Simple Models of Quasihomogeneous Projective 3-Folds

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Abstract. Let X be a projective complex 3-fold, quasihomogeneous with respect to an action of a linear algebraic group. We show that X is a compactification of SL_2/Γ, Γ a finite subgroup, or that X can be equivariantly transformed into $\mathbb{P}_3$, the quadric $\mathbb{Q}_3$, or into certain quasihomogeneous bundles with very simple structure.

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1 Introduction

Call a variety X quasihomogeneous if there is a connected algebraic group G acting algebraically on X with an open orbit. A rational map $X \dashrightarrow Y$ is said to be equivariant if G acts on Y and if the graph is stable under the induced action on $X \times Y$.

The class of varieties having an equivariant birational map to X is generally much smaller then the full birational equivalence class. The minimal rational surfaces are good examples: they are all quasihomogeneous with respect to an action of SL_2, but no two have an SL_2-equivariant birational map between them. On the other hand, if X is any rational SL_2-surface, then the map to a minimal model is always equivariant.

Generally, one may ask for a list of (minimal) varieties such that every quasihomogeneous X has an equivariant birational map to a variety in this list.

We give an answer for $\dim X = 3$ and G linear algebraic:

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THEOREM 1.1. *Let X be a 3-dimensional projective complex variety. Let G be a connected linear algebraic group acting algebraically and almost transitively on X. Assume that the ineffectivity, i.e. the kernel of the map $G \to \mathrm{Aut}(X)$, is finite. Then either $G \cong SL_2$, and X is a compactification of SL_2/Γ, where Γ is finite and not cyclic, or there exists an equivariant birational map $X \dashrightarrow^{eq} Z$, where Z is one of the following:*

- *$\mathbb{P}_3$ or $\mathbb{Q}_3$, the 3-dimensional quadric*
- *a $\mathbb{P}_2$-bundle over $\mathbb{P}_1$ of the form $\mathbb{P}(\mathcal{O}(e) \oplus \mathcal{O}(e) \oplus \mathcal{O})$.*
- *a linear $\mathbb{P}_1$-bundle over a smooth quasihomogeneous surface Y, i.e. $Z \cong \mathbb{P}(E)$, where E is a rank-2 vector bundle over Y. If G is solvable, then E can be chosen to be split.*

If G is not solvable, then the map $X \dashrightarrow^{eq} Z$ factors into a sequence $X \leftarrow \tilde{X} \to Z$, where the arrows denote sequences of equivariant blow ups with smooth center.

A fine classification of the (relatively) minimal varieties involving SL_2 will be given in a forthcoming paper.

The result presented here is contained the author's thesis. The author would like to thank his advisor, Prof. Huckleberry, and Prof. Peternell for support and valuable discussions.

2 EXISTENCE OF EXTREMAL CONTRACTIONS

The main tool we will use is MORI-theory. In order to utilize it, we show that in our context extremal contractions always exist.

LEMMA 2.1. *Let X and G be as in 1.1, but allow for $\mathbb{Q}$-factorial terminal singularities. Then there exists a* MORI*-contraction.*

Proof. Let $\pi : \tilde{X} \to X$ be an equivariant resolution of the singularities of X, let $H < G$ be a (linear) algebraic subgroup and let $v_1 \in Lie(G)$ be the associated element of the LIE-algebra. Since $\tilde{X}$ is quasihomogeneous, we can find elements $v_2, v_3 \in Lie(G)$ such that the associated vector fields

$$\tilde{v_i}(x) = \left.\frac{d}{dt}\right|_{t=0} \exp(tv_i)x \quad \in H^0(\tilde{X}, T\tilde{X})$$

are linearly independent at generic points of $\tilde{X}$. In other words,

$$\sigma := \tilde{v_1} \wedge \tilde{v_2} \wedge \tilde{v_3}$$

is a non-trivial holomorphic section of the anticanonical bundle $-K_{\tilde{X}}$. Because H is linear algebraic, the closure of a generic H-orbit is a rational curve, and H has a fixed point on this curve. Therefore $\tilde{v_1}$ has zeros, and the divisor given as the zero-set of σ is not trivial. In effect, we have shown that $-K_{\tilde{X}}$ is effective and not trivial.

