Invariant Inner Product in Spaces of Holomorphic Functions on Bounded Symmetric Domains

Jonathan Arazy and Harald Upmeier[1]

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Abstract. We provide new integral formulas for the invariant inner product on spaces of holomorphic functions on bounded symmetric domains of tube type.

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0 Introduction

Our main concern in this work is to provide *concrete formulas for the invariant inner products and hermitian forms on spaces of holomorphic functions on Cartan domains D of tube type*. As will be explained below, the group $Aut(D)$ of all holomorphic automorphisms of D acts transitively. $Aut(D)$ acts projectively on function spaces on D via $f \mapsto U^{(\lambda)}(\varphi)f := (f \circ \varphi)\,(J\varphi)^{\lambda/p},\ \ \varphi \in Aut(D),\ \lambda \in \mathbf{C}$, but these actions are not irreducible in general. The inner products we consider are those obtained from the holomorphic discrete series by analytic continuation. The associated Hilbert spaces generalize the weighted Bergman spaces, the Hardy and the Dirichlet space. By "concrete" formulas we mean Besov-type formulas, namely integral formulas involving the functions and some of their derivatives. Possible applications include the study of operators (Toeplitz, Hankel) acting on function spaces and the theory of invariant Banach spaces of analytic functions (where the pairing between an invariant space and its invariant dual is computed via the corresponding invariant inner product).

Our problem is closely related to finding concrete realizations (by means of integral formulas) of the analytic continuation of the *Riesz distribution*. [Ri], [Go], [FK2], Chapter VII.

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In principle, the analytic continuation is obtained from the integral formulas associated with the weighted Bergman spaces (i.e. the holomorphic discrete series) by "partial integration with respect to the radial variables". This program has been successful in the case of rank 1 (i.e. when D is the open unit ball of $\mathbf{C}^d$, see [A3]). The case of rank $r > 1$ is more difficult, and concrete formulas are known only in special cases, see [A2], [Y4], [Y1], [Y2].

This paper consists of two main parts. In the first part (Sections 2, 3, and 4) we develop in full generality the techniques of [A2], [Y4], and obtain integral formulas for the invariant inner products associated with the so-called Wallach set and pole set. In the second part (section 5) we introduce new techniques (integration on boundary orbits), to obtain new integral formulas for the invariant inner products in the important special cases of Cartan domains of type I and IV. This approach has the potential for further generalizations and applications, including the infinite dimensional setup.

The paper is organized as follows. Section 1 provides background information on Cartan domains, the associated symmetric cones and Siegel domains and the Jordan theoretic approach to the study of bounded symmetric domains [Lo], [FK2], [U2]. We also explain some general facts concerning invariant Hilbert spaces of analytic functions on Cartan domains and the connection to the Riesz distribution. Section 2 is devoted to the study of invariant differential operators on symmetric cones. We study the "shifting operators" introduced by Z. Yan and their derivatives with respect to the "spectral parameter". Section 3 is devoted to our generalization of Yan's shifting method, to find explicit integral formulas for the invariant inner products obtained by analytic continuation of the holomorphic discrete series. In section 4 we study the expansion of Yan's operators, and obtain applications to concrete integral formulas for the invariant inner products. Some of these results were obtained independently by Z. Yan, J. Faraut and A. Korányi, [FK2], [Y4]. We include these results and our proofs, in order to make the paper self contained, and also because in most cases our results go beyond the results in [FK2], [Y4].

In section 5 we propose a new type of integral formulas for the invariant inner products. These formulas involve integration on boundary orbits and the application of the localized versions of the radial derivative associated with the boundary components of Cartan domains. We are able to establish the desired formulas in the important special cases of type I and IV. The techniques established in this section can be used in the study of the remaining cases.

Finally, in the short section 6 we use the quasi-invariant measures on the boundary orbits of the associated symmetric cone in order to obtain integral formulas for some of the invariant inner products in the context of the *unbounded* realization of the Cartan domains (tube domains). These results are essentially implicitly contained in [VR], where the authors use the Lie-theoretic and Fourier-analytic approach to analysis on symmetric Siegel domains. We use the Jordan-theoretic approach which yields simpler formulation of the results and simpler proofs.

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1 Preliminaries

A *Cartan domain* $D \subset \mathbf{C}^d$ is an irreducible bounded symmetric domain in its Harish-Chandra realization. Thus D is the open unit ball of a Banach space $Z = (\mathbf{C}^d, \|\cdot\|)$ which admits the structure of a *JB*-triple*, namely there exists a continuous mapping $Z \times Z \times Z \ni (x,y,z) \to \{x,y,z\} \in Z$ (called the *Jordan triple product*) which is bilinear and symmetric in x and z, conjugate-linear in y, and so that the operators $D(x,x) : Z \to Z$ defined for every $x \in Z$ by $D(x,x)z := \{x,x,z\}$ are hermitian, have positive spectrum, satisfy the "C*-axiom" $\|D(x,x)\| = \|x\|^2$, and the operators $\delta(x) := iD(x,x)$ are *triple derivations*, i.e. the *Jordan triple identity holds*

$$\delta(x)\{y,z,w\} = \{\delta(x)y,z,w\} + \{y,\delta(x)z,w\} + \{y,z,\delta(x)w\}, \quad \forall y,z,w \in Z.$$

The norm $\|\cdot\|$ is called the *spectral norm.* We put also $D(x,y)z := \{x,y,z\}$. An element $v \in Z$ is called a *tripotent* if $\{v,v,v\} = v$. Every tripotent $v \in Z$ gives rise to a direct-sum *Peirce decomposition*

$$Z = Z_1(v) + Z_{\frac{1}{2}}(v) + Z_0(v), \;\text{ where }\; Z_\nu(v) := \{z \in Z;\; D(v,v)z = \nu z\}, \;\; \nu = 1, \frac{1}{2}, 0.$$

The associated *Peirce projections* are defined for $z_\kappa \in Z_\kappa(v)$, $\kappa = 1, \frac{1}{2}, 0$, by

$$P_\nu(v)(z_1 + z_{\frac{1}{2}} + z_0) = z_\nu, \quad \nu = 1, \frac{1}{2}, 0.$$

In this paper we are interested in the important special case where Z contains a *unitary* tripotent e, for which $Z = Z_1(e)$. In this case Z has the structure of a *JB*-algebra* with respect to the binary product $x \circ y := \{x,e,y\}$ and the involution $z^* := \{e,z,e\}$, and e is the unit of Z. The binary Jordan product is commutative, but in general non-associative. The triple product is expressed in terms of the binary product and the involution via $\{x,y,z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$. In this case the open unit ball D of Z is a *Cartan domain of tube-type.* This terminology is related to the *unbounded realization of* D, to be explained later.

Let $X := \{x \in Z; x^* = x\}$ be the real part of Z. It is a formally-real (or euclidean) Jordan algebra. Every $x \in X$ has a *spectral decomposition* $x = \sum_{j=1}^r \lambda_j e_j$, where $\{e_j\}_{j=1}^r$ is a *frame* of pairwise orthogonal minimal idempotents in X, and $\{\lambda_j\}_{j=1}^r$ are real numbers called the *eigenvalues* of x. The *trace* and *determinant* (or, "norm") are defined in X via

$$tr(x) := \sum_{j=1}^r \lambda_j, \qquad N(x) := \prod_{j=1}^r \lambda_j$$

respectively, and they are polynomials on X. The maximal number r of pairwise orthogonal minimal idempotents in X is called the *rank* of X. The positive-definite inner product in X, $\langle x,y\rangle = tr(x \circ y), \quad x,y \in X$, satisfies

$$\langle x \circ y, z\rangle = \langle x, y \circ z\rangle, \quad x,y,z \in X.$$

Equivalently, the multiplication operators $L(x)y := x \circ y, \;\; x,y \in X$, are self-adjoint. The trace and determinant polynomials, as well as the multiplication operators, have unique extensions to the complexification $X^{\mathbf{C}} := X + iX = Z$. Let

$$\Omega := \{x^2; x \in X, N(x) \neq 0\}.$$

Then Ω is a *symmetric, open convex cone*, i.e. Ω is *self polar* and *homogeneous* with respect to the group $GL(\Omega)$ of linear automorphisms of Ω. We denote the connected component of the identity in $GL(\Omega)$ by $G(\Omega)$. Define

$$P(x) := 2L(x)^2 - L(x^2), \quad x \in X, \tag{1.1}$$

then $P(x) \in G(\Omega)$ for every $x \in \Omega$, and $x = P(x^{1/2})e$. Thus $G(\Omega)$ is transitive on Ω. The map $x \to P(x)$ from X into $End(X)$ is called the *quadratic representation* because of the identity

$$P(P(x)y) = P(x)P(y)P(x), \quad \forall x, y \in X. \tag{1.2}$$

The domain $T(\Omega) := X + i\Omega$, called the *tube over* Ω. It is an irreducible symmetric domain which is biholomorphically equivalent to D by means of the *Cayley transform* $c : D \to T(\Omega)$, defined by

$$c(z) := i\frac{e+z}{e-z}, \quad z \in Z.$$

This explains why D is called a tube-type Cartan domain.

Let $e_1, e_2, \dots, e_r$ be a fixed frame of minimal, pairwise orthogonal idempotents satisfying $e_1 + e_2 + \dots + e_r = e$, where e is the unit of Z. Let

$$Z = \sum_{1 \le i \le j \le r} Z_{i,j}$$

be the associated *joint Peirce decomposition*, namely $Z_{i,j} := Z_{\frac{1}{2}}(e_i) \cap Z_{\frac{1}{2}}(e_j)$ for $1 \le i < j \le r$ and $Z_{i,i} := Z_1(e_i)$ for $1 \le i \le r$. The *characteristic multiplicity* is the common dimension $a = \dim(Z_{i,j})$, $1 \le i < j \le r$, and $d = r + r(r-1)a/2$. The number $p := (r-1)a + 2$ is called the *genus* of D. It is known that

$$Det(P(x)) = N(x)^p, \quad \forall x \in X,$$

where "Det" is the usual determinant polynomial in $End(Z)$. From this and (1.2) it follows that

$$N(P(x)y) = N(x)^2 N(y) \quad \forall x, y \in X. \tag{1.3}$$

Let $u_j := e_1 + e_2 + \dots + e_j$ and let $Z_j := \sum_{1 \le i \le k \le j} Z_{i,k}$ be the JB*- subalgebra of Z whose unit is u_j. Let N_j be the determinant polynomials of the $Z_j, 1 \le j \le r$; they are called the *principal minors* associated with the frame $\{e_j\}_{j=1}^r$. Notice that $Z_r = Z$ and $N_r = N$. For an r-tuple of integers $\mathbf{m} = (m_1, m_2, \dots, m_r)$ write $\mathbf{m} \ge 0$ if $m_1 \ge m_2 \ge \dots \ge m_r \ge 0$. Such r-tuples $\mathbf{m}$ are called *signatures* (or, "partitions"). The *conical polynomial* associated with the signature $\mathbf{m}$ is

$$N_{\mathbf{m}}(z) := N_1(z)^{m_1 - m_2}\, N_2(z)^{m_2 - m_3}\, N_3(z)^{m_3 - m_4} \dots N_r(z)^{m_r}, \quad z \in Z.$$

Notice that $N_{\mathbf{m}}(\sum_{j=1}^r t_j e_j) = \prod_{j=0}^r t_j^{m_j}$, thus the conical polynomials are natural generalizations of the monomials. Let $Aut(D)$ be the group of all biholomorphic automorphisms of D, and let G be its connected component of the identity. Let $K := \{g \in G; g(0) = 0\} = G \cap GL(Z)$ be the maximal compact subgroup of G. For any signature $\mathbf{m}$ let $P_{\mathbf{m}} := span\{N_{\mathbf{m}} \circ k; k \in K\}$. Clearly, $P_{\mathbf{m}} \subset \mathcal{P}_\ell$, where

$\ell = |\mathbf{m}| = \sum_{j=1}^r m_j$ and $\mathcal{P}_\ell$ is the space of homogeneous polynomials of degree ℓ. By definition, $P_{\mathbf{m}}$ are invariant under the composition with members of K. Let

$$\langle f, g\rangle_{\mathcal{F}} := \partial_f(g^\sharp)(0) = \frac{1}{\pi^d}\int_Z f(z)\overline{g(z)}\, e^{-|z|^2} dm(z) \tag{1.4}$$

be the *Fock-Fischer inner product* on the space $\mathcal{P}$ of polynomials, where $g^\sharp(z) := \overline{g(z^*)}$, $\partial_f = f(\frac{\partial}{\partial z})$, $|z|$ is the unique K-invariant Euclidean norm on Z normalized so that $|e_1| = 1$, and $dm(z)$ is the corresponding Lebesgue volume measure. (Thus $\langle 1, 1\rangle_{\mathcal{F}} = 1$). The following result (*Peter-Weyl decomposition*) is proved in [Sc], see also [U1]. Here the group K acts on functions on D via $\pi(k)f := f \circ k^{-1}$, $k \in K$. Notice that $\mathcal{P}_\ell$, $\ell = 0, 1, 2, \ldots$ and $\mathcal{P}$ are invariant under this action.

Theorem 1.1 (i) *The spaces $\{P_{\mathbf{m}}\}_{\mathbf{m}\geq 0}$, are K-invariant and irreducible. The representations of K on the spaces $P_{\mathbf{m}}$ are mutually inequivalent, the $P_{\mathbf{m}}$'s are mutually orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathcal{F}}$, and $\mathcal{P} = \sum_{\mathbf{m}\geq 0} P_{\mathbf{m}}$.*
(ii) *If $\mathcal{H}$ is a Hilbert space of analytic functions on D with a K-invariant inner product in which the polynomials are dense, then $\mathcal{H}$ is the orthogonal direct sum $\mathcal{H} = \sum_{\mathbf{m}\geq 0} \oplus P_{\mathbf{m}}$. Namely, every $f \in \mathcal{H}$ is expanded in the norm convergent series $f = \sum_{\mathbf{m}\geq 0} f_{\mathbf{m}}$, with $f_{\mathbf{m}} \in P_{\mathbf{m}}$, and the spaces $P_{\mathbf{m}}$ are mutually orthogonal in $\mathcal{H}$. Moreover, there exist positive numbers $\{c_{\mathbf{m}}\}_{\mathbf{m}\geq 0}$ so that for every $f, g \in \mathcal{H}$ with expansions $f = \sum_{\mathbf{m}\geq 0} f_{\mathbf{m}}$ and $g = \sum_{\mathbf{m}\geq 0} g_{\mathbf{m}}$ we have*

$$\langle f, g\rangle_{\mathcal{H}} = \sum_{\mathbf{m}\geq 0} c_{\mathbf{m}}\, \langle f_{\mathbf{m}}, g_{\mathbf{m}}\rangle_{\mathcal{F}}.$$

For every signature $\mathbf{m}$ let $K_{\mathbf{m}}(z, w)$ be the reproducing kernel of $P_{\mathbf{m}}$ with respect to (1.4). Clearly, the reproducing kernel of the *Fock-Fischer space* $\mathcal{F}$ (the completion of $\mathcal{P}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{F}}$) is

$$F(z, w) := \sum_{\mathbf{m}} K_{\mathbf{m}}(z, w) = e^{\langle z, w\rangle}.$$

An important property of the norm polynomial N is its transformation rule under the group K

$$N(k(z)) = \chi(k)N(z), \quad k \in K,\ z \in Z \tag{1.5}$$

where $\chi : K \to \mathbf{T} := \{\lambda \in \mathbf{C}; |\lambda| = 1\}$ is a character. In fact, $\chi(k) = N(k(e)) = Det(k)^{2/p}$ $\forall k \in K$. Notice that (1.5) implies that $P_{(m,m,\ldots,m)} = \mathbf{C}\, N^m$ for $m = 0, 1, 2, \ldots$.

The subgroup L of K defined via

$$L := \{k \in K; k(e) = 1\} \tag{1.6}$$

plays an important role in the theory.

Lemma 1.1 *For every signature $\mathbf{m} \geq 0$ the function*

$$\phi_{\mathbf{m}}(z) := \int_L N_{\mathbf{m}}(\ell(z))d\ell \tag{1.7}$$

is the unique spherical (i.e., L-invariant) polynomial in $P_{\mathbf{m}}$ satisfying $\phi_{\mathbf{m}}(e) = 1$.

For example, $\phi_{(m,m,\ldots,m)} = N^m$ by (1.5). The L-invariant real polynomial on X

$$h(x) = h(x,x) := N(e - x^2)$$

admits a unique K-invariant, hermitian extension $h(z,w)$ to all of Z. Thus, $h(k(z),k(w)) = h(z,w)$ for all $z,w \in Z$ and $k \in K$, $h(z,w)$ is holomorphic in z and anti-holomorphic in w, and $h(z,w) = \overline{h(w,z)}$, [FK1]. The transformation rule of $h(z,w)$ under $Aut(D)$ is

$$h(\varphi(z),\varphi(w)) = J\varphi(z)^{\frac{1}{p}}\; h(z,w)\; \overline{J\varphi(w)}^{\frac{1}{p}}, \;\; \varphi \in Aut(D),\; z,w \in D, \tag{1.8}$$

where $J\varphi(z) := Det(\varphi'(z))$ is the complex Jacobian of φ, and p is the genus of D.

For $\mathbf{s} = (s_1, s_2, \ldots, s_r) \in \mathbf{C}^r$ one defines the *conical function* $N_{\mathbf{s}}$ on Ω via

$$N_{\mathbf{s}}(x) := N_1^{s_1 - s_2}(x)\, N_2^{s_2 - s_3}(x)\, N_3^{s_3 - s_4}(x) \ldots \cdot N_r^{s_r}(x), \;\; x \in \Omega,$$

which generalize the conical polynomials $N_{\mathbf{m}}$. In what follows use the following notation:

$$\lambda_j := (j-1)\frac{a}{2}, \;\; 1 \le j \le r.$$

The *Gindikin - Koecher Gamma function* is defined for $\mathbf{s} = (s_1, s_2, \ldots, s_r) \in \mathbf{C}^r$ with $\Re(s_j) > \lambda_j,\; 1 \le j \le r$, via

$$\Gamma_\Omega(\mathbf{s}) := \int_\Omega e^{-tr(x)} N_{\mathbf{s}}(x) d\mu_\Omega(x).$$

Here $tr(x) = \langle x, e\rangle$ is the Jordan-theoretic trace of x, and

$$d\mu_\Omega(x) := N(x)^{-\frac{d}{r}} dx$$

is the (unique, up to a multiplicative constant) $G(\Omega)$-*invariant measure* on Ω. The following formula [Gi] reduces the computation of $\Gamma_\Omega(\mathbf{s})$ to that of ordinary Gamma functions:

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{(d-r)/2} \prod_{1 \le j \le r} \Gamma(s_j - \lambda_j), \tag{1.9}$$

and provides a meromorphic continuation of Γ_Ω to all of $\mathbf{C}^r$. In particular, $\Gamma_\Omega(\lambda) := \Gamma_\Omega(\lambda, \lambda, \ldots, \lambda)$ is given by

$$\Gamma_\Omega(\lambda) = \int_\Omega e^{-tr(x)}\; N(x)^\lambda\; d\mu_\Omega(x) = (2\pi)^{(d-r)/2} \prod_{1 \le j \le r} \Gamma(\lambda - \lambda_j),$$

and it is an entire meromorphic function. The *pole set* of $\Gamma_\Omega(\lambda)$ is precisely

$$\mathbf{P}(D) := \cup_{1 \le j \le r} (\lambda_j - \mathbf{N}) = \{\lambda_j - n;\; 1 \le j \le r,\; n \in \mathbf{N}\}. \tag{1.10}$$

For $\lambda \in \mathbf{C}$ and a signature $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ one defines

$$(\lambda)_{\mathbf{m}} := \frac{\Gamma_\Omega(\mathbf{m} + \lambda)}{\Gamma_\Omega(\lambda)} = \prod_{j=1}^r (\lambda - \lambda_j)_{m_j} = \prod_{j=1}^r \prod_{n=0}^{m_j - 1} (n + \lambda - \lambda_j),$$

where $\mathbf{m}+\lambda := (m_1+\lambda, m_2+\lambda, \dots, m_r+\lambda)$.

We recall two important formulas for *integration in polar coordinates* [FK2], Chapters VI and IX. The first formula uses the fact that $K\cdot\Omega = Z$, namely the fact that every $z\in Z$ can be written (not uniquely) in the form $z=k(x)$, where $x\in\Omega$ and $k\in K$. This is the first (or "conical") type of *polar decomposition* of x, and it generalizes the polar decomposition of matrices. This leads to the formula

$$\int_Z f(z)dm(z) = \frac{\pi^d}{\Gamma_\Omega(\frac{d}{r})}\int_\Omega \left(\int_K f(k(x^{\frac{1}{2}}))\,dk\right)\,dx \tag{1.11}$$

which holds for every $f\in L^1(Z,m)$. Next, fix a frame $e_1,\dots,e_r$, and define

$$R := \mathrm{span}_{\mathbf{R}}\{e_j\}_{j=1}^r \quad\text{and}\quad R_+ := \{\sum_{j=1}^r t_je_j;\ t_1>t_2>\ldots>t_r>0\}$$

and

$$\mathbf{R}_+^r := \{t=(t_1,\ldots t_r);\ t_1>t_2>\ldots>t_r>0\}.$$

Then $Z=K\cdot R$, namely every $z\in Z$ has a representation $z=k(\sum_{j=1}^r t_je_j)$ for some (again, not unique) $\sum_{j=1}^r t_je_j\in R$ and $k\in K$. This representation is the second type of polar decomposition of z. Moreover, $m(Z\setminus K\cdot R_+)=0$, namely up to a subset of measure zero, every $z\in Z$ has a representation $z=k(\sum_{j=1}^r t_j^{1/2}e_j)$ with $t_1>t_2>\ldots>t_r>0$. This leads to the formula

$$\int_Z f(z)dm(z) = c_0\int_{\mathbf{R}_+^r}\left(\int_K f(k(\sum_{j=1}^r t_j^{\frac{1}{2}}e_j))\,dk\right)\prod_{1\le i<j\le r}(t_i-t_j)^a\,dt_1\,dt_2\,\ldots dt_r, \tag{1.12}$$

which holds for every $f\in L^1(Z,m)$. The constant c_0 will be determined as a by-product of our work in section 5 below. For convenience, we can write (1.12) in the form

$$\int_Z f(z)dm(z) = c_0\int_{\mathbf{R}_+^r} f^\#(t)\,w(t)^a\,dt, \tag{1.13}$$

where

$$f^\#(t) := \int_K f(k(\sum_{j=1}^r t_j^{\frac{1}{2}}e_j))\,dk,\quad t=(t_1,t_2,\ldots,t_r)\in\mathbf{R}_+^r$$

is the *radial part* of F and

$$w(t) := \prod_{1\le i<j\le r}(t_i-t_j),\quad t=(t_1,t_2,\ldots,t_r)\in\mathbf{R}_+^r \tag{1.14}$$

is the *Vandermonde polynomial*.

By [Hu], [Be], [La1], [FK1], we have the *binomial formula*:

Theorem 1.2 *For $\lambda\in\mathbf{C}$ we have*

$$N(e-x)^{-\lambda} = \sum_{\mathbf{m}\ge 0}(\lambda)_{\mathbf{m}}\frac{\phi_{\mathbf{m}}(x)}{\|\phi_{\mathbf{m}}\|_{\mathcal{F}}^2},\quad \forall x\in\Omega\cap(e-\Omega), \tag{1.15}$$

and

$$h(z,w)^{-\lambda} = \sum_{\mathbf{m}\geq 0} (\lambda)_{\mathbf{m}}\, K_{\mathbf{m}}(z,w), \quad \forall z,w \in D. \tag{1.16}$$

The two series converge absolutely, (1.15) converges uniformly on compact subsets of $(\lambda,x) \in \mathbf{C} \times (\Omega \cap (e-\Omega))$, and (1.16) converges uniformly on compact subsets of $(\lambda,z,w) \in \mathbf{C} \times D \times D$.

