Selmer Groups and Torsion Zero Cycles on the Selfproduct of a Semistable Elliptic Curve

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Abstract. In this paper we extend the finiteness result on the p-primary torsion subgroup in the Chow group of zero cycles on the selfproduct of a semistable elliptic curve obtained in joint work with S. Saito to primes p dividing the conductor. On the way we show the finiteness of the Selmer group associated to the symmetric square of the elliptic curve for those primes. The proof uses p-adic techniques, in particular the Fontaine-Jannsen conjecture proven by Kato and Tsuji.

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Introduction.

In this note we extend the main finiteness result on p-primary torsion zero-cycles on the selfproduct of a semistable elliptic curve in [L-S] to primes $p \geq 3$ where E has (bad) multiplicative reduction, at least under a certain standard assumption. In the course of the proof we will also derive the finiteness of the Selmer group of the symmetric square $\operatorname{Sym}^2 H^1(E)(1)$ for these primes. However, this latter result has already been proven, under the additional condition that the Galois representation

$$\varrho_p : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \operatorname{Aut}(E_p)$$

is absolutely irreducible (here $E_p = E_p(\overline{\mathbb{Q}})$ is the subgroup of p-torsion elements of E), in a much more general context by Wiles in his main paper ([W] Theorem 3.1) for Selmer groups associated to deformation theories.

To state the Theorems, let E be a semistable elliptic curve over $\mathbb{Q}$ with conductor N and let $X = E \underset{\mathbb{Q}}{\times} E$ be its self-product. Consider the Chow group $CH_0(X)$ of zero-cycles on X modulo rational equivalence and let $CH_0(X)\{p\}$ be — for a fixed prime p — its p-primary torsion subgroup. For a prime p dividing N consider the following hypothesis:

H 1) The Gersten-Conjecture holds for the Quillen-(Milnor)-sheaf $\mathcal{K}_2$ on a regular model $\mathcal{X}$ of X over $\mathbb{Z}_p$.

Then we have

THEOREM A: *Let E be a semistable elliptic curve and $p \geq 3$ a prime such that $p \mid N$, i.e., E has (bad) multiplicative reduction at p. Assume that the condition H 1) is satisfied. Then $CH_0(X)\{p\}$ is a finite group.*

Let $A = H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$ be the $\mathbb{Q}_p/\mathbb{Z}_p$-realization of the motive $H^2(X)(2)$ with its $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. Then we have

THEOREM B: *Let E be a semistable elliptic curve over $\mathbb{Q}$ and $p \geq 3$ a prime such that $p \mid N$. Then the Selmer group $S(\mathbb{Q}, A)$ is finite.*

REMARKS:

— In [L-S] we showed the finiteness of $CH_0(X)\{p\}$ for primes p such that $p \not| 6$ and E has good reduction at p. We also proved that $CH_0(X)\{p\}$ is zero for almost all p. Therefore Theorem A extends this result to bad primes and provides a further step towards a proof that the full torsion subgroup $CH_0(X)_{tors}$ is finite. In order to find a first example where this is true it remains to consider the 2- and 3-primary torsion in $CH_0(X)$.

— The Selmer group $S(\mathbb{Q}, A)$ coincides with $S(\mathbb{Q}, \mathrm{Sym}^2 H^1(\overline{E}, \mathbb{Q}_p/\mathbb{Z}_p(1)))$ that was studied by [Fl], because $S(\mathbb{Q}, \mathbb{Q}_p/\mathbb{Z}_p(1))$ is zero. In [Fl] Flach proved the finiteness of $S(\mathbb{Q}, A)$ for primes $p \geq 5$ such that E has good reduction at p and the representation ϱ_p is surjective. We were able to remove the latter hypothesis by using a rank-argument of Bloch-Kato and reproved Flach's finiteness result for primes p such that $p \not| 6N$ (compare [L-S]). In the proof of Theorem B we combine the criterium of Bloch-Kato with Kolyvagin's argument that was used in Flach's paper. Flach's additional condition on the surjectivity of ϱ_p can be avoided by applying a certain lemma, due to J. Nekovář, that bounds the order of $H^1(\mathrm{Gal}(\mathbb{Q}(E_{p^n})/\mathbb{Q}), (\mathrm{Sym}^2 H^1(\overline{E}, \mathbb{Z}/p^n(1)))(-1))$ independently of n.