If r is the index of X, then the line bundle $-rK_X$ is effective. We are finished if we exclude the possibility that $-rK_X$ is trivial. Assume that this is the case. The section σ not vanishing on the smooth points of X implies that $X \setminus \mathrm{Sing}(X)$ is G-homogeneous. But the terminal singularities are isolated. Thus, by [HO80, thm. 1 on p. 113], X is a cone over a rational homogeneous surface, a contradiction to $-rK_X$ trivial.

Consequently $-rK_X$ is effective and not trivial. So there is always a curve C intersecting an element of $|-rK_X|$ transversally. Hence $C.K_X < 0$ and there must be an extremal contraction. $\square$

COROLLARY 2.2. *Let X and G be as in theorem 1.1 with the exception that X is allowed to have $\mathbb{Q}$-factorial terminal singularities. Let $\phi : X \to Y$ be an equivariant morphism with* $\dim Y < 3$. *Then there is a relative contraction over Y.*

Proof. If Y is a point, this follows directly from lemma 2.1. Otherwise, if $\eta \in Y$ generic, we know that the fiber X_η is smooth, does not intersect the singular set and is quasihomogeneous with respect to the isotropy group G_η. So there exists a curve $C \subset X_\eta$ with $C.K_{X_\eta} < 0$. Note that the adjunction formula holds, since X has isolated singularities and X_η does not intersect the singular set. Hence $K_{X_\eta} = K_X|_{X_\eta}$, and there must be an extremal ray $C \subset \overline{NE(X)}$ such that $\phi_*(C) = 0$. Thus, there exists a relative contraction. $\square$

Recall that all the steps of the MORI minimal model program (i.e. extremal contractions and flips) can be performed in an equivariant way. For details, see [Keb96, chap. 3].

3 EQUIVARIANT RATIONAL FIBRATIONS

In this section we employ group-theoretical considerations in order to find equivariant rational maps from X to varieties of lower dimension. These will later be used to direct the minimal model program.

We start with the case that G is solvable.

LEMMA 3.1. *Let X and G be as in 1.1. Assume additionally that G is solvable. Then there exists an equivariant rational map $X \dashrightarrow^{eq} Y$ to a projective surface Y.*

Proof. Since G is solvable, there exists a one-dimensional algebraic normal subgroup N. Let H be the isotropy group of a generic point, so that $\Omega \cong G/H$, and consider the map

$$\Omega \cong G/H \to G/(N.H)$$

Recall that $N.H$ is algebraic. Since N is not contained in H (or else G acted with positive dimensional ineffectivity), the map has one-dimensional fibers. Now $\dim G/(N.H) > 0$ and $G/(N.H)$ can always be equivariantly compactified to a projective variety Y. This yields an equivariant rational map $X \dashrightarrow^{eq} Y$. $\square$

Now consider the cases where G is not solvable.

LEMMA 3.2. *Let X and G be as above. Assume that G is neither reductive nor solvable. Then there exists an equivariant rational map $X \dashrightarrow^{eq} Y$ such that either*

1. *$Y \cong \mathbb{P}_3$, and $X \dashrightarrow^{eq} Y$ is birational, or* $\dim Y = 2$, *or*

2. $\dim Y = 1$, *and there exists a normal unipotent group A and a semisimple group $S < G$, acting trivially on Y. The unipotent part A acts almost transitively on generic fibers.*

Proof. Let $G = U \rtimes L$ be the LEVI decomposition of G, i.e. U is unipotent and L reductive and define A to be the center of U. Note that A is non-trivial. Since A is canonically defined, it is normalized by L, hence it is normal in G. Let H be the isotropy group of a generic point, Ω the open G-orbit, so that $\Omega \cong G/H$, and consider the map

$$\Omega \cong G/H \to G/(A.H)$$

There are two things to note. The first is that A is not contained in H (or else G acted with positive dimensional ineffectivity). So $\dim G/(A.H) < 3$. If $\dim G/(A.H) > 0$, it can always be equivariantly compactified $G/(A.H)$ to a variety Y yielding an equivariant rational map $X \dashrightarrow Y$. If $\dim G/(A.H) = 2$, we can stop here. If $\dim G/(A.H) = 1$, then note that A acts transitively on the fiber $A.H/H$. If $A.H$ does not contain a semi-simple group, we argue as in lemma 3.1 to find a subgroup H', $H < H' < A.H$ such that $\dim H'/H = 1$. Then $\dim G/H' = 2$, and again we are finished.

