Chow Groups with Coefficients

Markus Rost

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Abstract. We develop a generalization of the classical Chow groups in order to have available some standard properties for homology theories: long exact sequences, spectral sequences for fibrations, homotopy invariance and intersections. The basis for our constructions is Milnor's K-theory.

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Contents

SUMMARY

The paper considers generalities for localization complexes for varieties. Examples of these complexes are given by the Gersten resolutions in various contexts, in particular in K-theory and in étale cohomology. The paper gives a general notion of coefficient systems for such complexes, the so called cycle modules. There are the corresponding "complexes of cycles with coefficients" and their homology groups, the "Chow groups with coefficients". For these some general constructions are developed: proper push-forward, flat pull-back, spectral sequences for fibrations, homotopy invariance and intersection theory.

If one specializes the material to the case of Milnor's K-theory as coefficient system, one obtains in particular an elementary development of intersections for the classical Chow groups. This treatment is somewhat different to former approaches. The main tool is still the deformation to the normal cone. The major difference is that homotopy invariance is not established alone for the Chow groups, but for the "cycle complex with coefficients in Milnor's K-Theory". This enables one to keep control in fibered situations. The proof of associativity of intersections is based on a doubled version of the deformation to the normal cone.

CONVENTIONS AND NOTATIONS

We work over a ground field k and a base scheme $B \to \operatorname{Spec} k$. The word scheme means a localization of a separated scheme of finite type over k. (This includes schemes of finite type over a field finitely generated over k.) From Section 8 on all schemes are of finite type over a field. Moreover all schemes and morphisms are defined over B (with exceptions in Section 14). The letter M stands from Section 3 on for a cycle module. If not mentioned otherwise, it is defined over B (in Sections 3–5) or over X (in Sections 7–13).

For $x \in X$ we denote by $\dim(x, X)$ the dimension of the closure $\overline{\{x\}}$ of x in X and by $\operatorname{codim}(x, X)$ the dimension of the localization $X_{(x)}$. The set of points of X of dimension (resp. codimension) p is denoted by $X_{(p)}$ (resp. $X^{(p)}$). We make free use of some basic facts from commutative algebra and refer for this to (Hartshorne 1977; Matsumura 1980) and, in particular, to (Fulton 1984, App. A, App. B).

In Sections 6, 8, 9, and 11–13 we use the special notation $X \looparrowright Y$ for certain maps between the cycle complexes. This is explained in (3.8).

Introduction

The classical Chow groups $\mathrm{CH}_p(X)$ of p-dimensional cycles on a variety X may be defined as the cokernel of the divisor map

$$\coprod_{x \in X_{(p+1)}} \kappa(x)^* \xrightarrow{d} \coprod_{x \in X_{(p)}} \mathbb{Z}.$$

Here $X_{(p)}$ is the set of points of X of dimension p and $\kappa(x)$ is the residue class field of x. This paper studies complexes $C_*(X;M)$ of the following type:

$$\cdots \xrightarrow{d} \coprod_{x \in X_{(p+1)}} M\big(\kappa(x)\big) \xrightarrow{d} \coprod_{x \in X_{(p)}} M\big(\kappa(x)\big) \xrightarrow{d} \coprod_{x \in X_{(p-1)}} M\big(\kappa(x)\big) \xrightarrow{d} \cdots .$$

Here M is what we call a cycle module. This is a functor $F \to M(F)$ on fields to abelian groups equipped with four structural data (the even ones: restriction and corestriction; the odd ones: multiplication with K_1 and residue maps for discrete valuations). Moreover there is imposed a list of certain rules and axioms. A particular example of a cycle module is $M = K_*$, given by Milnor's (or Quillen's) K-ring

$$K_*F = \mathbb{Z} \oplus F^* \oplus K_2F \oplus \cdots .$$

Other examples are provided by Galois cohomology, specifically

$$M(F) = \coprod_{n \geq 0} H^n(F; D \otimes \mu_r^{\otimes n})$$

with D a Galois module over a ground field k with $\operatorname{char} k$ prime to r.

The complex $C_*(X;M)$ is called the chain complex of cycles on X and its homology groups $A_p(X;M)$ are called the Chow groups of X (with coefficients in M).

The Chow groups $A_p(X;M)$ enclose various familiar objects. The classical Chow group $\mathrm{CH}_p(X)$ is a direct summand of $A_p(X;K_*)$. The E^2-terms of the local-global spectral sequences in étale cohomology and in Quillen's K-theory are of type $A_p(X;M)$. For proper smooth X of dimension d the group $A_d(X;M)$ is a birational invariant—the "M-valued" analogue of unramified Galois cohomology.

The paper develops some basic constructions for the cycle complexes $C_*(X;M)$ and the Chow groups $A_*(X;M)$ for schemes X of finite type over a field. There are proper push-forward, flat pull-back and homotopy invariance. Moreover intersection theory is available: for regular imbeddings and morphisms to smooth varieties there is a pull-back map. Finally for a morphism $\pi\colon X \to Z$ there is a spectral sequence

$$E^2_{p,q} = A_p(Z; A_q[\pi;M]) \Longrightarrow A_{p+q}(X;M).$$

Here the $A_q[\pi;M]$ are certain cycle modules obtained from taking homology in the fibers. All the mentioned functorial behavior extends for appropriate fiber diagrams to the cycle modules $A_q[\pi;M]$ and the spectral sequences.

The constructions are carried out on complex level in a pointwise manner. The treatment has some parallels to a standard development of homology of CW-complexes. This analogy should not be taken too serious, but may give a first impression about the sort of technicalities. In this picture our "cells" are just all points of the variety in question. The patching data for the "cells" are given by the (geometric) valuations on the residue class field of one point having center in another point. The appropriate local coefficient systems are the cycle modules. However, the nature of these coefficient systems is more complicated than in topology. First of all, their ground ring is provided by Milnor's K-theory of fields. Moreover, besides the usual functorial behavior, there is need for transfer maps (basically because one has to deal with non algebraically closed fields) and there are residue maps for valuations (to give passage from one point of a variety to its specializations).

The material of this paper grew out from considerations concerning the bijectivity of the norm residue homomorphism and Hilbert's Satz 90 for Milnor's K_n. There the computation of the Chow groups of certain norm varieties and quadrics plays an important role. As a general technique (see also Karpenko and Merkurjev 1991) we used a spectral sequence for morphisms $\pi\colon X \to Z$ relating the Chow groups of the total space to something like "the Chow groups of the base with coefficients in the Chow groups of the fibers"; moreover these spectral sequences should be compatible with intersection operations. The goal of the paper was to present an appropriate framework in a fairly direct manner.

With the remarks following, we have tried to draw the line of development of the paper. In the discussion of intersection theory, we restrict for simplicity to typical situations and with Milnor's K-theory as coefficient system, although the actual treatment is more general.

Even if one is interested in classical Chow groups alone, one is led to consider some more general versions of Chow groups. To start with a simple situation, let $Y \subset X$ be a closed subvariety. Then there is an exact sequence

$$\mathrm{CH}_p(Y) \to \mathrm{CH}_p(X) \to \mathrm{CH}_p(X \setminus Y) \to 0.$$

For concrete computations as well as for general considerations, there appears the problem to extend this sequence to the left in a reasonable way by a sort of higher variants of Chow groups. Similarly, let $\pi\colon X \to Z$ be a morphism of varieties and try to relate the Chow groups of X with the Chow groups of Z and of the fibers. When working within the classical Chow groups alone, there will be no good answer in general.

In this paper the approach to these problems is provided by Milnor's K-theory. For a variety X one forms for $n \in \mathbb{Z}$ the complex[◊] $C_*(X;n)$ with

$$C_p(X;n) = \coprod_{x \in X_{(p)}} K_{n+p}\kappa(x)$$

[◊] The complex of cycles with coefficients in Milnor's K-theory to be considered later splits up as a direct sum $C_*(X;K_*) = \coprod_n C_*(X;n)$ according to the grading of Milnor's K-ring.

where K_nF is Milnor's n-th K-group of a field F. The homology groups of the complex $C_*(X;n)$ are denoted by $A_p(X;n)$. For $n=-p\leq 0$ it ends up with

$$\cdots \xrightarrow{d} \coprod_{x\in X_{(p+2)}} K_2\kappa(x) \xrightarrow{d} \coprod_{x\in X_{(p+1)}} K_1\kappa(x) \xrightarrow{d} \coprod_{x\in X_{(p)}} K_0\kappa(x) \longrightarrow 0$$

and one has $\mathrm{CH}_p(X)=A_p(X;-p)$.

Then for a subvariety $Y\subset X$ there is a long exact sequence

$$\cdots\to A_{p+1}(X\setminus Y;n)\to A_p(Y;n)\to A_p(X;n)\to A_p(X\setminus Y;n)\to\cdots.$$

Moreover let $\pi\colon X\to Z$ be a morphism. The filtration of the set $X_{(p)}$ given by the dimension of the image gives rise to a filtration of the complex $C_*(X;n)$. The corresponding E^1-spectral sequence looks like

$$E^1_{p,q}=\coprod_{z\in Z_{(p)}} A_q(X_z;n+p)\Longrightarrow A_{p+q}(X;n) \tag{1}$$

with $X_z=X\times_Z \operatorname{Spec}\kappa(z)$.

A major problem in intersection theory is to produce for a regular imbedding $f\colon X'\to X$ a pull-back map $f^\bullet$ on the Chow groups having the geometric meaning of intersecting cycles on X with X'. (For a general account on intersections we refer to Fulton 1984)

These maps are in the actual context of type

$$f^\bullet\colon A_p(X;n)\to A_{p-d}(X';n+d)$$

with $d=\operatorname{codim}(f)$.

In the paper the maps $f^\bullet$ are defined by first constructing homomorphisms of complexes

$$I(f)\colon C_*(X;n)\to C_{*-d}(X';n+d)$$

and then passing to homology. In a fibered situation (that is f lies over some map $Z'\to Z$ with appropriate smoothness conditions), the maps $I(f)$ can be chosen to respect the filtrations, thereby inducing homomorphisms on the corresponding spectral sequences.

As the reader might guess, the maps $I(f)$ cannot be defined canonically in terms of f. Namely, $I(f)$ gives in particular a lift of the classical pull-back map $f^\bullet\colon \mathrm{CH}_p(X)\to\mathrm{CH}_{p-d}(X')$ to the cycle groups. But if a cycle W on X does not meet X' properly, there is in general no way to define $W\cap X'$ by a canonical cycle.

It may be surprising that one can handle with such pull-back maps $I(f)$ on complex level in a reasonable way. Therefore we will discuss here the nature of these maps in some detail.

When working with the complexes $C_*(X;n)$, it turns out that the necessary constructions can be described in terms of four basic operations. These, called the "four basic maps", are of the following type.

For a morphism $f\colon X \to Y$, there is a push-forward map

$$f_*\colon C_p(X;n) \to C_p(Y;n).$$

For a morphism $g\colon X \to Y$ with fiber dimension s, there is a pull-back map

$$g^*\colon C_p(Y;n) \to C_{p+s}(X;n-s).$$

Moreover there is "multiplication with K_1": for a global unit a on X, there is a map

$$\{a\}\colon C_p(X;n) \to C_p(X;n+1)$$

given by pointwise multiplication with $a(x) \in \kappa(x)^* = K_1\kappa(x)$.

Finally for a closed immersion there is a canonical "boundary map"

$$\partial\colon C_p(X \setminus Y;n) \to C_{p-1}(Y;n).$$

All these maps are defined in a pointwise manner. If f is proper and g is flat, the maps f_* and g^* commute with the differentials of the complexes. One uses f_* also for open immersions f and g^* also for closed immersions g (then f_* and g^* are just the corresponding projections and don't commute with the differentials). The maps $\{a\}$ and ∂ anti-commute with the differentials.

In fact, the four basic maps are enough to define intersections on complex level: by their very definition, the maps $I(f)$ are sums of compositions of the four basic maps. For the construction of the $I(f)$, the first major tool is the deformation to the normal cone. This yields a canonical "deformation map"

$$C_*(X;n) \to C_*(N;n)$$

where N is the normal cone of f. The next step is to define for a vector bundle $\pi\colon V \to X$ of dimension d a homotopy inverse

$$C_*(V;n) \to C_{*-d}(X;n+d)$$

to the pull-back map π^*. It is at this place where one needs some extra noncanonical choices. The choice to be made is (at most) that of what we call a "coordination" of π. This is a stratification of X together with bundle trivializations on the strata.

In the end there is a canonical procedure which starts from the choice of a coordination of the normal bundle of f and yields a map $I(f)$ as desired, defined in terms of the four basic maps. Different choices lead to homotopic maps $I(f)$, with the homotopies again expressible in terms of the four basic maps. In a fibered situation, one may arrange things to end up with filtration preserving maps $I(f)$. Once having made the necessary choices, the construction is quite functorial. For example, it is compatible with respect to base change and localization. In order to establish functoriality (namely $I(f \circ f')$ should be homotopic to $I(f') \circ I(f)$, if necessary under a filtration preserving homotopy), we use a kind of doubled deformation space.

The viewpoint of the paper is to put the four basic maps in the center. In particular the maps $\{a\}$, ∂ are treated as if they were a kind of morphisms in their own right, of equal rank as the more familiar push-forward and pull-back maps. This has at least technical advantages. For example, in order to check various compatibilities concerning the maps $f^\bullet$, it is very convenient to reduce to a separate treatment of the four basic maps.

The reader may ask why we insist to stay on complex level although one is interested mainly in the Chow groups. Over some range this is quite natural from the material. However, the proof of homotopy invariance with respect to vector bundles is much simpler for the Chow groups (using the spectral sequences) than for the cycle complexes themselves (where one has to construct explicit homotopy inverses).

The major motive for keeping the complex level throughout was to keep control on the filtrations in fibered situations.

Besides this, we hope that our method is of some interest concerning questions for correspondences between arbitrary varieties. To give an example let $f\colon \widetilde{X} \to X$ be a proper birational morphism with X smooth. Then there are pull-back maps

$$I(f)\colon C_*(X;n) \to C_*(\widetilde{X};n)$$

similar to the $I(f)$ above. The $I(f)$ are unique up to homotopy and have the standard push-forward map f_* as left inverse. In particular, $I(f)$ identifies $C_*(X;n)$ as a subcomplex of $C_*(\widetilde{X};n)$. In the case of a blow up in a point x, the choice to be made in the construction of $I(f)$ is (at most) that of a system of parameters around x.

We think of the maps $I(f)$ as a sort of generalized correspondences. One can make this more precise in a further development which we call bivariant theory of cycles. There the four basic maps find their place as morphisms of varieties in an appropriate differential category and (the homotopy classes of) the maps $I(f)$ appear rather as morphisms in a category of varieties admitting products, than just as homomorphisms of complexes (as in this paper).

The motive of introducing a general notion of coefficient systems for cycles appears when looking at the spectral sequence (1). Its E^2-terms are the homology groups of complexes of type

$$\cdots \xrightarrow{d} \coprod_{z \in Z_{(p)}} A_q(X_z; n+1) \xrightarrow{d} \coprod_{z \in Z_{(p-1)}} A_q(X_z; n) \xrightarrow{d} \cdots .$$

We interpret this by saying that the collection of functors (with $n \in \mathbb{Z}$)

$$A_q[\pi;n]\colon F \mapsto A_q(X \times_Z \operatorname{Spec} F; n),$$

defined on fields F over Z, appear as new coefficient systems. This process of creating coefficient systems may in fact be iterated.

Therefore it seems convenient to have available some appropriate general notion of coefficient systems. The class considered in this paper is provided by the notion of what we call cycle modules. Its definition is formal and somewhat ad hoc. The important thing for us is, that it contains standard functors like Milnor's (or Quillen's) K-theory and Galois cohomology (as indicated above), that it is closed under processes like $M \to A_q[\pi; M]$ and that it allows intersection theory. Anyway it might be of at least heuristic interest, that many general constructions (intersections, also the proof of acyclicity for smooth local rings) can be based on pure formal properties—at least if one starts from Milnor's K-theory of fields.

Milnor's K-theory is the fundamental base of the whole paper. This was at first suggested by our original problem, Hilbert's Satz 90 for Milnor's K_n. Besides this, Milnor's K-theory seems to give the minimal framework needed in order to express the considerations on intersections discussed above. By the way, it seems likely that the general method works also with Milnor's K-theory replaced by the Witt ring of quadratic forms of fields of characteristic different from 2.

Milnor's K-theory has a simple definition in terms of generators and relations. Despite this fact, it is by no means a simple and well understood functor. Already to define the norm homomorphisms takes some effort. An even more serious and in general an unsolved problem is for example the computation of the torsion in Milnor's K-groups. These problems are related with Hilbert's Satz 90 (Merkurjev and Suslin 1982, 1986) and are part of a broader picture (Beilinson conjectures, motivic cohomology). In this context there appear other and more general higher versions of the classical Chow groups than the groups $A_p(X;n)$ based on Milnor's K-theory, namely motivic cohomology (Bloch's higher Chow groups and Suslin's singular homology) and also K-cohomology (Bloch 1986; Quillen 1973; Suslin and Voevodsky 1996). Milnor's K-theory forms a central part of motivic cohomology and of Quillen's K-theory. In fact, in the smooth case there are natural maps from the motivic cohomology of X to the groups $A_*(X;n)$ and from $A_*(X;n)$ to the K-cohomology of X, both of which are isomorphisms in some low degrees. On the other hand, motivic cohomology and Quillen's K-theory give rise to cycle modules in our sense. (In the case of Quillen's K-theory this is made more precise in Sections 1, 2 and 5). These functors are definitely necessary for a full understanding of Milnor's K-theory. For the purpose of this paper however, it turned out to be enough to rely on elementary properties of Milnor's K-theory.

The paper may be roughly divided in four parts. In Sections 1–2 the notion of cycle modules is defined. Here we have lent some weight to a discussion of the axioms. In Sections 3–5 the cycle complexes, the Chow groups and their basic functorial behavior are established; Section 6 is a side remark concerning the acyclicity of Gersten-type resolutions. Sections 7–8 treat the spectral sequences. Sections 9–14 are concerned with intersection theory.

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1. Cycle Premodules

Cycle premodules are roughly said functors on fields which have transfer, are modules over Milnor's K-theory, are equipped with residue maps for discrete valuations and satisfy the "usual rules". The definition is quite formal. It forms the local dimension 1 part of the notion of cycle modules. A major difference to cycle modules is that cycle premodules do not have to obey laws involving an infinite number of valuations like the sum formula for $\mathbb{P}^1$.

Cycle premodules are defined by a list of data and rules. These are just usual properties, quite familiar to standard examples. Equivalently, one may define cycle premodules as the additive functors on a certain category which has an explicit description in terms of Milnor's K-theory and valuations (see Remark 1.10). This point of view is perhaps more satisfying. It tells that our list of data and rules is in a sense a complete list. However, it would take some effort to establish the composition rule in the category and we omit therefore a detailed discussion. Moreover, in order to establish certain functors as cycle premodules, it is more convenient to refer to the explicit lists of properties.

The viewpoint of the four basic maps mentioned in the introduction would at first lead to functors $F \to M(F)$, such that each $M(F)$ is a module over the tensor algebra TF^*. However, the existence of norm maps and the homotopy property leads one to pass to modules over Milnor's K-ring (see Remark 2.7).

We first recall basic facts from Milnor's K-theory. Let F be a field. By definition Milnor's K-ring (Milnor 1970) of F is$^\diamond$

$$K_*F = TF^*/J$$

where F^* is the multiplicative group of F, TF^* is the tensor algebra of F^* as abelian group and J is the two-sided ideal of TF^* generated by the set

$$\{\, a \otimes b \mid a, b \in F^*,\, a + b = 1\,\}.$$

The standard grading on TF^* induces a grading

$$K_*F = \coprod_{n \geq 0} K_nF.$$

K_nF is the n-th Milnor's K-group of F. By definition $K_0F = \mathbb{Z}$ and $K_1F = F^*$. The elements of K_nF represented by tensors $a_1 \otimes \cdots \otimes a_n$, $a_i \in F^*$, are called symbols and denoted by $\{a_1, \ldots, a_n\}$. The group law in K_nF is written additively, e.g., $\{ab\} = \{a\} + \{b\}$. There are the rules $\{a, -a\} = 0$ and $\{a, b\} + \{b, a\} = 0$, see (Milnor 1970). In particular, K_*F is an anti-commutative ring with respect to the natural $\mathbb{Z}/2$-grading.

For a homomorphism of fields $\varphi\colon F \to E$ there is the ring homomorphism

$$\varphi_*\colon K_*F \to K_*E,$$
$$\varphi_*(\{a_1, \ldots, a_n\}) = \{\varphi(a_1), \ldots, \varphi(a_n)\}.$$

$^\diamond$ In the literature one often uses the notation $K_*^{\mathrm{M}}F$ for Milnor's K-ring, while K_*F stands for Quillen's K-ring.

If φ is finite, there is the norm homomorphism

$$\varphi^*\colon K_*E \to K_*F.$$

φ^* preserves the $\mathbb{Z}$-grading. Its component $\mathbb{Z} \to \mathbb{Z}$ in degree 0 is multiplication with $\deg\varphi = [E:F]$. In degree 1 it is the usual norm map $N_\varphi\colon E^* \to F^*$ for the finite field extensions. φ^* has been defined by Bass and Tate (1972) with respect to a choice of generators of E over F; it is in fact independent of such a choice (Kato 1980). For a characterization of φ^* see the remark after Theorem 1.4.

For a valuation $v\colon F^* \to \mathbb{Z}$ we denote by $\mathcal{O}_v$, $\mathfrak{m}_v$, $\kappa(v)$ its ring, maximal ideal and residue class field, respectively. For nontrivial v there is the residue homomorphism $\partial_v\colon K_*F \to K_*\kappa(v)$, see (Milnor 1970). ∂_v is of degree -1. It has the characterizing properties

$$\begin{aligned} \partial_v(\{\pi, u_1, \ldots, u_n\}) &= \{\bar u_1, \ldots, \bar u_n\}, \\ \partial_v(\{u_1, \ldots, u_n\}) &= 0 \end{aligned}$$

for a prime π of v and for v-units u_i with residue classes $\bar u_i \in \kappa(v)^*$. Define

$$\begin{aligned} s_v^\pi &\colon K_*F \to K_*\kappa(v), \\ s_v^\pi(x) &= \partial_v(\{-\pi\} \cdot x). \end{aligned}$$

s_v^π is a ring homomorphism and is characterized by

$$\begin{aligned} s_v^\pi(\{u_1, \ldots, u_n\}) &= \{\bar u_1, \ldots, \bar u_n\}, \\ s_v^\pi(\{\pi, u_1, \ldots, u_n\}) &= 0. \end{aligned}$$

Rules between the maps φ_*, φ^*, ∂_v and the multiplicative structure of K_* are comprised below in Theorem 1.4.

Let B be a scheme over a field k (recall our conventions). In the following we mean by a field over B a field F together with a morphism $\operatorname{Spec} F \to B$ such that F is finitely generated over k. By a valuation over B we mean a discrete valuation v of rank 1 together with a morphism $\operatorname{Spec}\mathcal{O}_v \to B$ such that v is of geometric type over k. The latter means that $\mathcal{O}_v$ is the localization of an integral domain of finite type over k in a regular point of codimension 1. Alternatively, valuations of geometric type may be characterized by: $k \subset \mathcal{O}_v$, the quotient field F and the residue class field $\kappa(v)$ are finitely generated over k and $\operatorname{tr.deg}(F|k) = \operatorname{tr.deg}(\kappa(v)|k) + 1$.

This geometric setting is convenient for our later purposes. We impose its restrictive conditions from the beginning in order to keep things straight. For some purposes one may consider also arbitrary fields and valuations (discrete, of rank 1 and eventually not equicharacteristic) over an arbitrary scheme B.

In the following, the letters φ, ψ stand for homomorphisms of fields over B and all maps between various $M(F)$, $M(E)$, ... are understood as homomorphisms of graded abelian groups.

(1.1) Definition. *Let $\mathcal{F}(B)$ be the class of fields over B. A cycle premodule M consists of an object function $M\colon \mathcal{F}(B) \to \mathcal{A}$ to the class of abelian groups together with a $\mathbb{Z}/2$-grading $M = M_0 \oplus M_1$ or a $\mathbb{Z}$-grading $M = \coprod_n M_n$ and with the following data* D1–D4 *and rules* R1a–R3e.

D1: For each $\varphi\colon F \to E$ there is $\varphi_*\colon M(F) \to M(E)$ of degree 0.
D2: For each finite $\varphi\colon F \to E$ there is $\varphi^*\colon M(E) \to M(F)$ of degree 0.
D3: For each F the group $M(F)$ is equipped with a left K_*F-module structure denoted by $x \cdot \rho$ for $x \in K_*F$ and $\rho \in M(F)$. The product respects the gradings: $K_nF \cdot M_m(F) \subset M_{n+m}(F)$.
D4: For a valuation v on F there is $\partial_v\colon M(F) \to M\big(\kappa(v)\big)$ of degree -1.

For a prime π of v on F we put

$$s_v^\pi\colon M(F) \to M\big(\kappa(v)\big),$$
$$s_v^\pi(\rho) = \partial_v(\{-\pi\} \cdot \rho).$$

R1a: For $\varphi\colon F \to E$, $\psi\colon E \to L$ one has $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.
R1b: For finite $\varphi\colon F \to E$, $\psi\colon E \to L$ one has $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.
R1c: Let $\varphi\colon F \to E$, $\psi\colon F \to L$ with φ finite. Put $R = L \otimes_F E$. For $p \in \operatorname{Spec} R$ let $\varphi_p\colon L \to R/p$, $\psi_p\colon E \to R/p$ be the natural maps. Moreover let l_p be the length of the localized ring $R_{(p)}$. Then

$$\psi_* \circ \varphi^* = \sum_p l_p \cdot (\varphi_p)^* \circ (\psi_p)_*.$$

R2: For $\varphi\colon F \to E$, $x \in K_*F$, $y \in K_*E$, $\rho \in M(F)$, $\mu \in M(E)$ one has (with φ finite in the projection formulae R2b and R2c):
R2a: $\varphi_*(x \cdot \rho) = \varphi_*(x) \cdot \varphi_*(\rho)$,
R2b: $\varphi^*\big(\varphi_*(x) \cdot \mu\big) = x \cdot \varphi^*(\mu)$,
R2c: $\varphi^*\big(y \cdot \varphi_*(\rho)\big) = \varphi^*(y) \cdot \rho$.