The paper is organized as follows:
In the first paragraph we reduce the proof of Theorem A to two Lemmas I and II. Lemma I was already proven in ([L-S], Lemma A). Lemma II is similar to ([L-S], Lemma B), but the statement is different. The difference is caused by the particular semistable situation. In the second paragraph we derive Lemma II and Theorem B from a key proposition that bounds the possible corank (at most 1 !) of the cokernel of the map defining the Selmer group. Finally this proposition is proven in the last paragraph. The methods of the proof are similar to those developed in [L-S]. At the

point where the crystalline conjecture was used in the good reduction case, we now use the Fontaine-Jannsen conjecture (proven by Kato/Tsuji for $p \geq 3$) that relates the log-crystalline cohomology to the p-adic étale cohomology. The role of the syntomic cohomology in the context of Schneider's p-adic points conjecture is now replaced by a semistable analog relating log-syntomic cohomology to $H^1_g(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_p(2)))$ (compare [L]). When we apply this argument we will also need the computation, due to Hyodo and used by Tsuji, on a filtration on the sheaf of p-adic vanishing cycles in terms of modified logarithmic Hodge-Witt sheaves.

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§1

We first fix some notations.

For an Abelian group M let M_{div} be the maximal divisible subgroup of M and $M\{p\}$ its p-primary torsion subgroup. For a scheme Z over a field k let $\overline{Z} = Z \underset{k}{\times} \overline{k}$ where k is an algebraic closure of k. Denote by $G_k = \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group of k. We will consider the Zariski sheaf $\mathcal{K}_2$ associated to the presheaf $U \to \mathcal{K}_2(U)$ of Quillen (-Milnor) K-groups on Z and let $H^j_{Zar}(Z, \mathcal{K}_2)$ be its Zariski cohomology. Let E be a semistable elliptic curve over $\mathbb{Q}$ with conductor N, $\phi : X_0(N) \to E$ a modular parametrization of E, $X = E \underset{\mathbb{Q}}{\times} E$. Let T, A, V be the following $G = G_{\mathbb{Q}}$-modules:

$$T = H^2(\overline{X}, \mathbb{Z}_p(2)) \quad , \quad A = H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \quad , \quad V = H^2(\overline{X}, \mathbb{Q}_p(2)) \quad .$$

Note that as Abelian groups $T \cong \mathbb{Z}_p^6$, $A \cong \mathbb{Q}_p/\mathbb{Z}_p^6$, because the integral cohomology of an Abelian variety is torsion-free and the second Betti number of X b_2 is 6.

Let K be the function field of X. For a prime p let

$$NH^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) := \ker(H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2)))$$

and

$$K_N H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) := \ker(NH^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^3(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))$$

By results of Bloch and Merkurjev-Suslin ([Bl], §5 and [M-S] we have the following exact sequence

$$(1-1) \qquad 0 \to H^1(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to NH^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to CH_0(X)\{p\} \to 0$$

Since $H^1(\overline{X}, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 0$ we get an exact sequence

$$(1-2) \qquad \begin{aligned} 0 &\longrightarrow H^1(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow K_N H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ &\longrightarrow \ker(CH_0(X)\{p\} \longrightarrow CH_0(\overline{X})\{p\}^G) \longrightarrow 0 \end{aligned}$$

Since X is identified with its Albanese variety, the map $CH_0(X)_{tors} \longrightarrow CH_0(\overline{X})^G_{tors}$ is the Albanese map and therefore $(CH_0(\overline{X})\{p\})^G \cong X(\mathbb{Q})\{p\}$ is finite. Consider the Hochschild-Serre spectral sequence

$$E_2^{a,b} = H^a(\mathbb{Q}, H^b(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \Longrightarrow H^{a+b}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \quad .$$

Then we have

LEMMA I: *Let the assumptions be as above. Then the composite map*

$$E_2^{2,1} \longrightarrow H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

is injective.

This is shown in ([L-S], Lemma (A)) without any assumption on the prime p.

COROLLARY (1.3) *The composite map*

$$\varphi : K_N H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^1(G_{\mathbb{Q}}, A)$$

that is obtained by the Hochschild-Serre spectral sequence is injective.