In particular, it follows that for fixed $z,w \in D$, the function $\lambda \to h(z,w)^{-\lambda}$ is analytic in all of $\mathbf{C}$ (this can be proved also by showing that $h(z,w) \neq 0$ for $z,w \in D$).

The *Wallach set* of D, denoted by $\mathbf{W}(D)$, is the set of all $\lambda \in \mathbf{C}$ for which the function $(z,w) \to h(z,w)^{-\lambda}$ is non-negative definite in $D \times D$, namely

$$\sum_{i,j} a_i \overline{a}_j\, h(z_i,z_j)^{-\lambda} \geq 0$$

for all finite sequences $\{a_j\} \subseteq \mathbf{C}$ and $\{z_j\} \subseteq D$. For $\lambda \in \mathbf{W}(D)$ let $\mathcal{H}_\lambda$ be the completion of the linear span of the functions $\{h(\cdot,w)^{-\lambda};\ w \in D\}$ with respect to the inner product $\langle\cdot,\cdot\rangle_\lambda$ determined by

$$\langle h(\cdot,w)^{-\lambda}, h(\cdot,z)^{-\lambda}\rangle_\lambda = h(z,w)^{-\lambda}, \ \ z,w \in D.$$

Since $h(z,w)^{-\lambda}$ is continuous in $D \times D$, it is the reproducing kernel of $\mathcal{H}_\lambda$. The transformation rule (1.8) implies that $\langle\cdot,\cdot\rangle_\lambda$ is K-invariant, namely $\langle f \circ k, g \circ k\rangle_\lambda = \langle f,g\rangle_\lambda$ for all $f,g \in \mathcal{H}_\lambda$ and $k \in K$. Thus, by Theorems 1.1 and 1.2, for every $f,g \in \mathcal{H}_\lambda$ with Peter-Weyl expansions $f = \sum_{\mathbf{m}\geq 0} f_{\mathbf{m}}$, $g = \sum_{\mathbf{m}\geq 0} g_{\mathbf{m}}$, we have

$$\langle f,g\rangle_\lambda = \sum_{\mathbf{m}\geq 0} \frac{\langle f_{\mathbf{m}}, g_{\mathbf{m}}\rangle_{\mathcal{F}}}{(\lambda)_{\mathbf{m}}}. \tag{1.17}$$

This formula defines $\lambda \mapsto \langle f,g\rangle_\lambda$ as a meromorphic function in all of $\mathbf{C}$, whose poles are contained in the pole set $\mathbf{P}(D)$ of Γ_Ω, see (1.10) and (1.16). Of course, for $\lambda \in \mathbf{C} \setminus \mathbf{W}(D)$ (1.17) is not an inner product, but merely a sesqui-linear form; it is hermitian precisely when $\lambda \in \mathbf{R}$.

Using (1.16) and (1.17) one obtains a complete description of the Wallach set $\mathbf{W}(D)$ and the Hilbert spaces $\mathcal{H}_\lambda$ for $\lambda \in \mathbf{W}(D)$.

Theorem 1.3 (i) *The Wallach set is given by $\mathbf{W}(D) = \mathbf{W}_d(D) \cup \mathbf{W}_c(D)$ where $\mathbf{W}_d(D) := \{\lambda_j = (j-1)\frac{a}{2}; 1 \leq j \leq r\}$ is the discrete part, and $\mathbf{W}_c(D) := (\lambda_r,\infty)$ is the continuous part.*

(ii) *For $\lambda \in \mathbf{W}_c(D)$ the polynomials are dense in $\mathcal{H}_\lambda$ and $\mathcal{H}_\lambda = \sum_{\mathbf{m}\geq 0} \oplus P_{\mathbf{m}}$ as in Theorem 1.1;*

(iii) *For $1 \leq j \leq r$, let $S_0(\lambda_j) := \{\mathbf{m} \geq 0; m_j = m_{j+1} = \ldots = m_r = 0\}$. Then $\mathcal{H}_{\lambda_j} = \sum_{\mathbf{m}\in S_0(\lambda_j)} P_{\mathbf{m}}$ and*

$$h(z,w)^{-\lambda_j} = \sum_{\mathbf{m}\in S_0(\lambda_j)} (\lambda_j)_{\mathbf{m}}\, K_{\mathbf{m}}(z,w), \ \ z,w \in D.$$

For $\lambda \in \mathbf{C}$, $\varphi \in G$ and a functions f on D, we define

$$U^{(\lambda)}(\varphi)f := (f \circ \varphi) \cdot (J\varphi)^{\frac{\lambda}{p}}$$

Then, $U^{(\lambda)}(id_D) = I$ and for $\varphi, \psi \in G$ we have

$$U^{(\lambda)}(\varphi \circ \psi) = c_\lambda(\varphi, \psi)\, U^{(\lambda)}(\psi)\, U^{(\lambda)}(\varphi),$$

where $c_\lambda(\varphi, \psi)$ is a unimodular scalar which transforms as a cocycle (*projective representation* of G). In particular, $U^{(\lambda)}(\varphi^{-1}) = U^{(\lambda)}(\varphi)^{-1}$.

Using (1.8) we see that

$$J\varphi(z)^{\frac{\lambda}{p}}\, h(\varphi(z), \varphi(w))^{-\lambda}\, \overline{J\varphi(w)}^{\frac{\lambda}{p}} = h(z, w)^{-\lambda}, \quad \forall z, w \in D, \ \ \forall \varphi \in G.$$

From this it follows that *the hermitian forms* $\langle \cdot, \cdot \rangle_\lambda$ given by (1.17) are $U^{(\lambda)}$-invariant:

$$\langle U^{(\lambda)}(\varphi)f\,,\, U^{(\lambda)}(\varphi)g\,\rangle_\lambda = \langle f, g\rangle_\lambda, \quad \forall f, g \in \mathcal{H}_\lambda, \quad \forall \varphi \in G.$$

In particular, for $\lambda \in \mathbf{W}(D)$ the inner products $\langle \cdot, \cdot \rangle_\lambda$ are $U^{(\lambda)}$-invariant and $U^{(\lambda)}(\varphi)$, $\varphi \in G$, are unitary operators on $\mathcal{H}_\lambda$.

There are other spaces of analytic functions on D which carry $U^{(\lambda)}$-invariant hermitian forms, some of which are non-negative. For any signature $\mathbf{m}$ and $\lambda \in \mathbf{C}$ let $q(\lambda, \mathbf{m}) := \deg_\lambda(\cdot)_\mathbf{m}$ be the multiplicity of λ as a zero of the polynomial $\xi \mapsto (\xi)_\mathbf{m}$. Notice that $0 \le q(\lambda, \mathbf{m}) \le r$ for all λ and $\mathbf{m}$. Let

$$q(\lambda) := \max\{q(\lambda, \mathbf{m}); \mathbf{m} \ge 0\}. \tag{1.18}$$

Let

$$\mathcal{P}^{(\lambda)} := \mathrm{span}\{U^{(\lambda)}(\varphi)f\ ;\ f \text{ polynomial } \ , \varphi \in G\}$$

For $0 \le j \le q(\lambda)$ set

$$S_j(\lambda) := \{\mathbf{m} \ge 0; q(\lambda, \mathbf{m}) \le j\} \ \ \mathcal{M}_j^{(\lambda)} := \{f \in \mathcal{P}^{(\lambda)}; f = \sum_{\mathbf{m} \in S_j(\lambda)} f_\mathbf{m}, \ \ f_\mathbf{m} \in P_\mathbf{m}\}. \tag{1.19}$$

The following result is established in [FK1], see also [A1], [O].

Theorem 1.4 *Let* $\lambda \in \mathbf{C}$ *and let* $0 \le j \le q(\lambda)$.

(i) *The spaces* $\mathcal{M}_j^{(\lambda)}, 0 \le j \le q(\lambda)$, *are* $U^{(\lambda)}$*-invariant,*

$$\mathcal{M}_0^{(\lambda)} \subset \mathcal{M}_1^{(\lambda)} \subset \mathcal{M}_2^{(\lambda)} \subset \ldots \subset \mathcal{M}_{q(\lambda)}^{(\lambda)} = \mathcal{P}^{(\lambda)}, \tag{1.20}$$

and every non-zero $U^{(\lambda)}$*-invariant subspace of* $\mathcal{P}^{(\lambda)}$ *is one of the spaces* $\mathcal{M}_j^{(\lambda)}$, $0 \le j \le q(\lambda)$.

(ii) *The quotients* $\mathcal{M}_j^{(\lambda)}/\mathcal{M}_{j-1}^{(\lambda)}$, $1 \le j \le q(\lambda)$, *are* $U^{(\lambda)}$*-irreducible.*

(III) *The sesqui-linear forms* $\langle\cdot,\cdot\rangle_{\lambda,j}$ *on* $\mathcal{M}_j^{(\lambda)}$, $1 \le j \le q(\lambda)$, *defined for* $f, g \in \mathcal{M}_j^{(\lambda)}$ *by*

$$\langle f,g\rangle_{\lambda,j} := \lim_{\xi\to\lambda} (\xi-\lambda)^j \langle f,g\rangle_\xi$$

are $U^{(\lambda)}$*-invariant and* $\{f \in \mathcal{M}_j^{(\lambda)}; \langle f,g\rangle_{\lambda,j} = 0, \forall g \in \mathcal{M}_j^{(\lambda)}\} = \mathcal{M}_{j-1}^{(\lambda)}$.

(IV) *For* $f, g \in \mathcal{M}_j^{(\lambda)}$ *with Peter-Weyl expansions* $f = \sum_{\mathbf{m}} f_{\mathbf{m}}$ *and* $g = \sum_{\mathbf{m}} g_{\mathbf{m}}$, *we have*

$$\langle f,g\rangle_{\lambda,j} = \sum_{\mathbf{m}\in S_j(\lambda)\setminus S_{j-1}(\lambda)} \frac{\langle f_{\mathbf{m}}, g_{\mathbf{m}}\rangle_{\mathcal{F}}}{(\lambda)_{\mathbf{m},j}}$$

where

$$(\lambda)_{\mathbf{m},j} := \lim_{\xi\to\lambda} \frac{(\xi)_{\mathbf{m}}}{(\xi-\lambda)^j} = \frac{1}{j!}\left(\frac{d}{d\xi}\right)^j (\xi)_{\mathbf{m}}{}_{|\xi=\lambda}. \tag{1.21}$$

(V) *The forms* $\langle\cdot,\cdot\rangle_{\lambda,j}$ *are hermitian if and only if* $\lambda \in \mathbf{R}$.

(VI) *The quotient* $\mathcal{M}_j^{(\lambda)}/\mathcal{M}_{j-1}^{(\lambda)}$ *is unitarizable (namely,* $\langle\cdot,\cdot\rangle_{\lambda,j}$ *is either positive definite or negative definite on* $\mathcal{M}_j^{(\lambda)}/\mathcal{M}_{j-1}^{(\lambda)}$*) if and only if either:* $\lambda \in \mathbf{W}(D)$ *and* $j = 0$, *or:* $\lambda \in \mathbf{P}(D)$, $j = q(\lambda)$, *and* $\lambda_r - \lambda \in \mathbf{N}$.

The sequence (1.20) is called the *composition series* of $\mathcal{P}^{(\lambda)}$.

DEFINITION 1.1 $\mathcal{H}_{\lambda,j} = \mathcal{H}_{\lambda,j}(D)$ *is the completion of* $\mathcal{M}_j^{(\lambda)}/\mathcal{M}_{j-1}^{(\lambda)}$ *with respect to* $\langle\cdot,\cdot\rangle_{\lambda,j}$.

Observe that $\mathcal{H}_{\lambda,0} = \mathcal{H}_\lambda$ for $\lambda \in \mathbf{W}(D)$. Also, $q(\lambda) > 0$ if and only if $\lambda \in \mathbf{P}(D)$.

The *main objective* of this work is to provide natural integral formulas for the $U^{(\lambda)}$-invariant hermitian forms $\langle\cdot,\cdot\rangle_{\lambda,j}$, with special emphasis on the case where the forms are definite, namely the case where $\mathcal{H}_{\lambda,j}$ is a $U^{(\lambda)}$-invariant Hilbert space. These integral formulas provide a characterization of the membership in the spaces $\mathcal{H}_{\lambda,j}$ in terms of finiteness of some weighted L^2-norms of the functions or of some of their derivatives. We discuss now some examples which motivate our study.

The weighted Bergman spaces: It is known [FK1] that for $\lambda \in \mathbf{R}$ the integral $c(\lambda)^{-1} := \int_D h(z,z)^{\lambda-p} dm(z)$ is finite if and only if $\lambda > p-1$, and in this case

$$c(\lambda) = \frac{\Gamma_\Omega(\lambda)}{\pi^d\, \Gamma_\Omega(\lambda - \frac{d}{r})}\ . \tag{1.22}$$

For $\lambda > p-1$ we consider the probability measure

$$d\mu_\lambda(z) := c(\lambda)\, h(z,z)^{\lambda-p}\, dm(z) \tag{1.23}$$

on D. The *weighted Bergman space* $L_a^2(D,\mu_\lambda)$ consists of all analytic functions in $L^2(D,\mu_\lambda)$. Using (1.8) one obtains the transformation rule of μ_λ under composition with $\varphi \in G$:

$$d\mu_\lambda(\varphi(z)) = |J\varphi(z)|^{\frac{2\lambda}{p}}\, d\mu_\lambda(z).$$

(The same argument yields the invariance of the measure $d\mu_0(z) := h(z,z)^{-p} dm(z)$). From this it follows that the operators $U^{(\lambda)}(\varphi)$ are isometries of $L^2(D,\mu_\lambda)$ which leave $L^2_a(D,\mu_\lambda)$ invariant. It is easy to verify that point evaluations are continuous linear functionals on $L^2_a(D,\mu_\lambda)$ and that the reproducing kernel of $L^2_a(D,\mu_\lambda)$ is $h(z,w)^{-\lambda}$. (For $w = 0$ this is trivial, and the general case follows by invariance.) It follows that $\mathcal{H}_\lambda = L^2_a(D,\mu_\lambda)$.

The Hardy space: The *Shilov boundary* S of a general Cartan domain D is the set of all maximal tripotents in Z. S is invariant and irreducible under both of G and K. Let σ be the unique K-invariant probability measure on S, defined via

$$\int_S f(\xi)\, d\sigma(\xi) := \int_K f(k(e))\, dk.$$

The *Hardy space* $H^2(S)$ is the space of all analytic functions f on D for which

$$\|f\|^2_{H^2(S)} := \lim_{\rho\to 1-} \int_S |f(\rho\xi)|^2\, d\sigma(\xi)$$

is finite. The polynomials are dense in $H^2(S)$ and every $f \in H^2(S)$ has radial limits $\tilde{f}(\xi) := \lim_{\rho\to 1-} f(\rho\xi)$ at σ-almost every $\xi \in S$. Moreover, for $f \in H^2(S)$, $\|f\|_{H^2(S)} = \|\tilde{f}\|_{L^2(S,\sigma)}$. This identifies $H^2(S)$ as the closed subspace of $L^2(S,\sigma)$ consisting of those functions $f \in L^2(S,\sigma)$ which extend analytically to D by means of the Poisson integral. Again, the point evaluations $f \mapsto f(z)$, $z \in D$, are continuous linear functionals on $H^2(S)$. The corresponding reproducing kernel is called the *Szegö kernel* and is given (as a function on S) by $\mathcal{S}_z(\xi) = \mathcal{S}(\xi,z) := h(\xi,z)^{-d/r}$. See [Hu], [FK1]. This non-trivial fact implies that $\mathcal{H}_{d/r} = H^2(S)$. The transformation rule of the measure σ under the automorphisms $\varphi \in G$ is

$$d\sigma(\varphi(\xi)) = |J\varphi(\xi)|\, d\sigma(\xi).$$

Hence, $U^{(d/r)}(\varphi)f = (f\circ\varphi)\,(J\varphi)^{1/2}$, $\varphi \in G$, are isometries of $L^2(S,\sigma)$ which leave $H^2(S)$ invariant.

The Dirichlet space: The classical *Dirichlet space* B_2 consists of those analytic functions f on the open unit disk $\mathbf{D} \subset \mathbf{C}$ for which the *Dirichlet integral*

$$\|f\|^2_{B_2} := \int_{\mathbf{D}} |f'(z)|^2\, dA(z) \tag{1.24}$$

is finite. Here $dA(z) := \frac{1}{\pi} dx\, dy$. Clearly, B_2 is a Hilbert space modulo constant functions, and $\|f\circ\varphi\|_{B_2} = \|f\|_{B_2}$ for every $f \in B_2$ and $\varphi \in Aut(\mathbf{D})$. Thus, B_2 is $U^{(0)}$-invariant. The composition series corresponding to $\lambda = \lambda_1 = 0$ is $\mathbf{C}1 = M^{(0)}_0 \subset M^{(0)}_1 = \mathcal{P}^{(0)}$. Hence $B_2 = \mathcal{H}_{0,1}(\mathbf{D})$. The inner product in B_2 can be computed also via integration on the boundary $\mathbf{T} := \partial\mathbf{D}$ (which coincides with the Shilov boundary in this simple case):

$$\langle f, g\rangle_{B_2} = \frac{1}{2\pi} \int_{\mathbf{T}} \xi f'(\xi)\, \overline{g(\xi)}\, |d\xi|. \tag{1.25}$$

Motivated by this example we call the spaces $\mathcal{H}_{0,q(0)}$ for a general Cartan domain D the *(generalized) Dirichlet space* of D. The paper [A2] provides integral formulas generalizing (1.24) and (1.25) for the norms in $\mathcal{H}_{\lambda,q(\lambda)}$ for $\lambda \in \mathbf{W}_d(D)$, in the context of a Cartan domain of tube type (in [A1] these formulas are extended to all $\lambda \in \mathbf{P}(D)$). Formula (1.24) says that $f \in B_2 = \mathcal{H}_{0,1}$ if and only if $f' \in \mathcal{H}_2$. Namely, differentiation "shifts" the space corresponding to $\lambda = 0$ to the one corresponding to $\lambda = 2$. This shifting technique is developed in [Y3] in order to get integral formulas for the inner products in certain spaces $\mathcal{H}_\lambda$ with $\lambda \leq p-1$. The general idea is to obtain such integral formulas via "partial integration in the radial directions", see [Ri], [Go], and [FK2], Chapter VII. (For the open unit ball of $\mathbf{C}^d$, the simplest (i.e. rank-one) non-tube Cartan domain, cf. [A3], [Pel]).

Finally, we describe the *relationship between the invariant inner product and the Riesz distribution.* The Riesz distribution was introduced in [Ri] for the Lorentz cone, i.e. the symmetric cone associated with the Cartan domain of type IV (the "Lie ball"). It was studied in [Go] for the cone of symmetric, positive definite real matrices (associated with the Cartan domain of type III) and for a general symmetric cone in [FK2], chapter VII. Let Ω be the symmetric cone associated with the Cartan domain of tube type D. For $\alpha \in \mathbf{C}$ with $\Re\alpha > (r-1)\frac{a}{2}$ let R_α be the linear functional on the Schwartz space $S(X)$ of X defined via

$$R_\alpha(f) := \frac{1}{\Gamma_\Omega(\alpha)} \int_\Omega f(x) N(x)^{\alpha - \frac{d}{r}}\, dx.$$

Then R_α is a tempered distribution satisfying $\partial_N R_\alpha = R_{\alpha-1}$, $R_\alpha \star R_\beta = R_{\alpha+\beta}$, $R_0 = \delta$, i.e. R_1 is the fundamental solution for the "wave operator" $\partial_N := N(\frac{\partial}{\partial x})$. These formulas permit analytic continuation of $\alpha \mapsto R_\alpha$ to an entire meromorphic function. It is very interesting to find the explicit description of the action of R_α for general α, but this is still open. What is known is that *the Riesz distribution R_α is represented by a positive measure if and only if $\alpha \in W(D)$.*

Writing the inner products $\langle \cdot,\cdot \rangle_\lambda$ in conical polar coordinates (1.11), we get for $\lambda > p-1$

$$\langle f,g \rangle_\lambda = \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega(\frac{d}{r})\, \Gamma_\Omega(\lambda - \frac{d}{r})} \int_{\Omega \cap (e-\Omega)} (f\overline{g})\tilde{}(x)\; N(e-x)^{\lambda-p}\, dx, \;\; \forall f,g \in \mathcal{H}_\lambda(D),$$

where $(f\bar{g})\tilde{}(x) := \int_K f(k(x^{\frac{1}{2}}))\, \overline{g(k(x^{\frac{1}{2}}))}\, dk$. Thus

$$\langle f,g \rangle_\lambda = \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega(\frac{d}{r})} \left(R_{\lambda - \frac{d}{r}} \star (f\bar{g})\tilde{} \right)(e),$$

where the convolution of functions u and v on Ω is

$$(u \star v)(x) := \int_{\Omega \cap (x-\Omega)} u(y)\, v(x-y)\, dy.$$

Also, the inner product $\langle \cdot,\cdot \rangle_\lambda$, $\lambda > p-1$, in the context of the tube domain $T(\Omega) := X + i\Omega$ (holomorphically equivalent to D) is

$$\langle f,g \rangle_\lambda := c(\lambda) \int_\Omega \left(\int_X f(x+iy)\, \overline{g(x+iy)}\, dx \right) N(2y)^{\lambda-p}\, dy.$$

See section 6 for the details. Thus

$$\langle f, g\rangle_\lambda = \pi^{-d}\, 2^{\lambda-p}\, \Gamma_\Omega(\lambda)\; R_{\lambda-\frac{d}{r}}\left((f\,\bar g)^\flat\right),$$

where $(f\,\bar g)^\flat)(y) := \int_X f(x+iy)\,\overline{g(x+iy)}\,dx,\;\; y\in\Omega$.

In view of these formulas the problem of obtaining an explicit description of the analytic continuation of the maps $\lambda \mapsto \langle f, g\rangle_\lambda$ is equivalent to the problem of determining the analytic continuation of the maps $\lambda \mapsto R_{\lambda-\frac{d}{r}}(u)$.

2 $G(\Omega)$-INVARIANT DIFFERENTIAL OPERATORS

Let Ω be the symmetric cone associated with the Cartan domain of tube type D, i.e. the interior of the cone of squares in the Euclidean Jordan algebra X. In this section we study $G(\Omega)$-invariant differential operators that will be used later for the invariant inner products. The ring $\mathrm{Diff}(\Omega)^{G(\Omega)}$ of $G(\Omega)$-invariant differential operators is a (commutative) polynomial ring $\mathbf{C}[X_1, X_2, \ldots, X_r]$, [He], [FK2]. By [FK2], Proposition IX.1.1, Ω is a set of uniqueness for analytic functions on Z (namely, if an analytic function on Z vanishes identically on Ω, it vanishes identically on Z). Similarly, $\Omega\cap D = \Omega\cap(e-\Omega)$ is a set of uniqueness for analytic functions on D. Thus, if f, g and q are polynomials on Z so that $\partial_f(g)(x) = f(\frac{d}{dx})g(x) = q(x)$ for every $x\in\Omega$, then $\partial_f(g)(z) = f(\frac{\partial}{\partial z})g(z) = q(z)$ for every $z\in Z$. We begin with the following known result [FK2], Proposition VII.1.6.

