarizable Hodge complexes is canonically isomorphic, under the identification of A.1.5, to*

$$a_* M^\cdot \in D^b\,\mathrm{MHS}_A\ .$$

c) *Let $f : Y \to X$ be a (proper) morphism of smooth and proper schemes over $\mathbb{C}$, and let b denote the structure morphism of Y, such that*

$$b = a \circ f\ .$$

For any $N^\cdot \in D^b\,\mathrm{MHM}_A(Y)$ together with a morphism $\eta : M^\cdot \to f_* N^\cdot$ in $D^b\,\mathrm{MHM}_A(X)$, the morphism

$$a_*\eta : a_* M^\cdot = \underline{R\Gamma} M^\cdot \longrightarrow \underline{R\Gamma} N^\cdot = b_* N^\cdot = a_* f_* N^\cdot$$

equals, under the isomorphism of a), the morphism

$$(R\Gamma\eta, R\Gamma\eta, R\Gamma\eta)$$

of A–Hodge complexes.

Proof. a) We may assume that $M^{\cdot}$ is pure of some weight. Using [S2], (4.5.4), we are reduced to the case where $M^{\cdot} = M$ is a Hodge module of weight n, and we have to show that $\underline{R\Gamma}M$ is a polarizable Hodge complex of the same weight. Axiom (CH 1) of [D3], (8.1.1) follows from [S2], Proposition 2.16, in particular (2.16.5), applied to $\mathrm{pr}^* M$, where

$$\mathrm{pr} : X \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} \longrightarrow X \ .$$

Furthermore, by the remark following [S2], (4.2.9), and by loc. cit., 2.15, we have isomorphisms in $\mathrm{MF}_h\mathrm{W}(\mathcal{D}_{\mathrm{Spec}(\mathbb{C})}, A)$

$$\underline{R^i\Gamma}M := \left(R^i\Gamma(M, F^{\cdot}, W_{\cdot}[i]), R^i\Gamma(K, W_{\cdot}[i]), R^i\Gamma\alpha\right) \xrightarrow{\sim} \mathcal{H}^i a_* M \ .$$

Since the right hand side is a polarizable Hodge structure of weight $i+n$ ([S2], (4.5.2)), we have (CH 2), and in addition, polarizability.
b) In the proof of a), we constructed a functor

$$a_*^{\sim} := \underline{R\Gamma} : D^b\,\mathrm{MHM}_A(X) \longrightarrow D^b\,\mathrm{MHS}_A \ ,$$

such that

$$\mathcal{H}^i a_*^{\sim} = \mathcal{H}^i a_* : \mathrm{MHM}_A(X) \hookrightarrow D^b\,\mathrm{MHM}_A(X) \longrightarrow \mathrm{MHS}_A$$

for all i. Composition with $j_* : D^b\,\mathrm{MHM}_A(U) \to D^b\,\mathrm{MHM}_A(X)$ for open immersions $j : U \hookrightarrow X$ defines

$$(a \circ j)_*^{\sim} := a_*^{\sim} \circ j_* : D^b\,\mathrm{MHM}_A(U) \longrightarrow D^b\,\mathrm{MHS}_A \ .$$

But for affine U, $(a \circ j)_*$ is the left derived functor of

$$\mathcal{H}^0(a \circ j)_* : \mathrm{MHM}_A(U) \longrightarrow \mathrm{MHS}_A$$

([S2], proof of Theorem 4.3.). If U is affine, then so is $j : U \hookrightarrow X$, and hence j_* is exact. Therefore,

$$\mathcal{H}^0(a \circ j)_* = \mathcal{H}^0 a_* \circ j_* : \mathrm{MHM}_A(U) \xrightarrow{j_*} \mathrm{MHM}_A(X) \xrightarrow{\mathcal{H}^0 a_*} \mathrm{MHS}_A$$

coincides with $\mathcal{H}^0(a \circ j)_*^{\sim}$, and we get a natural transformation

$$(a \circ j)_* \longrightarrow (a \circ j)_*^{\sim} \ ,$$

which is an isomorphism, since this is true on the level of cohomology objects, as one checks on the level of vector spaces. Observe that this natural transformation is compatible with restriction to smaller affine subschemes of X. Now recall ([S2], proof of 4.3) that the functor a_* is constructed using the Čech complex associated to an affine covering of X (for details, see [B3], 3.4). In the same way, the functor $a_*^{\sim}$ is recoverable from the $(a \circ j)_*^{\sim}$. We end up with an isomorphism of a_* and $a_*^{\sim}$, which is independent of the covering.
c) In the proof of b), we constructed a natural isomorphism

$$\kappa : a_* \xrightarrow{\sim} a_*^{\sim}$$

of functors from $D^b\,\mathrm{MHM}_A(X)$ to $D^b\,\mathrm{MHS}_A$. For $f = \mathrm{id}$, our claim is therefore proved. For the general situation, we use the same techniques as in the proof of b) to first construct a natural isomorphism

$$b_* \xrightarrow{\sim} a_*^{\sim} \circ f_*$$

of functors from $D^b\,\mathrm{MHM}_A(Y)$ to $D^b\,\mathrm{MHS}_A$, and then to see that the triangle

$$\begin{tikzcd} b_* \arrow[r] \arrow[dr] & a_* \circ f_* \arrow[d, "\kappa"] \\ & a_*^{\sim} \circ f_* \end{tikzcd}$$

commutes. ◻

Corollary A.1.7 (cf. [S3], (2.8)). *Let $j : U \hookrightarrow X$ be a smooth compactification of a smooth and separated scheme $a : U \to \mathrm{Spec}(\mathbb{C})$, such that $Y := X \setminus U$ is a divisor with normal crossings.*

a) $a_* A(0)_U \in D^b\,\mathrm{MHS}_A$ *is isomorphic, under the identification of A.1.5, to the class of the mixed polarizable A–Hodge complex*

$$\underline{R\Gamma}(U, A) := \underline{R\Gamma}(DR^{-1}\Omega^{\cdot}_X(\log Y), (j_{\mathrm{top}})_* A_U, \alpha)$$

of [D3], (8.1.12) and [B1], § 4 (with the same notation).

b) *If $f : X \to X'$ is a morphism of compactifications $j : U \hookrightarrow X$ and $j' : U \hookrightarrow X'$ of U as in a), then f induces an isomorphism*

$$\underline{R\Gamma}(DR^{-1}\Omega^{\cdot}_{X'}(\log Y'), (j'_{\mathrm{top}})_* A_U) \xrightarrow{\sim} \underline{R\Gamma}(DR^{-1}\Omega^{\cdot}_X(\log Y), (j_{\mathrm{top}})_* A_U)$$

([D3], remark preceding (8.1.17)), so $\underline{R\Gamma}(U, A)$ depends only on U.
The isomorphism in a) also depends only on U.

c) *In particular, the Hodge structures on*

$$\mathrm{rat}(\mathcal{H}^n a_* A(n)_U) = H^n_B(U(\mathbb{C}), (2\pi i)^n A)$$

given by Deligne's and Saito's constructions coincide.

Proof. a) Combine A.1.3 and A.1.6.b).
b) Use A.1.6.c).
c) follows from a) and b). ◻

Actually, the statement A.1.6.c) implies the functoriality property we were after: we have two functors

$$(Sm/\mathbb{C})^0 \longrightarrow D^b\,\mathrm{MHS}_A\ ,$$

where $(Sm/\mathbb{C})$ denotes the category of smooth separated schemes over $\mathbb{C}$:

$$\begin{aligned} \underline{R\Gamma}(\_, A) &: U \longmapsto \underline{R\Gamma}(U, A)\,, \\ {}_*(A) &: (a : U \longrightarrow \mathrm{Spec}(\mathbb{C})) \longmapsto a_*\,(A(0)_U)\ . \end{aligned}$$

Corollary A.1.8. *The isomorphism of A.1.7.a) is functorial in $U \in Sm/\mathbb{C}$. In other words, there is a natural isomorphism*

$$_*(A) \xrightarrow{\sim} \underline{R\Gamma}(\_, A)$$

of functors from $(Sm/\mathbb{C})^0$ to $D^b\,\mathrm{MHS}_A$.

Proof. Let

$$\begin{array}{ccc} U' & \overset{j'}{\hookrightarrow} & X' \\ f\downarrow & & \downarrow f \\ U & \overset{j}{\hookrightarrow} & X \end{array}$$

be a commutative diagram of smooth and separated schemes over $\mathbb{C}$, where X' and X are proper, and $Y' := X' \setminus U'$ and $Y := X \setminus U$ are divisors with normal crossings. We have a morphism

$$(*) \qquad j_* A(0)_U \longrightarrow f_*(j'_* A(0)_{U'}) \; .$$

Application of $(a_X)_*$ gives the morphism

$$(a_U)_* A(0)_U \longrightarrow (a_{U'})_* A(0)_{U'}$$

belonging to the functoriality requirement for $_*(A)$. Our claim follows from A.1.6.c), applied to a shift of the morphism $(*)$. □

Definition A.1.9. *Let $X/\mathbb{C}$ be separated, reduced and of finite type, and $M^\cdot$ an object of $D^b\,\mathrm{MHM}_A(X)$.*

a) *The absolute Hodge complex of X with coefficients in $M^\cdot$ is*

$$R\Gamma_{\mathfrak{H}^p}(X, M^\cdot) := R\,\mathrm{Hom}_{D^b\,\mathrm{MHM}_A(X)}(A(0)_X, M^\cdot) \; .$$

b) *Its cohomology groups*

$$H^i_{\mathfrak{H}^p}(X, M^\cdot) := H^i R\Gamma_{\mathfrak{H}^p}(X, M^\cdot)$$

are called absolute Hodge cohomology groups of X with coefficients in $M^\cdot$.

c) *We denote absolute Hodge cohomology with coefficients in Tate twists by*

$$H^i_{\mathfrak{H}^p}(X, n) := H^i_{\mathfrak{H}^p}(X, A(n)_X) \; .$$

d) *For a closed reduced subscheme Z of X with complement $j : U \hookrightarrow X$, we define relative absolute Hodge cohomology with coefficients in Tate twists as*

$$H^i_{\mathfrak{H}^p}(X \text{ rel } Z, n) := H_{\mathfrak{H}^p}(X, j_! A(n)_U) \; .$$

Note that if X is smooth and of pure dimension d, and if

$$M^{\cdot} = M \in \mathrm{MHM}_A(X)\ ,$$

then the right hand side of A.1.9.b), being equal to

$$\mathrm{Hom}_{D^b\,\mathrm{MHM}_A(X)}(A(0)_X[d], M[d+i])\ ,$$

admits an interpretation as the group of $(d+i)$–extensions of Hodge modules modulo Yoneda equivalence.

Corollary A.1.10. *If X is smooth and separated over $\mathbb{C}$, and $n \in \mathbb{Z}$, then*

$$R\Gamma_{\mathfrak{H}^p}(X,n) = R\Gamma_{\mathfrak{H}^p}(X, A(n)_X) \quad \textit{and} \quad H^{\cdot}_{\mathfrak{H}^p}(X,n) = H^{\cdot}_{\mathfrak{H}^p}(X, A(n)_X)$$

coincide functorially with the same noted objects of [B1], § 5.

Proof. This follows from A.1.8 and the adjunction formula

$$R\,\mathrm{Hom}_{D^b\,\mathrm{MHM}_A(X)}(A(0)_X, M^{\cdot}) = R\,\mathrm{Hom}_{D^b\,\mathrm{MHS}_A}(A(0), a_* M^{\cdot})\ .$$

□

Remark: The Leray spectral sequence for $a : X \to \mathrm{Spec}(\mathbb{C})$ yields exact sequences

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}_A}\left(A(0), H^{i-1}\right) \to H^i_{\mathfrak{H}^p}(X, A(n)_X) \to \mathrm{Hom}_{\mathrm{MHS}_A}\left(A(0), H^i\right) \to 0$$

(with $H^k := H^k_B(X(\mathbb{C}), (2\pi i)^n A))$ since MHS_A has cohomological dimension one ([B1], Corollary 1.10). Comparing them with the analogous sequences for $H^i_{\mathfrak{H}}$, we see that

$$H^i_{\mathfrak{H}^p}(X, A(n)_X) = H^i_{\mathfrak{H}}(X, A(n)_X)$$

(in the notation of [B1], § 5) if $H^{i-1}_B(X(\mathbb{C}), (2\pi i)^n A)$ has weights smaller than zero, which is the case if $i \le n$ ($i \le 2n$ if X is proper).

Observe that this is the same range of indices where Deligne cohomology coincides with $H^i_{\mathfrak{H}}(X, \mathbb{R}(n)_X)$ ([N], (7.1)): we have natural morphisms

$$H^i_{\mathfrak{H}^p}(X, \mathbb{R}(n)_X) \longrightarrow H^i_{\mathfrak{H}}(X, \mathbb{R}(n)_X) \longrightarrow H^i_{\mathcal{D}}(X, \mathbb{R}(n)_X)\ ,$$

both of which are isomorphisms if $i \le n$ ($i \le 2n$ if X is proper).

A.2 Algebraic Mixed Hodge Modules over $\mathbb{R}$

Algebraic Hodge modules over $\mathbb{R}$ are defined as the category of Hodge modules fixed under a certain involution given by complex conjugation. We start by constructing this involution:

Let $X/\mathbb{C}$ be smooth, and let ${}^{\iota}X$ denote the complex conjugate scheme. We have an equivalence

$$\iota^* : \mathrm{Var}_A({}^{\iota}X) \xrightarrow{\sim} \mathrm{Var}_A(X)$$

of the categories of admissible variations, induced by complex conjugation

$$\iota : X(\mathbb{C}) \longrightarrow {}^{\iota}X(\mathbb{C}),$$

and defined as follows:

The local system and the weight filtration on $X(\mathbb{C})$ are the pullbacks via ι of the local system and the weight filtration on ${}^{\iota}X(\mathbb{C})$, and the Hodge filtration on $X(\mathbb{C})$ is the pullback of the conjugate of the Hodge filtration on ${}^{\iota}X(\mathbb{C})$.

ι^* preserves admissibility, and behaves, in an obvious sense, involutively.
In particular, if X is defined over $\mathbb{R}$, we get an involution ι^* on $\mathrm{Var}_A(X \otimes_{\mathbb{R}} \mathbb{C})$.

Definition A.2.1. *Let $X/\mathbb{R}$ be smooth and separated.*

a) *The category $\mathrm{Var}_A^{\sim}(X/\mathbb{R})$ consists of pairs $(\mathbb{V}, F_\infty)$, where $\mathbb{V}$ is an object of $\mathrm{Var}_A(X \otimes_{\mathbb{R}} \mathbb{C})$, and F_∞ is an isomorphism*

$$\mathbb{V} \xrightarrow{\sim} \iota^*\mathbb{V}$$

of variations such that $\iota^* F_\infty = F_\infty^{-1}$.

In the category $\mathrm{Var}_A^{\sim}(X/\mathbb{R})$, we may define Tate twists $A(n)$: F_∞ acts via multiplication by $(-1)^n$.

b) *$\mathrm{Var}_A(X/\mathbb{R})$, the category of admissible variations of mixed A–Hodge structure over $\mathbb{R}$, is the full subcategory of $\mathrm{Var}_A^{\sim}(X/\mathbb{R})$ of pairs $(\mathbb{V}, F_\infty)$ which are graded–polarizable: for $n \in \mathbb{Z}$, there is a morphism*

$$\mathrm{Gr}_n^W(\mathbb{V}, F_\infty) \otimes_A \mathrm{Gr}_n^W(\mathbb{V}, F_\infty) \longrightarrow A(-n)$$

in $\mathrm{Var}_A^{\sim}(X/\mathbb{R})$, such that the induced morphism

$$\mathrm{Gr}_n^W \mathbb{V} \otimes_A \mathrm{Gr}_n^W \mathbb{V} \longrightarrow A(-n)$$

is a polarization in the usual sense.

Remark: We note that implicit in our definition is a descent datum over $\mathbb{R}$ of the bifiltered flat vector bundle on $X \otimes_{\mathbb{R}} \mathbb{C}$ underlying any admissible variation $(\mathbb{V}, F_\infty)$ of mixed A–Hodge structure over $\mathbb{R}$:

For this claim to make sense, recall first ([D1], II, Théorème 5.9) that any flat analytic vector bundle on $X(\mathbb{C})$ carries a canonical algebraic structure. If the vector bundle underlies an admissible variation, then the Hodge filtration is a filtration by algebraic subbundles ([Ks], Proposition 1.11.3).

Now the descent datum is given by the anti-linear isomorphism

$$c_{DR} := F_{\mathrm{diff}}(F_\infty) \circ c_\infty = c_\infty \circ F_{\mathrm{diff}}(F_\infty) : F_{\mathrm{diff}}(\mathbb{V}) \xrightarrow{\sim} F_{\mathrm{diff}}(\iota^*\mathbb{V})$$

of the C^∞–bundles underlying $\mathbb{V}$ and $\iota^*\mathbb{V}$. Here, c_∞ denotes the anti-linear involutions given by complex conjugation of coefficients, and F_{diff} is the forgetful functor to C^∞–bundles.

Lemma A.2.2. *The category $\mathrm{Var}_A(\mathrm{Spec}(\mathbb{R})/\mathbb{R})$ equals the category MHS_A^+ of mixed polarizable A–Hodge structures over $\mathbb{R}$ ([B1], § 7).*

Proof. Straightforward. □

Our aim is to generalize our definition of sheaves over $\mathbb{R}$ to algebraic Hodge modules.

For smooth and separated $X/\mathbb{C}$, recall that $\mathrm{MHM}_A(X)$ is an abelian subcategory of $\mathrm{MF}_h\mathrm{W}(\mathcal{D}_X, A)$. Objects of the latter are

$$((M, F^{\cdot}, W_{\cdot}), (K, W_{\cdot}), \alpha) \;\; ,$$

where $(M, F^{\cdot})$ is an object of the category $\mathrm{MF}_h(\mathcal{D}_X)$ of regular holonomic algebraic $\mathcal{D}_X$–modules with a good filtration, and $K \in \mathrm{Perv}_A(X)$. $W_{\cdot}$ is a locally finite ascending filtration, and α is an isomorphism

$$DR(M) \xrightarrow{\sim} K \otimes_A \mathbb{C}$$

respecting $W_{\cdot}$.

The equivalence

$$\iota^* : \mathrm{MF}_h\mathrm{W}(\mathcal{D}_{{}^{\iota}X}, A) \xrightarrow{\sim} \mathrm{MF}_h\mathrm{W}(\mathcal{D}_X, A)$$

is constructed componentwise:

The perverse sheaf and the weight filtration on $X(\mathbb{C})$ are the pullbacks via $\iota : X(\mathbb{C}) \to {}^{\iota}X(\mathbb{C})$ of the perverse sheaf and the weight filtration on ${}^{\iota}X(\mathbb{C})$.

The equivalence

$$\iota^* : \mathrm{Mod}_{\mathcal{D}_{{}^{\iota}X}} \xrightarrow{\sim} \mathrm{Mod}_{\mathcal{D}_X} \; ,$$

which by construction will respect holonomicity, comes about as follows:

Given a $\mathcal{D}_{{}^{\iota}X}$–module N, we may form the inverse image (in the sense of sheaves of abelian groups) $\iota^{-1}N$, which is a $\iota^{-1}\mathcal{D}_{{}^{\iota}X}$–module. All we therefore need is an isomorphism $c_\infty : \iota^{-1}\mathcal{D}_{{}^{\iota}X} \xrightarrow{\sim} \mathcal{D}_X$ of sheaves of rings extending the isomorphism $c_\infty : \iota^{-1}\mathcal{O}_{{}^{\iota}X} \xrightarrow{\sim} \mathcal{O}_X$ given by complex conjugation of coefficients – we then define

$$\iota^* N := \iota^{-1} N \otimes_{\iota^{-1}\mathcal{D}_{{}^{\iota}X}} \mathcal{D}_X \; .$$

Of course, the map c_∞ is itself given by conjugation of coefficients: in local coordinates $x_1, \ldots, x_n$, we have

$$c_\infty \left(\sum_\alpha f_\alpha \partial^\alpha_{\overline{x}} \right) = \sum_\alpha (c_\infty \circ f_\alpha \circ \iota) \partial^\alpha_x \; .$$

Altogether, we get

$$\iota^* : \mathrm{MF}_h\mathrm{W}(\mathcal{D}_{{}^{\iota}X}, A) \xrightarrow{\sim} \mathrm{MF}_h\mathrm{W}(\mathcal{D}_X, A) \, ,$$

which again behaves involutively.