The Corollary will play an important role in the proof of

LEMMA II: *Under the above assumptions let $p \geq 3$ be a prime such that $p \mid N$ and assume that the condition H 1) in the introduction is satisfied. Then we have*

$$H^1(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p = K_N H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{div}} \quad .$$

REMARK:

Lemma II was proven for primes $p \not| 6N$ in ([L-S, Lemma (B)) because in this case $K_N H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{div}}$ coincides with $H^1(\mathbb{Q}, A)_{\mathrm{div}}$. This is not stated there explicitly but follows from the proof of Lemma (B) in [L-S].

Now we deduce Theorem A from Lemma II.

The exact sequence (1-1) also holds for a smooth proper model $\mathcal{X}$ of X over $\mathbb{Z}\left[\frac{1}{Np}\right]$. So $CH_0(\mathcal{X})\{p\}$ is a subquotient of $H^3(\mathcal{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$ and one knows that the latter group is co-finitely generated. Therefore $CH_0(\mathcal{X})\{p\}$ is co-finitely generated as $\mathbb{Z}_p$-module. Since the kernel of the canonical map

$$CH_0(\mathcal{X})\{p\} \longrightarrow CH_0(X)\{p\}$$

is a torsion group by the main result in [Mi], the localization sequence in the Zariski K-cohomology over $\mathcal{X}$ yields a surjection

$$CH_0(\mathcal{X})\{p\} \twoheadrightarrow CH_0(X)\{p\} \quad .$$

So we also know that $CH_0(X)\{p\}$ is co-finitely generated.

On the other hand, by (1-2), the finiteness of $CH_0(\overline{X})\{p\}^G$ and Lemma II we conclude that the maximal divisible subgroup of $CH_0(X)\{p\}$ is zero. Therefore $CH_0(X)\{p\}$ is a finite group.

To complete the proof of Theorem A it remains to show Lemma II.

§2

For each prime ℓ let

$$H^1_e(\mathbb{Q}_\ell, V) \subset H^1_f(\mathbb{Q}_\ell, V) \subset H^1_g(\mathbb{Q}_\ell, V) \subset H^1(\mathbb{Q}_\ell, V)$$

be defined as in ([BK], 3.7)). Let

$$H^1_f(\mathbb{Q}_\ell, T) \subset H^1_g(\mathbb{Q}_\ell, T) \subset H^1(\mathbb{Q}_\ell, T)$$

be the inverse image of $H^1_f(\mathbb{Q}_\ell, V)$ and $H^1_g(\mathbb{Q}_\ell, V)$. Put

$$H^1_f(\mathbb{Q}_\ell, A) := H^1_f(\mathbb{Q}_\ell, T) \otimes \mathbb{Q}_p/\mathbb{Z}_p \subset H^1(\mathbb{Q}_\ell, A)$$

and

$$H^1_g(\mathbb{Q}_\ell, A) := H^1_g(\mathbb{Q}_\ell, T) \otimes \mathbb{Q}_p/\mathbb{Z}_p \subset H^1(\mathbb{Q}_\ell, A)$$

Write $\wedge_\ell = H^1(\mathbb{Q}_\ell, T)/H^1_f(\mathbb{Q}_\ell, T)$. Then we have

$$\wedge_\ell \otimes \mathbb{Q}_p/\mathbb{Z}_p = H^1(\mathbb{Q}_\ell, A)_{\mathrm{div}}/H^1_f(\mathbb{Q}_\ell, A)$$

Consider as in ([L-S], §3) the composite map

$$\psi : H^1(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow K_N H^3(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{div}} \xrightarrow{\alpha'} \bigoplus_{\ell} \wedge_\ell \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

where α' is the restriction of the map

$$\alpha : H^1(\mathbb{Q}, A) \longrightarrow \bigoplus_{all\ \ell} \frac{H^1(\mathbb{Q}_\ell, A)}{H^1_f(\mathbb{Q}_\ell, A)}$$

the kernel of which defines the Selmer group $S(\mathbb{Q}, A)$.

In analogy to ([L-S], Lemma 3.1) we will prove the following

Proposition (2.1): *Let the notations be as in §1. Let $p \geq 3$ a prime, such that E has multiplicative reduction at p. Assume that condition (H 1) holds. Then we have*

a) $$\operatorname{coker}\psi = H^1(\mathbb{Q}_p, A)_{\mathrm{div}}/H^1_g(\mathbb{Q}_p, A)$$

b) $$\operatorname{Im}\psi = \operatorname{Im}\alpha'$$

We will give the proof of Proposition 2.1 in the next section.