R3a: Let $\varphi\colon E \to F$ and let v be a valuation on F which restricts to a nontrivial valuation w on E with ramification index e. Let $\bar\varphi\colon \kappa(w) \to \kappa(v)$ be the induced map. Then

$$\partial_v \circ \varphi_* = e \cdot \bar\varphi_* \circ \partial_w.$$

R3b: Let $\varphi\colon F \to E$ be finite and let v be a valuation on F. For the extensions w of v to E let $\varphi_w\colon \kappa(v) \to \kappa(w)$ be the induced maps. Then

$$\partial_v \circ \varphi^* = \sum_w \varphi_w^* \circ \partial_w.$$

R3c: Let $\varphi\colon E \to F$ and let v be a valuation on F which is trivial on E. Then

$$\partial_v \circ \varphi_* = 0.$$

R3d: Let φ, v be as in R3c, let $\bar\varphi\colon E \to \kappa(v)$ be the induced map and let π be a prime of v. Then

$$s_v^\pi \circ \varphi_* = \bar\varphi_*.$$

R3e: For a valuation v on F, a v-unit u and $\rho \in M(F)$ one has

$$\partial_v(\{u\} \cdot \rho) = -\{\bar u\} \cdot \partial_v(\rho).$$

□

The maps φ_*, φ^* are called the restriction and corestriction homomorphisms, respectively. We use the notations $\varphi_* = r_{E|F}$, $\varphi^* = c_{E|F}$ if there is no ambiguity.
Note that R2c with $y = 1 \in K_0 E$ gives

R2d: For finite $\varphi\colon F \to E$ one has

$$\varphi^* \circ \varphi_* = (\deg \varphi) \cdot \mathrm{id}.$$

Moreover R1c implies

R2e: For finite totally inseparable $\varphi\colon F \to E$ one has

$$\varphi_* \circ \varphi^* = (\deg \varphi) \cdot \mathrm{id}.$$

We consider $M(F)$ also as a right K_*F-module via

$$\rho \cdot x = (-1)^{nm} x \cdot \rho$$

for $x \in K_n F$ and $\rho \in M_m(F)$.
The maps ∂_v are called the residue homomorphisms and the maps s_v^π are called the specialization homomorphisms. It is easy to check that R3e implies

R3f: For a valuation v on F, $x \in K_n F$, $\rho \in M(F)$ and a prime π of v one has

$$\begin{aligned} \partial_v(x \cdot \rho) &= \partial_v(x) \cdot s_v^\pi(\rho) + (-1)^n s_v^\pi(x) \cdot \partial_v(\rho) + \{-1\} \cdot \partial_v(x) \cdot \partial_v(\rho), \\ s_v^\pi(x \cdot \rho) &= s_v^\pi(x) \cdot s_v^\pi(\rho). \end{aligned}$$

If π' is another prime and u is the v-unit with $\pi' = \pi u$, then

$$s_v^{\pi'}(x) = s_v^\pi(x) - \{\bar{u}\} \cdot \partial(x).$$

From this and the rule R3c it follows in particular that the rule R3d holds for every prime π.

More remarks concerning these formulae and the residue homomorphisms in general are given below.

All relevant cycle premodules M known to us are $\mathbb{Z}$-graded with $M_n = 0$ for $n < 0$. Within the general theory however there is need only for a $\mathbb{Z}/2$-grading and we will understand this case if not mentioned otherwise.

A morphism $f\colon B' \to B$ defines a transformation $\mathcal{F}(B') \to \mathcal{F}(B)$ and the restriction of a cycle premodule M over B to $\mathcal{F}(B')$ is a cycle premodule over B'. It will be sometimes denoted by f^*M but mostly by M as well. If $B = \operatorname{Spec} R$ is affine, we call a cycle premodule over B a cycle premodule over R. If R is a field, we speak of a constant cycle premodule. The reference to the base B will be often dropped.

(1.2) Definition. *A pairing $M \times M' \to M''$ of cycle premodules over B is given by bilinear maps for each F in $\mathcal{F}(B)$*

$$\begin{gathered} M(F) \times M'(F) \to M''(F), \\ (\rho, \mu) \mapsto \rho \cdot \mu \end{gathered}$$

which respect the gradings and which have the properties P1–P3 *stated below.*

A ring structure on a cycle premodule M is a pairing $M \times M \to M$ which induces on each $M(F)$ an associative and anti-commutative ring structure.

P1: For $x \in K_*F$, $\rho \in M(F)$, $\mu \in M'(F)$ one has
P1a: $(x \cdot \rho) \cdot \mu = x \cdot (\rho \cdot \mu)$,
P1b: $(\rho \cdot x) \cdot \mu = \rho \cdot (x \cdot \mu)$.

P2: For $\varphi\colon F \to E$, $\eta \in M(F)$, $\nu \in M(E)$, $\rho \in M'(F)$, $\mu \in M'(E)$ one has (with φ finite in P2b, P2c)
P2a: $\varphi_*(\eta \cdot \rho) = \varphi_*(\eta) \cdot \varphi_*(\rho)$,
P2b: $\varphi^*\big(\varphi_*(\eta) \cdot \mu\big) = \eta \cdot \varphi^*(\mu)$,
P2c: $\varphi^*\big(\nu \cdot \varphi_*(\rho)\big) = \varphi^*(\nu) \cdot \rho$.

P3: For a valuation v on F, $\eta \in M_n(F)$, $\rho \in M'(F)$ and a prime π of v one has

$$\partial_v(\eta \cdot \rho) = \partial_v(\eta) \cdot s_v^\pi(\rho) + (-1)^n s_v^\pi(\eta) \cdot \partial_v(\rho) + \{-1\} \cdot \partial_v(\eta) \cdot \partial_v(\rho).$$

□

Note that P3 implies

$$s_v^\pi(\eta \cdot \rho) = s_v^\pi(\eta) \cdot s_v^\pi(\rho).$$

(1.3) Definition. *A homomorphism $\omega\colon M \to M'$ of cycle premodules over B of even resp. odd type is given by homomorphisms*

$$\omega_F\colon M(F) \to M'(F)$$

which are even resp. odd and which satisfy (with the signs corresponding to even resp. odd type)

(1) $\varphi_* \circ \omega_F = \omega_E \circ \varphi_*$,
(2) $\varphi^* \circ \omega_E = \omega_F \circ \varphi^*$,
(3) $\{a\} \cdot \omega_F(\rho) = \pm\omega_F\left(\{a\} \cdot \rho\right)$,
(4) $\partial_v \circ \omega_F = \pm\omega_{\kappa(v)} \circ \partial_v$.

A unit a on B provides a simple example of a homomorphism of odd type, namely $\{a\}\colon M \to M$ given by $\{a\}_F(\rho) = \{a_F\} \cdot \rho$ where $a_F \in F^*$ is the restriction of a.

The cycle premodules over B together with the notion of homomorphism of Definition 1.3 form an ($\mathbb{Z}/2$-graded) abelian category.

(1.4) Theorem. *Milnor's K-theory K_* together with the data*

$$\varphi_*,\ \varphi^*,\ \text{multiplication},\ \partial_v$$

is a $\mathbb{Z}$-graded cycle premodule over any field k. With its multiplication, K_* is a cycle premodule with ring structure. □

This statement is a compact form of results in (Bass and Tate 1972; Kato 1980; Milnor 1970); we omit a detailed deduction.

Theorem 1.4 holds also in the setting of arbitrary fields and valuations (discrete of rank 1 and with a restriction in R3b, see Remark 1.8 below).

Given the rings K_*F for each F in $\mathcal{F}(\operatorname{Spec} k)$, the maps φ_*, φ^* and ∂_v are uniquely determined by R1b, R1c, P2, P3 and

$$\begin{array}{lll}
(1) & \varphi_*(1) & = 1, \\
(2) & \varphi_*(\{a\}) & = \{\varphi(a)\}, \\
(3) & \varphi^*(1) & = \deg\varphi \cdot 1, \\
(4) & \varphi^*(\{a\}) & = \big\{N\big(\varphi(a)\big)\big\}, \\
(5) & \partial_v(1) & = 0, \\
(6) & \partial_v(\{a\}) & = v(a), \\
(7) & \partial_v\big(\{a,b\}\big) & = \big\{(-1)^{v(a)v(b)} b^{v(a)} a^{-v(b)} \bmod \mathfrak{m}_v\big\}.
\end{array}$$

Here v denotes a normalized valuation: $v(\pi) = 1$.

This statement is trivial for the maps φ_* and ∂_v; for the uniqueness of the maps φ^* see in particular (Bass and Tate 1972, p. 40).

The multiplication maps of the K_*F-module structures on $M(F)$ for each F give rise to a pairing of cycle premodules

$$K_* \times M \to M.$$

Here the axioms P1, P2, and P3 follow from D3, R2, and R3f.

In order to establish a cycle premodule it is convenient to use the following reduction.

(1.5) Lemma. *For the validity of* R3d *it suffices* (*under presence of the other rules of* Definition 1.1) *to require* R3d *for the case* $E = \kappa(v)$.

Proof: By R1a the rule R3d holds for E if it holds for some extension E' of E with $E' \subset \mathcal{O}_v$. Moreover by R3a we may replace $\mathcal{O}_v$ by any unramified extension $\mathcal{O}'_v$ with the same residue class field (we don't want to pass to the henselization $\lim \mathcal{O}'_v$, since our fields should be finitely generated over k). Now by lifting a transcendence base of $\kappa(v)$ over E to $\mathcal{O}_v$ we may assume that $\kappa(v)$ is finite over E. Moreover we may assume that E is algebraically closed in any $\mathcal{O}'_v$ as above. Then $\kappa(v)$ is totally inseparable over E. Suppose $p = \operatorname{char} F > 0$. We argue by induction on $[\kappa(v):E]$. Let $a \in E^*$ such that $E_1 = E(\sqrt[p]{a})$ is contained in $\kappa(v)$ but not in any $\mathcal{O}'_v$. Then the extension v_1 of v to $F_1 = F(\sqrt[p]{a})$ has ramification index p, has the same residue class field and $[\kappa(v_1):E_1] < [\kappa(v):E]$. Using R2c, R3b, R3c and R3e it is now easy to see that R3d holds for the pair (v,E) if it holds for the pair (v_1,E_1) (use the fact that the norm of a prime for v_1 is a prime for v). □

The rest of this section will not be used later within the general theory. However the following remarks may be of at least heuristic interest and we will refer to them partially in later side-remarks.

(1.6) Remark. There is the following point of view concerning R3f. See also (Bass and Tate 1972; Milnor 1970, remark at the end of p. 323).

For a valuation $v\colon F^* \to \mathbb{Z}$ let

$$K_*(v) = K_*F \,\big/\, \{1 + \mathfrak{m}_v\} \cdot K_*F.$$

Consider the ring homomorphisms

$$\tilde{p}\colon K_*F \to K_*(v),$$
$$i\colon K_*\kappa(v) \to K_*(v)$$

given by projection resp. by the formula

$$i(\{\bar{u}_1,\ldots,\bar{u}_n\}) = \tilde{p}(\{u_1,\ldots,u_n\})$$

for v-units u_i. There is an exact sequence

$$0 \longrightarrow K_*\kappa(v) \xrightarrow{i} K_*(v) \xrightarrow{\partial} K_*\kappa(v) \longrightarrow 0$$

with $\partial_v = \partial \circ \tilde{p}$. Any prime π gives rise to a section $y \mapsto \tilde{p}(\{\pi\}) \cdot i(y)$ of ∂.

We put

$$M(v) = K_*(v) \otimes_{K_*\kappa(v)} M\big(\kappa(v)\big).$$

Then there is an exact sequence

$$0 \longrightarrow M\big(\kappa(v)\big) \xrightarrow{i} M(v) \xrightarrow{\partial} M\big(\kappa(v)\big) \longrightarrow 0,$$

and the splittings above give for every π a decomposition of $K_*\kappa(v)$-modules

$$M(v) = M\big(\kappa(v)\big) \oplus M\big(\kappa(v)\big).$$

We define

$$p\colon M(F) \to M(v),$$
$$p(\rho) = 1 \otimes s_v^\pi(\rho) + \tilde{p}(\{\pi\}) \otimes \partial_v(\rho).$$

Note that p is independent of the choice of π. One has $\partial_v = (\partial \otimes 1) \circ p$.

Now R3f may be reformulated by saying that p is a module homomorphism over the ring homomorphism $\tilde{p}$. Similarly one may understand P3 via pairings

$$M(v) \otimes_{K_*(v)} M'(v) \to M''(v).$$

(1.7) Remark. A particular consequence of R3e is the fact that the subgroup

$$\{1 + \mathfrak{m}_v\} \cdot M(F)$$

is killed whenever one passes to $M\big(\kappa(v)\big)$. This seems to be a reasonable condition from a geometric point of view. However note that the continuous Steinberg symbol $K_2\mathbb{Q} \to \mathbb{Z}/2$ corresponding to the 2-adic valuation on $\mathbb{Q}$ (Milnor 1971, § 11) maps $\{5,2\}$ to the nontrivial element.

(1.8) Remark. If one wants to consider arbitrary valuations (discrete and of rank 1), one has to require in R3b that the integral closure of $\mathcal{O}_v$ in E is finite over $\mathcal{O}_v$. This condition holds for geometric and for complete valuation rings, see (Serre 1968). By looking at completions and using R1c and R3a one may then derive for arbitrary valuations a formula

$$\partial_v \circ \varphi^* = \sum_w l_w \cdot \varphi_w^* \circ \partial_w$$

with certain integers l_w. This remark applies in particular to Milnor's K-theory.

(1.9) LEMMA. *In the situation of* R3a *let* π *be a prime of* v, *let* τ *be a prime of* w *and let* u *be the* v*-unit with* $\pi^e = \tau u$. *Then*

$$s_v^\pi \circ \varphi_* = \bar{\varphi}_* \circ s_w^\tau - \{\bar{u}\} \cdot \bar{\varphi}_* \circ \partial_w.$$

Proof: First note that the validity of the statement does not depend on the choices of π and τ. Moreover, if $E \subset K \subset F$ is an intermediate field, we may restrict to consider the extensions $K|E$ and $F|K$.

If $F|E$ is unramified ($e = 1$), we may take $\pi = \tau$ and the claim follows from R2a and R3a.

After lifting a transcendence base of $\kappa(v)$ over $\kappa(w)$ to $\mathcal{O}_v$ we may therefore assume that F is finite over E.

If $e = [F:E]$ (case of total ramification, see Serre 1968, Chap. I, § 6), we may take $\tau = -N_\varphi(-\pi)$; then $\bar{u} = 1$ and the claim follows from R2c and R3b. We have now already covered totally inseparable extensions.

For a separable finite extension $F|E$, let $L|E$ be a Galois extension containing F, fix some extension of v to L and let $D(L|F) \subset D(L|E) \subset \mathrm{Gal}(L|E)$ be the decomposition groups.

Then $F' = L^{D(L|F)}$ is unramified over F with the same residue class field; by R3a we are reduced to consider the extension $F'|E$. The field $E' = L^{D(L|E)}$ is contained in F'; since it is unramified over E, we know the claim for $E'|E$ and we are reduced to consider $F'|E'$. Let $K = L^U$ where

$$U = \{\, g \in \mathrm{Gal}(L|E') \mid g \text{ acts trivially on } \kappa(v) \,\}$$

is the inertia group. Then $K|E'$ is unramified and $F'|K$ is totally ramified. □

(1.10) REMARK. The rules and Lemma 1.9 show that every composite of maps between various $M(F)$ given by the data D1–D4 is a sum of composites of the form

$$\psi^* \circ (x \cdot \underline{\quad}) \circ \varphi_* \circ \partial_{v_r} \circ \cdots \circ \partial_{v_1} \circ (y \cdot \underline{\quad}).$$

This kind of normal form for composites can be made more precise as follows. There is a category $\widetilde{\mathcal{F}}$ with objects the class of arbitrary fields and with morphism groups

$$\mathrm{Hom}(F, E) = \coprod_v \coprod_H K_* H \widehat{\otimes}_{K_*\kappa(v)} K_*(v).$$

Here v runs through the valuations on F with value groups $\mathbb{Z}^r$ with lexicographical order and with $r \geq 0$. The groups $K_*(v)$ are defined exactly as above for $r = 1$. Moreover H runs through those composites of $\kappa(v)$ and E which are finite over E (and $\widehat{\otimes}$ denotes the graded tensor product).

Restricting to the class $\mathcal{F}(B)$ and to geometric higher rank valuations one obtains a category $\widetilde{\mathcal{F}}(B)$. The cycle premodules over B may be then characterized as the additive functors on $\widetilde{\mathcal{F}}(B)$. In this alternative definition all the rules including Lemma 1.9 are hidden in the composition law of $\widetilde{\mathcal{F}}$.

(1.11) Remark. — *Galois cohomology as cycle premodule.* Any torsion étale sheaf on B (with the torsion prime to $\operatorname{char} k$) gives rise via Galois cohomology to a cycle premodule over B. For simplicity we restrict here to the case $B = \operatorname{Spec} k$ with k a field and to finite Galois modules over k. For generalities of Galois cohomology we refer to (Serre 1968, 1994; Shatz 1972).

Let $\bar{k}$ be a separable closure of k, let r be prime to $\operatorname{char} k$, let $\mu_r \subset \bar{k}^*$ be the group of r-th roots of unity and let D be a finite continous $\operatorname{Gal}(\bar{k}|k)$-module of exponent r. For a field F over k let $\bar{F}$ be a separable closure containing $\bar{k}$. Then μ_r and D are $\operatorname{Gal}(\bar{F}|F)$-modules via $\operatorname{Gal}(\bar{F}|F) \to \operatorname{Gal}(\bar{k}|k)$. Put

$$\widetilde{H}^*(F; D) = \coprod_{n \geq 0} H^n(F; D \otimes \mu_r^{\otimes n}).$$

Here we use for a finite Galois module C the notation

$$H^n(F; C) = H^n(\operatorname{Gal}(\bar{F}|F); C) = \varinjlim H^n(\operatorname{Gal}(L|F); C)$$

where L runs through the finite Galois subextensions of $\bar{F}|F$ such that $\operatorname{Gal}(\bar{F}|L)$ acts trivially on C.

$\widetilde{H}^*(F; \mathbb{Z}/r)$ is a ring and $\widetilde{H}^*(F; D)$ is a module over $\widetilde{H}^*(F; \mathbb{Z}/r)$ via cup products.

The object function $H^*[D]$ on $\mathcal{F}(k)$ given by $H^*[D](F) = \widetilde{H}^*(F; D)$ is in a natural way a $\mathbb{Z}$-graded cycle premodule over k. This statement is just a collection of well-known properties of Galois cohomology. In the following we restrict ourselves to a description of the data D1–D4. The rules follow from standard properties of the cohomology of finite groups and from standard ramification theory.

D1 and D2: For $\varphi\colon F \to E$ let $\bar{\varphi}\colon \bar{F} \to \bar{E}$ be some extension over $\bar{k}$ and let $\tilde{\varphi}\colon \operatorname{Gal}(\bar{E}|E) \to \operatorname{Gal}(\bar{F}|F)$ be the induced map. Define φ_* as the usual restriction homomorphism induced from $\tilde{\varphi}$. For finite φ define φ^* as the usual transfer homomorphism induced from $\tilde{\varphi}$ times the degree of inseparability $[E : E \cap \bar{\varphi}(\bar{F})]$ (cf. Serre 1992).

D3: The K_*F-module structure on $\widetilde{H}^*(F; D)$ is given by cup products and the norm residue homomorphism

$$h_F\colon K_*F/rK_*F \to \widetilde{H}^*(F; \mathbb{Z}/r).$$

h_F is the $\mathbb{Z}$-graded ring homomorphism which in degree 1 is given by the Kummer isomorphism $F^*/(F^*)^r \to H^1(F; \mu_r)$. For the rule $h_F(\{a\}) \smile h_F(\{1-a\}) = 0$ see for example (Tate 1976) or Remark 2.7.

D4: Let E be the completion of F with respect to v. Then there is a natural exact sequence

$$1 \to I \to \operatorname{Gal}(\bar{E}|E) \to \operatorname{Gal}(\bar{\kappa}|\kappa) \to 1$$

where I is the inertia group. Put $D_n = D \otimes \mu_r^{\otimes n}$ and consider the corresponding Hochschild-Serre spectral sequences

$$E_2^{p,q} = H^p(\kappa; H^q(I; D_n)) \Longrightarrow H^{p+q}(E; D_n).$$

The cohomology of the inertia group I is given by $H^0(I;D_n)=D_n$, $H^1(I;D_n)=\operatorname{Hom}(\mu_r,D_n)=D_{n-1}$ and $H^q(I;D_n)=0$ for $q\geq 2$ (Serre 1968, Chap. IV). Hence the spectral sequences give rise to homomorphisms

$$\tilde\partial_v\colon H^n(E;D_n)\to H^{n-1}(\kappa;D_{n-1}).$$

Composing with $H^n(F;D_n)\to H^n(E;D_n)$ defines the desired maps[◇]

$$\partial_v\colon H^n[D](F)\to H^{n-1}[D](\kappa).$$

(1.12) REMARK. — *Quillen's K-theory as cycle premodule.* We denote by $K'_*F=\coprod_n K'_nF$ Quillen's K-ring of a field F. Hereby we understand the definition $K'_nF=\pi_{n+1}\bigl(\mathrm{BQ}\,\mathit{Mod}(F)\bigr)$ of (Quillen 1973) with the product as defined in (Grayson 1978). (Here $\mathit{Mod}(F)$ is the category of finite dimensional F-modules. For generalities of Quillen's K-theory see also Grayson 1976; Srinivas 1991.)

The object function $F\to K'_*F$ defines a $\mathbb{Z}$-graded cycle premodule with ring structure over any field k. Its data are given as follows.

D1 and D2: One takes the pull-back map $\bar\varphi^*$ resp. the push-forward map $\bar\varphi_*$ of (Quillen 1973, § 7) where $\bar\varphi\colon \operatorname{Spec}E\to\operatorname{Spec}F$ is the morphism corresponding to φ.

D3: One uses the natural homomorphism $\omega\colon K_*F\to K'_*F$ from Milnor's to Quillen's K-theory. To define ω, one may refer to $K'_nF=\pi_n(\mathrm{BGL}(F)^+)$ and the computations $\pi_1(\mathrm{BGL}(F)^+)=H_1(\mathrm{GL}(F),\mathbb{Z})=K_1F$, $\pi_2(\mathrm{BGL}(F)^+)=H_2(\mathrm{E}(F);\mathbb{Z})=K_2F$ (Matsumoto's theorem, see Milnor 1971). Another possibility is to define directly a homomorphism $\omega_1\colon F^*\to K'_1F$ and then to check the rule $\omega_1(\{a\})\cdot\omega_1(\{1-a\})=0$ using the arguments of Remark 2.7.

D4: One uses the connecting map of the long exact localization sequence for $\mathcal{O}_v$ (Quillen 1973, § 7).

The verification of the rules is omitted. It is a lengthy but straightforward exercise to deduce them from (Grayson 1978; Quillen 1973).

[◇] According to the conventions made for the cup product and the spectral sequence, one may have different signs in the product rules for the differentials. This affects rule R3e, so if necessary, one should replace ∂_v by an appropriate sign (depending alone on n).

2. CYCLE MODULES

In this section we define the notion of a cycle module and derive important properties: the homotopy property for $\mathbb{A}^1$ and the sum formula for proper curves. Moreover we give a simplification of the axioms for a constant cycle module over a perfect field. The axioms of a cycle module are basic for all further considerations. Therefore we have included discussions on various related properties to a much larger extent than is actually needed in the following sections.

Throughout the section, M denotes a cycle premodule over some scheme B (recall our conventions).

For a scheme X over B we write $M(x) = M(\kappa(x))$ for $x \in X$. The generic point of an irreducible scheme X is denoted by ξ or ξ_X. If X is normal, then for $x \in X^{(1)}$ the local ring of X at x is a valuation ring; let $\partial_x\colon M(\xi_X) \to M(x)$ be the corresponding residue homomorphism.

For x, $y \in X$ we define

$$\partial^x_y\colon M(x) \to M(y)$$

as follows. Let $Z = \overline{\{x\}}$. If $y \notin Z^{(1)}$, then $\partial^x_y = 0$. Otherwise let $\tilde{Z} \to Z$ be the normalization and put

$$\partial^x_y = \sum_{z|y} c_{\kappa(z)|\kappa(y)} \circ \partial_z \qquad (2.1.0)$$

with z running through the finitely many points of $\tilde{Z}$ lying over y.

(2.1) DEFINITION. *A cycle module M over B is a cycle premodule M over B which satisfies the following conditions* (FD) *and* (C).

(FD): FINITE SUPPORT OF DIVISORS. Let X be a normal scheme and $\rho \in M(\xi_X)$. Then $\partial_x(\rho) = 0$ for all but finitely many $x \in X^{(1)}$.

(C): CLOSEDNESS. Let X be integral and local of dimension 2. Then

$$0 = \sum_{x \in X^{(1)}} \partial^x_{x_0} \circ \partial^\xi_x\colon M(\xi_X) \to M(x_0)$$

where ξ_X is the generic and x_0 is the closed point of X.

Many remarks and definitions of Section 1 are understood accordingly for cycle modules. For example a homomorphism of cycle modules is a homomorphism of the underlying cycle premodules.

Of course (C) has sense only under presence of (FD) which guarantees finiteness in the sum. More generally, note that if (FD) holds, then for any X, $x \in X$ and $\rho \in M(x)$ one has $\partial^x_y(\rho) = 0$ for all but finitely many $y \in X$.

If X is integral and (FD) holds for X, we put

$$d = (\partial^\xi_x)_{x \in X^{(1)}}\colon M(\xi_X) \longrightarrow \coprod_{x \in X^{(1)}} M(x).$$

In the following, F denotes a field over B and $\mathbb{A}^1_F = \operatorname{Spec} F[u]$ is the affine line over F with function field $F(u)$. Proofs of Proposition 2.2 and Theorem 2.3 are given after Remark 2.6.

(2.2) PROPOSITION. *Let M be a cycle module over B. Then the following properties* (H) *and* (RC) *hold for all fields F over B.*

(H): HOMOTOPY PROPERTY FOR $\mathbb{A}^1$. The sequence

$$0 \longrightarrow M(F) \xrightarrow{r} M\big(F(u)\big) \xrightarrow{d} \coprod_{x \in \mathbb{A}^1_{F(0)}} M(x) \longrightarrow 0$$

is an exact complex (with $r = r_{F(u)|F}$).