Going through the definition, one checks that ι^* induces

$$\iota^* : \mathrm{MHM}_A({}^{\iota}X) \xrightarrow{\sim} \mathrm{MHM}_A(X) \; .$$

Using local embeddings as in [S2], 2.1, we can define ι^* for any scheme X, which is separated, reduced and of finite type over $\mathbb{C}$. Furthermore, if X is defined over $\mathbb{R}$, we get an involution ι^* on $\mathrm{MHM}_A(X \otimes_{\mathbb{R}} \mathbb{C})$.

Theorem A.2.3. *Let X and Y be separated and reduced schemes of finite type over $\mathbb{C}$.*

a) *ι^* is compatible with $\underline{\mathrm{Hom}}$, $\otimes$, and $\mathbb{D}$: e.g., for $M^{\cdot}, N^{\cdot} \in D^b\,\mathrm{MHM}_A({}^{\iota}X)$, we have*

$$\underline{\mathrm{Hom}}_X(\iota^* M^{\cdot}, \iota^* N^{\cdot}) = \iota^* \underline{\mathrm{Hom}}_{{}^{\iota}X}(M^{\cdot}, N^{\cdot})\ .$$

b) *If $\pi : X \to Y$ is a morphism, then ι^* is compatible with $\pi_!$, $\pi^!$, π^*, π_*: e.g., for $M^{\cdot} \in D^b\,\mathrm{MHM}_A({}^{\iota}X)$, we have*

$$\iota^*({}^{\iota}\pi)_* M^{\cdot} = \pi_*(\iota^* M^{\cdot}) \in D^b\,\mathrm{MHM}_A(Y)\ .$$

Proof. This follows from the definitions. □

Definition A.2.4. a) *Let $a : X \to \mathrm{Spec}(\mathbb{R})$ be smooth and separated. The category $\mathrm{MHM}_A^{\sim}(X/\mathbb{R})$ consists of pairs (M, F_∞), where M is an object of $\mathrm{MHM}_A(X \otimes_{\mathbb{R}} \mathbb{C})$, and F_∞ is an isomorphism*

$$M \xrightarrow{\sim} \iota^* M$$

such that $\iota^* F_\infty = F_\infty^{-1}$.

By A.2.3.b), we have $a^! A(n) \in \mathrm{MHM}_A^{\sim}(X/\mathbb{R})$.

b) *Let $a : X \to \mathrm{Spec}(\mathbb{R})$ be smooth and separated. $\mathrm{MHM}_A(X/\mathbb{R})$, the category of algebraic mixed A–Hodge modules over $\mathbb{R}$ on X, is the full subcategory of $\mathrm{MHM}_A^{\sim}(X/\mathbb{R})$ of pairs (M, F_∞) which are graded–polarizable: for any $n \in \mathbb{Z}$, there is a morphism*

$$\mathrm{Gr}_n^W(M, F_\infty) \otimes_A \mathrm{Gr}_n^W(M, F_\infty) \longrightarrow a^! A(-n)$$

in $\mathrm{MHM}_A^{\sim}(X/\mathbb{R})$, such that the induced morphism

$$\mathrm{Gr}_n^W M \otimes_A \mathrm{Gr}_n^W M \longrightarrow a^! A(-n)$$

is a polarization in the sense of [S1], 5.2.10.

As in A.1.1, we identify the category of smooth objects in $\mathrm{MHM}_A(X/\mathbb{R})$ with $\mathrm{Var}_A(X/\mathbb{R})$.

c) *For an arbitrary separated and reduced scheme X of finite type over $\mathbb{R}$, one defines the category $\mathrm{MHM}_A(X/\mathbb{R})$ using local embeddings as in [S2], 2.1.*

Remark: a) As in the case of variations over $\mathbb{R}$, we get a descent datum over $\mathbb{R}$ for the bifiltered $\mathcal{D}_{X \otimes_{\mathbb{R}} \mathbb{C}}$–module underlying any Hodge module over $\mathbb{R}$ on a smooth and separated scheme X over $\mathbb{R}$.
b) As in [S2], (4.2.7), the category $\mathrm{MHM}_A(Z/\mathbb{R})$, for any closed reduced subscheme Z of X, is equivalent to the category of Hodge modules over $\mathbb{R}$ on X with support in Z.

Theorem A.2.5. *There is a formalism of Grothendieck's functors* $\pi_!$, $\pi^!$, π^*, π_*, $\underline{\mathrm{Hom}}$, $\otimes$, $\mathbb{D}$ *on* $D^b \operatorname{MHM}_A(\cdot/\mathbb{R})$. *It is compatible with the forgetful functor*

$$D^b \operatorname{MHM}_A(\cdot/\mathbb{R}) \longrightarrow D^b \operatorname{MHM}_A(\cdot \otimes_{\mathbb{R}} \mathbb{C}) \ .$$

Proof. By A.2.3, we may e.g. define

$$\pi_!(M^{\cdot}, F_{\infty}^{\cdot}) := (\pi_! M^{\cdot}, \pi_! F_{\infty}^{\cdot}) \ .$$

$\square$

Definition A.2.6. *Let* $X/\mathbb{R}$ *be separated, reduced and of finite type, and* $M^{\cdot}$ *an object of* $D^b \operatorname{MHM}_A(X/\mathbb{R})$.

a) *The absolute Hodge complex of* $X/\mathbb{R}$ *with coefficients in* $M^{\cdot}$ *is*

$$R\Gamma_{\mathfrak{H}^p}(X/\mathbb{R}, M^{\cdot}) := R\operatorname{Hom}_{D^b \operatorname{MHM}_A(X/\mathbb{R})}(A(0)_X, M^{\cdot}) \ .$$

b) *Its cohomology groups*

$$H^i_{\mathfrak{H}^p}(X/\mathbb{R}, M^{\cdot}) := H^i R\Gamma_{\mathfrak{H}^p}(X/\mathbb{R}, M^{\cdot})$$

are called absolute Hodge cohomology groups of $X/\mathbb{R}$ *with coefficients in* $M^{\cdot}$.

c) *We denote absolute Hodge cohomology with coefficients in Tate twists by*

$$H^i_{\mathfrak{H}^p}(X/\mathbb{R}, n) := H^i_{\mathfrak{H}^p}(X/\mathbb{R}, A(n)_X) \ .$$

d) *For a closed reduced subscheme* Z *of* X *with complement* $j : U \hookrightarrow X$, *we define relative absolute Hodge cohomology with coefficients in Tate twists as*

$$H^i_{\mathfrak{H}^p}(X \text{ rel } Z/\mathbb{R}, n) := H_{\mathfrak{H}^p}(X/\mathbb{R}, j_! A(n)_U) \ .$$

Again, if X is smooth and of pure dimension d, and $M^{\cdot} = M \in \operatorname{MHM}_A(X)$, we have

$$H^i_{\mathfrak{H}^p}(X/\mathbb{R}, M) = \operatorname{Ext}^{d+i}_{\operatorname{MHM}_A(X/\mathbb{R})}(A(0)_X[d], M) \ .$$

We have statements analogous to A.1.1–A.1.10 for the situation over $\mathbb{R}$. For reference, we note explicitly:

Theorem A.2.7. *If* X *is smooth and separated over* $\mathbb{R}$, *and* $n \in \mathbb{Z}$, *then*

$$R\Gamma_{\mathfrak{H}^p}(X/\mathbb{R}, n) \quad \text{and} \quad H^{\cdot}_{\mathfrak{H}^p}(X/\mathbb{R}, n)$$

coincide functorially with the absolute Hodge complex and cohomology groups of [B1], § 7.

Next, we have

Lemma A.2.8. *Let $X/\mathbb{R}$ be separated, reduced and of finite type, and $M^{\cdot}$ an object of $D^b\operatorname{MHM}_A(X/\mathbb{R})$. Then the forgetful functor*

$$D^b\operatorname{MHM}_A(X/\mathbb{R}) \longrightarrow D^b\operatorname{MHM}_A(X\otimes_{\mathbb{R}}\mathbb{C})$$

induces functorial isomorphisms

$$\begin{array}{rcl} R\Gamma_{\mathfrak{H}^p}(X/\mathbb{R},M^{\cdot}) & \xrightarrow{\sim} & R\Gamma_{\mathfrak{H}^p}(X\otimes_{\mathbb{R}}\mathbb{C},M^{\cdot})^+ \ , \\ H^{\cdot}_{\mathfrak{H}^p}(X/\mathbb{R},M^{\cdot}) & \xrightarrow{\sim} & H^{\cdot}_{\mathfrak{H}^p}(X\otimes_{\mathbb{R}}\mathbb{C},M^{\cdot})^+ \ . \end{array}$$

Here, the superscript $+$ denotes the fixed part of the action of the involution ι^* on

$$R\operatorname{Hom}_{D^b\operatorname{MHM}_A(X\otimes_{\mathbb{R}}\mathbb{C})}(A(0)_{X\otimes_{\mathbb{R}}\mathbb{C}},M^{\cdot}) \ .$$

In particular, the category MHS_A^+ has cohomological dimension one since this is true for MHS_A. Furthermore, observe that the above action of $\mathbb{Z}/2\mathbb{Z}$ on $R\Gamma_{\mathfrak{H}^p}(X\otimes_{\mathbb{R}}\mathbb{C},A(n)_{X\otimes_{\mathbb{R}}\mathbb{C}})$ is precisely that of [B1], § 7.

Corollary A.2.9. *Let $X/\mathbb{R}$ be separated, reduced and of finite type. The forgetful functor*

$$\operatorname{rat}:\operatorname{MHM}_A(X/\mathbb{R}) \longrightarrow \operatorname{Perv}_A(\overline{X\otimes_{\mathbb{R}}\mathbb{C}})$$

is faithful and exact.

Remark: Again we have

$$H^i_{\mathfrak{H}^p}(X/\mathbb{R},A(n)_X) = H^i_{\mathfrak{H}}(X/\mathbb{R},A(n)_X)$$

if $i\le n$ ($i\le 2n$ if X is proper). We have natural morphisms

$$H^i_{\mathfrak{H}^p}(X/\mathbb{R},\mathbb{R}(n)_X) \longrightarrow H^i_{\mathfrak{H}}(X/\mathbb{R},\mathbb{R}(n)_X) \longrightarrow H^i_{\mathcal{D}}(X/\mathbb{R},\mathbb{R}(n)_X)\,,$$

which are isomorphisms in the same range of indices.

We conclude with an explicit formula for Ext^1 in $\operatorname{MHM}_A(X/\mathbb{R})$ of a finite scheme $X/\mathbb{R}$.

Theorem A.2.10. *For any $H\in\operatorname{MHS}_A^+$, there is a canonical isomorphism*

$$\begin{array}{rcl} \big(W_0H_{\mathbb{C}}/(W_0H_A+W_0F^0H_{\mathbb{C}})\big)^+ & \xrightarrow{\sim} & \operatorname{Ext}^1_{\operatorname{MHS}_A^+}(A(0),H) \\ & = & H^1_{\mathfrak{H}^p}(\operatorname{Spec}(\mathbb{R})/\mathbb{R},H)\,, \end{array}$$

where the superscript $+$ on the left hand side denotes the fixed part of the de Rham–conjugation

$$\begin{array}{rcl} W_0H_{\mathbb{C}}/(W_0H_A+W_0F^0H_{\mathbb{C}}) & \xrightarrow{c_\infty} & W_0H_{\mathbb{C}}/(W_0H_A+W_0\overline{F}^0H_{\mathbb{C}}) \\ & = & W_0\,\iota^*H_{\mathbb{C}}/(W_0\,\iota^*H_A+W_0F^0\iota^*H_{\mathbb{C}}) \\ & \xrightarrow{F_\infty} & W_0H_{\mathbb{C}}/(W_0H_A+W_0F^0H_{\mathbb{C}})\,. \end{array}$$

The isomorphism is given by sending the class of $h \in W_0 H_{\mathbb{C}}$ to the extension described by the matrix

$$\begin{pmatrix} 1 & 0 \\ -h & \mathrm{id}_H \end{pmatrix} .$$

This means that we equip $\mathbb{C} \oplus H_{\mathbb{C}}$ with the diagonal weight and Hodge filtrations, and the A–rational structure extending the A–rational structure H_A of $H_{\mathbb{C}}$ by the vector

$$1 - h \in \mathbb{C} \oplus H_{\mathbb{C}} ,$$

thereby obtaining an extension E of $A(0)$ by H in the category MHS_A.

The conjugate extension $\iota^* E \in \mathrm{Ext}^1_{\mathrm{MHS}_A}(A(0), \iota^* H)$ is given, with the same notation, by the matrix

$$\begin{pmatrix} 1 & 0 \\ -F_\infty(h) & \mathrm{id}_{\iota^* H} \end{pmatrix} ,$$

and the extension of F_∞ to an isomorphism

$$F_\infty : E \xrightarrow{\sim} \iota^* E$$

sends $1 - h$ to $1 - F_\infty(h)$. Thus

$$(F_\infty)_{\mathbb{C}} = \mathrm{id} \oplus (F_\infty)_{\mathbb{C}} : \mathbb{C} \oplus H_{\mathbb{C}} \longrightarrow \mathbb{C} \oplus \iota^* H_{\mathbb{C}} .$$

Proof. Using [B1], § 1 or [Jn3], Lemma 9.2 and Remark 9.3.a), we see that there is an isomorphism

$$W_0 H_{\mathbb{C}} / (W_0 H_A + W_0 F^0 H_{\mathbb{C}}) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{MHS}_A}(A(0), H) .$$

Note that our normalization follows that of Jannsen, and therefore differs from that of Beilinson by the factor -1.
In general, if $h \in W_0 H_{\mathbb{C}}$ corresponds to an extension E in MHS_A, then $c_\infty h \in W_0 \iota^* H_{\mathbb{C}}$ corresponds to $\iota^* E$, and its pullback via

$$F_\infty : \iota^* H \longrightarrow H ,$$

is described by $F_\infty c_\infty h$. The action of the involution on $\mathrm{Ext}^1_{\mathrm{MHS}_A}(A(0), H)$ therefore corresponds to $F_\infty c_\infty$ on the left hand side of the above isomorphism. $\square$

Corollary A.2.11. *Let $X/\mathbb{R}$ be finite and reduced, and $M \in \mathrm{MHM}_A(X/\mathbb{R})$. Then there is a canonical isomorphism*

$$\begin{aligned} & \left(\bigoplus_{x \in X(\mathbb{C})} W_0 M_{x,\mathbb{C}} / (W_0 M_{x,A} + W_0 F^0 M_{x,\mathbb{C}}) \right)^+ \\ \xrightarrow[A.2.10]{\sim} \; & \mathrm{Ext}^1_{\mathrm{MHS}_A^+} \left(A(0), \bigoplus_{x \in X(\mathbb{C})} M_x \right) \\ = \; & H^1_{\mathfrak{H}^p}(X/\mathbb{R}, M) . \end{aligned}$$

Proof. The last isomorphism is given by the observation that we have

$$\mathrm{MHM}_A(X) = \bigoplus_{x\in X(\mathbb{C})} \mathrm{MHS}_A \ .$$

□

Corollary A.2.12. *For $X/\mathbb{R}$ finite and reduced, and $n \geq 1$, we have*

$$\begin{aligned}\left(\bigoplus_{x\in X(\mathbb{C})} \mathbb{C}/(2\pi i)^n A\right)^+ &\xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{MHM}_A(X/\mathbb{R})}(A(0)_X, A(n)_X) \\ &= H^1_{\mathfrak{H}^p}(X/\mathbb{R}, n) \ .\end{aligned}$$

Here, the superscript $+$ denotes the fixed part with respect to the conjugation on both $X(\mathbb{C})$ and $\mathbb{C}/(2\pi i)^n A$, and the isomorphism associates to $(z_x)_{x\in X(\mathbb{C})}$ the extension, whose stalk at $x \in X(\mathbb{C})$ is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{(2\pi i)^n}\cdot z_x & 1 \end{pmatrix} \ :$$

if e_0 and e_n are the base vectors $1 \in F \subset \mathbb{C}$ and $(2\pi i)^n \in (2\pi i)^n A \subset \mathbb{C}$, then the Hodge structure is specified by

$$F^0 := \langle e_0\rangle_{\mathbb{C}} \ , \quad W_{-2n} \otimes_A \mathbb{C} = \langle e_n\rangle_{\mathbb{C}} \ ,$$

and the A–rational structure is generated by e_n and

$$e_0 - \frac{1}{(2\pi i)^n}\cdot z_x e_n \ .$$

Proof. This is A.2.11 and A.2.10, using the basis (e_n) of $A(n)$. □

B K-Theory of Simplicial Schemes and Regulators

We start with a presentation of K-theory (B.2.1) for simplicial schemes in terms of generalized cohomology. Applied to a regular scheme, we get back its K-groups (cf. B.2.3.a)). Next we define λ-operations on K-cohomology (cf. B.2.10). Motivic cohomology of simplicial schemes, in particular relative motivic cohomology (B.2.11) is introduced as graded pieces of the γ-filtration with respect to these λ-operations. This discussion is based on the extremely useful (unfortunately unpublished) paper [GSo1] by Gillet and Soulé. More often than not the results in B.1 and B.2 will be due to them. The wish for a complete published reference made us go over the material again. Meanwhile an alternative approach to K-theory of simplicial schemes and λ-operations was also worked out by Levine [Le]. De Jeu was the first to use the setting of [GSo1] to define motivic cohomology of simplicial objects. In his article [Jeu] he proves Riemann-Roch in this setting. We give a more general version in B.2.18.

We then construct regulators (i.e., Chern classes) from K-cohomology to continuous étale cohomology (B.4) and to absolute Hodge cohomology (B.5) in this situation. Our main interest is the construction of a long exact sequence for relative

K-cohomology of simplicial schemes as well as for their motivic cohomology which is mapped to the corresponding long exact sequences in sheaf cohomology (B.3.8).

We would like to thank the referee for her or his competent and detailed comments and corrections.

B.1 Generalized Cohomology Theories

We need a framework which is general enough to treat K-theory and the usual cohomology theories in parallel. It turns out such a framework is given by homotopical algebra as axiomatized by Quillen in [Q1].

We define cohomology of spaces (=simplicial sheaves of sets) with coefficients in another space (B.1.4). We then construct a long exact sequence for relative cohomology in this context (B.1.6). Finally we deduce the spectral sequence relating generalized cohomology of a space to generalized cohomology of its components (B.1.7).

A systematic investigation of generalized cohomology for Grothendieck topologies was carried out by Jardine, in particular [Jr2]. We recapitulate the definitions for the convenience of the reader. A first introduction to the necessary simplicial methods is [M].

We fix a regular affine irreducible base scheme B of finite Krull dimension. In our applications B is either a field or an open subscheme of the ring of integers of a number field. We fix a small category of noetherian finite dimensional B–schemes which is closed under finite disjoint unions and contains all open subschemes of all its objects. We turn it into a site using the Zariski topology. Typically this will be a subcategory of all smooth schemes over the base B.

Let $\mathbf{T}$ be the topos of sheaves of sets on our Zariski site over B. Let $s\mathbf{T}$ be the category of pointed simplicial $\mathbf{T}$-objects. Its objects will be called *spaces* in the sequel. We denote the final and initial object of $s\mathbf{T}$ by $\star$.

Remark: A space is given by a simplicial sheaf of sets $X_.$ and a simplicial map ι from $\star$ (the constant simplicial sheaf all of whose components are given by the constant sheaf $\tilde{\star}$ attached to the set with one element) to $X_.$. Equivalently we can consider it as a simplicial object in the category of sheaves pointed by $\tilde{\star}$.

Let X be a scheme. We can also see it as an object of $\mathbf{T}$. The corresponding constant simplicial object pointed by a disjoint base point,

$$U \mapsto \mathrm{Mor}_B(U, X) \cup \{\star\} \qquad \text{for connected } U \in \mathbf{T},$$

will also be denoted X.

Definition B.1.1. *A space is said to be constructed from schemes if all components are representable by a scheme in the site plus a disjoint base point.*

Note that any simplicial scheme (whose components are schemes in the site) gives rise to a space constructed from schemes but there are many spaces constructed from schemes which do not come from simplicial schemes. The main example is the mapping cone of a map of schemes taken in $s\mathbf{T}$ (cf. B.1.5 below).

If P is a property of schemes and if the space X is constructed from schemes, we say X has P if the scheme parts of the components have P.