In the following we will compute the coranks of $H^1(\mathbb{Q}_p, A)_{\mathrm{div}}/H^1_g(\mathbb{Q}_p, A)$ and $H^1_g(\mathbb{Q}_p, A)/H^1_f(\mathbb{Q}_p, A)$. Let

$$\Omega_p = H^1_g(\mathbb{Q}_p, V)/H^1_f(\mathbb{Q}_p, V) \quad \text{and} \quad \theta_p = H^1(\mathbb{Q}_p, V)/H^1_g(\mathbb{Q}_p, V)$$

as in ([L-S], §4). It is well known that $X_{\mathbb{Q}_p} = E \times E_{\mathbb{Q}_p}$ has a regular proper model $\mathcal{X}$ over $\mathbb{Z}_p$ with semistable reduction. Let X_p be its closed fiber. By local Tate-Duality ([B-K], §3.8), Ω_p is the $\mathbb{Q}_p$-dual of $H^1_f(\mathbb{Q}_p, V(-1))/H^1_e(\mathbb{Q}_p, V(-1))$ and this quotient is — by the computations in [B-K], 3.8 — isomorphic to $(B_{\mathrm{crys}} \otimes V(-1))^{G_{\mathbb{Q}_p}}/1-f$, which is by Kato's and Tsuji's proof of the Fontaine-Jannsen-Conjecture ([Ka], §6), ([Tsu]) isomorphic to $(D_2)^{N=0}/1-f)$, where

$$D_2 = H^2_{\log\ \mathrm{crys}}((X_p, M_1)/W(\mathbb{F}_p), W(L), O^{\mathrm{crys}}) \otimes \mathbb{Q}_p$$

denotes the log-crystalline cohomology introduced by Hyodo-Kato [H-K], $N=0$ denotes the kernel under the action of the monodromy operator N, and f acts as $p^{-1}\varphi$, where φ is the Frobenius acting on D_2. Therefore we have by Poincaré duality for Hyodo-Kato cohomology that Ω_p is isomorphic to $(\operatorname{coker} N : D_2 \to D_2)^{\varphi=p}$. Since the functor $D_{st}(\cdot) = (B_{st} \otimes \cdot)^{G_{\mathbb{Q}_p}}$ commutes with tensor products and a Tate-elliptic curve has ordinary semistable reduction in the sense of ([Il], Definition 1.4) we have a Hodge-Witt-decomposition ([Il], Proposition 1.5)

$$D_2 = \bigoplus_{i+j=2} H^i(X_p, Ww^j) \otimes \mathbb{Q}_p \quad .$$

Here $H^i(X_p, Ww^j)$ is the cohomology of the modified Hodge-Witt-sheaves. From the action of the Frobenius φ on D_2 it is clear that $(D_2)_{\varphi=p}$ is contained in $H^1(X_p, Ww^1)_{\mathbb{Q}_p}$. By ([Mo], §6) we know that the monodromy filtration and the weight filtration on D_2 coincide. Using the formula $N\varphi = p\varphi N$ we have that

$$N(H^0(X_p, Ww^2)) \subset H^1(X_p, Ww^1)$$

and the map

$$N^2 : H^0(X_p, Ww^2) \longrightarrow H^2(X_p, Ww^0)$$

is an isomorphism. Since $\dim H^i(X_p, Ww^j)_{\mathbb{Q}_p} = \dim H^i(X_{\mathbb{Q}_p}\Omega^j)$ by ([Il], Corollaire 2.6), we see that

$$\dim(\operatorname{coker} N : D_2 \to D_2)^{\varphi=p} = \dim(D_2)^{N=0}_{\varphi=p} \leq 3 \quad .$$

On the other hand the B_{St}-comparison-isomorphism provides an injection

$$\operatorname{Pic}(X) \otimes \mathbb{Q}_p \hookrightarrow H^2(\overline{X}, \mathbb{Q}_p(1))^{G_{\mathbb{Q}_p}} \hookrightarrow (D_2)^{N=0}_{\varphi=p} \quad .$$

Since $\operatorname{Pic}(X)$ has rank 3 we have

Lemma (2.2):

$$\dim \Omega_p = \dim (D_2)^{N=0}_{\varphi=p} = 3 \quad .$$

By the same methods and the proof of ([L-S], Lemma 4.4) we get

Lemma (2.3):

$$\dim \theta_p = 1 \quad .$$

From Lemma (2.2) and ([L-S], Lemma 4.1) we get

Lemma (2.4): *The image of the composite map*

$$(\mathrm{Pic}(X) \otimes \mathbb{Q}^*) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow H^1(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\psi_p} \wedge_p \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

is

$$H^1_g(\mathbb{Q}_p, A)/H^1_f(\mathbb{Q}_p, A) \quad .$$

Now we will give the proof of Theorem B and we distinguish between two cases.