(RC): RECIPROCITY FOR CURVES. Let X be a proper curve over F. Then

$$M(\xi_X) \xrightarrow{d} \coprod_{x \in X_{(0)}} M(x) \xrightarrow{c} M(F)$$

is a complex: $c \circ d = 0$ (with $c = \sum c_{\kappa(x)|F}$).

The properties (FD), (C), (H), (RC) are all what we need in further sections. Axiom (FD) enables one to write down the differentials d of the complexes $C_*(X;M)$, axiom (C) guarantees that $d \circ d = 0$, property (H) yields the homotopy invariance of the Chow groups $A_*(X;M)$ and finally property (RC) is needed to establish proper push-forward. For the material from Section 3 on the reader may take (H) and (RC) just as additional axioms of cycle modules and skip without much harm everything after Remark 2.6 below.

For another example of the fundamental role of axioms like (RC) in formal definitions of functors on fields see also (Somekawa 1990).

For integral X we put

$$A^0(X;M) = \ker d = \bigcap_{x \in X^{(1)}} \ker \partial^\xi_x \ \subset\ M(\xi_X).$$

One may think of $A^0(X;M)$ as the group of "unramified M-valued functions" on X.

(2.3) THEOREM. *Let M be a cycle premodule over a perfect field k. Then M is a cycle module over k if and only if the following properties* (FDL) *and* (WR) *hold for all fields F over k.*

(FDL): FINITE SUPPORT OF DIVISORS ON THE LINE. Let $\rho \in M\big(F(u)\big)$. Then $\partial_v(\rho) = 0$ for all but finitely many valuations v of $F(u)$ over F.

(WR): WEAK RECIPROCITY. Let ∂_∞ be the residue map for the valuation of $F(u)|F$ at infinity. Then

$$\partial_\infty\big(A^0(\mathbb{A}^1_F;M)\big) = 0.$$

One implication here is obvious from Proposition 2.2, since trivially (FD) $\Rightarrow$ (FDL) and (RC) $\Rightarrow$ (WR). Conditions (FDL) and (WR) are comparatively weak: they deal only with the affine line and involve no corestriction maps. For nonconstant cycle modules (i.e., B is not the spectrum of a field) we don't know any similar simplification of the axioms (FD) and (C).

Further properties of cycle modules are

(Co): CONTINUITY. Let X be smooth and local and let $Y \to X$ be the blow up in the unique closed point x_0. Then

$$A^0(X;M) \subset A^0(Y;M).$$

In other words, if v is the valuation corresponding to the exceptional fiber over x_0, then

$$\partial_v\big(A^0(X;M)\big) = 0.$$

(E): EVALUATION. In the situation of (Co) there is a unique homomorphism

$$\mathrm{ev}\colon A^0(X;M) \to M(x_0)$$

("evaluation at x_0") such that

$$r_{\kappa(v)|\kappa(x_0)} \circ \mathrm{ev} = s_v^\pi \mid A^0(X;M)$$

for any prime π of v.

The validity of these two properties will follow from the construction of the pull-back maps $f^\bullet\colon A^0(X;M) \to A^0(Z;M)$ for morphisms $f\colon Z \to X$ in Section 12. Namely, the inclusion of (Co) is given by $f^\bullet$ with $f\colon Y \to X$ the blow up. Moreover in (E) one has $\mathrm{ev} = f^\bullet$ with $f\colon \operatorname{Spec}\kappa(x_0) \to X$ the inclusion. See also Remark 2.8 below.

(2.4) REMARK. A basic example of a cycle module over any field k is Milnor's K-ring K_*. Axiom (FD) follows as for classical divisors. For (H) see (Milnor 1970). The validity of (RC) for $X = \mathbb{P}^1$ is intrinsic to the definition of the norm homomorphisms in (Bass and Tate 1972). Kato (1986) has used (RC) to prove (C) by passing to completions.

(2.5) REMARK. The cycle premodules $H^*[D]$ and K'_* of Remarks 1.11 and 1.12 are cycle modules. Axioms (FD) and (C) are contained in (Bloch and Ogus 1974) and in (Quillen 1973, § 7, Sect. 5), respectively.

For $H^*[D]$ one may use here alternatively Theorem 2.3 and Tsen's Theorem as follows (see also Serre 1992). (FDL) follows from the fact that every finite extension of $F(u)$ is ramified only in finitely many places of $F(u)|F$. One has trivially (H) $\Rightarrow$ (WR). If F is separably closed, then (H) follows from Tsen's Theorem (i.e., $H^q(F(u);\mu_r) = 0$ for $q \geq 2$; see for example Shatz 1972) and the Kummer isomorphism $K_1F(u)/r = H^1(F(u);\mu_r)$. To deduce (H) for arbitrary F one applies the Hochschild-Serre spectral sequence for the extension $\bar{F}(u)|F(u)$.

(2.6) REMARK. Probably the considerations of this section (and of the whole paper) may be developed in characteristic $\neq 2$ also for a version of cycle modules which are modules over the Witt ring of quadratic forms instead over Milnor's K-ring. A transferring would be not at all formal because the residue maps for the Witt ring depend on choices of parameters.

In the following proofs of Proposition 2.2 and Theorem 2.3 we use the notations $\mathbb{A}^1 = \operatorname{Spec} F[u]$, $\mathbb{A}^2 = \operatorname{Spec} F[s,t]$ and $Z = \mathbb{A}^2_{(<s,t>)}$, the localization of $\mathbb{A}^2$ at 0. Moreover y, $z \in Z^{(1)} \subset \mathbb{A}^{2(1)}$ denote the points with parameters s, t, respectively. We proceed in several steps.

Step 1: (FD) + (C) $\Rightarrow$ (WR). Given $\rho \in A^0(\mathbb{A}^1; M)$ put

$$\eta = \{t\} \cdot \rho(t/s) \in M\big(F(s,t)\big)$$

or, more precisely, $\eta = \{t\} \cdot \varphi_*(\rho)$ with $\varphi\colon F(u) \to F(s,t)$, $\varphi(u) = t/s$. Using R2 and R3 one finds

$$\begin{aligned}
&\partial_x(\eta) = 0 \text{ for } x \in Z^{(1)} \setminus \{y,z\},\\
&\partial_y(\eta) = -\{t\} \cdot r_{\kappa(y)|F}\big(\partial_\infty(\rho)\big),\\
&\partial_z(\eta) = \partial_z\big(\varphi_*(\{u\}\cdot\rho) + \{s\}\cdot\varphi_*(\rho)\big)\\
&\qquad = r_{\kappa(z)|F} \circ \partial_0(\{u\}\cdot\rho) - \{s\}\cdot r_{\kappa(z)|F}\big(\partial_0(\rho)\big).
\end{aligned}$$

(C) and $\partial_0(\rho) = 0$ give

$$0 = \sum_{x \in Z^{(1)}} \partial_0^x \circ \partial_x(\eta) = \partial_0^y \circ \partial_y(\eta) = -\partial_\infty(\rho). \qquad \square$$

Step 2: (FDL) + (WR) $\Rightarrow$ (H). Note that $d \circ r = 0$ by R3c. Moreover any specialization map for an F-rational point on $\mathbb{P}^1$ is a left inverse to r by R3d.

Surjectivity of d: For a closed point $x \in \mathbb{A}^1$ let

$$\Phi^x\colon M(x) \to M\big(F(u)\big),$$

$$\Phi^x(\mu) = c_{\kappa(x)(u)|F(u)}\big(\{u - u(x)\} \cdot r_{\kappa(x)(u)|\kappa(x)}(\mu)\big)$$

and let

$$\Phi = \sum_x \Phi^x\colon \coprod_{x \in \mathbb{A}^1_{(0)}} M(x) \longrightarrow M\big(F(u)\big).$$

Then $d \circ \Phi = \mathrm{id}$ by R3b–R3e.

Exactness at $M(F(u))$: Given $\rho \in A^0(\mathbb{A}^1; M)$, put

$$\eta = \{t\} \cdot \big(\rho(u+t) - \rho(u)\big) \in M(F(u)(t)).$$

More precisely: Let $E = F(u)$, let $i, \varphi\colon E \to E(t)$ be the homomorphisms over F with $i(u) = u$, $\varphi(u) = u + t$ and put $\eta = \{t\} \cdot \big(\varphi_*(\rho) - i_*(\rho)\big)$.

We compute $\partial_w(\eta)$ for the valuations w of $E(t)$ over E. One finds easily $\partial_w(\eta) = 0$ for $w \neq 0, \infty$. But also $\partial_0(\eta) = 0$ by R3d, since the valuation at $t = 0$ restricts trivially under i and φ and since the induced homomorphisms $E \to \kappa(0)$ coincide. Hence (WR) tells $\partial_\infty(\eta) = 0$. On the other hand one has

$$\begin{aligned}
\partial_\infty\big(\{t\}\cdot i_*(\rho)\big) &= -\rho,\\
\partial_\infty\big(\{t\}\cdot\varphi_*(\rho)\big) &= \partial_\infty\big((\{t/(u+t)\} + \{u+t\})\cdot\varphi_*(\rho)\big)\\
&= -\{\overline{t/(u+t)}\}\cdot\partial_\infty\big(\varphi_*(\rho)\big) + \partial_\infty\big(\varphi_*(\{u\}\cdot\rho)\big)\\
&= 0 + r_{E|F}\big(\partial_\infty(\{u\}\cdot\rho)\big)
\end{aligned}$$

by making particular use of R3d and R3e (note that $t/(u+t)$ has residue class $1 \in \kappa(\infty)$). So

$$\rho = r_{E|F}\big(\partial_\infty(\{u\}\cdot\rho)\big) \in r_{E|F}\big(M(F)\big). \qquad \square$$

Step 3: (FD) + (H) $\Rightarrow$ (RC). There is a finite morphism $X \to \mathbb{P}^1$ over F. Using this and R3b one reduces to the case $X = \mathbb{P}^1$. Then it suffices to check

$$\sum_{u \in \mathbb{P}^1_{(0)}} c_{\kappa(u)|F} \circ \partial_u \circ \Phi^x = 0$$

for Φ^x as in Step 2. This equation follows from the computation $d \circ \Phi = \mathrm{id}$ and

$$\partial_\infty \circ \Phi^x = -c_{\kappa(x)|F},$$

a consequence of R3b and R3d. □

The proof of Proposition 2.2 is now complete. We next consider the nontrivial implication of Theorem 2.3. We will refer at some places to Sections 3 and 4, but only in a mild way. Note that (H) is available by Step 2.

Step 4: (FDL) $\Rightarrow$ (FD) for $X = \mathbb{A}^n$. Let $p_i\colon \mathbb{A}^n \to \mathbb{A}^{n-1}$ be the n standard projections. Then

$$\mathbb{A}^{n\,(1)} \subset \bigcup_i p_i^{-1}(\xi)$$

where ξ is the generic point of $\mathbb{A}^{n-1}$. (FD) follows from (FDL) applied to $F = \kappa(\xi)$. □

Step 5: (FD) for $X = \mathbb{A}^2$ + (WR) $\Rightarrow$ (C) for $X = Z$. As in Step 2 we have

$$M(\xi_Z) = M\big(F(s)\big) + \sum_{x \in \left(\mathbb{A}^1_{F(s)}\right)_{(0)}} \Phi^x\big(M(x)\big)$$

with

$$\Phi^x(\mu) = c_{\kappa(x)(t)|F(s,t)}\big(\{t - t(x)\} \cdot r_{\kappa(x)(t)|\kappa(x)}(\mu)\big).$$

(C) holds obviously on $M\big(F(s)\big)$. Let us verify (C) on the image of Φ^x for fixed x. By $d \circ \Phi = \mathrm{id}$ in Step 2 we are reduced to check

$$\partial_0^y \circ \partial_y \circ \Phi^x = -\partial_0^x.$$

Let v run through the valuations on $\kappa(x)(t)$ which restrict on $F[s,t]$ to the valuation with parameter s (and corresponding to y). Let $\bar v$ be the restriction of v to $\kappa(x)$ and let $c(\bar v) \in \overline{\{x\}}$ be the center of $\bar v$. If $c(\bar v) \neq 0$, then $t(x)$ is a $\bar v$-unit with residue $t(c(\bar v))$. Suppose $0 \in \overline{\{x\}}$ and let R_x be the residue class ring of x localized at 0. The valuations v with $c(\bar v) = 0$ restrict in a one-to-one manner to the valuations w of $\kappa(x)$ with $R_x \subset \mathcal{O}_w$. For these one has $\bar v(t(x)) > 0$ (since $t(x)$ is nilpotent in R_x/sR_x).

In the following, u runs through $\overline{\{x\}} \cap \overline{\{y\}} \setminus \{0\}$. One finds

$$\begin{aligned}\partial_y \circ \Phi^x(\mu) &= \sum_v c_{\kappa(v)|\kappa(y)} \circ \partial_v\big(\{t - t(x)\} \cdot r_{\kappa(x)(t)|\kappa(x)}(\mu)\big) \\ &= -\sum_{v, c(\bar v)\neq 0} c_{\kappa(v)|\kappa(y)}\Big(\{t - t\big(c(\bar v)\big)\} \cdot r_{\kappa(v)|\kappa(\bar v)} \circ \partial_{\bar v}(\mu)\Big) \\ &\quad - \{t(y)\} \cdot \sum_{v, c(\bar v)=0} c_{\kappa(v)|\kappa(y)} \circ r_{\kappa(v)|\kappa(\bar v)} \circ \partial_{\bar v}(\mu) \\ &= -\sum_u c_{\kappa(u)(t)|\kappa(y)}\big(\{t - t(u)\} \cdot r_{\kappa(u)(t)|\kappa(u)} \circ \partial_u^x(\mu)\big) \\ &\quad - \{t(y)\} \cdot r_{\kappa(y)|F} \circ \partial_0^x(\mu).\end{aligned}$$

Since $t(u) \neq 0$ the sum vanishes under ∂_0^y and we are done by R3d. $\square$

Step 6: REDUCTION OF (C) TO THE CASE $\kappa(x_0) \subset O_X$. Let X be as in (C) and write $X = \operatorname{Spec} R$. Lift a transcendence base of $\kappa(x_0)$ over k to elements $t_i \in R$ and put $K = k(t_1, \ldots, t_n) \subset R$. Then $\kappa(x_0)$ is a finite extension of K. Since k is perfect, we may take here a transcendence base such that $\kappa(x_0)$ is separable over K. Let $X' = \operatorname{Spec} R \otimes_K \kappa(x_0)$, let $u \in X'$ be the canonical lift of x_0 and let X'' be the localization of X' in u. We assume that (FD) holds for X and X''. Consider the pull-back along the flat map $X'' \to X$, see Section 3. The induced map $M(x_0) \to M(u)$ is injective, since x_0 and u have the same residue class fields. An application of R3a and R3b (see Proposition 4.6.2) shows that (C) holds for X if (C) holds for X''. $\square$

We know now in particular that (C) holds for every localization of $\mathbb{A}^2$ in some closed point.

Step 7: PROOF OF (FD). There exists a generically finite separable rational map $X \to \mathbb{A}^n_k$. All but finitely many $x \in X^{(1)}$ correspond to points of $\mathbb{A}^{n(1)}$. The argument of Step 4 yields a reduction to a plane curve X over some field K. So consider the case $X = \overline{\{x\}}$ for some $x \in \mathbb{A}^2_{(1)}$. We may assume that X maps dominantly to $\operatorname{Spec} F[s]$ so that Φ^x as in Step 5 is defined. Put $\eta = \Phi^x(\rho)$. We have $\partial_x(\eta) = \rho$. Moreover $\partial_u(\eta) \neq 0$ only for finitely many u (Step 4). The closure of $u \neq x$ meets X only in finitely many points. Now, since (C) holds for every localization of $\mathbb{A}^2$, we have $\partial_w^\xi(\rho) = 0$ for all but finitely many $w \in X^{(1)} \subset \mathbb{A}^2_{(0)}$. $\square$

Step 8: PROOF OF (C). By Step 6 we may assume that $F = \kappa(x_0)$ is contained in O_X. Choose a closed (2-dimensional) subscheme $Y \subset \mathbb{P}^n_F$ such that X is the localization of Y in a (F-rational) point y. We consider the generic projection from $\mathbb{P}^n_F$ to $\mathbb{P}^2_F$. More precisely: let T be the Grassmannian of 3-codimensional linear subspaces of $\mathbb{P}^n_F$, let $E = F(T)$, let $H \subset \mathbb{P}^n_E$ be the tautological subspace and let $\pi\colon \mathbb{P}^n_E \setminus H \to \mathbb{P}^2_E$ be a linear projection. Then $H \cap Y_E = \varnothing$ and π restricts to a proper map $p\colon Y_E \to \mathbb{P}^2_E$. Let $D = p^{-1}\big(p(y)\big)$. Then D is the intersection of Y_E with a generic 2-codimensional linear subspace passing through y. Hence

$$D \setminus \{y\} \subset Y_E \setminus Y.$$

In particular $D \cap Y_{(0)} = \{y\}$. Now we consider flat pull-back along the base change $q\colon Y_E \to Y$ followed by the push-forward along p, see Section 3. One finds (see (1) and (2) of Proposition 4.6) for $\rho \in M(\xi_X)$:

$$r_{E|F}\Big(\sum_{x \in X^{(1)}} \partial^x_{x_0} \circ \partial^\xi_x(\rho)\Big) = \sum_{u \in U^{(1)}} \partial^u_{p(u)} \circ \partial_u\big(p_* \circ q^*(\rho)\big)$$

where U is the localization of $\mathbb{P}^2_E$ in $p(y)$. The right hand side vanishes by Step 5 and $r_{E|F}$ is injective since $E|F$ is a rational extension. □

We conclude with some more considerations concerning the axioms of cycle modules. These have been included here more for illustration than for application. In order not to be too tiring, we have taken here the freedom to be a bit vague about our actual assumptions.

(2.7) Remark. In the datum D3 of cycle premodules there is hidden a strong rule, namely the relation $\{a, 1-a\} = 0$ of Milnor's K-theory. The main justification within this paper for using Milnor's K-theory is that it works well. Asking naively, one may try to weaken D3 by requiring only the existence of bilinear pairings

$$K_1F \times M(F) \to M(F),$$
$$\big(\{a\}, \rho\big) \mapsto \{a\} \cdot \rho$$

and restricting to $x \in K_1E$, $y \in K_1F$ in R2. Then $M(F)$ would be a TF^*-module.

However, if one wants to develop a geometric theory, one is in the end led to pass to modules over Milnor's K-theory. A reasoning for this is given by the following little game. It refers in a mild way to the rules of cycle premodules and to a part of the homotopy property (H).

Let $\rho \in M(F)$, let L be an overfield of F, let $u \in L \setminus \{0,1\}$ and consider

$$\eta(u) = \{u\} \cdot \big(\{1-u\} \cdot r_{L|F}(\rho)\big) \in M(L).$$

Our aim is to conclude $\eta(u) = 0$ for the case $L = F$. Assuming reasonable specialization maps, this follows from the generic case $L = F(u)$ with u a variable. To treat this case, our strategy is to argue that $\eta(u)$ is unramified on the whole affine line. Then, by homotopy invariance, $\eta(u)$ is constant. An extra argument finally shows $\eta(u) = 0$.

To be specific first a little calculation (which provides by the way already the divisibility of $\eta(u)$ referring only to the existence of norm maps and the projection formula). Let $L' = F(u')$ be the function field in the variable u' and let $L = F(u) \subset L'$ with $u = u'^n$. Then $1 - u = N_{L'|L}(1-u')$ and the projection formulae R2b and R2c give

$$\begin{aligned}
\eta(u) &= \{u\} \cdot \big(\{N_{L'|L}(1-u')\} \cdot r_{L|F}(\rho)\big) \\
&= \{u\} \cdot c_{L'|L}\big(\{1-u'\} \cdot r_{L'|F}(\rho)\big) \\
&= c_{L'|L}\Big(\{u'^n\} \cdot \big(\{1-u'\} \cdot r_{L'|F}(\rho)\big)\Big) \\
&= n \cdot c_{L'|L}\big(\eta(u')\big).
\end{aligned}$$

We want to conclude that $\partial_v\bigl(\eta(u)\bigr) = 0$ for all finite places v of $L|F$. This is quite natural to assume as long as u and $1-u$ are v-units. For the place at $u=0$ (and similarly at $u=1$) one may argue as follows.

Let $\alpha = \partial_0\bigl(\eta(u)\bigr) \in M(F)$ be the residue for the valuation of $L|F$ at $u=0$. Similarly let $\alpha' = \partial_0\bigl(\eta(u')\bigr)$, now with respect to the valuation of $L'|F$ at $u'=0$. A change of variables $u \to u'$ shows $\alpha = \alpha'$. But the above computation and rule R3b yields $\alpha = n\cdot\alpha'$. Taking $n=2$ gives $\alpha = 0$.

Now (H) tells that $\eta(u)$ comes from $M(F)$, that is $\eta(u) = r_{L|F}(\eta)$ for some $\eta \in M(F)$. Naturality with respect to the homomorphisms $L \to L$ over F with $u \to -u$ and $u \to u^2$ gives

$$\eta(u) = \eta(-u) = \eta(u^2).$$

On the other hand one has

$$\eta(u^2) = 2\eta(u) + 2\eta(-u),$$

just by linearity. One concludes $3\eta(u) = 0$. But then $3\eta(u') = 0$ as well and the above computation for $n=3$ tells $\eta(u) = 0$.

(2.8) REMARK. As already mentioned, the properties (Co) and (E) of cycle modules follow from the material in Section 12. The considerations there use the deformation to the normal cone and homotopy inverses. But things simplify considerably if one may pass to the limits $X = \operatorname{Spec} k[[t_1,\dots,t_n]]$. In the following we consider the case $X = \operatorname{Spec} k[[s,t]]$, tacitly assuming that our cycle modules are defined on an appropriate category of schemes. In fact we could have taken also more general schemes as basis for our notions, say excellent schemes over a perfect field (however one should then be careful with Theorem 2.3).

(C) $\Rightarrow$ (Co): Let $E = F(u)$ and $T = \operatorname{Spec} E[[s,t]]$. Given $\rho \in A^0(X;M)$ put

$$\eta = \{t-us\}\cdot r_{E((s,t))|F((s,t))}(\rho) \in M(\xi_T).$$

One may then calculate

$$\sum_{z\in T^{(1)}} \partial_0^z \circ \partial_z(\eta) = \partial_v(\rho).$$

Hence $\partial_v(\rho) = 0$ by (C).

(C) $\Rightarrow$ (E): By the last argument we may use (Co). It follows that for $\rho \in A^0(X;M)$ the value of $s_v^\pi(\rho)$ is independent of the choice of the prime π. Since $\kappa(v) = F(t/s)$ is rational, one is by (H) reduced to check

$$\partial_w \circ s_v^\pi(\rho) = 0$$

for all valuations w of $F(t/s)$ over F except the one with $w(t/s) = -1$. Every w defines a point in the exceptional fiber of the blow up $Y \to X$. One calculates for the w in question

$$0 = \sum_{y\in Y^{(1)}} \partial_w^y \circ \partial_y(\{s\}\cdot\rho) = \partial_w \circ \partial_v(\{s\}\cdot\rho).$$

Finally note $s_v^\pi(\rho) = \partial_v(\{s\}\cdot\rho)$ for the choice $\pi = -s$.

(2.9) Remark. In the case of a constant cycle premodule one may derive (C) from (Co) under presence of (FD). This tells that axiom (C) appears naturally in our framework if we require the existence of pull-back maps $f^\bullet$. As in Remark 2.8 we consider here the case $X = \operatorname{Spec} k[[s,t]]$.

(Co) $\Rightarrow$ (C): To derive (C) from (Co) for constant M and for $X = \operatorname{Spec} k[[s,t]]$ one argues first similarly as for Step 5 above as follows. Let y be the point with parameter s and define for $x \in X^{(1)} \setminus \{y\}$:

$$\Phi^x\colon M(x) \to M(\xi_X),$$
$$\Phi^x(\mu) = r_{F((s,t))|F((s))(t)} \circ c_{\kappa(x)(t)|F((s))(t)}\big(\{t - t(x)\} \cdot r_{\kappa(x)(t)|\kappa(x)}(\mu)\big).$$

As in Step 5 one has $\partial_x \circ \Phi^x = \mathrm{id}$ and $\partial_z \circ \Phi^x = 0$ for $z \neq x,\, y$; moreover one finds $\partial_y \circ \Phi^x = -\{t\} \cdot r_{F((t))|F} \circ \partial_0^x$. This shows that (C) holds on the image of the Φ^x.

In order to verify (C) for $\tilde\rho \in M(\xi_X)$ we may arrange things such that $\partial_y(\tilde\rho) = 0$. We are reduced to check (C) for

$$\rho = \tilde\rho - \sum_{x \neq y} \Phi^x \circ \partial_x(\tilde\rho).$$

Using the above computations one finds

$$\partial_x(\rho) = \begin{cases} 0 & \text{for} \quad x \neq y, \\ \{t\} \cdot r_{\kappa(y)|F}(\theta) & \text{for} \quad x = y \end{cases}$$

for some $\theta \in M(F)$.

We must show $\theta = 0$. Put $E = F(r)$ and—written in a somewhat sloppy form—

$$\eta = \rho(rs, rt) - \rho(s,t) - \{s, r\} \cdot \theta \in M\big(E((s,t))\big).$$

One computes $\eta \in A^0(\operatorname{Spec} E[[s,t]]; M)$ and

$$\partial_v(\eta) = -\{r\} \cdot \theta \in M\big(E(s/t)\big).$$

(Co) gives $\{r\} \cdot \theta = 0$ in $M\big(F(r, s/t)\big)$. Applying appropriate specialization and residue maps shows $\theta = 0$.

3. The Four Basic Maps

The purpose of this section is to introduce the cycle complexes and all the types of operations on them needed further on (except the cross products to be defined in Section 14).

Let M be a cycle module over X, let N be a cycle module over Y and let $U \subset X$, $V \subset Y$ be subsets. For a homomorphism

$$\alpha\colon \coprod_{x \in U} M(x) \longrightarrow \coprod_{y \in V} N(y)$$

we write $\alpha^x_y\colon M(x) \to N(y)$ for the components of α.

