Higher Index Theorems and the Boundary Map in Cyclic Cohomology

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Abstract. We show that the Chern–Connes character induces a natural transformation from the six term exact sequence in (lower) algebraic K–Theory to the periodic cyclic homology exact sequence obtained by Cuntz and Quillen, and we argue that this amounts to a general "higher index theorem." In order to compute the boundary map of the periodic cyclic cohomology exact sequence, we show that it satisfies properties similar to the properties satisfied by the boundary map of the singular cohomology long exact sequence. As an application, we obtain a new proof of the Connes–Moscovici index theorem for coverings.

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Contents

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Introduction

Index theory and K-Theory have been close subjects since their appearance [1, 4]. Several recent index theorems that have found applications to Novikov's Conjecture use algebraic K-Theory in an essential way, as a natural target for the generalized indices that they compute. Some of these generalized indices are "von Neumann dimensions"–like in the L^2–index theorem for coverings [3] that, roughly speaking, computes the trace of the projection on the space of solutions of an elliptic differential operator on a covering space. The von Neumann dimension of the index does not fully recover the information contained in the abstract (i.e., algebraic K-Theory index) but this situation is remedied by considering "higher traces," as in the Connes–Moscovici Index Theorem for coverings [11]. (Since the appearance of this theorem, index theorems that compute the pairing between higher traces and the K–Theory class of the index are called "higher index theorems.")

In [30], a general higher index morphism (i.e., a bivariant character) was defined for a class of algebras–or, more precisely, for a class of extensions of algebras–that is large enough to accommodate most applications. However, the index theorem proved there was obtained only under some fairly restrictive conditions, too restrictive for most applications. In this paper we completely remove these restrictions using a recent breakthrough result of Cuntz and Quillen.

In [16], Cuntz and Quillen have shown that periodic cyclic homology, denoted HP_*, satisfies excision, and hence that any two–sided ideal I of a complex algebra A gives rise to a periodic six-term exact sequence

(1)
$$\begin{array}{ccccc} \mathrm{HP}_0(I) & \longrightarrow & \mathrm{HP}_0(A) & \longrightarrow & \mathrm{HP}_0(A/I) \\ \partial\uparrow & & & & \downarrow\partial \\ \mathrm{HP}_1(A/I) & \longleftarrow & \mathrm{HP}_1(A) & \longleftarrow & \mathrm{HP}_1(I) \end{array}$$

similar to the topological K–Theory exact sequence [1]. Their result generalizes earlier results from [38]. (See also [14, 15].)

If M is a smooth manifold and $A = C^\infty(M)$, then $\mathrm{HP}_*(A)$ is isomorphic to the de Rham cohomology of M, and the Chern–Connes character on (algebraic) K–Theory generalizes the Chern–Weil construction of characteristic classes using connection and curvature [10]. In view of this result, the excision property, equation (1), gives more evidence that periodic cyclic homology is the "right" extension of de Rham homology from smooth manifolds to algebras. Indeed, if $I \subset A$ is the ideal of functions vanishing on a closed submanifold $N \subset M$, then

$$\mathrm{HP}_*(I) = \mathrm{H}^*_{DR}(M, N)$$

and the exact sequence for continuous periodic cyclic homology coincides with the exact sequence for de Rham cohomology. This result extends to (not necessarily smooth) complex affine algebraic varieties [22].

The central result of this paper, Theorem 1.6, Section 1, states that the Chern–Connes character

$$ch : \mathrm{K}_i^{\mathrm{alg}}(A) \to \mathrm{HP}_i(A),$$

where $i = 0, 1$, is a natural transformation from the six term exact sequence in (lower) algebraic K–Theory to the periodic cyclic homology exact sequence. In this formulation, Theorem 1.6 generalizes the corresponding result for the Chern character on the K–Theory of compact topological spaces, thus extending the list of common features of de Rham and cyclic cohomology.

The new ingredient in Theorem 1.6, besides the naturality of the Chern–Connes character, is the compatibility between the connecting (or index) morphism in algebraic K–Theory and the boundary map in the Cuntz–Quillen exact sequence (Theorem 1.5). Because the connecting morphism

$$\mathrm{Ind} : \mathrm{K}_1^{\mathrm{alg}}(A/I) \to \mathrm{K}_0^{\mathrm{alg}}(I)$$

associated to a two-sided ideal $I \subset A$ generalizes the index of Fredholm operators, Theorem 1.5 can be regarded as an abstract "higher index theorem," and the computation of the boundary map in the periodic cyclic cohomology exact sequence can be regarded as a "cohomological index formula."

We now describe the contents of the paper in more detail.

If τ is a trace on the two–sided ideal $I \subset A$, then τ induces a morphism

$$\tau_* : \mathrm{K}_0^{\mathrm{alg}}(I) \to \mathbb{C}.$$

More generally, one can–and has to–allow τ to be a "higher trace," while still getting a morphism $\tau_* : \mathrm{K}_1^{\mathrm{alg}}(I) \to \mathbb{C}$. Our main goal in Section 1 is to identify, as explicitly as possible, the composition $\tau_* \circ \mathrm{Ind} : \mathrm{K}_1^{\mathrm{alg}}(A/I) \to \mathbb{C}$. For traces this is done in Lemma 1.1, which generalizes a formula of Fedosov. In general,

$$\tau_* \circ \mathrm{Ind} = (\partial\tau)_*,$$

where $\partial : \mathrm{HP}^0(I) \to \mathrm{HP}^1(A/I)$ is the boundary map in periodic cyclic cohomology. Since ∂ is defined purely algebraically, it is usually easier to compute it than it is to compute Ind, not to mention that the group $\mathrm{K}_0^{\mathrm{alg}}(I)$ is not known in many interesting situations, which complicates the computation of Ind even further.

In Section 2 we study the properties of ∂ and show that ∂ is compatible with various product type operations on cyclic cohomology. The proofs use cyclic vector spaces [9] and the external product $\times$ studied in [30], which generalizes the cross-product in singular homology. The most important property of ∂ is with respect to the tensor product of an exact sequence of algebras by another algebra (Theorem 2.6). We also show that the boundary map ∂ coincides with the morphism induced by the odd bivariant character constructed in [30], whenever the later is defined (Theorem 2.10).

As an application, in Section 3 we give a new proof of the Connes–Moscovici index theorem for coverings [11]. The original proof uses estimates with heat kernels. Our proof uses the results of the first two sections to reduce the Connes–Moscovici index theorem to the Atiyah–Singer index theorem for elliptic operators on compact manifolds.

The main results of this paper were announced in [32], and a preliminary version of this paper has been circulated as "Penn State preprint" no. PM 171, March 1994. Although this is a completely revised version of that preprint, the proofs have not been changed in any essential way. However, a few related preprints and papers have appeared since this paper was first written; they include [12, 13, 33].

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1. Index theorems and Algebraic K–Theory

We begin this section by reviewing the definitions of the groups $\mathrm{K}_0^{\mathrm{alg}}$ and $\mathrm{K}_1^{\mathrm{alg}}$ and of the index morphism $\mathrm{Ind} : \mathrm{K}_1^{\mathrm{alg}}(A/I) \to \mathrm{K}_0^{\mathrm{alg}}(I)$ associated to a two-sided ideal $I \subset A$. There are easy formulas that relate these groups to Hochschild homology, and we review those as well. Then we prove an intermediate result that generalizes a formula of Fedosov in our Hochschild homology setting, which will serve both as a lemma in the proof of Theorem 1.5, and as a motivation for some of the formalisms developed in this paper. The main result of this section is the compatibility between the connecting (or index) morphism in algebraic K–Theory and the boundary morphism in cyclic cohomology (Theorem 1.5). An equivalent form of Theorem 1.5 states that the Chern–Connes character is a natural transformation from the six term exact sequence in algebraic K–Theory to periodic cyclic homology. These results extend the results in [30] in view of Theorem 2.10.

All algebras considered in this paper are complex algebras.

1.1. Pairings with traces and a Fedosov type formula. It will be convenient to define the group $\mathrm{K}_0^{\mathrm{alg}}(A)$ in terms of idempotents $e \in M_\infty(A)$, that is, in terms of matrices e satisfying $e^2 = e$. Two idempotents, e and f, are called *equivalent* (in writing, $e \sim f$) if there exist x, y such that $e = xy$ and $f = yx$. The direct sum of two idempotents, e and f, is the matrix $e \oplus f$ (with e in the upper–left corner and f in the lower–right corner). With the direct–sum operation, the set of equivalence classes of idempotents in $M_\infty(A)$ becomes a monoid denoted $\mathcal{P}(A)$. The group $\mathrm{K}_0^{\mathrm{alg}}(A)$ is defined to be the Grothendieck group associated to the monoid $\mathcal{P}(A)$. If $e \in M_\infty(A)$ is an idempotent, then the class of e in the group $\mathrm{K}_0^{\mathrm{alg}}(A)$ will be denoted $[e]$.

Let $\tau : A \to \mathbb{C}$ be a trace. We extend τ to a trace $M_\infty(A) \to \mathbb{C}$, still denoted τ, by the formula $\tau([a_{ij}]) = \sum_i \tau(a_{ii})$. If $e \sim f$, then $e = xy$ and $f = yx$ for some x and y, and then the tracial property of τ implies that $\tau(e) = \tau(f)$. Moreover $\tau(e \oplus f) = \tau(e) + \tau(f)$, and hence τ defines an additive map $\mathcal{P}(A) \to \mathbb{C}$. From the universal property of the Grothendieck group associated to a monoid, it follows that we obtain a well defined group morphism (or pairing with τ)

$$\mathrm{K}_0^{\mathrm{alg}}(A) \ni [e] \longrightarrow \tau_*([e]) = \tau(e) \in \mathbb{C}. \tag{2}$$

The pairing (2) generalizes to not necessarily unital algebras I and traces $\tau : I \to \mathbb{C}$ as follows. First, we extend τ to $I^+ = I + \mathbb{C}1$, the algebra with adjoint unit, to be zero on 1. Then, we obtain, as above, a morphism $\tau_* : \mathrm{K}_0^{\mathrm{alg}}(I^+) \to \mathbb{C}$. The morphism $\tau_* : \mathrm{K}_0^{\mathrm{alg}}(I) \to \mathbb{C}$ is obtained by restricting from $\mathrm{K}_0^{\mathrm{alg}}(I^+)$ to $\mathrm{K}_0^{\mathrm{alg}}(I)$, defined to be the kernel of $\mathrm{K}_0^{\mathrm{alg}}(I^+) \to \mathrm{K}_0^{\mathrm{alg}}(\mathbb{C})$.

The definition of $\mathrm{K}_1^{\mathrm{alg}}(A)$ is shorter:

$$\mathrm{K}_1^{\mathrm{alg}}(A) = \lim_{\to} \; GL_n(A)/[GL_n(A), GL_n(A)].$$

In words, $\mathrm{K}_1^{\mathrm{alg}}(A)$ is the abelianization of the group of invertible matrices of the form $1 + a$, where $a \in M_\infty(A)$. The pairing with traces is replaced by a pairing with Hochschild 1–cocycles as follows.

If $\phi : A \otimes A$ is a Hochschild 1-cocycle, then the the functional ϕ defines a morphism $\phi_* : \mathrm{K}_1^{\mathrm{alg}}(A) \to \mathbb{C}$, by first extending ϕ to matrices over A, and then by pairing it with the Hochschild 1–cycle $u \otimes u^{-1}$. Explicitly, if $u = [a_{ij}]$, with inverse $u^{-1} = [b_{ij}]$, then the morphism ϕ_* is

$$\mathrm{K}_1^{\mathrm{alg}}(A) \ni [u] \longrightarrow \phi_*([u]) = \sum_{i,j} \phi(a_{ij}, b_{ji}) \in \mathbb{C}. \tag{3}$$

The morphism ϕ_* depends only on the class of ϕ in the Hochschild homology group $\mathrm{HH}_1(A)$ of A.

If $0 \to I \to A \to A/I \to 0$ is an exact sequence of algebras, that is, if I is a two–sided ideal of A, then there exists an exact sequence [26],

$$\mathrm{K}_1^{\mathrm{alg}}(I) \to \mathrm{K}_1^{\mathrm{alg}}(A) \to \mathrm{K}_1^{\mathrm{alg}}(A/I) \xrightarrow{\mathrm{Ind}} \mathrm{K}_0^{\mathrm{alg}}(I) \to \mathrm{K}_0^{\mathrm{alg}}(A) \to \mathrm{K}_0^{\mathrm{alg}}(A/I),$$

of Abelian groups, called the *algebraic K–theory exact sequence*. The connecting (or *index*) morphism

$$\mathrm{Ind} : K_1^{alg}(A/I) \to K_0^{alg}(I)$$

will play an important role in this paper and is defined as follows. Let u be an invertible element in some matrix algebra of A/I. By replacing A/I with $M_n(A/I)$, for some large n, we may assume that $u \in A/I$. Choose an invertible element $v \in M_2(A)$ that projects to $u \oplus u^{-1}$ in $M_2(A/I)$, and let $e_0 = 1 \oplus 0$ and $e_1 = v e_0 v^{-1}$. Because $e_1 \in M_2(I^+)$, the idempotent e_1 defines a class in $\mathrm{K}_0^{\mathrm{alg}}(I^+)$. Since $e_1 - e_0 \in M_2(I)$, the difference $[e_1] - [e_0]$ is actually in $\mathrm{K}_0^{\mathrm{alg}}(I)$ and depends only on the class $[u]$ of u in $\mathrm{K}_1^{\mathrm{alg}}(A/I)$. Finally, we define

$$\mathrm{Ind}([u]) = [e_1] - [e_0]. \tag{4}$$

To obtain an explicit formula for e_1, choose liftings $a, b \in A$ of u and u^{-1} and let v, the lifting, to be the matrix

$$v = \begin{bmatrix} 2a - aba & ab - 1 \\ 1 - ba & b \end{bmatrix},$$

as in [26], page 22. Then a short computation gives

$$e_1 = \begin{bmatrix} 2ab - (ab)^2 & a(2-ba)(1-ba) \\ (1-ba)b & (1-ba)^2 \end{bmatrix}. \tag{5}$$

Continuing the study of the exact sequence $0 \to I \to A \to A/I \to 0$, choose an arbitrary linear lifting, $l : A/I^2 \to A$. If τ is a trace on I, we let

$$\phi_\tau(a, b) = \tau([l(a), l(b)] - l([a, b])). \tag{6}$$

Because $[a, xy] = [ax, y] + [ya, x]$, we have $\tau([A, I^2]) = 0$, and hence ϕ_τ is a Hochschild 1–cocycle on A/I^2 (i.e., $\phi_\tau(ab, c) - \phi_\tau(a, bc) + \phi_\tau(ca, b))$. The class of ϕ_τ in $\mathrm{HH}^1(A/I^2)$, denoted $\partial\tau$, turns out to be independent of the lifting l. If A is a locally convex algebra, then we assume that we can choose the lifting l to be continuous. If $\tau([A, I]) = 0$, then it is enough to consider a lifting of $A \to A/I$.

The morphisms $(\partial\tau)_* : \mathrm{K}_1^{\mathrm{alg}}(A/I^2) \to \mathbb{C}$ and $\tau_* : \mathrm{K}_0^{\mathrm{alg}}(I^2) \to \mathbb{C}$ are related through the following lemma.

LEMMA. 1.1. *Let τ be a trace on a two-sided ideal $I \subset A$. If*

$$\mathrm{Ind} : \mathrm{K}_1^{\mathrm{alg}}(A/I^2) \to \mathrm{K}_0^{\mathrm{alg}}(I^2)$$

is the connecting morphism of the algebraic K–Theory exact sequence associated to the two-sided ideal I^2 of A, then

$$\tau_* \circ \mathrm{Ind} = (\partial\tau)_*.$$

If $\tau([A, I]) = 0$, then we may replace I^2 by I.

Proof. We check that $\tau_* \circ \mathrm{Ind}([u]) = (\partial\tau)_*([u])$, for each invertible $u \in M_n(A/I^2)$. By replacing A/I^2 with $M_n(A/I^2)$, we may assume that $n = 1$.