The easiest way to define the homotopy sets $\pi_n(X, x)$ of a simplicial set X with basepoint $x \in X_0$ is to take the homotopy sets of its geometric realization. $\pi_n(X, x)$

is a group for $n \geq 1$, even abelian for $n \geq 2$. If X is a space and K a finite simplicial set (i.e., all K_n are finite), then we define the space $X \otimes K$ componentwise as the sum of pointed sheaves

$$n \mapsto \bigvee_{\sigma \in K_n} X_n \ .$$

Definition B.1.2 (Brown, Gersten, Gillet, Soulé). *Let X be a space and $f : X \to Y$ be a map of spaces.*

a) *f is called a weak equivalence if all stalks $f_P : X_P \to Y_P$ are weak equivalences of simplicial sets, i.e., if f_P induces an isomorphism on all homotopy sets for all choices of base point.*

b) *f is called a cofibration if for all schemes U in $\mathbf{T}$ the induced map $f(U) : X(U) \to Y(U)$ is injective.*

c) *f is called a fibration if it has the following lifting property: given a commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

where i is a cofibration and a weak equivalence, there exists a map $B \to X$ that makes the diagram commute.

d) *For two spaces X and Y, let $\mathrm{Hom}_.(X, Y)$ be the pointed simplicial set*

$$n \mapsto \mathrm{Hom}_{s\mathbf{T}}(X \otimes \Delta(n), Y)$$

where $\Delta(n)$ is the standard simplicial n-simplex (e.g. [M] 5.4) pointed by zero.

This is the pointed version of the global theory discussed in [Jr2] §2.

Quillen's notion of a closed model category axiomatizes the properties which are needed in order to pass to a homotopy category which behaves similar to the homotopy category of CW-spaces.

Proposition B.1.3 (Brown, Gersten, Joyal). *$s\mathbf{T}$ is a pointed closed simplicial model category in the sense of Quillen [Q1].*

Proof. For a model category we need fibrations, cofibrations and weak equivalences satisfying a set of axioms ([Q1] I Def. 1). This is [GSo1] Theorem 1. Gillet and Soulé attribute this theorem to Joyal (letter to Grothendieck). For simplicial sheaves a published proof of all properties can be found in [Jr2] Cor. 2.7. It is an abstract non-sense fact that with the category of simplicial sheaves the category of pointed simplicial sheaves is also a model category. It is pointed by $\star$. The simplicial structure ([Q1] II Def. 1) is given by B.1.2.d). $\square$

Technical Remark: Note that the unique map $\star \to X$ is always a cofibration, i.e., all spaces are cofibrant. A space will be called fibrant if the unique map $X \to \star$ is a fibration. If a space is fibrant, than its sections $X(U)$ over a scheme U form a simplicial set satisfying Kan's extension condition (cf. [M] 1.3). However, this property does not suffice to make X fibrant. Part of the proof of the proposition is the existence of fibrant resolutions. In fact, the construction in [Jr2] Lemma 2.5 is even functorial.

Let $\mathrm{Ho}(s\mathbf{T})$ be the homotopy category associated to the model category $s\mathbf{T}$ by localizing at the class of weak equivalences. As usual we will write $[X,Y]$ for the morphisms from X to Y in the homotopy category. If Y is fibrant, then this set is given by the set of morphisms from X to Y in $s\mathbf{T}$ up to simplicial homotopy. For general Y, we compute $[X,Y]$ by $[X,\tilde{Y}]$ where $\tilde{Y}$ is a fibrant resolution of Y.

Remark: The category of pointed presheaves with the same notions as in B.1.2 is also a pointed model category. By [Jr2], Lemma 2.6 the map from a presheaf to its sheafification is a weak equivalence and we get the same homotopy category from presheaves or sheaves.

If X is a space, then its suspension SX is given by $X \otimes \Delta(1)/\sim$ where $\sim$ is the usual equivalence relation generated by $(x,0)\sim(x,1)$. By [Q1] Ch. I 2, the loop space functor Ω is right adjoint to S on the homotopy category.

There are two natural ways of thinking about $\mathrm{Ho}(s\mathbf{T})$. From the point of view of algebraic topology it corresponds to the category of CW-complexes with morphisms up to homotopy. From the point of view of homology theory it corresponds to the category of homological complexes which are concentrated in positive degrees with morphisms up to homotopy. S and Ω shift the complexes. This second point of view is not quite precise - note that in general morphisms in $\mathrm{Ho}(s\mathbf{T})$ form pointed sets rather than groups.

Definition B.1.4. *For any space A we define cohomology of spaces with coefficients in A by setting*

$$H^{-m}_{s\mathbf{T}}(X,A) = [S^m X, A] \quad \text{for } m \geq 0 \quad .$$

This is a pointed set for $m=0$, a group for $m>0$ and even an abelian group for $m>1$. If A belongs to an infinite loop spectrum, i.e., if there are spaces A_i for $i\geq 0$ with $A_0 = A$ and weak equivalences $A_i \to \Omega A_{i+1}$, then we also define cohomology groups with positive indices by setting

$$H^{n-m}_{s\mathbf{T}}(X,A) = [S^m X, A_n] \quad \text{for } m,n \geq 0 \quad .$$

Note that the set only depends on $n-m$ because the suspension S and the loop functor Ω are adjoint.

Definition B.1.5. *Let $f: X\to Y$ be a map of spaces. Then the mapping cone of f is the space*

$$C(f) = X\otimes\Delta(1) \amalg Y/\sim$$

where $\sim$ is the usual equivalence relation of the mapping cone (i.e., $(x,1)\sim f(x)$, $(x,0)\sim\star$). For any map of spaces $f: X\to Y$, we define relative cohomology by

$$H^{-m}_{s\mathbf{T}}(Y \text{ rel } X, A) = H^{-m}_{s\mathbf{T}}(C(f),A) \quad .$$

$C(f)$ is the standard construction of the homotopy cofibre of a map.

Proposition B.1.6. *For any morphism $f : X \to Y$ of spaces there is a long exact cohomology sequence:*

$$\to H^{-m}_{s\mathbf{T}}(Y, A) \to H^{-m}_{s\mathbf{T}}(X, A) \to H^{-m+1}_{s\mathbf{T}}(Y \ rel\ X, A) \to H^{-m+1}_{s\mathbf{T}}(Y, A) \ .$$

Proof. By [Q1] Ch. I 3 we have the above long exact sequence attached to the triple of spaces

$$X \xrightarrow{i} Y' \longrightarrow Y' \vee_X \star$$

if i is a cofibration. The mapping cylinder of f is defined as $X \otimes \Delta(1) \vee_X Y$. It is weakly equivalent to Y, and the induced mapping $X \to X \otimes \Delta(1) \vee_X Y$ is a cofibration. The mapping cone of f is nothing but the cofibre of this inclusion. Hence the long exact sequence of the lemma is a special case of Quillen's with $Y' = X \otimes \Delta(1) \vee_X Y$. $\square$

If A is only a space, then the sequence will end at the index zero. There is no reason for the last arrow to be right exact. The $H^0_{s\mathbf{T}}$ are only pointed sets. The $H^{-1}_{s\mathbf{T}}$ are groups, all others are even abelian groups. However, if A is an infinite loop spectrum, then all cohomology groups will be abelian groups and the sequence is unbounded in both directions.

We will consider a couple of spectral sequences which are constructed by means of homotopical algebra. Their differentials are

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q+r-1} \quad .$$

We refer to this behaviour as homological spectral sequence as opposed to a cohomological spectral sequences with differentials

$$d_r : E_r^{p,q} \longrightarrow E_r^{p-r,q-r+1} \quad .$$

In the same way as with the long exact sequences which involve pointed sets we also have to be careful about our spectral sequences. They will be constructed by the method of Bousfield-Kan (cf. [BouK] Ch. IX §§4-5). We refer to them as spectral sequences of Bousfield-Kan type. We give an overview over their properties. They look like this:

$$E_r^{p,q} \Rightarrow L^{q-p} \qquad q \geq p \geq 0,\ r \geq 1$$

with homological differentials.

$$L^{q-p}, E_r^{p,q} = \begin{cases} \text{are abelian groups} & \text{if } q - p \geq 2; \\ \text{are groups} & \text{if } q - p = 1; \\ \text{are pointed sets} & \text{if } q - p = 0. \end{cases}$$

We have $E_{r+1}^{p,q} = \operatorname{Ker} d_r^{p,q} / \operatorname{im} d^{p-r,q-r+1}$. (Treat non-existing $E_r^{p,q}$ as zero for this formation.) By [BouK] IX 4.2.iv) this makes also sense for $p = q$. Let

$$E_\infty^{p,q} = \varprojlim_r E_r^{p,q} = \bigcap_{r > p} E_r^{p,q} \ .$$

There is a descending cofiltration Q_* on the limit term L^n (i.e., Q_iL^n is a quotient of L^n). Let

$$e_\infty^{p,q} = \operatorname{Ker}\left(Q_pL^{q-p} \longrightarrow Q_{p-1}L^{q-p}\right) \ .$$

In general, there will be an injection $e_\infty^{p,q} \longrightarrow E_\infty^{p,q}$. Convergence is a more complicated question. The spectral sequence stabilizes if all projective systems $(E_r^{p,q})_{r>p}$ become eventually stable. Then we have complete convergence ([BouK] IX 5.3). Hence the cofiltration on the limit term is exhaustive ($\varprojlim Q_sL^n = L^n$), and we have isomorphisms

$$e_\infty^{p,q} \xrightarrow{\cong} E_\infty^{p,q} \qquad \text{for } p-q>0.$$

Note that even then the case $p=q$ has to be discussed separately. We refer to this problem and more generally the fact that pointed sets rather then groups appear as the fringe effect.

Proposition B.1.7. a) *Let X and A be spaces. The filtration of X by its skeletons sq_nX induces a spectral sequence of Bousfield-Kan type for its A-cohomology*

$$E_1^{p,q} = H_{s\mathbf{T}}^{-q}(X_p,A) \Rightarrow H_{s\mathbf{T}}^{-(q-p)}(X,A) \qquad \text{for } q \geq p \geq 0 \ .$$

It converges completely if X is degenerate above some degree (i.e., if there is N such that for $n \geq N$, X_n is covered by the image of the degeneracy maps.).

b) *If A is an infinite loop spectrum and X as in a), then we have a converging homological spectral sequence*

$$E_1^{p,q} = H_{s\mathbf{T}}^{-q}(X_p,A) \Rightarrow H_{s\mathbf{T}}^{-(q-p)}(X,A) \qquad \text{for } p \geq 0 \ .$$

Proof. This is the hypercohomology spectral sequence of [GSo1] 1.2.3. We sketch their proof: We can assume A to be fibrant. We can construct a weak equivalence $X' \to X$ such that $sk_pX'/sk_{p-1}X' \cong S^pX_p$. The $\operatorname{Hom}_.(sk_pX',A)$ form a tower of fibrations of simplicial sets converging to $\operatorname{Hom}_.(X,A)$. The attached Bousfield-Kan spectral sequence ([BouK] §4 -§5) has starting terms

$$\begin{aligned} E_1^{p,q} &= \pi_{q-n} \operatorname{Hom}_.(sk_pX'/sk_{p-1}X',A) \\ &= \pi_{q-p} \operatorname{Hom}_.(S^pX_p,A) = H_{s\mathbf{T}}^{-q}(X_p,A) \ . \end{aligned}$$

This finishes the construction of the spectral sequence. In order to discuss convergence we consider the same spectral sequence attached to X itself. It stabilizes by the assumption on degeneracy (see [BouK] §5). Both spectral sequences agree from $r=2$ on.

For b) we consider the spectral sequence in a) for each space in the spectrum. By shifting q accordingly we get a direct system of spectral sequences whose limit is the one we are interested in. □

Remark: It would be much nicer to work with spectra and their homotopy category throughout. It would be a triangulated category. It would help to get rid of the fringe effects. However, the question of convergence of the spectral sequences does not get easier, the reason behind this being that all these spectral sequences are constructed for some kind of homotopy limit, and projective limits are not exact. However, the literature we want to use is in the setting of spaces. The reason is that we want to use the λ-ring structure in order to define motivic cohomology and the λ-operators do not deloop.

B.2 K-theory

We now introduce higher algebraic K–theory of spaces as a generalized cohomology theory. It gives back usual K-theory in the case of regular schemes (B.2.3). We then define λ-operators on these K-cohomology groups (B.2.10). This allows definition of motivic cohomology of spaces as graded parts of the γ-filtration (B.2.11). We then prove a Grothendieck-Riemann-Roch type theorem (B.2.18). As a consequence we get a long exact localization sequence for motivic cohomology (B.2.19).

Recall that all schemes in the site underlying $\mathbf{T}$ are assumed to be noetherian and finite dimensional.

Let $\mathbf{K}$ be the space $\mathbb{Z} \times \mathbb{Z}_\infty BGl$ where $\mathbb{Z}_\infty BGl$ is the simplicial sheaf associated to the simplicial presheaf $U \mapsto \mathbb{Z}_\infty BGl(U) = \varinjlim \mathbb{Z}_\infty BGl_n(U)$. $\mathbf{K}$ is pointed by $0 \times \varinjlim BGl_n(E_n)$. It is in fact part of an infinite loop spectrum. We also need the "unstable" spaces $\mathbf{K}^N = \mathbb{Z} \times \mathbb{Z}_\infty BGl_N$. There are natural transition maps $\mathbf{K}^N \to \mathbf{K}^{N+1} \to \mathbf{K}$. As K-groups commute with direct limits, the stalk of $\mathbf{K}$ in a point P on $U \in \mathbf{T}$ is weakly equivalent to

$$\mathbf{K}_P \cong \mathbb{Z} \times \mathbb{Z}_\infty BGl(\mathcal{O}_P) \ .$$

where $\mathcal{O}_P$ is the stalk of the structural sheaf.

Remark: Even though it is well-known that K-theory is defined by a spectrum, it is not completely trivial to define it as a functor from schemes to spectra (rather than just a functor up to homotopy). We refer to [GSo2], 5.1.2 for the details of this construction. For a different account of K-theory as a presheaf and its properties (including the product structure) we also refer to Jardine's book [Jr4].

Definition B.2.1 (Gillet, Soulé). *For any space X in $s\mathbf{T}$ we define its K-cohomology*

$$H^{-m}_{s\mathbf{T}}(X, \mathbf{K}) = [S^m X, \mathbf{K}] \qquad \text{for } m \in \mathbb{Z}$$

and the unstable K-groups $H^{-m}_{s\mathbf{T}}(X, \mathbf{K}^N)$ for $m \geq 0$. Following [GSo1] we call a space K-coherent if $\varinjlim H^{-m}_{s\mathbf{T}}(X, \mathbf{K}^N) \to H^{-m}_{s\mathbf{T}}(X, \mathbf{K})$ for $m \geq 0$ is an isomorphism.

Proposition B.2.2 (Brown). *Let $\mathcal{K}_q$ be the sheafification of the presheaf $Y \mapsto H^{-q}_{s\mathbf{T}}(Y, \mathbf{K})$. Let X be a scheme in $\mathbf{T}$. There is a homological spectral sequence*

$$E_2^{pq} \Rightarrow H^{-(q-p)}_{s\mathbf{T}}(X, \mathbf{K})$$

with

$$E_2^{pq} = H^p_{\mathrm{ZAR}}(X, \mathcal{K}_q)$$

It converges completely.

Proof. For $q-p \geq 0$ this is the spectral sequence [GSo1] Prop. 2. The basic version for the small Zariski site was constructed in [BrG] Theorem 3. Our generalization follows from the proof of [Jr2] 3.4 and 3.5, which deals with the étale topology. The key is to construct a Postnikov-tower for $\mathbf{K}$. This is done as in in the proof of [BrG] Thm 3. We then have to check that the homotopy sheaves of $\mathbf{K}$ are isomorphic to the homotopy sheaves of the limit of its Postnikov-tower. It suffices to check this for the small Zariski site Zar/Y for all schemes Y in $\mathbf{T}$. Hence we are reduced to the situation considered in loc. cit. Note that Y was assumed to be noetherian and finite dimensional. We extend to arbitrary p, q using the full K-theory spectrum. Convergence follows because X has finite cohomological dimension. □

REMARK: We could generalize the spectral sequence to arbitrary spaces X. $H^p_{\mathrm{ZAR}}(X, \mathcal{K}_q)$ would have to be understood as in B.3. Convergence would not be guaranteed anymore.

The most important application of this proposition is that it allows to transport properties which are well-known for cohomology with coefficients in an abelian sheaf to cohomology with coefficients in a space. One such property is the comparison between different Zariski sites.

PROPOSITION B.2.3 (GILLET, SOULÉ, DE JEU). a) *Let X be a noetherian regular finite dimensional scheme in the site. Then one has the equality $H^{-m}_{s\mathbf{T}}(X, \mathbf{K}) = K_m(X)$, where the right hand side means Quillen K-theory of the scheme X. In particular, $H^{-m}_{s\mathbf{T}}(X, \mathbf{K}) = 0$ for $m < 0$.*

b) *Let X be a space constructed from schemes. Assume that all components are regular Noetherian finite dimensional schemes and that X is degenerate above some simplicial degree. Then X is K-coherent.*

Proof. The constant case is proved in [GSo1] 2.2.2 Prop. 5. We sketch a slightly different argument: We use the converging Brown spectral sequence and comparison theorems for sheaf cohomology to show that it suffices to prove the proposition in the case of $\mathbf{T} = \mathrm{Zar}/X$. (Note that the existence of the whole spectrum means we do not have to worry about fringe effects.) In this case we have a Mayer-Vietoris sequence for K-theory ([Q2] Rem. 3.5) and hence the presheaf defining K-cohomology is pseudo-flasque in the sense of Brown and Gersten ([BrG] p. 285). By loc. cit. Thm. 4 this implies a) for the site Zar/X.
The vanishing follows because the K-theory spectrum is connective. The generalization to spaces constructed from schemes using the skeletal spectral sequence was carried out in [Jeu] 2.1 (1) and Lemma 2.1. □

COROLLARY B.2.4. *If X is a space meeting the conditions of part b) of the proposition, then its K-cohomology does not depend on the category of schemes underlying the topos.*

Proof. If X is constant, then we always get its K-theory. For more general X we have to use the converging skeletal spectral sequence. There are no fringe problems because $\mathbf{K}$ is an infinite loop spectrum. □

The direct sum of matrices (cf. [Lo] 1.2.4) together with addition on $\mathbb{Z}$ induces a compatible system of maps

$$\mathbf{K}^N \times \mathbf{K}^N \to \mathbf{K}\ .$$

Our aim is to show that its direct limit defines an H-group structure on $\mathbf{K}$. It will be used to define addition on K-cohomology.

Lemma B.2.5. *Let G, G' be algebraic groups over $\mathbb{Z}$, E a subgroup of G with $E = [E,E]$. Let $f_1, f_2 : G' \to G$ be homomorphisms which differ by conjugation by a global section of E. Then the induced maps*

$$\mathbb{Z}_\infty BG' \xrightarrow{f_1, f_2} \mathbb{Z}_\infty BG$$

agree in the homotopy category of spaces.

Proof. The construction in [Lo] A.3. is functorial. Hence it yields a free homotopy η between Bf_1 and Bf_2. By construction we get a commutative diagram

$$\begin{array}{ccc} \Delta(1) \times \star & \xrightarrow{\eta|_{\Delta(1)\times\star}} & \mathbb{Z}_\infty BE \\ \downarrow & & \downarrow{\scriptstyle i} \\ \Delta(1) \times \mathbb{Z}_\infty BG' & \xrightarrow{\eta} & \mathbb{Z}_\infty BG\ . \end{array}$$

The composition of η with $d : \mathbb{Z}_\infty BG \to C(i)$ is a homotopy between df_1 and df_2. Now it suffices to show that d is a weak equivalence, i.e., that $\mathbb{Z}_\infty BE$ is contractible. This can be checked on stalks. As homotopy groups commute with direct limits it is enough to show that $\mathbb{Z}_\infty BE(U)$ is contractible for all affine schemes U. We consider the diagram

$$\begin{array}{ccc} BE(U) & \xrightarrow{\phi} & BE(U)^+ \\ \downarrow & & \downarrow \\ \mathbb{Z}_\infty BE(U) & \xrightarrow{\mathbb{Z}_\infty(\phi)} & \mathbb{Z}_\infty BE(U)^+\ . \end{array}$$

By definition of Quillen's +-construction (see [Lo] ch. 1.1) ϕ induces an isomorphism on homology. Hence $\mathbb{Z}_\infty(\phi)$ is a weak equivalence ([BlK] Ch. I, 5.5). $BE(U)^+$ is contractible because $[E(U), E(U)] = E(U)$ ([Lo] Proposition 1.1.7). Hence $\mathbb{Z}_\infty BE(U)^+$ is also contractible. □

The standard application of this lemma is with $G' = Gl_n$, $G = Gl$ and E the subgroup generated by elementary matrices (which contains all even permutation matrices), see [Lo] 1.1.10.

Proposition B.2.6. *The direct sum of matrices induces an H-group structure on K.*

Proof. The same proof as in [Lo] Theorem 1.2.6 allows to check the identities of an H-space. On finite level, they hold up to conjugation with a permutation matrix.