LEMMA 2.1 *For every $\mathbf{s} = (s_1, s_2, \ldots, s_r)\in\mathbf{C}^r$ and $\ell\in\mathbf{N}$, we have*

$$N^\ell(\frac{d}{dx})\, N_{\mathbf{s}}(x) = \mu_{\mathbf{s}}(\ell)\; N_{\mathbf{s}-\ell}(x), \quad \forall x\in\Omega,$$

where

$$\mu_{\mathbf{s}}(\ell) := \frac{(\frac{d}{r})_{\mathbf{s}}}{(\frac{d}{r})_{\mathbf{s}-\ell}} = \frac{\Gamma_\Omega(\mathbf{s}+\frac{d}{r})}{\Gamma_\Omega(\mathbf{s}+\frac{d}{r}-\ell)} = \prod_{j=1}^{r}\prod_{\nu=0}^{\ell-1}(s_j-\nu+(r-j)\frac{a}{2}),$$

and

$$\Gamma_\Omega(\mathbf{s})\, N(\frac{d}{dx})\, N_{\mathbf{s}}(x^{-1}) = (-1)^r\; \Gamma_\Omega(\mathbf{s}+1)\; N_{\mathbf{s}+1}(x^{-1}).$$

Let N_j^* be the norm polynomial of the JB*-subalgebra $V_j := \sum_{r-j+1\le j\le k\le r} Z_{i,k}$, where $Z_{i,k}$ are the Peirce subspaces of Z associated with the fixed frame $\{e_j\}_{j=1}^r$. For every $\mathbf{s} = (s_1, \ldots, s_r)\in\mathbf{C}^r$ let

$$N^*_{\mathbf{s}}(x) := N_1^*(x)^{s_1-s_2}\, N_2^*(x)^{s_2-s_3}\, \ldots N_r^*(x)^{s_r},\;\; x\in\Omega,$$

and

$$\mathbf{s}^* := (s_r, s_{r-1}, s_{r-2}, \ldots, s_1).$$

Then we have $N_{\mathbf{s}}(x^{-1}) = N^*_{-\mathbf{s}^*}(x)$ for $x\in\Omega$, [FK2],Proposition VII.1.5.

DEFINITION 2.1 *For $\ell\in\mathbf{N}$ and $\lambda\in\mathbf{C}$ let $D_\ell(\lambda)$ be the operator on $C^\infty(\Omega)$ defined by*

$$D_\ell(\lambda) = N^{\frac{d}{r}-\lambda}(x)\, N^\ell(\frac{d}{dx})\, N^{\ell+\lambda-\frac{d}{r}}(x). \tag{2.1}$$

In the special case of the Cartan domain of type II the operators $D_1(\lambda)$ have been considered by Selberg (see [T], p.208). The operators $D_\ell(\lambda)$ were studied in full generality in [Y3], see also [FK2], Chapter XIV. Notice that by Lemma 2.1 we have

$$D_\ell(\lambda)\, N_{\mathbf{s}} = \frac{\Gamma_\Omega(\mathbf{s}+\lambda+\ell)}{\Gamma_\Omega(\mathbf{s}+\lambda)}\, N_{\mathbf{s}}. \tag{2.2}$$

In section 4 below we will extend $D_\ell(\lambda)$ to a polynomial differential operator on Z, i.e. $D_\ell(\lambda) = Q_{\ell,\lambda}(z, \frac{\partial}{\partial z})$ for some polynomial $Q_{\ell,\lambda}$.

LEMMA 2.2 *The operator $D_\ell(\lambda)$ is K-invariant, i.e.*

$$D_\ell(\lambda)(f\circ k) = (D_\ell(\lambda)f)\circ k \;\; \forall f \in C^\infty(\Omega), \;\; \forall k \in K.$$

PROOF: We have $N(kz) = \chi(k)N(z)$ for every $z \in Z$. Since the operator $\partial_N = N(\frac{\partial}{\partial z})$ is the adjoint of the operator of multiplication by N with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{F}}$, K-invariance of $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ implies $\partial_N(f\circ k) = \chi(k)(\partial_N f)\circ k$. It follows that

$$\begin{aligned} D_\ell(\lambda)(f(kz)) &= \overline{\chi(k)}^{\ell+\lambda-\frac{d}{r}}\, N(z)^{\frac{d}{r}-\lambda}\, N^\ell(\frac{\partial}{\partial z})\left(N^{\ell+\lambda-\frac{d}{r}}(kz)f(kz)\right) \\ &= \overline{\chi(k)}^{\ell+\lambda-\frac{d}{r}}\, N(z)^{\frac{d}{r}-\lambda}\, \chi(k)^\ell\, \left(N^\ell(\frac{\partial}{\partial z})(N^{\ell+\lambda-\frac{d}{r}}\, f)\right)(kz) \\ &= N^{\frac{d}{r}-\lambda}(kz)\, \left(N^\ell(\frac{\partial}{\partial z})(N^{\ell+\lambda-\frac{d}{r}}\, f)\right)(kz) = (D_\ell(\lambda)f)(kz). \quad \blacksquare \end{aligned}$$

Using (2.2) and the fact that $\Omega\cap D = \Omega\cap(e-\Omega)$ is a set of uniqueness for analytic functions on D, we obtain the following result.

COROLLARY 2.1 *The spaces $P_{\mathbf{m}}$ are eigenspaces of $D_\ell(\lambda)$ with eigenvalues*

$$\mu_{\ell,\mathbf{m}}(\lambda) := \frac{\Gamma_\Omega(\mathbf{m}+\lambda+\ell)}{\Gamma_\Omega(\mathbf{m}+\lambda)}. \tag{2.3}$$

Thus for every analytic function f on D with Peter-Weyl expansion $f = \sum_{\mathbf{m}\geq 0} f_{\mathbf{m}}$,

$$D_\ell(\lambda) f = \sum_{\mathbf{m}\geq 0} \frac{\Gamma_\Omega(\mathbf{m}+\lambda+\ell)}{\Gamma_\Omega(\mathbf{m}+\lambda)}\, f_{\mathbf{m}} = (\lambda)_{(\ell,\ell,\ldots,\ell)} \sum_{\mathbf{m}\geq 0} \frac{(\lambda+\ell)_{\mathbf{m}}}{(\lambda)_{\mathbf{m}}}\, f_{\mathbf{m}}. \tag{2.4}$$

Indeed, for every signature $\mathbf{m}$ and every $k \in K$,

$$D_\ell(\lambda)(N_{\mathbf{m}}\circ k) = (D_\ell(\lambda)N_{\mathbf{m}})\circ k = \frac{\Gamma_\Omega(\mathbf{m}+\lambda+\ell)}{\Gamma_\Omega(\mathbf{m}+\lambda)}\, N_{\mathbf{m}}\circ k.$$

Since $P_{\mathbf{m}} = span\{N_{\mathbf{m}}\circ k; k\in K\}$, (2.4) follows from the continuity of $D_\ell(\lambda)$ with respect to the topology of compact convergence on D.

COROLLARY 2.2 *Let $\lambda \in \mathbf{C}\setminus \mathbf{P}(D)$, $\ell\in\mathbf{N}$, and $w\in D$. Then*

$$D_\ell(\lambda)\, h(\cdot,w)^{-\lambda} = (\lambda)_{(\ell,\ell,\ldots,\ell)}\, h(\cdot,w)^{-(\lambda+\ell)}. \tag{2.5}$$

Proof: Using (1.16) and Corollary 2.2, we get

$$\begin{aligned}
D_\ell(\lambda)\, h(\cdot,w)^{-\lambda} &= \sum_{\mathbf{m}\geq 0} (\lambda)_{\mathbf{m}}\, D_\ell(\lambda)\, K_{\mathbf{m}}(\cdot,w) \\
&= (\lambda)_{(\ell,\ldots,\ell)} \sum_{\mathbf{m}\geq 0} (\lambda)_{\mathbf{m}} \frac{(\lambda+\ell)_{\mathbf{m}}}{(\lambda)_{\mathbf{m}}}\, K_{\mathbf{m}}(\cdot,w) \\
&= (\lambda)_{(\ell,\ldots,\ell)} \sum_{\mathbf{m}\geq 0} (\lambda+\ell)_{\mathbf{m}}\, K_{\mathbf{m}}(\cdot,w) = (\lambda)_{(\ell,\ldots,\ell)}\; h(\cdot,w)^{-(\lambda+\ell)}. \quad \blacksquare
\end{aligned}$$

Notice that the assumption that λ is not in $\mathbf{P}(D)$ is used in the above proof to ensure that $(\lambda)_{\mathbf{m}} \neq 0$ for every $\mathbf{m} \geq 0$. This is due to the fact that the zero set of the polynomial $(\lambda)_{\mathbf{m}}$ is

$$Z((\cdot)_{\mathbf{m}}) = \cup_{j=1}^{r} \{\lambda_j - k; k = 0,1,\ldots,m_j-1\}, \tag{2.6}$$

while $\mathbf{P}(D) = \cup_{j=1}^{r} (\lambda_j - \mathbf{N}) = \cup_{\mathbf{m}\geq 0} Z((\cdot)_{\mathbf{m}})$. Similarly, for each $\mathbf{m} \geq 0$ the zero set of the polynomial defined by (2.3) is given by

$$Z(\mu_{\ell,\mathbf{m}}(\cdot)) = \cup_{j=1}^{r} \{\lambda_j - k;\; m_j \leq k \leq m_j + \ell - 1\}. \tag{2.7}$$

The multiplicities of the zeros are equal to the number of their appearances on the right hand side of (2.7).

Corollary 2.3 *Let $\lambda \in \mathbf{C}$, $\ell \in \mathbf{N}$ be so that $\{\mathbf{m} \geq 0; (\lambda)_{\mathbf{m}} = 0\} \subseteq \{\mathbf{m} \geq 0; (\lambda + \ell)_{\mathbf{m}} = 0\}$. Then (2.5) holds.*

Proof: Notice first that $(\lambda)_{(\ell,\ell,\ldots,\ell)}(\lambda+\ell)_{\mathbf{m}} = (\lambda)_{\mathbf{m}+\ell}$ for all $\lambda \in \mathbf{C}$, $\ell \in \mathbf{N}$, and $\mathbf{m} \geq 0$. Hence, using the fact that $\{\mathbf{m}; (\lambda+\ell)_{\mathbf{m}} \neq 0\} \subseteq \{\mathbf{m}; (\lambda)_{\mathbf{m}} \neq 0\}$, we get for every $w \in D$

$$\begin{aligned}
D_\ell(\lambda) h(\cdot,w)^{-\lambda} &= D_\ell(\lambda) \sum_{(\lambda)_{\mathbf{m}}\neq 0} (\lambda)_{\mathbf{m}} K_{\mathbf{m}}(\cdot,w) \\
&= (\lambda)_{(\ell,\ell,\ldots,\ell)} \sum_{(\lambda)_{\mathbf{m}}\neq 0} (\lambda+\ell)_{\mathbf{m}} K_{\mathbf{m}}(\cdot,w) \\
&= (\lambda)_{(\ell,\ldots,\ell)} \sum_{(\lambda+\ell)_{\mathbf{m}}\neq 0} (\lambda+\ell)_{\mathbf{m}} K_{\mathbf{m}}(\cdot,w) \\
&= (\lambda)_{(\ell,\ldots,\ell)} h(\cdot,w)^{-(\lambda+\ell)}. \quad \blacksquare
\end{aligned}$$

For $\lambda \in \mathbf{P}(D)$ let $q = q(\lambda)$ be as in (1.18), and for $0 \leq j \leq q$ consider $S_j(\lambda)$ and $\mathcal{M}_j^{(\lambda)}$ as in (1.19).

Lemma 2.3 *Let λ, and $q = q(\lambda)$ be as above, and choose an integer ℓ so that $\lambda + \ell \geq \frac{d}{r} = \lambda_r + 1$. Then*

(i) *$deg_\lambda((\cdot)_{(\ell,\ell,\ldots,\ell)}) = q$.*

(II) *For every* $j = 0, 1, 2, \ldots, q$ *and every* $\mathbf{m} \in S_j(\lambda) \setminus S_{j-1}(\lambda)$, $deg_\lambda(\mu_{\ell,\mathbf{m}}) = q - j$.

(III) *If* $0 \le j \le q$ *and* $\mathbf{m} \in S_{j-1}(\lambda)$, *then* $deg_\lambda(\mu_{\ell,\mathbf{m}}) \ge q - j + 1$.

PROOF: Using (2.6) it is clear that

$$q(\lambda, \mathbf{m}) = q \quad \Leftrightarrow \quad \lambda_j - m_j + 1 \le \lambda \ \ \forall j \quad \Leftrightarrow \quad \lambda_r - m_r + 1 \le \lambda.$$

Since $\lambda_r + 1 \ge \lambda + \ell$, we see that $\mathbf{m} = (\ell, \ell, \ldots, \ell)$ satisfies the above condition, namely $deg_\lambda((\cdot)_{(\ell,\ldots,\ell)}) = q(\lambda, (\ell, \ldots, \ell)) = q$. This establishes (i). Next, $\mathbf{m} \in S_j^{(\lambda)} \setminus S_{j-1}^{(\lambda)}$ is equivalent to $q(\lambda, \mathbf{m}) = j$. By the argument given above, $q(\lambda, \mathbf{m} + \ell) = q$. Since $deg_\lambda(f/g) = deg_\lambda(f) - deg_\lambda(g)$, we get

$$deg_\lambda(\mu_{\ell,\mathbf{m}}) = deg_\lambda\left(\frac{(\cdot)_{\mathbf{m}+\ell}}{(\cdot)_{\mathbf{m}}}\right) =$$

$$= deg_\lambda((\cdot)_{\mathbf{m}+\ell}) - deg_\lambda((\cdot)_{\mathbf{m}}) = q(\lambda, \mathbf{m} + \ell) - q(\lambda, \mathbf{m}) = q - j.$$

This yields (ii). Finally, (iii) follows by similar computations. ∎

Let $\lambda \in \mathbf{P}(D)$, $\ell \in \mathbf{N}$, and $q = q(\lambda)$ as above. For every $\mathbf{m} \ge 0$ and $\nu \in \mathbf{N}$ we define

$$\mu^\nu_{\ell,\mathbf{m}}(\lambda) := \frac{1}{\nu!}\left(\frac{\partial}{\partial \xi}\right)^\nu \mu_{\ell,\mathbf{m}}(\xi)_{|\xi=\lambda}.$$

Using Lemma 2.3 (ii), we have

COROLLARY 2.4 (I) *If* $\mathbf{m} \in S_j(\lambda) \setminus S_{j-1}(\lambda)$ *then*

$$\mu^{q-j}_{\ell,\mathbf{m}}(\lambda) = \prod_{i=1}^r {\prod}'^{\,m_i+\ell-1}_{k=m_i} (\lambda + k - \lambda_i),$$

where the product ${\prod}'^{\,m_j+\ell-1}_{k=m_j}$ *ranges over all non-zero terms. In particular,* $\mu^{q-j}_{\ell,\mathbf{m}}(\lambda) \ne 0$.

(II) *If* $\mathbf{m} \in S_{j-1}(\lambda)$ *then* $\mu^{q-j}_{\ell,\mathbf{m}}(\lambda) = 0$.

DEFINITION 2.2 *For* $\lambda \in \mathbf{C}$ *and* $\nu, \ell \in \mathbf{N}$ *let* $D^\nu_\ell(\lambda)$ *be the operator on* $C^\infty(D)$ *defined by*

$$D^\nu_\ell(\lambda) f := \frac{1}{\nu!}\left(\frac{\partial}{\partial \xi}\right)^\nu (D_\ell(\xi) f)_{|\xi=\lambda}. \tag{2.8}$$

Notice that if $f = \sum_{\mathbf{m}\ge 0} f_{\mathbf{m}}$ is analytic in D, then $D^\nu_\ell(\lambda) f := \sum_{\mathbf{m}\ge 0} \mu^\nu_{\ell,\mathbf{m}}(\lambda)\, f_{\mathbf{m}}$.

By [FK2], Chapter VI the group $G(\Omega)$ admits an Iwasawa decomposition $G(\Omega) = NAL$, where L is the group defined via (1.6), and NA is a maximal solvable subgroup of $G(\Omega)$ (called the *triangular subgroup* with respect to the frame $\{e_i\}_{i=1}^r$) which acts simply transitively on Ω and for which all the conical functions $N_{\mathbf{s}}$, $\mathbf{s} \in \mathbf{C}^r$, are eigenfunctions:

$$N_{\mathbf{s}}(\tau(x)) = N_{\mathbf{s}}(\tau(e))\, N_{\mathbf{s}}(x), \quad \forall \tau \in NA, \quad \forall x \in \Omega. \tag{2.9}$$

Lemma 2.4 *The operators $D_\ell(\lambda)$ are $G(\Omega)$-invariant, i.e.* $D_\ell(\lambda)(f\circ\varphi) = (D_\ell(\lambda)f)\circ\varphi,\ \forall f\in C^\infty(\Omega),\ \forall\varphi\in G(\Omega)$.

Proof: By the L-invariance of $D_\ell(\lambda)$ (see Lemma 2.2) it is enough to verify the NA-invariance of $D_\ell(\lambda)$ for functions f of the form $f = N_{\mathbf{s}}\circ\ell$ for some $\mathbf{s}\in\mathbf{C}^r$ and $\ell\in L$. Let $\tau\in NA$, and decompose $\ell\circ\tau$ uniquely as $\ell\circ\tau = \tau'\circ\ell'$ with $\tau'\in NA$ and $\ell'\in L$. Then, using (2.2) and (2.9), we get

$$\begin{aligned}
D_\ell(\lambda)(f\circ\tau) &= D_\ell(\lambda)(N_{\mathbf{s}}\circ\ell\circ\tau) = D_\ell(\lambda)(N_{\mathbf{s}}\circ\tau'\circ\ell')\\
&= (D_\ell(\lambda)(N_{\mathbf{s}}\circ\tau'))\circ\ell' = N_{\mathbf{s}}(\tau'(e))(D_\ell(\lambda)N_{\mathbf{s}})\circ\ell'\\
&= N_{\mathbf{s}}(\tau'(e))\,\frac{\Gamma_\Omega(\mathbf{s}+\lambda+\ell)}{\Gamma_\Omega(\mathbf{s}+\lambda)}\,N_{\mathbf{s}}\circ\ell' \;=\; \frac{\Gamma_\Omega(\mathbf{s}+\lambda+\ell)}{\Gamma_\Omega(\mathbf{s}+\lambda)}\,N_{\mathbf{s}}\circ\tau'\circ\ell'\\
&= \frac{\Gamma_\Omega(\mathbf{s}+\lambda+\ell)}{\Gamma_\Omega(\mathbf{s}+\lambda)}\,N_{\mathbf{s}}\circ\ell\circ\tau \;=\; \frac{\Gamma_\Omega(\mathbf{s}+\lambda+\ell)}{\Gamma_\Omega(\mathbf{s}+\lambda)}\,f\circ\tau\\
&= (D_\ell(\lambda)f)\circ\tau.
\end{aligned}$$

∎

Corollary 2.5 *The operators $D_\ell^\nu(\lambda)$ are $G(\Omega)$-invariant.*

3 Integral formulas via the shifting method

In this section we develop general shifting techniques (introduced in [Y3], for the case of integer shifts). The simplest case where this technique is applied is the case of the Dirichlet space $\mathcal{D} = \mathcal{H}_{0,1}$ over the unit disk $\mathbf{D}$ (see Section 2). For any $\alpha\in\mathbf{C}$ and $\beta\in\mathbf{C}\setminus\mathbf{P}(D)$ we define an operator $S_{\alpha,\beta}$ on $\mathcal{H}(D)$ via

$$S_{\alpha,\beta}(\sum_{\mathbf{m}\geq 0} f_{\mathbf{m}}) := \sum_{\mathbf{m}\geq 0}\frac{(\alpha)_{\mathbf{m}}}{(\beta)_{\mathbf{m}}}\,f_{\mathbf{m}}.$$

Theorem 5 of [A4] and the known estimate

$$\frac{(x)_{\mathbf{m}}}{(y)_{\mathbf{m}}}\approx\prod_{j=1}^r (m_j+1)^{x-y},\quad \forall x,y\in\mathbf{R}$$

(an easy consequence of (1.9) and Stirling's formula) ensures that $S_{\alpha,\beta}$ is continuous on $\mathcal{H}(D)$. For $\beta\in\mathbf{P}(D)$ we define operators $S_{\alpha,\beta,j}$, $0\leq j\leq q(\beta)$, on the space of analytic functions on D of the form $f = \sum_{\mathbf{m}\in S_j(\beta)} f_{\mathbf{m}}$ by

$$S_{\alpha,\beta,j}f := \lim_{\xi\to\beta}(\xi-\beta)^j S_{\alpha,\beta}f = \sum_{\mathbf{m}\in S_j(\beta)\setminus S_{j-1}(\beta)}\frac{(\alpha)_{\mathbf{m}}}{(\beta)_{\mathbf{m},j}}\,f_{\mathbf{m}},$$

where $(\beta)_{\mathbf{m},j}$ are defined by (1.21). Again, $S_{\alpha,\beta,j}$ is continuous in the topology of $\mathcal{H}(D)$. Also, $S_{\alpha,\beta,0} = S_{\alpha,\beta}$.

PROPOSITION 3.1 *Let $\alpha, \beta > (r-1)\frac{a}{2}$. Then $\langle f,g\rangle_\beta = \langle S_{\alpha,\beta} f, g\rangle_\alpha$ for every $f,g \in \mathcal{H}_\beta$.*

PROOF: By (1.17) the operator $S^{\frac{1}{2}}_{\alpha,\beta} : \mathcal{H}_\beta \to \mathcal{H}_\alpha$ defined by

$$S^{\frac{1}{2}}_{\alpha,\beta}(\sum_{\mathbf{m}\geq 0} f_{\mathbf{m}}) := \sum_{\mathbf{m}\geq 0} \left(\frac{(\alpha)_{\mathbf{m}}}{(\beta)_{\mathbf{m}}}\right)^{\frac{1}{2}} f_{\mathbf{m}}$$

is a surjective isometry, and $\|f\|^2_\beta = \|S^{\frac{1}{2}}_{\alpha,\beta} f\|^2_\alpha = \langle S_{\alpha,\beta} f, f\rangle_\alpha$. Now the result follows by polarization. ∎

In a similar way one proves the following result.

PROPOSITION 3.2 *Let $\alpha > (r-1)\frac{a}{2}$ and let $\beta \in \mathbf{P}(D)$. Then for every $0 \leq j \leq q(\beta)$ and all $f,g \in \mathcal{H}_{\beta,j}$,*

$$\langle f,g\rangle_{\beta,j} = \langle S_{\alpha,\beta,j} f, g\rangle_\alpha. \tag{3.1}$$

The operators $S_{\alpha,\beta,j}$ allow the computation of the invariant hermitian forms $\langle f,g\rangle_{\beta,j}$ by "shifting" the point β to the point α. This is the "*shifting method*". One typically chooses either $\alpha = \frac{d}{r}$ or $\alpha > p-1$, so the forms $\langle f,g\rangle_{\beta,j}$ can be computed by the integral-type inner products of $H^2(D)$ or $L^2_a(D,\mu_\alpha)$. In order for the shifting method to be useful, one has to identify the operators $S_{\alpha,\beta,j}$ as differential or pseudo-differential operators. Essentially, this is our aim in the rest of the paper. Yan's paper [Y3] deals with the case where $\ell := \alpha - \beta$ is a sufficiently large natural number. The following result is a minor generalization of a result of [Y3].

THEOREM 3.1 *Let $\lambda > \lambda_r = \frac{d}{r} - 1$ and let $\ell \in \mathbf{N}$. Then for all $f,g \in \mathcal{H}_\lambda$*

$$\langle f,g\rangle_\lambda = \alpha(\lambda,\ell)\langle D_\ell(\lambda) f, g\rangle_{\lambda+\ell}, \tag{3.2}$$

where

$$\alpha(\lambda,\ell) = \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega(\lambda+\ell)} = \frac{1}{(\lambda)_{(\ell,\ell,\dots,\ell)}}.$$

We include a short proof for the sake of completeness.