If $\dim G/(A.H) = 0$, then A acts transitively on Ω. In this case $A \cong \mathbb{C}^n$, and hence (because the G-action is algebraic) $\Omega \cong \mathbb{C}^3$. The theorem on MOSTOW fibration (see e.g. [Hei91, p. 641]) yields that L has to have a fixed point in Ω. Therefore, without loss of generality, $L < H$. As a next step, consider the group $B := (U \cap H)^0$. Since both U and H are normalized by L, B is as well. Elements in A commute with all elements of U, hence $A.B$ normalizes B as well. Then B is a normal subgroup of $U \rtimes L = G$. Note that $A.B = U$, because $A.B = A.(H \cap U) = (A.H) \cap U = G \cap U = U$. Consequently B acts trivially. Therefore $B = \{e\}$.

We are now in a position where we may write $G = A \rtimes_\rho L$, where ρ is the action of L on A (L acting by conjugation). Now $H = L$, hence $A \cong \Omega \cong \mathbb{C}^3$ and the L-action on $A \cong (\mathbb{C}^3, +)$ is linear. So G is a subgroup of the affine group and Ω can be equivariantly compactified to $\mathbb{P}_3$, yielding an equivariant rational map $X \dashrightarrow^{eq} \mathbb{P}_3$. □

We study case (1) of the preceding proposition in more detail.

LEMMA 3.3. *Let X be as above and assume that G is reductive. Assume furthermore that G is not semisimple. Then there is an equivariant rational map $X \dashrightarrow^{eq} Z$, where* $\dim Z = 2$.

Proof. As a first step, recall that $G = T.S$, where S is semisimple, T is a torus, and S and T commute and have only finite intersection. If η is a point in the open orbit and G_η the associated isotropy group, then $T \not\subset G_\eta$, or otherwise T would not act at all. For that reason we will be able to find a 1-parameter group $T_1 < T$, $T_1 \not\subset G_\eta$ and consider the map

$$\Omega := G/G_\eta \to G/(T_1.G_\eta).$$

Since T_1 has non-trivial orbits, $\dim G/(T_1.G_\eta) = 2$. If we compactify the latter in an equivariant way to a variety Z, we automatically obtain an equivariant rational map $X \dashrightarrow^{eq} Z$ as claimed. □

LEMMA 3.4. *Suppose G is semisimple. Then one of the following holds:*

1. *$G \cong SL_2$ and the open orbit Ω is isomorphic to SL_2/Γ, where Γ is finite and not contained in a* BOREL *subgroup.*
2. *$X \cong \mathbb{P}_3$*
3. *X is isomorphic to $F_{1,2}(3)$, the full flag variety*
4. *X is homogeneous and either $X \cong \mathbb{Q}_3$, the 3-dimensional quadric or X is a direct product involving only $\mathbb{P}_1$ and $\mathbb{P}_2$.*
5. *X admits an equivariant rational map $X \dashrightarrow^{eq} Y$ onto a surface.*

Proof. If $G \cong SL_2$, and Γ is embeddable into a BOREL group B, then Γ is in fact embeddable into a 1-dimensional torus T. Consider the map $G/\Gamma \to G/T$, and we are finished.

Assume for the rest of this proof that $G \ncong SL_2$. Then the claim is already true in the complex analytic category: see [Win95, p. 3]. One must exclude torus bundles by the fact that they never allow an algebraic action of a *linear* algebraic group. □

We summarize a partial result:

COROLLARY 3.5. *Let X and G be as above. If there exists an equivariant map $X \dashrightarrow^{eq} \mathbb{P}_1$ and no such map to $\mathbb{P}_3$ or to a surface, then G is not solvable and there exist subgroups S and A as in lemma 3.2.*

4 THE CASE THAT Y IS A CURVE

In this section we investigate relatively minimal models over $\mathbb{P}_1$. The main proposition is:

PROPOSITION 4.1. *Let X and G be as in 1.1 with the exception that X is allowed to have $\mathbb{Q}$-factorial terminal singularities. Assume that $\phi : X \to \mathbb{P}_1$ is an extremal contraction. Assume additionally that there does not exist an equivariant rational map $X \dashrightarrow^{eq} Y$, where $\dim Y = 2$ or $Y \cong \mathbb{P}_3$. Then*

$$X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}_1}(e) \oplus \mathcal{O}_{\mathbb{P}_1}(e) \oplus \mathcal{O}_{\mathbb{P}_1}),$$

with $e > 0$. In particular, X is smooth.