By the previous lemma this implies that they hold in the homotopy category. We use that the transition maps $K^N \to K^{N+1}$ are cofibrations in order to show that the maps on finite level define one on $\mathbf{K}$. For the existence of a homotopy inverse we argue differently. An H-space is an H-group if and only if the shear map

$$\mathbf{K} \times \mathbf{K} \to \mathbf{K} \times \mathbf{K} \ , (k_1, k_2) \mapsto (k_1, k_1 + k_2)$$

is a weak equivalence. This can be checked on stalks. But the stalks of $\mathbf{K}$ are the simplicial sets computing K-theory of local rings. They are H-groups with the same addition by the affine case [Lo] 1.2.6. □

Remark: We now have two H-group structures on $\mathbf{K}$: the explicit one we just have constructed and one because $\mathbf{K}$ is a loop space as part of a spectrum. We expect them to be equal but have not been able to prove it. They certainly induce the same addition on higher K-cohomology groups. On $H^0_{s\mathbf{T}}(X, \mathbf{K})$ they agree at least if X is represented by a scheme because they do for K-theory of schemes. This is enough for our needs. In the sequel the addition on K-cohomology is the one of the proposition.

The next aim is the definition of a multiplicative structure on $\mathbf{K}$. We start with the operation of $\mathbb{Z}$ on $\mathbf{K}$. The H-group structure on $\mathbf{K}$ allows to define a map of spaces

$$\mu_{\mathbb{Z}} : \mathbb{Z} \times \mathbf{K} \to \mathbf{K} \quad .$$

It vanishes on $\mathbb{Z} \times \star \vee \star \times \mathbf{K}$ and hence factors over $\mathbb{Z} \wedge \mathbf{K}$.

The construction of the Loday product [Lo] 2.1.5

$$\mathbb{Z}_\infty BGl_N(U) \wedge \mathbb{Z}_\infty BGl_N(U) \to \mathbb{Z}_\infty BGl(U)$$

is functorial in U. Together with the product $\mu_{\mathbb{Z}}$ on the factor $\mathbb{Z}$ it defines a system of maps

$$\mu_{\mathbf{K}} : \mathbf{K}^N \wedge \mathbf{K}^N \to \mathbf{K}$$

(compatible up to homotopy), which defines a product

$$[Y, \mathbf{K}] \times [Y, \mathbf{K}] \to [Y, \mathbf{K}]$$

for all K–coherent spaces Y. It turns all $H^{-n}_{s\mathbf{T}}(Y, \mathbf{K})$ for $n \geq 0$ into a ring, possibly without unity.

Remark: Note that this product on $[Y, \mathbf{K}]$ is zero on $H^{-n}_{s\mathbf{T}}(Y, \mathbf{K})$ for $n > 0$ (cf. [Kr] Ex. 1 p. 243). The same map $\mu_{\mathbf{K}}$ of spaces also induces a non-trivial product

$$[S^n Y, \mathbf{K}] \times [S^m Y, \mathbf{K}] \to [S^{n+m} Y, \mathbf{K}] \quad .$$

This is the one which is usually called Loday product. We do not need it in the sequel.

Let S^0 be the simplicial version of the 0-sphere, i.e., the constant simplicial sheaf associated to $\{0, 1\}$ pointed by 0. We will use the notation $K_0(s\mathbf{T})$ for $H^0_{s\mathbf{T}}(S^0, \mathbf{K})$. It is a ring with unity where the ring structure is induced by the ring structure on $\mathbb{Z}$.

Lemma B.2.7. *If the site underlying* $\mathbf{T}$ *has a final object* X, *then*

$$K_0(X) \cong K_0(s\mathbf{T}) \ .$$

Proof. If X is the final object of the site, then the space we denote by X is equal to S^0. □

The following lemma generalizes an operation of $K_0(X)$ which was explained to us by de Jeu in the case where Y is constructed from X–schemes.

Lemma B.2.8. *Let* Y *be a space in* $s\mathbf{T}$. *Then the ring* $K_0(s\mathbf{T})$ *operates on* $H^{-n}_{s\mathbf{T}}(Y, \mathbf{K})$ *for* $n \geq 0$ *and makes it into an* $K_0(s\mathbf{T})$*-algebra.*

Proof. If Y is a space in $s\mathbf{T}$, then there is canonical isomorphism $Y \cong S^0 \wedge Y$. The product $\alpha \in K_0(s\mathbf{T})$ with $\beta \in H^{-n}_{s\mathbf{T}}(Y, \mathbf{K})$ is defined by the composition

$$Y \longrightarrow S^0 \wedge Y \xrightarrow{\alpha \wedge \beta} \mathbf{K} \wedge \mathbf{K} \xrightarrow{\mu_{\mathbf{K}}} \mathbf{K} \quad .$$

□

Lemma B.2.9 (Gillet, Soulé). *Let* G *be a group over* $\mathbb{Z}$. *Let* $R_{\mathbb{Z}}(G)$ *be the Grothendieck group of representations of* G *on free* $\mathbb{Z}$*–modules of finite type.*

a) *Let* A *be an* N*-dimensional representation of* G. *There is a canonical class in* $[\mathbb{Z} \times \mathbb{Z}_\infty BG, \mathbf{K}]$ *which depends only on the equivalence class of* A. *The direct sum of representations is mapped to the sum of classes.*

b) *The map in a) induces an algebra homomorphism*

$$r : R_{\mathbb{Z}}(G) \to [\mathbb{Z} \times \mathbb{Z}_\infty BG, \mathbf{K}] \ .$$

Proof. We follow [GSo1] 3.2 or the affine case [Kr] 3. By choice of a basis of an N-dimensional representation A induces a map of sheaves

$$A : G \to Gl_N$$

and hence by functoriality a map

$$r'(A) : \mathbb{Z}_\infty BG \to \{N\} \times \mathbb{Z}_\infty BGl_N \to \mathbf{K}^N \ .$$

For different choices of basis the maps differ by conjugation with an element of $\alpha \in Gl_N$. The matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ is in the perfect subgroup $E = [Gl, Gl]$ hence by Lemma B.2.5 the image of $r'(A)$ in $[\mathbb{Z}_\infty BG, \mathbf{K}^{2N}]$ does not depend on the choice of matrix. Viewed as map to $\mathbf{K}$, this $r'(A)$ extends to the factor $\mathbb{Z}$ using the above product $\mu_{\mathbb{Z}}$. The last statement of a) follows by definition of the H-group structure on $\mathbf{K}$.
For b) we have to check that the relations of the Grothendieck-group are mapped to zero and that the multiplicative structure is well-behaved. We first prove the analogue of [Kr] Theorem 3.1: The canonical maps

$$Gl \begin{pmatrix} ? & 0 \\ 0 & ? \end{pmatrix} \leftrightarrows Gl \begin{pmatrix} ? & ? \\ 0 & ? \end{pmatrix}$$

induce weak equivalences of simplicial sheaves after applying $\mathbb{Z}_\infty B$. This can be checked on stalks and is hence reduced to the affine case. From now, the proof works precisely as in the affine case, see [Kr] Cor. 3.2. □

$K_0(s\mathbf{T})$ is a λ–ring, i.e., the axioms in [Kr] Def. 4.1 are satisfied. If R is a $K_0(s\mathbf{T})$–algebra, then it is called a $K_0(s\mathbf{T})$–λ–algebra if it is equipped with operators λ^i for $i \geq 1$ such that $K_0(s\mathbf{T}) \oplus R$ is a λ–ring (cf. [Kr] 5.). Note that λ^0 has to have the constant value 1. If R itself does not have a unity, then it cannot be a λ–ring.

Theorem B.2.10 (Gillet, Soulé). *Let Y be a K–coherent space. For $k \geq 1$ and $m \geq 0$ there are maps*

$$\lambda^k : H^{-m}_{s\mathbf{T}}(Y, \mathbf{K}) \longrightarrow H^{-m}_{s\mathbf{T}}(Y, \mathbf{K}) \ .$$

They turn $H^{-m}_{s\mathbf{T}}(Y, \mathbf{K})$ into a $K_0(s\mathbf{T})$–λ–algebra.

Proof. This is essentially [GSo1] Prop. 8. Put $G = Gl_n$ in the previous lemma. Let $\tilde{\mathbb{Z}}^n = [\mathbb{Z}^n_{id}] - [n \cdot 1] \in R_{\mathbb{Z}}(Gl_n)$ where $\mathbb{Z}^n_{id}$ is the canonical representation of Gl_n on $\mathbb{Z}^n$ and 1 is the trivial representation. We define $\lambda^k_n = r(\lambda^k(\tilde{\mathbb{Z}}^n))$. By composition it induces a map $\lambda^k_n : H^{-m}_{s\mathbf{T}}(Y, \mathbf{K}^n) \to H^{-m}_{s\mathbf{T}}(Y, \mathbf{K})$. These form a projective system and hence define an operation on K–cohomology of a K–coherent space. Well-definedness and all properties of a λ–ring are checked on the universal level (i.e., on $\mathbf{K}^n$ for varying n) and hence as in the affine case [Kr] Thm 5.1. For example, we want to show

$$\lambda^k(x + y) = \sum_{i=0}^{k} \lambda^i(x)\lambda^j(y) \ .$$

Assume that x, y are represented by elements in $[Y, \mathbf{K}^n]$. On $R_{\mathbb{Z}}(Gl_n \times Gl_n)$ we have the λ-ring identity

$$\lambda^k \circ \bigoplus = \sum_{i=0}^{k} \lambda^i \otimes \lambda^j \ .$$

We evaluate this identity in $\tilde{\mathbb{Z}}^n$ and get an equality of elements in $R_{\mathbb{Z}}(Gl_n \times Gl_n)$. By the previous lemma it induces the same equality of elements in $[\mathbf{K}^n \times \mathbf{K}^n, \mathbf{K}]$. Composed with (x, y) this is the required equality. $\square$

Remark: A more conceptual proof was suggested to us by Soulé and the referee. One should use the integral completion functor constructed by Goerss and Jardine [GoeJr]. It has a universal property similar to the one of the +-construction and hence allows to copy directly Kratzer's arguments.

Technical Remark: When we try to define λ^0 in the same way, then we still get a map

$$\lambda^0 : \mathbb{Z}_\infty BGl_N \longrightarrow \mathbb{Z} \times \mathbb{Z}_\infty BGl \ .$$

It does not extend to the factor $\mathbb{Z}$ because $\lambda^0 : \mathbb{Z} \to \mathbb{Z}$ does not respect the base point - in fact it maps 0 to 1. This reflects the fact that the ring $K_0(Y)$ does not have a unity for a general space Y. The most striking example is $Y = C(i)$ where $i : Z \to X$ is a morphism between regular schemes (cf. [Sou4] 4.3). Then $K_0(Y) = \mathrm{Ker}\,(K_0(X) \to K_0(Z))$ does not contain 1.

Gillet and Soulé ([GSo1] Prop. 8) consider the structure as a $H^0_{s\mathbf{T}}(Y, \mathbf{K})$–$\lambda$–algebra. This only makes sense if $H^0_{s\mathbf{T}}(Y, \mathbf{K})$ happens to have a unity. However, we can check in general that the operation of $H^0_{s\mathbf{T}}(Y,\mathbf{K})$ on $H^{-m}_{s\mathbf{T}}(Y,\mathbf{K})$ is compatible with the $K_0(s\mathbf{T})$–λ–algebra structure of both groups.

Note that the λ-structure is compatible with the contravariant functoriality of K-cohomology. This means that the long exact sequences for relative K-theory are compatible with the λ-operation where it is defined.

Once we have λ-operations we get as usual a γ-filtration and Adams-operators on the λ-module $H^n_{s\mathbf{T}}(Y,\mathbf{K})$ for $n \leq 0$. If the γ-filtration is locally finite, then we have in particular the Chern character

$$ch : H^n_{s\mathbf{T}}(Y,\mathbf{K})_{\mathbb{Q}} \longrightarrow \bigoplus_{j\in\mathbb{N}_0} \mathrm{Gr}^j_\gamma\, H^n_{s\mathbf{T}}(Y,\mathbf{K})_{\mathbb{Q}} \qquad \text{for } n \leq 0,$$

which is an isomorphism. For a quick survey cf. [T] pp. 117–123.

Definition B.2.11. *Let Y be a K-coherent space. Suppose that the γ-filtration is locally finite and hence that rationally K-cohomology splits into Adams-eigenspaces. Then we put for $j \geq n/2$*

$$H^n_{\mathcal{M}}(Y,j) = \mathrm{Gr}^j_\gamma\, H^{n-2j}_{s\mathbf{T}}(Y,\mathbf{K})_{\mathbb{Q}}\ ,$$

the motivic cohomology of the space Y. If $i : X \to Y$ is a morphism of spaces then we define relative motivic cohomology by

$$H^n_{\mathcal{M}}(Y \text{ rel } X,j) = H^n_{\mathcal{M}}(\mathrm{Cone}(i),j)\ .$$

Remark: We restrict to this range of indices because we did not define Adams-eigenspaces for K-cohomology with positive indices (=K-theory with negative indices). However, if these K-groups vanish we can simply define the corresponding motivic cohomology groups to be zero. This is the case if X is a regular scheme.

The long exact sequence for relative cohomology (B.1.6) together with the above remarks on the λ-operation give a long exact sequence for relative motivic cohomology

$$\to H^{-m}_{\mathcal{M}}(Y,A) \to H^{-m}_{\mathcal{M}}(X,A) \to H^{-m+1}_{\mathcal{M}}(Y \text{ rel } X,A) \to H^{-m+1}_{\mathcal{M}}(Y,A)\ .$$

Lemma B.2.12. *Let X be a space degenerate above some simplicial degree. We assume the conditions of the previous definition. Fix an integer j. There is a cohomological spectral sequence with starting terms*

$$E_1^{s,t} = \begin{cases} H^t_{\mathcal{M}}(X_s,j) & \text{for } s \geq 0,\ 2j \geq t, \\ 0 & \text{else.} \end{cases}$$

It converges to $H^{s+t}_{\mathcal{M}}(X,j)$ for $2j \geq s+t$.

Proof. Consider the skeletal spectral sequence B.1.7.a) with coefficients in the space $\mathbf{K}$. It reads

$$E_1^{p,q} = H^{-q}_{s\mathbf{T}}(X_p,\mathbf{K}) \Rightarrow H^{-(q-p)}_{s\mathbf{T}}(X,\mathbf{K})$$

for $p \geq 0$. By carefully checking the construction of the spectral sequence, we see that all differentials $d_r^{p,q}$ are induced by functoriality in the first argument. Hence they

are morphisms of λ-modules. For $q-p\geq 0$ the limit terms are also λ-modules and by construction the morphisms $e_\infty^{p,q}\to E_\infty^{p,q}$ are compatible with this structure. They are isomorphisms for $q>p$. Note, however, that we do not get enough information on the limit terms on the $p=q$-line. Convergence only implies that $e_\infty^{p,p}$ injects into $E_\infty^{p,p}$. We want to show that it is even a bijection. In order to see this we consider the skeletal spectral sequence with coefficients in the spectrum $\mathbf{K}$. The spectral sequences agree where the first is defined, in particular convergence of the second spectral sequence implies our isomorphism. (There is an issue here with the H-group structure. A priori the two spectral sequences use different group laws. But on all initial terms they give the same addition and hence also on all higher terms.)
Now we take Adams-eigenspaces. By re-indexing $s=p,t=-q+2j$ we get a cohomological spectral sequence as stated. Note that we use the terms below the $p=q$-diagonal to compute the terms on it but we do not consider their limit terms. □

The same spectral sequence also shows that the conditions in the definition of motivic cohomology hold if X is a space constructed from schemes and degenerate above some degree.

The next thing we need is pushout at least for certain closed immersions and a Riemann-Roch theorem. Over a field push-forward was defined by de Jeu in [Jeu] 2.2. We adapt his method to more general bases and formalize the geometric situation.

Definition B.2.13. *Let S be a regular irreducible Noetherian affine scheme. Let X be smooth and quasi-projective over S. A finite diagram $\mathcal{D}_X$ over X is a category of finitely many smooth quasi-projective S-schemes with final object X such that all $\mathrm{Mor}_{\mathcal{D}_X}(Y,Y')$ are finite sets and such that all morphisms in $\mathcal{D}_X$ are of finite Tor-dimension.*
By the small Zariski site $\mathrm{Zar}_{\mathcal{D}_X}$ we mean the category of all finite disjoint unions of open subschemes of objects in $\mathcal{D}_X$ with the induced morphisms between them. It is equipped with the Zariski-topology. The corresponding topos will be denoted $\mathbf{T}_X$.

An easy case of such a diagram is a single morphism $Y\to X$ that meets the conditions.

We consider the following situation: Let $i:Z\to X$ be a closed immersion of smooth quasi-projective S-schemes and $\mathcal{D}_X$ a finite diagram over X. We assume the following conditions, corresponding to the ones formulated by de Jeu in [Jeu] 2.2:

(TC) For all X' in $\mathcal{D}_X$, the pullback $X'\times_X Z$ is S–smooth. If $f:X_1\to X_2$ is a morphism in $\mathcal{D}_X$, then in the cartesian diagram

$$\begin{array}{ccc} Z_1=X_1\times_X Z & \longrightarrow & X_1 \\ {\scriptstyle f\times_X Z}\downarrow & & \downarrow{\scriptstyle f} \\ Z_2=X_2\times_X Z & \xrightarrow{i} & X_2 \end{array}$$

the maps f and i are tor-independent, i.e.,

$$\underline{\mathrm{Tor}}^k_{\mathcal{O}_{X_2}}(\mathcal{O}_{Z_2},\mathcal{O}_{X_1})=0$$

for $k>0$. ($\underline{\mathrm{Tor}}^k$ denotes the sheaf of tor-groups.)

Lemma B.2.14. *The pullback $\mathcal{D}_Z$ of $\mathcal{D}_X$ by Z satisfies the conditions for a finite diagram over Z.*

Proof. Finite Tor-dimension in $\mathcal{D}_Z$ follows from Tor-independence and the same property in $\mathcal{D}_X$. $\square$

Let $Y_.$ be a space in $s\mathbf{T}_X$. Let $j : U \to X$ be the open complement of Z in X. Let $Y_. \times_X U$ be the pointed version of $j_! j^* Y_.$, i.e., the sheaf associated to the presheaf

$$V \mapsto \begin{cases} Y_.(V) & \text{if } V \to U \subset X, \\ 0 & \text{else.} \end{cases}$$

It is a space in $s\mathbf{T}_X$. Let $Y_. \times_X Z = i^{-1}Y_.$, a space in $s\mathbf{T}_Z$. If $Y_.$ is constructed from schemes, then so are $Y_. \times_X U$ and $Y_. \times_X Z$. The scheme components are given by the base change with U or Z respectively. Note that $i^{-1}(Y_. \times_X U)$ is empty, i.e., only consists of the base point.

Proposition B.2.15 (de Jeu). *Let $i : Z \to X$ be a closed immersion with open complement U. Let $\mathcal{D}_X$ be a finite diagram over X such that (TC) holds with respect to i. Then for $Y_. \in s\mathbf{T}$:*

a) *There is a natural pushout map*

$$H^k_{s\mathbf{T}_Z}(Y_. \times_X Z, \mathbf{K}) \longrightarrow H^k_{s\mathbf{T}_X}(Y_., \mathbf{K}).$$

b) *Let $Y_.$ be a space in $s\mathbf{T}_X$ which is constructed from schemes. We assume that it is degenerate above some simplicial degree. Then*

$$Y_. \times_X Z = C(Y_. \times_X U \subset Y_.) \times_X Z$$

and the pushout

$$H^k_{s\mathbf{T}_Z}(Y_. \times_X Z, \mathbf{K}) \longrightarrow H^k_{s\mathbf{T}_X}(Y_. \text{ rel } Y_. \times_X U, \mathbf{K})$$

is an isomorphism.

Proof. For an object V of the site $\mathrm{Zar}_{\mathcal{D}_X}$ let $M(V)$ be the category of all coherent sheaves on V. In it let $P(V, \mathcal{D}_X)$ be the subcategory of those sheaves $\mathcal{F}$ satisfying

$$\underline{\mathrm{Tor}}^j_{\mathcal{O}_V}(\mathcal{O}_{V'}, \mathcal{F}) = 0$$

for all $j > 0$ and all $V' \to V$ in $\mathcal{D}_X$. Note that there are only finitely many conditions as our diagram is finite. The nice thing about $P(V, \mathcal{D}_X)$ is that it is contravariantly functorial. Hence Quillen's $\Omega BQP(\,\cdot\,, \mathcal{D}_X)$ (loop space of the classifying space of the Q-construction) defines a presheaf of simplicial sets on the site by [Q2] §7 2.5. It is here where we use the fact that all schemes are quasi-projective. Let $\Omega BQP'_X$ be the space in $s\mathbf{T}_X$ defined by its sheafification. By Quillen's Resolution Theorem ([Q2] Thm 3, Cor 3, p. 27) there is a weak equivalence of spaces $\Omega BQP'_X \to \mathbf{K}_X$. (Basically this is the fact that K'-theory and K-theory agree for regular schemes.)