PROOF: Let $f,g \in \mathcal{H}_\lambda$ with expansions $f = \sum_{\mathbf{m}\geq 0} f_{\mathbf{m}}$ and $g = \sum_{\mathbf{m}\geq 0} g_{\mathbf{m}}$ respectively. Then

$$\begin{aligned} \langle D_\ell(\lambda) f, g\rangle_{\lambda+\ell} &= \sum_{\mathbf{m}\geq 0} \frac{\mu_{\ell,\mathbf{m}}(\lambda)}{(\lambda+\ell)_{\mathbf{m}}} \langle f_{\mathbf{m}}, g_{\mathbf{m}}\rangle_{\mathcal{F}} \\ &= \frac{\Gamma_\Omega(\lambda+\ell)}{\Gamma_\Omega(\lambda)} \sum_{\mathbf{m}\geq 0} \frac{\langle f_{\mathbf{m}}, g_{\mathbf{m}}\rangle_{\mathcal{F}}}{(\lambda)_{\mathbf{m}}} = \alpha(\lambda,\ell)^{-1} \langle f,g\rangle_\lambda. \end{aligned}$$ ∎

COROLLARY 3.1 *Let $\lambda > \lambda_r = \frac{d}{r} - 1$, and $\ell \in \mathbf{N}$ be so that $\lambda + \ell > p - 1$. Then $\mathcal{H}_{\lambda+\ell} = L^2_a(D,\mu_{\lambda+\ell})$, and for every $f,g \in L^2_a(D,\mu_{\lambda+\ell})$,*

$$\langle f,g\rangle_\lambda = \alpha(\lambda,\ell)\, c(\lambda+\ell) \int_D (D_\ell(\lambda) f)(z)\, \overline{g(z)}\, h(z,z)^{\lambda+\ell-p}\, dm(z).$$

Our main result in this section is a generalization of both Theorem 3.1 and the other results of [Y3] to the case of invariant hermitian forms associated with the pole set $\mathbf{P}(D) = \cup_{j=1}^{r}(\lambda_j - \mathbf{N})$. Since $\mathbf{W}(D) \subset \mathbf{P}(D)$, this covers cases not studied in [A1].

THEOREM 3.2 *Let $\lambda \in \mathbf{P}(D)$, $\ell \in \mathbf{N}$ and assume that $\lambda + \ell \geq \frac{d}{r} = \lambda_r + 1$. Let $q = q(\lambda)$, $0 \leq j \leq q$, and $\nu = q - j$. Then for all $f, g \in \mathcal{H}_{\lambda,j}$,*

$$\langle f, g\rangle_{\lambda,j} = \gamma \langle D_\ell^\nu(\lambda) f, g\rangle_{\lambda+\ell}, \tag{3.3}$$

where $\gamma = \gamma(\lambda, \ell)$ is the non-zero constant

$$\gamma := \frac{1}{q!}(\frac{\partial}{\partial \xi})^q \left((\xi)_{(\ell,\ell,\ldots,\ell)}\right)_{|\xi=\lambda}. \tag{3.4}$$

In particular, if $\lambda + \ell > p - 1$, then

$$\langle f, g\rangle_{\lambda,j} = \gamma\, c(\lambda+\ell) \int_D (D_\ell^\nu(\lambda) f)(z)\, \overline{g(z)}\, dm(z). \tag{3.5}$$

Moreover, if $\lambda_r - \lambda \in \mathbf{N}$ and ℓ is chosen so that $\lambda + \ell = \frac{d}{r} = \lambda_r + 1$, then

$$\langle f, g\rangle_{\lambda,j} = \gamma \int_S (D_\ell^\nu(\lambda) f)(\xi)\, \overline{g(\xi)}\, d\sigma(\xi). \tag{3.6}$$

We shall also give a new proof of the following known result (see [FK1], Theorem 5.3) and of a part of Theorem 1.4 above, based on our analysis of the structure of zeros of the polynomials $(\cdot)_{\mathbf{m}}$. Recall that $\mathcal{H}_{\lambda,j}$ is said to be *unitarizable* if $\langle \cdot, \cdot\rangle_{\lambda,j}$ is either positive definite or negative definite.

THEOREM 3.3 *Let λ, ℓ, q, and j be as in Theorem 3.2. Then $\mathcal{H}_{\lambda,j}$ is unitarizable if and only if either*
(a) $j = q$ and $\lambda_r - \lambda \in \mathbf{N}$, or
(b) $j = 0$ and $\lambda \in \mathbf{W}_d(D) = \{\lambda_j\}_{j=1}^r$.

For the proof of Theorems 3.2 and 3.3 we consider separately the cases $j = 0$, $j = q$, and $1 \leq j \leq q - 1$.

CASE 1: $\mathbf{j = 0}$. Since $\lambda \in \mathbf{P}(D)$, there is a smallest $k \in \{1, 2, \ldots, r\}$ and a unique $s \in \mathbf{N}$ so that $\lambda = \lambda_k - s$. We claim that $S_0(\lambda) = \{\mathbf{m} \geq 0; m_k \leq s\}$. Indeed, if $\mathbf{m} \geq 0$, then $\prod_{i=1}^{k-1}\prod_{\nu=0}^{m_i-1}(\lambda + \nu - \lambda_i) \neq 0$, by the minimality of k. The term $\prod_{\nu=0}^{m_k-1}(\lambda + \nu - \lambda_k) = \prod_{\nu=0}^{m_k-1}(\nu - s)$ is non-zero if and only if $m_k \leq s$. If $m_k \leq s$ and $k < n \leq r$ then

$$\prod_{\nu=0}^{m_k-1}(\lambda + \nu - \lambda_k) = \prod_{\nu=0}^{m_k-1}((\lambda_k - \lambda_n) + (\nu - s)) \neq 0$$

because $m_n \le m_k \le s$. This establishes the claim. Notice that since $\lambda + \ell \ge \lambda_r + 1$, we have $(\lambda+\ell)_{\mathbf{m}} > 0$ for any $\mathbf{m} \ge 0$. Also, $deg_\lambda((\cdot)_{(\ell,\ell,\ldots,\ell)}) = q$ by Lemma 2.3. It follows that for $\mathbf{m} \in S_0(\lambda)$, $deg_\lambda(\mu_{\ell,\mathbf{m}}) = q$, and

$$\begin{aligned}\mu^q_{\ell,\mathbf{m}}(\lambda) &= \frac{1}{q!}(\frac{\partial}{\partial \xi})^q \mu_{\ell,\mathbf{m}}(\xi)_{\,|\xi=\lambda} = \frac{1}{q!}(\frac{\partial}{\partial \xi})^q \left(\frac{(\xi+\ell)_{\mathbf{m}}}{(\xi)_{\mathbf{m}}}(\xi)_{(\ell,\ell,\ldots,\ell)}\right)_{|\xi=\lambda} \\ &= \frac{(\lambda+\ell)_{\mathbf{m}}}{(\lambda)_{\mathbf{m}}}\frac{1}{q!}(\frac{\partial}{\partial \xi})^q (\xi)_{(\ell,\ell,\ldots,\ell)\,|\xi=\lambda} = \gamma\,\frac{(\lambda+\ell)_{\mathbf{m}}}{(\lambda)_{\mathbf{m}}}.\end{aligned}$$

Hence, for $f, g \in \mathcal{H}_{\lambda,0}$,

$$\begin{aligned}\langle D^q_\ell(\lambda) f, g\rangle_{\lambda+\ell} &= \sum_{\mathbf{m}\in S_0(\lambda)} \mu^q_{\ell,\mathbf{m}}(\lambda)\,\frac{\langle f_{\mathbf{m}}, g_{\mathbf{m}}\rangle_{\mathcal{F}}}{(\lambda+\ell)_{\mathbf{m}}} \\ &= \gamma \sum_{\mathbf{m}\in S_0(\lambda)} \frac{\langle f_{\mathbf{m}}, g_{\mathbf{m}}\rangle_{\mathcal{F}}}{(\lambda)_{\mathbf{m}}} \;=\; \gamma\langle f, g\rangle_{\lambda,0}.\end{aligned}$$

This proves Theorem 3.2 in case $j = 0$. If $\lambda \in \mathbf{W}_d(D)$, i.e. $\lambda = \lambda_k$ and $s = 0$, then $(\lambda)_{\mathbf{m}} > 0$ for every $\mathbf{m} \in S_0(\lambda)$, namely $0 = m_k = m_{k+1} = \cdots = m_r$. If $\lambda \in \mathbf{P}(D) \setminus \mathbf{W}_d(D)$, then $\lambda = \lambda_k - s$ with $1 \le s$. In this case $(\lambda)_{\mathbf{m}}$ assumes both positive and negative values as $\mathbf{m}$ ranges over $S_0(\lambda)$. Indeed, if $\mathbf{m}$ and $\mathbf{n}$ are defined by $m_i = n_i = 0$ for $1 \le i \le k-1$ and $k < i \le r$, and $m_k = 0$, $n_k = s-1$, then $(\lambda)_{\mathbf{m}}$ and $(\lambda)_{\mathbf{n}}$ have different signs. Thus $\langle\cdot,\cdot\rangle_{\lambda,0}$ is not definite (positive or negative), and thus $\mathcal{H}_{\lambda,0}$ is not unitarizable. This proves Theorem 3.3 in case $j = 0$.

Case 2: $\mathbf{j} = \mathbf{q}$. In this case $\nu = q - j = 0$. Also, Lemma 2.3 yields $deg_\lambda(\mu_{\ell,\mathbf{m}}) = 0$ if $\mathbf{m} \in S_q(\lambda)$ and $deg_\lambda(\mu_{\ell,\mathbf{m}}) \ge 1$ if $\mathbf{m} \in S_{q-1}(\lambda)$. It follows that for $f, g \in \mathcal{H}_{\lambda,q}$,

$$\langle D_\ell(\lambda) f, g\rangle_{\lambda+\ell} = \sum_{\mathbf{m}\in S_q(\lambda)} \mu_{\ell,\mathbf{m}}(\lambda)\frac{\langle f_{\mathbf{m}}, g_{\mathbf{m}}\rangle_{\mathcal{F}}}{(\lambda+\ell)_{\mathbf{m}}}.$$

Now,

$$\mu_{\ell,\mathbf{m}}(\lambda) = \lim_{\xi\to\lambda}\frac{(\xi+\ell)_{\mathbf{m}}}{(\xi)_{\mathbf{m}}}\,(\xi)_{(\ell,\ell,\ldots,\ell)} = (\lambda+\ell)_{\mathbf{m}}\lim_{\xi\to\lambda}\frac{(\xi)_{(\ell,\ell,\ldots,\ell)}}{(\xi)_{\mathbf{m}}} = \gamma\frac{(\lambda+\ell)_{\mathbf{m}}}{(\lambda)_{\mathbf{m},q}},$$

where γ is the non-zero constant defined in (3.4). It follows that

$$\langle D_\ell(\lambda) f, g\rangle_{\lambda+\ell} = \gamma\sum_{\mathbf{m}\in S_q(\lambda)}\frac{\langle f_{\mathbf{m}}, g_{\mathbf{m}}\rangle_{\mathcal{F}}}{(\lambda)_{\mathbf{m},q}} = \gamma\langle f, g\rangle_{\lambda,q}.$$

This proves Theorem 3.2 in case $j = q$. To prove Theorem 3.3 in this case, assume first that $\lambda = \lambda_r - s$ for some $s \in \mathbf{N}$. We claim now that

$$S_q(\lambda) \setminus S_{q-1}(\lambda) = \{\mathbf{m} \ge 0; m_r \ge s+1\}. \tag{3.7}$$

Indeed, if $m_r \ge s+1$ then $\prod_{u=0}^{m_r-1}(\lambda+u-\lambda_r) = 0$. If $\lambda \in \lambda_i - \mathbf{N}$, then $\prod_{u=0}^{m_i-1}(\lambda+u-\lambda_r) = 0$ because $m_i \ge m_r \ge s+1$. Thus $deg_\lambda((\cdot)_{\mathbf{m}}) = q$. Conversely, if $deg_\lambda((\cdot)_{\mathbf{m}}) = q$, then in order that $\prod_{u=0}^{m_r-1}(\lambda + u - \lambda_r) = 0$ it is necessary that $s \leq m_r - 1$. This establishes (3.7).

Next, let $\mathbf{m} \in S_q(\lambda)$, and let $1 \leq i \leq r$ be so that $\lambda \in \lambda_i - \mathbf{N}$, say $\lambda = \lambda_i - k_i$. Then

$$\lim_{\xi\to\lambda}(\xi-\lambda)^{-1}\prod_{u=0}^{m_i-1}(\xi+u-\lambda_i) = \prod_{u=0}^{k_i-1}(\lambda+u-\lambda_i)\prod_{u=k_i+1}^{m_i-1}(\lambda+u-\lambda_i) = \gamma_{i,\mathbf{m}}\,\beta_i$$

with $\beta_i \neq 0$ and $\gamma_{i,\mathbf{m}} > 0$. If $\lambda \notin \lambda_i - \mathbf{N}$ we let $\beta_i = \prod_{u<\lambda_i-\lambda}(\lambda+u-\lambda_i) \neq 0$ and $\gamma_{i,\mathbf{m}} = \prod_{u>\lambda_i-\lambda}(\lambda+u-\lambda_i) > 0$. Then

$$(\lambda)_{\mathbf{m},q} = \lim_{\xi\to\lambda}\frac{(\xi)_{\mathbf{m}}}{(\xi-\lambda)^q} = \prod_{i=1}^{r}\gamma_{i,\mathbf{m}}\,\beta_i.$$

Hence, all the numbers $\{(\lambda)_{\mathbf{m},q}\}_{\mathbf{m}\in S_q(\lambda)}$ have constant sign (equal to $sgn(\prod_{i=1}^r \beta_i)$), and thus $\mathcal{H}_{\lambda,q}$ is unitarizable. Assume now that $\lambda \notin \lambda_r - \mathbf{N}$. Then, necessarily, the characteristic multiplicity a is odd and $\lambda \in \lambda_{r-1} - \mathbf{N}$. Writing $\lambda = \lambda_{r-1} - s$, $s \in \mathbf{N}$, it is clear by the above arguments that

$$S_q(\lambda) \setminus S_{q-1}(\lambda) = \{\mathbf{m} \geq 0;\ m_{r-1} \geq s+1\}.$$

Let $\mathbf{m} = (s+1, s+1, \ldots, s+1, 1)$ and $\mathbf{n} = (s+1, s+1, \ldots, s+1, 0)$. Then $\mathbf{m}, \mathbf{n} \in S_q(\lambda)$ and $(\lambda)_{\mathbf{m},q} = (\lambda - \lambda_r)(\lambda)_{\mathbf{n},q}$. Thus $(\lambda)_{\mathbf{m},q}$ and $(\lambda)_{\mathbf{n},q}$ have different signs, and so $\mathcal{H}_{\lambda,q}$ is not unitarizable. This proves Theorem 3.3 in case $j = q$.

Case 3: $\mathbf{1 \leq j \leq q-1}$. Put $\nu = q - j$. As before, $\ell \in \mathbf{N}$ is chosen so that $\lambda + \ell \geq \lambda_r + 1$, and this guarantees that $deg_\lambda((\cdot)_{\mathbf{m}+\ell}) = q$ and $(\lambda+\ell)_{\mathbf{m}} > 0$ for all signatures $\mathbf{m} \geq 0$. Let $f, g \in \mathcal{H}_{\lambda,j}$. Then

$$\langle D_\ell^\nu(\lambda)f, g\rangle_{\lambda+\ell} = \sum_{\mathbf{m}\in S_j(\lambda)}\mu_{\ell,\mathbf{m}}^\nu(\lambda)\,\frac{\langle f_{\mathbf{m}}, g_{\mathbf{m}}\rangle_{\mathcal{F}}}{(\lambda+\ell)_{\mathbf{m}}}.$$

If $\mathbf{m} \in S_j(\lambda) \setminus S_{j-1}(\lambda)$, then

$$deg_\lambda(\mu_{\ell,\mathbf{m}}) = deg_\lambda\left(\frac{(\cdot)_{\mathbf{m}+\ell}}{(\cdot)_{\mathbf{m}}}\right) = q - j = \nu.$$

Thus,

$$\mu_{\ell,\mathbf{m}}^\nu(\lambda) = \lim_{\xi\to\lambda}\frac{\mu_{\ell,\mathbf{m}}(\xi)}{(\xi-\lambda)^\nu} = \lim_{\xi\to\lambda}\frac{(\xi+\ell)_{\mathbf{m}}(\xi-\lambda)^{-q}(\xi)_{(\ell,\ell,\ldots,\ell)}}{(\xi-\lambda)^{-j}(\xi)_{\mathbf{m}}} = \gamma\,\frac{(\lambda+\ell)_{\mathbf{m}}}{(\lambda)_{\mathbf{m},j}}.$$

If $\mathbf{m} \in S_{j-1}(\lambda)$, then $deg_\lambda(\mu_{\ell,\mathbf{m}}) \geq q - j + 1 = \nu + 1$, and so $\mu_{\ell,\mathbf{m}}^\nu(\lambda) = 0$. Thus

$$\begin{aligned}\langle D_\ell^\nu(\lambda)f, g\rangle_{\lambda+\ell} &= \gamma\sum_{\mathbf{m}\in S_j(\lambda)\setminus S_{j-1}(\lambda)}\frac{(\lambda+\ell)_{\mathbf{m}}}{(\lambda)_{\mathbf{m},j}}\,\frac{\langle f_{\mathbf{m}}, g_{\mathbf{m}}\rangle_{\mathcal{F}}}{(\lambda+\ell)_{\mathbf{m}}}\\ &= \gamma\sum_{\mathbf{m}\in S_j(\lambda)\setminus S_{j-1}(\lambda)}\frac{\langle f_{\mathbf{m}}, g_{\mathbf{m}}\rangle_{\mathcal{F}}}{(\lambda)_{\mathbf{m},j}} = \gamma\langle f, g\rangle_{\lambda,j}.\end{aligned}$$

This proves Theorem 3.2 in case $1 \leq j \leq q-1$. To prove Theorem 3.3 in this case we need to show that as $\mathbf{m}$ varies in $S_j(\lambda) \setminus S_{j-1}(\lambda)$, $(\lambda)_{\mathbf{m},j}$ assumes both positive and negative values. Notice first that there exists a unique pair (k,s) of integers with $1 \leq k < s \leq r$ so that $\lambda_k - \lambda$ and $\lambda_s - \lambda$ are positive integers and

$$\mathbf{m} \in S_j(\lambda) \setminus S_{j-1}(\lambda) \quad \Longleftrightarrow \quad m_k \geq \lambda_k - \lambda + 1 \ \textit{and} \ m_s \leq \lambda_s - \lambda.$$

In fact, $s = k+1$ if the characteristic multiplicity a is even, and $s = k+2$ if a is odd. Next, $\lambda_s - \lambda = \lambda_k - \lambda + (s-k)\frac{a}{2} \geq 1$. Define $\mathbf{m}, \mathbf{n}$ by $m_i = n_i = \lambda_k - \lambda + 1$ if $1 \leq i \leq k$, $m_i = n_i = 0$ if $k+2 \leq i \leq r$, and $m_{k+1} = 0$, $n_{k+1} = 1$. Then $\mathbf{m}, \mathbf{n} \in S_j(\lambda) \setminus S_{j-1}(\lambda)$ and $(\lambda)_{\mathbf{n},j} = (\lambda)_{\mathbf{m},j}(\lambda - \lambda_s)$. Thus $(\lambda)_{\mathbf{n},j}$ and $(\lambda)_{\mathbf{m},j}$ have different signs, and so $\mathcal{H}_{\lambda,j}$ is not unitarizable. This proves Theorem 3.3 in case $1 \leq j \leq q-1$. ∎

A special case of Theorem 3.2 is the following essentially known result.

Corollary 3.2 *Let $\lambda \in \mathbf{P}(D)$ be so that $s = s(\lambda) := \frac{d}{r} - \lambda \in \mathbf{N}$. Then*

(i) *$\mathcal{H}_{\lambda,q}$ is unitarizable, and*

$$\langle f, g \rangle_{\lambda,q} = \gamma \int_S N^s(\xi)(\partial_N^s f)(\xi)\, \overline{g(\xi)}\, d\sigma(\xi), \quad \forall f, g \in \mathcal{H}_{\lambda,q}.$$

Thus, an analytic function f on D belongs to $\mathcal{H}_{\lambda,q}$ if and only if $(N^s \partial_N^s)^{1/2} f \in H^2(S)$.

(ii) *Moreover, if $\ell \in \mathbf{N}$ is chosen so that $\lambda + \ell > p-1$, then*

$$\langle f, g \rangle_{\lambda,q} = \gamma' \int_D (D_\ell(\lambda) f)(z)\, \overline{g(z)}\, h(z,z)^{\lambda+\ell-p}\, dm(z), \quad \forall f, g \in \mathcal{H}_{\lambda,q}.$$

Consequently, an analytic function f on D belongs to $\mathcal{H}_{\lambda,q}$ if and only if $(D_\ell(\lambda))^{1/2} f \in L^2_a(D, \mu_{\lambda+\ell})$.

In the last statement $(D_\ell(\lambda))^{1/2}$ is the positive square root of the positive operator $D_\ell(\lambda)$, see Corollary 2.1 Indeed, part (i) follows from Theorem 3.2 with $j = q$, $\nu = q - j = 0$, $\ell = s$ and $D_s(\lambda) = N^s \partial_N^s$. In this case $\mathcal{H}_{\lambda+s} = \mathcal{H}_{\frac{d}{r}}$ is the Hardy space $H^2(S)$ on the Shilov boundary S. Corollary 3.2 (i) for $\lambda \in \mathbf{W}_d(D)$ was proved in [A2]. The proof of part (ii) is similar.

The case where $\lambda \in \mathbf{P}(D)$ and $s := \frac{d}{r} - \lambda \in \mathbf{N}$ (i.e. the highest quotient of the composition series of $U^{(\lambda)}$-invariant spaces is unitarizable) is of particular interest.