Case I:

The map α'_p, i.e. the p-component of α' is surjective.

This case is actually obstructed by the Gersten-conjecture as we will see in the proof of Proposition (2.1). Since we do not assume (H 1) in Theorem B we also consider this case. Using the surjectivity-property of ψ_ℓ, i.e. the ℓ-component of ψ, for $\ell \neq p$ that follows from Prop. 2.1, and where the condition (H 1) is not needed, we see that $\operatorname{coker} \alpha$ has $\mathbb{Z}_p$-corank 0. Now apply the modified version of ([B-K], Lemma 5.16) that is given in ([L-S], Lemma (3.3)): All the assumptions there are also satisfied for our choice of p:

— V is a de Rham representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ by Falting's proof of the de Rham conjecture.

— For the characteristic polynomial $P_\ell(V,t)$ we have $P_\ell(V,1) \neq 0$. For $\ell \neq p$ the proof is the same as in ([L-S], §3). For $\ell = p$, we have $\mathrm{Crys}(V)^{f=1} = (D_2)^{N=0}_{\varphi=p^2}$. By the same methods as in the proof of Lemma (2.2) we have $(D_2)^{N=0}_{\varphi=p^2} = 0$.
By the same arguments as in the proof of ([L-S], Theorem 3.2) we get the formula $\operatorname{corank}(\ker \alpha) = \operatorname{corank}(\operatorname{coker} \alpha) = 0$. Therefore $S(\mathbb{Q}, A) = \ker \alpha$ is finite.

Case II:

$\operatorname{Im} \alpha'_p = H^1_g(\mathbb{Q}_p, A)/H^1_f(\mathbb{Q}_p, A)$

By Lemmas (2.3) and (2.4) this is the only remaining case to consider.

Let $T' = \mathrm{Sym}^2 H^1(\overline{E}, \mathbb{Z}_p(1))$. By Lemma (2.2) and Lemma (2.4) we have $H^1_g(\mathbb{Q}_p, T')/H^1_f(\mathbb{Q}_p, T') = 0$. Let $c(\ell)$ for $\ell \not| N$ be the elements in $H^1(X, \mathcal{K}_2)$ that were constructed by Mildenhall and Flach. In the notation of ([Fl], Prop. (1.1)) we therefore have $\mathrm{res}_{r=p} c(\ell) \in H^1_f(\mathbb{Q}_p, T')$. We get this property with little effort whereas in [Fl] this was one of the harder parts in the whole paper. It is now easy to check that all the other required properties on the elements $c(\ell)$ in ([Fl], Prop. (1.1)) are also satisfied for our choice of p. Thus we apply Kolyvagin's argument in ([Fl], Prop. (1.1)). At the point where Flach needs the surjectivity of the Galois representation ϱ_p in order to derive the finiteness of $S(\mathbb{Q}, A(-1))$, we use the following Lemma, due to Nekovář, that finishes, after applying Poitou-Tate Duality, the proof of Theorem B.

LEMMA (2.5): *Let $\mathbb{Q}(E_{p^n})/\mathbb{Q}$ be the Galois extension obtained by adjoining the coordinates of all p^n-torsion points on E and let T' be as above. Then there exists a $c > 0$, such that the exponent of $H^1(\mathrm{Gal}(\mathbb{Q}(E_{p^n})/\mathbb{Q}), T'(-1)/p^n)$ divides p^c for all $n \geq 0$.*

REMARK: Flach uses the vanishing of this cohomology group that follows from his additional assumption on the surjectivity of ϱ_p.