Proof. As a first step, we show that the generic fiber X_η is isomorphic to $\mathbb{P}_2$. As ϕ is a MORI-contraction, X_η is a smooth FANO surface. By corollary 3.5, the stabilizer $G_\eta < G$ of X_η contains a unipotent group A acting almost transitively on X_η and a semisimple part S. This already rules out all FANO surfaces other than $\mathbb{P}_2$. Furthermore, $S \cong SL_2$. Note that G_η stabilizes a unique line $L \subset X_\eta$ and that S acts transitively on L.

Set $D' := \overline{G.L}$ and remark that D' intersects the generic ϕ-fiber in the unique G_η-stable line: $D' \cap X_\eta = L$. We claim that D' is CARTIER. The desingularization $\tilde{D}'$ has a map to $\mathbb{P}_1$, the generic fiber is isomorphic to $\mathbb{P}_1$ and S acts non-trivially on all the fibers. Thus, $\tilde{D}'$ is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$, and S does not have a fixed point on D'. Consequently, $\tilde{D}'$ does not intersect the singular set of X and is CARTIER.

Take D'' to be an ample divisor on Y. As ϕ is a MORI-contraction, the line bundle L associated to $D := D' + n\phi^*(D'')$, $n >> 0$, is ample on X. In this setting, a theorem of FUJITA (cf. [BS95, Prop. 3.2.1]) yields that X is of the form $\mathbb{P}(E)$, where E is a vector bundle on $\mathbb{P}_1$.

The transition functions of E must commute with S, but the only matrices commuting with SL_2 are $Diag(\lambda, \lambda, \mu)$, hence $E = \mathcal{O}(e) \oplus \mathcal{O}(e) \oplus \mathcal{O}(f)$ and $X \cong \mathbb{P}(\mathcal{O}(e-f) \oplus \mathcal{O}(e-f) \oplus \mathcal{O})$. □

For future use, we note

LEMMA 4.2. *Let X and G be as in proposition 4.1. Then, by equivariantly blowing up and down, $X \dashrightarrow^{eq} \mathbb{P}(\mathcal{O}(e') \oplus \mathcal{O}(e') \oplus \mathcal{O})$ where the latter does not contain a G-fixed point.*

Proof. The semisimple group S fixes a unique point of each ϕ-fiber, so that there exists a curve C of S-fixed points. Suppose that G has a fixed point f. Then $f \in C$, and we can perform an elementary transformation $X \dashrightarrow^{eq} X'$ with center f, i.e. if X_μ is the ϕ-fiber containing f, then we blow up f and blow down the strict transform of the X_μ, again obtaining a linear $\mathbb{P}_2$-bundle of type $\mathbb{P}(\mathcal{O}(e) \oplus \mathcal{O}(e) \oplus \mathcal{O})$. This transformation exists, as has been shown in [Mar73]. Since all the centers of the blow-up and -down are G-stable, the transformation is equivariant.

We will use this transformation in order to remove G-fixed points. Let $g \in G$ be an element not stabilizing C. The curves gC and C meet in f. We know that after finitely many blow-ups of the intersection points of C and gC, the curves become disjoint, so that there no longer exists a G-fixed point! This, however, is exactly what we do when applying the elementary transformation. □

5 THE CASE THAT Y IS A SURFACE

The cases that G is solvable or not solvable are in many respects quite different. Here we have to treat them separately.

5.1 THE CASE G SOLVABLE

We will show that in this situation the open G-orbit can be compactified in a particularly simple way.

PROPOSITION 5.1. *Let X and G be as in theorem 1.1. Assume additionally that G is solvable and $\phi : X \to Y$ is an equivariant map with connected fibers onto a smooth surface. Then there exists a splitting rank-2 vector bundle E on Y and an equivariant birational map $X \dashrightarrow^{eq} \mathbb{P}(E)$.*

We remark that if $y \in Y$ is contained in the open G-orbit, then it's preimage is quasihomogeneous with respect to the isotropy group G_y, hence isomorphic to $\mathbb{P}_1$. As a first step in the proof of proposition 5.1, we show the existence of very special divisors in X.

Notation 5.2. We call a divisor $D \subset X$ a "rational section" if it intersects the generic ϕ-fiber with multiplicity one.