(3.1) Change of coefficients. Let $\omega\colon M \to N$ be a homomorphism of cycle modules over X and let $U \subset X$ be a subset. We put

$$\omega_\#\colon \coprod_{x \in U} M(x) \longrightarrow \coprod_{x \in U} N(x)$$

where $(\omega_\#)^x_x = \omega_{\kappa(x)}$ and $(\omega_\#)^x_y = 0$ for $x \neq y$.

(3.2) Cycle complexes. For a cycle module M over X and an integer p let

$$C_p(X;M) = \coprod_{x \in X_{(p)}} M(x).$$

We define

$$d = d_X\colon C_p(X;M) \to C_{p-1}(X;M)$$

by $d^x_y = \partial^x_y$ with ∂^x_y as in (2.1.0). This definition has sense by axiom (FD).

(3.3) Lemma. $d_X \circ d_X = 0$.

Proof: One has to check $(d \circ d)^x_z = 0$ for $x \in X_{(p+1)}$, $z \in X_{(p-1)}$. This is trivial if $z \notin \overline{\{x\}}$. Otherwise let Y be the localization of $\overline{\{x\}}$ in z. Since our schemes are catenary, we have $X_{(p)} \cap Y = Y_{(1)}$ and $\dim Y = 2$. Now apply axiom (C) to Y. □

The complex $C_*(X;M) = (C_p(X;M), d_X)_{p \geq 0}$ is called the *complex of cycles on X with coefficients in M*.

When developing a theory of cycles, first natural questions are the following. Given a proper morphism $f\colon X \to Y$, what is the push-forward map f_* on cycles? Or, given a flat morphism $g\colon Y \to X$, what is the pull-back map g^* on cycles? In fact, we will define such maps. However these questions are not our guiding point of view. We rather fix schemes X, Y and numbers p, q and then ask: what is the class of maps

$$C_p(X;M) \to C_q(Y;M)$$

which we should consider? Our answer is then motivated by what we want to do with the complexes, namely developing intersection theory etc. This leads to the "four basic maps" as defined in (3.4)–(3.7).

The definitions of the basic maps "multiplication with K_1" and "boundary maps" in (3.6) and (3.7) are easy to understand. However our way of introducing push-forward and pull-back maps as in (3.4) and (3.5) deserves some words of comment. It turns out that these maps (denoted by f_* and $[\mathcal{A}, g, s]$) are sums of compositions of maps of simpler type, namely push-forward maps f_* for proper morphisms f, pull-back maps g^* for flat morphisms g and the projections i^* and inclusions j_* corresponding to closed (or open) subvarietes (see 3.10). This fact (which we will not prove) seems to be however only of heuristic interest. In fact it would be a nuisance if we had to consider at each step such a reduction of the language expressing the maps between the cycle complexes.

(3.4) Push-forward. For a morphism $f\colon X \to Y$ of schemes of finite type over a field we define

$$f_*\colon C_p(X;M) \to C_p(Y;M)$$

as follows. If $y = f(x)$ and if $\kappa(x)$ is finite over $\kappa(y)$, then $(f_*)^x_y = c_{\kappa(x)|\kappa(y)}$. Otherwise $(f_*)^x_y = 0$.

(3.5) Pull-back. Our main interest is to define the particular types of pull-back maps as considered in (3.5.5) below. In our general definition in (3.5.3) we define pull-back maps $C_p(X;M) \to C_q(Y;M)$ associated to any morphism $g\colon Y \to X$ of relative dimension $\leq q - p$. Moreover we use coherent sheaves $\mathcal{A}$ on Y as modifiers of the arising multiplicities. This construction gives great technical flexibility and is useful in Section 4.

(1) For a morphism $g\colon Y \to X$ let

$$s(g) = \max\left\{ \dim(y, Y) - \dim\bigl(g(y), X\bigr) \mid y \in Y \right\}.$$

Moreover let $Y_x = Y \times_X \operatorname{Spec} \kappa(x)$ for $x \in X$.

Note that if $x \in X_{(p)}$, $y \in Y_{(q)}$, $g(y) = x$ and $s(g) \leq q - p$, then necessarily $y \in Y_x^{(0)}$.

(2) Let $g\colon Y \to X$ be a morphism and let $\mathcal{A}$ be a coherent sheaf on Y. For $x \in X$ and $y \in Y_x^{(0)}$ we define an integer

$$[\mathcal{A}, g]^x_y \in \mathbb{Z}$$

as follows. The localization $Y_{x,(y)}$ of Y_x in y is the spectrum of an artinian ring R with only residue class field $\kappa(y)$. Let $\tilde{\mathcal{A}}$ be the pull-back of $\mathcal{A}$ via $Y_{x,(y)} \to Y_x \to Y$ and define $[\mathcal{A}, g]^x_y = l_R(\tilde{\mathcal{A}})$ as the length of $\tilde{\mathcal{A}}$ considered as R-module (for the notion of length and further properties we refer to Fulton 1984, App. A).

(3) Fix $s \in \mathbb{Z}$. Let $g\colon Y \to X$ be a morphism with $s(g) \leq s$ and let $\mathcal{A}$ be a coherent sheaf on Y. We define homomorphisms

$$[\mathcal{A}, g, s]\colon C_p(X; M) \to C_{p+s}(Y; M)$$

by

$$[\mathcal{A}, g, s]_y^x = \begin{cases} [\mathcal{A}, g]_y^x \cdot r_{\kappa(y)|\kappa(x)} & \text{if } g(y) = x, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\kappa(x)$ is considered as a subfield of $\kappa(y)$ via g.

(4) Let F be a field, let $g\colon Y \to \operatorname{Spec} F$ be a morphism and let

$$0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A}'' \to 0$$

be an exact sequence of coherent sheaves over Y. Then

$$[\mathcal{A}', g, s] - [\mathcal{A}, g, s] + [\mathcal{A}'', g, s] = 0.$$

This follows from the additivity of length with respect to short exact sequences.

(5) For some particularly interesting cases we use the following notations. Let $F \to E$ be a homomorphism of fields, let X be of finite type over F and let $g\colon Y = X \times_{\operatorname{Spec} F} \operatorname{Spec} E \to X$ be the base change. Then we put $g^* = [\mathcal{O}_Y, g, 0]$.

A morphism $g\colon Y \to X$ of schemes of finite type over a field is said to have (constant) relative dimension s if all fibers are either empty or equidimensional of dimension s. In this case we write $\dim(g) = s$ and put

$$g^* = [\mathcal{O}_Y, g, \dim(g)].$$

Particular cases are here open and closed immersions (with $s = 0$).

(3.6) Multiplication with units. For global units $a_1, \ldots, a_n \in O_X^*$ we define homomorphisms

$$\{a_1, \ldots, a_n\}\colon C_p(X; M) \to C_p(X; M)$$

by

$$\{a_1, \ldots, a_n\}_y^x(\rho) = \begin{cases} \{a_1(x), \ldots, a_n(x)\} \cdot \rho & \text{for } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

This definition turns $C_p(X; M)$ into a module over the tensor algebra of O_X^*. If X is defined over some field F, then $C_p(X; M)$ becomes via $F^* \subset O_X^*$ a module over K_*F.

(3.7) Boundary maps. Let X be of finite type over a field, let $i\colon Y \to X$ be a closed immersion and let $j\colon U = X \setminus Y \to X$ be the inclusion of the open complement. We will refer to (Y, i, X, j, U) as a boundary triple and define

$$\partial = \partial_Y^U\colon C_p(U; M) \to C_{p-1}(Y; M)$$

by taking for ∂_y^x the definition in (2.1.0) with respect to X. The map ∂_Y^U is called the boundary map associated to the boundary triple, or just the boundary map for the closed immersion $i\colon Y \to X$.

We conclude this section with a few notations and remarks concerning the four basic maps.

(3.8) Generalized correspondences. We introduce the notation

$$\alpha\colon X \leftrightarrow Y$$

to denote homomorphisms

$$\alpha\colon C_*(X;M) \to C_*(Y;M)$$

which are sums of composites of the four basic maps f_*, g^*, $\{a\}$ and ∂ for schemes of finite type over a field.

This notation is made for the sake of simplification. It also stresses the fact that we think of the maps in question rather as a sort of morphisms of varieties than just maps of complexes associated to every M. As mentioned in the introduction, this can be made more precise in a further development. (The differential d_X is not subject to this notation convention—we rather think of d_X as a part of the inner structure of X. Similarly for homomorphisms induced by a change of coefficients.)

(3.9) Gradings. The $\mathbb{Z}/2$-gradings on M induces a $\mathbb{Z}/2$-grading on $C_*(X;M)$ by

$$C_p(X;M,n) = \coprod_{x \in X_{(p)}} M_{n+p}(x)$$

with $n \in \mathbb{Z}/2$. Suppose $\alpha\colon X \leftrightarrow Y$ respects this grading in the sense that

$$\alpha\big(C_*(X;M,n)\big) \subset C_*(Y;M,n+r)$$

for some $r \in \mathbb{Z}/2$. In this case we write $\operatorname{sgn}(\alpha) = (-1)^r$. One has $\operatorname{sgn}(f_*) = \operatorname{sgn}(g^*) = +1$, $\operatorname{sgn}(\{a_1,\dots,a_n\}) = (-1)^n$ and $\operatorname{sgn}(\partial) = -1$. Moreover we put

$$\delta(\alpha) = d \circ \alpha - \operatorname{sgn}(\alpha) \cdot \alpha \circ d.$$

Then

$$\begin{aligned} \operatorname{sgn}\big(\delta(\alpha)\big) &= -\operatorname{sgn}(\alpha), \\ \delta \circ \delta(\alpha) &= 0, \\ \delta(\alpha \circ \beta) &= \delta(\alpha) \circ \beta + \operatorname{sgn}(\alpha) \cdot \alpha \circ \delta(\beta). \end{aligned}$$

All the maps α to be considered will respect the $\mathbb{Z}/2$-grading. Moreover, if M is $\mathbb{Z}$-graded, then the α will respect the corresponding $\mathbb{Z}$-gradings on the complexes. Additionally they respect the natural $\mathbb{Z}$-gradings given by dimension. So if M is $\mathbb{Z}$-graded, there is an underlying $\mathbb{Z} \times \mathbb{Z}$-grading (see also Section 5). In the general treatment however there is need only for the $\mathbb{Z}/2$-grading.

(3.10) BOUNDARY TRIPLES. Let (Y, i, X, j, U) be a boundary triple. The set theoretic union $X_{(p)} = Y_{(p)} \cup U_{(p)}$ yields a natural decomposition

$$C_p(X; M) = C_p(Y; M) \oplus C_p(U; M) \tag{3.10.1}$$

of abelian groups. Here the complex $C_*(Y; M)$ is a subcomplex of the complex $C_*(X; M)$ with $C_*(U; M)$ as quotient complex. The maps i_*, j_* and i^*, j^* are the corresponding inclusions and projections, respectively. In a formal way, the situation is described by the following formulae:

$$\begin{aligned}
&\partial = i^* \circ d_X \circ j_*, \\
&i^* \circ i_* = \mathrm{id}_Y, \quad j^* \circ j_* = \mathrm{id}_U, \\
&i^* \circ j_* = 0, \quad j^* \circ i_* = 0, \\
&i_* \circ i^* + j_* \circ j^* = \mathrm{id}_X, \\
&\delta(j_*) = i_* \circ \partial, \quad \delta(i_*) = -\partial \circ j^*, \\
&\partial = i^* \circ \delta(j_*) = -\delta(i^*) \circ j_*, \\
&\delta(i_*) = 0, \quad \delta(j^*) = 0, \quad \delta(\partial) = 0.
\end{aligned}$$

Later on we will make free use of these simple rules, in particular in Sections 6 and 9. The canonical decomposition (3.10.1) is the source of our formal treatment of intersection theory on complex level.

4. COMPATIBILITIES

In this section we establish the basic compatibilities for the maps considered in the last section. All arguments are simple in nature or at least familiar to cycle theories. They are basically of local nature. As usual the treatment of flat pull-back causes most of the technicalities.

Rules among the maps of (3.4)–(3.7) are formulated in (4.1)–(4.5). Proposition 4.6 is concerned with the compatibility with the differentials. The compatibilities with change of coefficients are obvious and we don't make a point of them here and further.

(4.1) PROPOSITION.

(1) *For $f\colon X \to Y$, $f'\colon Y \to Z$ as in* (3.4) *one has $(f' \circ f)_* = f'_* \circ f_*$.*

(2) *Let $g\colon Y \to X$ and $g'\colon Z \to Y$ be morphisms. Let $s \geq s(g)$ and $s' \geq s(g')$ and let $\mathcal{A}$, $\mathcal{A}'$ be coherent sheaves on Y, Z, respectively, with $\mathcal{A}'$ flat over Y. Then $s + s' \geq s(g \circ g')$ and*

$$[g'^* \mathcal{A} \otimes_{\mathcal{O}_Z} \mathcal{A}', g \circ g', s + s'] = [\mathcal{A}', g', s'] \circ [\mathcal{A}, g, s].$$

In particular

$$(g \circ g')^* = g'^* \circ g^*$$

for g, g' as in (3.5.5) *with g' flat.*

(3) *Consider a pull-back diagram*

$$\begin{array}{ccc} U & \xrightarrow{g'} & Z \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & X \end{array}$$

with f and f' as in (3.4). *Let $s \geq s(g)$, $s(g')$ and let $\mathcal{A}$ be a coherent sheaf on Y. Then*

$$[\mathcal{A}, g, s] \circ f_* = f'_* \circ [f'^*\mathcal{A}, g', s].$$

In particular

$$g^* \circ f_* = f'_* \circ g'^*$$

for g as in (3.5.5).

Proof: (1) is immediate from the definitions and R1a.

Proof of (2): The inequality is obvious. Let $x \in X$, $y \in Y_x$, $z \in Z_y$ with $\dim(y, Y) = \dim(x, X) + s$, $\dim(z, Z) = \dim(y, Y) + s'$. We have to check

$$[g'^*\mathcal{A} \otimes_{\mathcal{O}_Z} \mathcal{A}', g \circ g']^x_z = [\mathcal{A}', g']^y_z \cdot [\mathcal{A}, g]^x_y.$$

We may assume $X = \operatorname{Spec} \kappa(x)$ and $Y = \operatorname{Spec} R$ with R as in (3.5.2). By devisage using the flatness of $\mathcal{A}'$ over R and (3.5.4) we may reduce to the case $\mathcal{A} = \kappa(y)$. Now the claim is trivial.

Proof of (3): Let $\delta = [\mathcal{A}, g, s] \circ f_* - f'_* \circ [f'^*\mathcal{A}, g', s]$. We have to show $\delta^z_y = 0$ for $z \in Z_{(p)}$ and $y \in Y_{(p+s)}$.

This obvious if $f(z) \neq g(y)$. Otherwise let $x = f(z) = g(y)$. Our assumptions give $\dim(x, X) \leq \dim(z, Z)$ and $\dim(x, X) \geq \dim(y, Y) - s(g) \geq p$; hence $\dim(x, X) = \dim(z, Z) = p$ and $\kappa(z)$ is finite over $\kappa(x)$.

Let $u \in U_z$ be a maximal point of the fiber over z. Our assumptions give $\dim(u, U) \geq \dim(y, Y) = p + s$ and $\dim(u, U) \leq \dim(z, Z) + s(g') \leq p + s$; hence $u \in U_{(p+s)}$. This shows that δ^z_y remains unchanged if we replace X by $\operatorname{Spec} \kappa(x)$, Z by $\operatorname{Spec} \kappa(z)$, Y by $Y_{x,(y)} = \operatorname{Spec} R$ (see 3.5.2) and p, s by 0. Then f is finite and flat. Hence f' is flat and by devisage using (3.5.4) we may assume $\mathcal{A} = \kappa(y)$ as R-module. But then it suffices to consider the case $Y = \operatorname{Spec} \kappa(y)$ and the claim follows from rule R1c. $\square$

(4.2) Lemma. *Let $f \colon Y \to X$ be as in* (3.4).

(1) *If a is a unit on X, then*

$$f_* \circ \{f^*(a)\} = \{a\} \circ f_*.$$

(2) *Let f be finite and flat and let a be a unit on Y. Then*

$$f_* \circ \{a\} \circ f^* = \{\tilde{f}_*(a)\}.$$

Here $\tilde{f}_* \colon O^*_Y \to O^*_X$ is the standard transfer map.

Proof: (1) is immediate from R2b. For (2) we may assume $X = \operatorname{Spec} F$ with F a field. Then for $y \in Y$ let l_y be the length of $\mathcal{O}_{y,Y}$. By R2c we have

$$f_* \circ \{a\} \circ f^* = \sum_y l_y \cdot c_{\kappa(y)|F}\big(\{a(y)\}\big)$$

and the claim follows. $\square$

(4.3) LEMMA. *Let a be a unit on X.*

(1) *For $g\colon Y \to X$ as in* (3.5.5) *one has*

$$g^* \circ \{a\} = \{g^* a\} \circ g^*.$$

(2) *For a boundary triple (Y, i, X, j, U) one has*

$$\partial_Y^U \circ \{j^*(a)\} = -\{i^*(a)\} \circ \partial_Y^U.$$

Proof: (1) follows from R2a and (2) from R2b and R3e. □

Let $h\colon X \to X'$ be a morphism of schemes of finite type over a field and let $Y' \hookrightarrow X'$ be a closed immersion. Consider the induced diagram given by $U' = X' \setminus Y'$ and pull-back:

$$\text{(4.4.0)} \qquad \begin{array}{ccccc} Y & \hookrightarrow & X & \hookleftarrow & U \\ \downarrow{\scriptstyle \bar{h}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \bar{\bar{h}}} \\ Y' & \hookrightarrow & X' & \hookleftarrow & U' \end{array}$$

(4.4) PROPOSITION.

(1) *If h is proper, then*

$$\bar{h}_* \circ \partial_Y^U = \partial_{Y'}^{U'} \circ \bar{\bar{h}}_*.$$

(2) *If h is flat (of constant relative dimension), then*

$$\bar{h}^* \circ \partial_{Y'}^{U'} = \partial_Y^U \circ \bar{\bar{h}}^*.$$

Proof: Immediate from Proposition 4.6.1 and 4.6.2 below. □

(4.5) LEMMA. *Let $g\colon Y \to X$ be a smooth morphism of schemes of finite type over a field of constant fiber dimension 1, let $\sigma\colon X \to Y$ be a section to g and let $t \in O_Y$ be a global parameter defining the subscheme $\sigma(X)$. Moreover let $\tilde{g}\colon Y \setminus \sigma(X) \to X$ be the restriction of g and let ∂ be the boundary map associated to σ. Then*

$$\partial \circ \tilde{g}^* = 0 \quad \text{and} \quad \partial \circ \{t\} \circ \tilde{g}^* = (\mathrm{id}_X)_*.$$

Proof: One reduces to $X = \operatorname{Spec} E$ and applies R3c and R3d. □

(4.6) PROPOSITION.

(1) *For proper $f\colon X \to Y$ as in* (3.4) *one has*

$$d_Y \circ f_* = f_* \circ d_X.$$

(2) *Let $g\colon Y \to X$ be a morphism and let $\mathcal{A}$ be a coherent sheaf on Y flat over X. Then*

$$d_Y \circ [\mathcal{A}, g, s] = [\mathcal{A}, g, s] \circ d_X$$

for $s \geq s(g)$. In particular

$$g^* \circ d_X = d_Y \circ g^*$$

for flat g as in (3.5.5).

(3) *For a unit a on X one has*

$$d_X \circ \{a\} = -\{a\} \circ d_X.$$

(4) *For a boundary triple (Y, i, X, j, U) one has*

$$d_Y \circ \partial_Y^U = -\partial_Y^U \circ d_U.$$

Proof: (3) follows as Lemma 4.3.2 and (4) follows from Lemma 3.3.

Proof of (1): Let $\delta(f_*) = d_Y \circ f_* - f_* \circ d_X$. We have to show $\delta(f_*)_y^x = 0$ for $x \in X_{(p)}$ and $y \in Y_{(p-1)}$. Let $z = f(x)$ and $q = \dim(z, Y)$. If $y \notin \overline{\{z\}}$, the claim is obvious. If $y = z$, we first replace Y by $\operatorname{Spec} \kappa(y)$ and then X by $\overline{\{x\}}$. This is the case of a proper curve over a field considered in (RC) of Section 2. If $y \in \overline{\{z\}}$ and $y \neq z$, we must have $q = p$ and $\kappa(x)$ is finite over $\kappa(z)$. We may assume $Y = \overline{\{z\}}$ and $X = \overline{\{x\}}$. Consider the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{g} & X \\ \Big\downarrow{\scriptstyle \tilde{f}} & & \Big\downarrow{\scriptstyle f} \\ \tilde{Y} & \xrightarrow{h} & Y \end{array}$$

where g and h are the normalizations. Let $\tilde{x} \in \tilde{X}$ and $\tilde{z} \in \tilde{Y}$ be the generic points (lying over x and z, respectively). We have $\delta(g_*) \mid M(\tilde{x}) = 0$ by the very definition of the differentials; similarly $\delta(h_*) \mid M(\tilde{z}) = 0$. This and 4.1.1 show

$$\begin{aligned} \delta(f_*) \circ g_* \mid M(\tilde{x}) &= (d_Y \circ h_* \circ \tilde{f}_* - f_* \circ g_* \circ d_{\tilde{X}}) \mid M(\tilde{x}) \\ &= h_* \circ \delta(\tilde{f}_*) \mid M(\tilde{x}). \end{aligned}$$

Since $g_* \mid M(\tilde{x})$ is an isomorphism onto $M(x)$ we are reduced to show $\delta(\tilde{f}_*)_{\tilde{y}}^{\tilde{x}} = 0$ for $\tilde{y} \in \tilde{Y}_{(p-1)}$. Let $\tilde{u} \in \tilde{X}$ be a point over $\tilde{y}$. We have $\tilde{u} \in \tilde{X}^{(1)}$. Now $\delta(\tilde{f}_*)_{\tilde{y}}^{\tilde{x}} = 0$ follows from rule R3b, the properness of $\tilde{f}$ and the fact that the local rings of $\tilde{y}$ and of all the preimages $\tilde{u}$ are valuation rings.

Proof of (2): Let $\delta = d_Y \circ [\mathcal{A}, g, s] - [\mathcal{A}, g, s] \circ d_X$. We have to show $\delta_y^x = 0$ for $x \in X_{(p)}$, $y \in Y_{(p+s-1)}$. Put $z = g(y)$. If $z \notin \overline{\{x\}}$, the claim is obvious. If $z = x$, then for $u \in Y_x$ all valuations on $\kappa(u)$ with center y are trivial on $\kappa(x)$; the claim follows from rule R3c. We are now reduced to the case $z \in \overline{\{x\}}$, $z \neq x$. Then $z \in X_{(p-1)}$ since $\dim(z, X) \geq \dim(y, Y) - s = p - 1$. We may assume $X = \overline{\{x\}}$. Moreover by Propositions 4.1.3 and 4.6.1 we may additionally assume that X is normal. Let $U = \{\, u \in Y_x^{(0)} \mid y \in \overline{\{u\}} \,\}$. Then

$$\delta_y^x = \sum_{u \in U} \partial_y^u \circ [\mathcal{A}, g, s]_u^x - [\mathcal{A}, g, s]_y^z \circ \partial_z^x.$$

We may replace X and Y by its localizations in z and y, respectively. Then $X = \operatorname{Spec} R$ with R a valuation ring, $Y = \operatorname{Spec} S$ with S local of dimension ≤ 1 and $U = Y_{(1)}$.

In this case we have by definition

$$\delta_y^x = \sum_{u\in U} l_{S_{(u)}}(\mathcal{A}_{(u)}) \cdot \partial_y^u \circ r_{\kappa(u)|\kappa(x)} - l_S(\mathcal{A}/\pi\mathcal{A}) \cdot r_{\kappa(y)|\kappa(z)} \circ \partial_z^x$$

where $S_{(u)}$, $\mathcal{A}_{(u)}$ are the localizations at u and where π is a prime element of R.

For $u \in U$ let $\tilde{S}_u$ be the normalization of S/u. For a $\tilde{S}_u$-module H of finite length we define

$$L_u(H) = \sum_w l_{\tilde{S}_{(w)}}(H_{(w)}) \cdot [\kappa(w) : \kappa(y)]$$

where w runs through the maximal prime ideals of $\tilde{S}_u$ and where $\tilde{S}_{(w)}$ resp. $H_{(w)}$ are the localizations of $\tilde{S}_u$ resp. H at w. We claim that $L_u(H)$ is the length of H as S-module:

$$L_u(H) = l_S(H).$$

To prove this use devisage to reduce to the trivial case $H = \kappa(w)$ for some w.

Moreover we have

$$l_S(\tilde{S}_u/\pi\tilde{S}_u) = l_S(S/u + \pi S).$$

This follows from the fact that the cokernel and the (trivial) kernel of $S/u \to \tilde{S}_u$ have finite S-length and π is a nonzero divisor of S/u and $\tilde{S}_u$ (see Fulton 1984, Lemma A.2.4).

We have for fixed u:

$$\begin{aligned}
\partial_y^u \circ r_{\kappa(u)|\kappa(x)} &= \sum_w c_{\kappa(w)|\kappa(y)} \circ \partial_w^u \circ r_{\kappa(u)|\kappa(x)} \\
&= \sum_w l_{\tilde{S}_{(w)}}(\tilde{S}_{(w)}/\pi\tilde{S}_{(w)}) \cdot c_{\kappa(w)|\kappa(y)} \circ r_{\kappa(w)|\kappa(z)} \circ \partial_z^x \\
&= L_u(\tilde{S}_u/\pi\tilde{S}_u) \cdot r_{\kappa(y)|\kappa(z)} \circ \partial_z^x.
\end{aligned}$$

Here we used the definition of ∂_y^u and R3a, R1b, R2d.

Putting things together one finds that $\delta_y^x = 0$ follows from

$$l_S(\mathcal{A}/\pi\mathcal{A}) = \sum_{u\in U} l_{S_{(u)}}(\mathcal{A}_{(u)}) \cdot l_S(S/u + \pi S).$$

This formula is exactly the formula of (Fulton 1984, Lemma A.2.7) because the map $\mathcal{A} \to \mathcal{A}$, $a \mapsto \pi a$ is injective by the flatness of $\mathcal{A}$ over R. $\square$

5. Cycle Complexes and Chow Groups

This section contains notations and a few remarks and examples. In Section 3 we have introduced for a cycle module M over X the complexes

$$C_p(X;M) = \coprod_{x \in X_{(p)}} M(x)$$

with differentials

$$d = d_X \colon C_p(X;M) \to C_{p-1}(X;M).$$

Sometimes it is convenient to use the codimension index instead of the dimension index. We put

$$C^p(X;M) = \coprod_{x \in X^{(p)}} M(x)$$

and define

$$d = d_X \colon C^p(X;M) \to C^{p+1}(X;M)$$

again by $d^x_y = \partial^x_y$ with ∂^x_y as in (2.1.0). Similar as in Lemma 3.3 one finds $d \circ d = 0$.