PROOF: Put $G := \mathrm{Im}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}_{\mathbb{Z}_p}(T_p(E)))$. Since E is without complex multiplication over $\overline{\mathbb{Q}}$, G is of finite index in $\mathrm{Aut}_{\mathbb{Z}_p}(T_p(E)) = GL_2(\mathbb{Z}_p)$. Put $G_n := \ker(G \to GL_2(\mathbb{Z}/p^n)), T' := \mathrm{Sym}^2(T_p(E))$, $\tilde{G} := \mathrm{Im}(G \to \mathrm{Aut}_{\mathbb{Z}_p}(T')) = G/Z \cap G$, where $Z = $ center of $GL_2(\mathbb{Z}_p) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \lambda \in \mathbb{Z}_p^* \right\}$.
Consider the following diagram with horizontal and vertical exact sequences:
(note that $G/G_n \cong \mathrm{Gal}(\mathbb{Q}(E_{p^n})/\mathbb{Q})$.

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & H^1(G, T'(-1)) \otimes \mathbb{Z}/p^n & & \\
 & & \downarrow & & \\
0 \to H^1(G/G_n, T'(-1)/p^n) & \xrightarrow{\mathrm{inf}} & H^1(G, T'(-1)/p^n) & \xrightarrow{\mathrm{res}} & H^1(G_n, T'(-1)/p^n)^{G/G_n} \\
 & & \downarrow & & \\
 & & H^2(G, T'(-1))_{p^n} & &
\end{array}$$

It is clear that $H^i(G, T'(-1)) = H^i_{\mathrm{cont}}(G, T'(-1)) = H^i_{\mathrm{naive}}(G, T'(-1))$ are $\mathbb{Z}_p$-modules of finite type. Therefore $H^2(G, T'(-1))_{p^\infty}$ is finite. We have an exact sequence

$$\begin{array}{c}
0 \to H^1(\tilde{G}, T'(-1)) \xrightarrow{\mathrm{inf}} H^1(G, T'(-1)) \xrightarrow{\mathrm{res}} H^1(Z \cap G, T'(-1))^{G/Z\cap G} \\
\hfill = \\
\hfill \mathrm{Hom}_{\mathrm{cont}}(Z \cap G, (T'(-1))^{G/Z\cap G})
\end{array}$$

But $(T'(-1))^{G/Z\cap G}$ is zero (E has no CM). Thus $H^1(\tilde{G},T'(-1)) = H^1(G,T'(-1))$. By result of Lazard there is an injection

$$\begin{aligned} H^1(\tilde{G},T'(-1))\otimes\mathbb{Q} \quad \hookrightarrow \quad & H^1(Lie(\tilde{G}),T'(-1)\otimes\mathbb{Q}) \\ & = H^1(sl(2),T'(-1)\otimes\mathbb{Q}) \end{aligned}$$

and H^1 vanishes for semisimple Lie-algebras (and every representation). So $H^1(G,T'(-1))$ is finite and Lemma 2.5 follows.

Finally it is easy to see that Corollary (1.3), Proposition (2.1) b) and Theorem B imply Lemma II and as a consequence also Theorem A. It remains to show Proposition (2.1). This will be accomplished in the next paragraph.

§3

The surjectivity of the map

$$\psi' = \bigoplus_{\ell\neq p}\psi_\ell : H^1(X,\mathcal{K}_2)\otimes\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \bigoplus_{\ell\neq p} H^1(\mathbb{Q}_\ell,A)_{\mathrm{div}}/H^1_f(\mathbb{Q}_\ell,A)$$

follows from ([L-S], Lemmas (4.1), (4.3), (4.4) and (4.5)). On the other hand the composite map

$$\mathrm{Pic}(X)\otimes p^{\mathbb{Z}})\otimes\mathbb{Q}_p/\mathbb{Z}_p \to H^1(X,\mathcal{K}_2)\otimes\mathbb{Q}_p/\mathbb{Z}_p \stackrel{\psi_p}{\to} H^1_g(\mathbb{Q}_p,A)/H^1_f(\mathbb{Q}_p,A)$$

is surjective by Lemma (2.2), whereas the image of $(\mathrm{Pic}(X)\otimes p^{\mathbb{Z}})\otimes\mathbb{Q}_p/\mathbb{Z}_p$ under the map ψ' is zero. To finish the proof of Proposition (2.1) we therefore have to show that the image of α'_p, the p-component of α' is contained in $H^1_g(\mathbb{Q}_p,A)/H^1_f(\mathbb{Q}_p,A)$.