Let $l : A/I^2 \to A$ be the linear lifting used to define the 1–cocycle ϕ_τ representing $\partial\tau$, equation (6), and choose $a = l(u)$ and $b = l(u^{-1})$ in the formula for e_1, equation (5). Then, the left hand side of our formula becomes

$$\tau_*\big(\mathrm{Ind}([u])\big) = \tau\big((1-ba)^2\big) - \tau\big((1-ab)^2\big) = 2\tau([a,b]) - \tau([a,bab]). \tag{7}$$

Because $(1 - ba)b$ is in I^2, we have $\tau([a, bab]) = \tau([a, b])$, and hence

$$\tau_*(\mathrm{Ind}([u])) = \tau_*([e_1] - [e_0]) = \tau(e_1 - e_0) = \tau([a, b]).$$

Since the right hand side of our formula is

$$(\partial\tau)_*([u]) = (\partial\tau)(u, u^{-1}) = \tau([l(u), l(u^{-1})] - l([u, u^{-1}])) = \tau([a, b]),$$

the proof is complete. □

Lemma 1.1 generalizes a formula of Fedosov in the following situation. Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on a fixed separable Hilbert space $\mathcal{H}$ and $\mathcal{C}_p(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ be the (non-closed) ideal of p–summable operators [36] on $\mathcal{H}$:

$$\mathcal{C}_p(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}),\ Tr(A^*A)^{p/2} < \infty\}. \tag{8}$$

(We will sometimes omit $\mathcal{H}$ and write simply $\mathcal{C}_p$ instead of $\mathcal{C}_p(\mathcal{H})$.) Suppose now that the algebra A consists of bounded operators, that $I \subset \mathcal{C}_1$, and that a is an element of A whose projection u in A/I is invertible. Then a is a *Fredholm* operator, and, for a suitable choice of a lifting b of u^{-1}, the operators $1 - ba$ and $1 - ab$ become the orthogonal projection onto the kernel of a and, respectively, the kernel of a^*. Finally, if $\tau = Tr$, this shows that

$$Tr_*\big(\mathrm{Ind}([u])\big) = \dim\ker(a) - \dim\ker(a^*)$$

and hence that $Tr_* \circ \mathrm{Ind}$ recovers the Fredholm index of a. (The *Fredholm index* of a, denoted $ind(a)$, is by definition the right-hand side of the above formula.) By equation (7), we see that we also recover a form of Fedosov's formula:

$$ind(a) = Tr\big((1-ba)^k\big) - Tr\big((1-ab)^k\big)$$

if b is an inverse of a modulo $\mathcal{C}_p(\mathcal{H})$ and $k \geq p$.

The connecting (or boundary) morphism in the algebraic K–Theory exact sequence is usually denoted by '∂'. However, in the present paper, this notation becomes unsuitable because the notation '∂' is reserved for the boundary morphism in the periodic cyclic cohomology exact sequence. Besides, the notation 'Ind' is supposed to suggest the name 'index morphism' for the connecting morphism in the algebraic K–Theory exact sequence, a name justified by the relation that exists between Ind and the indices of Fredholm operators, as explained above.

1.2. "Higher traces" and excision in cyclic cohomology. The example of $A = C^\infty(M)$, for M a compact smooth manifold, shows that, in general, few morphisms $\mathrm{K}_0^{\mathrm{alg}}(A) \to \mathbb{C}$ are given by pairings with traces. This situation is corrected by considering 'higher-traces,' [10].

Let A be a unital algebra and

$$(9)\qquad \begin{aligned} b'(a_0 \otimes \ldots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \\ b(a_0 \otimes \ldots \otimes a_n) &= b'(a_0 \otimes \ldots \otimes a_n) + (-1)^n a_n a_0 \otimes \ldots \otimes a_{n-1}, \end{aligned}$$

for $a_i \in A$. The *Hochschild homology* groups of A, denoted $\mathrm{HH}_*(A)$, are the homology groups of the complex $(A \otimes (A/\mathbb{C}1)^{\otimes n}, b)$. The *cyclic homology* groups [10, 24, 37] of a unital algebra A, denoted $\mathrm{HC}_n(A)$, are the homology groups of the complex $(\mathcal{C}(A), b + B)$, where

$$(10)\qquad \mathcal{C}_n(A) = \bigoplus_{k \geq 0} A \otimes (A/\mathbb{C}1)^{\otimes n-2k}.$$

b is the Hochschild homology boundary map, equation (9), and B is defined by

$$(11)\qquad B(a_0 \otimes \ldots \otimes a_n) = s \sum_{k=0}^{n} t^k (a_0 \otimes \ldots \otimes a_n).$$

Here we have used the notation of [10], that $s(a_0 \otimes \ldots \otimes a_n) = 1 \otimes a_0 \otimes \ldots \otimes a_n$ and $t(a_0 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}$.

More generally, Hochschild and cyclic homology groups can be defined for "mixed complexes," [21]. A *mixed complex* $(\mathcal{X}, b, B)$ is a graded vector space $(\mathcal{X}_n)_{n \geq 0}$, endowed with two differentials b and B, $b : \mathcal{X}_n \to \mathcal{X}_{n-1}$ and $B : \mathcal{X}_n \to \mathcal{X}_{n+1}$, satisfying the compatibility relation $b^2 = B^2 = bB + Bb = 0$. The *cyclic complex*, denoted $\mathcal{C}(\mathcal{X})$, associated to a mixed complex $(\mathcal{X}, b, B)$ is the complex

$$\mathcal{C}_n(\mathcal{X}) = \mathcal{X}_n \oplus \mathcal{X}_{n-2} \oplus \mathcal{X}_{n-4} \ldots = \bigoplus_{k \geq 0} \mathcal{X}_{n-2k},$$

with differential $b + B$. The cyclic homology groups of the mixed complex $\mathcal{X}$ are the homology groups of the cyclic complex of $\mathcal{X}$:

$$\mathrm{HC}_n(\mathcal{X}) = \mathrm{H}_n(\mathcal{C}(\mathcal{X}), b + B).$$

Cyclic cohomology is defined to be the homology of the complex

$$(\mathcal{C}(\mathcal{X})' = \mathrm{Hom}(\mathcal{C}(\mathcal{X}), \mathbb{C}), (b + B)'),$$

dual to $\mathcal{C}(\mathcal{X})$. From the form of the cyclic complex it is clear that there exists a morphism $S : \mathcal{C}_n(\mathcal{X}) \to \mathcal{C}_{n-2}(\mathcal{X})$. We let

$$\mathcal{C}_n(\mathcal{X}) = \lim_{\leftarrow} \mathcal{C}_{n+2k}(\mathcal{X})$$

as $k \to \infty$, the inverse system being with respect to the periodicity operator $\mathbb{S}$. Then the *periodic cyclic homology* of $\mathcal{X}$ (respectively, the *periodic cyclic cohomology* of $\mathcal{X}$), denoted $\mathrm{HP}_*(\mathcal{X})$ (respectively, $\mathrm{HP}^*(\mathcal{X})$) is the homology (respectively, the cohomology) of $\mathcal{C}_n(\mathcal{X})$ (respectively, of the complex $\lim\limits_{\rightarrow} \mathcal{C}_{n+2k}(\mathcal{X})'$).

If A is a unital algebra, we denote by $\mathcal{X}(A)$ the mixed complex obtained by letting $\mathcal{X}_n(A) = A \otimes (A/\mathbb{C}1)^{\otimes n}$ with differentials b and B given by (9) and (11). The various homologies of $\mathcal{X}(A)$ will not include $\mathcal{X}$ as part of notation. For example, the periodic cyclic homology of $\mathcal{X}$ is denoted $\mathrm{HP}_*(A)$.

For a topological algebra A we may also consider *continuous* versions of the above homologies by replacing the ordinary tensor product with the projective tensor product. We shall be especially interested in the continuous cyclic cohomology of A, denoted $\mathrm{HP}^*_{\mathrm{cont}}(A)$. An important example is $A = C^\infty(M)$, for a compact smooth manifold M. Then the Hochschild-Kostant-Rosenberg map

$$\chi : A^{\hat{\otimes} n+1} \ni a_0 \otimes a_1 \otimes \ldots \otimes a_n \longrightarrow (n!)^{-1} a_0 da_1 \ldots da_n \in \Omega^n(M) \tag{12}$$

to smooth forms gives an isomorphism

$$\mathrm{HP}_i^{\mathrm{cont}}(C^\infty(M)) \simeq \bigoplus_k \mathrm{H}_{DR}^{i+2k}(M)$$

of *continuous* periodic cyclic homology with the de Rham cohomology of M [10, 24] made $\mathbb{Z}_2$–periodic. The normalization factor $(n!)^{-1}$ is convenient because it transforms B into the de Rham differential d_{DR}. It is also the right normalization as far as Chern characters are involved, and it is also compatible with products, Theorem 3.5. From now on, we shall use the de Rham's Theorem

$$\mathrm{H}^i_{DR}(M) \simeq \mathrm{H}^i(M)$$

to identify de Rham cohomology and singular cohomology *with complex coefficients* of the compact manifold M.

Sometimes we will use a version of continuous periodic cyclic cohomology for algebras A that have a locally convex space structure, but for which the multiplication is only partially continuous. In that case, however, the tensor products $A^{\otimes n+1}$ come with natural topologies, for which the differentials b and B are continuous. This is the case for some of the groupoid algebras considered in the last section. The periodic cyclic cohomology is then defined using continuous multi-linear cochains.

One of the original descriptions of cyclic cohomology was in terms of "higher traces" [10]. A *higher trace*–or cyclic cocycle–is a continuous multilinear map $\phi : A^{\otimes n+1} \to \mathbb{C}$ satisfying $\phi \circ b = 0$ and $\phi(a_1, \ldots, a_n, a_0) = (-1)^n \phi(a_0, \ldots, a_n)$. Thus cyclic cocycles are, in particular, Hochschild cocycles. The last property, the cyclic invariance, justifies the name "cyclic cocycles." The other name, "higher traces" is justified since cyclic cocycles on A define traces on the universal differential graded algebra of A.

If $I \subset A$ is a two–sided ideal, we denote by $\mathcal{C}(A, I)$ the kernel of $\mathcal{C}(A) \to \mathcal{C}(A/I)$. For possibly non-unital algebras I, we define the cyclic homology of I using the complex $\mathcal{C}(I^+, I)$. The cyclic cohomology and the periodic versions of these groups are defined analogously, using $\mathcal{C}(I^+, I)$. For topological algebras we replace the algebraic tensor product by the projective tensor product.

An equivalent form of the excision theorem in periodic cyclic cohomology is the following result.

THEOREM. 1.2 (Cuntz–Quillen). *The inclusion $\mathcal{C}(I^+, I) \hookrightarrow \mathcal{C}(A, I)$ induces an isomorphism, $\mathrm{HP}^*(A, I) \simeq \mathrm{HP}^*(I)$, of periodic cyclic cohomology groups.*

This theorem is implicit in [16], and follows directly from the proof there of the Excision Theorem by a sequence of commutative diagrams, using the Five Lemma each time.[2]

This alternative definition of excision sometimes leads to explicit formulae for ∂. We begin by observing that the short exact sequence of complexes $0 \to \mathcal{C}(A,I) \to \mathcal{C}(A) \to \mathcal{C}(A/I) \to 0$ defines a long exact sequence

$$.. \leftarrow \mathrm{HP}^n(A,I) \leftarrow \mathrm{HP}^n(A) \leftarrow \mathrm{HP}^n(A/I) \xleftarrow{\partial} \mathrm{HP}^{n-1}(A,I) \leftarrow \mathrm{HP}^{n-1}(A) \leftarrow ..$$

in cyclic cohomology that maps naturally to the long exact sequence in periodic cyclic cohomology.

Most important for us, the boundary map $\partial : \mathrm{HP}^n(A,I) \to \mathrm{HP}^{n+1}(A/I)$ is determined by a standard algebraic construction. We now want to prove that this boundary morphism recovers a previous construction, equation (6), in the particular case $n = 0$. As we have already observed, a trace $\tau : I \to \mathbb{C}$ satisfies $\tau([A, I^2]) = 0$, and hence defines by restriction an element of $\mathrm{HC}^0(A, I^2)$. The traces are the cycles of the group $\mathrm{HC}^0(I)$, and thus we obtain a linear map $\mathrm{HC}^0(I) \to \mathrm{HC}^0(A, I^2)$. From the definition of $\partial : \mathrm{HP}^0(A,I) \to \mathrm{HP}^1(A/I)$, it follows that $\partial[\tau]$ is the class of the cocycle $\phi(a,b) = \tau([l(a), l(b)] - l([a,b]))$, which is cyclically invariant, by construction. (Since our previous notation for the class of ϕ was $\partial\tau$, we have thus obtained the paradoxical relation $\partial[\tau] = \partial\tau$; we hope this will not cause any confusions.)

Below we shall also use the natural map (transformation)

$$\mathrm{HC}^n \to \mathrm{HP}^n = \lim_{k \to \infty} \mathrm{HC}^{n+2k}.$$

LEMMA. 1.3. *The diagram*

$$\begin{tikzcd}
\mathrm{HC}^0(I) \arrow[r] \arrow[d] & \mathrm{HC}^0(A,I^2) \arrow[r, "\partial"] \arrow[d] & \mathrm{HC}^1(A/I^2) \arrow[d] & \mathrm{HC}^1(A/I) \arrow[l] \arrow[d] \\
\mathrm{HP}^0(I) \arrow[r, "\sim"] & \mathrm{HP}^0(A,I^2) \arrow[r, "\partial"] & \mathrm{HP}^1(A/I^2) & \mathrm{HP}^1(A/I) \arrow[l, "\sim"']
\end{tikzcd}$$

commutes. Consequently, if $\tau \in \mathrm{HC}^0(I)$ is a trace on I and $[\tau] \in \mathrm{HP}^0(I)$ is its class in periodic cyclic homology, then $\partial[\tau] = [\partial\tau] \in \mathrm{HP}^1(A/I)$, where $\partial\tau \in \mathrm{HC}^1(A/I^2)$ is given by the class of the cocycle ϕ defined in equation (6) *(see also above).*

Proof. The commutativity of the diagram follows from definitions. If we start with a trace $\tau \in \mathrm{HC}^0(I)$ and follow counterclockwise through the diagram from the upper–left corner to the lower–right corner we obtain $\partial[\tau]$; if we follow clockwise, we obtain the description for $\partial[\tau]$ indicated in the statement. □

1.3. AN ABSTRACT "HIGHER INDEX THEOREM". We now generalize Lemma 1.1 to periodic cyclic cohomology. Recall that the pairings (2) and (3) have been generalized to pairings

$$\mathrm{K}_i^{\mathrm{alg}}(A) \otimes \mathrm{HC}^{2n+i}(A) \longrightarrow \mathbb{C}, \quad i = 0, 1.$$

[10]. Thus, if ϕ be a higher trace representing a class $[\phi] \in \mathrm{HC}^{2n+i}(A)$, then, using the above pairing, ϕ defines morphisms $\phi_* : \mathrm{K}_i^{\mathrm{alg}}(A) \to \mathbb{C}$, where $i = 0, 1$. The explicit formulae for these morphisms are $\phi_*([e]) = (-1)^n \frac{(2n)!}{n!} \phi(e, e, \ldots, e)$, if $i = 0$ and e

[2] I am indebted to Joachim Cuntz for pointing out this fact to me.

is an idempotent, and $\phi_*([u]) = (-1)^n n!\phi_*(u, u^{-1}, u, \dots, u^{-1})$, if $i = 1$ and u is an invertible element. The constants in these pairings are meaningful and are chosen so that these pairings are compatible with the periodicity operator.

Consider the standard orthonormal basis $(e_n)_{n\geq 0}$ of the space $l^2(\mathbb{N})$ of square summable sequences of complex numbers; the *shift* operator S is defined by $Se_n = e_{n+1}$. The adjoint S^* of S then acts by $S^*e_0 = 0$ and $S^*e_{n+1} = e_n$, for $n \geq 0$. The operators S and S^* are related by $S^*S = 1$ and $SS^* = 1-p$, where p is the orthogonal projection onto the vector space generated by e_0.

Let $\mathcal{T}$ be the algebra generated by S and S^* and $\mathbb{C}[w, w^{-1}]$ be the algebra of Laurent series in the variable w, $\mathbb{C}[w, w^{-1}] = \{\sum_{n=-N}^{N} a_k w^k,\ a_k \in \mathbb{C}\} \simeq \mathbb{C}[\mathbb{Z}]$. Then there exists an exact sequence

$$0 \to M_\infty(\mathbb{C}) \to \mathcal{T} \to \mathbb{C}[w, w^{-1}] \to 0,$$

called *the Toeplitz extension*, which sends S to w and S^* to w^{-1}.