The choice between the dimension and codimension index depends on the matter. Our basic interest is in schemes X of finite type over a field F. In this case the dimension setting is in general appropriate, since then the dimension of a point x is an absolute notion independent of the ambient space: $\dim(x,X) = \mathrm{tr.deg}(\kappa(x)|F)$.

If X is in addition equidimensional of dimension d, then $X^{(p)} = X_{(d-p)}$ and $C^p(X;M) = C_{d-p}(X;M)$. Then we will freely switch between the two notions if it is convenient (in particular if we consider intersections in the smooth case). The codimension setting will also be used for certain schemes not necessarily of finite type over a field, e.g., for spectra of local rings. In this case we understand the material of Section 4 to be transferred from the dimension to the codimension setting via finite type models.

In practice, all M have a $\mathbb{Z}$-grading and one likes to keep track on it. We put for $\mathbb{Z}$-graded M

$$C_p(X;M,n) = \coprod_{x \in X_{(p)}} M_{n+p}(x),$$

$$C^p(X;M,n) = \coprod_{x \in X^{(p)}} M_{n-p}(x).$$

with $n \in \mathbb{Z}$. Then there are decompositions of complexes

$$C_*(X;M) = \coprod_n C_*(X;M,n),$$

$$C^*(X;M) = \coprod_n C^*(X;M,n).$$

(In the introduction we have used the notation $C_*(X;n) = C_*(X;K_*,n)$).

The *Chow group of p-dimensional cycles with coefficients in M* is defined as the p-th homology group of the complex $C_*(X;M)$ and denoted by $A_p(X;M)$. Similarly we define $A^p(X;M)$, $A_p(X;M,n)$ and $A^p(X;M,n)$ according to the notations used for the complexes.

The homomorphisms f_* for proper f, g^* for flat g, $\{a_1,\dots,a_n\}$, ∂^U_Y and $\omega_\#$ of Section 3 (anti-)commute with the differentials (see Proposition 4.6). The induced maps on the (co-)homology groups will be denoted by the same letters. The compatibilities of (4.1)–(4.5) carry over (for proper f, f' and flat g).

It is obvious from (3.10) that for a boundary triple (Y,i,X,j,U) there is the long exact sequence

$$\cdots \xrightarrow{\partial} A_p(Y;M) \xrightarrow{i_*} A_p(X;M) \xrightarrow{j_*} A_p(U;M) \xrightarrow{\partial} A_{p-1}(Y;M) \xrightarrow{i_*} \cdots .$$

We conclude by mentioning a few examples. $H^*[D]$ and K'_* denote the cycle modules given by Galois cohomology and Quillen's K-theory as considered in Sections 1–2.

(5.1) REMARK. — *Classical Chow groups.* We understand here $\mathrm{CH}_p(X)$ as the group of p-cycles modulo rational equivalence as defined in (Fulton 1984, Sect. 1.3; denoted by $A_p(X)$). From this definition$^\diamond$ it is obvious that

$$A_p(X;K_*,-p) = \mathrm{CH}_p(X).$$

For the Chow group $\mathrm{CH}^p(X)$ of p-codimensional cycles (for smooth irreducible X say) our notations give

$$A^p(X;K_*,p) = \mathrm{CH}^p(X).$$

(5.2) REMARK. — *Unramified cohomology.* For a proper smooth variety X over a field k and a cycle module M over k, the group

$$A^0(X;M) \subset M\big(k(X)\big)$$

is a birational invariant of the field extension $k(X)|k$ (see Corollary 12.10). A well-known example here is the unramified Brauer group of $k(X)|k$. Its n-torsion subgroup is in our notations given by $A^0(X;H^*[\mu_n^{\otimes -1}],2)$.

(5.3) REMARK. — *Relations with local-global spectral sequences.* In Quillen's K-theory as well as in étale cohomology there are spectral sequences given by codimension of support (see Quillen 1973, Sect. 5; Bloch and Ogus 1974). The corresponding E_1-terms together with the d^1-differentials may be identified with the complexes $C^*(X;K'_*,n)$ and $C^*(X;H^*[D],n)$. The corresponding E_2-terms are of the form $E_2^{p,q} = A^p(X;K'_*,-q)$ and $E_2^{p,q} = A^p(X;H^*[D\otimes\mu_r^{\otimes -q}],q)$, respectively (where $r\cdot D=0$).

$^\diamond$ Namely the definition of $\mathrm{CH}_p(X)$ mentioned in the first sentence of the introduction.

(5.4) Remark. — *The map from Milnor's to Quillen's K-theory.* The natural homomorphisms $K_*F \to K'_*F$ form a homomorphism of cycle modules. It is an isomorphism in degrees ≤ 2. Moreover for a valuation v on F one has $\partial_v(K_3F) = \partial_v(K'_3F)$, see (Merkurjev and Suslin 1987). It follows that the induced homomorphisms

$$A_p(X; K_*, n) \to A_p(X; K'_*, n)$$

are bijective for $n + p \leq 2$.

6. Acyclicity for Smooth Local Rings

The following observations have been included to underpin the notion of cycle modules. They are not needed in further sections. M is a cycle module over a field k.

(6.1) Theorem. *Let X be smooth and semi-local. Then*

$$A^p(X; M) = 0 \quad \textit{for} \quad p > 0.$$

This theorem is known in Quillen's K-theory (Quillen 1973, § 7, Theorem 5.11) and in étale cohomology (Bloch and Ogus 1974) and has been proved by O. Gabber for Milnor's K-theory. The main step in these proofs is sometimes called "Quillen's trick" and carries over to cycle modules as well. Here we follow essentially this method but with a simplification due to I. Panin.

Let V be a vector space over k and let $\mathbb{A}(V)$ be the associated affine space. For a linear subspace W of V let

$$\pi_W \colon \mathbb{A}(V) \to \mathbb{A}(V/W),$$
$$\pi_W(v) = v + W$$

be the projection.

(6.2) Lemma. *Let $X \subset \mathbb{A}(V)$ be an equidimensional closed subvariety with* $\dim X = d$ *and let $Y \subset X$ be a closed subvariety with* $\dim Y < d$. *Moreover let $S \subset Y$ be a finite subset such that X is smooth in S. Then for a generic $(d-1)$-codimensional linear subspace W of V the following conditions hold.*

(1) *The restriction*

$$\pi_W \mid Y : Y \to \mathbb{A}(V/W)$$

is finite.

(2) *The restriction*

$$\pi_W \mid X : X \to \mathbb{A}(V/W)$$

is locally around S smooth of relative dimension 1.

Proof: (Panin) We extend the situation to the projective closure $\mathbb{A}(V) \subset \mathbb{P}(V \oplus k)$ with $\mathbb{P}(V) \subset \mathbb{P}(V \oplus k)$ as hyperplane at infinity. Let

$$\bar{\pi}_W : \mathbb{P}(V \oplus k) \setminus \mathbb{P}(W) \to \mathbb{P}(V/W \oplus k),$$
$$\bar{\pi}_W([v,t]) = [v + W, t]$$

be the projection. $\bar{\pi}_W$ is an affine bundle over $\mathbb{P}(V/W \oplus k)$ which extends the affine bundle π_W over $\mathbb{A}(V/W)$.

Let $\overline{Y} \subset \mathbb{P}(V \oplus k)$ be the closure of Y and let $Y_\infty = \overline{Y} \cap \mathbb{P}(V)$. Then $\dim Y_\infty < d-1$. Hence for generic $(d-1)$-codimensional W we have $Y_\infty \cap \mathbb{P}(W) = \varnothing$. Therefore there is the map

$$\bar{\pi}_W \mid \overline{Y} : \overline{Y} \to \mathbb{P}(V/W \oplus k).$$

This map is finite since it is proper and since $\bar{\pi}_W$ is an affine bundle. This shows (1) because $\pi_W \mid Y$ is the pull-back of $\bar{\pi}_W \mid \overline{Y}$ along $\mathbb{A}(V) \subset \mathbb{P}(V \oplus k)$.

For condition (2) one just needs to guarantee that π_W maps for each $s \in S$ the tangent space $T_s X \subset V$ of X in s epimorphically onto the tangent space V/W of $\mathbb{A}(V/W)$ in $\pi_W(s)$. This is again an open condition for W. $\square$

This lemma is very close to (Quillen 1973, § 7, Lemma 5.12.) and suffices for all applications I know. The existence of such a space W is not clear over finite ground fields and needs some extra discussion. However, if one is in the end interested in (co-)homology groups, there is usually no problem with replacing the ground field k by a rational extension $k(t_1, \ldots, t_r)$. In this case one may take for W for example the tautological subspace of V defined over the function field of the Grassmannian of $(d-1)$-codimensional subspaces of V. In our situation we refer here to the following remark.

(6.3) Lemma. *Let X be a variety over k and let $g: X_{k(t)} \to X$ be the base change. Then*

$$g^* : A_q(X; M) \to A_q(X_{k(t)}; M)$$

is injective.

This lemma will be become obvious in the next section where we show that $F \to A_q(X_F; M)$ is a cycle module. Then g^* is just the restriction map $r_{k(t)|k}$ for this cycle module and any specialization s_v^π at a rational point of $\mathbb{P}^1$ yields a left inverse. (What one really uses here is Lemma 4.5 with $Y = X \times \mathbb{P}^1$ and Proposition 4.6.2.)

(6.4) Proposition. *Let X be a smooth variety over a field and let $Y \subset X$ be a closed subscheme of codimension ≥ 1. Then for any finite subset $S \subset Y$ there is an open neighborhood X' of S in X such that the map*

$$i_* : A_*(Y \cap X'; M) \to A_*(X'; M)$$

is the trivial map. Here $i: Y \cap X' \to X'$ is the inclusion.

Proof: We may assume that X is affine. By Lemma 6.2 we find a diagram (at least after replacing k by a rational extension)

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr] & & X \arrow[dl] \\ & A & \end{tikzcd}$$

with $Y \to A$ finite and with $X \to A$ smooth of relative dimension 1 in S.

Put $Z = Y \times_A X$ and consider the diagram

$$\begin{tikzcd} & Z \arrow[dl, shift right, "g"'] \arrow[dr, "\pi"] & \\ Y \arrow[ur, shift right, "\sigma"'] \arrow[rr, "i"] & & X \end{tikzcd}$$

where g and π are the projections and σ is the diagonal. Note that π is finite, g is smooth of relative dimension 1 in S and that σ is a section to g and a lift of the immersion i. Moreover after a localization to an open subset $X' \subset X$ containing S we may assume that there is a global parameter $t \in O_Z$ defining the closed subscheme $\sigma(Y)$.

Let (Y, σ, Z, j, Q) be the boundary triple given by σ (with $Q = Z \setminus \sigma(Y)$) and let $\tilde{g}$: $Q \to Y$ be the restriction of g. Now consider the composite

$$H \colon Y \overset{\tilde{g}^*}{\bullet\!\!\longrightarrow} Q \overset{\{t\}}{\bullet\!\!\longrightarrow} Q \overset{j_*}{\bullet\!\!\longrightarrow} Z \overset{\pi_*}{\bullet\!\!\longrightarrow} X.$$

One finds

$$\delta(H) = \pi_* \circ \sigma_* \circ \partial_Y^Q \circ \{t\} \circ \tilde{g}^* = \pi_* \circ \sigma_* = i_*$$

by Lemma 4.5. Therefore i_* is nullhomotopic. $\square$

Proof of Theorem 6.1: We may assume that X is connected. Put $d = \dim X$. Consider pairs (U, S) where U is a smooth d-dimensional variety of finite type over k and $S \subset U$ is a finite subset such that X is the localization of U in S. Then

$$C^p(X; M) = \varinjlim_{(U,S)} C^p(U; M).$$

Moreover

$$C^p(U; M) = C_{d-p}(U; M) = \varinjlim_{Y} C_{d-p}(Y; M)$$

where Y runs over the closed p-codimensional subsets of U. Hence

$$A^p(X; M) = \varinjlim_{(U,S)} A^p(U; M) = \varinjlim_{(U,S)} \varinjlim_{Y} A_{d-p}(Y; M).$$

But Proposition 6.4 tells that $A_{d-p}(Y; M) \to A^p(U; M) \to A^p(U'; M)$ is the trivial map for small enough $U' \subset U$. $\square$

In the smooth case we sheafify cycle modules as follows. For a smooth variety X let $\mathcal{M}_X$ be the Zariski sheaf on X given by

$$\mathcal{M}_X(U) = A^0(U; M) \subset M(\xi_X)$$

for open subsets U of X.

(6.5) COROLLARY. *For a smooth variety X over k there are natural isomorphisms*

$$A^p(X; M) = H^p_{\mathrm{Zar}}(X; \mathcal{M}_X).$$

Proof: Define the Zariski sheaves $\mathcal{C}^p$ on X by

$$\mathcal{C}^p(U) = C^p(U; M).$$

Then there is a complex of sheaves

$$0 \longrightarrow \mathcal{M}_X \longrightarrow \mathcal{C}^0 \xrightarrow{d} \mathcal{C}^1 \xrightarrow{d} \cdots .$$

The complex is exact. This holds at $\mathcal{M}_X$ and at $\mathcal{C}^0$ by the very definitions. Theorem 6.1 implies exactness at positive dimensions. The corollary follows, since the $\mathcal{C}^p$ are flasque. □

The resolution of $\mathcal{M}_X$ considered in this proof has nice functorial properties. Namely, we will define for morphisms $f\colon Y \to X$ maps of complexes (Section 12)

$$I(f)\colon C^p(X; M) \to C^p(Y; M)$$

and, under presence of a ring structure for M, a pairing of complexes (Section 14)

$$C^*(X; M) \times C^*(X; M) \to C^*(X; M).$$

These are functorial with respect to localizations. Therefore the isomorphisms of Corollary 6.5 are compatible with pull-backs and with products.

The following example is a nice illustration of Corollary 6.5. Let X be smooth and define the Zariski sheaf $\mathcal{K}_n$ on X by

$$\mathcal{K}_n(U) = A^0(U; K_*, n) \subset K_n k(X).$$

The sheaf $\mathcal{K}_n$ has a comparatively simple definition: it just refers to the definition of Milnor's K-groups for fields and of the residue maps for valuations. Corollary 6.5 yields the following interpretation of the classical Chow groups on a smooth variety:

$$\mathrm{CH}^p(X) = H^p_{\mathrm{Zar}}(X; \mathcal{K}_p). \tag{6.6}$$

The same result holds with Milnor's K-theory replaced by Quillen's K-theory. The corresponding sheaf $\mathcal{K}'_n$ coincides with the sheaf induced from the presheaf $U \to K^Q_n(U)$ where $K^Q_n(U)$ denotes the n-th Quillen's K-group of the category of vector bundles on U. In this context (6.6) is known as Bloch's formula (see Quillen 1973, Thm. 5.19; Grayson 1978).

Another special case of Corollary 6.5 for $M = K_*$ is

$$A_0(X; K_*, n) = H^d_{\mathrm{Zar}}(X; \mathcal{K}_{n+d}),$$

with $d = \dim X$. This interpretation of the "Chow groups of zero cycles on X with coefficients in K_n" was obtained already in (Kato 1986).

7. The Cycle Modules $A_q[\rho; M]$

In this section we show that new cycle modules can be obtained from the Chow groups of the fibers of a morphism. It was in fact this process of forming local coefficient systems for cycles which motivated the notion of cycle modules.

Let $\rho: Q \to B$ be a morphism of finite type and let M be a cycle module over Q. For fields F over B let $Q_F = Q \times_B \operatorname{Spec} F$. We define an object function $A_q[\rho, M]$ on $\mathcal{F}(B)$ by

$$A_q[\rho; M](F) = A_q(Q_F; M).$$

Our aim is to show that $A_q[\rho, M]$ is in a natural way a cycle module over B.

All the properties of cycle modules except axiom (C) hold already on complex level, i.e., for the groups $C_q(Q_F; M)$. It is appropriate to establish first the corresponding object function as a cycle premodule.

So let $\widehat{M}$ be the object function on $\mathcal{F}(B)$ defined by

$$\widehat{M}(F) = C_q(Q_F; M).$$

We first describe its data as a cycle premodule. These will be denoted by $\widehat{\varphi}_*$, $\widehat{\varphi}^*$, $\widehat{\partial}_v$, $\widehat{r}_{E|F}$, $\widehat{c}_{E|F}$, etc. in order to distinguish them from the data φ_*, φ^*, ∂_v, etc. of M.

For a homomorphism of fields $\varphi: F \to E$ let $\bar{\varphi}: Q_E \to Q_F$ be the induced morphism. We define the data D1 and D2 by

$$\widehat{\varphi}_* = \bar{\varphi}^*: C_q(Q_F; M) \to C_q(Q_E; M),$$
$$\widehat{\varphi}^* = \bar{\varphi}_*: C_q(Q_E; M) \to C_q(Q_F; M).$$

For D3 we take the K_*F-module structure on $C_q(Q_F; M)$ described in (3.6). To establish D4 put $\widetilde{Q}_v = Q \times_B \operatorname{Spec} \mathcal{O}_v$. It has over $\operatorname{Spec} \mathcal{O}_v$ the generic fiber Q_F and the special fiber $Q_{\kappa(v)}$. Define

$$\widehat{\partial}_v: C_q(Q_F; M) \to C_q(Q_{\kappa(v)}; M)$$

by $(\widehat{\partial}_v)^x_y = \partial^x_y$ with ∂^x_y as in (2.1.0) with respect to the scheme $\widetilde{Q}_v$.

(7.1) Theorem. *Together with these data, $\widehat{M}$ is a cycle premodule over B.*

Proof: All the required properties follow from the rules and axioms for M and from Section 4.

Below we consider R3a in detail. Here is a sketch for the other (less complicated) cases:
for R1a use (4.1.2); for R1b use (4.1.1);
for R1c use (4.1.3) and a length consideration;
for R2a use (4.3.1); for R2b use (4.2.1); for R2c use R1c and R2c;
for R3b use (4.6.1); for R3c use R3c;
for R3d use (1.5) and R3d; for R3e use R2b and R3e.

Proof of R3a: Let $g_\xi\colon Q_F \to Q_E$ and $g_0\colon Q_{\kappa(v)} \to Q_{\kappa(w)}$ be the projections. We have to show that the following diagram is commutative:

$$\begin{array}{ccc} C_q(Q_F;M) & \xrightarrow{\widehat{\partial}_v} & C_q(Q_{\kappa(v)};M) \\ \Big\uparrow{\scriptstyle g_\xi^*} & & \Big\uparrow{\scriptstyle e\cdot g_0^*} \\ C_q(Q_E;M) & \xrightarrow{\widehat{\partial}_w} & C_q(Q_{\kappa(w)};M). \end{array}$$

We want to apply Proposition 4.6.2 to the projection $g\colon \widetilde{Q}_v \to \widetilde{Q}_w$. The pull-back of g along $\operatorname{Spec} E \to \operatorname{Spec} \mathcal{O}_w$ is g_ξ. Let

$$\bar{g}_0\colon \bar{Q}_{\kappa(v)} = \widetilde{Q}_v \times_{\operatorname{Spec}\mathcal{O}_w} \operatorname{Spec}\kappa(w) \to Q_{\kappa(w)}$$

be the pull-back of g along $\operatorname{Spec}\kappa(w) \to \operatorname{Spec}\mathcal{O}_w$. Note that $\bar{Q}_{\kappa(v)}$ and $Q_{\kappa(v)}$ have the same reductions and therefore the same cycle groups.

We claim $\bar{g}_0^* = e\cdot g_0^*$. Let $R = \mathcal{O}_v \otimes_{\mathcal{O}_w} \kappa(w)$. Note that g_0, $\bar{g}_0$ are the pull-backs along $Q_{\kappa(w)} \to \operatorname{Spec}\kappa(w)$ of the morphisms $\operatorname{Spec}\kappa(v) \to \operatorname{Spec}\kappa(w)$, $\operatorname{Spec} R \to \operatorname{Spec}\kappa(w)$, respectively. The claim follows from $e = l_R(R)$ and a standard length consideration.

It remains to show that the diagram commutes with $e\cdot g_0^*$ replaced by $\bar{g}_0^*$. This follows (cum grano salis, see the following remark) from Proposition 4.6.2.

Remark. When applying here Proposition 4.6 in a formal way, there appears an artificial problem caused by the fact that the dimension index does not behave perfectly well for schemes over local rings like $\widetilde{Q}_v$. However, note that to check a commutativity like $\partial_v \circ g_\xi^* = \bar{g}_0^* \circ \partial_w$ it suffices to restrict to the components corresponding to points $x \in Q_{E(q)}$ with $\overline{\{x\}} \cap \big(Q_{\kappa(w)}\big)_{(q)} \neq \varnothing$. For these points one has $x \in (\widetilde{Q}_w)_{(q+1)}$ by the dimension inequality (Matsumura 1980, p. 85). A similar remark applies to $\widetilde{Q}_v$. Therefore the desired identity follows from $d\circ g^* = g^*\circ d$ on $C_{q+1}(\widetilde{Q}_w;M)$. One may avoid these considerations by looking more closely to the proof of Proposition 4.6. $\square$

We have to relate the differentials for the cycle premodule $\widehat{M}$ to the differentials for the cycle module M.

Let $X \to B$ be a scheme over B and let $\widetilde{X} = Q \times_B X$. Then for $x, y \in X$ there is the map

$$\widehat{\partial}^x_y\colon \widehat{M}(x) \to \widehat{M}(y)$$

according to (2.1.0). By definition this is a map

$$\widehat{\partial}^x_y\colon C_q(Q_{\kappa(x)};M) \to C_q(Q_{\kappa(y)};M)$$

between cycle groups with coefficients in M.

(7.2) Proposition. *Let $\tilde{x}$, $\tilde{y} \in \widetilde{X}$ be points lying over x, $y \in X$, respectively, and suppose $\tilde{x} \in \big(Q_{\kappa(x)}\big)_{(q)}$ and $\tilde{y} \in \big(Q_{\kappa(y)}\big)_{(q)}$. Denote by $(\widehat{\partial}^x_y)^{\tilde{x}}_{\tilde{y}}$ the component of $\widehat{\partial}^x_y$ with respect to $\tilde{x}$ and $\tilde{y}$. Then*

$$(\widehat{\partial}^x_y)^{\tilde{x}}_{\tilde{y}} = \partial^{\tilde{x}}_{\tilde{y}}\colon M(\tilde{x}) \to M(\tilde{y}).$$

Proof: We may assume $\tilde{y} \in \overline{\{\tilde{x}\}}^{(1)}$, since otherwise both sides are trivial. The dimension inequality (Matsumura 1980, p. 85) shows then $y \in \overline{\{x\}}^{(1)}$. Let v run through the valuations of $\kappa(x)$ with center y in X. Moreover let w run through the valuations on $\kappa(\tilde{x})$ with center $\tilde{y}$ in $\tilde{X}$. The restriction of any w to $\kappa(x)$ is one of the valuations v. Let $\tilde{w} \in Q_{\kappa(v)}$ be the center of w in $\widetilde{X} \times_X \operatorname{Spec} \mathcal{O}_v$. Now the claim follows from

$$\begin{aligned}
(\widehat{\partial}^x_y)^{\tilde{x}}_{\tilde{y}} &= (\sum_v \widehat{c}_{\kappa(v)|\kappa(y)} \circ \widehat{\partial}_v)^{\tilde{x}}_{\tilde{y}} \\
&= \sum_v \sum_{w|v} (\widehat{c}_{\kappa(v)|\kappa(y)})^{\tilde{w}}_{\tilde{y}} \circ (\widehat{\partial}_v)^{\tilde{x}}_{\tilde{w}} \\
&= \sum_v \sum_{w|v} c_{\kappa(\tilde{w})|\kappa(\tilde{y})} \circ c_{\kappa(w)|\kappa(\tilde{w})} \circ \partial_w \\
&= \sum_w c_{\kappa(w)|\kappa(\tilde{y})} \circ \partial_w = \partial^{\tilde{x}}_{\tilde{y}}.
\end{aligned}$$ □

It follows from Proposition 4.6 that the data of the cycle premodules $\widehat{M}$ (for various q) commute resp. anti-commute with the differentials of the complexes $C_*(Q_F; M)$. Passing to homology we obtain data D1–D4 for the object functions $A_q[\rho; M]$.

(7.3) Theorem. *Together with these data, $A_q[\rho; M]$ is a cycle module over B.*

Proof: The rules for the data of the cycle premodule $A_q[\rho; M]$ are immediate from the rules for $\widehat{M}$. Moreover axiom (FD) for M and Proposition 7.2 show that (FD) holds for $\widehat{M}$—consequently also for $A_q[\rho; M]$. It remains to verify axiom (C).

Consider the maps[◇]

$$C_q(Q_{\kappa(\xi)}) \xrightarrow{\Theta} C_{q-1}(Q_{\kappa(\xi)}) \oplus \coprod_{x \in X^{(1)}} C_q(Q_{\kappa(x)}) \oplus C_{q+1}(Q_{\kappa(x_0)}) \xrightarrow{\Theta} C_q(Q_{\kappa(x_0)})$$

defined by $\Theta^z_y = \partial^z_y$ with ∂^z_y as in (2.1.0) with respect to the scheme $Q \times_B X$.

By Proposition 7.2 we are reduced to show $\Theta \circ \Theta = 0$. It suffices to check $(\Theta \circ \Theta)^z_y = 0$ for $z \in \big(Q_{\kappa(\xi)}\big)_{(q)}$ and $y \in \big(Q_{\kappa(x_0)}\big)_{(q)}$ with $y \in \overline{\{z\}}^{(2)}$ (here $\overline{\{z\}}$ is the closure of z in $\widetilde{X}$). The dimension inequality (Matsumura 1980, p. 85) shows

$$Z^{(1)} \subset \big(Q_{\kappa(\xi)}\big)_{(q-1)} \cup \bigcup_x \big(Q_{\kappa(x)}\big)_{(q)} \cup \big(Q_{\kappa(x_0)}\big)_{(q+1)}$$

with $Z = \overline{\{z\}}_{(y)}$. We are done by axiom (C) for M. □

In the following proposition we formulate some functorial properties of the construction $\rho \to A_q[\rho; M]$. Let

(7.4.0)
$$\begin{array}{ccc} Y & \xrightarrow{h} & X \\ & {}_{\eta}\searrow \quad \swarrow_{\rho} & \\ & B & \end{array}$$

◇ Here the indication of the cycle module M has been dropped.

be a commutative diagram with η and ρ of finite type and let M be a cycle module over X. For a field F over B let

$$h_F\colon Y_F \to X_F$$

be the morphism induced by h.