By the theory of Bloch-Ogus and the work of Merkurjev-Suslin [M-S] we have an isomorphism

$$H^1(X,\mathcal{K}_2/p^n) \cong NH^3_{et}(X,\mathbb{Z}/p^n(2)) \quad .$$

Let $\mathcal{X}$ be a proper regular semistable model of $X_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$, $i: X_p \to \mathcal{X}$ and $j: X_{\mathbb{Q}_p} \hookrightarrow \mathcal{X}$ the inclusions of the closed and generic fiber.

Let $H^3_{et}(\mathcal{X},\tau_{\leq 2}Rj_*\mathbb{Z}/p^n(2))$ be the cohomology of the truncated complex of p-adic vanishing cycles. Then we have

Lemma (3.1): *Assume that the Gersten-Conjecture holds for the Zariski sheaf $\mathcal{K}_2$ on the regular scheme $\mathcal{X}$. Then we have the inclusion*

$$H^1(X_{\mathbb{Q}_p},\mathcal{K}_2/p^n) \subset H^3_{et}(\mathcal{X},\tau_{\leq 2}Rj_*\mathbb{Z}/p^n(2)) \quad .$$

Proof:

This follows from the proof of ([L-S], Lemma (5.4)).

LEMMA (3.2): $H^3(\overline{X}_{\mathbb{Q}_p}, \mathbb{Q}_p(2))^{G_{\mathbb{Q}_p}} = 0$.

PROOF:

Using the Künneth formula and the fact that $H^2(\overline{E}, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p$ (the Brauer group of a curve over an algebraically closed field is zero), it suffices to show that $H^1(\overline{E}, \mathbb{Q}_p(1))^{G_{\mathbb{Q}_p}} = 0$. This follows from ([J], Theorem 5a).

Using Lemma (3.2) and the Hochschild-Serre spectral sequence we get a canonical map

$$\sigma : \varprojlim_n H^1(X_{\mathbb{Q}_p}, \mathcal{K}_2/p^n) \otimes \mathbb{Q}_p \longrightarrow H^1(\mathbb{Q}_p, V) \quad .$$

When we deal with a variety over a local field, all cohomology groups under consideration are (co-)finitely generated. The map α_p' certainly factors through $\varinjlim_n H^1(X_{\mathbb{Q}_p}, \mathcal{K}_2/p^n)_{\text{div}}$. The assertion that $\varinjlim_n H^1(X_{\mathbb{Q}_p}, \mathcal{K}_2/p^n)_{\text{div}}$ is contained in $H^1_g(\mathbb{Q}_p, A)$ is therefore equivalent to the assertion that the image of σ is contained in $H^1_g(\mathbb{Q}_p, V)$. In view of Lemma (3.1) we see that Proposition (2.1) follows from the following

LEMMA (3.3): *Under the condition H1) we have:* $\operatorname{Im}\sigma \subset H^1_g(\mathbb{Q}_p, V)$.

To prove Lemma (3.3) it suffices to show that the image of the map

$$H^3(\mathcal{X}, \tau_{\leq 2} Rj_* \mathbb{Q}_p(2)) \longrightarrow H^1(\mathbb{Q}_p, V)$$

is contained in $H^1_g(\mathbb{Q}_p, V)$.

Let $s_n^{\log}(2)$ be the log-syntomic complex in $D_{et}(\mathcal{X})$ constructed by Kato ([Ka], §6) and Tsuji [Tsu] together with a canonical map

$$s_n^{\log}(2) \longrightarrow \tau_{\leq 2} i_* i^* Rj_* \mathbb{Z}/p^n(2) \quad .$$

This gives rise to a composite map

$$\eta : H^3_{et}(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(2)) \longrightarrow H^1(\mathbb{Q}_p, V) \quad .$$

Since $(D_2)^{N=0}_{\varphi=p^2} = (D_3)^{N=0}_{\varphi=p^2} = 0$ (D_i denotes the i-th log-crystalline cohomology of X_p) we may apply the main result in [L] on a semistable analogue of Schneider's p-adic points conjecture to get

Lemma (3.4) $\operatorname{Im}\eta = H^1_g(\mathbb{Q}_p, V)$.