In our context, such divisors always exist:

LEMMA 5.3. *Let $\phi : X \to Y$ be as in lemma 5.1 and assume additionally that there exists a group $H^* \cong \mathbb{C}^*$ acting trivially on Y. Let D'_X be the fixed point set of the H^*-action. Then D'_X contains two rational sections as irreducible components.*

Proof. Let D_X be the union of those irreducible divisors in D'_X which are not preimages of curves or points by ϕ. The subvariety D_X intersects every generic ϕ-fiber at least once. Hence $D_X \neq 0$.

We claim that the set of branch points

$$M := \{y \in Y : \#(\phi^{-1}(y) \cap D_X) = 1\}$$

is discrete. Linearization of the H^*-action yields that for any point $f \in D_X \setminus \mathrm{Sing}(X)$, there is a unique H^*-stable curve intersecting D_X at f. Furthermore, the intersection is transversal. Assume $\dim M \geq 1$ and let y be a generic point in M. Then $\dim \phi^{-1}(y) = 1$ and $\phi^{-1}(y) = 1$ contains a smooth curve C as an irreducible component intersecting D_X. Now $C.D_X = 1$ and, because $C \cap D_X$ was the only intersection point by assumption, $\phi^{-1}(y).D_X = 1$. This is contrary to D_X intersecting the generic ϕ-fiber twice.

Set

$$N := \{\mu \in Y \,|\, \dim(X_\mu \cap D_X) > 0\} \cup M \cup \phi(\mathrm{Sing}(X)).$$

By definition N is finite and D_X is a 2-sheeted cover over $Y \setminus N$. Now Y is smooth and quasihomogeneous with respect to an algebraic action of the linear algebraic group G. Hence it is rational. This implies that $Y \setminus N$ is simply connected. Hence D_X has two connected components over $Y \setminus N$. Now the set $D_X \cap \phi^{-1}(N)$ is just a curve. Therefore D_X cannot be irreducible. □

LEMMA 5.4. *Under the assumptions of lemma 5.1, there exists a G-stable rational section $E_1 \subset X$.*

Proof. If G is a torus, then there exists a subgroup T_1 acting trivially on Y. In this case we are finished by applying lemma 5.3. Thus we may assume that the unipotent part U of G is non-trivial. Let $\eta \in Y$ be a generic point and $x \in X_\eta \setminus \Omega$, where Ω denotes the open G-orbit in X. If x is unique, then the divisor $E_1 := \overline{G.x}$ has the required properties. Similarly, if U acts almost transitively on Y, then it's isotropy at η is connected and we may set $E_1 := \overline{U.x}$.

If neither holds, then necessarily $\dim U = 1$, and we can assume that U acts non-trivially on Y. Otherwise $X_\eta \setminus \Omega$ consists of a single point and we are finished as above. Let T_1 be a 1-dimensional subgroup of a maximal torus such that $I := U.T_1$ acts almost transitively on Y. If $\eta \in Y$ is generic, the isotropy group I_η is cyclic: I_η has two fixed points in X_η. Consequently, there exist at least two I-orbits whose closures D_i are rational sections.

Note that I is normal in G, i.e. all elements of G map I-orbits to I-orbits. If D_i are the only rational sections occurring as closures of I-orbits, they are automatically G-stable. Otherwise, all I-orbits are mapped injectively to Y, and at least one of these is G-stable. □

The existence of E_1 already yields a map to a $\mathbb{P}_1$-bundle.

LEMMA 5.5. *Under the assumptions of lemma 5.1, there exists a rank-2 vector bundle E on Y (not necessarily split) and an equivariant birational map $X \dashrightarrow^{eq} \mathbb{P}(E)$.*

Proof. Set $E := (\phi_*(\mathcal{O}_X(E)))^{**}$. Since a reflexive sheaf on a smooth surface is locally free, E is a vector bundle. If $\Omega_Y \subset Y$ is the open orbit, $\phi^{-1}(\Omega_Y) \cong \mathbb{P}(E|_{\Omega_Y})$ (cf. [BS95, Prop. 3.2.1]), inducing a birational map $\psi : X \dashrightarrow \mathbb{P}(E)$. Note that $\phi_*(\mathcal{O}_X(E))$ is torsion free. In particular, $\phi_*(\mathcal{O}_X(E))$ is locally free over a G-stable cofinite set $Y_0 \subset Y$ so that, by the universal property of *Proj*, ψ is regular over Y_0. As $\psi|_{Y_0}$ is proper, it is equivariant. The automorphisms over Y_0 extend to the whole of $\mathbb{P}(E)$ by the RIEMANN extension theorem. Hence ψ is equivariant as claimed. □

In order to show that E can be chosen to be split we need to find another rational section. We will frequently deal with the following situation, for which we fix some notation.