Let $\mathbb{C}\langle\, a, b\,\rangle$ be the free *non-commutative* unital algebra generated by the symbols a and b and $J = \ker(\mathbb{C}\langle\, a, b\,\rangle \to \mathbb{C}[w, w^{-1}])$, the kernel of the unital morphism that sends $a \to w$ and $b \to w^{-1}$. Then there exists a morphism $\psi_0 : \mathbb{C}\langle\, a, b\,\rangle \to \mathcal{T}$, uniquely determined by $\psi_0(a) = S$ and $\psi_0(b) = S^*$, which defines, by restriction, a morphism $\psi : J \to M_\infty(\mathbb{C})$, and hence a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d, "\psi"'] & \mathbb{C}\langle\, a, b\,\rangle \arrow[r] \arrow[d, "\psi_0"] & \mathbb{C}[w, w^{-1}] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_\infty(\mathbb{C}) \arrow[r] & \mathcal{T} \arrow[r] & \mathbb{C}[w, w^{-1}] \arrow[r] & 0
\end{tikzcd}$$

LEMMA. 1.4. *Using the above notations, we have that* $\mathrm{HC}^*(J)$ *is singly generated by the trace* $\tau = Tr \circ \psi$.

Proof. We know that $\mathrm{HP}^i(\mathbb{C}[w, w^{-1}]) \simeq \mathbb{C}$, see [24]. Then Lemma 1.1, Lemma 1.3, and the exact sequence in periodic cyclic cohomology prove the vanishing of the *reduced* periodic cyclic cohomology groups:

$$\widetilde{\mathrm{HC}}^*(\mathcal{T}) = \ker(\mathrm{HP}^*(\mathcal{T}) \to \mathrm{HP}^*(\mathbb{C})).$$

The algebra $\mathbb{C}\langle\, a, b\,\rangle$ is the tensor algebra of the vector space $\mathbb{C}a \oplus \mathbb{C}b$, and hence the groups $\widetilde{\mathrm{HC}}^*(T(V))$ also vanish [24]. It follows that the morphism ψ_0 induces (trivially) an isomorphism in cyclic cohomology. The comparison morphism between the Cuntz–Quillen exact sequences associated to the two extensions shows, using "the Five Lemma," that the induced morphisms $\psi^* : \mathrm{HP}^*(M_\infty(\mathbb{C})) \to \mathrm{HP}^*(J)$ is also an isomorphism. This proves the result since the canonical trace Tr generates $\mathrm{HP}^*(M_\infty(\mathbb{C}))$. $\square$

We are now ready to state the main result of this section, the compatibility of the boundary map in the periodic cyclic cohomology exact sequence with the index (i.e., connecting) map in the algebraic K–Theory exact sequence. The following theorem generalizes Theorem 5.4 from [30].

THEOREM. 1.5. *Let* $0 \to I \to A \to A/I \to 0$ *be an exact sequence of complex algebras, and let* $\mathrm{Ind} : \mathrm{K}_1^{\mathrm{alg}}(A/I) \to \mathrm{K}_0^{\mathrm{alg}}(I)$ *and* $\partial : \mathrm{HP}^0(I) \to \mathrm{HP}^1(A/I)$ *be the* *connecting morphisms in algebraic K–Theory and, respectively, in periodic cyclic cohomology. Then, for any $\varphi \in \mathrm{HP}^0(I)$ and $[u] \in \mathrm{K}_1^{\mathrm{alg}}(A/I)$, we have*

$$\varphi_*(\mathrm{Ind}[u]) = (\partial\varphi)_*[u]\,. \tag{13}$$

Proof. We begin by observing that if the class of φ can be represented by a trace (that is, if φ is the equivalence class of a trace in the group $\mathrm{HP}^0(I)$) then the boundary map in periodic cyclic cohomology is computed using the recipe we have indicated, Lemma 1.3, and hence the result follows from Lemma 1.1. In particular, the theorem is true for the exact sequence

$$0 \longrightarrow\ J\ \to \mathbb{C}\langle\, a, b\,\rangle \to \mathbb{C}[w, w^{-1}\,] \longrightarrow 0,$$

because all classes in $\mathrm{HP}^0(J)$ are defined by traces, as shown in Lemma 1.4. We will now show that this particular case is enough to prove the general case "by universality."

Let u be an invertible element in $M_n(A/I)$. After replacing the algebras involved by matrix algebras, if necessary, we may assume that $n = 1$, and hence that u is an invertible element in A/I. This invertible element then gives rise to a morphism $\eta : \mathbb{C}[w, w^{-1}] \to A/I$ that sends w to u. A choice of liftings $a_0, b_0 \in A$ of u and u^{-1} defines a morphism $\psi_0 : \mathbb{C}\langle\, a, b\,\rangle \to A$, uniquely determined by $\psi_0(a) = a_0$ and $\psi_0(b) = b_0$, which restricts to a morphism $\psi : J \to I$. In this way we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d, "\psi"'] & \mathbb{C}\langle\, a, b\,\rangle \arrow[r] \arrow[d, "\psi_0"] & \mathbb{C}[w, w^{-1}\,] \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & I \arrow[r] & A \arrow[r] & A/I \arrow[r] & 0
\end{tikzcd}$$

of algebras and morphisms.

We claim that the naturality of the index morphism in algebraic K–Theory and the naturality of the boundary map in periodic cyclic cohomology, when applied to the above exact sequence, prove the theorem. Indeed, we have

$$\psi_* \circ \mathrm{Ind} = \mathrm{Ind} \circ \eta_* : \mathrm{K}_1^{\mathrm{alg}}(\mathbb{C}[w, w^{-1}]) \to \mathrm{K}_0^{\mathrm{alg}}(I), \quad \text{and}$$

$$\partial \circ \psi^* = \eta^* \circ \partial : \mathrm{HP}^*(I) \to \mathrm{HP}^{*+1}(\mathbb{C}[w, w^{-1}]).$$

As observed in the beginning of the proof, the theorem is true for the cocycle $\psi^*(\varphi)$ on J, and hence $(\psi^*(\varphi))_*(\mathrm{Ind}\,[w]) = (\partial \circ \psi^*(\varphi))_*[w]$. Finally, from definition, we have that $\eta_*[w] = [u]$. Combining these relations we obtain

$$\begin{aligned}
\varphi_*(\mathrm{Ind}\,[u]) &= \varphi_*(\mathrm{Ind} \circ \eta_*[w]) = \varphi_*(\psi_* \circ \mathrm{Ind}[w]) = (\psi^*(\varphi))_*(\mathrm{Ind}\,[w]) = \\
&= (\partial \circ \psi^*(\varphi))_*[w] = (\eta^* \circ \partial(\varphi))_*[w] = (\partial\varphi)_*(\eta_*[w]) = (\partial\varphi)_*[u].
\end{aligned}$$

The proof is complete. □

The theorem we have just proved can be extended to topological algebras and topological K–Theory. If the topological algebras considered satisfy Bott periodicity, then an analogous compatibility with the other connecting morphism can be proved and one gets a natural transformation from the six-term exact sequence in topological K–Theory to the six-term exact sequence in periodic cyclic homology. However, a factor of $2\pi\imath$ has to be taken into account because the Chern-Connes character is not directly compatible with periodicity [30], but introduces a factor of $2\pi\imath$. See [12] for details.

So far all our results have been formulated in terms of cyclic cohomology, rather than cyclic homology. This is justified by the application in Section 3 that will use this form of the results. This is not possible, however, for the following theorem, which states that the Chern character in periodic cyclic homology (i.e., the Chern–Connes character) is a natural transformation from the six term exact sequence in (lower) algebraic K–Theory to the exact sequence in cyclic homology.

THEOREM. 1.6. *The diagram*

$$\begin{tikzcd}
\mathrm{K}_1^{\mathrm{alg}}(I) \arrow[r] \arrow[d] & \mathrm{K}_1^{\mathrm{alg}}(A) \arrow[r] \arrow[d] & \mathrm{K}_1^{\mathrm{alg}}(A/I) \arrow[r, "\mathrm{Ind}"] \arrow[d] & \mathrm{K}_0^{\mathrm{alg}}(I) \arrow[r] \arrow[d] & \mathrm{K}_0^{\mathrm{alg}}(A) \arrow[r] \arrow[d] & \mathrm{K}_0^{\mathrm{alg}}(A/I) \arrow[d] \\
\mathrm{HP}_1(I) \arrow[r] & \mathrm{HP}_1(A) \arrow[r] & \mathrm{HP}_1(A/I) \arrow[r, "\partial"] & \mathrm{HP}_0(I) \arrow[r] & \mathrm{HP}_0(A) \arrow[r] & \mathrm{HP}_0(A/I),
\end{tikzcd}$$

in which the vertical arrows are induced by the Chern characters $ch : \mathrm{K}_i^{\mathrm{alg}} \to \mathrm{HP}_i$, for $i = 0, 1$, commutes.

Proof. Only the relation $ch \circ \mathrm{Ind} = \partial \circ ch$ needs to be proved, and this is dual to Theorem 1.5. ☐

2. PRODUCTS AND THE BOUNDARY MAP IN PERIODIC CYCLIC COHOMOLOGY

Cyclic vector spaces are a generalization of simplicial vector spaces, with which they share many features, most notably, for us, a similar behavior with respect to products.

2.1. CYCLIC VECTOR SPACES. We begin this section with a review of a few needed facts about the cyclic category Λ from [9] and [30]. We will be especially interested in the $\times$–product in bivariant cyclic cohomology. More results can be found in [23].

DEFINITION. 2.1. *The cyclic category, denoted Λ, is the category whose objects are $\Lambda_n = \{0, 1, \dots, n\}$, where $n = 0, 1, \dots$ and whose morphisms $\mathrm{Hom}_\Lambda(\Lambda_n, \Lambda_m)$ are the homotopy classes of increasing, degree one, continuous functions $\varphi : S^1 \to S^1$ satisfying $\varphi(\mathbb{Z}_{n+1}) \subseteq \mathbb{Z}_{m+1}$.*

A *cyclic vector space* is a contravariant functor from Λ to the category of complex vector spaces [9]. Explicitly, a cyclic vector space X is a graded vector space, $X = (X_n)_{n\geq 0}$, with structural morphisms $d_n^i : X_n \to X_{n-1}$, $s_n^i : X_n \to X_{n+1}$, for $0 \leq i \leq n$, and $t_{n+1} : X_n \to X_n$ such that (X_n, d_n^i, s_n^i) is a simplicial vector space ([25], Chapter VIII,§5) and t_{n+1} defines an action of the cyclic group $\mathbb{Z}_{n+1}$ satisfying $d_n^0 t_{n+1} = d_n^n$ and $s_n^0 t_{n+1} = t_{n+2}^2 s_n^n$, $d_n^i t_{n+1} = t_n d_n^{i-1}$, and $s_n^i t_{n+1} = t_{n+2} s_n^{i-1}$ for $1 \leq i \leq n$. Cyclic vector spaces form a category.

The cyclic vector space associated to a unital locally convex complex algebra A is $A^\natural = (A^{\otimes n+1})_{n\geq 0}$, with the structural morphisms

$$\begin{aligned}
s_n^i(a_0 \otimes \dots \otimes a_n) &= a_0 \otimes \dots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \dots \otimes a_n, \\
d_n^i(a_0 \otimes \dots \otimes a_n) &= a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n, \quad \text{ for } 0 \leq i < n, \text{ and} \\
d_n^n(a_0 \otimes \dots \otimes a_n) &= a_n a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n-1}, \\
t_{n+1}(a_0 \otimes \dots \otimes a_n) &= a_n \otimes a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}.
\end{aligned}$$

If $X = (X_n)_{n\geq 0}$ and $Y = (Y_n)_{n\geq 0}$ are cyclic vector spaces, then we can define on $(X_n \otimes Y_n)_{n\geq 0}$ the structure of a cyclic space with structural morphisms given by the diagonal action of the corresponding structural morphisms, s_n^i, d_n^i, and t_{n+1}, of X and Y. The resulting cyclic vector space will be denoted $X \times Y$ and called the *external product* of X and Y. In particular, we obtain that $(A \otimes B)^\natural = A^\natural \times B^\natural$ for all unital algebras A and B, and that $X \times \mathbb{C}^\natural \simeq X$ for all cyclic vector spaces X. There is an obvious variant of these constructions for locally convex algebras, obtained by using the complete projective tensor product.

The cyclic cohomology groups of an algebra A can be recovered as *Ext*–groups. For us, the most convenient definition of *Ext* is using exact sequences (or resolutions). Consider the set $E = (M_k)_{k=0}^n$ of resolutions of length $n+1$ of X by cyclic vector spaces, such that $M_n = Y$. Thus we consider exact sequences

$$E: \quad 0 \to Y = M_n \to M_{n-1} \to \cdots \to M_0 \to X \to 0,$$

of cyclic vector spaces. For two such resolutions, E and E', we write $E \simeq E'$ whenever there exists a morphism of complexes $E \to E'$ that induces the identity on X and Y. Then $\mathrm{Ext}^n_\Lambda(X, Y)$ is, by definition, the set of equivalence classes of resolutions $E = (M_k)_{k=0}^n$ with respect to the equivalence relation generated by $\simeq$. The set $\mathrm{Ext}^n_\Lambda(X, Y)$ has a natural group structure. The equivalence class in $\mathrm{Ext}^n_\Lambda(X, Y)$ of a resolution $E = (M_k)_{k=0}^n$ is denoted $[E]$. This definition of *Ext* coincides with the usual one–using resolutions by projective modules–because cyclic vector spaces form an Abelian category with enough projectives.

Given a cyclic vector space $X = (X_n)_{n\geq 0}$ define $b, b' : X_n \to X_{n-1}$ by $b' = \sum_{j=0}^{n-1} (-1)^j d_j$, $b = b' + (-1)^n d_n$. Let $s_{-1} = s_n^n \circ t_{n+1}$ be the 'extra degeneracy' of X, which satisfies $s_{-1} b' + b' s_{-1} = 1$. Also let $\epsilon = 1 - (-1)^n t_{n+1}$, $N = \sum_{j=0}^n (-1)^{nj} t_{n+1}^j$ and $B = \epsilon s_{-1} N$. Then (X, b, B) is a mixed complex and hence $\mathrm{HC}_*(X)$, *the cyclic homology of* X, is the homology of $(\oplus_{k\geq 0} X_{n-2k}, b + B)$, by definition. Cyclic cohomology is obtained by dualization, as before.

The *Ext*–groups recover the cyclic cohomology of an algebra A via a natural isomorphism,

$$\mathrm{HC}^n(A) \simeq \mathrm{Ext}^n_\Lambda(A^\natural, \mathbb{C}^\natural), \tag{14}$$

[9]. This isomorphism allows us to use the theory of derived functors to study cyclic cohomology, especially products.

The *Yoneda product*,

$$\mathrm{Ext}^n_\Lambda(X, Y) \otimes \mathrm{Ext}^m_\Lambda(Y, Z) \ni \xi \otimes \zeta \to \zeta \circ \xi \in \mathrm{Ext}^{n+m}_\Lambda(X, Z),$$

is defined by splicing [18]. If $E = (M_k)_{k=0}^n$ is a resolution of X, and $E' = (M'_k)_{k=0}^m$ a resolution of Y, such that $M_n = Y$ and $M'_m = Z$, then $E' \circ E$ is represented by

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & Z = M'_m \arrow[r] & M'_{m-1} \arrow[r] & \cdots \arrow[r] & M'_0 \arrow[r] \arrow[d] & M_{n-1} \arrow[r] & \cdots \arrow[r] & M_0 \arrow[r] & X \arrow[r] & 0 \\
 & & & & Y \arrow[ur] & & & & &
\end{tikzcd}$$

The resulting product generalizes the composition of functions. Using the same notation, the *external product* $E \times E'$ is the resolution

$$E \times E' = \left(\sum_{k+j=l} M'_k \times M_j \right)_{l=0}^{n+m}.$$

Passing to equivalence classes, we obtain a product

$$\mathrm{Ext}_\Lambda^m(X,Y) \otimes \mathrm{Ext}_\Lambda^n(X_1,Y_1) \xrightarrow{\times} \mathrm{Ext}_\Lambda^{m+n}(X \times X_1, Y \times Y_1).$$

If $f : X \to X'$ is a morphism of cyclic vector spaces then we shall sometimes denote $E' \circ f = f^*(E')$, for $E' \in \mathrm{Ext}_\Lambda^n(X', \mathbb{C}^\natural)$.