(7.4) Proposition. *The following transformations are homomorphisms of cycle modules over B:*

(1) *For proper h let*

$$[h_*]\colon A_q[\eta; M] \to A_q[\rho; M]$$

with $[h_*]_F = (h_F)_*$.

(2) *For flat h of relative dimension s let*

$$[h^*]\colon A_q[\rho; M] \to A_{q+s}[\eta; M]$$

with $[h^*]_F = [h_F^*]$.

(3) *For a global unit a on X let*

$$[\{a\}]\colon A_q[\rho; M] \to A_q[\rho; M]$$

with $[\{a\}]_F = \{a \mid X_F\}$.

(4) *For a boundary triple (Y, i, X, j, U) let*

$$[\partial] = [\partial_Y^U]\colon A_q[\rho \circ j; M] \to A_{q-1}[\rho \circ i; M]$$

with $[\partial]_F$ the boundary map for $Y_F \to X_F$.

Proof: One has to check the compatibility with D1–D4. This follows for (1) from (4.1.3), (4.1.1), (4.2.1) and (4.6.1); for (2) from (4.1.2), (4.1.3), (4.3.1) and (4.6.2); for (3) from (4.2.1), (4.3.1), the anti-commutativity of K_* and (4.6.3); for (4) from (4.6.1), (4.6.2), (4.6.3) and (C). □

Let $\rho\colon Q \to B$ be flat and not necessarily of finite type. One may then define cycle modules $A^q[\rho; M]$ with

$$A^q[\rho; M](F) = A^q(Q_F; M).$$

To establish these cycle modules one proceeds analogous to the $A_q[\rho; M]$ above. Alternatively one may reduce to the consideration of the $A_q[\rho; M]$ as follows. If ρ is of finite type, one may assume that it is of constant dimension s. In this case one has $A^q[\rho; M] = A_{s-q}[\rho; M]$. For the general case note that (at least locally with respect to B) one has $A^q(Q_F; M) = \varinjlim A^q(Q'_F; M)$ where $\rho'\colon Q' \to B$ runs through the flat finite type models of ρ.

8. Fibrations

In this section we consider the spectral sequence associated to a morphism and formulate some basic functorial properties. A first application yields the homotopy property for vector bundles.

From now on all schemes are assumed to be of finite type over a field and M is (with exceptions in Section 14) a cycle module over X.

For a morphism $\rho\colon X \to X'$ we put

$$C_{p,l}(\rho) = \coprod_{x \in X_{(p,l)}} M(x) \subset C_l(X; M)$$

where

$$X_{(p,l)} = \left\{ x \in X_{(l)} \mid \dim\bigl(\rho(x), X'\bigr) \leq p \right\}.$$

Then

$$\cdots \subset C_{p-1,*}(\rho) \subset C_{p,*}(\rho) \subset \cdots \subset C_*(X; M)$$

is a finite filtration of $C_*(X; M)$ by subcomplexes. This filtration has the subquotients

$$\coprod_{u \in X'_{(p)}} C_*(X_{\kappa(u)}; M).$$

Let $\bigl(E^n_{p,q}(\rho)\bigr)_n$ be the associated spectral sequence (see e.g. Hilton and Stammbach 1971). The differential for X restricts on $C_*(X_{\kappa(u)}; M)$ to the differential for $X_{\kappa(u)}$. Therefore

$$E^1_{p,q}(\rho) = \coprod_{u \in X'_{(p)}} A_q(X_{\kappa(u)}; M).$$

(8.1) Proposition. *The differential $d^1_{p,q}$ of this spectral sequence equals the differential $d_{X'}$ for the cycle module $A_q[\rho; M]$.*

Proof: For $u' \in X'_{(p)}$, $y' \in X'_{(p-1)}$ we have to check equality of the corresponding components of $d^1_{p,q}$ and $d_{X'}$:

$$\bigl(d^1_{p,q}\bigr)^{u'}_{y'} = (d_{X'})^{u'}_{y'} : A_q(X_{\kappa(u')}; M) \to A_q(X_{\kappa(y')}; M).$$

The map $(d^1_{p,q})^{u'}_{y'}$ is by definition induced from the map

$$\Theta\colon C_q(X_{\kappa(u')}; M) \to C_q(X_{\kappa(y')}; M)$$

where $\Theta^u_y = \partial^u_y$ for u, $y \in X$ lying over u', y', respectively.
The claim follows from Proposition 7.2. $\square$

(8.2) COROLLARY. *There is a convergent spectral sequence*

$$E^2_{p,q}(\rho) = A_p(X'; A_q[\rho; M]) \Longrightarrow A_{p+q}(X; M).$$

If X' is equidimensional and ρ is flat, then there is a convergent spectral sequence

$$E_2^{p,q}(\rho) = A^p(X'; A^q[\rho; M]) \Longrightarrow A^{p+q}(X; M). \qquad \square$$

Here the second statement follows from the first by a formal switch to codimension index. In this codimension setting one may drop the finite type hypotheses.

(8.3) REMARK. We will use the following dictions. Let $\rho\colon X \to X'$, $\eta\colon Y \to Y'$ be morphisms. Then $\alpha\colon X \bullet\!\!\to Y$ is called filtration preserving (with respect to ρ, η) of degree (r,t), if

$$\alpha\big(C_{p,l}(\rho)\big) \subset C_{p+r,l+t}(\eta).$$

If $\delta(\alpha) = 0$ (see Sec. 3 for the definition of δ), then α is homomorphism of filtered complexes and induces maps (denoted by the same letter)

$$\alpha\colon E^n_{p,q}(\rho) \to E^n_{p+r,q+t-r}(\eta).$$

Two filtration preserving maps $\alpha, \beta\colon X \bullet\!\!\to Y$ of degree (r,t) are called homotopic, $\alpha \simeq \beta$, if there is a filtration preserving $H\colon X \bullet\!\!\to Y$ of degree $(r+1, t+1)$ such that $\alpha - \beta = \delta(H)$. If $\delta(\alpha) = \delta(\beta) = 0$ and $\alpha \simeq \beta$, then the induced maps on the E^2-terms coincide. If the homotopy H can be chosen of degree $(r, t+1)$, then already the induced maps on the E^1-terms coincide. This follows from a little calculation working for arbitrary filtered complexes.

Let

(8.4.0)
$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\rho} \\ Y' & \xrightarrow{f'} & X' \end{array}$$

be a commutative diagram of morphisms. The following statement is trivial.

(8.4) LEMMA.

(1) *One has*

$$f_*\big(C_{p,l}(\eta)\big) \subset C_{p,l}(\rho).$$

(2) *Suppose f has relative dimension t and let $s \geq s(f')$, see* (3.5.1). *Then*

$$f^*\big(C_{p,l}(\rho)\big) \subset C_{p+s,l+t}(\eta).$$

(3) *If a is a unit on X, then*

$$\{a\} \cdot \big(C_{p,l}(\rho)\big) \subset C_{p,l}(\rho).$$

(4) *Let* (Y, i, X, j, U) *be a boundary triple. Then*

$$\partial_Y^U\big(C_{p,l}(\rho \circ j)\big) \subset C_{p,l-1}(\rho \circ i).$$

(5) *For the diagram* (4.4.0) *one has*

$$\partial_Y^U\big(C_{p,l}(\bar{\bar{h}})\big) \subset C_{p-1,l-1}(\bar{h}). \qquad \square$$

Let

$$\begin{array}{ccccc} Y & \xrightarrow{\hat{f}} & \widehat{Y} & \xrightarrow{\bar{f}} & X \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \hat{\rho}} & & \downarrow{\scriptstyle \rho} \\ Y' & = & Y' & \xrightarrow{f'} & X' \end{array} \tag{8.5.0}$$

be the natural decomposition of diagram (8.4.0) with $\widehat{Y} = Y' \times_{X'} X$ and $f = \bar{f} \circ \hat{f}$. We call the diagram (8.4.0) a flat square if $\hat{f}$ and f' are flat of some constant relative dimensions. This holds then also for f.

We use the natural identification

$$A_q[\hat{\rho}; (\bar{f})^* M] = (f')^* A_q[\rho; M]$$

of cycle modules over Y'.

(8.5) Proposition.

(1) *If f and f' are proper, then the map*

$$f_*\colon E^2_{p,q}(\eta) \to E^2_{p,q}(\rho)$$

corresponding to (8.4.1) *equals the composite*

$$A_p(Y'; A_q[\eta; M]) \xrightarrow{[\hat{f}_*]_\#} A_p(Y'; A_q[\rho; M]) \xrightarrow{f'_*} A_p(X'; A_q[\rho; M]).$$

(2) *Suppose the square* (8.4.0) *is flat and put* $r = \dim(f')$, $s = \dim(\hat{f})$. *Then the map*

$$f^*\colon E^2_{p,q}(\rho) \to E^2_{p+r,q+s}(\eta)$$

corresponding to (8.4.2) *equals the composite*

$$A_p(X'; A_q[\rho; M]) \xrightarrow{(f')^*} A_{p+r}(Y'; A_q[\rho; M]) \xrightarrow{[\hat{f}^*]_\#} A_{p+r}(Y'; A_{q+s}[\eta; M]).$$

(3) *For a global unit a on X the map*

$$\{a\}\colon E^2_{p,q}(\rho) \to E^2_{p,q}(\rho)$$

corresponding to (8.4.3) *equals* $[\{a\}]_\#$.

(4) *The map*

$$\partial\colon E^2_{p,q}(\rho\circ j)\to E^2_{p,q-1}(\rho\circ i)$$

corresponding to (8.4.4) *equals* $[\partial^U_Y]_\#$.

(5) *The map*

$$\partial\colon E^2_{p,q}(\bar{\bar{h}})\to E^2_{p-1,q}(\bar h)$$

corresponding to (8.4.5) *equals the map*

$$\partial^{U'}_{Y'}\colon A_p(U';A_q[h;M])\to A_{p-1}(Y';A_q[h;M]).$$

Proof: (3) is trivial. (5) follows from Proposition 8.1. In (1) and (2) one may suppose either $f=\hat f$ or $f=\bar f$.

Proof of (1) for $f=\hat f$: Here $X'=Y'$ and the map

$$f_*\colon C_l(Y;M)\to C_l(X;M)$$

is the family of maps

$$(f_{\kappa(u)})_*\colon C_q(Y_{\kappa(u)};M)\to C_q(X_{\kappa(u)};M)$$

with $u\in X'_{(p)}$ and $p+q=l$. On the other hand

$$[\hat f_*]_\#\colon C_p(X';A_q[\eta;M])\to C_p(X';A_q[\rho;M])$$

is componentwise induced by the maps $(f_{\kappa(u)})_*$.

Proof of (1) for $f=\bar f$: Here we have a pull-back diagram, $Y=Y'\times_{X'}X$. We consider the maps induced by

$$f_*\colon C_l(Y;M)\to C_l(X;M)$$

on the E^1-terms. These are maps (with $p+q=l$)

$$\coprod_{y'\in Y'_{(p)}} C_q(X_{\kappa(y')};M)\longrightarrow \coprod_{x'\in X'_{(p)}} C_q(X_{\kappa(x')};M).$$

Their components are the corestrictions $c_{\kappa(y)|\kappa(x)}$ with $y\in\big(X_{\kappa(y')}\big)_{(q)}$, $y'\in Y'_{(p)}$ and $f(y)\in\big(X_{\kappa(x')}\big)_{(q)}$, $x'=f'(y')\in X_{(p)}$. Here $\kappa(y')$ is necessarily finite over $\kappa(x')$, since both fields have the same transcendence degree. Therefore the maps on the E^1-terms are given by the maps

$$(f'_{y'})_*\colon C_q(X_{\kappa(y')};M)\to C_q(X_{\kappa(f'(y'))};M)$$

with $y'\in Y'_{(p)}$ such that $\kappa(y')|\kappa(f'(y'))$ is finite and where

$$f'_{y'}\colon X_{\kappa(y')}\to X_{\kappa(f'(y'))}$$

is the associated finite morphism. On the other hand

$$f'_*\colon C_p(Y';A_q[\rho;M])\to C_p(X';A_q[\rho;M])$$

is induced exactly by the maps $f'_{y'}$.

Proof of (2) for $f = \hat{f}$: One argues as for (1) and notes that

$$f^*\colon C_l(X;M) \to C_{l+s}(Y;M)$$

is the family of maps

$$(f_{\kappa(u)})^*\colon C_q(X_{\kappa(u)};M) \to C_{q+s}(Y_{\kappa(u)};M)$$

with $u \in X'_{(p)}$ and $p+q=l$.

Proof of (2) for $f = \bar{f}$: The map

$$f^*\colon C_l(X;M) \to C_{l+r}(Y;M)$$

is the family of maps

$$[\mathcal{O}_Y, f]_y^x \cdot r_{\kappa(y)|\kappa(x)}\colon M(x) \to M(y)$$

with $y \in Y_{(l+r)}$, $x \in X_{(l)}$ and $f(y) = x$.

The map

$$(f')^*\colon C_p(X';A_q[\rho;M]) \to C_{p+r}(Y';A_q[\rho;M])$$

is the family of maps induced by the maps

$$[\mathcal{O}_{Y'}, f']_{y'}^{x'} \cdot (f'_{y'})^*\colon C_q(X_{\kappa(x')};M) \to C_q(X_{\kappa(y')};M)$$

with $y' \in Y'_{(p+r)}$, $x' \in X'_{(p)}$, $f'(y') = x'$ and where $f'_{y'}\colon X_{\kappa(y')} \to X_{\kappa(x')}$ is the natural map. Moreover $(f'_{y'})^*$ is the family of maps

$$[\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} \kappa(y'), f'_{y'}]_y^x \cdot r_{\kappa(x)|\kappa(y)}\colon M(x) \to M(y)$$

with $y \in Y_{(p+q+r)}$ lying over $x \in X_{(p+q)}$ and over y'.

The claim amounts to show for such points y', x', y, x the equality

$$l\big((\kappa(x) \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{Y'})_{(y)}\big) = l\big((\kappa(x') \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{Y'})_{(y')}\big) \cdot l\big((\kappa(x) \otimes_{\mathcal{O}_{X'}} \kappa(y'))_{(y)}\big).$$

For this see (Fulton 1984, A.4.1).

Proof of (4): The map

$$\partial_Y^U\colon C_l(U;M) \to C_{l-1}(Y;M)$$

is on the subquotients of the filtrations given by the family of maps

$$\partial_u\colon C_q(U_{\kappa(u)};M) \to C_{q-1}(Y_{\kappa(u)};M)$$

with $u \in X'_{(p)}$, $p+q=l$ and where ∂_u is the boundary map for the closed immersion $Y_{\kappa(u)} \to X_{\kappa(u)}$. On the other hand

$$[\partial_Y^U]_\#\colon C_p\,(X';A_q[\rho \circ j;M]) \to C_p\,(X';A_q[\rho \circ i;M])$$

is componentwise induced by the maps ∂_u. $\square$

By an affine bundle of dimension n we mean a bundle $\pi\colon V \to X$ which is locally on X isomorphic to $X \times \mathbb{A}^n \to X$ with affine transition maps. (In applications we are mainly interested in the special case of vector bundles.)

A first application of the spectral sequence is

(8.6) PROPOSITION. *Let $\pi\colon V \to X$ be an affine bundle of dimension n. Then*

$$\pi^*\colon A_p(X;M) \to A_{p+n}(V;M)$$

is bijective for all p. If X is equidimensional, then

$$\pi^*\colon A^p(X;M) \to A^p(V;M)$$

is bijective for all p.

Here again the second statement follows from the first and one may drop in the codimension setting the finite type hypothesis.

Proof: By Corollary 8.2 and Proposition 8.5.2 applied to $Y' = X' = X$, $Y = V$, $f = \hat{f} = \pi$, all we need to show is

$$A_q[\pi;M] = 0 \quad \text{for} \quad q \neq n$$

and that

$$[\pi^*]\colon M = A_0[\mathrm{id}_X;M] \to A_n[\pi;M]$$

is an isomorphism of cycle modules over X.

Therefore we are reduced to the case $X = \operatorname{Spec} F$. Then V is a trivial bundle, $V = \mathbb{A}^n_F$. For $n = 1$ the claim is (H) of Section 2. So we know Proposition 8.6 for line bundles over an arbitrary base. But then the case $V = \mathbb{A}^n_F$ follows by induction on n. □

9. Homotopy

We have just observed the homotopy property for affine bundles. In this section we show that this fact can be made more precise on cycle level by means of a homotopy inverse.

A homomorphism $\alpha\colon X \bullet\!\!\to Y$ with $\delta(\alpha)=0$ is called a strong homotopy equivalence if there is $r\colon Y \bullet\!\!\to X$ and $H\colon Y \bullet\!\!\to Y$ such that

$$\delta(r)=0, \tag{9.0.1}$$
$$r\circ\alpha=\mathrm{id}, \tag{9.0.2}$$
$$H\circ\alpha=0, \tag{9.0.3}$$
$$\delta(H)=\mathrm{id}-\alpha\circ r. \tag{9.0.4}$$

The pair (r,H) will be called h-data for α.

Let $\pi\colon V\to X$ be an affine bundle. We will show that $\pi^*\colon X \bullet\!\!\to V$ is a strong homotopy equivalence. A crucial point here is the treatment of the case $V=X\times\mathbb{A}^1$. The general case is then more or less clear in view of the decomposition of the cycle complexes corresponding to boundary triples. We give here explicit formulas in order to make clear compatibility with base change and filtrations.

By a *coordination* $\tau=(X_i,\tau_i)$ of an affine bundle $\pi\colon V\to X$ of dimension n we mean a sequence $\varnothing=X_0\subset X_1\subset\cdots\subset X_k=X$ of closed subsets of X together with trivializations

$$\tau_i\colon V\mid(X_i\setminus X_{i-1})\to(X_i\setminus X_{i-1})\times\mathbb{A}^n.$$

(We use the notation $V\mid U=V\times_W U$ for $U\subset W$ and a scheme $V\to W$ over W). Coordinations clearly exist since X is noetherian. For a morphism $f\colon Y\to X$ we denote by $f^*\tau$ the induced coordination on the pull-back bundle f^*V.

In the following we construct in several steps h-data

$$r(\tau)\colon C_p(V;M)\to C_{p-n}(X;M),$$
$$H(\tau)\colon C_p(V;M)\to C_{p+1}(V;M)$$

for π^* depending on a coordination τ.

(9.1) The case $V=X\times\mathbb{A}^1$. h-data (r,H) for $\pi^*\colon X \bullet\!\!\to X\times\mathbb{A}^1$ are given by the composites

$$r\colon X\times\mathbb{A}^1 \xrightarrow{j^*} X\times(\mathbb{A}^1\setminus\{0\}) \xrightarrow{\{-1/t\}} X\times(\mathbb{A}^1\setminus\{0\}) \xrightarrow{\partial_\infty} X,$$

$$H\colon X\times\mathbb{A}^1 \xrightarrow{p_2^*} X\times(\mathbb{A}^1\times\mathbb{A}^1\setminus\Delta) \xrightarrow{\{s-t\}} X\times(\mathbb{A}^1\times\mathbb{A}^1\setminus\Delta) \xrightarrow{p_{1*}} X\times\mathbb{A}^1.$$

Here t is the coordinate of $\mathbb{A}^1=\operatorname{Spec}\mathbb{Z}[t]$ and s, t are the coordinates of $\mathbb{A}^1\times\mathbb{A}^1=\operatorname{Spec}\mathbb{Z}[s]\times\operatorname{Spec}\mathbb{Z}[t]$. Moreover $\Delta=\{s-t=0\}$ is the diagonal, j is the standard inclusion, p_1 and p_2 are given by the standard projections and ∂_∞ is induced by $X=X\times\infty\subset X\times(\mathbb{P}^1\setminus\{0\})$ with open complement $X\times(\mathbb{A}^1\setminus\{0\})$.

We have to verify for (r,H) the defining properties of h-data. (9.0.1) and (9.0.3) are immediate and (9.0.2) follows from Lemma 4.5. To check (9.0.4) consider the decomposition

$$(p_1)_*\colon X\times(\mathbb{A}^1\times\mathbb{A}^1\setminus\Delta) \overset{q_*}{\bullet\!\!\longrightarrow} X\times\mathbb{A}^1\times\mathbb{P}^1 \overset{\bar p_{1*}}{\bullet\!\!\longrightarrow} X\times\mathbb{A}^1$$

where q is the inclusion and $\bar p_1$ is the projection. $\bar p_1$ is proper and therefore

$$\delta(H)=(\bar p_1)_*\circ\delta(q_*)\circ\{s-t\}\circ p_2^*.$$

Moreover

$$\delta(q_*)=(i_\Delta)_*\circ\partial_\Delta+(i_\infty)_*\circ\partial_\infty$$

where $i_\Delta\colon X\times\Delta\to X\times\mathbb{A}^1\times\mathbb{P}^1$, $i_\infty\colon X\times\mathbb{A}^1\times\infty\to X\times\mathbb{A}^1\times\mathbb{P}^1$ are the inclusions and ∂_Δ, ∂_∞ are the boundary maps for $X\times\Delta\to X\times\mathbb{A}^1\times\mathbb{A}^1$, $X\times\mathbb{A}^1\times\infty\to X\times(\mathbb{A}^1\times\mathbb{P}^1\setminus\Delta)$, respectively.

Since $s-t$ is a parameter for Δ one finds

$$(\bar p_1)_*\circ(i_\Delta)_*\circ\partial_\Delta\circ\{s-t\}\circ p_2^*=\mathrm{id}$$

by Lemma 4.5.

Let $W=\mathbb{A}^1\times\mathbb{P}^1\setminus(\Delta\cup\mathbb{A}^1\times 0)$. Moreover let $\tilde p_2$ be the restriction of p_2 to $U=X\times(W\setminus\mathbb{A}^1\times\infty)$ and let $\tilde\partial_\infty$ be the boundary map corresponding to the inclusion $X\times\mathbb{A}^1\times\infty\to X\times W$. Then

$$\partial_\infty\circ\{s-t\}\circ p_2^*=\tilde\partial_\infty\circ\{s-t\}\circ\tilde p_2^*.$$

Since $(s-t)/(-t)$ is a unit on W with constant value 1 on $X\times\mathbb{A}^1\times\infty$ one has

$$\tilde\partial_\infty\circ\{s-t\}\circ\tilde p_2^*=\tilde\partial_\infty\circ\{-t\}\circ\tilde p_2^*.$$

The compatibility of the boundary maps with flat pull-back now gives

$$(\bar p_1)_*\circ(i_\infty)_*\circ\partial_\infty\circ\{s-t\}\circ p_2^*=-\pi^*\circ r.$$

Putting things together yields (9.0.4).

(9.2) The case $V=X\times\mathbb{A}^n$. Let $\pi_n\colon X\times\mathbb{A}^n\to X$ be the projection and put $\pi_X^n=\pi_n^*\colon X\bullet\!\!\to X\times\mathbb{A}^n$. By induction on n we define h-data (r_X^n,H_X^n) for π_X^n. Let $Y=X\times\mathbb{A}^1$ so that $Y\times\mathbb{A}^{n-1}=X\times\mathbb{A}^n$. Note that $\pi_X^n=\pi_Y^{n-1}\circ\pi^*$ where $\pi\colon Y\to X$ is the projection. Put

$$\begin{aligned} r_X^n&=r\circ r_Y^{n-1},\\ H_X^n&=H_Y^{n-1}+\pi_Y^{n-1}\circ H\circ r_Y^{n-1}. \end{aligned}$$

Here (r,H) are the h-data for π^* from (9.1); moreover, $r_Y^0=\pi_Y^0=\mathrm{id}_Y^*$ and $H_Y^0=0$.

The properties (9.0.1) and (9.0.2) can be easily verified. For (9.0.3) note that

$$\begin{aligned} H_X^n \circ \pi_X^n &= H_X^n \circ \pi_Y^{n-1} \circ \pi^* \\ &= (H_Y^{n-1} \circ \pi_Y^{n-1}) \circ \pi^* + \pi_Y^{n-1} \circ H \circ (r_Y^{n-1} \circ \pi_Y^n) \circ \pi^* \\ &= 0 + \pi_Y^{n-1} \circ (H \circ \pi^*) = 0. \end{aligned}$$

Finally (9.0.4) follows from

$$\begin{aligned} \delta(H_X^n) &= \delta(H_Y^{n-1}) + \pi_Y^{n-1} \circ \delta(H) \circ r_Y^{n-1} \\ &= 1 - \pi_Y^{n-1} \circ r_Y^{n-1} + \pi_Y^{n-1} \circ (1 - \pi^* \circ r) \circ r_Y^{n-1} \\ &= 1 - \pi_X^n \circ r_Y^n. \end{aligned}$$

(9.3) Glueing. Let $\pi\colon V \to X$ be an affine bundle, let $Y \subset X$ be closed, let $U = X \setminus Y$ and put $V' = V \mid Y$, $V'' = V \mid U$. For given h-data (r', H') for $(\pi \mid V')^*\colon Y \bullet\!\!\to V'$ and (r'', H'') for $(\pi \mid V'')^*\colon U \bullet\!\!\to V''$ we define h-data (r, H) for $\pi^*\colon X \bullet\!\!\to V$ by the formulae:

$$r = \begin{pmatrix} r' & -r' \circ \partial \circ H'' \\ 0 & r'' \end{pmatrix}, \qquad H = \begin{pmatrix} H' & -H' \circ \partial \circ H'' \\ 0 & H'' \end{pmatrix}.$$

Here the matrix notation refers to the natural decompositions

$$\begin{aligned} C_*(X; M) &= C_*(Y; M) \oplus C_*(U; M), \\ C_*(V; M) &= C_*(V'; M) \oplus C_*(V''; M). \end{aligned}$$

Moreover $\partial\colon V'' \bullet\!\!\to V'$ is the boundary map corresponding to $V' \subset V$. The verification of (9.0.1)–(9.0.4) is straightforward and omitted.