Notation 5.6. Let $\phi : X \to Y$ be as above and assume that there exists a map $\pi : Y \to Z \cong \mathbb{P}_1$, e.g. if Y is isomorphic to a (blown-up) HIRZEBRUCH surface Σ_n. Then, if $F \in Z$ is a generic point, set $F_Y := \pi^{-1}(F)$ and $F_X := \phi^{-1}(F_Y)$.

LEMMA 5.7. *In the setting of proposition 5.1, there exists a second rational section E_2. If E_1 is as constructed in lemma 5.4, then $E_1 \cap E_2$ is G-stable.*

Proof. If G is a torus, we are finished, as we have seen in the proof of lemma 5.4. Hence we may assume that $\dim U > 0$, where U is the unipotent part of G.

Suppose that U acts trivially on Y. Then we are able to choose a 2-dimensional torus $T < G$ such that T acts almost transitively on Y. If $\eta \in Y$ is generic, then the isotropy group T_η may not be cyclic, but since it has to fix the unique U-fixed point in X_η, its image $T_\eta \to \mathrm{Aut}(X_\eta)$ is contained in a BOREL group, hence cyclic. Consequently, T_η fixes another point x, and we may set $E_2 := \overline{T.x}$.

The other case is that U acts non-trivially on Y. We need to consider a mapping $\pi : Y \to Z \cong \mathbb{P}_1$. If $Y \cong \Sigma_n$, or a blow-up, there is no problem. If $Y \cong \mathbb{P}_2$, we note that, by G being solvable and BOREL's fixed point theorem (see [HO80, p. 32]), there exists a G-fixed point $y \in Y$. We can always blow up y and X_y in order to obtain a new $\mathbb{P}_1$-bundle over Σ_1. If we are able to construct our rational sections here, then we can simply take their images to be the desired rational sections in the variety we started with. So let us assume that $Y \ncong \mathbb{P}_2$.

There exists a 1-dimensional normal unipotent subgroup $U_1 < G$. Assume first that U_1 acts non-trivially on Z. Using notation 5.6, F_Y is isomorphic to $\mathbb{P}_1$, F_X to a HIRZEBRUCH surface Σ_n. Choose a section $\sigma \subset F_X$ with the property that $\phi(\sigma \cap E_1)$ does not meet the open G-orbit in Y. As the stabilizer of F_X in G stabilizes E_1, so that $E_1 \cap F_X$ is either the infinity- or zero-section in $F_X \cong \Sigma_n$ or the diagonal in $F_X \cong \Sigma_0$, and G stabilizes a section of $Y \to \mathbb{P}_1$, this can always be accomplished. Set $E_1 := \overline{U_1.\sigma}$.

Secondly, we must consider the case that U_1 acts trivially on Z. We proceed similarly to the above. Choose a 1-dimensional group $G_1 < G$ such that the G_1-orbit in Z coincides with that of G. Now G_1 stabilizes at least one section $\sigma_Y \subset Y$ over Z which is not U_1-stable! Set $\sigma_X := \phi^{-1}(\sigma_Y)$ and consider a section $\sigma \subset \sigma_X$ over σ_Y such that $\phi(\sigma \cap E_1)$ is disjoint from the open G-orbit in Y. Then $E_1 := \overline{U_1.\sigma}$ is the divisor we were looking for.

We still have to show that the intersection $E_1 \cap E_2$ is G-stable. Note that by construction, $\phi(E_1 \cap E_2)$ does not meet the open G-orbit in Y. This, together with E_1 being G-stable, yields the claim. □

We shall use the second rational section in order to transform E into a splitting bundle.