We also have the space $\Omega BQP'_Z$ in $s\mathbf{T}_Z$. For the closed immersion $i : V \times_X Z \to V$ the pushout i_* is exact on the category of coherent sheaves. Because of (TC), it maps the the subcategory $P(V \times Z, \mathcal{D}_Z)$ to $P(V, \mathcal{D}_X)$. In fact we get a morphism of spaces in $s\mathbf{T}_X$

$$i_*(\Omega BQP'_Z) \xrightarrow{i_*} \Omega BQP'_X \ .$$

Using the weak equivalences to $\mathbf{K}_?$ this defines a map in the homotopy category

$$i_*(\mathbf{K}_Z) \xrightarrow{i_*} \mathbf{K}_X \ .$$

If $Y_.$ is a space in $s\mathbf{T}_X$, then we get the map in a) as

$$H^k_{s\mathbf{T}_Z}(i^{-1}Y_., \mathbf{K}_Z) \longrightarrow H^k_{s\mathbf{T}_X}(i_* i^{-1}Y_., i_*\mathbf{K}_Z) \longrightarrow H^k_{s\mathbf{T}_X}(Y_., \mathbf{K}_X) \ .$$

In the special case of a scheme Y part b) is nothing but Quillen's pushout isomorphism

$$K_n(i^{-1}Y) \longrightarrow K_n(Y \text{ rel } Y \times_X U)$$

for regular schemes [Q2] §7 Prop. 3.2 (recall that all schemes in the site are regular). This generalizes to the case of spaces constructed from schemes by the skeletal spectral sequence. □

Lemma B.2.16. *Consider a cartesian diagram of smooth quasi-projective S-schemes*

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f_Z"'] & X' \arrow[d, "f_X"'] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

where i is a closed immersion. Let $\mathcal{D}_X$ be a finite diagram on X. Assume that the pullback $\mathcal{D}_{X'}$ defines a finite diagram over X' and that both i and i' satisfy (TC). We also assume that for all V in $\mathcal{D}_X$ the maps

$$V \times_X X' \longrightarrow V$$

and

$$V \times_X Z \longrightarrow V$$

are tor-independent.

Then for all spaces $Y_.$ in $s\mathbf{T}_X$ there is a commutative diagram

$$\begin{tikzcd} H^k_{s\mathbf{T}_{Z'}}(f_Z^* i^* Y_., \mathbf{K}) \arrow[r, "i'_*"] & H^k_{s\mathbf{T}_{X'}}(f_X^* Y_., \mathbf{K}) \\ H^k_{s\mathbf{T}_Z}(i^* Y_., \mathbf{K}) \arrow[r, "i_*"] \arrow[u, "f_Z^*"] & H^k_{s\mathbf{T}_X}(Y_., \mathbf{K}) \arrow[u, "f_X^*"'] \end{tikzcd} .$$

Proof. We have to refine the categories $P(V, \mathcal{D}_Z)$ used in the proof of B.2.15 further. Let $P''(V, \mathcal{D}_Z)$ be the subcategory of $P'(V, \mathcal{D}_Z)$ of those coherent sheaves $\mathcal{F}$ satisfying

$$\underline{\mathrm{Tor}}^j_{\mathcal{O}_Z}(\mathcal{O}_{Z'}, \mathcal{F}) = 0 \ .$$

The induced space $\Omega BQP''_Z$ is again weakly equivalent to $\mathbf{K}_Z$. By [Q2] §7 2.11 there is a commutative diagram of spaces in $s\mathbf{T}_X$

$$\begin{array}{ccc} i'_* f_{X,*}\Omega BQP'_{Z'} & \longrightarrow & f_{X,*}\Omega P'_{X'} \\ \uparrow & & \uparrow \\ i_*\Omega BQP''_Z & \longrightarrow & \Omega BQP'_X \end{array} .$$

This proves the lemma. □

We also need the following lemma from algebraic geometry.

Lemma B.2.17. *Suppose we are given a cartesian diagram*

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ \downarrow & & \downarrow f \\ Z & \xrightarrow{i} & X \end{array}$$

of smooth S-schemes where i is a closed embedding, then the blow-up of X' in Z' is the base change by f of the blow-up of X in Z provided i and f are tor-independent.

Proof. In order to see this, note that by [EGAII] 3.5.3 we have to check that $f^*(\mathcal{I}^n) = \mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ is isomorphic to $\mathcal{J}^n$ where $\mathcal{I}$ is the sheaf of ideals of Z in X and $\mathcal{J}$ the one of Z' in X'. This follows from tor-independence in the case $n = 1$. Note that in general we have a surjection $f^*\mathcal{I}^n \to \mathcal{J}^n$. Let K_n be the kernel. Pull-back by f^* is right exact, i.e., we have an exact sequence

$$f^*\mathcal{I}^2 \to \mathcal{J} \to f^*(\mathcal{I}/\mathcal{I}^2) \to 0 .$$

Together with the above surjectivity this implies $f^*(\mathcal{I}/\mathcal{I}^2) \cong \mathcal{J}/\mathcal{J}^2$. As X respectively X' are regular and Z respectively Z' are locally given by regular sequences, the structural theorem [Ha] II Theorem 8.21A e) implies

$$f^*(\mathcal{I}^n/\mathcal{I}^{n+1}) \cong \mathcal{J}^n/\mathcal{J}^{n+1} .$$

By the snake lemma $K_{n+1} \to K_n$ is surjective and hence $f^*(\mathcal{I}^n/\mathcal{I}^{n+k}) \cong \mathcal{J}^n/\mathcal{J}^{n+k}$ for all k. But then

$$\mathcal{J}^n \cong \varprojlim \mathcal{J}^n/\mathcal{J}^{n+k} \cong \varprojlim f^*\mathcal{I}^n/\operatorname{im} f^*\mathcal{I}^{n+k} \cong \varprojlim f^*\mathcal{I}^n/\mathcal{J}^k f^*\mathcal{I}^n \cong f^*\mathcal{I}^n .$$

□

Push-forward is not a λ-ring morphism but it does respect the γ-filtration up to a shift, at least under good conditions. This is made precise in the following Riemann-Roch Theorem, which is a slight generalization of de Jeu's in [Jeu] 2.3. He considers a special type of diagram and restricts to a base field. De Jeu imitates the proof in [T] Theorem 1.1, which is over a field. However, his arguments work for our base as well. Indeed, the original article [Sou4] Thm 3 treated the more general case.

THEOREM B.2.18 (GROTHENDIECK-RIEMANN-ROCH). *Let S be a regular irreducible Noetherian affine scheme S. Let $i : Z \to X$ be a closed immersion of constant codimension d of quasi-projective smooth S-schemes. For $? = X, Z$ let $td(?) \in \mathrm{Gr}_\gamma^* K_0(?)_\mathbb{Q}$ be the usual Todd classes (e.g. [T] p. 135). Let a finite diagram $\mathcal{D}_X$ be given that satisfies the conditions (TC) with respect to i. Finally let $Y_.$ be a space constructed from schemes in $s\mathbf{T}_X$.*

a) *The homomorphism $i_* : K_n(i^{-1}Y_.)_\mathbb{Q} \to K_n(Y_.)_\mathbb{Q}$ has degree $-d$ with respect to the γ-filtration, i.e.,*

$$F^j K_n(i^{-1}Y_.)_\mathbb{Q} \xrightarrow{i_*} F^{j-d} K_n(Y_.)_\mathbb{Q} \quad .$$

b) *The following diagram commutes:*

$$\begin{array}{ccc} K_n(i^{-1}Y_.)_\mathbb{Q} & \xrightarrow{td(Z)ch} & \mathrm{Gr}_\gamma^* K_n(i^{-1}Y_.)_\mathbb{Q} \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ K_n(Y_.)_\mathbb{Q} & \xrightarrow{td(X)ch} & \mathrm{Gr}_\gamma^* K_n(Y_.)_\mathbb{Q} \end{array}$$

REMARK: $td(?)$ is a unit with augmentation 1. Hence the horizontal maps in b) are isomorphisms.

Proof. We essentially have to prove classical Riemann-Roch for the inclusion $Z \to X$. The conditions on our situation are chosen in a way that the diagrams we drag along do not make any difficulties. Note also that we can replace $Y_.$ by the cone of $Y_. \times U \to Y_.$, i.e., we can assume that all pushout maps are isomorphisms. Having observed this we can follow de Jeu's arguments in [Jeu] 2.3.

The first step is to prove the analogue of [T] Theorem 1.2 or [Jeu] Proposition 2.5 ("Riemann-Roch without denominators"). We only sketch the idea: Because of functoriality B.2.16 and the homotopy property of K'-theory we can make the transformation to the normal cone. Hence we can assume without loss of generality that i is a section of a projective bundle over Z. The existence of the projection p which is a left-inverse of i allows to make explicit calculations. All details of the argument can be found in [Jeu] 2.5 when replacing $K_0(Y_0)$ $(= K_0(X_0)$ there) by $K_0(X) = K_0(s\mathbf{T}_X)$. The necessary compatibility of blow-up and base change is guaranteed by the previous lemma.

We then show that up to multiplication with the appropriate Todd class i_* has the required behaviour with respect to Adams eigenspaces. The argument is the same as in [Jeu] Proposition 2.3 or [T] Lemma 2.2. Now the theorem follows by the same formal manipulations as in the proof of [T] Lemma 2.3. $\square$

COROLLARY B.2.19. *Let $i : Z \to X$ (closed immersion of constant codimension d) and $Y_.$ be as in the theorem. Let $U = X \smallsetminus Z$. Then there is a natural localization sequence*

$$\begin{aligned} \ldots \longrightarrow K_m(Z \times_X Y_.)_\mathbb{Q} \longrightarrow K_m(Y_.)_\mathbb{Q} \longrightarrow K_m(U \times_X Y_.)_\mathbb{Q} \\ \longrightarrow K_{m-1}(Z \times_X Y_.)_\mathbb{Q} \longrightarrow \ldots \end{aligned}$$

or in terms of motivic cohomology

$$\begin{aligned} \ldots \longrightarrow H^{i-2d}_{\mathcal{M}}(Z \times_X Y_., j-d) \longrightarrow H^i_{\mathcal{M}}(Y_., j) \longrightarrow H^i_{\mathcal{M}}(U \times Y_., j) \\ \longrightarrow H^{i-2d+1}_{\mathcal{M}}(Z \times_X Y_., j-d) \longrightarrow \ldots \end{aligned}$$

Proof. Part b) of Theorem B.2.18 implies that

$$i_* : \bigoplus_{j \in \mathbb{N}_0} \mathrm{Gr}^j_\gamma K_m(Y_. \text{ rel } Y_. \times U) \longrightarrow \bigoplus_{j \in \mathbb{N}_0} \mathrm{Gr}^{j-d}_\gamma K_m(Y_. \times Z)$$

is an isomorphism, i.e., $H^i_{\mathcal{M}}(Y_. \text{ rel } Y_. \times U, j) \cong H^{i-2d}_{\mathcal{M}}(Z \times_X Y_., j-d)$.
We consider the long exact sequence of relative K-cohomology or relative motivic cohomology for the open embedding $U \times Y_. \subset Y_.$. We can use i_* to identify the relative cohomology with cohomology of the closed complement. □

Only a few K-groups are known. However, the ranks of the K-groups of number fields are understood.

THEOREM B.2.20 (BOREL). *Let K be a number field with ring of S-integers $\mathfrak{o}_S$ where S is a finite set of primes of K. Let $B = \operatorname{Spec} \mathfrak{o}_S$. As usual r_1 is the number of real places of K and r_2 the number of complex places. Then the motivic cohomology has the following ranks:*

$H^0_{\mathcal{M}}(B,0)$	1	
$H^1_{\mathcal{M}}(B,1)$	$\#S + r_1 + r_2 - 1$	
$H^1_{\mathcal{M}}(B,n)$	r_2	$n > 1$, *even* ;
$H^1_{\mathcal{M}}(B,n)$	$r_1 + r_2$	$n > 1$, *odd* ;
$H^i_{\mathcal{M}}(B,j)$	0	*else* .

Proof. The computation of $K_0(B)$ and $K_1(B)$ is classical ([Ba] Ch. IX, Prop. 3.2 and Ch. X, Cor. 3.6). The higher K-groups for the ring of integers $\mathfrak{o}_K$ were calculated by Borel ([Bo1], Prop 12.2). It follows from Quillen's computation of the K-groups of finite fields that the ranks are not changed by localizing at finite primes. □

B.3 COHOMOLOGY OF ABELIAN SHEAVES

We now show how the usual cohomology theories fit in the set-up of generalized cohomology. This is well documented in the literature [BrG], [G], [Jeu]. In the case of a cohomology theory defined by a pseudo-flasque complex of presheaves $\mathcal{F}$, we compare the different possible points of view. These are Zariski-cohomology of the associated complex of sheaves, generalized cohomology of the associated space or simply cohomology of the sections. We always get the same cohomology groups (B.3.2 and B.3.4). If the complex of presheaves $\mathcal{F}$ is part of a twisted duality theory (B.3.7), we define Chern classes from K–cohomology of spaces to cohomology with coefficients in $\mathcal{F}$. Finally we check compatibility of the localization sequence in K–cohomology with the one for cohomology of spaces with coefficients in $\mathcal{F}$ (B.3.8).

By a complex we always mean a cohomological complex. Of course it can also be considered as a homological complex by inverting the signs of the indices.

The Dold-Puppe functor [M] Thm 22.4 attaches to a complex of abelian groups G which is concentrated in non-positive degrees a simplicial abelian group $K(G)$ pointed by 0 whose homotopy groups $\pi_i(K(G), 0)$ agree with the cohomology groups $h^{-i}(G)$. It induces an equivalence between the homotopy category of simplicial abelian groups and the homotopy category of complexes of abelian groups concentrated in non-positive degrees. By construction of the functor K there is a natural weak equivalence of spaces

$$\mathrm{Cone}(K(G) \to *) \longrightarrow K(\mathrm{Cone}(G \to 0)) = K(G[1])$$

and hence a natural map $\Omega K(G[1]) \to K(G)$ in the homotopy category of pointed simplicial sets, which is a homotopy equivalence. If G is an arbitrary complex of abelian groups, let $\tau_{\leq N} G$ be the canonical sub-complex in degrees less or equal to N. We put

$$K(G)_N = K(\tau_{\leq N} G[N]) \; .$$

The natural map $\tau_{\leq N-1} G[N] \to \tau_{\leq N} G[N]$ induces

$$K(G)_{N-1} \cong \Omega K(\tau_{\leq N-1} G[N]) \longrightarrow \Omega K(G)_N \; ,$$

which is a weak equivalence. This means the $K(G)_N$ form an infinite loop spectrum whose homotopy groups reflect all cohomology groups of the complex.

Definition B.3.1. *Let $\mathcal{G}$ be a cohomological complex of sheaves of abelian groups on the big Zariski site. The sheafified version of the above construction yields an infinite loop spectrum of spaces $K(\mathcal{G})$ with*

$$h^{-i}(\mathcal{G}) \cong \underline{\pi}_i(K(\mathcal{G}), 0)$$

where the right hand side is the sheafification of the presheaf

$$U \mapsto \pi_i(K(\mathcal{G})(U), 0) \; .$$

As a spectrum $K(\mathcal{G})$ defines generalized cohomology groups with indices in $\mathbb{Z}$ for any space X.

Proposition B.3.2. *Let $\mathcal{G}$ be a bounded below complex of sheaves on the big Zariski site. Let X be a scheme. Then*

$$H^i_{s\mathbf{T}}(X, K(\mathcal{G})) \cong H^i_{\mathrm{ZAR}}(X, \mathcal{G}) \; .$$

Proof. As $\mathcal{G}$ is bounded below it has a bounded below resolution by flasque sheaves. Now the proof proceeds as in [BrG] Prop. 2. The main ingredient is that $K(\mathcal{I})$ is a fibrant space if $\mathcal{I}$ is a flasque sheaf. □

Definition B.3.3. a) *Following [BrG], Sect. 2 a complex $\mathcal{F}$ of abelian presheaves on the big Zariski site is called pseudo-flasque if it has the Mayer-Vietoris property, i.e., for open subschemes U and V of some scheme X, we have a long exact sequence of abelian groups*

$$\begin{aligned} \ldots \longrightarrow h^i\left(\mathcal{F}(U \cup V)\right) \longrightarrow h^i\left(\mathcal{F}(U) \oplus \mathcal{F}(V)\right) \longrightarrow h^i\left(\mathcal{F}(U \cap V)\right) \\ \longrightarrow h^{i+1}\left(\mathcal{F}(U \cup V)\right) \longrightarrow \ldots \end{aligned}$$

More precisely, the square

$$\begin{array}{ccc} \mathcal{F}(U \cap V) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U \cup V) \end{array}$$

is homotopically cartesian.

b) *Let $\mathcal{F}$ be a complex of abelian presheaves. For the object $\star \amalg U$ in $\mathbf{T}$ where U is a scheme, we put*

$$\mathcal{F}(\star \amalg U) = \mathcal{F}(U) \ .$$

Let X be a space constructed from schemes. Then we put

$$\mathcal{F}(X) = \mathrm{Tot}_i \, \mathcal{F}(X_i) \ .$$

the total complex of the cosimplicial complex $\mathcal{F}(X_i)_{i \in \mathbb{N}_0}$.

Taking the total complex of a bicomplex as in b) of course involves a choice of signs which we fix once and for all. Different choices of signs differ by a canonical isomorphism of the total complex.

LEMMA B.3.4. *Let $\mathcal{F}$ be a bounded below pseudo-flasque complex of abelian presheaves. Let $\tilde{\mathcal{F}}$ be its sheafification. Then*

$$H^i_{s\mathbf{T}}(X, K(\tilde{\mathcal{F}})) = h^i(\mathcal{F}(X))$$

for all spaces X constructed from schemes.

Proof. Let $\mathcal{I}$ be a (bounded below) flasque resolution of $\tilde{\mathcal{F}}$. This is in particular a pseudo-flasque complex of presheaves that is quasi-isomorphic to $\mathcal{F}$ as a complex of presheaves because both compute Zariski-cohomology of $\tilde{\mathcal{F}}$. As in the proof of [BrG] Theorem 4, the simplicial sheaf $K(\mathcal{I})$ is a fibrant resolution of $K(\tilde{\mathcal{F}})$. Hence we can assume without loss of generality that $\mathcal{F}$ itself is a complex of flasque sheaves.
For the case of a scheme X the lemma is the reformulation of [BrG] Theorem 4 in the easier case of simplicial presheaves that come from a complex of abelian presheaves. In the general case

$$\begin{aligned} H^i_{s\mathbf{T}}(X, K(\tilde{\mathcal{F}})) &= \pi_{-i} \, \mathrm{Hom}_{\cdot}(X, K(\tilde{\mathcal{F}})) \\ &= \pi_{-i} \, \mathrm{Hom}_{\cdot}(\mathrm{hocolim}\, X_j, K(\tilde{\mathcal{F}})) \\ &= \pi_{-i} \, \mathrm{holim}\, \mathrm{Hom}_{\cdot}(X_j, K(\tilde{\mathcal{F}})) \qquad \text{[BouK] XII Prop. 4.1} \\ &= h^i(\mathrm{Tot}\, \mathcal{F}(X_i)) = h^i(\mathcal{F}(X)) \ . \end{aligned}$$

□

This means if we define a cohomology theory by a pseudo-flasque complex of presheaves on the big Zariski site we can freely change from the point of view of generalized cohomology to ordinary Zariski-cohomology or cohomology of the sections of the presheaf.

If $X \to Y$ is a morphism of schemes, we consider as usual its Čech-nerve $\mathrm{cosk}_0(X/Y)$, i.e., the simplicial Y-scheme given by

$$\mathrm{cosk}_0(X/Y)_n = (X \times_Y \cdots \times_Y X) \qquad n+1\text{-fold product}$$

with the natural boundary and degeneracy morphisms.

Definition B.3.5. *We say that a morphism $X \to Y$ of schemes has cohomological descent for the cohomology theory given by the complex of abelian Zariski-sheaves $\mathcal{G}$ if the natural morphisms*

$$H^i_{s\mathbf{T}}(Y, K(\mathcal{G})) \longrightarrow H^i_{s\mathbf{T}}(\mathrm{cosk}_0(X/Y), K(\mathcal{G}))$$

are isomorphisms for all $i \in \mathbb{Z}$.