Tsuji has proven that there is a canonical isomorphism between the cohomology $\mathcal{H}^2(i^* s_n^{\log}(2))$ and the sheaf $M_n^2 = i^* R^2 j_* \mathbb{Z}/p^n(2)$ of p-adic vanishing cycles ([Tsu], Theorem 3.2). His proof relies on a filtration $\mathrm{Fil}^{\cdot}$ on M_n^2 that was defined by Hyodo ([H], (1.4)) and is induced by a symbol map on Milnor K-Theory. Hyodo has shown ([H], Theorem (1.6)) that the highest graded quotient $gr^0 M_n^2$ sits in an extension (change of notation: $Y := X_p$, the closed fiber of $\mathcal{X}$)

$$0 \longrightarrow W_n w^1_{Y,log} \longrightarrow gr^0 M_n^2 \longrightarrow W_n w^2_{Y,log} \longrightarrow 0$$

where $W_n w^i_{Y,log}$ are the modified logarithmic Hodge-Witt-sheaves ([H] (1.5)). On the other hand Hyodo and Kato ([H-K] Prop. 1.5) constructed an exact sequence of Hodge-Witt-sheaves

$$0 \longrightarrow W_n w^1_Y \longrightarrow W_n \tilde{w}^2_Y \longrightarrow W_n w^2_Y \longrightarrow 0$$

and used the connecting homomorphism on the level of cohomology to define the monodromy operator on log-crystalline cohomology. It follows from the work of Tsuji ([Tsu], §2.4) that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & W_n w^1_{Y,log} & \to & gr^0 M_n^2 & \to & W_n w^2_{Y,log} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & W_n w^1_Y & \to & W_n \tilde{w}^2_Y & \to & W_n w^2_Y & \to & 0 \end{array}$$

such that the upper exact sequence is obtained by taking the kernel of $1-F$ acting on the lower exact sequence, where F is the Frobenius. From the Hodge-Witt-decomposition of $H^r(Y, Ww^{\cdot})$ ([Il], Proposition (1.5)) it is easy to derive a Hodge-Witt-decomposition for $H^r(Y, W\tilde{w}^{\cdot}_Y)$

$$H^r(Y, W\tilde{w}^{\cdot}_Y) = \bigoplus_{i+j=r} H^i(Y, W\tilde{w}^j_Y) \quad .$$

From the action of the Frobenius φ on $H^r(Y, W\tilde{w}^{\cdot}_Y)$ we get

$$H^3(Y, W\tilde{w}^{\cdot}_Y)_{\varphi=p^2} = H^1(Y, W\tilde{w}^2_Y)^{F=1} \quad .$$

On the other hand it is shown in the proof of the semistable analogue of the p-adic points conjecture on log-syntomic cohomology [L], (2.6), Prop. (2.9), Prop. (2.13) that there is a surjection

$$H^3_{et}(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(2)) \twoheadrightarrow (H^3(Y, W\tilde{w}^{\cdot}_Y)_{\mathbb{Q}_p})_{\varphi=p^2}$$

and the above arguments yield a commutative diagram

$$\begin{tikzcd}
H^3(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(2)) \arrow[d] \arrow[dr] & & \\
H^3_{et}(\mathcal{X}, \tau_{\leq 2} Rj_* \mathbb{Q}_p(2)) \arrow[r] \arrow[d] & H^1(Y, gr^0 M^2_{\mathbb{Q}_p}) \arrow[r] & (H^3(Y, W\tilde{\omega}^{\cdot}_Y)_{\mathbb{Q}_p})_{\varphi = p^2} \arrow[d] \\
H^1(\mathbb{Q}_p, V) \arrow[rr] & & H^1(\mathbb{Q}_p, B_{\text{crys}} \otimes V)
\end{tikzcd}$$

It follows from ([L], (2.10)) that the composite

$$(H^3(Y, W\tilde{\omega}^{\cdot}_Y)_{\mathbb{Q}_p})_{\varphi=p^2} \longrightarrow H^1(\mathbb{Q}_p, B_{\text{crys}} \otimes V) \longrightarrow H^1(\mathbb{Q}_p, B_{st} \otimes V)$$

is the zero map. Using the fact that $H^1_{st} = H^1_g$ (unpublished result of Hyodo, see also Nekovář ([Ne](1.24)) we conclude that the image of the map

$$H^3_{et}(\mathcal{X}, \tau_{\leq 2} Rj_* \mathbb{Q}_p(2)) \longrightarrow H^1(\mathbb{Q}_p, V)$$

is $H^1_g(\mathbb{Q}_p, V)$ in view of Lemma (3.4). This finishes the proof of Lemma (3.3) and Proposition (2.1).

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