(9.4) The general case. Given a coordination τ one uses iteratively the recipe of (9.3) to construct h-data $\big(r(\tau), H(\tau)\big)$ for π^*.

It turns out that the glueing process of (9.3) is "associative" in the sense that $\big(r(\tau), H(\tau)\big)$ does not depend on the ordering in which the different pieces are glued together. However, this is not at all important for us; one should just decide oneself for some fixed standard ordering.

(9.5) Functoriality. The construction of $\big(r(\tau), H(\tau)\big)$ is compatible with manipulations on the base given by the four types of maps f_*, g^*, $\{a\}$ and ∂. We omit a formulation, since this will be used only in the trivial case of open immersions g.

(9.6) Proposition.

(1) *Let $\pi\colon V \to X$ be an affine bundle of dimension n with coordination τ and let $\rho\colon X \to X'$ be a morphism. Then*

$$\begin{aligned} r(\tau)\big(C_{p,l}(\rho \circ \pi)\big) &\subset C_{p,l-n}(\rho), \\ H(\tau)\big(C_{p,l}(\rho \circ \pi)\big) &\subset C_{p,l+1}(\rho). \end{aligned}$$

(2) *Let*

$$\begin{array}{ccc} V & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\rho} \\ V' & \xrightarrow{\pi'} & X' \end{array}$$

be a pull-back diagram with π' an affine bundle of dimension n and let τ' be a coordination for π'. Then

$$r(\rho^*\tau')\big(C_{p,l}(\eta)\big) \subset C_{p-n,l-n}(\rho),$$
$$H(\rho^*\tau')\big(C_{p,l}(\eta)\big) \subset C_{p+1,l+1}(\rho).$$

Proof: This is straightforward (but nevertheless tedious) by following the constructions. □

In order to define h-data as above one needs less than the choice of a coordination. For example, in (9.2) one refers alone to trivializations of the one-dimensional bundles $X \times \mathbb{A}^{m+1} \to X \times \mathbb{A}^m$.

We have not tried to describe the precise amount of information of a coordination needed in order to perform the above construction.

10. DEFORMATION TO THE NORMAL CONE

This section describes three general constructions associated to closed imbeddings: the normal cone, the deformation space and the double deformation space.

For the general role of the deformation space in intersection theory, we refer to (Fulton 1984). The double deformation space will be our tool to verify associativity of the intersection operations.

We first fix notations and describe some significant properties. Explicit descriptions are given in (10.3)–(10.5) below.

Let $Z \to Y \to X$ be closed imbeddings.

The normal cone to Y in X is denoted by $N = N_Y X = N(X,Y)$. There is the projection $N_Y X \to Y$ and the inclusion $Y \to N_Y X$. If $Y \to X$ is a regular imbedding, then $N_Y X$ is a vector bundle over Y with the inclusion as zero section.

The deformation space $D = D(X,Y)$ is a scheme over $X \times \mathbb{A}^1$. It is flat over $\mathbb{A}^1$. Over $\mathbb{A}^1 \setminus \{0\} \subset \mathbb{A}^1$ one has

$$D \mid (\mathbb{A}^1 \setminus \{0\}) = X \times (\mathbb{A}^1 \setminus \{0\}).$$

Furthermore the projection $D \mid \{0\} \to X \times \{0\}$ factors through $Y \to X \times \{0\}$ and one has

$$D \mid \{0\} = N_Y X$$

as schemes over Y. (Our D is in Fulton 1984, Chap. 5 denoted by M^0; moreover we have taken 0 instead of ∞ as the basepoint of the special fiber.)

The double deformation space $\overline{D} = \overline{D}(X,Y,Z)$ is a scheme over $X \times \mathbb{A}^2$. It is flat over $\mathbb{A}^2$ and one has the following canonical identifications of schemes over $\mathbb{A}^2$, assuming in (10.0.3)–(10.0.5) that $Z \to Y \to X$ are regular imbeddings.

$$\overline{D} \mid \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\}) = D(X,Y) \times (\mathbb{A}^1 \setminus \{0\}), \tag{10.0.1}$$

$$\overline{D} \mid (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1 = (\mathbb{A}^1 \setminus \{0\}) \times D(X,Z), \tag{10.0.2}$$

$$\overline{D} \mid \mathbb{A}^1 \times \{0\} = D(N_Z X, N_Z Y), \tag{10.0.3}$$

$$\overline{D} \mid \{0\} \times \mathbb{A}^1 = D(N_Y X, N_Y X \mid Z). \tag{10.0.4}$$

Moreover the projection $\overline{D} \mid \{(0,0)\} \to X \times \{(0,0)\}$ factors through $Z \to X$ and one has

$$\overline{D} \mid \{(0,0)\} = N(N_Z X, N_Z Y) = N(N_Y X, N_Y X \mid Z) \tag{10.0.5}$$

as schemes over Z.

There is a more symmetric but less general version of the double deformation space. Namely, let Y, Y' be closed subschemes of X and let Z be the intersection of Y and Y', i.e., $Z = Y \times_X Y'$. Then there is a double deformation space $\widetilde{D} = \widetilde{D}(X;Y,Y') \to X \times \mathbb{A}^2$ relating (in the transversal case) all five inclusions induced from $Z \subset Y, Y' \subset X$. The deformation space $\widetilde{D}$ is flat over $\mathbb{A}^2$ and symmetric with respect to a simultaneous interchange of Y, Y' and of the factors of $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$. Suppose that Y and Y' meet transversally. Then

$$\widetilde{D} \mid (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1 = D(X,Y) \times \mathbb{A}^1,$$
$$\widetilde{D} \mid \{0\} \times \mathbb{A}^1 = D(N_Y X, N_Y X \mid Z).$$

Moreover one has $\widetilde{D} \mid L = D(X,Z)$ for any line $L \subset \mathbb{A}^2$ through the origin as long as L is different from the two axes.

In the case $Z = Y \times_X Y'$, the space $\overline{D}(X,Y,Z)$ is the pull-back of the space $\widetilde{D}(X;Y,Y')$ along the affine blow up $\mathbb{A}^2 \to \mathbb{A}^2$, $(t,s) \to (ts,s)$. We don't need $\widetilde{D}$, but we have included below its definition, since it might be a bit simpler to understand than $\overline{D}$.

We have to recall facts from local algebra. Remark 10.1 is a special version of the local criterion of flatness (Matsumura 1980, (20.G), p. 152). Remark 10.2 may be deduced by considering locally regular sequences for J containing regular sequences for I (see Serre 1957). For a compact account of other facts needed in the following we refer to (Fulton 1984, App. A, App. B).

(10.1) Remark. A morphism $U \to V \times \mathbb{A}^1$ is flat if and only if the morphisms $U \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus \{0\}) \to V \times (\mathbb{A}^1 \setminus \{0\})$, $U \times_{\mathbb{A}^1} \{0\} \to V \times \{0\}$ and $U \to \mathbb{A}^1$ are flat. $\square$

(10.2) Remark. If $Z \to Y$ and $Y \to X$ are regular imbeddings, then $Z \to X$ is a regular imbedding. If X is affine, and if $X = \operatorname{Spec} A$ and $I \subset J \subset A$ are the ideals corresponding to Y and Z, respectively, then

$$I^n J^m \cap J^{m+n+1} = I^n J^{m+1}, \tag{1}$$

$$I^n J^m \cap I^{n+1} = I^{n+1} J^{m-1}. \tag{2}$$

Here we understand n, $m \in \mathbb{Z}$ with $I^n = J^n = A$ for $n \leq 0$. □

We next give the definitions of N, D, $\overline{D}$ and $\widetilde{D}$ for affine X. From the naturality of the affine constructions it will be obvious that they extend to global ones.

We keep the notations of Remark 10.2. Moreover we use the coordinates $\mathbb{A}^1 = \operatorname{Spec} k[t]$ and $\mathbb{A}^2 = \operatorname{Spec} k[t,s]$. The indices n, m always run in $\mathbb{Z}$.

(10.3) The normal cone. $N = N_Y X$ is defined as the spectrum of the ring

$$O_N = \coprod_n I^n / I^{n+1}.$$

O_N is a ring over $O_Y = A/I$ and projection to the degree zero summand gives a homomorphism $O_N \to O_Y$.

(10.4) The deformation space. $D = D(X,Y)$ is defined as the spectrum of the subring

$$O_D = \sum_n I^n \cdot t^{-n} \subset A[t, t^{-1}].$$

O_D is a finitely generated ring over $A[t]$ (with generators $x_i t^{-1}$ if x_i are generators of I). After inverting t one has

$$O_D[t^{-1}] = A[t, t^{-1}].$$

Since t is not a zero divisor, it follows that O_D is flat over $k[t]$. Moreover

$$O_D \,/\, t \cdot O_D = \coprod_n I^n / I^{n+1} = O_N.$$

For later purposes we are very precise about this identification: for $x \in I^n$ the residue of $x \cdot t^{-n} \bmod I^{n+1} \cdot t^{-n}$ corresponds to $(-1)^n x \bmod I^{n+1}$. (This sign convention will avoid some other signs later on.)

(10.5) The double deformation space. $\overline{D} = \overline{D}(X,Y,Z)$ is the spectrum of the subring

$$O_{\overline{D}} = \sum_{n,m} I^n J^{m-n} \cdot t^{-n} s^{-m} \subset A[t, s, t^{-1}, s^{-1}].$$

$O_{\overline{D}}$ is finitely generated over $A[s,t]$. After inverting s or t one has (with $D' = D(X,Z)$)

$$O_{\overline{D}}[s^{-1}] = \sum_{n,m} I^n \cdot t^{-n} s^{-m} = O_D[s, s^{-1}],$$

$$O_{\overline{D}}[t^{-1}] = \sum_{n,m} J^m \cdot t^{-n} s^{-m} = O_{D'}[t, t^{-1}].$$

This shows (10.0.1) and (10.0.2). For (10.0.3) note first

$$O_{\overline{D}} \,/\, s \cdot O_{\overline{D}} = \coprod_{n,m} [I^n J^{m-n} / I^n J^{m-n+1}] \cdot t^{-n} s^{-m}.$$

In order to make clear the ring structures (in particular as ring over $k[t,s]$) we keep here the terms $t^{-n}s^{-m}$, having now merely the meaning of symbols.

Moreover $N_Z X$ and $N_Z Y$ are the spectra of

$$R = \coprod_m [J^m / J^{m+1}] \cdot s^{-m},$$

$$R' = \coprod_m [(J^m + I)/(J^{m+1} + I)] \cdot s^{-m},$$

The projection $R \to R'$ yields an inclusion $N_Z Y \to N_Z X$.

Let $\widetilde{I} = \ker(R \to R')$. By Remark 10.2.2 one has $J^m \cap I \subset I \cdot J^{m-1}$ and therefore

$$\widetilde{I} = \coprod_m [(I \cdot J^{m-1} + J^{m+1})/J^{m+1}] \cdot s^{-m}$$

and

$$\widetilde{I}^n = \coprod_m [(I^n \cdot J^{m-n} + J^{m+1})/J^{m+1}] \cdot s^{-m}.$$

Hence $D(N_Z X, N_Z Y)$ is the spectrum of

$$\coprod_n \widetilde{I}^n = \coprod_{n,m} [(I^n \cdot J^{m-n} + J^{m+1})/J^{m+1}] \cdot t^{-n} s^{-m}.$$

(10.0.3) follows now from Remark 10.2.1.

For (10.0.4) note first

$$O_{\overline{D}} \,/\, t \cdot O_{\overline{D}} = \coprod_{n,m} [I^n J^{m-n} / I^{n+1} J^{m-n-1}] \cdot t^{-n} s^{-m}.$$

For the ring of $N_Y X$ we write now

$$O_N = \coprod_n [I^n / I^{n+1}] \cdot u^{-n}.$$

Let $\widetilde{J} \subset O_N$ be the ideal corresponding to the closed subscheme $N_Y X \mid Z$. Its powers are

$$\widetilde{J}^m = \coprod_n [(J^m I^n + I^{n+1})/I^{n+1}] \cdot u^{-n}.$$

Hence $D(N_Y Z, N_Y X \mid Z)$ is the spectrum of

$$R'' = \coprod_{n,m} [(J^m I^n + I^{n+1})/I^{n+1}] \cdot u^{-n} s^{-m}.$$

Define

$$\varphi\colon O_{\overline{D}} \,/\, t \cdot O_{\overline{D}} \to R'',$$
$$(x \bmod I^{n+1} J^{m-n-1}) \cdot t^{-n} s^{-m} \to (x \bmod I^{n+1}) \cdot u^{-n} s^{-m+n}.$$

It is easy to see that φ is a surjective ring homomorphism over $k[s]$. Moreover φ is injective by Remark 10.2.1. The map φ gives the identification of (10.0.4). Now (10.0.5) is obvious. The flatness over $\mathbb{A}^2$ (not needed in the following) may be deduced from Remark 10.1.

(10.6) The double deformation space $\widetilde{D}$. We just give the definition. Let $I' \subset A$ be the ideal corresponding to $Y' \to X$. One puts

$$O_{\widetilde{D}} = \sum_{n,m} I^n I'^m \cdot v^{-n} s^{-m} \subset A[v, s, v^{-1}, s^{-1}].$$

One may handle with $\widetilde{D}$ similar as with $\overline{D}$ in (10.5). In the transversal case one has

$$O_{\overline{D}} = O_{\widetilde{D}} \otimes_{k[v,s]} k[t, s]$$

where $k[v, s] \subset k[t, s]$ via $v \to ts$, $s \to s$ and with $J = I + I'$.

11. The Basic Construction

For a closed immersion $i\colon Y \to X$ we define

$$J(i) = J(X, Y)\colon X \bullet\!\!\to N_Y X$$

as the composite of

$$X \xrightarrow{\pi^*} X \times (\mathbb{A}^1 \setminus \{0\}) \xrightarrow{\{t\}} X \times (\mathbb{A}^1 \setminus \{0\}) \xrightarrow{\partial} N_Y X.$$

Here $\pi\colon X \times (\mathbb{A}^1 \setminus \{0\}) \to X$ is the projection, $\mathbb{A}^1 = \operatorname{Spec} k[t]$ and ∂ is the boundary map for $N_Y X \to D(X, Y)$. One has $\delta\big(J(X, Y)\big) = 0$ so that $J(X, Y)$ is a homomorphism of complexes $C_*(X; M) \to C_*(N_Y X; M)$.

If $M = K_*$, then the restriction of $J(X, Y)$ to the classical cycle groups coincides with the specialization homomorphisms σ of (Fulton 1984, Chap. 5.2); this may be deduced from the description of σ in (Fulton 1984, Prop. 5.2) via Cartier divisors. As for classical cycles, one may think of $J(X, Y)$ as the pull-back along tubular neighborhoods followed by a linearization process. In the following we have collected the remarks on $J(X, Y)$ which are needed in further sections. We have not tried to give a detailed geometrical description.

The construction of $J(X, Y)$ is local in the sense that

$$J(U, Y \cap U) \circ j^* = \tilde{\jmath}^* \circ J(X, Y)$$

where $j\colon U \to X$ is an open immersion and $\tilde{\jmath}\colon N_Y X \mid (Y \cap U) \to N_Y X$ is the induced inclusion.

(11.1) Lemma. *Let $\sigma\colon Y \to N_Y X$ be the inclusion. Then*

$$J(X, Y) \circ i_* = \sigma_*.$$

Proof: The statement follows from Lemma 4.5 and the fact that the closure of $Y \times (\mathbb{A}^1 \setminus \{0\})$ in $D(X, Y)$ is $Y \times \mathbb{A}^1$. □

Let X be normal, $y \in X^{(1)}$ and $Y = \overline{\{y\}}$. Moreover let F and E be the function fields of X and $N_Y X$, respectively. We want to compute the codimension 0 component

$$J^0\colon M(F) \to M(E)$$

of $J(X,Y)$. The problem is purely local in y. Let v be the valuation on F corresponding to y and let $\kappa = \kappa(y) = \kappa(v)$. Moreover let $\mathfrak{m}$ be the ideal of y, let $\pi \in \mathfrak{m}$ be a prime and let $\bar{\pi} \in \mathfrak{m}/\mathfrak{m}^2$ be its image. The normal cone $N_Y X$ is the spectrum of

$$\kappa[\bar{\pi}] = \coprod_n \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

and one has $E = \kappa(\bar{\pi})$.

The following lemma shows that there is a factorization

$$J^0\colon M(F) \xrightarrow{p} M(v) \to M(E)$$

where p is from Remark 1.6.

(11.2) Lemma.

$$J^0 = r_{E|\kappa} \circ s_v^{\pi} + \{\bar{\pi}\} \cdot r_{E|\kappa} \circ \partial_v.$$

Proof: We may suppose $X = \operatorname{Spec} A$ and that the ideal I corresponding to y is generated by π. Then $D(X,Y)$ is the spectrum of

$$A[t, \pi t^{-1}] \subset A[t, t^{-1}].$$

By definition we have

$$J^0 = \partial_w \circ \{t\} \circ r_{F(t)|F}$$

where w is the valuation on $F(t)$ corresponding to the principal ideal $t \cdot A[t, \pi t^{-1}]$. Note that $E = \kappa(w)$ and that $\bar{\pi}$ is the residue of the w-unit $-\pi t^{-1}$ (by the sign convention in 10.4). The claim follows now from

$$\begin{aligned}\partial_w \circ \{t\} \circ r_{F(t)|F}(\rho) &= \partial_w \circ \{-\pi\} \circ r_{F(t)|F}(\rho) - \partial_w \circ \{-\pi t^{-1}\} \circ r_{F(t)|F}(\rho) \\ &= \partial_w \circ r_{F(t)|F}(\{-\pi\} \cdot \rho) + \{\bar{\pi}\} \circ \partial_w \circ r_{F(t)|F}(\rho)\end{aligned}$$

and the fact that w restricts on F to v. $\square$

The preceding remarks yield a complete description of $J(X,Y)$ for smooth curves X.

The rest of the section contains a series of technical lemmata.

(11.3) Lemma. *Let $Y \to X$ be a closed immersion, let $g\colon V \to X$ be flat (of constant relative dimension) and let*

$$N(g)\colon N(V, Y \times_X V) = N(X,Y) \times_X V \to N(X,Y)$$

be the projection. Then

$$J(V, Y \times_X V) \circ g^* = N(g)^* \circ J(X,Y).$$

Proof: This follows from the flatness of $D(V, Y \times_X V) \to D(X,Y)$. $\square$

(11.4) Lemma. *Let $U \to V$ be a closed immersion and let $p\colon V \to W$ be flat. Suppose that the composite*

$$q\colon N_U V \to U \to V \to W$$

is flat of the same relative dimension as p. Then

$$J(V,U) \circ p^* = q^*\colon W \bullet\!\!\to N_U V.$$

Proof: Let $\pi\colon W \times (\mathbb{A}^1 \setminus \{0\}) \to W$ be the projection and let f be the composite

$$f\colon D(V,U) \longrightarrow V \times \mathbb{A}^1 \xrightarrow{p \times \mathrm{id}} W \times \mathbb{A}^1.$$

Then, by definition,

$$J(V,U) \circ p^* = \partial \circ \{t\} \circ \big(f \mid V \times (\mathbb{A}^1 \setminus \{0\})\big)^* \circ \pi^*.$$

Now f is flat by Remark 10.1 and $f \mid N_U V = q$. Hence

$$J(V,U) \circ p^* = q^* \circ \partial' \circ \{t\} \circ \pi^*$$

where ∂' is the boundary map corresponding to $W \times \{0\} \hookrightarrow W \times \mathbb{A}^1$. But $\partial' \circ \{t\} \circ \pi^* =$ id by Lemma 4.5. $\square$

(11.5) Lemma. *Let $U \to V$ be a regular imbedding, let $p\colon V \to W$ be smooth of constant relative dimension and suppose $p \circ i$ is a regular imbedding. Then the projection*

$$q\colon N_U V \to N_U W$$

is an epimorphism of vector bundles and

$$J(V,U) \circ p^* = q^* \circ J(W,U).$$

Proof: Use the flatness of $D(V,U) \to D(W,U)$. $\square$

(11.6) Lemma. *Let $\rho\colon T \to T'$ be a morphism, let $T_1', T_2' \subset T'$ be closed subschemes and let $T_i = T \times_{T'} T_i'$ for $i = 1,2$.*

Put $T_3 = T \setminus (T_1 \cup T_2)$, $T_0 = T_1 \cap T_2$, $\widetilde{T}_1 = T_1 \setminus T_0$, $\widetilde{T}_2 = T_2 \setminus T_0$ and let ∂_1^3, ∂_0^1, ∂_2^3, ∂_0^2 be the boundary maps for the closed immersions

$$\widetilde{T}_1 \to T \setminus T_2, \quad T_0 \to T_1, \quad \widetilde{T}_2 \to T \setminus T_1, \quad T_0 \to T_2,$$

respectively. Then

$$0 \simeq \partial_0^1 \circ \partial_1^3 + \partial_0^2 \circ \partial_2^3\colon T_3 \bullet\!\!\to T_0$$

under a filtration preserving homotopy of degree $(-1,-1)$.

Proof: Corresponding to the set theoretic decomposition of T we have

$$C_*(T;M) = C_*(T_0;M) \oplus C_*(\widetilde{T}_1;M) \oplus C_*(\widetilde{T}_2;M) \oplus C_*(T_3;M).$$

Let

$$\partial_0^3 \colon T_3 \hookrightarrow T_0$$

be the corresponding component of d_T. Then $d_T \circ d_T = 0$ gives

$$\delta(\partial_0^3) + \partial_0^1 \circ \partial_1^3 + \partial_0^2 \circ \partial_2^3 = 0.$$

Hence $-\partial_0^3$ is a homotopy as required. □

Let $T = \overline{D} = \overline{D}(X,Y,Z)$, $T_1 = \overline{D}\,|\,(\{0\} \times \mathbb{A}^1)$, $T_2 = \overline{D}\,|\,(\mathbb{A}^1 \times \{0\})$. We keep the notations of Lemma 11.6. Then $T_3 = X \times (\mathbb{A}^1 \setminus \{0\}) \times (\mathbb{A}^1 \setminus 0)$ and $T_0 = \overline{D}\,|\,\{(0,0)\}$. Let $\pi\colon T_3 \to X$ be the projection and let t, s be the coordinates of $\mathbb{A}^2$ (as in (10.5), so that $T_1 = \{t = 0\}$, $T_2 = \{s = 0\}$).

(11.7) Lemma. *Let $Z \to Y \to X$ be regular imbeddings. Then*

$$\partial_0^1 \circ \partial_1^3 \circ \{t,s\} \circ \pi^* = J(N_Y X, N_Y X \mid Z) \circ J(X,Y),$$
$$\partial_0^2 \circ \partial_2^3 \circ \{s,t\} \circ \pi^* = J(N_Z X, N_Z Y) \circ J(X,Z).$$

Proof: Let

$$\pi_i\colon X \times (\mathbb{A}^1 \setminus \{0\}) \times (\mathbb{A}^1 \setminus \{0\}) \to X \times (\mathbb{A}^1 \setminus \{0\}), \quad \pi^i\colon X \times (\mathbb{A}^1 \setminus \{0\}) \to X$$

be the projections with

$$\pi_1(x,t,s) = (x,s), \quad \pi^1(x,s) = x,$$
$$\pi_2(x,t,s) = (x,t), \quad \pi^2(x,t) = x.$$

One finds (using in particular Lemma 11.3):

$$\begin{aligned}
\partial_0^1 \circ \partial_1^3 \circ \{t,s\} \circ \pi^* &= \partial_0^1 \circ \{s\} \circ \partial_1^3 \circ \{t\} \circ \pi_1^* \circ \pi^{1*} \\
&= \partial_0^1 \circ \{s\} \circ J\big(X \times (\mathbb{A}^1 \setminus 0); Y \times (\mathbb{A}^1 \setminus \{0\})\big) \circ \pi^{1*} \\
&= \partial_0^1 \circ \{s\} \circ \big(N_Y X \times (\mathbb{A}^1 \setminus \{0\}) \to N_Y X\big)^* \circ J(X,Y) \\
&= J(N_Y X, N_Y X \mid Z) \circ J(X,Y),
\end{aligned}$$

$$\begin{aligned}
\partial_0^2 \circ \partial_2^3 \circ \{s,t\} \circ \pi^* &= \partial_0^2 \circ \{t\} \circ \partial_1^2 \circ \{s\} \circ \pi_2^* \circ \pi^{2*} \\
&= \partial_0^2 \circ \{t\} \circ J\big(X \times (\mathbb{A}^1 \setminus \{0\}), Z \times (\mathbb{A}^1 \setminus \{0\})\big) \circ \pi^{2*} \\
&= \partial_0^2 \circ \{t\} \circ \big(N_Z X \times (\mathbb{A}^1 \setminus \{0\}) \to N_Z X\big)^* \circ J(X,Z) \\
&= J(N_Z X, N_Z Y) \circ J(X,Z).
\end{aligned}$$

□

12. The Pull-back Map

In this section we define the pull-back maps for morphisms to smooth varieties. Some properties are formulated, in particular the functoriality of the spectral sequences. We conclude with applications and discussions concerning birational questions. The proofs of Theorems 12.1 and 12.7 are given in the next section.

In the following all schemes $X, Y, X', \dots$ are flat over B of some constant relative dimension denoted by $\dim_B X, \dots$. All products $Y \times X$, $Y' \times X'$, ... are taken over B and cycle modules will be induced via projection to the second factor (projection to the first factor does not exist for us). We use the notations $T_S X = N_X(X \times_S X)$ and $TX = T_B X$. We are primarily interested in the case $B = \operatorname{Spec} k$, but we don't have to pay much for considering arbitrary B. M is a cycle module over X.

Let X be smooth over B. Then TX is a vector bundle on X. For a morphism $f\colon Y \to X$ let

$$f\colon Y \xrightarrow{i} Y \times X \xrightarrow{p} X$$

be the factorization with $i(y) = \big(y, f(y)\big)$ and $p(y,x) = x$. Then i is a regular imbedding and $N_Y(Y \times X) = f^* TX$.

We choose a coordination τ on TX and define

$$I(f) = I(f;\tau) = r(f^*\tau) \circ J(Y \times X, Y) \circ p^*\colon X \bullet\!\!\rightarrow Y.$$

Note that the construction is local in the sense that for an open immersion $j\colon U \to X$ one has

$$I(\tilde f; j^*\tau) \circ j^* = \tilde\jmath^* \circ I(f;\tau)$$

where $\tilde f\colon f^{-1}(U) \to U$ is the restriction of f and $\tilde\jmath\colon f^{-1}(U) \to Y$ is the inclusion.