Theorem 3.4 *Let $\lambda \in \mathbf{P}(D)$ and assume that $s := \frac{d}{r} - \lambda \in \mathbf{N}$. Then, for each $\varphi \in Aut(D)$ and $f \in \mathcal{H}(D)$*

$$\partial_N^s (U^{(\lambda)}(\varphi) f) = U^{(p-\lambda)}(\varphi)(\partial_N^s f). \tag{3.8}$$

Namely, the operator ∂_N^s intertwines the actions $U^{(\lambda)}$ and $U^{(p-\lambda)}$ of $Aut(D)$. Moreover,

$$\langle f, g \rangle_{\lambda,q} = c_1 \, \langle \partial_N^s f, \partial_N^s g \rangle_{p-\lambda}, \quad \forall f, g \in \mathcal{H}_{\lambda,q}, \tag{3.9}$$

where

$$c_1^{-1} := (\frac{d}{r})_{(s,s,\ldots,s)} \prod_{j=1}^{r} {\prod'}_{u=0}^{s-1} (\lambda + u - \lambda_j), \tag{3.10}$$

and the product ${\prod'}_{u=0}^{s-1}$ *ranges over all non-zero terms. In particular, if* $\lambda < 1$*, then*

$$\langle f, g \rangle_{\lambda,q} = c_1\, c(p-\lambda) \int_D (\partial_N^s f)(z)\, \overline{(\partial_N^s g)(z)}\, h(z,z)^{-\lambda}\, dm(z), \quad \forall f, g \in \mathcal{H}_{\lambda,q}. \tag{3.11}$$

Proof: (3.8) is proved in [A1], Theorem 6.4. For the proof of (3.9) and (3.11) we define an inner product on the polynomials modulo $\mathcal{M}_{q-1}^{(\lambda)}$ by

$$[f, g] := \langle \partial_N^s f, \partial_N^s g \rangle_{p-\lambda}, \quad f, g \in \mathcal{H}_{\lambda,q}.$$

We claim that $[\cdot,\cdot]$ is $U^{(\lambda)}$-invariant. Indeed, using (3.8) we see that for every $\varphi \in Aut(D)$ and polynomials f and g,

$$\begin{aligned} [U^{(\lambda)}(\varphi) f, U^{(\lambda)}(\varphi) g] &= \langle \partial_N^s (U^{(\lambda)}(\varphi) f), \partial_N^s (U^{(\lambda)}(\varphi) g) \rangle_{p-\lambda} \\ &= \langle U^{(p-\lambda)}(\varphi)(\partial_N^s f), U^{(p-\lambda)}(\varphi)(\partial_N^s g) \rangle_{p-\lambda} \\ &= \langle \partial_N^s f, \partial_N^s g \rangle_{p-\lambda} = [f, g]. \end{aligned}$$

Since polynomials are dense in $\mathcal{H}_{\lambda,q}$, the fact that its inner product is the unique $U^{(\lambda)}$-invariant inner product (see [AF], [A1]) implies that

$$\langle f, g \rangle_{\lambda,q} = c_1\, [f, g], \quad \forall f, g \in \mathcal{H}_{\lambda,q}.$$

The value (3.10) of c_1 is found by taking $f = g = N^s$, and using the facts that $\langle N^s, N^s \rangle_{\mathcal{F}} = (\frac{d}{r})_{(s,s,\ldots,s)}$, $[N^s, N^s] = (\partial_N^s N^s)^2 = \langle N^s, N^s \rangle_{\mathcal{F}}^2$, and

$$\langle N^s, N^s \rangle_{\lambda,q} = \lim_{\xi \to \lambda} (\xi - \lambda)^q \frac{\langle N^s, N^s \rangle_{\mathcal{F}}}{(\xi)_{(s,s\ldots,s)}} = \frac{\langle N^s, N^s \rangle_{\mathcal{F}}}{\prod_{j=1}^r {\prod'}_{u=0}^{s-1} (\lambda + u - \lambda_j)}. \qquad \blacksquare$$

Example: In the special case where $\lambda = 0$ and $s := \frac{d}{r} \in \mathbf{N}$, $\mathcal{H}_{0,q}$ is the generalized Dirichlet space, and formula (3.11) is the generalized Dirichlet inner product

$$\langle f, g \rangle_{0,q} = c_1\, c(p-\lambda) \int_D (\partial_N^s f)(z)\, \overline{(\partial_N^s g)(z)}\, dm(z), \quad \forall f, g \in \mathcal{H}_{0,q}.$$

4 The expansion of the operators $D_\ell(\lambda)$

Yan's operators $D_\ell(\lambda) = N^{\frac{d}{r} - \lambda} \partial_N^\ell N^{\lambda + \ell - \frac{d}{r}}$ and their derivatives play an important role in the previous section. In this section we obtain an expansion of $D_\ell(\lambda)$ in powers of λ. This expansion will exhibit $D_\ell(\lambda)$ as a polynomial in z, $\frac{\partial}{\partial z}$, and λ, showing that $D_\ell(\lambda)$ is a differential operator (with parameters λ and ℓ) in the ordinary sense. It also facilitates the computation of the derivatives

$$D_\ell^\nu(\lambda) = \frac{1}{\nu!} (\frac{\partial}{\partial \xi})^\nu D_\ell(\xi)_{|\xi = \lambda},$$

needed in formulas (3.3), (3.5) and (3.6) for the forms $\langle f,g\rangle_{\lambda,j}$. Another consequence will be that for any r distinct complex numbers $\alpha_1,\ldots,\alpha_r$ the operators $D_1(\alpha_1),\ldots,D_1(\alpha_r)$ are algebraically independent generators of the ring of invariant differential operators on the cone Ω, a result obtained independently also by Korányi and Yan (see [FK2], Chapter XIV). We shall work in the framework of the cone Ω, but all the results will be valid for Z, because Ω is a set of uniqueness for analytic functions on Z.

EXAMPLE 4.1. Let $D\subset \mathbf{C}^d$, $d\geq 3$ be a Cartan domain of rank $r=2$ (called the *Lie ball*). The associated JB*-algebra $Z=\mathbf{C}^d$, called the *complex spin factor*, is defined via

$$zw := (z_1w_1 - z'\cdot w', z_1w' + w_1z'),\quad z^* := (\overline{z_1}, -\overline{z'}),$$

where $z=(z_1,z')$, $z'=(z_2,z_3,\ldots,z_d)$, and $z\cdot w := \sum_{j=1}^d z_jw_j$. The unit of Z is $e:=(1,0,0,\ldots,0)$, and the canonical frame is $\{e_1,e_2\}$, where $e_1 := \frac{1}{2}(1,i,0,0,\ldots,0)$, $e_2 := \frac{1}{2}(1,-i,0,0,\ldots,0)$. The norm polynomial and the associated differential operator are given by

$$N(z) := z\cdot z = \sum_{j=1}^d z_j^2 \quad\text{and}\quad \partial_N = N(\frac{\partial}{\partial z}) = \frac{1}{4}\sum_{j=1}^d \frac{\partial^2}{\partial z_j^2}$$

respectively, since $(z|w)=2z\cdot\overline{w}$ is the normalized inner product. Since $r=2$ and $a=d-2$, the Wallach set is

$$\mathbf{W}(D) = \mathbf{W}_d(D)\cup\mathbf{W}_c(D),\ \ \mathbf{W}_d(D) = \{0,\frac{d-2}{2}\},\ \ \mathbf{W}_c(D) = (\frac{d-2}{2},\infty).$$

One can show that D is given by

$$D = \{z\in Z;\left((\sum_{j=1}^d |z_j|^2)^2 - |N(z)|^2\right)^{\frac{1}{2}} \ <\ 1-\sum_{j=1}^n |z_j|^2\}. \tag{4.1}$$

For every $\alpha\in\mathbf{C}$

$$\begin{aligned}\frac{\partial^2}{\partial z_k^2}N^\alpha &= \frac{\partial}{\partial z_k}(2\alpha N^{\alpha-1}z_k + N^\alpha\frac{\partial}{\partial z_k})\\ &= 2\alpha N^{\alpha-1} + 4\alpha N^{\alpha-1}z_k\frac{\partial}{\partial z_k} + 4\alpha(\alpha-1)N^{\alpha-2}z_k^2 + N^\alpha\frac{\partial^2}{\partial z_k^2}.\end{aligned}$$

Since $R=\sum_{j=1}^d z_j\frac{\partial}{\partial z_j}$, we obtain

$$\partial_N N^\alpha = \frac{1}{4}(\sum_{j=1}^d\frac{\partial^2}{\partial z_j^2})N^\alpha = \alpha(\alpha-\frac{a}{2})N^{\alpha-1} + \alpha N^{\alpha-1}R + N^\alpha\partial_N.$$

It follows that for every $\alpha\in\mathbf{C}$ and $\ell\in\mathbf{N}$,

$$N^{1-\alpha}\partial_N N^\alpha = N\partial_N + \alpha R + \alpha(\alpha+\frac{d-2}{2})I = N\partial_N + (\alpha)_{(1,0)}R + (\alpha)_{(1,1)}I. \tag{4.2}$$

Since

$$D_\ell(\lambda) = \left(N^{\frac{d}{r}-\lambda}\partial_N N^{1+\lambda-\frac{d}{r}}\right)\left(N^{\frac{d}{r}-\lambda-1}\partial_N N^{2+\lambda-\frac{d}{r}}\right)\cdots\left(N^{\frac{d}{r}+1-\ell-\lambda}\partial_N N^{\ell+\lambda-\frac{d}{r}}\right),$$

we finally obtain

$$D_\ell(\lambda) = \prod_{j=1}^{\ell}(N\partial_N + (\lambda - \frac{d}{2} + j)R + (\lambda - 1 + j)(\lambda - \frac{d}{2} + j)I). \tag{4.3}$$

Note that the factors on the right hand sides of (4.2) and (4.3) commute, since they are $G(\Omega)$-invariant, and the entire ring of $G(\Omega)$-invariant operators is commutative. Also, the operators R and $N\partial_N$ are K-invariant. Hence the factors on the right hand sides of (4.2) and (4.3) are multipliers of the Peter-Weyl decomposition of analytic functions on D (see Corollary 2.1).

Consider a general Cartan domain of tube-type $D \subset \mathbf{C}^d$ with rank r. Let Ω be the associated symmetric cone in the Euclidean Jordan algebra X and fix a frame $\{e_1, \ldots, e_r\}$ of pairwise orthogonal primitive idempotents in X, whose sum is the unit element e. For $1 \le \nu \le r$, let $\phi_\nu := \phi_{\mathbf{1}_\nu}$ be the spherical polynomial associated with the signature $\mathbf{1}_\nu := (1, 1, \ldots, 1, 0, 0, \ldots, 0)$, where there are ν "1"'s and $r - \nu$ "0"'s. Put also $\phi_0(z) \equiv 1$. Let $\{\Delta_\nu\}_{\nu=0}^r$ be the differential operators on Ω defined via

$$(\Delta_\nu)f(a) := \phi_\nu(\frac{d}{dx})(f(P(a^{\frac{1}{2}})x))_{|x=e}, \tag{4.4}$$

where for $b \in X$, $P(b)$ is defined via (1.1). Recall that $P(b) \in G(\Omega)$ for every $b \in \Omega$, and that $\Omega = \{P(b)e; b \in \Omega\}$ since $P(a^{\frac{1}{2}})e = a$. Moreover, the L-invariance of the ϕ_ν's and the "polar decomposition" for Ω imply that

$$(\Delta_\nu)f(a) := \phi_\nu(\frac{d}{dx})(f(\psi(x)))_{|x=e}, \quad a \in \Omega \tag{4.5}$$

for every $\psi \in G(\Omega)$ for which $\psi(e) = a$. This implies that the operators $\{\Delta_\nu\}_{\nu=0}^r$ are $G(\Omega)$-invariant, namely

$$\Delta_\nu(f \circ \psi) = (\Delta_\nu f) \circ \psi, \quad \forall \psi \in G(\Omega), \quad \forall f \in C^\infty(\Omega).$$

We remark that (4.4) and (4.5) are equivalent to

$$\Delta_\nu e^{\langle x,y\rangle}{}_{|x=a} = \phi_\nu(\psi^*(y))\, e^{\langle a,y\rangle} = \phi_\nu(P(a^{\frac{1}{2}})y)\, e^{\langle a,y\rangle}, \quad a, y \in \Omega, \tag{4.6}$$

where $\psi \in G(\Omega) \subset GL(X)$ satisfies $\psi(e) = a$, ψ^* is the adjoint of ψ with respect to the inner product $\langle\cdot,\cdot\rangle$ on X, and Δ_ν differentiates the coordinate x. Notice also that the operators Δ_ν can be written as

$$\Delta_\nu = c_{\mathbf{m}} K_{\mathbf{m}}(x, \frac{\partial}{\partial x}),$$

where $\mathbf{m} = (1, 1, \ldots, 1, 0, \ldots, 0)$ (ν "ones" and $r - \nu$ zeros), and $c_{\mathbf{m}}$ is an appropriate constant.

For $\nu = 0, 1, r$ it is easy to compute Δ_ν. Clearly, $\Delta_0 = I$. Since N is L-invariant, $\phi_r = N$. Using (4.6) and (1.3), we find that

$$\Delta_r = N\,\partial_N.$$

Also, $\phi_1(x) = \frac{1}{r}\,tr(x) = \frac{1}{r}\,\langle x, e\rangle$. Indeed, using $N_1(x) = \langle x, e_1\rangle$ and the fact that L is transitive on the frames, we get

$$\begin{aligned} \phi_1(x) &= \int_L \langle \ell x, e_1\rangle\, d\ell = \frac{1}{r}\sum_{j=1}^{r}\int_L \langle \ell x, e_j\rangle\, d\ell \\ &= \frac{1}{r}\int_L \langle \ell x, e\rangle\, d\ell = \frac{1}{r}\int_L \langle x, \ell e\rangle\, d\ell = \frac{1}{r}\langle x, e\rangle. \end{aligned}$$

Using the fact that $tr(P(a^{\frac{1}{2}})y) = \langle P(a^{\frac{1}{2}})y, e\rangle = \langle y, P(a^{\frac{1}{2}})e\rangle = \langle y, a\rangle, \quad \forall a, y \in \Omega$, we find that

$$\Delta_1 = \frac{1}{r}R,$$

where $Rf(x) := \frac{\partial}{\partial t} f(tx)_{|t=1}$ is the radial derivative.

Our main result in this section is the expansion of $D_1(\lambda) = N^{\frac{d}{r}-\lambda}\,\partial_N\, N^{1+\lambda-\frac{d}{r}}$. This result was obtained independently by A. Korányi, see [FK2], Proposition XIV.1.5.

THEOREM 4.1 *For every $\lambda \in \mathbf{C}$,*

$$D_1(\lambda) = \sum_{\nu=0}^{r}\binom{r}{\nu}\prod_{j=\nu+1}^{r}(\lambda - \lambda_j)\,\Delta_\nu. \tag{4.7}$$

PROOF: For $x \in \Omega$, the function $\alpha \to N(x)^\alpha$ is entire in α. Hence both sides of (4.7) are entire in λ, and it is therefore enough to prove (4.7) for λ with $\Re\lambda < 0$. Let $\alpha = \lambda_r - \lambda$. Since $\Re\lambda > \lambda_r$, we get for every $x \in \Omega$

$$N(x)^{-\alpha} = \frac{1}{\Gamma_\Omega(\alpha)}\int_\Omega e^{-\langle x,t\rangle} N(t)^\alpha\, d\mu_\Omega(t),$$

where $d\mu_\Omega(t) := N(t)^{-\frac{d}{r}}dt$ is the $G(\Omega)$-invariant measure on Ω. Fix $a, y \in \Omega$ and put $f_y(x) := e^{\langle x,y\rangle}$. Then

$$\begin{aligned} &(N^{\alpha+1}\partial_N N^{-\alpha} f_y)(a) \\ &= \frac{N(a)^{\alpha+1}}{\Gamma_\Omega(\alpha)} N(\frac{d}{dx})\int_\Omega e^{\langle x, y-t\rangle} N(t)^\alpha d\mu_\Omega(t)\,_{|x=a} \\ &= \frac{N(a)^{\alpha+1}}{\Gamma_\Omega(\alpha)}\int_\Omega e^{\langle a, y-t\rangle} N(y-t) N(t)^\alpha\, d\mu_\Omega(t) \\ &= \frac{f_y(a)}{\Gamma_\Omega(\alpha)}\int_\Omega e^{-\langle e, P(a^{\frac{1}{2}})t\rangle} N(P(a^{\frac{1}{2}})(y-t)) N(P(a^{\frac{1}{2}})t)^\alpha\, d\mu_\Omega(t). \end{aligned}$$

Letting $b = P(a^{\frac{1}{2}})y$, the substitution $t := P(a^{-\frac{1}{2}})P(b^{\frac{1}{2}})\tau$ gives

$$(N^{\alpha+1}\partial_N N^{-\alpha} f_y)(a) = \frac{f_y(a)}{\Gamma_\Omega(\alpha)} N(y)^{1+\alpha} N(a)^{1+\alpha} \int_\Omega e^{-\langle b,\tau\rangle} N(e-\tau) N(\tau)^\alpha \, d\mu_\Omega(\tau).$$

Now, the well-known "binomial formula"

$$N(e+x) = \sum_{\nu=0}^{r} \binom{r}{\nu} \phi_\nu(x), \quad x \in X \tag{4.8}$$

(which follows from Theorem 1.2 and the knowledge of the norms of the ϕ_ν's) and the fact that for every $\mathbf{s} \in \mathbf{C}^r$ and $b \in \Omega$

$$\frac{1}{\Gamma_\Omega(\mathbf{s})} \int_\Omega e^{-\langle b,\tau\rangle} \phi_{\mathbf{s}}(\tau) \, d\mu_\Omega(\tau) = \phi_{\mathbf{s}}(b^{-1}) \tag{4.9}$$

(which follows from the analogous formula for the conical functions), imply

$$\int_\Omega e^{-\langle b,\tau\rangle} N(e-\tau) N(\tau)^\alpha \, d\mu(\tau) = \sum_{\nu=0}^{r} \binom{r}{\nu} \int_\Omega e^{-\langle b,\tau\rangle} \phi_{\mathbf{1}_\nu+\alpha}(\tau) \, d\mu_\Omega(\tau)$$

$$= \sum_{\nu=0}^{r} \binom{r}{\nu} \Gamma_\Omega(\mathbf{1}_\nu+\alpha) \, \phi_{\mathbf{1}_\nu+\alpha}(b^{-1}) = N(b)^{-\alpha} \sum_{\nu=0}^{r} \binom{r}{\nu} \Gamma_\Omega(\mathbf{1}_\nu+\alpha) \, \phi_\nu(b^{-1}).$$

We claim that for every $b \in \Omega$ and $1 \leq \nu \leq r$,

$$\phi_\nu(b^{-1}) = \phi_{r-\nu}(b) \, N(b)^{-1}. \tag{4.10}$$

Indeed, using (4.8) we have $N(e+tb^{-1}) = \sum_{\nu=0}^{r} \binom{r}{\nu} \phi_\nu(b^{-1}) \, t^\nu$, as well as

$$\begin{aligned} N(e+tb^{-1}) &= N(P(b^{-\frac{1}{2}})(b+te)) = N(b)^{-1} \, t^r \, N(e+t^{-1}b) \\ &= N(b)^{-1} \, t^r \sum_{k=0}^{r} \binom{r}{k} \phi_k(b) \, t^{-k}. \end{aligned}$$

Comparing the coefficients of t^ν in the two expansions, we obtain (4.10). It follows that

$$\begin{aligned} &(N^{\alpha+1} \, \partial_N N^{-\alpha} f_y)(a) \\ &\quad= \frac{f_y(a) \, N(y)^{1+\alpha} \, N(a)^{1+\alpha}}{\Gamma_\Omega(\alpha) \, N(b)^{1+\alpha}} \sum_{\nu=0}^{r} (-1)^\nu \binom{r}{\nu} \Gamma_\Omega(\mathbf{1}_\nu+\alpha) \, \phi_{r-\nu}(b) \\ &\quad= f_y(a) \sum_{\nu=0}^{r} (-1)^\nu \binom{r}{\nu} \frac{\Gamma_\Omega(\mathbf{1}_\nu+\alpha)}{\Gamma_\Omega(\alpha)} \, \phi_{r-\nu}(b) \\ &\quad= f_y(a) \sum_{\nu=0}^{r} \binom{r}{\nu} \prod_{j=1}^{\nu} (\lambda_j - \alpha) \, \phi_{r-\nu}(P(a^{\frac{1}{2}})y). \end{aligned}$$

Comparing this with (4.6), we conclude that

$$N^{\alpha+1}\,\partial_N N^{-\alpha} = \sum_{\nu=0}^{r} \binom{r}{\nu} \prod_{j=1}^{\nu}(\lambda_j-\alpha)\;\Delta_{r-\nu} = \sum_{k=0}^{r}\binom{r}{k}\prod_{j=1}^{r-k}(\lambda_j-\alpha)\;\Delta_k .$$

Using the relations $\alpha = \lambda_r - \lambda$ and $\frac{d}{r} = 1+\lambda_r$, we obtain (4.7). ∎

REMARK: The "binomial formula" (4.8) yields that for every $\nu = 1, 2, \ldots, r$ and every $x \in X$,

$$\phi_\nu(x) = \sum_{1\le i_1<i_2<\ldots<i_\nu\le r} \lambda_{i_1}\,\lambda_{i_2}\,\cdots\,\lambda_{i_\nu}/\binom{r}{\nu} = S_{r,\nu}(\lambda)/\binom{r}{\nu},$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is the sequence of eigenvalues of x, and $S_{r,\nu}$ is the elementary symmetric polynomial of degree ν in r variables.

Combining the definition $D_\ell(\lambda) = \prod_{k=0}^{\ell-1} D_1(\lambda+k)$ with Theorem 4.1, we obtain

COROLLARY 4.1 *For every $\lambda \in \mathbf{C}$ and $\ell \in \mathbf{N}$,*

$$D_\ell(\lambda) = \prod_{k=0}^{\ell-1}\sum_{\nu=0}^{r}\binom{r}{\nu}\prod_{j=\nu+1}^{r}(\lambda+k-\lambda_j)\;\Delta_\nu. \tag{4.11}$$

For any signature $\mathbf{m} \ge 0$ let $\Delta_{\mathbf{m}}$ be the differential operator associated with the spherical polynomial $\phi_{\mathbf{m}}$ via

$$(\Delta_{\mathbf{m}} f)(a) := \phi_{\mathbf{m}}(\frac{d}{dx})\, f(P(a^{\frac12}))_{|x=e}, \quad a \in \Omega. \tag{4.12}$$

Equivalently,

$$\Delta_{\mathbf{m}} e^{\langle x,y\rangle}{}_{|x=a} = \phi_{\mathbf{m}}(P(a^{\frac12})y)\; e^{\langle a,y\rangle}, \quad a \in \Omega. \tag{4.13}$$

Again, one can replace in (4.12) and (4.13) $P(a^{\frac12})$ by any $\psi \in G(\Omega)$ satisfying $\psi(e) = a$. Hence the operators $\Delta_{\mathbf{m}}$ are $G(\Omega)$-invariant, namely

$$\Delta_{\mathbf{m}}(f\circ\psi) = (\Delta_{\mathbf{m}} f)\circ\psi, \quad \forall \psi \in G(\Omega).$$

THEOREM 4.2 *For every $\lambda \in \mathbf{C}$ and $\ell \in \mathbf{N}$,*

$$\begin{aligned} D_\ell(\lambda) &= \sum_{\mathbf{m}\ge 0}^{(\ell)} \frac{\Gamma_\Omega(\frac{d}{r}+\ell)\,\Gamma_\Omega(\frac{d}{r}-\lambda-\mathbf{m}^*)}{\Gamma_\Omega(\frac{d}{r}+\ell-\mathbf{m}^*)\,\Gamma_\Omega(\frac{d}{r}-\ell-\lambda)}\;\frac{d_{\mathbf{m}}}{(\frac{d}{r})_{\mathbf{m}}}\;\Delta_{\mathbf{m}} \\ &= (\frac{d}{r}-\lambda-\ell)_{(\ell,\ldots,\ell)}\sum_{\mathbf{m}\ge 0}^{(\ell)}\frac{(-\ell)_{\mathbf{m}}}{(\lambda)_{\mathbf{m}}}\;\frac{d_{\mathbf{m}}}{(\frac{d}{r})_{\mathbf{m}}}\;\Delta_{\mathbf{m}}. \end{aligned} \tag{4.14}$$

Here $\mathbf{m}^* := (m_r, m_{r-1}, \ldots, m_1)$, $d_{\mathbf{m}} = dim(P_{\mathbf{m}})$, and the summation $\sum_{\mathbf{m}\ge0}^{(\ell)}$ extends over all $\mathbf{m} = (m_1, m_2, \ldots, m_r) \in \mathbf{N}^r$ with $\ell \ge m_1 \ge m_2 \ge \ldots \ge m_r \ge 0$.