The Yoneda product, "$\circ$," and the external product, "$\times$," are both associative and are related by the following identities, [30], Lemma 1.2.

LEMMA. 2.2. *Let $x \in \mathrm{Ext}_\Lambda^n(X,Y)$, $y \in \mathrm{Ext}_\Lambda^m(X_1,Y_1)$, and τ be the natural transformation $\mathrm{Ext}_\Lambda^{m+n}(X_1 \times X, Y_1 \times Y) \to \mathrm{Ext}_\Lambda^{m+n}(X \times X_1, Y \times Y_1)$ that interchanges the factors. Then*

$$x \times y = (id_Y \times y) \circ (x \times id_{X_1}) = (-1)^{mn}\ (x \times id_{Y_1}) \circ (id_X \times y),$$

$$id_X \times (y \circ z) = (id_X \times y) \circ (id_X \times z),$$

$$x \times y = (-1)^{mn} \tau(y \times x), \qquad and \qquad x \times id_{\mathbb{C}^\natural} = x = id_{\mathbb{C}^\natural} \times x.$$

We now turn to the definition of the periodicity operator. A choice of a generator σ of the group $\mathrm{Ext}_\Lambda^2(\mathbb{C}^\natural, \mathbb{C}^\natural)$, defines a periodicity operator

$$\mathrm{Ext}_\Lambda^n(X,Y) \ni x \to \mathbb{S}x = x \times \sigma \in \mathrm{Ext}_\Lambda^{n+2}(X,Y). \tag{15}$$

In the following we shall choose the standard generator σ that is defined 'over $\mathbb{Z}$', and then the above definition extends the periodicity operator in cyclic cohomology. This and other properties of the periodicity operator are summarized in the following Corollary ([30], Corollary 1.4)

COROLLARY. 2.3. *a) Let $x \in \mathrm{Ext}_\Lambda^n(X,Y)$ and $y \in \mathrm{Ext}_\Lambda^m(X_1,Y_1)$. Then $(\mathbb{S}x) \times y = \mathbb{S}(x \times y) = x \times (\mathbb{S}y)$.*

b) If $x \in \mathrm{Ext}_\Lambda^n(\mathbb{C}^\natural, X)$, then $\mathbb{S}x = \sigma \circ x$.

c) If $y \in \mathrm{Ext}_\Lambda^m(Y, \mathbb{C}^\natural)$, then $\mathbb{S}y = y \circ \sigma$.

d) For any extension x, we have $\mathbb{S}x = \sigma \times x$.

Using the periodicity operator, we extend the definition of periodic cyclic cohomology groups from algebras to cyclic vector spaces by

$$\mathrm{HP}^i(X) = \lim_{\rightarrow} \mathrm{Ext}_\Lambda^{i+2n}(X, \mathbb{C}^\natural), \tag{16}$$

the inductive limit being with respect to $\mathbb{S}$; clearly, $\mathrm{HP}^i(A^\natural) = \mathrm{HP}^i(A)$. Then Corollary 2.3 a) shows that the external product $\times$ is compatible with the periodicity morphism, and hence defines an external product,

$$\mathrm{HP}^i(A) \times \mathrm{HP}^j(B) \xrightarrow{\otimes} \mathrm{HP}^{i+j}(A \otimes B), \tag{17}$$

on periodic cyclic cohomology.

2.2. Extensions of algebras and products. Cyclic vector spaces will be used to study exact sequences of algebras. Let $I \subset A$ be a two–sided ideal of a complex unital algebra A (recall that in this paper all algebras are complex algebras.) Denote by $(A,I)^\natural$ the kernel of the map $A^\natural \to (A/I)^\natural$, and by $[A,I] \in \mathrm{Ext}^1_\Lambda((A/I)^\natural,(A,I)^\natural)$ the (equivalence class of the) exact sequence

$$0 \to (A,I)^\natural \to A^\natural \to (A/I)^\natural \to 0 \tag{18}$$

of cyclic vector spaces.

Let $\mathrm{HC}^i(A,I) = \mathrm{Ext}^i_\Lambda((A,I)^\natural,\mathbb{C}^\natural)$, then the long exact sequence of *Ext*–groups associated to the short exact sequence (18) reads

$$\cdots \to \mathrm{HC}^i(A/I) \to \mathrm{HC}^i(A) \to \mathrm{HC}^i(A,I) \to \mathrm{HC}^{i+1}(A/I) \to \mathrm{HC}^{i+1}(A) \to \cdots$$

By standard homological algebra, the boundary map of this long exact sequence is given by the product

$$\mathrm{HC}^i(A,I) \ni \xi \to \xi \circ [A,I] \in \mathrm{HC}^{i+1}(A/I).$$

For an arbitrary algebra I, possibly without unit, we let $I^\flat = (I^+,I)^\natural$. Then the isomorphism (14) becomes $\mathrm{HC}^n(I) \simeq \mathrm{Ext}^n_\Lambda(I^\flat,\mathbb{C}^\natural)$, and the excision theorem in periodic cyclic cohomology for cyclic vector spaces takes the following form.

Theorem. 2.4 (Cuntz–Quillen). *The inclusion $j_{I,A} : I^\flat \hookrightarrow (A,I)^\natural$ of cyclic vector spaces induces an isomorphism $\mathrm{HP}^*(A,I) \simeq \mathrm{HP}^*(I)$.*

It follows that every element $\xi \in \mathrm{HP}^*(I)$ is of the form $\xi = \xi_0 \circ j_{I,A}$, and that the boundary morphism $\partial_{A,I} : \mathrm{HP}^*(I) \to \mathrm{HP}^{*+1}(A/I)$ satisfies

$$\partial_{A,I}(\xi_0 \circ j_{I,A}) = \xi_0 \circ [A,I] \tag{19}$$

for all $\xi_0 \in \mathrm{HC}^i(A,I) = \mathrm{Ext}^i_\Lambda((A,I)^\natural,\mathbb{C}^\natural)$. Formula (19) then uniquely determines $\partial_{I,A}$.

We shall need in what follows a few properties of the isomorphisms $j_{I,A}$. Let B be an arbitrary unital algebra and I an arbitrary, possibly non–unital algebra. The inclusion $(I\otimes B)^+ \to I^+ \otimes B$, of unital algebras, defines a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (I\otimes B)^\flat \arrow[r] \arrow[d, "\eta_{I,B}"'] & (I\otimes B)^{+\natural} \arrow[r] \arrow[d] & \mathbb{C}^\natural \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^\flat \times B^\natural \arrow[r] & (I^+\otimes B)^\natural \arrow[r] & B^\natural \arrow[r] & 0
\end{tikzcd}$$

with exact lines. The morphism $\eta_{I,B}$, defined for possibly non-unital algebras I, will replace the identification $A^\natural \times B^\natural = (A\otimes B)^\natural$, valid only for unital algebras A.

Using the notation of Theorem 2.4, we see that $\eta_{I,B} = j_{I\otimes B, I^+\otimes B}$, and hence, by the same theorem, it follows that $\eta_{I,B}$ induces an isomorphism

$$\mathrm{HP}^*(I^\flat \times B^\natural) \ni \alpha \to \alpha \circ \eta_{I,B} \in \mathrm{HP}^*(I\otimes B).$$

Using this isomorphism, we extend the external product

$$\otimes : \mathrm{HP}^*(I) \otimes \mathrm{HP}^*(B) \to \mathrm{HP}^*(I\otimes B)$$

to a possibly non-unital algebra I by

$$\mathrm{HP}^i(I)\otimes\mathrm{HP}^j(B)=\lim_{\to}\mathrm{Ext}_\Lambda^{i+2n}(I^\flat,\mathbb{C}^\natural)\otimes\lim_{\to}\mathrm{Ext}_\Lambda^{j+2m}(B^\natural,\mathbb{C}^\natural)$$
$$\xrightarrow{\times}\lim_{\to}\mathrm{Ext}_\Lambda^{i+j+2l}(I^\flat\times B^\natural,\mathbb{C}^\natural)=\mathrm{HP}^*(I^\flat\times B^\natural)\simeq\mathrm{HP}^{i+j}(I\otimes B).$$

This extension of the external tensor product $\otimes$ to possibly non-unital algebras will be used to study the tensor product by B of an exact sequence $0\to I\to A\to A/I\to 0$ of algebras.

Tensoring by B is an exact functor, and hence we obtain an exact sequence

$$0\to I\otimes B\to A\otimes B\to (A/I)\otimes B\to 0. \tag{20}$$

LEMMA. 2.5. *Using the notation introduced above, we have the relation*

$$[A\otimes B,I\otimes B]=[A,I]\times\mathrm{id}_B\in\mathrm{Ext}_\Lambda^1((A/I\otimes B)^\natural,(A\otimes B,I\otimes B)^\natural).$$

Proof. We need only observe that the relation $A^\natural\times B^\natural=(A\times B)^\natural$ and the exactness of the functor $X\to X\times B^\natural$ imply that $(A,I)^\natural\times B^\natural=(A\otimes B,I\otimes B)^\natural$. □

2.3. PROPERTIES OF THE BOUNDARY MAP. The following theorem is a key tool in establishing further properties of the boundary map in periodic cyclic homology.

THEOREM. 2.6. *Let A and B be complex unital algebras and $I\subset A$ be a two-sided ideal. Then the boundary maps*

$$\partial_{I,A}:\mathrm{HP}^*(I)\to\mathrm{HP}^{*+1}(A/I)$$

and

$$\partial_{I\otimes B,A\otimes B}:\mathrm{HP}^*(I\otimes B)\to\mathrm{HP}^{*+1}((A/I)\otimes B)$$

satisfy

$$\partial_{I\otimes B,A\otimes B}(\xi\otimes\zeta)=\partial_{I,A}(\xi)\otimes\zeta$$

for all $\xi\in\mathrm{HP}^*(I)$ and $\zeta\in\mathrm{HP}^*(B)$.

Proof. The groups $\mathrm{HP}^k(I)$ is the inductive limit of the groups $\mathrm{Ext}_\Lambda^{k+2n}(I^\flat,\mathbb{C}^\natural)$ so ξ will be the image of an element in one of these Ext–groups. By abuse of notation, we shall still denote that element by ξ, and thus we may assume that $\xi\in\mathrm{Ext}_\Lambda^k(I^\flat,\mathbb{C}^\natural)$, for some large k. Similarly, we may assume that $\zeta\in\mathrm{Ext}_\Lambda^j(B^\natural,\mathbb{C}^\natural)$. Moreover, by Theorem 2.4, we may assume that $\xi=\xi_0\circ j_{I,A}$, for some $\xi_0\in\mathrm{Ext}_\Lambda^i((A,I)^\natural,\mathbb{C}^\natural)$.

We then have

$$\begin{aligned}
&\partial_{I,A}(\xi)\otimes\zeta=\partial(\xi_0\circ j_{I,A})\times\zeta= && \\
&= (\xi_0\circ[A,I])\times\zeta && \text{by equation (19)}\\
&= (\mathrm{id}_{\mathbb{C}^\natural}\times\zeta)\circ((\xi_0\circ[A,I])\times\mathrm{id}_B) && \text{by Lemma 2.2}\\
&= (\mathrm{id}_{\mathbb{C}^\natural}\times\zeta)\circ(\xi_0\times\mathrm{id}_B)\circ([A,I]\times\mathrm{id}_B) && \text{by Lemma 2.2}\\
&= (\xi_0\times\zeta)\circ[A\otimes B,I\otimes B] && \text{by Lemma 2.2 and Corollary 2.3}\\
&= \partial_{A\otimes B,I\otimes B}((\xi_0\times\zeta)\circ j_{I\otimes B,A\otimes B}) && \text{by equation (19).}
\end{aligned}$$

By definition, the morphism $j_{I,A}$ introduced in Theorem 2.4 satisfies

$$j_{I\otimes B,A\otimes B}=(j_{I,A}\times\mathrm{id}_B)\circ\eta_{I,B}. \tag{21}$$

Equation (21) then gives

$$\partial_{I,A}(\xi)\otimes\zeta=\partial_{I\otimes B,A\otimes B}((\xi\times\zeta)\circ\eta_{I,B})$$

in $\mathrm{Ext}_\Lambda^{i+j+1}((A/I\otimes B)^\natural,\mathbb{C}^\natural)$. This completes the proof in view of the definition of the external product $\otimes$ in the non-unital case: $\xi\otimes\zeta=(\xi\times\zeta)\circ\eta_{I,B}$. □

We now consider crossed products. Let A be a unital algebra and Γ a discrete group acting on A by $\Gamma \times A \ni (\gamma, a) \to \alpha_\gamma(a) \in A$. Then the (algebraic) crossed product $A \rtimes \Gamma$ consists of finite linear combinations of elements of the form $a\gamma$, with the product rule $(a\gamma)(b\gamma_1) = a\alpha_\gamma(b)\gamma\gamma_1$. Let $\delta(a\gamma) = a\gamma \otimes \gamma$, which defines a morphism $\delta : A \rtimes \Gamma \to A \rtimes \Gamma \otimes \mathbb{C}[\Gamma]$. Using δ, we define on $\mathrm{HP}^*(A \rtimes \Gamma)$ a $\mathrm{HP}^*(\mathbb{C}[\Gamma])$–module structure [28] by

$$\mathrm{HP}^*(A \rtimes \Gamma) \otimes \mathrm{HP}^*(\mathbb{C}[\Gamma]) \xrightarrow{\otimes} \mathrm{HP}^*((A \rtimes \Gamma) \otimes \mathbb{C}[\Gamma]) \xrightarrow{\delta^*} \mathrm{HP}^*(A \rtimes \Gamma).$$

A Γ–invariant two-sided ideal $I \subset A$ gives rise to a "crossed product exact sequence"

$$0 \to I \rtimes \Gamma \to A \rtimes \Gamma \to (A/I) \rtimes \Gamma \to 0$$

of algebras. The following theorem describes the behavior of the boundary map of this exact sequence with respect to the $\mathrm{HP}^*(\mathbb{C}[\Gamma])$–module structure on the corresponding periodic cyclic cohomology groups.

Theorem. 2.7. *Let Γ be a discrete group acting on the unital algebra A, and let I be a Γ-invariant ideal. Then the boundary map*

$$\partial_{I \rtimes \Gamma, A \rtimes \Gamma} : \mathrm{HP}^*(I \rtimes \Gamma) \to \mathrm{HP}^{*+1}((A/I) \rtimes \Gamma)$$

is $\mathrm{HP}^*(\mathbb{C}[\Gamma])$-linear.

Proof. The proof is based on the previous theorem, Theorem 2.6, and the naturality of the boundary morphism in periodic cyclic cohomology.

From the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I \rtimes \Gamma \arrow[r] \arrow[d] & A \rtimes \Gamma \arrow[r] \arrow[d] & (A/I) \rtimes \Gamma \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (I \rtimes \Gamma) \otimes \mathbb{C}[\Gamma] \arrow[r] & (A \rtimes \Gamma) \otimes \mathbb{C}[\Gamma] \arrow[r] & (A/I) \rtimes \Gamma \otimes \mathbb{C}[\Gamma] \arrow[r] & 0,
\end{tikzcd}$$

we obtain that $\delta^* \partial = \partial \delta^*$ (we have omitted the subscripts). Then, for each $x \in \mathrm{HP}^*(\mathbb{C}[\Gamma])$ and $\xi \in \mathrm{HP}^*(I \rtimes \Gamma)$, we have $\xi x = \delta^*(\xi \otimes x)$, and hence, using also Theorem 2.6, we obtain

$$\partial(\xi x) = \partial(\delta^*(\xi \otimes x)) = \delta^*(\partial(\xi \otimes x)) = \delta^*((\partial \xi) \otimes x) = (\partial \xi) x \ .$$

The proof is complete. □

For the rest of this subsection it will be convenient to work with *continuous* periodic cyclic homology. Recall that this means that all algebras have compatible locally convex topologies, that we use complete projective tensor products, and that the projections $A \to A/I$ have continuous linear splittings, which implies that $A \simeq A/I \oplus I$ as locally convex vector spaces. Moreover, since the excision theorem is known only for m–algebras [13], we shall also assume that our algebras are m–algebras, that is, that their topology is generated by a family of sub-multiplicative seminorms. Slightly weaker results hold for general topological algebras and *discrete* periodic cyclic cohomology.