5.1.1 ELIMINATING VERTICAL CURVES

If $S \subset \phi(E_1 \cap E_2)$ is an irreducible curve which is a ϕ-fiber, then we say that E_1 and E_2 intersect vertically in S. We know that after blowing up S we obtain a $\mathbb{P}_1$-bundle over the blow-up of Y. Furthermore, the process is equivariant. The proper transforms of E_1 and E_2 are again rational sections. If they still intersect vertically, the blow-up procedure can be applied again. So we eventually obtain a sequence of blow-ups. The strict transforms of the E_1 and E_2 are again rational sections in X_i. We denote them by E_1^i or E_2^i, respectively. By the theorem on embedded resolution, we have:

LEMMA 5.8. *The sequence described above terminates, i.e. there exists a number $i \in \mathbb{N}$ such that the strict transforms E_1^i and E_2^i do not intersect vertically.*

5.1.2 ELIMINATING HORIZONTAL CURVES

We may now assume that E_1 and E_2 do not intersect vertically. Let $S \subset \phi(E_1 \cap E_2)$ be an irreducible curve. Then S gives rise to an elementary transformation as ensured by [Mar73]. Again, the transformation is equivariant and the strict transforms of E_1 and E_2 are rational sections. If they still intersect over S, we transform as before. Again one may use the embedded resolution to show (cf. [Keb96, thm. 5.30] for details):

LEMMA 5.9. *The sequence described above terminates after finitely many transformations, i.e. there exists a $j \in \mathbb{N}$ such that for all curves $C \subset E_1^{(j)} \cap E_2^{(j)}$ it follows that $\phi^{(j)}(C) \neq S$. Furthermore, if E_1 and E_2 do not intersect vertically, then $E_1^{(i)}$ and $E_2^{(i)}$ do not intersect vertically for all i.*

5.1.3 THE CONSTRUCTION OF INDEPENDENT SECTIONS

By lemma 5.8 the variety X can be transformed into a $\mathbb{P}_1$-bundle such that the strict transforms of E_1 and E_2 do not intersect in fibers. A second transformation will rid us of curves in $E_1 \cap E_2$ which are not contained in fibers. Since the latter transformation does not create new curves in the intersection, the strict transforms of E_1 and E_2 eventually become disjoint. The resulting space is the compactification of a line bundle.

LEMMA 5.10. *If E_1 and E_2 do not intersect, X is the compactification of a line bundle.*

Proof. Since E_1 and E_2 are disjoint, neither contains a fiber. Thus they are sections. □

As a net result, we have shown proposition 5.1.

5.2 The case G not solvable

As first step, we show that X is again a linear $\mathbb{P}_1$-bundle. We do this under an additional hypothesis which will not impose problems in the course of the proof of theorem 1.1.

Lemma 5.11. *Let X and G be as in theorem 1.1, with the exception that X is allowed to have $\mathbb{Q}$-factorial terminal singularities. Let $\phi : X \to Y$ be a* Mori*-contraction to a surface and assume additionally that G is not solvable and that there exists an equivariant morphism $\psi : Y \to Y'$, where Y' is a smooth surface. Then X and Y are smooth and X is a linear $\mathbb{P}_1$-bundle over Y.*

Proof. First, we show that all ϕ-fibers are of dimension 1. If there exists a fiber X_μ which is not 1-dimensional, then $\dim X_\mu = 2$. Take a curve $C \subset Y$ so that $\mu \in C$. Set $D := \overline{\phi^{-1}(C \setminus \mu)}$. The divisor D intersects an irreducible component of X_μ. Now take a curve $R \subset X_\mu$ intersecting D in finitely many points. We have $R.D > 0$. However, all generic q-fibers X_η are homologous to R (up to positive multiples). So $X_\eta.D > 0$, contradicting the definition of D.

Secondly, we claim that X is smooth. Assume to the contrary and let $x \in X$ be a singular point, $\mu := \phi(x)$. Recall that terminal singularities in 3-dimensional varieties are isolated. Thus, if S is the semisimple part of G, then the fiber X_μ through x is pointwise S-fixed. Linearizing the S-action at a generic point $y \in X_\mu$, the complete reducibility of the S-representation yields an S-quasihomogeneous divisor D which intersects X_μ transversally at y and is Cartier in a neighborhood of y. The induced map $D \to Y'$ must be unbranched: Y' contains an S-fixed point and is therefore isomorphic to $\mathbb{P}_2$; but there is no equivariant cover of this other than the identity. So D is a rational section which is Cartier over a neighborhood of μ. If $H \in Pic(Y)$ is sufficiently ample, then $D + \phi^*(H)$ is ample, and [BS95, Prop. 3.2.1] applies, contradicting the assumption that X is singular.