This is of course a very special case of the general notion of cohomological descent.

Lemma B.3.6. *Let $j : U \to X$ be an open immersion with closed complement Y. Let $\mathcal{F}$ be a pseudo-flasque complex of presheaves on ZAR_X with sheafification $\tilde{\mathcal{F}}$.*

a) *There are natural isomorphisms*

$$H^i_{s\mathbf{T}}(X \ \mathit{rel}\ Y, K(\tilde{\mathcal{F}})) \longrightarrow H^i_{\mathrm{ZAR}}(X, j_!j^*\tilde{\mathcal{F}}) \ .$$

b) *If $\tilde{Y} \to Y$ is a morphism with cohomological descent for $\tilde{\mathcal{F}}$, then we get a natural isomorphism*

$$H^i_{s\mathbf{T}}(X \ \mathit{rel}\ \mathrm{cosk}_0(\tilde{Y}/Y), K(\tilde{\mathcal{F}})) \xrightarrow{\cong} H^i_{\mathrm{ZAR}}(X, j_!j^*\tilde{\mathcal{F}}) \ .$$

Proof. By B.3.4 the left-hand side of a) is canonically isomorphic to the cohomology of

$$\mathcal{F}(C(Y \xrightarrow{i} X)) \cong \mathrm{Cone}\left(\mathcal{F}(X) \xrightarrow{\mathcal{F}(i)} \mathcal{F}(Y)\right)[-1]$$

where the right hand side is the cone in the category of cohomological complexes. We assume without loss of generality that $\tilde{\mathcal{F}}$ is a flasque complex. The key point is the short exact sequence of complexes of sheaves on X

$$0 \longrightarrow j_!j^*\tilde{\mathcal{F}} \longrightarrow \tilde{\mathcal{F}} \longrightarrow i_*i^*\tilde{\mathcal{F}} \longrightarrow 0 \ \ .$$

It induces a canonical quasi-isomorphism of complexes

$$j_!j^*\tilde{\mathcal{F}} \longrightarrow \mathrm{Cone}\left(\tilde{\mathcal{F}} \to i_*i^*\tilde{\mathcal{F}}\right)[-1] \ .$$

We now take $R\Gamma_{\mathrm{Zar}}(X, \ \cdot\ )$ of the right-hand side. Because $\mathcal{F}$ was assumed to be pseudo-flasque the morphism

$$\mathrm{Cone}\left(\mathcal{F}(X) \to \mathcal{F}(Y)\right) \longrightarrow \mathrm{Cone}\left(\tilde{\mathcal{F}}(X) \to \tilde{\mathcal{F}}(Y)\right) \ .$$

is a quasi-isomorphism. This last fact follows from B.3.4 and B.3.2. (Of course it can also be proved, even more easily, in terms of complexes of abelian groups rather than simplicial abelian groups.) In the case of a morphism $\tilde{Y} \to Y$ with cohomological descent the left hand side of the statement is by B.3.4 given by the cohomology of

$$\mathrm{Cone}\left(\mathcal{F}(X) \to \mathcal{F}(\mathrm{cosk}_0(\tilde{Y}/Y))\right)[-1] \ .$$

The natural morphism $\mathcal{F}(Y) \longrightarrow \mathcal{F}(\mathrm{cosk}_0(\tilde{Y}/Y))$ is a quasi-isomorphism by definition and Lemma B.3.4. $\square$

Theorem B.3.7 (Gillet, de Jeu). *Let $\mathcal{F} = \bigoplus_{i\in\mathbb{Z}} \mathcal{F}(i)$ be a pseudo-flasque complex of abelian presheaves on the big Zariski site. Assume that $\mathcal{F}$ defines a twisted duality theory, i.e., the extra data of [G] Def. 1.1 exist and all conditions of loc. cit. Def. 1.2 are fulfilled. Then:*

- *There are Chern class maps of spaces*

$$c_j : \mathbf{K} \longrightarrow K(\tilde{\mathcal{F}}(j)[2j]) \ .$$

They induce morphisms

$$c_j : H^i_{s\mathbf{T}}(Y, \mathbf{K}) \longrightarrow H^{i+2j}_{s\mathbf{T}}(Y, K(\tilde{\mathcal{F}}(j)))$$

for all spaces Y in $s\mathbf{T}$.

- *If Y is a K-coherent space, then the total Chern class c_Γ is a morphism of λ-algebras on K–cohomology of Y.*

- *Let $i : Z \to X$ a closed immersion of smooth S–schemes with open complement U. The map $i_! : i_*\mathcal{F}(r)\mid_Z \to \mathcal{F}(r+d)\mid_X [2d]$ required in [G] Def. 1.2. induces push-forward on generalized cohomology. If $Y_.$ is a space over X as in B.2.18, then the diagram*

$$\begin{array}{ccc} \mathrm{Gr}^j_\gamma K_n(Y_. \times_X Z)_{\mathbb{Q}} & \xrightarrow{i_*} & \mathrm{Gr}^{j+d}_\gamma K_n(Y_.)_{\mathbb{Q}} \\ {\scriptstyle c_j}\downarrow & & \downarrow{\scriptstyle c_{j+d}} \\ H^{2j-n}_{s\mathbf{T}}(Y_. \times_X Z, K(\tilde{\mathcal{F}}(j)))_{\mathbb{Q}} & \xrightarrow{i_!} & H^{2j+2d-n}_{s\mathbf{T}}(Y_., K(\tilde{\mathcal{F}}(j+d)))_{\mathbb{Q}} \end{array}$$

is commutative.

Proof. The construction of the Chern classes is [G] Thm 2.2. Gillet's formulation is for schemes but he constructs in fact a morphism of spaces (loc. cit. p. 225) so the results hold for more general spaces (see also [GSo1] 4.1). The assertion on the λ-ring structure is [GSo1] Thm. 7. We sketch the idea: Everything is defined on the level of coefficients, so it does not depend on Y. Compatibility with multiplication is [G] 2.3.2. Compatibility with γ-operators can be checked on the level of universal Chern classes, i.e., for elements $C_{i,N} \in H^{2i}_{s\mathbf{T}}(BGl_n, \tilde{\mathcal{F}}(i))$. Now use the splitting principle ([G] 2.4).
The last part of the proposition is a generalization of Gillet's Riemann-Roch Theorem [G] 4.1 to spaces of our special type. The proof carries over by the same method as in the proof of Riemann-Roch for K-cohomology B.2.18. Mutis mutanda the statement can be found in [Jeu] Lemma 2.13. $\square$

Remark: This will allow to define regulator maps from K-cohomology to the cohomology theories we are interested in.

Corollary B.3.8. *Let X, Z, d, $Y_.$ and $\mathcal{F}$ be as in the theorem. In addition assume that $\mathcal{F}$ is pseudo-flasque. Let U be the complement of Y in X. We abbreviate $Y_U = Y_. \times_X U$, $Y_Z = Y_. \times_X Z$ and $F_j = K(\tilde{\mathcal{F}}(j))$. Then there is a natural morphism of long exact sequences*

$$\begin{tikzcd}
H^{i-1}_{\mathcal{M}}(Y_U, j) \arrow[r] \arrow[d] & H^{i-2d}_{\mathcal{M}}(Y_Z, j-d) \arrow[r] \arrow[d] & H^i_{\mathcal{M}}(Y_., j) \arrow[r] \arrow[d] & H^i_{\mathcal{M}}(Y_U, j) \arrow[d] \\
H^{i-1}_{s\mathbf{T}}(Y_U, F_j) \arrow[r] \arrow[d, "="'] & H^{i-2d}_{s\mathbf{T}}(Y_Z, F_{j-d}) \arrow[r] \arrow[d, "="'] & H^i_{s\mathbf{T}}(Y_., F_j) \arrow[r] \arrow[d, "="'] & H^i_{s\mathbf{T}}(Y_U, F_j) \arrow[d, "="'] \\
h^{i-1}\mathcal{F}(j)(Y_U) \arrow[r] & h^{i-2d}\mathcal{F}(j-d)(Y_Z) \arrow[r] & h^i\mathcal{F}(j)(Y_.) \arrow[r] & h^i\mathcal{F}(j)(Y_U)
\end{tikzcd}$$

Proof. We start with the long exact sequences for relative cohomology (B.1.7) with coefficients in the spectrum $\mathbf{K}$ and in the spectrum $K(\tilde{\mathcal{F}})$. Their compatibility is nothing but functoriality. Relative cohomology is replaced by cohomology of $Y_. \times_X Z$ using B.3.7. Finally we pass to graded pieces of the γ-filtration. Note that the indices in the definition of motivic cohomology are chosen in a way that they agree with the indices of other cohomology theories under Chern class maps. Equality of the last two lines is B.3.4 $\square$

Note that the last line has nothing to do with generalized cohomology or spaces.

B.4 Continuous Etale Cohomology

There are different ways of defining continuous étale cohomology. We will see that they all give the same thing.

Fix a number field K and a prime l. Let B be an open subscheme of $\operatorname{Spec} \mathfrak{o}_K[1/l]$ where $\mathfrak{o}_K$ is the ring of integers of K.

Proposition B.4.1 (Deligne, Ekedahl). *Let $f : Y \to X$ be a morphism of B-schemes of finite type. Then there are triangulated categories $D^b_c(X - \mathbb{Z}_l)$ and $D_c(Y - \mathbb{Z}_l)$ admitting the following: there is a t-structure whose heart are the constructible l-adic systems. There are functors*

$$f_!, f_* : D^b_c(Y - \mathbb{Z}_l) \longrightarrow D^b_c(X - \mathbb{Z}_l)$$

and

$$f^*, f^! : D^b_c(X - \mathbb{Z}_l) \longrightarrow D^b_c(Y - \mathbb{Z}_l)$$

having all the usual properties of Grothendieck functors.

Proof. This is [Ek] Thm 6.3. In the case $B = \operatorname{Spec} \mathfrak{o}_K[1/l]$ the category was already constructed in [D4], 1.1.2. $\square$

REMARK: $D^b_c(X - \mathbb{Z}_l)$ should be thought of as the bounded derived categories of constructible l-adic sheaves on X_{et}. By Ekedahl's construction $D^b_c(X - \mathbb{Z}_l)$ is a subcategory of a localization of a subcategory of the derived category of the abelian category $(X_{et})^{\mathbb{N}} - \mathbb{Z}_l$. By this notation Ekedahl means the category of projective systems of étale sheaves on X ringed by the projective system $\mathbb{Z}/l^n$. The four functors are defined on the level of this last derived category. Ekedahl then shows that they induce well-defined functors on $D^b_c(X - \mathbb{Z}_l)$. In the case B open in $\operatorname{Spec}\mathfrak{o}_K[1/l]$, we get away with Deligne's more straightforward construction.

DEFINITION B.4.2 (1. VERSION). a) *For $k \in \mathbb{Z}$ let $\mathbb{Z}_l(k)$ be the constructible l-adic sheaf on B given by the projective system $\mu_{l^n}^{\otimes k}$.*

b) *We define continuous étale cohomology of $s : X \to B$ by*

$$H^i_{cont}(X, k) = \operatorname{Hom}_{D^b_c(X - \mathbb{Z}_l)}(s^*\mathbb{Z}_l(0), s^*\mathbb{Z}_l(k)[i]) \quad .$$

c) *If $j : U \to X$ is an open immersion with complement Y we define relative continuous étale cohomology by*

$$H^i_{cont}(X \text{ rel } Y, k) = \operatorname{Hom}_{D^b_c(X - \mathbb{Z}_l)}(s^*\mathbb{Z}_l(0), j_!(s \circ j)^*\mathbb{Z}_l(k)[i]) \quad .$$

d) *More generally, let $\mathcal{M}$ be an object of $D^b_c(X - \mathbb{Z}_l)$. We define continuous étale cohomology of X with coefficients in $\mathcal{M}$ as*

$$H^i_{cont}(X, \mathcal{M}) = \operatorname{Hom}_{D^b_c(X - \mathbb{Z}_l)}(s^*\mathbb{Z}_l(0), \mathcal{M}[i]) \quad .$$

This definition allows to derive all the usual spectral sequences from the calculus of the Grothendieck functors.

REMARK: As checked in [H2] §4 this definition coincides with Jannsen's original one in [Jn1] sect. 3. In our case continuous étale cohomology with coefficients in a constructible l-adic sheaf $(\tilde{\mathcal{F}}_n)_n$ is nothing but the naive $\varprojlim H^n_{et}(X, \tilde{\mathcal{F}}_n)$ because all $H^n_{et}(X, \tilde{\mathcal{F}}_n)$ are finite.

Let us now define continuous étale cohomology in a way that fits in with the setting of the previous section.

DEFINITION B.4.3 (2. VERSION). *Consider the projective system of sheaves $\left(\mu_{l^n}^{\otimes k}\right)_{n \in \mathbb{N}}$ on the big étale site over B. Let $\mathcal{I}$ be an injective resolution in the category of projective systems. It is given by a projective system $\mathcal{I}_n$ of injective resolutions of $\mu_{l^n}^{\otimes k}$ on the big étale site with split surjective transition morphisms ([Jn1] 1.1). By taking sections we get a projective system of complexes of Zariski-presheaves $R\Gamma\left(\mu_{l^n}^{\otimes k}\right)_{n \in \mathbb{N}}$. The functor $R\varprojlim$ turns it into a complex $\mathcal{F}_l(k)$ of Zariski-presheaves. For any space X put*

$$H^i_{cont}(X, k) = H^i_{s\mathbf{T}}(X, K(\tilde{\mathcal{F}}_l(k))) \quad .$$

In particular if $\iota : Y \to X$ is a morphism of spaces, then we put

$$H^i_{cont}(X \text{ rel } Y, k) = H^i_{s\mathbf{T}}(C(\iota), K(\tilde{\mathcal{F}}_l(k))) \quad .$$

LEMMA B.4.4. *If X is a B-scheme, then both versions of the definition of continuous étale cohomology agree canonically. If $Z \to X$ is a closed immersion, then the same is true for both definitions of relative continuous étale cohomology.*

Proof. $\mathcal{F}_l(X)$ is nothing but an explicit version of the derived functor $R \varprojlim R\Gamma(X, \cdot)$ from the derived category of projective systems of étale sheaves to the derived category of abelian groups. Hence the complex $\mathcal{F}_l(X)$ computes the first version of continuous étale cohomology. In particular it has the Mayer-Vietoris property. Hence we can apply the lemmas of the previous section (B.3.4) and get

$$H^i_{s\mathbf{T}}(X, K(\tilde{\mathcal{F}}_l)) = h^i(\mathcal{F}_l(X)) \quad .$$

To extend the result to relative étale cohomology we use essentially the same argument as in B.3.6.b). $\square$

REMARK: When we say that the isomorphism is canonical, we think in particular of the following situation: The cartesian diagram of schemes

$$\begin{tikzcd}
U' \arrow[r, "j'"] \arrow[d] & X' \arrow[d, "f"'] & Y' \arrow[l, "i'"'] \arrow[d] \\
U \arrow[r, "j"] & X & Y \arrow[l, "i"'] \\
U'' \arrow[r, "j''"] \arrow[u] & X'' \arrow[u, "g"] & Y'' \arrow[l, "i''"'] \arrow[u]
\end{tikzcd}$$

(f, j open, g, i closed complements) induces a map

$$H^i_{cont}(X \text{ rel } Y, n) \xrightarrow{f^*} H^i_{cont}(X' \text{ rel } Y', n) \quad ,$$

which is compatible with the identification. If all schemes are smooth and X'' intersects Y transversally, then we also get the same long exact sequence

$$\begin{aligned}
\cdots \to H^{i-2d}_{cont}(X'' \text{ rel } Y'', n-d) &\to H^i_{cont}(X \text{ rel } Y, n) \to H^i_{cont}(X' \text{ rel } Y', n) \\
\to H^{i+1-2d}_{cont}(X'' \text{ rel } Y'', n-d) &\to \cdots
\end{aligned}$$

using either definition of relative cohomology.

LEMMA B.4.5. *If $\tilde{Y} \to Y$ is a proper covering (i.e., a proper and surjective map), then it has cohomological descent for continuous étale cohomology. In particular if $Y \to X$ is a closed embedding and $\tilde{Y}$ a proper covering of Y, then there is a natural isomorphism*

$$H^i_{cont}(X \textit{ rel } Y, j) \longrightarrow H^i_{cont}(X \textit{ rel } \operatorname{cosk}_0(\tilde{Y}/Y), j)$$

where the right hand side is taken in the sense of spaces.

Proof. Cohomological descent is a consequence of the same descent for étale cohomology with torsion coefficients prime to the characteristic of the schemes ([SGA4,II], Exp. Vbis, 4.1.6). By B.3.6.b) the second part follows. $\square$

Proposition B.4.6. *On the Zariski site of smooth schemes over B, the presheaf $\tilde{\mathcal{F}}_l$ has the properties of a twisted duality theory. There are regulator maps from K-cohomology to continuous étale cohomology*

$$H^i_{\mathcal{M}}(Y,j) \longrightarrow H^i_{cont}(Y,j)$$

for all K–coherent spaces Y. They are compatible with pullback, i.e., if $f : Y \to Y'$ is a map of K–coherent spaces, we get commutative diagrams

$$\begin{array}{ccc} H^i_{\mathcal{M}}(Y',j) & \xrightarrow{f^*} & H^i_{\mathcal{M}}(Y,j) \\ {\scriptstyle c_j}\downarrow & & \downarrow{\scriptstyle c_j} \\ H^i_{cont}(Y',j) & \xrightarrow{f^*} & H^i_{cont}(Y,j) \end{array} .$$

If $i : Z \to X$ is a closed immersion of smooth schemes (constant codimension d) with open complement U and $Y_.$ a space constructed form schemes over X as in B.2.18, then the regulator is compatible with pushout, i.e., the diagram

$$\begin{array}{ccc} H^{n-2d}_{\mathcal{M}}(Y_. \times_X Z, j-d) & \xrightarrow{i_*} & H^n_{\mathcal{M}}(Y_.,j) \\ {\scriptstyle c_{j-d}}\downarrow & & \downarrow{\scriptstyle c_j} \\ H^{n-2d}_{cont}(Y_. \times_X Z, j-d) & \xrightarrow{i_!} & H^n_{cont}(Y_.,j) \end{array}$$

is commutative.

Proof. We restrict to smooth schemes for simplicity. We have to define the extra-structure from [G] 1.1 and 1.2. We put

$$H_i(X,j) = H^{2d-i}_{cont}(X, d-j)$$

for a d-dimensional smooth connected scheme. Pull-back on cohomology and pushout on homology are induced from the functors on sheaves on the étale site. We do not work out the details. For a single étale sheaf μ_{l^n} this is actually one of Gillet's examples 1.4 (iii). ☐

There is really only one case when this regulator is understood.

Lemma B.4.7. *Let K be a number field, $\mathfrak{o}_K$ be its ring of integers and l a prime. Assume $2i-k \geq 2$, then Soulé's l-adic regulator*

$$K_{2i-k}(\mathfrak{o}_K[1/l]) \otimes \mathbb{Z}_l \longrightarrow H^k_{cont}(\operatorname{Spec} \mathfrak{o}_K[1/l], i)$$

agrees with the one obtained from Prop. B.4.6.

Proof. Put $A = \mathfrak{o}_K[1/l]$. Soulé's definition in [Sou2] is the composition

$$K_{2i-k}(A) \longrightarrow \varprojlim K_{2i-k}(A, \mathbb{Z}/l^\nu) \xrightarrow{\varprojlim \overline{c}_{i,k}} \varprojlim H^k_{et}(A, \mathbb{Z}/l^\nu(i))$$

where $\overline{c}_{i,k}$ is as in [Sou1] II.2.3. There is a natural map of presheaves $\mathcal{F}_l(i) \to R\Gamma(\,\cdot\,, \mathbb{Z}/l^\nu(i))$. Hence in Gillet's definition of Chern classes, we get a commutative diagram

$$\begin{array}{ccc} K_{2i-k}(A) & \xrightarrow{c_i} & H^k_{cont}(\mathrm{Spec}\, A, i) \\ & \searrow & \downarrow \\ & & H^k_{et}(\mathrm{Spec}\, A, \mathbb{Z}/l^\nu(i)) \end{array} .$$

Hence we only have to consider finite coefficients. Furthermore in this simple case of a regular commutative ring, we do not really need to consider the sheafified versions and generalized cohomology. Gillet's construction boils down to a composition of the Hurewicz-map with universal Chern classes.

For $2i - k \geq 2$, the map $\overline{c}_{i,k}$ is defined by the same type of composition ([Sou2] II 2.3.) with the same universal Chern classes.