Proof: The general binomial formula (1.15) and the relations

$$K_{\mathbf{m}}(x,e) = \frac{\phi_{\mathbf{m}}}{\|\phi_{\mathbf{m}}\|_F^2}, \quad \|\phi_{\mathbf{m}}\|_F^2 = \frac{(\frac{d}{r})_{\mathbf{m}}}{d_{\mathbf{m}}}$$

(see [FK2], Chapter XI) imply for $\ell \in \mathbf{N}$ and $x \in X$

$$N(e+x)^\ell = c \sum_{\mathbf{m}\geq 0}^{(\ell)} \frac{d_{\mathbf{m}}}{(\frac{d}{r})_{\ell-\mathbf{m}^*}\,(\frac{d}{r})_{\mathbf{m}}}\ \phi_{\mathbf{m}}(x), \tag{4.15}$$

where $c := (\frac{d}{r})_{(\ell,\ell,\ldots,\ell)}$, and $\mathbf{m}^*$ and $\sum_{\mathbf{m}\geq 0}{}^{(\ell)}$ are as in Theorem 4.2. Indeed, by (1.15),

$$N(e+x)^\ell = \sum_{\mathbf{m}\geq 0} (-\ell)_{\mathbf{m}} \frac{(-1)^{|\mathbf{m}|}\,d_{\mathbf{m}}}{(\frac{d}{r})_{\mathbf{m}}}\ \phi_{\mathbf{m}}(x).$$

From this (4.15) follows by the fact that $(-\ell)_{\mathbf{m}} = 0$ if $m_1 > \ell$, whereas in case $m_1 \leq \ell$,

$$(-\ell)_{\mathbf{m}}\,(-1)^{|\mathbf{m}|} = \frac{(\frac{d}{r})_{(\ell,\ell,\ldots,\ell)}}{(\frac{d}{r})_{\ell-\mathbf{m}^*}}.$$

As in the proof of Theorem 4.1, it is enough to prove that for every $\alpha \in \mathbf{C}$ with $\Re\alpha > \lambda_r$ and every $\ell \in \mathbf{N}$,

$$N^{\alpha+\ell}\,\partial_N^\ell\,N^{-\alpha} = c \sum_{\mathbf{m}\geq 0}^{(\ell)} \frac{(\alpha)_{\ell-\mathbf{m}^*}\,d_{\mathbf{m}}}{(\frac{d}{r})_{\ell-\mathbf{m}^*}\,(\frac{d}{r})_{\mathbf{m}}}\ \Delta_{\mathbf{m}}. \tag{4.16}$$

From this one obtains (4.14) by the substitution $\alpha = \frac{d}{r} - \ell - \lambda$. To prove (4.16), fix $a, y \in \Omega$ and let $f_y(x) := e^{\langle x,y\rangle}$. Then

$$\begin{aligned}(N^{\alpha+\ell}\,\partial_N^\ell\,N^{-\alpha} f_y)(a) &= \frac{N(a)^{\alpha+\ell}\,f_y(a)}{\Gamma_\Omega(\alpha)} \int_\Omega e^{-\langle a,t\rangle}\,N(y-t)^\ell\,N(t)^\alpha\,d\mu_\Omega(t)\\ &= \frac{N(b)^{\alpha+\ell}\,f_y(a)}{\Gamma_\Omega(\alpha)} \int_\Omega e^{-\langle b,u\rangle}\,N(e-u)^\ell\,N(u)^\alpha\,d\mu_\Omega(u),\end{aligned}$$

by the substitutions $b = P(a^{\frac12})y$ and $u = P(b^{-\frac12})P(a^{\frac12})t$. Using (4.15), (4.9), and

$$\phi_{\mathbf{m}}(x^{-1}) = \phi_{\ell-\mathbf{m}^*}(x)\,N(x)^{-\ell} \tag{4.17}$$

(a consequence of [FK2], Proposition VII.1.5), we obtain

$$(N^{\alpha+\ell}\,\partial_N^\ell\,N^{-\alpha} f_y)(a) = c\,\frac{f_y(a)}{\Gamma_\Omega(\alpha)} \sum_{\mathbf{m}\geq 0}^{(\ell)} \frac{\Gamma_\Omega(\mathbf{m}+\alpha)\,d_{\mathbf{m}}}{(\frac{d}{r})_{\ell-\mathbf{m}^*}\,(\frac{d}{r})_{\mathbf{m}}}\ \phi_{\ell-\mathbf{m}^*}(P(a^{\frac12})y).$$

With the change of variables $\mathbf{n} := \ell - \mathbf{m}^*$, the fact that $d_{\mathbf{m}} = d_{\mathbf{n}}$ (use (4.17) or the general formula for $d_{\mathbf{m}}$ in [U1]), the definition (4.12), and

$$(N^{\alpha+\ell}\,\partial_N^\ell\,N^{-\alpha} f_y)(a) = c\,f_y(a) \sum_{\mathbf{n}\geq 0}^{(\ell)} \frac{(\alpha)_{\ell-\mathbf{n}^*}\,d_{\mathbf{n}}}{(\frac{d}{r})_{\ell-\mathbf{n}^*}\,(\frac{d}{r})_{\mathbf{n}}}\ \phi_{\mathbf{n}^*}(P(a^{\frac12})y),$$

we obtain (4.16). ∎

COROLLARY 4.2 *The operators $\{\Delta_k\}_{k=1}^r$ are algebraically independent generators of the ring Diff$(\Omega)^{G(\Omega)}$ of $G(\Omega)$-invariant differential operators on Ω.*

PROOF: Comparing the two expansions (4.11) and (4.14) of $D_\ell(\lambda)$, we see that

$$\Delta_{\mathbf{m}} \in \mathbf{C}[\Delta_1, \Delta_2, \ldots, \Delta_r]$$

for every signature $\mathbf{m} \geq 0$. Since $\{\phi_{\mathbf{m}}\}_{\mathbf{m}\geq 0}$ is a basis for the space of spherical polynomials, the one-to-one correspondence between spherical polynomials and the elements of Diff$(\Omega)^{G(\Omega)}$ (see [FK2], Chapter XIV) implies that $\{\Delta_{\mathbf{m}}\}_{\mathbf{m}\geq 0}$ is a basis of Diff$(\Omega)^{G(\Omega)}$. Thus Diff$(\Omega)^{G(\Omega)} = \mathbf{C}[\Delta_1, \Delta_2, \ldots, \Delta_r]$. Since the minimal number of algebraic generators of Diff$(\Omega)^{G(\Omega)}$ is $r = rank(\Omega)$ [He], it follows that $\Delta_1, \Delta_2, \ldots, \Delta_r$ are algebraically independent. ∎

The *divided differences* of a C^1-function f on $\mathbf{R}$ are defined by

$$f^{[1]}(t_0, t_1) := \frac{f(t_0) - f(t_1)}{t_0 - t_1}$$

for $t_0 \neq t_1$, and $f^{[1]}(t_0, t_0) := f'(t_0)$. The higher order divided differences of a smooth enough function f are defined inductively by

$$f^{[n]}(t_0, t_1, \ldots, t_n) := g^{[1]}(t_{n-1}, t_n),$$

where $g(x) := f^{[n-1]}(t_0, t_1, \ldots, t_{n-2}, x)$. Then $f^{[n]}(t_0, t_1, \ldots, t_n)$ is symmetric in $t_0, t_1, \ldots, t_n$, and

$$f^{[n]}(t, t, \ldots, t) = \frac{1}{n!}\,\frac{d^n}{dt^n} f(t).$$

Moreover, if f is analytic in a domain $\mathcal{D} \subset \mathbf{C}$, then

$$f^{[n]}(t_0, t_1, \ldots, t_n) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\prod_{j=0}^n (\xi - t_j)}\, d\xi$$

for all $t_0, t_1, \ldots, t_n \in \mathcal{D}$ and every Jordan curve Γ in $\mathcal{D}$ whose interior contains $t_0, t_1, \ldots, t_n$ and is contained in $\mathcal{D}$. The divided differences of vector-valued maps are defined in the same way and have analogous properties. For convenience we put also $f^{[0]}(t) := f(t)$.

THEOREM 4.3 *Let $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbf{C}$ be distinct. Then $\{D_1(\alpha_j)\}_{j=1}^r$ are algebraically independent generators of Diff$(\Omega)^{G(\Omega)}$. Moreover, for $\ell = 1, 2, \ldots, r$,*

$$\Delta_\ell = D_1^{[r-\ell]}(\lambda_\ell, \lambda_{\ell+1}, \ldots, \lambda_r) / \binom{r}{\nu}, \tag{4.18}$$

where $D_1^{[r-\ell]}(\lambda_\ell, \ldots, \lambda_r)$ are the divided differences of order $r - \ell$ of $D_1(\lambda)$, evaluated at $(\lambda_\ell, \lambda_{\ell+1}, \ldots, \lambda_r)$.

Proof: Let $h_k(x) := \binom{r}{\ell}\prod_{j=k+1}^{r}(x-\lambda_j)$, $0 \le k \le r$. Then $h_k^{[m]}(x_0, x_1, \dots, x_m) \equiv 0$ whenever $m > r-k$, and $h_k^{[r-k]}(x_0, x_1, \dots, x_{r-k}) \equiv \binom{r}{\ell}$ for all choices of $x_0, x_1, \dots, x_{r-k}$. By Theorem 4.2, $D_1(\alpha) = \sum_{k=0}^{r} h_k(\alpha)\,\Delta_k$. Hence, for $1 \le \ell \le r$,

$$D_1^{[r-\ell]}(\alpha_\ell, \alpha_{\ell+1}, \dots, \alpha_r) = \sum_{k=0}^{\ell} h_k^{[r-\ell]}(\alpha_\ell, \alpha_{\ell+1}, \dots, \alpha_r)\,\Delta_k.$$

Solving this system of equations for the Δ_k's, we see that $\mathrm{Diff}(\Omega)^{G(\Omega)} = \mathbf{C}[\Delta_1, \Delta_2, \dots, \Delta_r]$ coincides with the ring generated by the operators $\{D_1^{[r-\ell]}(\alpha_\ell, \alpha_{\ell+1}, \dots, \alpha_r)\}_{\ell=1}^{r}$. If the $\{\alpha_j\}_{j=1}^{r}$ are distinct, then

$$D_1^{[r-\ell]}(\alpha_\ell, \alpha_{\ell+1}, \dots, \alpha_r) \;\in\; \mathbf{C}[D_1(\alpha_1), D_1(\alpha_2), \dots, D_1(\alpha_r)].$$

Hence,

$$\mathrm{Diff}(\Omega)^{G(\Omega)} \;=\; \mathbf{C}[\Delta_1, \Delta_2, \dots, \Delta_r] \;=\; \mathbf{C}[D_1(\alpha_1), D_1(\alpha_2), \dots, D_1(\alpha_r)].$$

The operators $\{D_1(\alpha_j\}_{j=1}^{r}$ are algebraically independent, since $\mathrm{Diff}(\Omega)^{G(\Omega)}$ cannot be algebraically generated by less than r elements. If $\alpha_j = \lambda_j$ for $j = 1, 2, \dots, r$, then $h_k^{[r-\ell]}(\alpha_\ell, \dots, \alpha_r) = 0$ for $k < \ell$. Thus, for $\ell = 1, 2, \dots, r$,

$$D_1^{[r-\ell]}(\alpha_\ell, \alpha_{\ell+1}, \dots, \alpha_r) \;=\; h_\ell^{[r-\ell]}(\alpha_\ell, \alpha_{\ell+1}, \dots, \alpha_r)\,\Delta_\ell = \binom{r}{\ell}\Delta_\ell. \qquad \blacksquare$$

Remark: The first statement in Theorem 4.3 was proved independently also by A. Korányi [FK2] and Z. Yan [Y1]. Our result is slightly stronger, giving the exact formula (4.18).

Combining Theorems 3.2 and 4.2 (or, 4.1) we obtain integral formulas for the invariant hermitian forms $\langle\cdot,\cdot\rangle_{\lambda,j}$, $\lambda \in \mathbf{P}(D)$, $0 \le j \le q(\lambda)$.

Corollary 4.3 *Let $\lambda \in \mathbf{P}(D)$, $\ell \in \mathbf{N}$ and assume that $\lambda + \ell \ge \frac{d}{r} = \lambda_r + 1$. Let $q = q(\lambda)$, $0 \le j \le q$, and $\nu = q - j$. Consider the $G(\Omega)$-invariant differential operator*

$$T_{\lambda,j} := \gamma \sum_{\mathbf{m}\ge 0}^{(\ell)} c_{\mathbf{m}}(\lambda,\ell)\,\frac{d_{\mathbf{m}}}{(\frac{d}{r})_{\mathbf{m}}}\;\Delta_{\mathbf{m}}, \tag{4.19}$$

where γ is given by (3.4), and for every $\mathbf{m} \ge 0$ with $m_1 \le \ell$

$$c_{\mathbf{m}}(\lambda,\ell) := \frac{1}{\nu!}\Big(\frac{\partial}{\partial\xi}\Big)^{\ell}\left(\frac{\Gamma_\Omega(\frac{d}{r}+\ell)\,\Gamma_\Omega(\frac{d}{r}-\xi-\mathbf{m}^*)}{\Gamma_\Omega(\frac{d}{r}+\ell-\mathbf{m}^*)\,\Gamma_\Omega(\frac{d}{r}-\ell-\xi)}\right)_{|\xi=\lambda}. \tag{4.20}$$

Then $T_{\lambda,j}$ is defined on all analytic functions on D, and for all $f, g \in \mathcal{H}_{\lambda,j}$

$$\langle f, g\rangle_{\lambda,j} \;=\; \langle T_{\lambda,j} f, g\rangle_{\lambda+\ell}. \tag{4.21}$$

In particular, if $\lambda + \ell > p - 1$ or $\lambda + \ell = \frac{d}{r}$ then we have

$$\langle f, g\rangle_{\lambda,j} \;=\; \int_D (T_{\lambda,j} f)(z)\,\overline{g(z)}\,d\mu_{\lambda+\ell}(z) \;\text{ and }\; \langle f, g\rangle_{\lambda,j} \;=\; \int_S (T_{\lambda,j} f)(\xi)\,\overline{g(\xi)}\,d\sigma(\xi) \tag{4.22}$$

respectively.

The case $\lambda = \lambda_r$ is particularly simple, since then $\frac{d}{r} - \lambda_r = 1$, and we can use (4.7) rather than (4.14).

COROLLARY 4.4 *Let D be a Cartan domain of tube type and rank $r \geq 2$ in $\mathbf{C}^d$, $d \geq 3$. Then*

$$\langle f, g\rangle_{\lambda_r,0} = \langle \beta \sum_{\nu=0}^{r-1} \binom{r}{\nu} \prod_{i=2}^{r-\nu} \lambda_i \; \Delta_\nu f, g\rangle_{H^2(S)}, \quad \textit{where} \quad \beta := \prod_{i=2}^{r} \lambda i. \tag{4.23}$$

PROOF: In this case $q = q(\lambda_r) = 1$, $j = 0$, and $\nu = q - j = 1$. We choose $\ell = 1$, so $\lambda_r + \ell = \frac{d}{r}$. In order to apply Theorem 3.2 we use Theorem 4.1, and compute

$$\begin{aligned} D_1^1(\lambda_r) = \frac{\partial}{\partial \xi} D_1(\xi)_{|\xi=\lambda} &= \frac{\partial}{\partial \xi} \left(\sum_{\nu=0}^{r} \binom{r}{\nu} \prod_{i=\nu+1}^{r} (\xi - \lambda_i) \; \Delta_\nu \right)_{|\xi=\lambda_r} \\ &= \sum_{\nu=0}^{r-1} \binom{r}{\nu} \prod_{i=\nu+1}^{r-1} (\lambda_r - \lambda_i) \; \Delta_\nu = \sum_{\nu=0}^{r-1} \binom{r}{\nu} \prod_{i=2}^{r-\nu} \lambda_i \; \Delta_\nu . \end{aligned}$$

Using this, (4.23) follows from

$$\beta := \frac{\partial}{\partial \xi} \left(\prod_{i=1}^{r} (\xi - \lambda_i) \right)_{\xi=\lambda} = \prod_{i=1}^{r-1} (\lambda_r - \lambda_i) = \prod_{i=2}^{r} \lambda_i . \qquad \blacksquare$$

EXAMPLE 4.2. Let D be the Cartan domain of rank $r = 2$ in $\mathbf{C}^d$ (the Lie ball), $d \geq 3$. Then

$$\langle f, g\rangle_{\frac{d-2}{2},0} = \langle (\frac{2}{d-2} R + I) f, g\rangle_{H^2(S)}. \tag{4.24}$$

Namely, in this case $\lambda = \lambda_2 = \frac{d-2}{2}$, $q = q(\lambda) = 1$, $j = 0$, and $\nu = q - j = 1$. With $\ell = 1$, $\lambda + \ell = \frac{d}{2} = \lambda_2 + 1 = \frac{d}{r}$ we get by using Theorem 3.2 and Corollary 3.2,

$$\begin{aligned} \langle f, g\rangle_{\frac{d-2}{2},0} &= \gamma \langle D_1^1(\frac{d-2}{2}) f, g\rangle_{\frac{d}{2}} \\ &= \gamma \langle (R + \frac{d-2}{2} I) f, g\rangle_{H^2(S)} = \langle (\frac{2}{d-2} R + I) f, g\rangle_{H^2(S)}. \end{aligned}$$

Since the Shilov boundary S of D is given by

$$S = \{ e^{i\theta}(x_1, ix_2, ix_3, \ldots, ix_d); \theta \in \mathbf{R}, \; \sum_{j=1}^{d} x_j^2 = 1 \} \equiv S^1 \cdot S^{d-1},$$

the unique K-invariant probability measure on S is $d\sigma(e^{i\theta}(x_1, ix')) = \frac{d\theta}{2\pi} d\nu_{d-1}(x)$, where ν_{d-1} is the unique $O(d-1)$-invariant probability measure on S^{d-1}. Thus (4.24) provides a very concrete formula for the inner product $\langle \cdot, \cdot \rangle_{\frac{d-2}{2},0}$.

5 Integration over boundary orbits of $Aut(D)$

In this section we obtain formulas for the invariant inner products in terms of integration over an orbit of $Aut(D)$ on the boundary ∂D. We focus on the inner products $\langle\cdot,\cdot\rangle_{\lambda_2,0} = \langle\cdot,\cdot\rangle_{\frac{a}{2}}$, and conjecture that our method can be generalized for the derivation of similar formulas for the inner products $\langle\cdot,\cdot\rangle_{\lambda_j,0} = \langle\cdot,\cdot\rangle_{\lambda_j}$, $\lambda_j = (j-1)\frac{a}{2}$, $j = 3,4,\ldots,r$, in terms of integration on an appropriate boundary orbit. (Notice that the case $j=1$ is trivial, since $\lambda_1 = 0$ and $\mathcal{H}_{0,0} = \mathcal{H}_0 = \mathbf{C}1$).

In order to describe the *facial structure* of a Cartan domain of tube-type $D \subset \mathbf{C}^d$ [Lo], [A1], let S_ℓ be the compact, real analytic manifold of tripotents in Z of rank $\ell = 1,2,\ldots,r$. The group K acts transitively and irreducibly on S_ℓ. Let σ_ℓ be the unique K-invariant probability measure on S_ℓ given by

$$\int_{S_\ell} f \, d\sigma_\ell = \int_K f(k(v_\ell)) \, dk, \tag{5.1}$$

where v_ℓ is any fixed element of S_ℓ. For any tripotent v let $Z = Z_1(v) + Z_{\frac{1}{2}}(v) + Z_0(v)$ be the corresponding Peirce decomposition. Then $D_v := D \cap Z_0(v)$ is a Cartan domain of tube-type, which is the open unit ball of the JB*-algebra $Z_0(v)$. If $v \in S_\ell$ then the rank of D_v is $r_v := r - \ell$, its characteristic multiplicity is $a_v := a$ if $\ell \le r-2$ and $a_v = 0$ if $\ell = r-1$, and the genus is $p_v = p - \ell a$. The set $v + D_v$ is a face of the closure $\overline{D}$ of D. For any function f on $\overline{D}$ let f_v be the function on $\overline{D_v}$ defined by

$$f_v(z) := f(v+z), \quad z \in \overline{D_v}. \tag{5.2}$$

The fundamental polynomial "h" of $Z_0(v)$ is defined by

$$h_v(z,w) := h(z,w), \quad z,w \in Z_0(v). \tag{5.3}$$

For $\ell = 1,2,\ldots,r$, $\partial_\ell D := \cup_{v \in S_\ell}(v + D_v)$ is an orbit of G: $\partial_\ell D = G(v_\ell)$. If $v \in S_r$ is a maximal tripotent, then $D_v = Z_0(v) = \{0\}$. Hence $\partial_r D = S_r = S$ is the Shilov boundary. In particular, S is a G-orbit. The only tripotent of rank 0 is $0 \in Z$, and $D = D_0$ is also a G-orbit. Thus the decomposition of $\overline{D}$ into G-orbits is

$$\overline{D} = D \cup \bigcup_{\ell=1}^{r} \partial_\ell D.$$

For every tripotent $v \in Z$ and $\lambda > p_v - 1$ consider the probability measure $\mu_{v,\lambda}$ on D_v, defined via

$$\int_{D_v} f \, d\mu_{v,\lambda} := c_{v,\lambda} \int_{D_v} f(z) \, h_v(z,z)^{\lambda - p_v} \, dm_v(z), \tag{5.4}$$

where m_v is the Lebesgue measure on D_v and $c_{v,\lambda}$ is the normalization factor. Similarly, one defines a probability measure σ_v on the Shilov boundary S_v of D_v, via

$$\int_{S_v} f \, d\sigma_v := \int_{K_v} f(k(v')) \, dk,$$

where v' is any tripotent orthogonal to v and $K_v := \{k \in K; k(Z_\nu(v)) = Z_\nu(v)\}$, $\nu = 0, 1/2, 1$, so that $K_v(v') = S_v$. The combination of $\mu_{v,\lambda}$ and σ_ℓ yields K-invariant probability measures $\mu_{\ell,\lambda}$ on $\partial_\ell D$, $1 \le \ell \le r-1$, $\lambda > p - \ell\, a - 1$, via

$$\int_{\partial_\ell D} f d\mu_{\ell,\lambda} := \int_{S_\ell} \left(\int_{D_v} f_v(z)\, d\mu_{v,\lambda}(z) \right) d\sigma_\ell(v).$$

Next, consider the "sphere bundle" B_ℓ, $1 \le \ell \le r$, whose base is S_ℓ and the fiber at each $v \in S_\ell$ is $v + S_v$ (where $S_v := \partial_{r-\ell} D_v$ is the Shilov boundary of D_v). The group K acts on B_ℓ naturally, and this action is transitive. The combination of the measures σ_v, $v \in S_\ell$ and σ_ℓ yields K-invariant probability measures ν_ℓ on B_ℓ via

$$\int_{B_\ell} f\, d\nu_\ell := \int_{S_\ell} \left(\int_{S_v} f(v+\xi)\, d\sigma_v(\xi) \right) d\sigma_\ell(v).$$

For $v \in S_\ell$, consider the symmetric cone Ω_v in $Z_0(v)$, and let $\Delta_1^{(v)}, \Delta_2^{(v)}, \ldots, \Delta_{r-\ell}^{(v)}$ be the canonical generators of the ring $\mathrm{Diff}(\Omega_v)^{G(\Omega_v)}$ as in section 4. We also denote

$$\Delta_0^{(v)} = I,\ \mathbf{\Delta}^{(v)} := (\Delta_1^{(v)}, \Delta_2^{(v)}, \ldots, \Delta_{r-\ell}^{(v)}), \text{ and } \lambda_j = (j-1)\frac{a}{2},\ 0 \le j \le r.$$

CONJECTURE: *For every $2 \le j \le r$ and every $\lambda > \lambda_{j-1}$ there exists a positive function $p_{j,\lambda} \in C^\infty([0,\infty)^{j-1})$, so that the inner product $\langle \cdot, \cdot \rangle_{\lambda_j} = \langle \cdot, \cdot \rangle_{\lambda_j, 0}$ is given by*

$$\langle f, g \rangle_{\lambda_j} = \int_{S_{r-j+1}} \langle p_{j,\lambda}(\mathbf{\Delta}^{(v)}) f_v, g_v \rangle_{\mathcal{H}_\lambda(D_v)}\, d\sigma_{r-j+1}(v). \tag{5.5}$$

Moreover, if $\lambda = \lambda_{j-1} + 1 = dim(D_v)/rank(D_v)$, then $p_j := p_{j,\lambda}$ is a polynomial with positive coefficients.