Since X is smooth, the same theorem shows that in order to prove the lemma it is sufficient to show that there exists a rational section. If all the simple factors of S have orbits of dimension ≤ 2, then, after replacing the factors by their Borel groups, we obtain a solvable group G', acting almost transitively as well. In this case lemma 5.4 applies.

If $S' < S$ is a simple factor acting with 3-dimensional orbit on X, its action on Y is almost transitively. In particular, there exists a 2-dimensional group $B < S$, isomorphic to a Borel group in SL_2, which also acts almost transitively on Y. As in the proof of lemma 5.4, B has cyclic isotropy at a generic point of Y and so there exist two rational sections which are compactifications of B-orbits. □

6 Proof of theorem 1.1

Prior to proving theorem 1.1, we still need to describe equivariant maps to $\mathbb{P}_3$ in more detail:

Lemma 6.1. *Let $X \dashrightarrow^{eq} \mathbb{P}_3$ be an equivariant birational map. Then either X has an equivariant rational fibration with 2-dimensional base variety or X and $\mathbb{P}_3$ are equivariantly linked by a sequence of blowing ups of X followed by a sequence of blow-downs.*

Proof. If the G-action on $\mathbb{P}_3$ has a fixed point, we can blow up this point and obtain a map from the blown-up $\mathbb{P}_3$ to $\mathbb{P}_2$. If there is no such G-fixed point in $\mathbb{P}_3$, then after replacing X by an equivariant blow-up, there is a regular equivariant map $\phi : X \to \mathbb{P}_3$. Recall that such a map factors through an extremal contraction. Since the base does not contain a fixed point, the classification of extremal contractions of smooth varieties yields the claim. □

Now we compiled all the results needed to finish the

Proof of theorem 1.1. Given X, we apply lemmata 3.1–3.4. Unless $X \cong Q_3$, $F_{1,2}(3)$ or a compactification of SL_2/Γ, Γ not cyclic, there exists an equivariant map $X \dashrightarrow^{eq} Y$, where Y is smooth and $Y \cong \mathbb{P}_3$, $\dim(Y) = 2$ or, if no other case applies, $\dim(Y) = 1$.

If $Y \cong \mathbb{P}_3$, then, by lemma 6.1, we may replace $\mathbb{P}_3$ by a surface, or else we are finished.

In the case of a map to Y with $\dim Y < 3$, we can blow up X equivariantly to obtains a morphism $\tilde{X} \to Y$. Recalling that all steps in the minimal model program (i.e. contractions and flips) are equivariant, we may perform a relative minimal model program over Y. In this situation corollary 2.2 shows that the program does not stop unless we encounter a contraction of fiber type $X' \to Y'$ and $\dim Y' < 3$. Note that $\dim Y' \geq \dim Y$.

In case that Y' is a surface, X' is the projectivization of a line bundle or can be equivariantly transformed into one (cf. lemma 5.5 and 5.11). If G is solvable, proposition 5.1 allows us to transform X into the projectivization of a splitting bundle over a surface.

If $\dim Y' = 1$ and there does not exist a map to one of the other cases, $X \cong \mathbb{P}(\mathcal{O}(e) \oplus \mathcal{O}(e) \oplus \mathcal{O})$ over $\mathbb{P}_1$, as was shown in proposition 4.1.

We still have to show that if G is not solvable, the map to one of the models in our list factors into equivariant monoidal transformations. Recall that it suffices to show that, after equivariantly blowing up, if necessary, the minimal models do not have a G-fixed point. We do a case-by-case checking:

$\mathbb{P}_2$-BUNDLES OVER $\mathbb{P}_1$: By lemma 4.2, these can be chosen not to contain a fixed point.

$\mathbb{P}_1$-BUNDLES OVER A SURFACE Y: If the semisimple part S of G acts trivially on Y, we can stop. Otherwise, if the S-action on Y has a fixed point f, we blow up f and the fiber over f and obtain a $\mathbb{P}_1$-bundle over Σ_1. Recall that actions of semisimple groups on Σ_n never have fixed points.

$\mathbb{P}_3$: This case has already been handled in lemma 6.1.

SL_2/Γ: After desingularizing and blowing up all fixed points, if any, the compactification of SL_2/Γ is fixed point free. Otherwise, linearization at a fixed point yields a contradiction to S acting almost transitively.

OTHER CASES: The remaining cases occur only when X is homogeneous (cf. lemma 3.4).

□

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