There is an analog of Theorem 2.7 for actions of compact Lie groups. If G is a compact Lie group acting smoothly on a complete locally convex algebra A by

$\alpha : G \times A \to A$, then the *smooth crossed product algebra* is $A \rtimes G = C^\infty(G, A)$, with the convolution product $*$,

$$f_0 * f_1(g) = \int_G f_0(h)\alpha_h(f_1(h^{-1}g))dh,$$

the integration being with respect to the normalized Haar measure on G. As before, if $I \subset A$ is a *complemented* G-invariant ideal of A, we get an exact sequence of smooth crossed products

$$0 \to I \rtimes G \to A \rtimes G \to (A/I) \rtimes G \to 0. \tag{22}$$

Still assuming that G is compact, let $R(G)$ be the representation ring of G. Then the group $\mathrm{HP}^*(A \rtimes G)$ has a natural $R(G)$–module structure defined as follows (see also [31]). The diagonal inclusion $A \rtimes G \hookrightarrow M_n(A) \rtimes G$ induces an isomorphism in cyclic cohomology, with inverse induced by the morphism

$$\frac{1}{n}Tr : M_n(A \rtimes G)^\natural \to (A \rtimes G)^\natural$$

of cyclic objects. Then, for any representation $\pi : G \to M_n(\mathbb{C})$, we obtain a unit preserving morphism

$$\mu_\pi : A \rtimes G \to M_n(A \rtimes G),$$

defined by $\mu_\pi(f)(g) = f(g)\pi(g) \in C^\infty(G, M_n(A))$, for any $f \in C^\infty(G, A)$. Finally, if $\pi \in R(G)$, we define the multiplication by π to be the morphism

$$(Tr \circ \mu_\pi)^* : \mathrm{HP}^*_{\mathrm{cont}}(A \rtimes G) \to \mathrm{HP}^*_{\mathrm{cont}}(A \rtimes G).$$

Thus, $\pi x = x \circ Tr \circ \mu_\pi$.

THEOREM. 2.8. *Let A be a locally convex m–algebra and $I \subset A$ a complemented G–invariant two-sided ideal. Then the boundary morphism associated to the exact sequence* (22)*,*

$$\partial_{I\rtimes G, A\rtimes G} : \mathrm{HP}^*_{\mathrm{cont}}(I \rtimes G) \to \mathrm{HP}^{*+1}_{\mathrm{cont}}((A/I) \rtimes G),$$

is $R(G)$-linear.

Proof. First, we observe that the morphism $Tr : M_n(A)^\natural \to A^\natural$ is functorial, and, consequently, that it gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d] & M_n(A \rtimes G)^\natural \arrow[r] \arrow[d] & (M_n(A/I) \rtimes G)^\natural \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (A \rtimes G, I \rtimes G)^\natural \arrow[r] & (A \rtimes G)^\natural \arrow[r] & ((A/I) \rtimes G)^\natural \arrow[r] & 0
\end{tikzcd}$$

where $X = (M_n(A \rtimes G), M_n(I \rtimes G))^\natural$ and whose vertical arrows are given by Tr.

Regarding this commutative diagram as a morphism of extensions, we obtain that

$$Tr \circ [M_n(A) \rtimes G, M_n(I) \rtimes G] = [A \rtimes G, I \rtimes G] \circ Tr. \tag{23}$$

Then, using a similar reasoning, we also obtain that

$$[M_n(A) \rtimes G, M_n(I) \rtimes G] \circ \mu_\pi = \mu_\pi \circ [A \rtimes G, I \rtimes G]. \tag{24}$$

Now let $\xi \in \mathrm{HP}^*_{\mathrm{cont}}(I \rtimes G)$, which we may assume, by Theorem 2.4, to be an element of the form $\xi = \xi_0 \circ j_{I\rtimes G, A\rtimes G}$, for some $\xi_0 \in \mathrm{Ext}^i_\Lambda((A \rtimes G, I \rtimes G)^\natural, \mathbb{C}^\natural)$. Using equations (23) and (24) and that the inclusion $j = j_{I\rtimes G, A\rtimes G}$, by the naturality of μ_π, is $R(G)$-linear, we finally get

$$\begin{aligned}
& \partial(\pi\xi) = \partial(\pi(\xi_0 \circ j)) = \partial((\pi\xi_0)\circ j) = \\
= \quad & \partial(\xi_0 \circ Tr \circ \mu_\pi \circ j) = \xi_0 \circ Tr \circ \mu_\pi \circ [A \rtimes G, I \rtimes G] = \\
= \quad & \xi_0 \circ [M_n(A\rtimes G), M_n(I\rtimes G)] \circ Tr \circ \mu_\pi = \partial(\xi)\circ Tr \circ \mu_\pi = \pi\partial(\xi)
\end{aligned}$$

The proof is now complete. □

In the same spirit and in the same framework as in Theorem 2.8, we now consider the action of Lie algebra cohomology on the periodic cyclic cohomology exact sequence.

Assume that G is compact and connected, and denote by $\mathfrak{g}$ its Lie algebra and by $H_*(\mathfrak{g})$ the Lie algebra homology of $\mathfrak{g}$. Since G is compact and connected, we can identify $\mathrm{H}_*(\mathfrak{g})$ with the bi-invariant currents on G. Let $\mu : G\times G \to G$ be the multiplication. Then one can alternatively define the product on $\mathrm{H}_*(\mathfrak{g})$ as the composition

$$\begin{aligned}
&\mathrm{H}_*(\mathfrak{g}) \otimes \mathrm{H}_*(\mathfrak{g}) \simeq \mathrm{HP}^*_{\mathrm{cont}}(C^\infty(G)) \otimes \mathrm{HP}^*_{\mathrm{cont}}(C^\infty(G)) \\
&\xrightarrow{\times} \mathrm{HP}^*_{\mathrm{cont}}(C^\infty(G\times G)) \xrightarrow{\mu^*} \mathrm{HP}^*_{\mathrm{cont}}(C^\infty(G)) \simeq \mathrm{H}_*(\mathfrak{g}).
\end{aligned}$$

We now recall the definition of the product $\mathrm{H}_*(\mathfrak{g}) \otimes \mathrm{HP}^*_{\mathrm{cont}}(A) \to \mathrm{HP}^*_{\mathrm{cont}}(A)$. Denote by $\varphi : A \to C^\infty(G,A)$ the morphism $\varphi(a)(g) = \alpha_g(a)$, where, this time, $C^\infty(G,A)$ is endowed with the *pointwise* product. Then $x\times\xi \in \mathrm{HP}^*_{\mathrm{cont}}(C^\infty(G)\widehat{\otimes}A)$ is a (continuous) cocycle on $C^\infty(G,A) \simeq C^\infty(G)\widehat{\otimes}A$, and we define $x\xi = \varphi^*(x\otimes\xi)$. The associativity of the $\times$-product shows that $\mathrm{HP}^*_{\mathrm{cont}}(A)$ becomes a $\mathrm{H}_*(\mathfrak{g})$–module with respect to this action.

THEOREM. 2.9. *Suppose that a compact connected Lie group G acts smoothly on a complete locally convex algebra A and that I is a closed invariant two-sided ideal of A, complemented as a topological vector space. Then*

$$\partial(x\xi) = x(\partial\xi),$$

for any $x \in \mathrm{H}_*(\mathfrak{g})$ and $\xi \in \mathrm{HP}^*_{\mathrm{cont}}(I)$.

Proof. The proof is similar to the proof of Theorem 2.8, using the morphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (A,I)^\natural & \longrightarrow & A^\natural & \longrightarrow & (A/I)^\natural & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & C^\infty(G,A)^\natural & \longrightarrow & C^\infty(G,A/I)^\natural & \longrightarrow & 0
\end{array}$$

where $X = (C^\infty(G,A), C^\infty(G,I))^\natural$. □

2.4. RELATION TO THE BIVARIANT CHERN–CONNES CHARACTER. A different type of property of the boundary morphism in periodic cyclic cohomology is its compatibility (effectively an identification) with the bivariant Chern-Connes character [30]. Before we can state this result, need to recall a few constructions from [30].

Let A and B be unital locally convex algebras and assume that a continuous linear map

$$\beta : A \to \mathcal{B}(\mathcal{H})\widehat{\otimes}B$$

is given, such that the cocycle $\ell(a_0, a_1) = \beta(a_0)\beta(a_1) - \beta(a_0a_1)$ factors as a composition $A\widehat{\otimes}A \to \mathcal{C}_p(\mathcal{H})\widehat{\otimes}B \to \mathcal{B}(\mathcal{H})\widehat{\otimes}B$ of continuous maps. (Recall that $\mathcal{C}_p(\mathcal{H})$ is the ideal of p–summable operators and that $\widehat{\otimes}$ is the complete projective tensor product.) Using the cocycle ℓ, we define on $E_\beta = A \oplus \mathcal{C}_p(\mathcal{H})\hat{\otimes}B$ an associative product by the formula

$$(a_1, x_1)(a_2, x_2) = (a_1a_2, \beta(a_1)x_2 + x_1\beta(a_2) + \ell(a_1, a_2)).$$

Then the algebra E_β fits into the exact sequence

$$0 \to \mathcal{C}_p(\mathcal{H})\widehat{\otimes}B \to E_\beta \to A \to 0. \tag{25}$$

An exact sequence

$$[E]: \quad 0 \to \mathcal{C}_p(\mathcal{H})\widehat{\otimes}B \to E \to A \to 0. \tag{26}$$

that is isomorphic to an exact sequence of the form (25) will be called an *admissible* exact sequence. If $[E]$ is an admissible exact sequence and $n \geq p - 1$, then [30, Theorem 3.5] associates to $[E]$ an element

$$ch_1^{2n+1}([E]) \in \mathrm{Ext}^{2n+1}_{\Lambda,\mathrm{cont}}(A^\natural, B^\natural), \tag{27}$$

which for $B = \mathbb{C}$ recovers Connes' Chern character in K-homology [10]. (The subscript "cont" stresses that we are considering the version of the Yoneda Ext defined for locally convex cyclic objects.)

Let $Tr : \mathcal{C}_1(\mathcal{H}) \to \mathbb{C}$ be the ordinary trace, i.e., $Tr(T) = \sum_n (Te_n, e_n)$ for any orthonormal basis $(e_n)_{n\geq 0}$ of the Hilbert space $\mathcal{H}$. Using the trace Tr we define $Tr_n \in \mathrm{HC}^{2n}(\mathcal{C}_p(\mathcal{H}))$, for $2n \geq p - 1$, to be the class of the cyclic cocycle

$$Tr_n(a_0, a_1, \ldots, a_{2n}) = (-1)^n \frac{n!}{(2n)!} Tr(a_0a_1 \ldots a_{2n}). \tag{28}$$

The normalization factor was chosen such that $Tr_n = \mathbb{S}^n Tr_1 = \mathbb{S}^n Tr$ on $C_1(\mathcal{H})$. We have the following compatibility between the bivariant Chern-Connes character and the Cuntz–Quillen boundary morphism.

Let $\mathrm{HP}^*_{\mathrm{cont}} \ni \xi \to \xi_{\mathrm{disc}} \in \mathrm{HP}^*_{\mathrm{disc}} := \mathrm{HP}^*$ be the natural transformation that "forgets continuity" from continuous to ordinary (or discrete) periodic cyclic cohomology. We include the subscript "disc" only when we need to stress that discrete homology is used. By contrast, the subscript "cont" will always be included.

THEOREM. 2.10. *Let $0 \to \mathcal{C}_p(\mathcal{H})\widehat{\otimes}B \to E \to A \to 0$ be an admissible exact sequence and $ch_1^{2n+1}([E]) \in \mathrm{Ext}^{2n+1}_{\Lambda,\mathrm{cont}}(A^\natural, B^\natural)$ be its bivariant Chern–Connes character, equation (27). If Tr_n is as in equation (28) and $n \geq p - 1$, then*

$$\partial(Tr_n \otimes \xi)_{\mathrm{disc}} = (\xi \circ ch_1^{2n+1}([E]))_{\mathrm{disc}} \in \mathrm{HP}^{q+1}(A),$$

for each $\xi \in \mathrm{HP}^q_{\mathrm{cont}}(B)$.

This theorem provides us–at least in principle–with formulæ to compute the boundary morphism in periodic cyclic cohomology, see [29] and [30], Proposition 2.3.

Before proceeding with the proof, we recall a construction implicit in [30]. The algebra $RA = \oplus_{j\geq 0} A^{\hat{\otimes} j}$ is the tensor algebra of A, and rA is the kernel of the map $RA \to A^+$. Because A has a unit, we have a canonical isomorphism $A^+ \simeq \mathbb{C} \oplus A$. We do not consider any topology on RA, but in addition to $(RA)^\natural$, the cyclic object associated to RA, we consider a completion of it in a natural topology with respect to which all structural maps are continuous. The new, completed, cyclic object is denoted $(RA)^{\natural}_{\mathrm{cont}}$ and is obtained as follows. Let $R_kA = \oplus_{j=0}^{k} A^{\hat{\otimes} j}$. Then

$$(RA)^{\natural}_{\mathrm{cont},n} = \lim_{k\to\infty} (R_kA)^{\hat{\otimes} n+1},$$

with the inductive limit topology.

Proof. We begin with a series of reductions that reduce the proof of the Theorem to the proof of (29).

Since $[E]$ is an admissible extension, there exists by definition a continuous linear section $s : A \to E$ of the projection $\pi : E \to A$ (i.e., $\pi \circ s = id$). Then s defines a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & rA \arrow[r] \arrow[d, "\varphi"] & RA \arrow[r] \arrow[d, "\psi"] & A^+ \arrow[r] \arrow[d, "\pi_A"] & 0 \\ 0 \arrow[r] & \mathcal{C}_p \widehat{\otimes} B \arrow[r] & E \arrow[r] & A \arrow[r] & 0 \end{tikzcd},$$

where the right hand vertical map is the projection $A^+ \simeq \mathbb{C} \oplus A \to A$.

By increasing q if necessary, we may assume that the cocycle $\xi \in \mathrm{HP}^q_{\mathrm{cont}}(B)$ comes from a cocycle, also denoted ξ, in $\mathrm{HC}^q_{\mathrm{cont}}(B)$. Let

$$\xi_1 = (Tr_n \otimes \xi)_{\mathrm{disc}} \in \mathrm{HC}^{q+2n}_{\mathrm{disc}}(\mathcal{C}_p \widehat{\otimes} B) := \mathrm{HC}^{q+2n}(\mathcal{C}_p \widehat{\otimes} B)$$

be as in the statement of the theorem.

We claim that it is enough to show that

$$\partial(\varphi^*\xi_1) \circ j_A = (\xi \circ ch_1^{2n+1}([E]))_{\mathrm{disc}}, \tag{29}$$

where $j_A = A^{\natural} \to (A^+)^{\natural}$ is the inclusion.

Indeed, assuming (29) and using the above commutative diagram and the naturality of the boundary morphism, we obtain

$$(\xi \circ ch_1^{2n+1}([E]))_{\mathrm{disc}} = \partial(\varphi^*\xi_1) \circ j_A = \pi_A^*(\partial\xi_1) \circ j_A = \partial\xi_1 \circ \pi_A \circ j_A = \partial\xi_1,$$

as stated in theorem, because $\pi_A \circ j_A = id$.

Let $j_{rA,RA} : (rA)^{\flat} \hookrightarrow (RA, rA)^{\natural}$ be the morphism (inclusion) considered in Theorem 2.4. Also, let $\xi_2 \in \mathrm{HC}^n_{\mathrm{disc}}((RA,rA)^{\natural}) = \mathrm{Ext}^n_{\Lambda}((RA,rA)^{\natural}, \mathbb{C}^{\natural})$ satisfy

$$\xi_2 \circ j_{rA,RA} = \varphi^*\xi_1 \in \mathrm{HC}^n_{\mathrm{disc}}((rA)^{\flat}) = \mathrm{Ext}^n_{\Lambda}((rA)^{\flat}, \mathbb{C}^{\natural}). \tag{30}$$

(In words: "ξ_2 restricts to $\varphi^*\xi_1$ on $(rA)^{\flat}$.") Then, using equation (19), we have

$$\partial(\varphi^*\xi_1) = \xi_2 \circ [RA, rA]. \tag{31}$$

The rest of the proof consists of showing that the construction of the odd bivariant Chern-Connes character [30] provides us with ξ_2 satisfying equations (30) and (32):

$$\xi_2 \circ [RA, rA] \circ j_A = (\xi \circ ch_1^{2n+1}([E]))_{\mathrm{disc}}. \tag{32}$$

This is enough to complete the proof because equations (31) and (32) imply (29) and, as we have already shown, equation (30) implies equation (31). So, to complete the proof, we now proceed to construct ξ_2 satisfying (30) and (32).