If λ is chosen appropriately then (5.5) becomes an integral formula for $\langle f, g \rangle_{\lambda_j}$. For instance, if $\lambda = \lambda_{j-1} + 1$ in (5.5), then we have $\mathcal{H}_\lambda(D_v) = H^2(S_v)$, and (5.5) becomes

$$\langle f, g \rangle_{\lambda_j} = \int_{S_{r-j+1}} \left(\int_{S_v} (p_{j,\lambda}(\mathbf{\Delta}^{(v)}) f_v)(\xi)\, \overline{g_v(\xi)} d\sigma_v(\xi) \right) d\sigma_{r-j+1}(v). \tag{5.6}$$

Also, if $\lambda > (j-2)a + 1$ in (5.5) then $\mathcal{H}_\lambda(D_v) = L_a^2(D_v, \mu_{v,\lambda})$, and (5.5) becomes

$$\langle f, g \rangle_{\lambda_j} = \int_{S_{r-j+1}} \left(\int_{D_v} (p_j(\mathbf{\Delta}^{(v)}) f_v)(z)\, \overline{g_v(z)}\, d\mu_{v,\lambda}(z) \right) d\sigma_{r-j+1}(v). \tag{5.7}$$

Note that the integral in (5.7) can be expressed as an integral on $\partial_{r-j+1} D$ with respect to $d\mu_{r-j+1,\lambda}$. Similarly, (5.6) is an integral on B_{r-j+1} with respect to ν_{r-j+1}.

Integral formulas for $\langle f,g\rangle_{a/2}$ via integration on $\partial_{r-1}D$

In what follows we shall establish (5.5) for $j=2$ (i.e. $\lambda_2=\frac{a}{2}$) in two important special cases, namely for Cartan domains of type I and IV. Our method suggests an approach for the general case. For $j=2$ (5.5) becomes

$$\langle f,g\rangle_{\frac{a}{2}}=\int_{S_{r-1}} d\sigma_{r-1}(v)\langle p_\lambda(R^{(v)})f_v,g_v\rangle_{\mathcal{H}_\lambda(\mathbf{D})}, \tag{5.8}$$

where $p_\lambda(x)=p_{2,\lambda}(x)\in C^\infty([0,\infty))$ is a positive function, $\Delta_1^{(v)}=R^{(v)}$, where $R^{(v)}$ is the localized radial derivative (i.e. the radial derivative in $Z_0(v)$), and $D_v\equiv\mathbf{D}=\{z\in\mathbf{C};|z|<1\}$. We will show that in our two cases

$$p_\lambda(x)=\frac{\Gamma(x+\lambda)}{\Gamma(\lambda)\Gamma(x+1)}\ q(x),$$

where $q(x)$ is a polynomial with positive rational coefficients. In particular, for $\lambda=1,2,\ldots$, $p_\lambda(x)$ itself is a polynomial with positive rational coefficients. If λ is chosen appropriately, then (5.8) becomes an integral formula analogous to (5.6) or (5.7). For $\lambda=1$, (5.8) becomes

$$\langle f,g\rangle_{\frac{a}{2}}=\int_{S_{r-1}} d\sigma_{r-1}(v)\langle p_1(R^{(v)})f_v,g_v\rangle_{H^2(\mathbf{T})}, \tag{5.9}$$

and for $\lambda>1$, (5.8) becomes

$$\langle f,g\rangle_{\frac{a}{2}}=\int_{S_{r-1}} d\sigma_{r-1}(v)\langle p_\lambda(R^{(v)})f_v,g_v\rangle_{L^2(\mathbf{D},\mu_\lambda)}. \tag{5.10}$$

Lemma 5.1 *The right hand side of (5.5) is K-invariant. Consequently, the right hand sides of (5.6), (5.7), (5.8), (5.9), and (5.10) are K-invariant.*

Proof: Let $\ell=r-j+1$, and note that for each fixed smooth function f the maps $S_\ell\ni v\mapsto\Delta_i^{(v)}(f_v)$, $1\le i\le j-1$, are K-invariant, in the sense that

$$\Delta_i^{(k(v))}(f_{k(v)})\circ k=\Delta_i^{(v)}((f\circ k)_v),\quad \forall k\in K,\ \forall v\in S_\ell.$$

From this it follows that if $v_\ell\in S_\ell$ is any fixed element, then

$$\begin{aligned}&\int_{S_\ell}\langle p_{j,\lambda}(\mathbf{\Delta}^{(v)})f_v,g_v\rangle_{\mathcal{H}_\lambda(D_v)}d\sigma_\ell(v)\\ =&\int_K\langle p_{j,\lambda}(\mathbf{\Delta}^{(v_\ell)})(f\circ k)_{v_\ell},(g\circ k)_{v_\ell}\rangle_{\mathcal{H}_\lambda(D_{v_\ell})}dk.\end{aligned}$$

The K-invariance of the right hand side of (5.5) follows from the invariance of the Haar measure dk. ∎

Since $\mathcal{M}_0^{(\frac{a}{2})}=\sum_{m=0}^\infty P_{(m,0,0,\ldots)}$ and

$$\langle f,g\rangle_{\frac{a}{2}}=\sum_{\mathbf{m}=(m,0,\ldots,0),0\le m<\infty}\frac{\langle f_\mathbf{m},g_\mathbf{m}\rangle_\mathcal{F}}{(\frac{a}{2})_\mathbf{m}},$$

in order to establish (5.8) it is enough, by the K-invariance of both sides, to find positive functions $p_\lambda(x) \in C^\infty([0,\infty))$ so that (5.8) holds for the functions $f(z) = g(z) = N_1^m(z)$, $m \geq 0$. This is equivalent to

$$\int_{S_{r-1}} d\sigma_{r-1}(v) \langle p_\lambda(R^{(v)})(N_1^m)_v, (N_1^m)_v\rangle_{H_\lambda(\mathbf{D})} = \frac{m!}{(\frac{a}{2})_m}. \tag{5.11}$$

Fix a frame $e_1, e_2, \ldots, e_r$ in Z. Then $N_1(z) = (z, e_1)$, where $(\cdot,\cdot)$ is the unique K-invariant inner product on Z for which $(v,v) = 1$ for every minimal tripotent v. Let $e' := e_2 + e_3 + \ldots + e_r$. Then for $z = k(\xi e_1 + e')$ with $k \in K$ and $\xi \in \mathbf{T}$, we have

$$N_1^m(z) = (\xi k(e_1) + k(e'), e_1)^m = \sum_{\ell=0}^{m} \binom{m}{\ell} (k(e_1), e_1)^\ell \, (k(e'), e_1)^{m-\ell} \, \xi^\ell.$$

Thus, for $v = k(e')$, $m \geq 0$ and any continuous function f we have

$$(f(R^{(v)})N_1^m)(z) = \sum_{\ell=0}^{m} \binom{m}{\ell} (k(e_1), e_1)^\ell \, (k(e'), e_1)^{m-\ell} \, f(\ell) \, \xi^\ell.$$

Let us define

$$J_{m,\ell} := \int_K |(k(e_1), e_1)|^{2\ell} \, |(k(e'), e_1)|^{2(m-\ell)} \, dk, \quad 0 \leq \ell \leq m < \infty. \tag{5.12}$$

It follows that the function p_λ should satisfy

$$\int_{S_{r-1}} d\sigma_{r-1}(v) \, \langle p_\lambda(R^{(v)})(N_1^m)_v, (N_1^m)_v\rangle_{\mathcal{H}_\lambda(\mathbf{D})} = \sum_{\ell=0}^{m} J_{m,\ell} \binom{m}{\ell}^2 \frac{\ell!}{(\lambda)_\ell} \, p_\lambda(\ell).$$

Thus (5.11) becomes

$$\sum_{\ell=0}^{m} J_{m,\ell} \binom{m}{\ell}^2 q_\ell = \frac{m!}{(\frac{a}{2})_m}, \quad m = 0, 1, 2, \ldots, \tag{5.13}$$

where the numbers

$$q_\ell := \frac{\ell!}{(\lambda)_\ell} \, p_\lambda(\ell), \quad \ell = 0, 1, 2, \ldots \tag{5.14}$$

do not depend on λ. The infinite system of linear equations (5.13) in the unknowns $\{q_\ell\}_{\ell=0}^\infty$ corresponds to the lower triangular matrix $A = (a_{m,\ell})_{m,\ell=0}^\infty$, where $a_{m,\ell} = J_{m,\ell}\binom{m}{\ell}^2$ for $m \geq \ell$, and $a_{m,\ell} = 0$ for $m < \ell$. Since $a_{m,m} > 0$ for $m = 0, 1, 2, \ldots$, there exists a unique solution $\{q_\ell\}_{\ell=0}^\infty$ to (5.13). There are many smooth functions which interpolate the values $\{q_\ell\}_{\ell=0}^\infty$. We will show that $q_\ell > 0$ for every $\ell \geq 0$, and prove that $\{q_\ell\}_{\ell=0}^\infty$ can be interpolated by a polynomial of degree $r-1$ with positive coefficients. For Cartan domains of type I and IV, we will solve the system (5.13) by calculating explicitly the numbers $J_{m,\ell}$ and applying powers of the difference operator

$$\delta(f)(t) := f(t) - f(t-1), \quad t \in \mathbf{R}.$$

If f is defined only on $[0,\infty)$ then we define $\delta(f) := \delta(F)$, where $F(t) := f(t)$ for $0 \le t$ and $F(t) = 0$ for $0 > t$. Similarly, δ can be defined on two-sided sequences (i.e. on functions on $\mathbf{Z}$) or on sequences (i.e. functions on $\mathbf{N}$). The powers of δ are defined inductively by $\delta^{n+1} := \delta \circ \delta^n$.

CASE 1: CARTAN DOMAINS OF TYPE I. Let $D = D(I_{r,r}) := \{z \in M_{r,r}(\mathbf{C}); \|z\| < 1\}$. The rank of D is r, the dimension is $d = r^2$, the genus is $p = 2r$, and the characteristic multiplicity is $a = 2$. To every $k \in K$ there correspond $u, w \in U(r)$ (the *unitary group*) so that $det(u) = det(w)$, and

$$k(z) = uzw^*, \quad z \in D. \tag{5.15}$$

Thus $\int_K f(k(z))\, dk = \int_{U(r)} \int_{U(r)} f(uzw^*)\, du\, dw$, where dk is the Haar measure of K. Choose the canonical frame of matrix units $e_j := e_{j,j},\ j = 1, 2, \ldots, r$, and denote $e = \sum_{j=1}^r e_j$ and $e' := e - e_1 = \sum_{j=2}^r e_j$.

PROPOSITION 5.1 *Let $D = D(I_{r,r})$. Then for every integers m, ℓ with $0 \le \ell \le m < \infty$, we have*

$$J_{m,\ell} = \frac{(r-1)\,(\ell\,!)^2\,(m-\ell)!\,(m-\ell+r-2)!}{(r)_m\,(m+r-1)!}. \tag{5.16}$$

PROOF: Let $k \in K$ be given by (5.15). Then $(k(e_1), e_1) = u_{1,1}\overline{w_{1,1}}$ and $(k(e'), e_1) = \sum_{j=2}^r u_{1,j}\overline{w_{1,j}}$. Thus, for $0 \le \ell \le m < \infty$,

$$J_{m,\ell} = \int_{U(r)} \int_{U(r)} |u_{1,1}|^{2\ell} |w_{1,1}|^{2\ell} |\sum_{j=2}^r u_{1,j}\overline{w_{1,j}}|^{2(m-\ell)}\, du\, dw.$$

This integral can be written as an integral on the product of the unit spheres $\partial\mathbf{B}_r \subset \mathbf{C}^r$ with respect to the $U(r)$-invariant probability measure σ:

$$J_{m,\ell} = \int_{\partial\mathbf{B}_r} \int_{\partial\mathbf{B}_r} |\xi_1^\ell|^2 |\eta_1^\ell|^2 |(\xi', \eta')|^{2(m-\ell)}\, d\sigma(\xi)\, d\sigma(\eta),$$

where $\xi' := (\xi_2, \ldots, \xi_r)$ and $\eta' := (\eta_2, \ldots, \eta_r)$. Now, by the $U(r)$-invariance,

$$\begin{aligned} &\int_{\partial\mathbf{B}_r} |\xi_1^\ell|^2 |(\xi', \eta')|^{2(m-\ell)}\, d\sigma(\xi) \\ &= \|\eta'\|^{2(m-\ell)} \int_{\partial\mathbf{B}_r} |\xi_1^\ell|^2 |\xi_2^{m-\ell}|^2\, d\sigma(\xi) \\ &= \|\eta'\|^{2(m-\ell)} \|\xi_1^\ell \xi_2^{m-\ell}\|^2_{\mathcal{H}_r(D)} = \|\eta'\|^{2(m-\ell)} \frac{\ell!(m-\ell)!}{(r)_m}. \end{aligned}$$

It follows by using [Ru], 1.4.5, that

$$\begin{aligned} J_{m,\ell} &= \frac{\ell!(m-\ell)!}{(r)_m} \int_{\partial\mathbf{B}_r} |\eta_1^\ell|^2 (1 - |\eta_1|^2)^{m-\ell}\, d\sigma(\eta) \\ &= \frac{\ell!(m-\ell)!}{(r)_m} (r-1) \int_0^1 t^\ell (1-t)^{m-\ell+r-2}\, dt \end{aligned}$$

$$= \frac{\ell!(m-\ell)!}{(r)_m}(r-1)B(\ell+1, m-\ell+r-1)$$

$$= \frac{(r-1)(\ell!)^2(m-\ell)!(m-\ell+r-2)!}{(r)_m(m+r-1)!}. \qquad \blacksquare$$

COROLLARY 5.1 *For $D = D(I_{r,r})$ the system of equations (5.13) is equivalent to the system*

$$\sum_{\ell=0}^{m} \frac{(m-\ell+r-2)!}{(m-\ell)!}\, q_\ell = (r-2)! \binom{m+r-1}{r-1}^2, \quad m = 0,1,2,\ldots. \tag{5.17}$$

PROPOSITION 5.2 *For every $r \geq 2$ there exists a polynomial $q(x) = q_r(x)$ of degree $r-1$ with positive rational coefficients, so that $q(\ell) = q_\ell$ for $\ell = 0,1,2,\ldots$, where $\{q_\ell\}_{\ell=0}^{\infty}$ is the unique solution of (5.17).*

For small values of r it is easy to solve (5.17) explicitly by applying powers of δ. Thus,

$$q_2(x) = 2x+1, \quad q_3(x) = 3x^2+3x+1, \quad \text{and} \quad q_4(x) = \frac{1}{3}(10x^3+15x^2+11x+3).$$

The proof in the general case requires more preparation. Define

$$f_n(x) := (x+1)_n = \prod_{j=1}^{n}(x+j), \quad n \geq 1, \text{ and } g_n(x) := \prod_{j=0}^{n}(x+j)^2, \ n \geq 0. \tag{5.18}$$

Then $g_n(x+1) = f_{n+1}(x)^2$, and

$$(\delta^k f_n)(x) = n(n-1)\cdots(n-k+1)\, f_{n-k}(x), \quad k \geq 0, \tag{5.19}$$

where δ is defined by $\delta(f)(x) := f(x) - f(x-1)$. Indeed, (5.19) is trivial for $k = 0$. For $k = 1$ and all n we have

$$\delta(f_n)(x) = \prod_{j=1}^{n}(x+j) - \prod_{j=1}^{n}(x+j-1) = \prod_{j=1}^{n-1}(x+j)\,(x+n-x) = n\, f_{n-1}(x).$$

Assuming (5.19) for k, let $n > k$ and compute $\delta^{k+1}(f_n)(x) = n(n-1)\cdots(n-k+1)\,\delta(f_{n-k})(x) = n(n-1)\cdots(n-k+1)(n-k)f_{n-k-1}(x)$. This establishes (5.19).

Next, define an operator σ, analogous to δ, via

$$(\sigma f)(x) := f(x) + f(x-1), \quad x \in \mathbf{R}.$$

Clearly, $\delta\sigma = \sigma\delta$, and both σ and δ commute with all the translation operators

$$(\tau_c f)(x) := f(x+c).$$

Denote by $\mathcal{P}_+$ the set of polynomials in one variable with non-negative coefficients.

Lemma 5.2 *Let $f(x)$ be a polynomial and let $n, m \in \mathbf{N}$. If $\delta^n f \in \mathcal{P}_+$, then $\delta^{n+j}\tau_{m/2} f \in \mathcal{P}_+$ for every integer $0 \le j \le m$.*

Proof: Since δ commutes with translations, we may assume that $n = 0$ and $m = 1$. It is therefore enough to check that $\delta\tau_{1/2}x^k \in \mathcal{P}_+$ for every $k \in \mathbf{N}$. This follows from the binomial expansion:

$$\delta\tau_{1/2}x^k = (x+\frac{1}{2})^k - (x-\frac{1}{2})^k = \sum_{j=0}^{[\frac{k-1}{2}]} \binom{k}{2j+1} 2^{-2j} x^{k-2j-1}. \qquad \blacksquare$$

Lemma 5.3 *Let $f(x)$ be a polynomial and let $n \in \mathbf{N}$. Assume that $\delta^j\sigma^{n-j} f \in \mathcal{P}_+$ for every $0 \le j \le n$. Then $\delta^j\sigma^{n-j}\left((x+c)^k f(x)\right) \in \mathcal{P}_+$ for every $k \in \mathbf{N}$, $c \ge \frac{n}{2}$ and $0 \le j \le n$.*

Proof: Again, since δ and σ commute with translations, it is enough to assume that $k = 1$. We shall prove the assertion by induction on n. The case $n = 0$ is trivial since $\mathcal{P}_+$ is closed under sums and products. Assume that $n > 0$ and that the assertion holds for $n - 1$. A computation yields

$$\delta\left((x+\frac{n}{2})f(x)\right) = (x+\frac{n-1}{2})(\delta f)(x) + \frac{1}{2}(\sigma f)(x) \tag{5.20}$$

and

$$\sigma\left((x+\frac{n}{2})f(x)\right) = (x+\frac{n-1}{2})(\sigma f)(x) + \frac{1}{2}(\delta f)(x). \tag{5.21}$$

If $0 < j \le n$ then using (5.20) we get

$$\delta^j\sigma^{n-j}\left((x+\frac{n}{2})f(x)\right) = \delta^{j-1}\sigma^{(n-1)-(j-1)}\left((x+\frac{n-1}{2})(\delta f)(x) + \frac{1}{2}(\sigma f)(x)\right).$$

By assumption,

$$\delta^{j-1}\sigma^{(n-1)-(j-1)}\sigma f = \delta^{j-1}\sigma^{n-(j-1)} f \in \mathcal{P}_+, \quad \text{for } 0 < j \le n.$$

Similarly,

$$\delta^{j-1}\sigma^{(n-1)-(j-1)}\delta f = \delta^j\sigma^{n-j} f \in \mathcal{P}_+ \quad \text{for } 0 < j \le n.$$

Thus, by the induction hypothesis on $n-1$,

$$\delta^{j-1}\sigma^{(n-1)-(j-1)}\left((x+\frac{n-1}{2})\delta f(x)\right) \in \mathcal{P}_+, \quad \text{for } 0 < j \le n.$$

Next, using (5.21) we get

$$\sigma^n\left((x+\frac{n}{2})f(x)\right) = \sigma^{n-1}\left((x+\frac{n-1}{2})\sigma f(x) + \frac{1}{2}\delta f(x)\right).$$

By assumption, $\sigma^{n-1}\delta f(x) \in \mathcal{P}_+$ and $\delta^\ell\sigma^{n-1-\ell}\sigma f(x) \in \mathcal{P}_+$ for $0 \le \ell \le n-1$. Thus, by the induction hypothesis, $\delta^\ell\sigma^{n-1-\ell}\left((x+\frac{n-1}{2})\sigma f(x)\right) \in \mathcal{P}_+$ for $0 \le \ell \le n-1$, and in particular $\sigma^{n-1}\left((x+\frac{n-1}{2})\sigma f(x)\right) \in \mathcal{P}_+$. It follows that $\sigma^n\left((x+\frac{n}{2})f(x)\right) \in \mathcal{P}_+$. This completes the induction step. $\blacksquare$

LEMMA 5.4 *Let $g_n(x)$ be the polynomial defined by (5.18). Then $\delta^i\sigma^j g_n \in \mathcal{P}_+$ whenever $i+j \le n$.*

PROOF: We proceed by induction on n. The case $n=0$ is trivial, since $g_0(x) = x^2 \in \mathcal{P}_+$. Assume that $n>0$ and that $\delta^i\sigma^j g_{n-1} \in \mathcal{P}_+$ whenever $i+j \le n-1$. A computation yields

$$\delta g_n(x) = 2(n+1)(x+\frac{n-1}{2})\, g_{n-1}(x) \tag{5.22}$$

and

$$\sigma g_n(x) = 2\left((x+\frac{n-1}{2})^2 + (\frac{n+1}{2})^2\right) g_{n-1}(x). \tag{5.23}$$

Now assume $i+j \le n$. If $i>0$, (5.22) yields

$$\delta^i\sigma^j g_n(x) = \delta^{i-1}\sigma^j(\delta g_n(x)) = 2(n+1)\delta^{i-1}\sigma^j\left((x+\frac{n-1}{2})g_{n-1}(x)\right),$$

and by induction hypothesis and Lemma 5.3

$$\delta^{i-1}\sigma^j\left((x+\frac{n-1}{2})g_{n-1}(x)\right) \in \mathcal{P}_+,$$

so that $\delta^i\sigma^j g_n \in \mathcal{P}_+$. If $i=0$ and $0 \le j \le n$, then (5.23) implies

$$\sigma^j g_n(x) = \sigma^{j-1}(\sigma g_n(x)) = 2\sigma^{j-1}\left(\left((x+\frac{n-1}{2})^2 + (\frac{n+1}{2})^2\right) g_{n-1}(x)\right).$$

The polynomial $\sigma^{j-1}g_{n-1}$ belongs to $\mathcal{P}_+$ by the induction hypothesis. Also, the induction hypothesis ($\delta^i\sigma^{j-1}g_{n-1} \in \mathcal{P}_+$ whenever $i+j \le n$) and Lemma 5.3 imply that

$$\delta^i\sigma^{j-1}\left((x+\frac{n-1}{2})g_{n-1}(x)\right) \in \mathcal{P}_+ \quad \text{whenever} \quad i+j \le n.$$

In particular, $\sigma^{j-1}\left((x+\frac{n-1}{2})g_{n-1}(x)\right) \in \mathcal{P}_+$. Hence $\sigma^j g_n \in \mathcal{P}_+ \ \ \forall\, 0 \le j \le n$. ∎

COROLLARY 5.2 (I) *$\delta^j g_n \in \mathcal{P}_+$ for all $j,n \in \mathbf{N}$ satisfying $0 \le j \le n$.*

(II) *$\delta^j\left((x+\frac{m}{2})g_n(x)\right) \in \mathcal{P}_+$ for all $j,n,m \in \mathbf{N}$ satisfying $0 \le j \le n+m$.*

(III) *$\delta^j f_n(x)^2 \in \mathcal{P}_+$ for all $j,n \in \mathbf{N}$ satisfying $0 \le j \le n+1$.*

PROOF: (i) is a special case of Lemma 5.4, and (ii) follows by (i) and Lemma 5.2. Since $f_n(x)^2 = g_{n-1}(x+1)$, (iii) follows from Lemma 5.2 with $m=2$. ∎

REMARK The result in part (iii) of Corollary 5.2 is best possible in the sense that $\delta^{n+2}(f_n^2)^2)$ need not be in $\mathcal{P}_+$. Indeed, $\delta^6(f_4^2)^2)$ is not in $\mathcal{P}_+$.