By the definition of K-theory with coefficients, we have a commutative diagram (loc. cit. II.2.2) with $X = \mathbb{Z}_\infty BGl(A)$:

$$\begin{array}{ccccccccc} \longrightarrow & \pi_n(X) & \xrightarrow{\times l^\nu} & \pi_n(X) & \longrightarrow & \pi_n(X, \mathbb{Z}/l^\nu) & \longrightarrow \\ & {\scriptstyle h}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h_q} & \\ \longrightarrow & H_n(X, \mathbb{Z}) & \xrightarrow{\times l^\nu} & H_n(X, \mathbb{Z}) & \longrightarrow & H_n(X, \mathbb{Z}/q) & \longrightarrow \end{array} .$$

For the prime 2 compare also [We]. □

Theorem B.4.8 (Soulé). *Let K be a number field, $\mathfrak{o}_K$ be its ring of integers and l any prime. Let S' be a finite set of prime ideals of $\mathfrak{o}_K$ and $S = S' \cup \{l\}$. Let $\mathfrak{o}_S$ be the localization of $\mathfrak{o}_K$ at S. The regulator map*

$$c_j : H^i_{\mathcal{M}}(\mathrm{Spec}\, \mathfrak{o}_{S'}, j) \otimes_{\mathbb{Q}} \mathbb{Q}_l \longrightarrow H^i_{cont}(\mathrm{Spec}\, \mathfrak{o}_S, j)_{\mathbb{Q}}$$

is always injective and an isomorphism for $i = 1$ and $j > 1$. We have the following behaviour for pairs of indices (i, j):

$(0, j)$	$j \in \mathbb{Z}$	*isomorphism*
$(1, j)$	$j < 1$	*mot. coh. vanishes, l-adic does not in general*
$(1, 1)$		*injective of finite codimension*
$(1, j)$	$j > 1$	*isomorphism*
$(2, j)$	$j < 1$	*conjectured to be isom., i.e., etale coh. to vanish*
$(2, 1)$		*injective of finite codimension*
$(2, j)$	$j > 1$	*isomorphism, i.e., both vanish*
(i, j)	*else*	*both vanish*

Proof. We have

$$H^i_{cont}(\mathrm{Spec}\, \mathfrak{o}_S, j)_{\mathbb{Q}} = H^i(G_S, \mathbb{Q}_l(j))$$

where G_S is the Galois group of the maximal extension of K that is unramified outside of S. We first check that these groups vanish for $i > 2$: By [Mi] I Cor. 4.15 all $H^i(G_S, \mu_{l^n}^{\otimes j})$ are finite. This means that the projective systems for varying n are

Artin-Rees. We do not get a $\varprojlim{}^1$-contribution to continuous cohomology. Moreover, by loc. cit. I. 4.10.c) the $H^i(G_S, \mu_{l^n}^{\otimes j})$ for $i \geq 3$ are 2-torsion. This implies that their projective limit is 2-torsion. In total we have vanishing cohomology $H^i(G_S, \mathbb{Q}_l(j))$ for $i \geq 3$.
The case $i = 0$ is trivial. $H^1(G_S, \mathbb{Q}_l(1)) = E_S \otimes \mathbb{Q}_l$ where E_S are the S-units, while $H^1_{\mathcal{M}}(\operatorname{Spec} \mathfrak{o}_{S'}, 1) = \mathfrak{o}^*_{S'} \otimes \mathbb{Q}_l$. For $H^2(G_S, \mathbb{Q}_l(1))$ (the S-Brauer-group) the codimension is the same as in the $(1,1)$–case by Euler-Poincaré duality (cf. the discussion in [Jn2] Lemma 2 and Cor. 1.). In the remaining cases, neither motivic (B.2.20) nor continuous étale cohomology ([Jn3] Lemma 4) is changed by the inversion of S', at least up to torsion. We assume $S' = \emptyset$. For odd l, the cases $(1,j)$ and $(2,j)$ for $j > 1$ are Soulé's result in [Sou2] Theorem 1. Note that we are in the range where the previous lemma applies.
For $l = 2$, we have to refine the argument. On the level of $\mathbb{Q}_2$-coefficients we may, by Galois descent, assume that K contains $\sqrt{-1}$ – note that the only prime which could possibly ramify in this quadratic extension has been inverted, and hence we get an étale extension of rings. By [DwF], Theorem 8.7 and the succeeding remark, we have surjectivity even for $l = 2$.
To conclude, we need to show that the $\mathbb{Q}_2$-vector spaces have the right dimension. Let $j > 1$. By [Jn2], proof of Lemma 1, the dimension of

$$H^i_{cont}(\operatorname{Spec} \mathfrak{o}_K[1/2], j)_{\mathbb{Q}}$$

equals the corank of

$$H^i_{cont}(\operatorname{Spec} \mathfrak{o}_K[1/2], \mathbb{Q}_2/\mathbb{Z}_2(j)) .$$

By [Sou3], 1.2 and Proposition 2, this corank, for $i = 1$, equals the rank of the K–group if and only if

$$H^2_{cont}(\operatorname{Spec} \mathfrak{o}_K[1/2], \mathbb{Q}_2/\mathbb{Z}_2(j))$$

is torsion. This in turn follows from [We], Theorem 7.3. $\square$

Finally we want to discuss Soulé's elements in K-theory with coefficients. Everything is in the setting of simplicial sets and spectra in the usual sense. Generalized cohomology does not enter. Let Σ be the sphere spectrum and l^r a prime power. By definition of the Moore spectrum there is a cofibration sequence

$$\Sigma \xrightarrow{l^r} \Sigma \xrightarrow{i_{l^r}} M_{l^r} \xrightarrow{j_{l^r}} S\Sigma .$$

Recall that for the ring of integers in a number field A

$$K_n(A, \mathbb{Z}_l) = \varprojlim K_n(A, \mathbb{Z}/l^\nu) = \varprojlim \pi_n(\mathbf{K} \wedge M_{l^\nu}) .$$

The Moore spectrum has a unique product for $l > 2$. For $l = 2, r \geq 2$ there are two projective systems of regular product structures on M_{l^r} ([O], Theorem 2 (a), (b) and Lemma 5). Together with the product structure on $\mathbf{K}$ this defines a product on $K_*(A, \mathbb{Z}_l)$ for $l \geq 2$.

For $d \geq 2$, we define $R = \mathbb{Z}(\mu_d, 1/dl)$. Recall ([Sou2], Lemma 1, [Sou5], 4.1–4.3) Soulé's construction of maps

$$\varphi_l : \text{primitive elements of } \mu_d \longrightarrow K_{2n+1}(R, \mathbb{Z}_l) = K_{2n+1}(R) \otimes_{\mathbb{Z}} \mathbb{Z}_l \ .$$

The original statement is for odd primes l, but using the above 2-adic product the construction works without any changes for $l = 2$. For a primitive d-th root of unity ω, choose some $(\alpha_r)_{r\geq 1} \in \varprojlim \mu_{dl^r}$ satisfying $\alpha_1^l = \omega$. Let $(\beta_r)_{r\geq 1} \in \varprojlim K_2(R, \mathbb{Z}/l^r)$ be the projective system of Bott elements with $j_{2^r}(\beta_r) = \alpha_r \in K_1(R)$. Using the formalism of norm compatible units developed in [Sou2], one lets $\varphi_l(\omega)$ denote the projective system

$$\left(N_r((1-\alpha_r) \cup (\beta_r^d)^{\cup n})\right)_r \in \varprojlim K_{2n+1}(R, \mathbb{Z}/l^r\mathbb{Z}) \ .$$

REMARK: It is not clear to the authors whether the 2-adic Soulé elements depend on the choice of product on the Moore spectrum. By [O] pp. 263–264, the difference between the two regular products μ and μ' on M_{2^r} is given by

$$M_{2^r} \wedge M_{2^r} \xrightarrow{j_{2^r} \wedge j_{2^r}} S\Sigma \wedge S\Sigma \xrightarrow{\eta^2} \Sigma \xrightarrow{i_{2^r}} M_{2^r} \ .$$

LEMMA B.4.9. *Let ζ be a root of unity and $n \geq 0$. The restriction map from $H^1_{cont}(\mathbb{Q}(\zeta), \mathbb{Q}_l(n+1))$ into*

$$\begin{aligned}
& H^1_{cont}(\mathbb{Q}(\mu_{l^\infty}, \zeta), \mathbb{Q}_l(n+1))^{\mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty},\zeta)/\mathbb{Q}(\zeta))} \\
= & \left(\varprojlim_{r\geq 1} \left(H^1_{cont}(\mathbb{Q}(\mu_{l^\infty}, \zeta), \mu_{l^r}) \otimes \mu_{l^r}^{\otimes n} \right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \right)^{\mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty},\zeta)/\mathbb{Q}(\zeta))} \\
= & \left(\varprojlim_{r\geq 1} \left(\mathbb{Q}(\mu_{l^\infty}, \zeta)^* / (\mathbb{Q}(\mu_{l^\infty}, \zeta)^*)^{l^r} \otimes \mu_{l^r}^{\otimes n} \right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \right)^{\mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty},\zeta)/\mathbb{Q}(\zeta))}
\end{aligned}$$

is injective.

Proof. Note that the argument given in the discussion preceding [WiIV], Theorem 4.5 is incorrect since the transition maps

$$H^1_{cont}\left(\mathbb{Q}(\mu_{l^m}, \zeta), \mu_{l^r}^{\otimes(n+1)}\right) \longrightarrow H^1_{cont}\left(\mathbb{Q}(\mu_{l^{m+1}}, \zeta), \mu_{l^r}^{\otimes(n+1)}\right)$$

are in general not injective. The kernel of the restriction map is given by

$$H^1_{cont}(\mathbb{Q}(\mu_{l^\infty}, \zeta)/\mathbb{Q}(\zeta), \mathbb{Q}_l(n+1)) \ .$$

Since $[\mathbb{Q}(\mu_l, \zeta) : \mathbb{Q}(\zeta)]$ is prime to l, we have to show that

$$H^1_{cont}(\mathbb{Q}(\mu_{l^\infty}, \zeta)/\mathbb{Q}(\mu_l, \zeta), \mathbb{Z}_l(n+1))$$

is torsion. But the Galois group G of $\mathbb{Q}(\mu_{l^\infty}, \zeta)/\mathbb{Q}(\mu_l, \zeta)$ is isomorphic to $\mathbb{Z}_l$, and hence its first cohomology equals the functor of coinvariants. Our claim follows since $n \geq 0$. $\square$

Proposition B.4.10. *Let ζ be a fixed d-th root of unity. The l-adic regulator*

$$r_l : K_{2n+1}(R) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to H^1_{cont}(\mathbb{Q}(\mu_d), \mathbb{Q}_l(n+1))$$

takes $\varphi_l(\zeta^b)$ to the cyclotomic element in continuous Galois cohomology

$$\left(\sum_{\alpha^{l^r} = \zeta^b} [1-\alpha] \otimes (\alpha^d)^{\otimes n} \right)_r$$

(in the description of the last lemma) defined by Soulé and Deligne (cf. [Sou2], page 384, [D5], 3.1, 3.3).

Proof. If l is odd, then this is [Sou1], Théorèmes 1 and 2. For $l = 2$ the same is true using the properties of the 2-adic regulator (see [We]). □

B.5 Absolute Hodge Cohomology

Let $B = \operatorname{Spec} \mathbb{C}$ or $B = \operatorname{Spec} \mathbb{R}$ in this section.

In A.1.9 a definition of absolute Hodge cohomology and relative cohomology for general varieties over $\mathbb{C}$ was given. The variant over $\mathbb{R}$ was A.2.6.

By A.1.10 resp. A.2.7 absolute Hodge cohomology of smooth varieties is given functorially by Beilinson's complexes $R\Gamma_{\mathfrak{H}^p}(\,\cdot\,/B, n)$.

Lemma B.5.1. *These form a pseudo-flasque complex of presheaves on the Zariski site of smooth B-schemes.*

Proof. By construction [B1] they form a presheaf on pairs $(U, \overline{U})$ where $\overline{U}$ is a compactification with complement an NC-divisor. (For more details cf. [H1] Prop. 8.3.3.) Taking the limit over all choices of $\overline{U}$ we get the desired presheaf. To say it is pseudo-flasque means that absolute Hodge cohomology has the Mayer-Vietoris property. In the context of A.1.9 and A.2.6 it is a formal consequence of the existence of triangles $(i_*i^!, id, j_*j^*)$ for open immersions j with closed complement i. In the context of [B1] it follows from the Mayer-Vietoris property of De Rham-cohomology and singular cohomology. □

We now consider the corresponding generalized cohomology.

Definition B.5.2 (2. Version). *If X is a space over B, then we define absolute Hodge cohomology by*

$$H^i_{\mathfrak{H}^p}(X/B, n) = H^i_{s\mathbf{T}}(X, K(\widetilde{R\Gamma}_{\mathfrak{H}^p}(\,\cdot\,/B, n)) \,.$$

If $f : Z \to X$ is a morphism of spaces, then we define relative cohomology

$$H^i_{\mathfrak{H}^p}(X \text{ rel } Z/B, n) = H^i_{s\mathbf{T}}(\operatorname{Cone}(f), K(\widetilde{R\Gamma}_{\mathfrak{H}^p}(\,\cdot\,/B, n)) \,.$$

Lemma B.5.3. *There is a functorial isomorphism between both definitions of absolute Hodge cohomology for a smooth variety X. If $Y \to X$ is a closed immersion of smooth schemes, then the same is true for relative cohomology.*

Proof. Lemma B.3.4 and Lemma B.3.6.a). □

In order to get the same equalities at least for some singular varieties we have to check a descent property for Hodge modules. For this we need functoriality of i_*i^* with values in complexes of Hodge modules rather than objects in the derived category.

Lemma B.5.4. *Let $X/\mathbb{C}$ be smooth and $i : Y \to X$ a closed reduced subscheme of pure codimension 1. Let $Y = \bigcup_{i=0}^n Y_i$. For $I \subset \{0, \dots, n\}$ and $M \in \mathrm{MHM}_F(X)$ let*

$$\begin{aligned} i_I : Y_I &= \bigcap_{i\in I} Y_i \longrightarrow X \\ j_I : U_I &= X \smallsetminus \bigcup_{i\in I} Y_i \longrightarrow X \\ M_I &= j_{I!}j_I^* M \in \mathrm{MHM}_F(X) \ . \end{aligned}$$

All Y_I are equipped with the reduced structure. Then $i_{I*}i_I^*M$ defines a functor

$$\{\textit{subsets of } \{0, \dots, n\}\} \longrightarrow C^b(\mathrm{MHM}_F(X)) \ .$$

Proof. As j_I is affine both j_I^* and $j_{I!}$ map Hodge modules to such. Note that locally each Y_i is given by a function f_i on X. The functor $i_{I*}i_I^*$ has an explicit description for closed subschemes of the type Y_I given in the proof of [S2] Prop. 2.19. In fact

$$i_{I*}i_I^*M = \ldots \longrightarrow \bigoplus_{I'\subset I; |I'|=2} M_{I'} \longrightarrow \bigoplus_{I'\subset I; |I'|=1} M_{I'} \longrightarrow M$$

where the complex sits in degrees less or equal to zero. □

Proposition B.5.5. *Let $X/\mathbb{C}$ be smooth and $i : Y \to X$ a closed subscheme as in the lemma. Let $\tilde{Y} = Y_0 \amalg \dots \amalg Y_n$ and*

$$\tilde{Y}_\cdot = \mathrm{cosk}_0(\tilde{Y}/Y) \xrightarrow{s} Y \ ,$$

i.e.,

$$\tilde{Y}_k = \tilde{Y} \times_Y \dots \times_Y \tilde{Y} \qquad (k+1 \textit{ factors}) \ .$$

Then the functor s_*s^* defined by the total complex of the cosimplicial complex $(s_{n*}s_n^*)_{n\in\mathbb{N}_0}$ is isomorphic to i_*i^*.

Proof. Note that

$$\tilde{Y}_k = \coprod_{I\in\{0,\dots,n\}^{k+1}} Y_I$$

where $Y_I = Y_{\{i_0,\dots,i_k\}}$ in the notation of the previous lemma. Let M be in $\mathrm{MHM}_F(X)$. By the previous lemma we get indeed a cosimplicial complex hence s_*s^*M is a well-defined complex of Hodge modules. Let $\tilde{Y}_\cdot^{\leq}$ be the simplicial subscheme given by

$$\tilde{Y}_k^{\leq} = \coprod_{I=(i_0\leq i_1\leq\dots\leq i_k)} Y_I \xrightarrow{s_k^{\leq}} Y \ .$$

By the Hodge module version of the combinatorial Lemma B.6.2, the morphism $s_* s^* M \to s_*^{\leq} s^{\leq *} M$ is a quasi-isomorphism. By definition ([S2] 2.19)

$$i_* i^* M = M_{\{0,\ldots,n\}} \to M \ ,$$

and this complex is canonically quasi-isomorphic to the total complex of the constant cosimplicial complex $i_* i^* M$. It is easy to see that the natural morphism

$$\operatorname{Tot} i_* i^* M \longrightarrow s_*^{\leq} s^{\leq *} M$$

is a quasi-isomorphism. □

Corollary B.5.6. *Let X/B be smooth. Suppose $Y \to X$ is an NC-divisor over B all of whose irreducible components are smooth over B. Then the group $H^i_{\mathfrak{H}^p}(Y/B, j)$ as defined in A.1.9 resp. A.2.6 is isomorphic to the generalized cohomology group $H^i_{\mathfrak{H}^p}(\tilde{Y}_\cdot/B, j)$ and to the same noted group in [B1].*

Proof. The condition on Y ensures that $\tilde{Y}_\cdot$ is indeed a smooth simplicial scheme. It gives rise to a space over B. Cohomological descent for the coefficients as in B.5.5 implies cohomological descent for their global sections in the sense of B.3.5. We can use $\tilde{Y}_\cdot$ as the smooth proper hyper-covering needed in Beilinson's definition. Equality to the generalized cohomology version is again B.3.4. □

This is of course cohomological descent for a closed Čech-covering. We have restricted to this case which is built into the very definition of Hodge modules for simplicity. There is no reason why there should not be cohomological descent in the same generality as for constructible sheaves.

Lemma B.5.7. *Let X/B be smooth, and $Z \subset X$ a closed immersion of an NC-divisor all of whose irreducible components are smooth over B. Let $\tilde{Z}_\cdot$ be the smooth simplicial scheme of B.5.5, then there is a canonical isomorphism*

$$H^i_{\mathfrak{H}^p}(X \text{ rel } Z/B, n) = H^i_{\mathfrak{H}^p}(X \text{ rel } \tilde{Z}_\cdot/B, n)$$

where we use the original definition on the left and the second on the right.

Proof. This follows by the general method of B.3.6.b) from the descent property that we have just established. □

Remark: If we had checked cohomological descent in general, then we would get B.5.6 for arbitrary varieties and B.5.7 for arbitrary closed immersions.

Theorem B.5.8. *On the site of smooth schemes over B, the presheaves $R\Gamma_{\mathfrak{H}^p}(\,\cdot\,/B, n)$ have the properties of a twisted duality theory. There are regulator maps from K-cohomology to absolute Hodge cohomology*

$$H^i_{\mathcal{M}}(Y, j) \longrightarrow H^i_{\mathfrak{H}^p}(Y/B, j)$$

for all K–coherent spaces Y. They are compatible with pullback, i.e., if $f : Y \to Y'$ is a map of K–coherent spaces, we get commutative diagrams

$$\begin{array}{ccc} H^i_{\mathcal{M}}(Y', j) & \xrightarrow{f^*} & H^i_{\mathcal{M}}(Y, j) \\ {\scriptstyle c_j}\downarrow & & \downarrow{\scriptstyle c_j} \\ H^i_{\mathfrak{H}^p}(Y'/B, j) & \xrightarrow{f^*} & H^i_{\mathfrak{H}^p}(Y/B, j) \end{array} .$$

If $i : Z \to X$ is a closed immersion of smooth schemes (constant codimension d) with open complement U and $Y_.$ a space constructed form schemes over X as in B.2.18, then the regulator is compatible with pushout, i.e., the diagram

$$\begin{array}{ccc}
H^{n-2d}_{\mathcal{M}}(Y_. \times_X Z, j-d) & \xrightarrow{i_*} & H^n_{\mathcal{M}}(Y_., j) \\
{\scriptstyle c_{j-d}}\downarrow & & \downarrow{\scriptstyle c_j} \\
H^{n-2d}_{\mathfrak{H}^p}(Y_. \times_X Z/B, j-d) & \xrightarrow{i_!} & H^n_{\mathfrak{H}^p}(Y_./B, j)
\end{array}$$

is commutative.