Recall from [30] that the ideal rA defines a natural increasing filtration of $(RA)^{\natural}_{\mathrm{cont}}$ by cyclic vector spaces:

$$(RA)^{\natural}_{\mathrm{cont}} = F_0(RA)^{\natural}_{\mathrm{cont}} \supset F_{-1}(RA)^{\natural}_{\mathrm{cont}} \supset \dots,$$

such that $(rA)^\flat \subset F_{-1}(RA)^\natural_{\rm cont} = (RA, rA)^\natural$. If $(rA)^\flat_k$ is the k–th component of the cyclic vector space $(rA)^\flat$ (and if, in general, the lower index stands for the $\mathbb{Z}_+$–grading of a cyclic vector space) then we have the more precise relation

$$(rA)^\flat_k \subset (F_{-n-1}(RA)^\natural_{\rm cont})_k\ , \qquad \text{for } k \geq n. \tag{33}$$

It follows that the morphism of cyclic vector spaces

$$\tilde{\tau}_n = Tr \circ F_{-n-1}(\psi) : F_{-n-1}(RA)^\natural_{\rm cont} \to B^\natural$$

(defined in [30], page 579) satisfies $\tilde{\tau}_n = Tr \circ \varphi$ on $(rA)^\flat_k$, for $k \geq n \geq p-1$. Fix then $k = q + 2n$, and conclude that $\xi_1 = Tr_n \otimes \xi_{\rm disc} \in \mathrm{HC}^{q+2n}(\mathcal{C}_p \widehat{\otimes} B)$ satisfies

$$\varphi^* \xi_1 = \varphi^*(Tr_n \otimes \xi) = \xi_{\rm disc} \circ \mathbb{S}^n \tilde{\tau_n} \tag{34}$$

on $(rA)^\flat_k \subset F_{-n-1}(RA)^\natural_{\rm cont}$, because Tr_n restricts to $\mathbb{S}^n Tr$ on $\mathcal{C}_1(\mathcal{H})$. Now recall the crucial fact that there exists an extension

$$C_0^{2n}(RA) \in \mathrm{Ext}^{2n}_{\Lambda, \rm cont}(F_{-1}(RA)^\natural_{\rm cont}, F_{-n-1}(RA)^\natural_{\rm cont})$$

that has the property that $C_0^{2n}(RA) \circ i = \mathbb{S}^n$, if $i : F_{-n-1}(RA)^\natural_{\rm cont} \to F_{-1}(RA)^\natural_{\rm cont}$ is the inclusion (see [30], Corollary 2.2). Using this extension, we finally define

$$\xi_2 = (\xi \circ \tilde{\tau}_n \circ C_0^{2n}(RA))_{\rm disc} \in \mathrm{Ext}^n_\Lambda(F_{-1}(RA)^\natural_{\rm cont}, \mathbb{C}^\natural).$$

Since ξ_2 has order $k = q + 2n \geq 2n \geq n$, we obtain from the equations (33) and (34) that ξ_2 satisfies (30) (i.e., that it restricts to $\varphi^*\xi_1$ on $(rA)^\flat_k \subset F_{-n-1}(RA)^\natural_{\rm cont}$), as desired.

The last thing that needs to be checked for the proof to be complete is that ξ_2 satisfies equation (32). By definition, the odd bivariant Chern-Connes character ([30], page 579) is

$$ch_1^{2n+1}([E]) = \tilde{\tau}_n \circ ch_1^{2n+1}(RA) \circ j_A, \tag{35}$$

where $ch_1^{2n+1}(RA) = C_1^{2n+1}(RA) = C_0^{2n}(RA) \circ q_0(RA)$, and $j_A : A^\natural \to (A^+)^\natural$ is the inclusion (see [30], page 568, definition 2.4. page 574, and the discussion on page 579). Moreover $q_0(RA)$ is nothing but a continuous version of $[RA, rA]$, that is

$$q_0(RA)_{\rm disc} = [RA, rA],$$

and hence

$$\xi_2 \circ [RA, rA] \circ j_A = (\xi \circ \tilde{\tau}_n \circ C_0^{2n}(RA) \circ q_0(RA) \circ j_A)_{\rm disc} = (\xi \circ ch_1^{2n+1}([E]))_{\rm disc}.$$

Since ξ_2 satisfies equation (30) and (32), which imply equation (29), the proof is complete. $\square$

For any locally convex algebra B and $\xi \in \mathrm{HP}^*(B)$, the *discrete* periodic cyclic cohomology of B, we say that ξ is a *continuous class* if it can be represented by a continuous cocycle on B. Put differently, this means that $\xi = \zeta_{\rm disc}$, for some $\zeta \in \mathrm{HP}^*_{\rm cont}(B)$. Since the bivariant Chern–Connes character can, at least in principle, be expressed by an explicit formula, it preserves continuity. This gives the following corollary.

COROLLARY. 2.11. *The periodic cyclic cohomology boundary map ∂ associated to an admissible extension maps a class of the form $Tr_n \otimes \xi$, for ξ a continuous class, to a continuous class.*

It is likely that recent results of Cuntz, see [12, 13], will give the above result for all continuous classes in $\mathrm{HP}^*(\mathcal{C}_p\hat{\otimes}B)$ (not just the ones of the form $Tr_n \otimes \xi$).

Using the above corollary, we obtain the compatibility between the bivariant Chern–Connes character and the index morphism in full generality. This result had been known before only in particular cases [30].

THEOREM. 2.12. *Let $0 \to \mathcal{C}_p(\mathcal{H})\widehat{\otimes}B \to E \to A \to 0$ be an admissible exact sequence and $ch_1^{2n+1}([E]) \in \mathrm{Ext}_\Lambda^{2n+1}(A^\natural, B^\natural)$ be its bivariant Chern–Connes character, equation* (27). *If Tr_n is as in equation* (28) *and* $\mathrm{Ind} : \mathrm{K}_1^{\mathrm{alg}}(A) \to \mathrm{K}_0^{\mathrm{alg}}(\mathcal{C}_p(\mathcal{H})\widehat{\otimes}B)$ *is the connecting morphism in algebraic K–Theory then, for any $\varphi \in \mathrm{HP}^0_{\mathrm{cont}}(B)$ and $[u] \in \mathrm{K}_1^{\mathrm{alg}}(A)$, we have*

$$\langle Tr_n \otimes \varphi, \mathrm{Ind}[u] \rangle = \langle ch_1^{2n+1}([E]) \circ \varphi, [u] \rangle . \tag{36}$$

3. THE INDEX THEOREM FOR COVERINGS

Using the methods we have developed, we now give a new proof of Connes–Moscovici's index theorem for coverings. To a covering $\widetilde{M} \to M$ with covering group Γ, Connes and Moscovici associated an extension

$$0 \longrightarrow \mathcal{C}_{n+1} \otimes \mathbb{C}[\Gamma] \longrightarrow E_{CM} \longrightarrow C^\infty(S^*M) \longrightarrow 0, \quad n = \dim M,$$

(the Connes–Moscovici exact sequence), defined using invariant pseudodifferential operators on $\widetilde{M}$; see equation (45). If $\varphi \in \mathrm{H}^*(\Gamma) \subset \mathrm{HP}^*_{\mathrm{cont}}(\mathcal{C}_{n+1} \otimes \mathbb{C}[\Gamma])$ is an even cyclic cocycle, then the Connes–Moscovici index theorem computes the morphisms

$$\varphi_* \circ \mathrm{Ind} : \mathrm{K}_1^{\mathrm{alg}}(C^\infty(S^*M)) \longrightarrow \mathbb{C},$$

where Ind is the index morphism associated to the Connes–Moscovici exact sequence. Our method of proof then is to use the compatibility between the connecting morphisms in algebraic K–Theory and ∂, the connecting morphism in periodic cyclic cohomology (Theorem 1.5), to reduce the proof to the computation of ∂. This computation is now a problem to which the properties of ∂ established in Section 2 can be applied.

We first show how to obtain the Connes–Moscovici exact sequence from another exact sequence, the Atiyah–Singer exact sequence, by a purely algebraic construction. Then, using the naturality of ∂ and Theorem 2.6, we determine the connecting morphism ∂_{CM} of the Connes–Moscovici exact sequence in terms of the connecting morphism ∂_{AS} of the Atiyah–Singer exact sequence. For the Atiyah–Singer exact sequence the procedure can be reversed and we now use the Atiyah-Singer Index Theorem and Theorem 1.5 to compute ∂_{AS}.

A comment about the interplay of continuous and discrete periodic cyclic cohomology in the proof below is in order. We have to use continuous periodic cyclic cohomology whenever we want explicit computations with the periodic cyclic cohomology of groupoid algebras, because only the continuous version of periodic cyclic cohomology is known for groupoid algebras associated to étale groupoids [7]. On the other hand, in order to be able to use Theorem 1.5, we have to consider ordinary (or discrete) periodic cyclic cohomology as well. This is not an essential difficulty because, using Corollary 2.11, we know that the index classes are represented by continuous cocycles.

3.1. GROUPOIDS AND THE CYCLIC COHOMOLOGY OF THEIR ALGEBRAS. Our computations are based on groupoids, so we first recall a few facts about groupoids.

A *groupoid* is a small category in which every morphism is invertible. (Think of a groupoid as a set of points joined arrows; the following examples should clarify this abstract definition of groupoids.) A *smooth étale* groupoid is a groupoid whose set of morphisms (also called *arrows*) and whose set of objects (also called *units*) are smooth manifolds such that the domain and range maps are étale (i.e., local diffeomorphisms). To any smooth étale groupoid $\mathcal{G}$, assumed Hausdorff for simplicity, there is associated the algebra $C_c^\infty(\mathcal{G})$ of compactly supported functions on the set of arrows of $\mathcal{G}$ and endowed with the convolution product $*$,

$$(f_0 * f_1)(g) = \sum_{r(\gamma)=r(g)} f_0(\gamma) f_1(\gamma^{-1} g).$$

Here r is the range map and $r(\gamma) = r(g)$ is the condition that γ^{-1} and g be composable. Whenever dealing with $C_c^\infty(\mathcal{G})$, we will use *continuous* cyclic cohomology, as in [7]. See [7] for more details on étale groupoids, and [35] for the general theory of locally compact groupoids.

Étale groupoids conveniently accommodate in the same framework smooth manifolds and (discrete) groups, two extreme examples in the following sense: the smooth étale groupoid associated to a smooth manifold M has only identity morphisms, whereas the smooth étale groupoid associated to the (discrete) group Γ has only one object, the identity of Γ. The algebras $C_c^\infty(\mathcal{G})$ associated to these groupoids are $C_c^\infty(M)$ and, respectively, the group algebra $\mathbb{C}[\Gamma]$. Here are other examples used in the paper.

The groupoid R_I associated to an equivalence relation on a discrete set I has I as the set of units and exactly one arrow for any ordered pair of equivalent objects. If I is a finite set with k elements and all objects of I are equivalent (i.e., if R_I is the *total equivalence relation* on I) then $C_c^\infty(R_I) \simeq M_k(\mathbb{C})$ and its classifying space in the sense of Grothendieck [34], the space $\mathrm{B}\, R_I$, is contractable [17, 34].

Another example, the *gluing groupoid* $\mathcal{G}_\mathcal{U}$, mimics the definition a manifold M in terms of "gluing coordinate charts." The groupoid $\mathcal{G}_\mathcal{U}$ is defined [7] using an open cover $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of M, i.e., $M = \cup_{\alpha\in I} U_\alpha$. Then $\mathcal{G}_\mathcal{U}$ has units $\mathcal{G}_\mathcal{U}^0 = \cup_{\alpha \in I} U_\alpha \times \{\alpha\}$ and arrows

$$\mathcal{G}_\mathcal{U}^{(1)} = \{(x, \alpha, \beta), \alpha, \beta \in I, x \in U_\alpha \cap U_\beta\}.$$

If R_I is the total equivalence relation on I, then there is an injective morphism $l : \mathcal{G}_\mathcal{U} \hookrightarrow M \times R_I$ of étale groupoids.

Let $f : \mathcal{G}_1 \to \mathcal{G}_2$ be an *étale morphism of groupoids*, that is, a morphism of étale groupoids that is a local diffeomorphism. Then the map f defines a continuous map, $Bf : B\mathcal{G}_2 \to B\mathcal{G}_1$, of classifying spaces and a group morphism, $f_{Tr} : \mathrm{HP}^*_{\mathrm{cont}}(C_c^\infty(\mathcal{G}_1)) \to \mathrm{HP}^*_{\mathrm{cont}}(C_c^\infty(\mathcal{G}_2))$. If f is *injective* when restricted to units, then there exists an algebra morphism $\iota(f) : C_c^\infty(\mathcal{G}_1) \to C_c^\infty(\mathcal{G}_2)$ such that $f_{TR} = \iota(f)^*$.

The following theorem, a generalization of [7], Theorem 5.7. (2), is based on the fact that all isomorphisms in the proof of that theorem are functorial with respect to étale morphisms. It is the reason why we use continuous periodic cyclic cohomology when working with groupoid algebras. Note that the cyclic object associated to $C_c^\infty(\mathcal{G})$, for G an étale groupoid, is an inductive limit of locally convex nuclear spaces.

THEOREM. 3.1. *If $\mathcal{G}$ is a Hausdorff étale groupoid of dimension n, and if $\mathfrak{o}$ is the complexified orientation sheaf of $\mathrm{B}\,\mathcal{G}$, then there exists a natural embedding $\Phi : \mathrm{H}^{*+n}(\mathrm{B}\,\mathcal{G},\mathfrak{o}) \hookrightarrow \mathrm{HP}^*_{\mathrm{cont}}(C_c^\infty(\mathcal{G}))$. Here "natural" means that if $f : \mathcal{G}_1 \to \mathcal{G}_2$ is an étale morphism of groupoids, then the diagram*

$$\begin{tikzcd} \mathrm{H}^{*+n}(\mathrm{B}\,\mathcal{G}_2,\mathfrak{o}_2) \arrow[r] \arrow[d, "(\mathrm{B}\,f)^*"'] & \mathrm{HP}^*_{\mathrm{cont}}(C_c^\infty(\mathcal{G}_2)) \arrow[d, "f_{Tr}"] \\ \mathrm{H}^{*+n}(\mathrm{B}\,\mathcal{G}_1,\mathfrak{o}_1) \arrow[r] & \mathrm{HP}^*_{\mathrm{cont}}(C_c^\infty(\mathcal{G}_1)), \end{tikzcd}$$

whose horizontal lines are the morphisms Φ, commutes.

For discrete groups, Theorem 3.1 recovers the embedding

$$\mathrm{H}^*(\Gamma) = \mathrm{H}^*(\mathrm{B}\,\Gamma,\mathbb{C}) \hookrightarrow \mathrm{HP}^*_{\mathrm{cont}}(\mathbb{C}[\Gamma])$$

of [8, 20].

For smooth manifolds, the embedding Φ of Theorem 3.1 is just the Poincaré duality–an isomorphism. This isomorphism has a very concrete form. Indeed, let $\xi \in \mathrm{H}^{n-i}(M,\mathfrak{o})$ be an element of the singular cohomology of M with coefficients in the orientation sheaf, let $\eta \in \mathrm{H}^i_c(M)$ be an element of the singular cohomology of M with compact supports (all cohomology groups have complex coefficients), and let

$$\chi : \mathrm{HP}_i^{\mathrm{cont}}(C_c^\infty(M)) \simeq \oplus_k \mathrm{H}^{i+2k}_{c,DR}(M) = \oplus_k \mathrm{H}^{i+2k}_c(M)$$

be the canonical isomorphism induced by the Hochschild-Kostant-Rosenberg map χ, equation (12). Then the isomorphism Φ is determined by

$$\langle \Phi(\xi),\eta\rangle = \langle \xi \wedge \chi(\eta),[M]\rangle \in \mathbb{C}, \tag{37}$$

where the first pairing is the map $\mathrm{HP}^*_{\mathrm{cont}}(C_c^\infty(M)) \otimes \mathrm{HP}_*^{\mathrm{cont}}(C_c^\infty(M)) \to \mathbb{C}$ and the second pairing is the evaluation on the fundamental class.