PROOF OF PROPOSITION 5.2: In terms of the polynomials (5.18), the system of equations (5.17) with unknowns q_ℓ has the form

$$\sum_{\ell=0}^{m} f_{r-2}(m-\ell)\, q_\ell = \frac{f_{r-1}(m)^2}{(r-1)\,(r-1)!}, \quad m \geq 0. \tag{5.24}$$

Applying powers of the operator δ with respect to the variable m and using (5.19), we get by induction on k that

$$\delta^k \left(\sum_{\ell=0}^{m} f_{r-2}(m-\ell)\, q_\ell \right) = (r-2)(r-3)\cdots(r-k-1) \sum_{\ell=0}^{m} f_{r-2-k}(m-\ell)\, q_\ell$$

for $0 \leq k \leq r-2$ (here $f_0(x) \equiv 1$). From this it follows that

$$\delta^{r-1} \left(\sum_{\ell=0}^{m} f_{r-2}(m-\ell)\, q_\ell \right) = (r-2)!\, q_m, \quad m \geq 0.$$

Applying δ^{r-1} to both sides of (5.24), Corollary 5.2 (iii) implies that there exists a polynomial $q(x)$ of degree $r-1$ with positive rational coefficients so that $q_m = q(m), \ \forall m \geq 0$. ∎

THEOREM 5.1 *Let $D = D(I_{r,r})$. Then for every $f, g \in \mathcal{H}_{\frac{a}{2}}(D)$ and $\lambda > 0$ we have*

$$\langle f, g \rangle_{\frac{a}{2}} = \int_{S_{r-1}} d\sigma_{r-1}(v) \langle p_\lambda(R^{(v)}) f_v, g_v \rangle_{\mathcal{H}_\lambda(\mathbf{D})},$$

where $p_\lambda(x) := \Gamma(x+\lambda)\,\Gamma(\lambda)^{-1}\,\Gamma(x+1)^{-1}\,q(x)$, and $q(x)$ is the polynomial of degree $r-1$ with positive rational coefficients as in Proposition 5.2.

CASE 2: CARTAN DOMAINS OF TYPE IV. Let $D \subset \mathbf{C}^d$, $d \geq 3$, be the Cartan domain of rank $r = 2$ (see Examples 4.1 and 4.2), and fix a frame $\{e_1, e_2\}$. Since $a = d-2$, (5.13) becomes

$$\sum_{\ell=0}^{m} \binom{m}{\ell}^2 J_{m,\ell}\, q_\ell = \frac{m!}{(\frac{a}{2}-1)_m}, \quad m \geq 0, \tag{5.25}$$

where for $0 \leq \ell \leq m$

$$J_{m,\ell} = \int_K |(k(e_1), e_1)|^{2\ell} |(k(e_2), e_1)|^{2(m-\ell)}\, dk.$$

Without computing the numbers $J_{m,\ell}$ explicitly we show that

$$J_{m,\ell} = J_{m,m-\ell}, \quad 0 \leq \ell \leq m. \tag{5.26}$$

Indeed, let $k' \in K$ satisfy $k'(e_1) = e_2$ and $k'(e_2) = e_1$. Then, by invariance of the Haar measure dk,

$$\begin{aligned} J_{m,\ell} &= \int_K |(k(k'(e_1)), e_1)|^{2\ell} |(k(k'(e_2)), e_1)|^{2(m-\ell)}\, dk \\ &= \int_K |(k(e_2), e_1)|^{2\ell} |(k(e_1), e_1)|^{2(m-\ell)}\, dk = J_{m,m-\ell}. \end{aligned}$$

THEOREM 5.2 *The polynomial*

$$q(x) = \frac{4}{a}\, x + 1 = \frac{4}{d-2}\, x + 1$$

satisfies $q(\ell) = q_\ell$ *for every* $\ell \geq 0$, *where* $\{q_\ell\}_{\ell=0}^\infty$ *is the unique solution of (5.25). Therefore, for every* $\lambda > 0$ *and every* $f, g \in \mathcal{H}_{\frac{a}{2}}(D)$,

$$\langle f, g\rangle_{\frac{a}{2}} = \int_{S_1} \langle p_\lambda(R^{(v)}) f_v, g_v\rangle_{\mathcal{H}_\lambda(D_v)}\, d\sigma_1(v),$$

where the functions p_λ, $0 < \lambda < \infty$, *are given by*

$$p_\lambda(x) = \frac{\Gamma(x+\lambda)}{\Gamma(\lambda)\,\Gamma(x+1)}\,(\frac{4}{a}\, x + 1). \tag{5.27}$$

In particular, for $\lambda = 1, 2, \ldots$ p_λ *is a polynomial of degree* λ *with positive rational coefficients.*

PROOF: We claim first that

$$\sum_{\ell=0}^m \binom{m}{\ell}^2 J_{m,\ell} = \frac{m!}{(\frac{d}{2})_m}, \quad m \geq 0. \tag{5.28}$$

Indeed, it is clear that

$$\sum_{\ell=0}^m \binom{m}{\ell}^2 J_{m,\ell} = \int_K \left(\int_{\mathbf{T}} |(k(e^{it}e_1 + e_2), e_1)^m|^2\, \frac{dt}{2\pi}\right) dk.$$

Interchanging the order of integration and using the transitivity of K on the frames, we get

$$\sum_{\ell=0}^m \binom{m}{\ell}^2 J_{m,\ell} = \int_K |(k(e), e_1)^m|^2\, dk = \|N_1^m\|^2_{H^2(D)} = \frac{m!}{(\frac{d}{2})_m}, \quad m \geq 0,$$

by using the well-known fact that $\|(\cdot, z)^m\|^2_{\mathcal{F}} = m!(z,z)^m$ for every $z \in Z$ and $m \geq 0$. Using (5.26) and (5.28) we see that

$$\begin{aligned}
\sum_{\ell=0}^m \ell \binom{m}{\ell}^2 J_{m,\ell} &= \sum_{\ell=0}^m (m-\ell) \binom{m}{m-\ell}^2 J_{m,m-\ell} \\
&= \sum_{\ell=0}^m (m-\ell) \binom{m}{\ell}^2 J_{m,\ell} = \frac{m \cdot m!}{(\frac{d}{2})_m} - \sum_{\ell=0}^m \ell \binom{m}{\ell}^2 J_{m,\ell}.
\end{aligned}$$

Thus

$$\sum_{\ell=0}^m \ell \binom{m}{\ell}^2 J_{m,\ell} = \frac{m \cdot m!}{2(\frac{d}{2})_m}, \quad m \geq 0. \tag{5.29}$$

Combining (5.28) and (5.29), and using the fact that $(\frac{d}{2})_m = (\frac{a}{2})_m \frac{(\frac{a}{2}+m)}{\frac{a}{2}}$, we get for $m \geq 0$

$$\sum_{\ell=0}^m \binom{m}{\ell}^2 J_{m,\ell}\,(\frac{4}{a}\,\ell + 1) = \frac{4}{a}\,\frac{m \cdot m!}{2(\frac{d}{2})_m} + \frac{m!}{(\frac{d}{2})_m} = \frac{m!}{(\frac{a}{2})_m}.$$

In view of (5.14), this completes the proof. ∎

—large The computation of $\langle f,g\rangle_{p-1}$ by integration on $\partial_1 D$

We conclude this section with the derivation of a formula for $\langle f,g\rangle_{p-1}$ via integration on $\partial_1 D$.

Proposition 5.3 *Let $F \in C(\overline{D})$. Then*

$$\lim_{\lambda\downarrow p-1}\int_D F(z)\,d\mu_\lambda(z) = \int_{S_1}\left(\int_{D_v} F_v(w)\,d\mu_{v,p-1}(w)\right)d\sigma_1(v), \tag{5.30}$$

where the measures $\mu_{v,p-1}$ are defined by (5.4).

Proof: Using (1.13) and (1.14) as well as (1.22), (1.23), and (1.9), we can write

$$\begin{aligned}\int_D F(z)\,d\mu_\lambda(z) &= c_0\,c(\lambda)\int_{\mathbf{R}_+^r} F^\#(t)w(t)^a\prod_{j=1}^r(1-t_j)^a\,dt\\ &= c_0\,c(\lambda)\int_0^1 \psi(t_1)\,(1-t_1)^{\lambda-p}\,dt_1,\end{aligned}$$

where

$$\psi(t_1) := \int_{[0,t_1)_+^{r-1}} F^\#(t_1,t')\prod_{1\le i<j\le r}(t_i-t_j)^a\prod_{j=2}^r(1-t_j)^{\lambda-p}\,dt',$$

and $c(\lambda)=c_D(\lambda)$ is given by (1.22). Here $t' := (t_2,t_3,\ldots,t_r)$, $dt' := dt_2\,dt_3\,\ldots dt_r$, and $[0,t_1)_+^{r-1} := \{t'\in\mathbf{R}^{r-1}; t_2>t_3>\ldots>t_r>0\}$. Since $\psi\in C([0,1])$, we have $\lim_{\epsilon\downarrow 0}\left(\epsilon\int_0^1\psi(t)(1-t)^{\epsilon-1}\,dt\right)=\psi(1)$. Since $\lim_{\lambda\downarrow p-1}\ \Gamma(\lambda-p+1)\,(\lambda-p+1)=1$ and $c(p-1)=0$, we get

$$\begin{aligned}\lim_{\lambda\downarrow p-1}\int_D F(z)\,d\mu_\lambda(z) &= b\,\psi(1)\\ &= b\int_{[0,1)_+^{r-1}} F^\#(1,t')\prod_{2\le i<j\le r}(t_i-t_j)^a\prod_{j=2}^r(1-t_j)^{a-1}\,dt',\end{aligned}$$

where $b := c_0\,c'(p-1)$. Using the definitions (5.1), (5.3) and the fact that for $v\in S_1$ the genus of D_v is $p-a$, we have (with the obvious meaning of the constants)

$$\begin{aligned}&\int_{S_1}\left(\int_{D_v} F_v(w)\,d\mu_{v,p-1}(w)\right)d\sigma_1(v)\\ &= c_{D_{e_1}}(p-1)\int_K\left(\int_{D_{e_1}} F_{k(e_1)}(k(\xi))\,h(k(\xi),k(\xi))^{a-1}\,dm(k(\xi))\right)dk\end{aligned}$$

$$= c_{D_{e_1}}(p-1)\, c_0(D_{e_1})$$

$$\times \int_K \left(\int_{[0,1)_+^{r-1}} \left(\int_{K_{e_1}} F(k(e_1 + k'(\sum_{j=2}^r t_j^{\frac{1}{2}} e_j\; dk')) \right) w(t')^a \prod_{j=2}^r (1-t_j)^{a-1}\, dt' \right) dk,$$

where $K_{e_1} := \{k \in K; k(e_1) = e_1\}$ and $w(t') := \prod_{2\le i<j\le r}(t_i - t_j)^a$. Interchanging the order of integration, and using the fact that $k'(e_1) = e_1$ and the invariance of the Haar measure dk, we see that the last expression is equal to

$$c_{D_{e_1}}(p-1)\, c_0(D_{e_1}) \int_{[0,1)_+^{r-1}} F^{\#}(1,t')\, w(t')^a \prod_{j=2}^r (1-t_j)^{a-1}\, dt'.$$

Comparing the computations for the left and right hand sides of (5.30), we see they are proportional. Taking $F(z) \equiv 1$, the proportionality constant is 1. ∎

COROLLARY 5.3 *The constant $c_0 = c_0(D)$ in the formula (1.12) is*

$$c_0(D) = \frac{\pi^d\, \Gamma(\frac{a}{2})^{r-2}}{(\prod_{\ell=1}^{r-1} \ell\, \frac{a}{2})\, \Gamma(r\,\frac{a}{2}) \prod_{\ell=2}^{r-1} \Gamma(\ell\,\frac{a}{2})^2}.$$

PROOF: Define $v_r = 0$, $v_\ell := e_1 + \ldots + e_{r-\ell}$, $\ell = 1, 2, \ldots, r-1$, and $\gamma_\ell := c_0(D_{v_\ell})$. Then the above proof (with r replaced by ℓ) yields

$$\frac{\gamma_\ell}{\gamma_{\ell-1}} = \frac{c_{D_{v_{\ell+1}}}((\ell-1)a+1)}{c'_{D_{v_\ell}}((\ell-1)a+1)} = \frac{\pi^{(\ell-1)a+1}\, \Gamma(\frac{a}{2})}{\Gamma((\ell-1)\frac{a}{2}+1)\, \Gamma(\frac{ra}{2})}$$

for $\ell = 2, 3, \ldots, r$. Therefore, using the easily proved fact that $\gamma_1 = \pi$, we get

$$\begin{aligned} c_0(D) &= \gamma_r = \frac{\gamma_r}{\gamma_{r-1}}\, \frac{\gamma_{r-1}}{\gamma_{r-2}} \cdots \frac{\gamma_2}{\gamma_1}\, \gamma_1 \\ &= \pi \prod_{\ell=2}^r \frac{\pi^{(\ell-1)a+1}\, \Gamma(\frac{a}{2})}{\Gamma((\ell-1)\frac{a}{2}+1)\, \Gamma(\frac{ra}{2})} = \frac{\pi^d\, \Gamma(\frac{a}{2})^{r-2}}{(\prod_{\ell=1}^{r-1} \ell\, \frac{a}{2})\, \Gamma(r\,\frac{a}{2}) \prod_{\ell=2}^{r-1} \Gamma(\ell\,\frac{a}{2})^2}. \end{aligned}$$ ∎

Proposition 5.3 allows the computation of the inner products $\langle f, g \rangle_{p-1}$ by integrating over the boundary orbit $\partial_1(D) = G(e_1)$ of G.

THEOREM 5.3 *Let $f, g \in \mathcal{H}_{p-1}$. Then*

$$\langle f, g \rangle_{p-1} = \int_{S_1} \left(\int_{D_v} f_v(w)\, \overline{g_v(w)}\, d\mu_{v,p-1}(w) \right) d\sigma_1(v), \tag{5.31}$$

PROOF: It is enough to establish (5.31) for polynomials f and g, and this case follows from Proposition 5.3 with $F(z) = f(z)\,\overline{g(z)}$. ∎

6 Integral formulas in the context of the associated Siegel domain

In what follows we shall use the fact [FK2] that D is holomorphically equivalent to the tube domain

$$T(\Omega) := X + i\Omega$$

via the *Cayley transform* $c : D \to T(\Omega)$, defined by $c(z) := i(e+z)(e-z)^{-1}$. For $\lambda \in W(D)$ the operator $V^{(\lambda)} f := (f \circ c^{-1})(Jc^{-1})^{\lambda/p}$ maps the space $\mathcal{H}_\lambda = \mathcal{H}_\lambda(D)$ isometrically onto a Hilbert space of analytic functions on $T(\Omega)$, denoted by $\mathcal{H}_\lambda(T(\Omega))$. We will denote $\langle f, g\rangle_{\mathcal{H}_\lambda(T(\Omega))}$ simply by $\langle f, g\rangle_\lambda$. It is known that the reproducing kernel of $\mathcal{H}_\lambda(T(\Omega))$ is

$$K_\lambda(z,w) = \left(N(\frac{z-w^*}{i})\right)^{-\lambda}, \quad z, w \in T(\Omega). \tag{6.1}$$

Recall that for $\lambda > p-1$ we have $\mathcal{H}_\lambda(D) = L^2_a(D, \mu_\lambda)$, where μ_λ is the measure on D defined via (1.23). Using the facts that $h(c^{-1}(w), c^{-1}(w)) = 4^r |N(w+ie)|^{-2} N(v)$ and $J(c^{-1})(w) = (2i)^d N(w+ie)^{-p}, \quad \forall w \in T(\Omega)$, we get by a change of variables that

$$\mathcal{H}_\lambda(T(\Omega)) = L^2_a(T(\Omega), \nu_\lambda) = L^2(T(\Omega), \nu_\lambda) \cap \{\text{analytic functions}\},$$

where

$$d\nu_\lambda(z) := c(\lambda) dx \;\; N(2y)^{\lambda - p}\, dy, \quad z = x+iy,\ x \in X,\ y \in \Omega, \tag{6.2}$$

and $c(\lambda)$ is defined by (1.22). In this case $V^{(\lambda)}$ extends to an isometry of $L^2(D, \mu_\lambda)$ onto $L^2(T(\Omega), \nu_\lambda)$.

In this section we obtain integral formulas for the invariant inner products in the spaces $\mathcal{H}_\lambda(T(\Omega))$. Using the isometry $V^{(\lambda)} : \mathcal{H}_\lambda(D) \to \mathcal{H}_\lambda(T(\Omega))$ one obtains integral formulas for the inner products in the spaces $\mathcal{H}_\lambda(D)$. Our results are essentially implicitly contained in [VR], where the authors determine the Wallach set for Siegel domains of type II, using Lie and Fourier theoretical methods. The Jordan-theoretical formalism allows us to formulate our results in a simpler way, avoiding the Lie-theoretical details. Since the Fourier-theoretical arguments in our proofs are contained in[VR], we omit all proofs.

For $\lambda > (r-1)\frac{a}{2}$ consider the measure σ_λ on Ω defined by $d\sigma_\lambda(v) := \beta_\lambda\, N(v)^{\frac{d}{r} - \lambda}\, dv$ where $\beta_\lambda := (2\pi)^{-2d} \Gamma_\Omega(\lambda)$.

Proposition 6.1 *Let $\lambda > (r-1)\frac{a}{2}$ and let f be a holomorphic function on $T(\Omega)$. Then the following conditions are equivalent:*

(i) $f \in \mathcal{H}_\lambda(T(\Omega))$;

(ii) *The boundary values $f(x) := \lim_{\Omega \ni y \to 0} f(x+iy)$ exist almost everywhere on X, and the Fourier transform $\hat{f}$ of $f(x)$ is supported in $\overline{\Omega}$ and belongs to $L^2(\Omega, \sigma_\lambda)$.*

Moreover, the map $f \mapsto \hat{f}$ is an isometry of $\mathcal{H}_\lambda(T(\Omega))$ onto $L^2_a(\Omega, \sigma_\lambda)$.

Proposition 6.1 yields the following result.

Theorem 6.1 *Let $\lambda > (r-1)\frac{a}{2}$ and let $f, g \in \mathcal{H}_\lambda(T(\Omega))$. Then*

$$\langle f, g\rangle_{\mathcal{H}_\lambda(T(\Omega))} = \langle \hat{f}, \hat{g}\rangle_{L^2(\Omega,\sigma_\lambda)} = \frac{\Gamma_\Omega(\lambda)}{(2\pi)^{2d}} \int_\Omega \hat{f}(t)\overline{\hat{g}(t)}\, N(t)^{\frac{d}{r}-\lambda}\, dt.$$

The group $GL(\Omega) := \{\varphi \in GL(X); \varphi(\Omega) = \Omega\}$ acts transitively on Ω . It acts also on the boundary $\partial\Omega$, but this action is not transitive. The orbits of $GL(\Omega)$ on $\partial\Omega$ are exactly the r disjoint sets

$$\partial_k\Omega := GL(\Omega)(e_k) = \{x \in \overline{\Omega};\ \mathrm{rank}(x) = k\}, \quad k = 0, 1, \ldots, r-1,$$

where $\{c_1, \ldots, c_r\}$ is a frame of pairwise orthogonal primitive idempotents, $e_0 := 0$, and $e_k := \sum_{j=1}^k c_j, \quad k = 1, 2, \ldots, r-1$. Consider the Peirce decomposition $X_\nu = X_\nu(e_k) = \{x \in X; e_k x = \nu x\}$, $\nu = 0, \frac{1}{2}, 1$. Let $\Omega(k)$ be the symmetric cone of $X_1(e_k)$, and let $\Gamma_{\Omega(k)}$ be the associated Gamma function. Let $GL(\Omega) = LN_\Omega A$ be the Iwasawa decomposition. Then $N_\Omega A(e_k) = \{x \in \partial_k\Omega; N_k(x) > 0\}$ is an open dense subset of $\partial_k\Omega$, and every $x \in N_\Omega A(e_k)$ has a Peirce decomposition of the form $x = x_1 + x_{\frac{1}{2}} + 2(e-e_k)(x_{\frac{1}{2}}(x_{\frac{1}{2}} x_1^{-1}))$ [La2]. Let us define a measure ν_k on $\partial_k\Omega$ with support $N_\Omega A(e_k)$ by

$$d\nu_k(x) := N_k(x_1)^{k\frac{a}{2}-\frac{d}{r}}\, dx_1\, dx_{\frac{1}{2}}. \tag{6.3}$$

It has the following fundamental properties (see[VR] and [La2]).

Theorem 6.2 *Let $1 \le k \le r-1$. Then the measure ν_k satisfies*

$$\int_{N_\Omega A(e_k)} e^{-\langle y,x\rangle}\, d\nu_k(x) = \gamma_k\, N(y)^{-k\frac{a}{2}}, \quad \forall y \in \Omega, \tag{6.4}$$

where $\gamma_k := (2\pi)^{k(r-k)\frac{a}{2}}\,\Gamma_{\Omega(k)}(k\frac{a}{2})$, and

$$d\nu_k(\varphi(x)) = Det(\varphi)^{(k\frac{a}{2})/\frac{d}{r}}\, d\nu_k(x), \quad \forall \varphi \in GL(\Omega). \tag{6.5}$$

Since Ω is a set of uniqueness for analytic functions on $T(\Omega)$, (6.4) implies by analytic continuation

$$\int_{N_\Omega A(e_k)} e^{-\langle \frac{z-w^*}{i}, x\rangle}\, d\nu_k(x) = \gamma_k\, 2^{-k\frac{a}{2}} \left(N(\frac{z-w^*}{i})\right)^{-k\frac{a}{2}}, \quad \forall z, w \in T(\Omega).$$

Thus $\left(N(\frac{z-w^*}{i})\right)^{-k\frac{a}{2}}$ is positive definite, and so $k\frac{a}{2}$ is in the Wallach set $W(D) = W(T(\Omega))$.

By complexification, $GL(\Omega)$ is realized as a subgroup of $Aut(T(\Omega))$ which normalizes the translations $\tau_x(z) := z + x$, i.e.

$$\varphi\, \tau_x\, \varphi^{-1} = \tau_{\varphi(x)}, \quad \forall x \in X, \quad \forall \varphi \in GL(\Omega).$$

Let $G \subset Aut(T(\Omega))$ be the semi-direct product of X and $GL(\Omega)$. It acts transitively on $T(\Omega)$. Let $N \subset G$ be the semi-direct product of X and N_Ω. Then the Iwasawa decomposition of $Aut(T(\Omega))_0$ is KAN. For

$$\alpha_k = \frac{d}{r} + k\frac{a}{2}, \quad k = 0, 1, 2, \ldots, r-1$$

let $\mathcal{H}_{\alpha_k} = \mathcal{H}_{\alpha_k}(T(\Omega))$ be the Hilbert space of analytic functions on $T(\Omega)$ whose reproducing kernel is $K_{\alpha_k}(z,w) := \left(N(\frac{z-w^*}{i})\right)^{-\alpha_k}$. Note that $\alpha_{r-1} = p-1$ and for $k=0$ we have $\alpha_0 = \frac{d}{r}$ and $\nu_0 = \delta_0$, the Dirac measure at 0.

Theorem 6.3 *For $k = 0, 1, \ldots, r-1$ $\mathcal{H}_{\alpha_k}(T(\Omega))$ consists of all analytic functions f on $T(\Omega)$ for which*

$$\|f\|^2_{\mathcal{H}_{\alpha_k}(T(\Omega))} := \beta_k \sup_{t\in\Omega} \int_{N_\Omega A(e_k)} \left(\int_X |f(x+i(y+t))|^2 \, dx \right) d\nu_k(y) \tag{6.6}$$

is finite, where

$$\beta_k = \frac{\Gamma_\Omega(\alpha_k) 2^{rk\frac{a}{2}}}{\Gamma_{\Omega(k)}(k\frac{a}{2})} \, (2\pi)^{-(d+k(r-k)\frac{a}{2})}.$$

Moreover, for every $f, g \in \mathcal{H}_{\alpha_k}(T(\Omega))$,

$$\langle f,g\rangle_{\alpha_k} = \beta_k \lim_{\Omega\ni t\to 0} \int_{N_\Omega A(e_k)} \left(\int_X f(x+i(y+t)) \, \overline{g(x+i(y+t))} \, dx \right) d\nu_k(y).$$

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Jonathan Arazy
Department of Mathematics
University of Haifa
Haifa 31905, Israel
jarazy@mathcs2.haifa.ac.il

Harald Upmeier
Fachbereich Mathematik
Universität Marburg
35032 Marburg, Germany
upmeier@mathematik.uni-marburg.de