Proof. We use Gillet's method B.3.7. All axioms of a twisted duality theory hold e.g. [H1] Ch. 15. Granted this the proof proceeds as in the l-adic case (B.4.6). □

Remark: Recall ([N], (7.1)) that there is a natural transformation from absolute Hodge to Deligne cohomology. The composition of the above regulator with this transformation was already constructed in [Jeu], 2.5.

Theorem B.5.9 (Borel). *Let K be a number field with r_1 real and r_2 pairs of complex embeddings into $\mathbb{C}$. We consider the ring of integers $\mathfrak{o}_K$ as a scheme over $\mathbb{Z}$. Then the Beilinson regulator*

$$H^i_{\mathcal{M}}(\operatorname{Spec}\mathfrak{o}_K, j) \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow H^i_{\mathfrak{H}^p}((\operatorname{Spec}\mathfrak{o}_K)_{\mathbb{R}}/\mathbb{R}, j)$$

is an isomorphism for all pairs $(i,j) \neq (0,0),(1,1)$. It is injective of codimension r_1+r_2-1 for $(i,j)=(0,0)$, and injective of codimension one in the case $(i,j)=(1,1)$.

Proof. Note that the cohomological dimension of the category of Hodge structures is 1. The case $i=0$ is trivial, and the case $(1,1)$ is Dirichlet's classical result. In [Bo2], the claim (and much more) is proved for the Borel regulator instead of the Beilinson regulator. By [Rp], Corollary 4.2, the two regulators coincide up to a non–vanishing rational factor. □

B.6 A Combinatorial Lemma

This section gives a purely combinatorial proof why two conceivable definitions of the Čech-nerve of a covering are homotopically equivalent. This is well-known at least for open coverings and Čech-cohomology (and probably in general). But for lack of finding an appropriate reference we work out the combinatorics here.

Let $C(n)$ be the following simplicial set:

$$C(n)_k = \{1, \dots, n\}^{k+1}$$

with the obvious face and degeneracy maps. Let $C(n)^{\leq}$ be the simplicial subset of simplices whose entries are ordered by $\leq$. In fact this is the simplicial version of the n-simplex.

Suppose we are given a covariant functor from the category of subsets of $\{1, \dots, n\}$ to the category of sets. We get simplicial sets by setting

$$A(n)_k = \dot{\bigcup}_{I \in C(n)_k} A_I$$
$$A(n)^{\leq}_k = \dot{\bigcup}_{I \in C(n)^{\leq}_k} A_I$$

where A_I is the value of our functor on the set $I = \{i_0, \ldots, i_k\}$. Note that the elements of $C(n)_k$ are ordered tuples but the value of A_I does not depend on the ordering.

Lemma B.6.1. *If the functor has constant value A, then both simplicial sets have the homotopy*

$$\pi_i\left(A(n)^?_\cdot, \star\right) = \begin{cases} A & \textit{if } i = 0, \\ 0 & \textit{else.} \end{cases}$$

Proof. Obviously it is enough to consider the case $A = \star$, i.e., of the simplicial sets $C(n)^{\leq} \to C(n)$ themselves. Both simplicial sets satisfy the extension condition [M] 1.3 rather trivially. Hence we can use the combinatorial computation of the homotopy groups given in [M] Def. 3.6. We immediately get the result. $\square$

Proposition B.6.2. *For a general functor A the injection $A(n)^{\leq} \to A(n)$ of simplicial sets is a weak homotopy equivalence.*

Proof. We filter the simplicial sets $C(n)^?$ by the simplicial subsets $F^iC(n)^?$ of simplices in which at most i different integers occur. This induces a filtration of the simplicial sets $A(n)^?$. Let $G^iA(n)^?$ be the cofibre of the cofibration $F^{i-1}A(n)^? \subset F^iA(n)^?$. It consists of simplices in which precisely i different integers occur. We argue by induction on i for all functors A at the same time. There is a long exact homotopy sequence attached to the cofibration sequence

$$F^{i-1}A(n)^? \longrightarrow F^iA(n)^? \longrightarrow G^iA(n)^? \,.$$

By induction it suffices to show that all cofibres $G^iA(n)/(G^iA(n)^{\leq})$ are weakly equivalent to the final object $\star$. The cofibre decomposes into a union of simplicial sets corresponding to a different choice of i elements in $\{1, \ldots, n\}$ each. If suffices to prove acyclicity for one choice e.g for the subset $\{1, \ldots, i\}$. Hence we only have to consider $G^iA(i)/G^iA(i)^{\leq}$. But this last cofibre is isomorphic to $G^iB(i)/G^iB(i)^{\leq}$ where B is the functor with constant value $A_{\{1,\ldots,i\}}$. For $i > 1$ it is easy to see that $\pi_0 G^iB(i)/G^iB(i)^{\leq} = \star$. By B.6.1 the quotients $B(n)/B(n)^{\leq}$ are acyclic for all n. Using the same cofibration sequence as for A and the inductive hypothesis this implies that all $G^iB(i)/G^iB(i)^{\leq}$ are acyclic. $\square$

Note that A could also be a functor to the category of abelian groups or to the dual of the category of abelian groups.

References

[Ba] H. Bass, "Algebraic K–theory", Benjamin Inc., 1968.

[B1] A.A. Beilinson, "Notes on absolute Hodge cohomology", in "Applications of Algebraic K–theory to Algebraic Geometry and Number Theory", Proceedings of a Summer Research Conference held June 12–18, 1983, in Boulder, Colorado, Contemp. Math., vol. 55, Part I, AMS, Providence, pp. 35–68.

[B2] A.A. Beilinson, "Higher regulators and values of L–functions", Jour. Soviet Math. 30 (1985), pp. 2036–2070.

[B3] A.A. Beilinson, "On the derived category of perverse sheaves", in Yu.I. Manin (ed.), "K–Theory, Arithmetic and Geometry", LNM 1289, Springer–Verlag 1987.

[B4] A.A. Beilinson, "Polylogarithm and Cyclotomic Elements", typewritten preprint, MIT 1989.

[BBD] A.A. Beilinson, J. Bernstein, P. Deligne, "Faisceaux pervers", in B. Teissier, J.L. Verdier, "Analyse et Topologie sur les Espaces singuliers" (I), Astérisque 100, Soc. Math. France 1982.

[BD1] A.A. Beilinson, P. Deligne, "Motivic Polylogarithm and Zagier Conjecture", preprint, 1992.

[BD1p] A.A. Beilinson, P. Deligne, "Polylogarithms and Regulators – the *very* preliminary version", handwritten preprint, 1991.

[BD2] A.A. Beilinson, P. Deligne, "Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs", in U. Jannsen, S.L. Kleiman, J.–P. Serre (eds.), "Motives", Proceedings of the Research Conference on Motives held July 20 – August 2, 1991, in Seattle, Washington, Proc. of Symp. in Pure Math. 55, Part II, AMS 1994, pp. 97–121.

[BL] A.A. Beilinson, A. Levin, "The Elliptic Polylogarithm", in U. Jannsen, S.L. Kleiman, J.–P. Serre, "Motives", Proceedings of the Research Conference on Motives held July 20 – August 2, 1991, in Seattle, Washington, Proc. of Symp. in Pure Math. 55, Part II, AMS 1994, pp. 123–190.

[BLp] A.A. Beilinson, A. Levin, "Elliptic Polylogarithm", typewritten preprint, MIT 1992.

[Bl] S. Bloch, "Application of the dilogarithm function in algebraic K–theory and algebraic geometry", in Proc. Int. Symp. Alg. Geometry, Kyoto, 1977, pp. 103–115.

[BlK] S. Bloch, K. Kato, "L–functions and Tamagawa Numbers of Motives", in P. Cartier et al. (eds.), "The Grothendieck Festschrift", Volume I, Birkhäuser 1990, pp. 333–400.

[Bo1] A. Borel, "Stable real cohomology of arithmetic groups", Ann. Sc. Ec. Norm. Sup. 7 (1974), pp. 235–272.

[Bo2] A. Borel, "Cohomologie de SL_n et valeurs de fonctions zêta", Ann. Scuola Normale Superiore di Pisa 4 (1977), pp. 613–636.

[Bo3] A. Borel et al., "Algebraic $\mathcal{D}$–modules", Perspectives in Mathematics 2, Academic Press 1987.

[BouK] A.K. Bousfield, D.M. Kan, "Homotopy Limits, Completions and Localizations", LNM 304, Springer 1972.

[Br] K.S. Brown,"Abstract homotopy theory and generalized sheaf cohomology", Trans. AMS 186 (1974), pp. 419–458.

[BrG] K.S. Brown, S.M. Gersten, "Algebraic K–theory as generalized sheaf cohomology", in Alg. K–Theory I, LNM 341, pp. 266–292, Springer 1973.

[Brw] W. Browder, "Algebraic K–theory with coefficients $\mathbb{Z}/p$", in M.G. Barratt, M.E. Mahowald (eds.), "Geometric Applications of Homotopy Theory I", Proceedings Evanston 1977, LNM 657, Springer–Verlag 1978, pp. 40–84.

[D1] P. Deligne, "Equations Différentielles à Points Singuliers Réguliers", LNM 163, Springer–Verlag 1970.

[D2] P. Deligne, "Théorie de Hodge, II", Publ. Math. IHES 40 (1971), pp. 5–57.

[D3] P. Deligne, "Théorie de Hodge, III", Publ. Math. IHES 44 (1974), pp. 5–77.

[D4] P. Deligne, "La Conjecture de Weil. II", Publ. Math. IHES 52 (1981), pp. 313–428.

[D5] P. Deligne, "Le Groupe Fondamental de la Droite Projective Moins Trois Points", in Y. Ihara, K. Ribet, J.–P. Serre, "Galois Groups over $\mathbb{Q}$", Proceedings of a Workshop held March 23–27, 1987, at the MSRI, Berkeley, California, Springer 1989, pp. 79–297.

[Den] C. Deninger, "Higher regulators and Hecke L-series of imaginary quadratic fields. II", Ann. of Math. 132 (1990), pp. 131–158.

[DwF] W.G. Dwyer, E.M. Friedlander, "Algebraic étale K-theory", Trans. Am. Math. Soc. 292, no. 1 (1985), pp. 247–280.

[E] H. Esnault, "On the Loday Symbol in the Deligne–Beilinson Cohomology", K–Theory 3 (1989), pp. 1–28.

[EGAII] A. Grothendieck, J. Dieudonné, "Eléments de géométrie algébrique II", Publ. Math. IHES 8 (1961).

[Ek] T. Ekedahl, "On the Adic Formalism", in P. Cartier et al. (e ds.), "The Grothendieck Festschrift", Volume II, Birkhäuser 1990, pp. 197–218.

[GH] J. Gamst, K. Hoechsmann, "Products in sheaf cohomology", Tôhoku Math. J. 22 (1970), pp. 143–162.

[G] H. Gillet, "Riemann–Roch theorems for higher algebraic K–theory", Adv. in Math. 40 (1981), pp. 203–289.

[GSo1] H. Gillet, C. Soulé, "Filtrations on higher algebraic K–theory", unpublished; revised version in preparation.

[GSo2] H. Gillet, C. Soulé, "Descent, Motives and K–Theory", Preprint 1995.

[GoeJr] P.G. Goerss, J.F. Jardine, "Localization theories for simplicial presheaves", preprint, 1997.

[Go] A.B. Goncharov, "Polylogarithms and Motivic Galois Group", in U. Jannsen, S.L. Kleiman, J.–P. Serre, "Motives", Proceedings of the Research Conference on Motives held July 20 – August 2, 1991, in Seattle, Washington, Proc. of Symp. in Pure Math. 55, Part II, AMS 1994, pp. 43–96.

[Gr] M. Gros, "Régulateurs syntomiques et valeurs de fonctions L p-adiques. II", Invent. Math. 115, pp. 61–79 (1994).

[HZ] R.M. Hain, S. Zucker, "Unipotent variations of mixed Hodge structure", Inv. math. 88 (1987), pp. 83–124.

[Ha] R. Hartshorne, "Algebraic Geometry", Graduate Texts in Mathematics 52, Springer Verlag 1977.

[H1] A. Huber, "Mixed Motives and Their Realization in Derived Categories", LNM 1604, Springer 1995.

[H2] A. Huber, "Mixed Perverse Sheaves for Schemes over Number Fields", Comp. Math. 108 (1997), pp. 107–121.

[HW] A. Huber, J. Wildeshaus, "The Classical Polylogarithm", Abstract of a series of lectures given at the workshop on polylogs in Essen, May 1–4, 1997, available under `http://www.math.uiuc.edu/K-theory/229`.

[Jeu] R. de Jeu, "Zagier's Conjecture and Wedge Complexes in Algebraic K–theory", Comp. Math. 96 (1995), pp. 197–247.

[Jn1] U. Jannsen, "Continuous étale cohomology", Math. Ann. 280 (1988), pp. 207-245.

[Jn2] U. Jannsen, "On the l–adic Cohomology of Varieties over Number Fields and its Galois Cohomology", in Y. Ihara, K. Ribet, J.-P. Serre, "Galois Groups over $\mathbb{Q}$", Proceedings of a Workshop held March 23–27, 1987, at the MSRI, Berkeley, California, Springer 1989, pp. 315–360.

[Jn3] U. Jannsen, "Mixed Motives and Algebraic K– Theory", LNM 1400, Springer–Verlag 1990.

[Jr1] J.F. Jardine, "Simplicial objects in a Grothendieck topos", in "Applications of Algebraic K–theory to Algebraic Geometry and Number Theory", Proceedings of a Summer Research Conference held June 12–18, 1983, in Boulder, Colorado, Contemp. Math., vol. 55, Part I, AMS, Providence, pp. 193–239.

[Jr2] J.F. Jardine, "Simplicial presheaves", Journal of Pure and Appl. Algebra 47 (1987), pp. 35–87.

[Jr3] J.F. Jardine, "Stable homotopy theory of simplicial presheaves", Can. J. Math. 39 (1987), pp. 733–747.

[Jr4] J.F. Jardine, "Generalized Etale Cohomology Theories", Progress in Mathematics 146, Birkhäuser–Verlag 1997.

[Kn] D. Kan, "On c. s. s. complexes", Am. Journ. Math. 79 (1957), pp. 449–476.

[KNF] M. Kolster, T. Nguyen Quang Do, V. Fleckinger, "Twisted S-units, p-adic class number formulas, and the Lichtenbaum conjectures", Duke Math. J. 84 (1996), pp. 679–717.

[Kr] C. Kratzer, "λ–strucure en K–théorie algébrique", Comm. Math. Helv. 55 (1980), no 2, pp. 233–254.

[Ks] M. Kashiwara, "A study of variation of mixed Hodge structure", Publ. RIMS, Kyoto Univ. 22 (1986), pp. 991–1024.

[Le] M. Levine, "Lambda-operations, K-theory and motivic cohomology", in V.P. Snaith (ed.), "Algebraic K-theory", papers from the 2nd Great Lakes conference, Canada, March 1996, in memory of R.W. Thomason, Providence, RI, AMS, Fields Inst. Comm. 16 (1997), pp. 131–184.

[Lo] J.-L. Loday, "K–théorie algébrique et représentations de groupes", Ann. scient. Ec. Norm. Sup., 4e série, t. 9 (1976), pp. 309–377.

[M] P. May, "Simplicial objects in algebraic topology", Van Nostrand 1967.

[Mi] J.S. Milne, "Arithmetic Duality theorems", Academic Press 1986.

[N] J. Nekovář, "Beilinson's Conjectures", in U. Jannsen, S.L. Kleiman, J.–P. Serre, "Motives", Proceedings of the Research Conference on Motives held July 20 – August 2, 1991, in Seattle, Washington, Proc. of Symp. in Pure Math. 55, Part I, AMS 1994, pp. 537–570.

[Neu] J. Neukirch, "The Beilinson Conjecture for Algebraic Number Fields", in M. Rapoport, N. Schappacher, P. Schneider (eds.), "Beilinson's Conjectures on Special Values of L–Functions", Perspectives in Mathematics 4, Academic Press 1988, pp. 193–247.

[O] S. Oka, "Multiplications on the Moore Spectrum", Memoirs of the Fac. of Sciences, Kyushu Univ., Ser. A, Vol. 38, No. 2 (1984), pp. 257–276.

[Q1] D. Quillen, "Homotopical Algebra", LNM 43, Springer 1967.

[Q2] D. Quillen, "Higher Algebraic K–Theory", in "Alg. K–Theory I", LNM 341, Springer 1973 (1988), pp. 207–245.

[Rm] D. Ramakrishnan, "Regulators, algebraic cycles, and values of L-functions", in M. R. Stein, R. Keith Dennis (eds.), "Algebraic K-Theory and Algebraic Number Theory", Proc. of a Seminar held at the East-West Center in Honolulu, Hawaii on Jan. 12-16, 1987, Cont. Math. 83 (1988), pp. 183–310.

[Rp] M. Rapoport, "Comparison of the regulators of Beilinson and of Borel", in M. Rapoport, N. Schappacher, P. Schneider (eds.), "Beilinson's Conjectures on Special Values of L–Functions", Perspectives in Mathematics 4, Academic Press 1988, pp. 169–192.

[S1] Morihiko Saito, "Modules de Hodge Polarisables", Publ. RIMS, Kyoto Univ. 24 (1988), pp. 849–995.

[S2] Morihiko Saito, "Mixed Hodge Modules", Publ. RIMS, Kyoto Univ. 26 (1990), pp. 221–333.

[S3] Morihiko Saito, "Hodge Conjecture and Mixed Motives. I", in J.A. Carlson, C.H. Clemens, D.R. Morrison, "Complex Geometry and Lie Theory", Proc. of Symp. in Pure Math. 53, AMS 1991, pp. 283–303.

[Sch] A.J. Scholl, "Height pairings and Special Values of L–functions", in U. Jannsen, S.L. Kleiman, J.–P. Serre, "Motives", Proceedings of the Research Conference on Motives held July 20 – August 2, 1991, in Seattle, Washington, Proc. of Symp. in Pure Math. 55, Part I, AMS 1994, pp. 571–598.

[SGA1] A. Grothendieck et al., "Revètements Etales et Groupe Fondamental", LNM 224, Springer–Verlag 1971.

[SGA4,II] M. Artin, A. Grothendieck, J.L. Verdier et al., "Théorie des Topos et Cohomologie Etale des Schémas", Tôme 2, LNM 270, Springer–Verlag 1973.

[SGA4,III] M. Artin, A. Grothendieck, J.L. Verdier et al., "Théorie des Topos et Cohomologie Etale des Schémas", Tôme 3, LNM 305, Springer–Verlag 1973.

[Sou1] C. Soulé, "K–théorie des anneaux d'entiers de corps de nombres et cohomologie étale", Inv. Math. 55 (1979), pp. 251–295.

[Sou2] C. Soulé, "On higher p–adic regulators", in E. M. Friedlander, M. R. Stein (eds.), "Algebraic K–theory", Proceedings Evanston 1980, LNM 854, Springer–Verlag 1981.

[Sou3] C. Soulé, "The rank of étale cohomology of varieties over p–adic and number fields", Comp. Math. 53 (1984), pp. 113–131.

[Sou4] C. Soulé, "Operations en K–théorie algébrique", Can. Jour. Math. 37 (1985), no 3, pp. 488–550.

[Sou5] C. Soulé, "Eléments cyclotomiques en K–théorie", Astérisque 147–148 (1987), pp. 225–257.

[T] G. Tamme, "The Theorem of Riemann-Roch", in M. Rapoport, N. Schappacher, P. Schneider (eds.), "Beilinson's Conjectures on Special Values of L–Functions", Perspectives in Mathematics 4, Academic Press 1988, pp. 103–168.

[We] C. Weibel, "Etale Chern classes at the prime 2", in P.G. Goerss and J.F. Jardine (eds.), "Algebraic K-Theory and Algebraic Topology", Proc. Lake Louise 1991, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 407, Kluwer Acad. Publ. 1993, pp. 249–286.

[Wi] J. Wildeshaus, "Realizations of Polylogarithms", LNM 1650, Springer-Verlag 1997.

[WiI] J. Wildeshaus, "Mixed structures on fundamental groups", in [Wi], pp. 23–76.

[WiIII] J. Wildeshaus, "Polylogarithmic Extensions on Mixed Shimura varieties. Part I: Construction and basic properties", in [Wi], pp. 141–197.

[WiIV] J. Wildeshaus, "Polylogarithmic Extensions on Mixed Shimura varieties. Part II: The classical polylogarithm", in [Wi], pp. 199–248.

Annette Huber
Math. Institut
Einsteinstr. 62
48149 Münster
Germany
huber@math.uni-muenster.de

Jörg Wildeshaus
Math. Institut
Einsteinstr. 62
48149 Münster
Germany
wildesh@math.uni-muenster.de