Typically, we shall use these results for the manifold S^*M, for which there is an isomorphism $\mathrm{H}^{*-1}(S^*M) \simeq \mathrm{HP}^*_{\mathrm{cont}}(C^\infty(S^*M))$, because S^*M is oriented. (The orientation of S^*M is the one induced from that of T^*M as in [5]. More precisely B^*M, the disk bundle of M, is given the orientation in which the "the horizontal part is real and the vertical part is imaginary," and S^*M is oriented as the boundary of an oriented manifold.) The shift in the $\mathbb{Z}_2$-degree is due to the fact that S^*M is odd dimensional.

3.2. MORITA INVARIANCE AND COVERINGS. Let M be a smooth compact manifold and $q : \widetilde{M} \to M$ be a covering with Galois group Γ; said differently, $\widetilde{M}$ is a principal Γ–bundle over M. We fix a finite cover $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ of M by trivializing open sets, i.e., $q^{-1}(U_\alpha) \simeq U_\alpha \times \Gamma$ and $M = \cup U_\alpha$. The transition functions between two trivializing isomorphisms on their common domain, the open set $U_\alpha \cap U_\beta$, defines a 1–cocycle $\gamma_{\alpha\beta}$ that completely determines the covering $q : \widetilde{M} \to M$.

In what follows, we shall need to lift the covering $q : \widetilde{M} \to M$ to a covering $q : S^*\widetilde{M} \to S^*M$, using the canonical projection $p : S^*M \to M$. All constructions then lift, from M to S^*M, canonically. In particular, $V_\alpha = p^{-1}(U_\alpha)$ is a finite covering of S^*M with trivializing open sets, and the associated 1–cocycle is (still) $\gamma_{\alpha\beta}$. Moreover, if $f_0 : M \to \mathrm{B}\,\Gamma$ classifies the covering $q : \widetilde{M} \to M$, then $f = f_0 \circ p$ classifies the covering $S^*\widetilde{M} \to S^*M$.

Suppose that the trivializing cover $\mathcal{V} = (V_\alpha)_{\alpha\in I}$ of S^*M consists of k open sets, and let $\sum \varphi_\alpha^2 = 1$ be a partition of unity subordinated to $\mathcal{V}$. The cocycle identity $\gamma_{\alpha\beta}\gamma_{\beta\delta} = \gamma_{\alpha\delta}$ ensures then that the matrix

$$p = [\varphi_\alpha \gamma_{\alpha\beta} \varphi_\beta]_{\alpha,\beta\in I} \in M_k(C^\infty(M)) \otimes \mathbb{C}[\Gamma] \tag{38}$$

is an idempotent, called *the Mishchenko idempotent*; a different choice of a trivializing cover and of a partition of unity gives an equivalent idempotent.

Using the Mishchenko idempotent p, we now define the morphism

$$\lambda : C^\infty(S^*M) \to M_k(C^\infty(S^*M)) \otimes \mathbb{C}[\Gamma]$$

by $\lambda(a) = ap$, for $a \in C^\infty(S^*M)$; explicitly,

$$\lambda(a)(x) = a(x)p(x) = [a(x)\varphi_\alpha(x)\varphi_\beta(x) \otimes \gamma_{\alpha\beta}]. \tag{39}$$

Because the morphism λ is used to define the Connes–Moscovici extension, equation (45) below, we need to identify the induced morphism

$$\lambda^* : \mathrm{HP}^*_{\mathrm{cont}}(C^\infty(S^*M) \otimes \mathbb{C}[\Gamma]) \to \mathrm{HP}^*_{\mathrm{cont}}(C^\infty(S^*M)).$$

The identification of λ, Proposition 3.3, is based on writing λ as a composition of three simpler morphisms, morphisms that will play an auxiliary role. The next few paragraphs before Proposition 3.3 will deal with the definition and properties of these morphisms.

We define the first auxiliary morphism $\iota(g)$ to be induced by an étale morphism of groupoids. Let $\mathcal{G}_\mathcal{V}$ be the gluing groupoid associated to the cover $\mathcal{V} = (V_\alpha)_{\alpha\in I}$ of S^*M. Using the cocycle $(\gamma_{\alpha\beta})_{\alpha,\beta\in I}$ associated to $\mathcal{V}$ that identifies the covering $S^*\widetilde{M} \to S^*M$, we define the étale morphism of groupoids g by

$$\mathcal{G}_\mathcal{V} \ni (x, \alpha, \beta) \xrightarrow{g} (x, \alpha, \beta, \gamma_{\alpha\beta}) \in \mathcal{G}_\mathcal{V} \times \Gamma,$$

which induces a morphism $\iota(g) : C_c^\infty(\mathcal{G}_\mathcal{V}) \to C_c^\infty(\mathcal{G}_\mathcal{V}) \otimes \mathbb{C}[\Gamma]$ and a continuous map $\mathrm{B}\, g : \mathrm{B}\,\mathcal{G}_\mathcal{V} \to \mathrm{B}(\mathcal{G}_\mathcal{V} \times \Gamma) = \mathrm{B}\,\mathcal{G}_\mathcal{V} \times \mathrm{B}\,\Gamma$.

The projection $t : \mathcal{G}_\mathcal{V} \to S^*M$ is an etale morphism of groupoids that induces a homotopy equivalence $\mathrm{B}\,\mathcal{G}_\mathcal{V} \to S^*M$ and hence also an isomorphism

$$t_{Tr} : \mathrm{HP}^*_{\mathrm{cont}}(C^\infty(S^*M)) \to \mathrm{HP}^*_{\mathrm{cont}}(C_c^\infty(\mathcal{G}_\mathcal{V})).$$

By definition, $t_{Tr} = Tr \circ \iota(l)^*$, where $l : \mathcal{G}_\mathcal{V} \to S^*M \times R_I$ is the natural inclusion considered also before, and Tr is the generic notation for the isomorphisms $Tr : \mathrm{HP}_*(M_n(A)) \simeq \mathrm{HP}_*(A)$, induced by the trace. In particular, $\iota(l)^*$ is also an isomorphism.

Using the homotopy equivalence $\mathrm{B}\, t$ of $\mathrm{B}\,\mathcal{G}_\mathcal{V}$ and S^*M, we obtain a continuous map

$$h_0 : S^*M \to S^*M \times \mathrm{B}\,\Gamma,$$

uniquely determined by the condition $h_0 \circ \mathrm{B}\, t = (\mathrm{B}\, t \times id) \circ \mathrm{B}\, g$.

Lemma. 3.2. *The map h_0 defined above coincides, up to homotopy, with the product function (id_{S^*M}, f), where $f : S^*M \to \mathrm{B}\,\Gamma$ classifies $S^*\widetilde{M} \to S^*M$.*

Proof. Denote by p_1 and p_2 the projections of $S^*M \times \mathrm{B}\,\Gamma$ onto components. The map $p_1 \circ h_0$ is easily seen to be the identity, so $h_0 = \mathrm{id}_{S*M} \times h_1$ where $h_1 : S^*M \to \mathrm{B}\,\Gamma$ is induced by the *non*-étale morphism of topological groupoids $\mathcal{G}_\mathcal{V} \ni (x, \alpha, \beta) \to \gamma_{\alpha\beta} \in \Gamma$. In order to show that h_1 coincides with f, up to homotopy, it is enough to show that the principal Γ-bundle (i.e., covering) that h_1 pulls back from $\mathrm{B}\,\Gamma$ to S^*M is isomorphic to the covering $S^*\widetilde{M} \to \widetilde{M}$.

Let $\mathcal{G}_{\mathcal{U}}$ be the gluing groupoid associated to the cover $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of M. It is seen from the definition that $\mathcal{G}_{\mathcal{V}} \to \Gamma$ factors as $\mathcal{G}_{\mathcal{V}} \to \mathcal{G}_{\mathcal{U}} \to \Gamma$, where the function $\mathcal{G}_{\mathcal{V}} \to \Gamma$ acts as $(m, \alpha, \beta) \to \gamma_{\alpha\beta}$. Thus we may replace S^*M by M everywhere in the proof.

Since the the covering $\widetilde{M} \to M$ is determined by its restriction to loops, we may assume that M is the circle S^1. Cover $M = S^1$ by two contractable intervals $I_0 \cap I_1$ which intersect in two small disjoint neighborhoods of 1 and -1: $I_0 \cap I_1 = (z, z^{-1}) \cup (-z, -z^{-1})$ where $z \in S'$ and $|z-1|$ is very small. We may also assume that the transition cocycle is the identity on (z, z^{-1}) and $\gamma \in \Gamma$ on $(-z, -z^{-1})$ (we have replaced constant Γ–cocycles with locally constant Γ–cocycles). The map h_1 maps each of the units of $\mathcal{G}_{\mathcal{U}}$ and each of the 1-cells corresponding to the right hand interval (z, z^{-1}) to the only 0-cell of $\mathrm{B}\,\Gamma$, the cell corresponding to the identity $e \in \Gamma$. (Recall that the classifying space of a topological groupoid is the geometrical realization of the simplicial space of composable arrows [34], and that that there is a 0 cell for each unit, a 1-cell for each non-identity arrow, a 2-cell for each pair of non-identity composable arrows, and so on). The other 1-cells (i.e., corresponding to the arrows leaving from a point on the left hand side interval) will map to the 1-cell corresponding γ. This shows that, on homotopy groups, the induced map $\mathbb{Z} = \pi_1(S^1) \to \Gamma = \pi_1(B\Gamma)$ sends the generator 1 to γ. This completes the proof of the lemma. $\square$

We need to introduce one more auxiliary morphism before we can determine λ^*. Using the partition of unity $\sum_\alpha \varphi_\alpha^2 = 1$ subordinated to $\mathcal{V} = (V_\alpha)_{\alpha \in I}$, we define $\nu : C^\infty(S^*M) \to C_c^\infty(\mathcal{G}_{\mathcal{V}})$ by

$$\nu(f)(x, \alpha, \beta) = f(x)\varphi_\alpha(x)\varphi_\beta(x),$$

which turns out to be a morphism of algebras. Because the composition

$$C^\infty(S^*M) \xrightarrow{\nu} C_c^\infty(\mathcal{G}_{\mathcal{V}}) \xrightarrow{\iota(l)} C_c^\infty(S^*M \times R_I) = M_k(C^\infty(S^*M))$$

is (unitarily equivalent to) the upper–left corner embedding, we obtain that the morphism $\nu^* : \mathrm{HP}^*_{\mathrm{cont}}(C_c^\infty(\mathcal{G}_{\mathcal{V}})) \to \mathrm{HP}^*_{\mathrm{cont}}(C^\infty(S^*M))$ is the inverse of t_{Tr}.

We are now ready to determine the morphism

$$\lambda^* : \mathrm{HP}^*_{\mathrm{cont}}(C^\infty(S^*M) \otimes \mathbb{C}[\Gamma]) \to \mathrm{HP}^*_{\mathrm{cont}}(C^\infty(S^*M)).$$

In order to simplify notation, in the statement of the following result we shall identify $\mathrm{HP}^*_{\mathrm{cont}}(M_k(C^\infty(S^*M)) \otimes \mathbb{C}[\Gamma])$ with $\mathrm{HP}^*_{\mathrm{cont}}(C^\infty(S^*M) \otimes \mathbb{C}[\Gamma])$, and we shall do the same in the proof.

Proposition. 3.3. *The composition*

$$\mathrm{H}^{*-1}(S^*M \times \mathrm{B}\,\Gamma; \mathbb{C}) \hookrightarrow \mathrm{HP}^*_{\mathrm{cont}}(C^\infty(S^*M) \otimes \mathbb{C}[\Gamma]) \xrightarrow{\lambda^*}$$
$$\to \mathrm{HP}^*_{\mathrm{cont}}(C^\infty(S^*M)) \simeq \mathrm{H}^{*-1}(S^*M; \mathbb{C})$$

is $\Phi^{-1} \circ \lambda^* \circ \Phi = (id \times f)^*$.

Proof. Consider as before the morphism $l : \mathcal{G}_{\mathcal{V}} \to S^*M \times R_I$ of groupoids, which defines an injective morphism of algebras $\iota(l) : C^\infty(\mathcal{G}_{\mathcal{V}}) \to C^\infty(S^*M \times R_I) = M_k(C^\infty(S^*M))$, and hence also a morphism

$$\iota(l) \otimes id = \iota(l \times id) : C^\infty(\mathcal{G}_{\mathcal{V}} \times \Gamma) \hookrightarrow C^\infty(S^*M \times R_I \times \Gamma) = M_k(C^\infty(S^*M)) \otimes \mathbb{C}[\Gamma].$$

Then we can write
$$\lambda = \iota(l \times id) \circ \iota(g) \circ \nu,$$
where $g : \mathcal{G}_{\mathcal{V}} \to \mathcal{G}_{\mathcal{V}} \times \Gamma$ is as defined before: $g(x, \alpha, \beta) = (x, \alpha, \beta, \gamma_{\alpha\beta})$.

Because $\nu^* = (t_{Tr})^{-1}$, we have that $\Phi^{-1} \circ \nu^* \circ \Phi = (\mathrm{B}\, t)^{*-1}$, by Theorem 3.1. Also by Theorem 3.1, we have $\iota(g)^* \circ \Phi = \Phi \circ (\mathrm{B}\, g)^*$ and $\iota(l \times id)^* \circ \Phi = \Phi \circ (\mathrm{B}\, l \times id)^*$. This gives then

$$\Phi^{-1} \circ \lambda^* \circ \Phi = \Phi^{-1} \circ \nu^* \circ \Phi \circ (\mathrm{B}\, g)^* \circ (\mathrm{B}\, l \times id)^* = (\mathrm{B}\, t)^{*-1} \circ \Phi \circ (\mathrm{B}\, g)^* \circ (\mathrm{B}\, l \times id)^* = h_0^*.$$

Since Lemma 3.2 states that $h_0 = id \times f$, up to homotopy, the proof is complete. $\square$

3.3. THE ATIYAH–SINGER EXACT SEQUENCE. Let M be a smooth compact manifold (without boundary). We shall denote by $\Psi^k(M)$ the space of classical, order at most k pseudodifferential operators on M. Fix a smooth, nowhere vanishing density on M. Then $\Psi^0(M)$ acts on $L^2(M)$ by bounded operators and, if an operator $T \in \Psi^0(M)$ is compact, then it is of order -1. More precisely, it is known that order -1 pseudodifferential operators satisfy $\Psi^{-1}(M) \subset \mathcal{C}_p = \mathcal{C}_p(L^2(M))$ for any $p > n$. (Recall that $C_p(\mathcal{H})$ is the ideal of p–summable operators on $\mathcal{H}$, equation (8)).

It will be convenient to include all $(n+1)$–summable operators in our calculus, so we let $E_{AS} = \Psi^0(M) + \mathcal{C}_{n+1}$, and obtain in this way an extension of algebras,

$$0 \to \mathcal{C}_{n+1} \to E_{AS} \xrightarrow{\sigma_0} C^\infty(S^*M) \to 0, \tag{40}$$

called *the Atiyah-Singer exact sequence*. The boundary morphisms in periodic cyclic cohomology associated to the Atiyah-Singer exact sequence defines a morphism

$$\partial_{AS} : \mathrm{HP}^*(\mathcal{C}_{n+1}) \to \mathrm{HP}^{*+1}(C^\infty(S^*M)).$$

Let $Tr_n \in \mathrm{HP}^0_{\mathrm{cont}}(\mathcal{C}_{n+1})$ be as in (28) (i.e., $Tr_n(a_0, \dots, a_{2n}) = CTr(a_0 \dots a_{2n})$, for some constant C), and denote

$$\mathcal{J}(M) = \partial_{AS}(Tr_n) \in \mathrm{HP}^1_{\mathrm{cont}}(C^\infty(S^*M)) \subset \mathrm{HP}^1(C^\infty(S^*M)), \tag{41}$$

which is justified by Corollary 2.11.

We shall determine $\mathcal{J}(M)$ using Theorem 1.5. In order to do this, we need to make explicit the relation between ch, the Chern character in cyclic homology, and Ch, the classical Chern character as defined, for example, in [27]. Let $E \to M$ be a smooth complex vector bundle, embedded in a trivial bundle: $E \subset M \times \mathbb{C}^N$, and let $e \in M_N(C^\infty(M))$ be the orthogonal projection on E. If we endow E with the connection $ed_{DR}e$, acting on $\Gamma^\infty(E) \subset C^\infty(M)^N$, then the curvature Ω of this connection turns out to be $\Omega = e(d_{DR}e)^2$. The classical Chern character $Ch(E)$ is then the cohomology class of the form $Tr(\exp(\frac{\Omega}{2\pi\imath}))$ in the even (de Rham) cohomology of M. Comparing this definition with the definition of the Chern character in cyclic cohomology via the Hochschild-Kostant-Rosenberg map, we see that the two of them are equal–up to a renormalization with a factor of $2\pi\imath$. (If $\xi \in \mathrm{H}^*(M) = \oplus_k \mathrm{H}^k(M)$ is a cohomology class, we denote by ξ_k its component in $\mathrm{H}^k(M)$.) Explicitly, let $\chi : \mathrm{HP}_i^{\mathrm{cont}}(C_c^\infty(S^*M)) \simeq \oplus_{k\in\mathbb{Z}} \mathrm{H}^{i+2k}(S^*M)$ be the canonical isomorphism induced by the Hochschild-Kostant-Rosenberg map χ, equation (12), then

$$\chi(ch(\xi)) = \sum_{k\in\mathbb{Z}} (2\pi\imath)^m Ch(\xi)_{2m-i} \in \mathrm{H}^{2m-i}(M) \tag{42}$$

for $i \in \{0, 1\}$ and $\xi \in \mathrm{K}_i^{\mathrm{alg}}(C^\infty(M))$. (Note the '$-i$').

Proposition. 3.4. *Let $\mathcal{T}(M) \in \mathrm{H}^{\mathrm{even}}(S^*M)$ be the Todd class of the complexification of T^*M, lifted to S^*M, and $\Phi : \mathrm{H}^{\mathrm{even}}(S^*M) \to \mathrm{HP}^1_{\mathrm{cont}}(C^\infty(S^*M))$ be the isomorphism of Theorem 3.1. Then*

$$\mathcal{J}(M) = (-1)^n \sum_k (2\pi\imath)^{n-k} \Phi(\mathcal{T}(M)_{2k}) \in \mathrm{HP}^1_{\mathrm{cont}}(C^\infty(S^*M)).$$

Proof. We need to verify the equality of two classes in $\mathrm{HP}^1_{\mathrm{cont}}(C^\infty(S^*M))$. It is hence enough to check that their pairings with $ch([u])$ are equal, for any $[u] \in \mathrm{K}_1^{\mathrm{alg}}(C^\infty(S^*M))$, because of the classical result that the Chern character

$$ch : \mathrm{K}_1^{\mathrm{alg}}(C^\infty(S^*M)) \to \mathrm{HP}_1^{\mathrm{cont}}(C^\infty(S^*M))$$

is onto.

If Ind is the index morphism of the Atiyah–Singer exact sequence then the Atiyah-Singer index formula [5] states the equality

$$\mathrm{Ind}[u] = (-1)^n \langle Ch[u], \mathcal{T}(M) \rangle. \tag{43}$$

Using equation (41) and Theorem 1.5 (see also the discussion following that theorem), we obtain that $\mathrm{Ind}[u] = \langle ch[u], \mathcal{J}(M) \rangle$. Equations (37) and (43) then complete the proof. □

3.4. The Connes–Moscovici exact sequence and proof of the theorem. We now extend the constructions leading to the Atiyah–Singer exact sequence, equation (40), to covering spaces.

Let M be a smooth compact manifold and let $E_1 = M_k(E) \otimes \mathbb{C}[\Gamma]$, which fits into the exact sequence

$$0 \longrightarrow M_k(\mathcal{C}_{n+1}) \otimes \mathbb{C}[\Gamma] \longrightarrow E_1 \xrightarrow{\sigma_0} M_k(C^\infty(S^*M)) \otimes \mathbb{C}[\Gamma] \longrightarrow 0. \tag{44}$$

Let $\Gamma \to \widetilde{M} \to M$ be a covering of M with Galois group Γ. Using the Mishchenko idempotent p associated to this covering and the injective morphism

$$\lambda : C^\infty(S^*M) \to p(M_k(C^\infty(S^*M)) \otimes \mathbb{C}[\Gamma])p,$$

equation 39, we define the Connes–Moscovici algebra E_{CM} as the fibered product

$$E_{CM} = \{(T, a) \in pE_1p \oplus C^\infty(S^*M), \sigma_0(T) = \lambda(a)\}.$$

By definition, the algebra E_{CM} fits into the exact sequence

$$0 \longrightarrow p\big(M_k(\mathcal{C}_{n+1}) \otimes \mathbb{C}[\Gamma]\big)p \longrightarrow E_{CM} \longrightarrow C^\infty(S^*M) \longrightarrow 0.$$

We now take a closer look at the algebra E_{CM} and the exact sequence it defines. Observe first that p acts on $(L^2(M) \otimes l^2(\Gamma))^k$ and that $p(L^2(M) \otimes l^2(\Gamma))^k \simeq L^2(\widetilde{M})$ via a Γ–invariant isometry. Since E_1 can be regarded as an algebra of operators on $(L^2(M) \otimes l^2(\Gamma))^k$ that commute with the (right) action of Γ, we obtain that E_{CM} can also be interpreted as an algebra of operators commuting with the action of Γ on $L^2(\widetilde{M})$. Using also [11], Lemma 5.1, page 376, this recovers the usual description of E_{CM} that uses properly supported Γ–invariant pseudodifferential operators on $\widetilde{M}$.

Also observe that "M_k" is superfluous in $M_k(\mathcal{C}_{n+1})$ because $M_k(\mathcal{C}_{n+1}) \simeq \mathcal{C}_{n+1}$; actually, even "$p$" is superfluous in $p\big(M_k(\mathcal{C}_{n+1}) \otimes \mathbb{C}[\Gamma]\big)p$ because

$$p\big(M_k(\mathcal{C}_{n+1}) \otimes \mathbb{C}[\Gamma]\big)p \simeq \mathcal{C}_{n+1} \otimes \mathbb{C}[\Gamma]$$

by an isomorphism that is uniquely determined up to an inner automorphism. Thus the Connes–Moscovici extension becomes

$$0 \longrightarrow \mathcal{C}_{n+1} \otimes \mathbb{C}[\Gamma] \longrightarrow E_{CM} \longrightarrow C^\infty(S^*M)) \longrightarrow 0, \tag{45}$$

up to an inner automorphism.

We now proceed as for the Atiyah–Singer exact sequence. The boundary morphisms in periodic cyclic cohomology associated to the Connes–Moscovici extensions defines a map

$$\partial_{CM} : \mathrm{HP}^*(\mathcal{C}_{n+1} \otimes \mathbb{C}[\Gamma]) \to \mathrm{HP}^{*+1}(C^\infty(S^*M)),$$

and the Connes–Moscovici Index Theorem amounts to the identification of the classes

$$\partial_{CM}(Tr_n \otimes \xi) \in \mathrm{HP}^{*+1}_{\mathrm{cont}}(C^\infty(S^*M)) \subset \mathrm{HP}^{*+1}(C^\infty(S^*M)),$$

for cocycles ξ coming from the cohomology of Γ.

In order to determine $\partial_{CM}(Tr_n \otimes \xi)$, we need the following theorem.

THEOREM. 3.5. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be smooth étale groupoids. Then the diagram*

$$\begin{tikzcd}
\mathrm{H}^{*+n}(\mathrm{B}\,\mathcal{G}_1, \mathfrak{o}_1) \otimes \mathrm{H}^{*+m}(\mathrm{B}\,\mathcal{G}_2, \mathfrak{o}_2) \arrow[r, "\Phi"] \arrow[d, "\times"'] & \mathrm{HP}^*_{\mathrm{cont}}(C_c^\infty(\mathcal{G}_2)) \otimes \mathrm{HP}^*_{\mathrm{cont}}(C_c^\infty(\mathcal{G}_2)) \arrow[d, "\otimes"] \\
\mathrm{H}^{*+n+m}(\mathrm{B}(\mathcal{G}_1 \times \mathcal{G}_2), \mathfrak{o}) \arrow[r, "\Phi"] & \mathrm{HP}^*_{\mathrm{cont}}(C_c^\infty(\mathcal{G}_1 \times \mathcal{G}_2))
\end{tikzcd}$$

is commutative. Here the left product $\times$ is the external product in cohomology and $\mathfrak{o}_1$, $\mathfrak{o}_2$, and $\mathfrak{o}$ are the orientation sheaves.

Proof. The proof is a long but straightforward verification that the sequence of isomorphisms in [7] is compatible with products.

Using [30], Proposition 1.5. (c), page 563, which states that the $\times$-products are compatible with the tensor products of mixed complexes, we replace everywhere cyclic vector spaces by mixed complexes. Then we go through the specific steps of the proof as in [7]. This amounts to verify the following facts:

(i) The Hochschild-Kostant-Rosenberg map χ (equation (12)) transforms the differential $B \otimes 1 + 1 \otimes B$ into the de Rham differential of the product.

(ii) By the Eilenberg-Zilber Theorem [25], the augmentation map ϵ ([7] Proposition 4.2 (1)), and the isomorphism it induces, are compatible with products.

(iii) The chain map f in the Moore isomorphism (see [6], Theorems 4.1 and 4.2, page 32) is compatible with products. This too involves the Eilenberg-Zilber theorem. □

We remark that the proof of the above theorem is easier if both groupoids are of the same "type," i.e., if they are both groups or smooth manifolds, in which case our theorem is part of folklore. However, in the case we shall use this theorem–that of a group and a manifold–there are no significant simplifications: one has to go through all the steps of the proof given above.

LEMMA. 3.6. *Let $\lambda : C^\infty(S^*M) \to M_k(C^\infty(S^*M)) \otimes \mathbb{C}[\Gamma]$ be as defined in (39) and $Tr_n \in \mathrm{HP}^0(\mathcal{C}_{n+1})$ be as in (28). Then, for any cyclic cocycle $\eta \in \mathrm{HP}^*_{\mathrm{cont}}(\mathbb{C}[\Gamma])$, we have*

$$\partial_{CM}(Tr_n \otimes \eta) = \lambda^*(\mathcal{J}(M) \otimes \eta) \in \mathrm{HP}^{*+1}_{\mathrm{cont}}(C^\infty(S^*M)) \subset \mathrm{HP}^{*+1}(C^\infty(S^*M)).$$

Proof. Denote by $\partial_1 : \mathrm{HP}^*_{\mathrm{cont}}(\mathcal{C}_{n+1}\otimes\mathbb{C}[\Gamma]) \to \mathrm{HP}^{*+1}(C^\infty(S^*M\otimes\mathbb{C}[\Gamma]))$ the boundary morphism of the exact sequence (44). Using Theorem 2.6, we obtain

$$\partial_1(Tr_n\otimes\eta) = \partial_{AS}(Tr_n)\otimes\eta = \mathcal{J}(M)\otimes\eta \in \mathrm{HP}^{*+1}_{\mathrm{cont}}(C^\infty(S^*M)\otimes\mathbb{C}[\Gamma]) \subset \mathrm{HP}^{*+1}(C^\infty(S^*M)\otimes\mathbb{C}[\Gamma]).$$

Then, the naturality of the boundary map and Theorem 2.10 show that $\partial_{CM} = \lambda^*\circ\partial_1$. This completes the proof. □

Let $\mathcal{T}(M) \in \mathrm{H}^{\mathrm{even}}(S^*M)$ be the Todd class of $TM\otimes\mathbb{C}$ lifted to S^*M and Ch be the classical Chern character on K–Theory, as before. Also, recall that Theorem 3.1 defines an embedding $\Phi : \mathrm{H}^*(\mathrm{B}\,\Gamma) = \mathrm{H}^*(\Gamma) \to \mathrm{HP}^*_{\mathrm{cont}}(\mathbb{C}[\Gamma]) = \mathrm{HP}^*(\mathbb{C}[\Gamma])$.

We are now ready to state Connes–Moscovici's Index Theorem for elliptic systems, see [11][Theorem 5.4], page 379, which computes the "higher index" of a matrix of P of properly supported, order zero, Γ-invariant elliptic pseudodifferential operators on $\widetilde{M}$, with principal symbol the invertible matrix $u = \sigma_0(P) \in M_m(C^\infty(S^*M))$.

THEOREM. 3.7 (Connes–Moscovici). *Let $\widetilde{M} \to M$ be a covering with Galois group Γ of a smooth compact manifold M of dimension n, and let $f : S^*M \to \mathrm{B}\,\Gamma$ the continuous map that classifies the covering $S^*\widetilde{M} \to S^*M$. Then, for each cohomology class $\xi \in \mathrm{H}^{2q}(\mathrm{B}\,\Gamma)$ and each $[u] \in K^1(S^*M)$, we have*

$$\tilde{\xi}_*(\mathrm{Ind}[u]) = \frac{(-1)^n}{(2\pi\imath)^q}\langle\, Ch(u)\wedge\mathcal{T}(M)\wedge f^*\xi, [S^*M]\,\rangle,$$

where $\tilde{\xi} = Tr_n\otimes\Phi(\xi) \in \mathrm{HP}^0(\mathcal{C}_{n+1}\otimes\mathbb{C}[\Gamma])$.

Proof. All ingredients of the proof are in place, and we just need to put them together. Let $\xi \in \mathrm{H}^{2q}(\mathrm{B}\,\Gamma)$ and $\tilde{\xi} = Tr_n\otimes\Phi(\xi)$ be as in the statement of the theorem. Then

$$\begin{aligned}
&(-1)^n\tilde{\xi}_*(\mathrm{Ind}[u]) = \\
&= (-1)^n\big(\partial_{CM}\tilde{\xi}\big)_*[u] && \text{by Theorem 1.5}\\
&= (-1)^n\big(\lambda^*(\mathcal{J}(M)\otimes\Phi(\xi))\big)_*[u] && \text{by Lemma 3.6}\\
&= (-1)^n\big(\lambda^*\circ\Phi(\Phi^{-1}(\mathcal{J}(M))\times\xi)\big)_*[u] && \text{by Theorem 3.5}\\
&= (-1)^n\big(\Phi\circ(id\times f)^*(\Phi^{-1}(\mathcal{J}(M))\times\xi)\big)_*[u] && \text{by Proposition 3.3}\\
&= (-1)^n\langle\Phi(\Phi^{-1}(\mathcal{J}(M))\wedge f^*\xi), ch([u])\rangle\\
&= (-1)^n\langle\,\Phi^{-1}(\mathcal{J}(M))\wedge f^*\xi)\wedge\chi(ch[u]), [S^*M]\,\rangle && \text{by equation (37)}\\
&= \sum_{k+j=n-q}(2\pi\imath)^{k-n}\langle\mathcal{T}(M)_{2k}\wedge f^*\xi\wedge\chi(ch[u])_{2j-1}, [S^*M]\rangle && \text{by Proposition 3.4}\\
&= \sum_{k+j=n-q}(2\pi\imath)^{-q}\langle\mathcal{T}(M)_{2k}\wedge f^*\xi\wedge Ch_{2j-1}[u], [S^*M]\rangle && \text{by equation (42)}\\
&= (2\pi\imath)^{-q}\langle Ch[u]\wedge\mathcal{T}(M)\wedge f^*\xi, [S^*M]\rangle.
\end{aligned}$$

The proof is now complete. □

For $q = 0$ and $\xi = 1 \in \mathrm{H}^0(\mathrm{B}\,\Gamma) \simeq \mathbb{C}$, we obtain that $\tau = \Phi(\xi)$ is the von Neumann trace on $\mathbb{C}[\Gamma]$, that is $\tau(\sum a_\gamma\gamma) = a_e$, the coefficient of the identity, and the above theorem recovers Atiyah's L^2–index theorem for coverings [2]. The reason for obtaining a different constant than in [11] is due to different normalizations. See [19] for a discussion on how to obtain the usual index theorems from the index theorems for elliptic systems.

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