One has $\delta\big(I(f)\big) = 0$ and

$$I(f)\big(C_p(X;M)\big) \subset C_{p+r}(Y;M)$$

where $r = \dim_B Y - \dim_B X$. If B is equidimensional, then

$$I(f)\big(C^p(X;M)\big) \subset C^p(Y;M).$$

We define

$$f^\bullet\colon A_p(X;M) \to A_{p+r}(Y;M)$$

and

$$f^\bullet\colon A^p(X;M) \to A^p(Y;M)$$

as the induced maps on (co-)homology. $f^\bullet$ does not depend on the choice of τ. One has the following properties.

(12.1) Theorem. *For $g\colon Z \to Y$ and $f\colon Y \to X$ with X and Y smooth over B one has $(f \circ g)^\bullet = g^\bullet \circ f^\bullet$.*

For the proof see the next section.

(12.2) Proposition. *If f is flat, then $I(f) = f^*$.*

Proof: It suffices to show

$$J(Y \times X, Y) \circ p^* = \pi^* \circ f^*$$

where $\pi\colon f^*TX \to Y$ is the projection. For this apply Lemma 11.4 with $U = Y$, $V = Y \times X$ and $W = X$. □

(12.3) Proposition. *If $i\colon Y \to X$ is a regular imbedding and X is smooth over B, then $I(i)$ is homotopic to $r \circ J(i)$ where r is any retraction to $Y \hookrightarrow N_Y X$.*

Proof: Apply Lemma 11.5 with $U = Y$, $V = Y \times X$ and $W = X$. □

The following corollary applied to the blow up at x_0 implies (together with Theorem 12.1) property (E) of Section 2.

(12.4) Corollary. *Let X be smooth over $B = \operatorname{Spec} k$, let $x \in X^{(p)}$ and let*

$$i_x\colon \overline{\{x\}} \to X$$

be the inclusion. Moreover let $\pi_1, \ldots, \pi_p$ be any regular sequence at x and let $v_1, \ldots, v_p$ be the induced sequence of valuations with the fraction fields $k(X)$, $\kappa(v_1), \ldots, \kappa(v_{p-1})$ and with the (residue classes of) $\pi_1, \ldots, \pi_p$ as primes. Then

$$i_x^\bullet\colon A^0(X; M) \to A^0(\overline{\{x\}}; M)$$

is the restriction of

$$s_{v_p}^{\pi_p} \circ \cdots \circ s_{v_1}^{\pi_1}\colon M\big(k(X)\big) \to M\big(\kappa(x)\big).$$

Proof: Let $X = X_0 \supset X_1 \supset \cdots \supset X_p$ be the sequence of smooth schemes locally around x with X_i defined by $\langle \pi_1, \ldots, \pi_i \rangle$. Using Theorem 12.1 one reduces to $p = 1$. This case follows from Proposition 12.3 and Lemma 11.2. □

(12.5) Proposition. (Projection formula.) *Consider a pull-back square*

$$\begin{array}{ccc}
\overline{Y} & \xrightarrow{\bar f} & \overline{X} \\
\Big\downarrow{\scriptstyle \tilde h} & & \Big\downarrow{\scriptstyle h} \\
Y & \xrightarrow{f} & X
\end{array}$$

with h smooth and proper and with X smooth over B. Then

$$\tilde h_* \circ \bar f^\bullet = f^\bullet \circ h_*.$$

Proof: One considers the diagram

$$\begin{array}{ccccc}
\overline{Y} & \longrightarrow & \overline{Y}\times\overline{X} & \longrightarrow & \overline{X}\\
\| & & \big\downarrow{\scriptstyle\hat h} & & \|\\
\overline{Y} & \longrightarrow & Y\times\overline{X} & \longrightarrow & \overline{X}\\
\big\downarrow{\scriptstyle\tilde h} & & \big\downarrow & & \big\downarrow{\scriptstyle h}\\
Y & \longrightarrow & Y\times X & \longrightarrow & X.
\end{array}$$

Here the bottom diagram is the pull-back along h and $\hat h = \tilde h \times \mathrm{id}_{\overline{X}}$. We have three maps $\overline{X} \bullet\!\!\rightarrow Y$: the first is constructed along $D(\overline{Y}\times\overline{X},\overline{Y})$ and is given by pull-back to $\overline{Y}\times\overline{X}$ and specialization to $\overline{Y}$ followed by push-forward; the second goes along $D(Y\times\overline{X},\overline{Y})$ and is given by pull-back to $Y\times\overline{X}$ and specialization to $\overline{Y}$ followed by push-forward; the third goes along $D(Y\times X,Y)$ and is given by push-forward, pull-back to $Y\times X$ and specialization to Y. The first two may be related using Lemma 11.5 (with $U=\overline{Y}$, $V=\overline{Y}\times\overline{X}$, and $W=Y\times\overline{X}$), the last two by the compatibility of the constructions with proper push-forward. □

Consider the triangle (7.4.0) and assume that ρ is smooth and η is flat. Define

$$[h/B]\colon A^p[\rho;M]\to A^p[\eta;M]$$

by $[h/B]_F=(h_F)^\bullet$. Here we understand $B=\operatorname{Spec}F$ in the definition of $(h_F)^\bullet$.

(12.6) PROPOSITION. *$[h/B]$ is a homomorphism of cycle modules over B.*

Proof: We apply Proposition 7.4. Since the projection $\pi\colon N(Y\times X,Y)\to Y$ is a vector bundle we know that

$$[\pi^*]\colon A^q[\eta]\to A^q[\eta\circ\pi]$$

is an isomorphism of cycle modules. Moreover

$$[\pi^*]\circ[h/B]=[\partial]\circ[\{t\}]\circ[p^*]$$

where $p\colon Y\times X\times(\mathbb{A}^1\setminus\{0\})\to X$ is the projection and ∂ is the boundary for $N(Y\times X,Y)\to D(Y\times X,Y)$. □

(12.7) THEOREM. *Consider the square* (8.4.0) *and its decomposition* (8.5.0). *Suppose that B is equidimensional, η is flat, ρ is smooth and X' (hence also X) is smooth over B. Then the spectral sequences*

$$E_2^{p,q}(\rho)=A^p(X';A^q[\rho;M])\Longrightarrow A^{p+q}(X;M),$$

$$E_2^{p,q}(\eta)=A^p(Y';A^q[\eta;M])\Longrightarrow A^{p+q}(Y;M)$$

commute with the maps

$$A^p(X';A^q[\rho;M])\xrightarrow{(f')^\bullet}A^p(Y';A^q[\rho;M])\xrightarrow{[\hat f/Y']_\#}A^p(Y';A^q[\eta;M]),$$

$$f^\bullet\colon A^p(X;M)\to A^p(Y;M).$$

For the proof see the next section. Switching to dimension indices this theorem holds without the equidimensionality assumption on B.

For the rest of the section we assume $B = \operatorname{Spec} k$.

(12.8) Lemma. *Let X be smooth, let Y be integral, let $f\colon Y \to X$ be a dominant morphism and let $\varphi\colon k(X) \to k(Y)$ be the induced homomorphism of the function fields. Then*

$$I(f) \mid M\big(k(X)\big) = \varphi_*\colon M\big(k(X)\big) \to M\big(k(Y)\big).$$

Proof: After replacing Y by an open subset we may assume that f is flat. The claim follows from Proposition 12.2. □

(12.9) Lemma. *Assume in* (12.8) *additionally that f is proper and that φ is an isomorphism. Then $f_* \circ I(f) = \mathrm{id}$.*

Proof: Let

$$\bar{f}\colon D(Y \times X, Y) \to D(X \times X, X)$$

be the proper map induced from f. There is the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\bullet" description] \arrow[d, "="] & Y \times X \times (\mathbb{A}^1 \setminus \{0\}) \arrow[r, "\partial\circ\{t\}"] \arrow[d, "\tilde{f}_*"] & f^*TX \arrow[r, "r(f^*\tau)"] \arrow[d, "\hat{f}_*"] & Y \arrow[d, "f_*"] \\
X \arrow[r, "\bullet" description] & X \times X \times (\mathbb{A}^1 \setminus \{0\}) \arrow[r, "\partial\circ\{t\}"] & TX \arrow[r, "r(\tau)"] & X
\end{tikzcd}$$

where $\tilde{f}$, $\hat{f}$ are the restrictions of $\bar{f}$. The diagram shows $f_* \circ I(f) = I(\mathrm{id})$. But $I(\mathrm{id}) = \mathrm{id}$ by Proposition 12.2. □

Lemma 12.9 shows in particular that for any blow up $Y \to X$ the complex $C_*(X; M)$ is a direct summand of $C_*(Y; M)$. This splitting via $I(f)$ depends alone on the choice of a coordination of TX near the singular locus and is unique up to homotopy.

(12.10) Corollary. *Let X be a proper smooth variety over k and let M be a cycle module over k. Then the group $A^0(X; M)$ is a birational invariant of X.*

Proof: If X_1, X_2 are proper and birational isomorphic there exist a proper Y and birational morphisms $Y \to X_i$ (take for Y the closure in $X_1 \times X_2$ of a common open subset of the X_i). Then as subgroups of $M(\xi_{X_i}) = M(\xi_Y)$ one has the trivial inclusions $A^0(Y; M) \subset A^0(X_i; M)$; Lemma 12.8 shows $A^0(X_i; M) \subset A^0(Y; M)$. □

For an illustration let X be a smooth and proper variety over k with function field F. Then for any geometric valuation v on F (of rank 1) there is a birational morphism $f\colon Y \to X$ such that v has center y in $Y^{(1)}$ with $\kappa(v) = \kappa(y)$. The map $I(f)$ yields a formula

$$\partial_v = \sum_{x \in X^{(1)}} \alpha_v^x \circ \partial_x \tag{12.11}$$

where

$$\alpha_v^x\colon M\big(\kappa(x)\big) \to M\big(\kappa(v)\big)$$

equals the component $I(f)_y^x$.

This formula is a sort of higher dimensional analogue of the sum formula for one-dimensional function fields. It has the following properties:

— it is local, that is x runs only through $X^{(1)}_{(z)}$ where z is the center of v in X (in other words: $\alpha^x_v = 0$ for $x \notin X^{(1)}_{(z)}$).

— it is not unique, but depends only on the choice of a coordination of the tangent bundle of X restricted to $X_{(z)}$.

— it is universal in the sense that the α^x_v can be written as sums of compositions of the data of cycle modules, independent of M. This is quite obvious from the construction of $I(f)$. One can make this more precise by interpreting the α^x_v as morphisms in the category $\widetilde{\mathcal{F}}$ of Remark 1.10. In this way the category $\widetilde{\mathcal{F}}$ appears as the natural place for the coefficients α^x_y of formulas like (12.11).

Exercise: Describe the α^x_v for $\dim X = 2$ and v the valuation corresponding to the exceptional fiber of the blow up in a closed point (see Remark 2.8).

Birational invariants like $A^0(X; M)$ have been considered in various contexts like étale cohomology and K-theory, see (Colliot-Thélène 1992) for a survey. The advantage of the method of proof of Corollary 12.10 lies in its general and essentially simple nature (after having established the functors in question as cycle modules); moreover the formula (12.11) makes things perhaps more enlightening. A similar method works probably for functors related with the Witt ring of quadratic forms.

To mention a particular example, let $\pi\colon Z \to \operatorname{Spec} k$ be proper and let M (resp. N) be the $\mathbb{Z}$-graded cycle module over k given as the cokernel (resp. image) of

$$[\pi_*]\colon A_0[\pi; K_*] \to A_0[\mathrm{id}_{\operatorname{Spec} k}; K_*] = K_*.$$

By Corollary 12.10 the group $A^0(X; M, 1)$ (which is a subquotient of $k(X)^*$) is a birational invariant for proper smooth X over k. The proof of this fact was the main aim of (Rost 1990). There it was achieved by a different method using the triviality of $A^1(\widetilde{X}; N, 1)$ for smooth local $\widetilde{X}$ (proved in this paper in more generality in Section 6).

13. Intersection Theory for Fibrations

The purpose of this section is to prove Theorems 12.1 and 12.7. We define pull-back maps on complex level for regular imbeddings and for morphisms to smooth varieties in fibered situations. Moreover we establish functoriality of the constructions. Most of the work has been done already in Sections 7–11.

Consider a commutative square

$$(\Delta)\qquad \begin{array}{ccc} Y & \xrightarrow{f} & X \\ \big\downarrow{\scriptstyle \eta} & & \big\downarrow{\scriptstyle \rho} \\ Y' & \xrightarrow{f'} & X'. \end{array}$$

The square Δ is called a regular imbedding, if f and f' are regular imbeddings of some constant codimensions and if the induced map

$$p\colon N_Y X \to \eta^* N_{Y'} X'$$

is an epimorphism of vector bundles over Y. The kernel bundle of p is denoted by N_Δ. We consider p also as vector bundle and identify it with $q^* N_\Delta$ where $q\colon \eta^* N_{Y'} X' \to Y$ is the projection.

Let Δ be a regular imbedding and let $\tilde\tau$ and $\tilde\tau'$ be coordinations of $N_\Delta \to Y$ and of $N_{Y'} X' \to Y'$, respectively. We define

$$\overline{J}(\Delta)\colon X \bullet\!\!\to Y$$

by

$$\overline{J}(\Delta) = \overline{J}(\Delta, \tilde\tau, \tilde\tau') = r(\eta^* \tilde\tau') \circ r(q^* \tilde\tau) \circ J(X, Y).$$

The following is clear from Sections 8–9. One has $\delta\big(\overline{J}(\Delta)\big) = 0$ and

$$\overline{J}(\Delta)\big(C_{p,l}(\rho)\big) \subset C_{p+s, l+t}(\eta)$$

with $s = -\operatorname{codim}(f')$ and $t = -\operatorname{codim}(f)$. Moreover the homotopy class of $\overline{J}(\Delta)$ (with respect to the degree (s,t)) does not depend on the choice of $\tilde\tau$ and $\tilde\tau'$.

In the definition of $\overline{J}(\Delta)$ we wanted to be as canonical as possible. If one is interested only in the homotopy class, one may put $\overline{J}(\Delta) = r \circ J(X, Y)$ for any filtration preserving retraction r to $N_Y X \to X$ ("filtration preserving" means always with respect to the natural degrees).

Next consider a diagram

$$(13.1.0)\qquad \begin{array}{ccccc} Z & \xrightarrow{f_1} & Y & \xrightarrow{f_2} & X \\ \big\downarrow{\scriptstyle \mu} & & \big\downarrow{\scriptstyle \eta} & & \big\downarrow{\scriptstyle \rho} \\ Z' & \xrightarrow{f_1'} & Y' & \xrightarrow{f_2'} & X'. \end{array}$$

We denote the left square by Δ_1, the right square by Δ_2 and the composed square by Δ_3.

(13.1) Theorem. *If Δ_1 and Δ_2 are regular imbeddings, then Δ_3 is a regular imbedding and*

$$\bar{J}(\Delta_1) \circ \bar{J}(\Delta_2) \simeq \bar{J}(\Delta_3)$$

under a filtration preserving homotopy.

Proof: The first statement is straightforward. In the following deduction of the second statement the letters r_i stand for some filtration preserving retractions. We make use of Lemma 11.3 for the homotopy (1), of Lemmata 11.6 and 11.7 for the homotopy (2) and of Lemma 11.4 for the equality (3).

$$\begin{aligned}
\bar{J}(\Delta_1) \circ \bar{J}(\Delta_2) &\simeq r_1 \circ J(Y,Z) \circ r_2 \circ J(X,Y) \\
&\simeq r_3 \circ \big(N(N_Y X, N_Y X \mid Z) \to N(Y,Z)\big)^* \circ J(Y,Z) \circ r_2 \circ J(X,Y) \\
&\overset{(1)}{\simeq} r_3 \circ J(N_Y X, N_Y X \mid Z) \circ (N(Y,X) \to Y)^* \circ r_2 \circ J(X,Y) \\
&\simeq r_3 \circ J(N_Y X, N_Y X \mid Z) \circ J(X,Y) \\
&\overset{(2)}{\simeq} r_3 \circ J(N_Z X, N_Z Y) \circ J(X,Z) \\
&\simeq r_3 \circ J(N_Z X, N_Z Y) \circ (N_Z X \to Z)^* \circ r_4 \circ J(X,Z) \\
&\overset{(3)}{=} r_3 \circ \big(N(N_Z X, N_Z Y) \to Z\big)^* \circ r_4 \circ J(X,Z) \\
&\simeq r_5 \circ J(X,Z) \\
&\simeq \bar{J}(\Delta_3). \qquad \square
\end{aligned}$$

The square Δ is called admissible if η is flat, ρ is smooth and X' is smooth over B. Consider the diagram

$$\begin{array}{ccccc}
Y & \xrightarrow{i} & Y \times X & \xrightarrow{\rho} & X \\
\big\downarrow{\scriptstyle\eta} & & \big\downarrow & & \big\downarrow{\scriptstyle\rho} \\
Y' & \xrightarrow{i'} & Y' \times X' & \xrightarrow{\rho'} & X'
\end{array}$$

and denote the left square by Δ_i and the right square by Δ_p. If Δ is admissible, then Δ_i is a regular imbedding and Δ_p is a flat square (see (8.5.0)). Moreover the normal bundles of i and i' are given by f^*TX and f'^*TX', respectively, and N_{Δ_i} is given by $f^*T_{X'}X$.

Let Δ be admissible and let τ and τ' be coordinations of $T_{X'}X$ and TX', respectively. We define

$$\bar{I}(\Delta)\colon X \bullet\!\!\to Y$$

by

$$\bar{I}(\Delta) = \bar{I}(\Delta, \tau, \tau') = \bar{J}(\Delta_i, f^*\tau, f'^*\tau') \circ p^*.$$

One has $\delta\big(\bar{I}(\Delta)\big) = 0$ and

$$\bar{I}(\Delta)\big(C_{p,l}(\rho)\big) \subset C_{p+s,l+t}(\eta)$$

with $s = \dim_B Y' - \dim_B X'$ and $t = \dim_B Y - \dim_B X$. Moreover the homotopy class of $\bar{I}(\Delta)$ (with respect to the degree (s,t)) does not depend on the choice of τ and τ'.

(13.2) Theorem. *If in* (13.1.0) *the squares* Δ_1 *and* Δ_2 *are admissible, then the square* Δ_3 *is admissible and*

$$\bar{I}(\Delta_3) \cong \bar{I}(\Delta_1) \circ \bar{I}(\Delta_2)$$

under a filtration preserving homotopy.

Proof: The first statement is trivial. For the second we consider the diagrams

$$\begin{tikzcd}
Z \arrow[r, "i_1"] & Z \times Y \arrow[r, "i_4"] \arrow[d, "p_1"] & (Z \times Y) \times X \arrow[d, "p_4"] \\
 & Y \arrow[r, "i_2"] & Y \times X \arrow[d, "p_2"] \\
 & & X,
\end{tikzcd}$$

$$Z \xrightarrow{i_3} Z \times X \xrightarrow{i_5} (Z \times Y) \times X \xrightarrow{p_5} X.$$

The regular imbeddings i_j lie over accordingly defined regular imbeddings i'_j; the corresponding squares are denoted by Σ_j.

The Σ_j are regular imbeddings. An application of Lemma 11.3 and Theorem 13.1 shows that (by noting $i_4 \circ i_1 = i_5 \circ i_3$ and $p_2 \circ p_4 = p_5$)

$$\bar{I}(\Delta_1) \circ \bar{I}(\Delta_2) \simeq \bar{J}(\Sigma_3) \circ \bar{J}(\Sigma_5) \circ p_5^*.$$

By definition we have

$$\bar{I}(\Delta_3) \simeq \bar{J}(\Sigma_3) \circ (p_5 \circ i_5)^*.$$

Finally Lemma 11.4 shows

$$\bar{J}(\Sigma_5) \circ p_5^* = (p_5 \circ i_5)^*. \qquad \square$$

Theorem 13.2 implies Theorem 12.1 by passing to homology. For a proof of Theorem 12.7 we consider six squares with the top arrows

$$\begin{tikzcd}
Y \arrow[r, "i_1"] & Y \times_{X'} X \arrow[r, "i_2"] \arrow[d, "p_1"] & Y \times X \arrow[d, "p_2"] & \\
 & Y' \times_{X'} X \arrow[r, "i_3"] & Y' \times X \arrow[r, "p_3"] & X
\end{tikzcd}$$

lying over the bottom arrows

$$\begin{tikzcd}
Y' \arrow[r, equal] & Y' \arrow[r, "i'"] \arrow[d, equal] & Y' \times X' \arrow[d, equal] & \\
 & Y' \arrow[r, "i'"] & Y' \times X' \arrow[r, "p'"] & X'.
\end{tikzcd}$$

Let Σ_j be the square corresponding to i_j for $j = 1, 2, 3$. The Σ_j are regular imbeddings. The map $f^\bullet$ is induced from $\bar{I}(\Delta)$. By the definition of $\bar{I}(\Delta)$ and Theorem 13.1 we have

$$\bar{I}(\Delta) \simeq \bar{J}(\Sigma_1) \circ \bar{J}(\Sigma_2) \circ p_2^* \circ p_3^*.$$

Lemma 11.3 shows that

$$\bar{J}(\Sigma_2) \circ p_2^* \simeq p_1^* \circ \bar{J}(\Sigma_3).$$

Therefore $f^\bullet$ is the composition of the maps induced by $\bar{J}(\Sigma_1) \circ p_1^*$ and by $\bar{J}(\Sigma_3) \circ p_3^*$.

Next note that $p_1 \circ i_1 = \hat{f}$. An application of Proposition 8.5 shows that $\bar{J}(\Sigma_1) \circ p_1^*$ induces on the E^2-terms the map $[\hat{f}/Y']_\#$. Finally note that $p' \circ i' = f'$ and that the squares under i_3 and p_3 are pull-back squares. An application of Proposition 8.5 shows that $\bar{J}(\Sigma_3) \circ p_3^*$ induces on the E^2-terms the map $(f')^\bullet$. □

14. PRODUCTS

In this section M is a cycle module over B and N is a cycle module over k. We assume that either $N = K_*$ or that $M = N$ is a cycle module with ring structure over $B = \operatorname{Spec} k$. So in any case we are given a pairing $N \times M \to M$ of cycle modules over B.

The restriction to these special cases are made for simplification. For example, in forming intersections of cycles with coefficients in a cycle module M with ring structure, one needs to know that its pairing factors through a cycle module over $B \times B$. However, the existence of a corresponding appropriate notion of tensor product of cycle modules is not clear to me (and a settling of this question would lead to far here anyway). The problem could be avoided in the following by assuming the necessary factorizations, but this is somewhat tiring.

(14.1) CROSS PRODUCTS. Let Y be a scheme over k and let Z be a scheme over B (all of finite type over k). We define the cross product

$$\times : C_p(Y; N) \times C_q(Z; M) \to C_{p+q}(Y \times Z; M)$$

as follows. For $y \in Y$ let $Z_y = \operatorname{Spec} \kappa(y) \times Z$, let $\pi_y \colon Z_y \to Z$ be the projection and let $i_y \colon Z_y \to Y \times Z$ be the inclusion. For $z \in Z$ we understand similarly $Y_z, \pi_z \colon Y_z \to Y$ and $i_z \colon Y_z \to Y \times Z$. We give the following two equivalent definitions:

$$\rho \times \mu = \sum_{y \in Y_{(p)}} (i_y)_* \big(\rho_y \cdot \pi_y^*(\mu)\big),$$

$$\rho \times \mu = \sum_{z \in Z_{(q)}} (i_z)_* \big(\pi_z^*(\rho) \cdot \mu_z\big).$$

Here $\rho_y \in N(y)$ is the y-component of ρ and the product is understood after pointwise restriction of ρ_y. The map

$$(i_y)_* \colon C_q(Z_y; M) \to C_{p+q}(Y \times Z; M)$$

is the inclusion corresponding to $Z_{y(q)} \subset (Y \times Z)_{(p+q)}$. Similarly we understand $\mu_z \in M(z)$ and

$$(i_z)_*\colon C_p(Y_z; M) \to C_{p+q}(Y \times Z; M).$$

To check equality of the two definitions consider the u-components for $u \in Y \times Z$. Let y, z be the images of u under the projections $Y \times Z \to Y$, Z and let $R = \kappa(y) \otimes_k \kappa(z)$. Then the u-components are either trivial or u is a minimal prime of R. In the latter case the u-components are given by

$$(\rho \times \mu)_u = r_{\kappa(u)|\kappa(y)}(\rho_y) \cdot r_{\kappa(u)|\kappa(z)}(\mu_z) \cdot l_R(R_{(u)}).$$

(14.2) Associativity. Additionally let X be of finite type over k and let $\eta \in C_r(X; N)$. Then

$$\eta \times (\rho \times \mu) = (\eta \times \rho) \times \mu.$$

For a proof consider the u-components for $u \in X \times Y \times Z$. Let x, y, z be the images of u in X, Y, Z, respectively, and let $R = \kappa(x) \otimes_k \kappa(y) \otimes_k \kappa(z)$. Then the u-components are either trivial or u is a minimal prime of R. In the latter case it follows from standard rules for length that the u-components are given by

$$(\eta \times \rho \times \mu)_u = r_{\kappa(u)|\kappa(x)}(\eta_x) \cdot r_{\kappa(u)|\kappa(y)}(\rho_y) \cdot r_{\kappa(u)|\kappa(z)}(\mu_z) \cdot l_R(R_{(u)}).$$

(14.3) Commutativity. Suppose $M = N$ is a cycle module with ring structure over $B = \operatorname{Spec} k$. Let $\tau\colon Y \times Z \to Z \times Y$ be the interchange of factors. For $\rho \in C_p(Y; M, n)$ and $\mu \in C_q(Z; M, m)$ one has

$$\tau_*(\rho \times \eta) = (-1)^{nm} \eta \times \rho \in C_{p+q}(Z \times Y; M, n+m).$$

This is immediate from the definitions.

(14.4) Chain rule. For $\rho \in C_p(Y; N, n)$ and $\mu \in C_q(Z; M, m)$ one has

$$d(\rho \times \mu) = d(\rho) \times \mu + (-1)^n \rho \times d(\mu).$$

For a proof we may assume $\rho \in M(y)$, $\mu \in M(z)$ for some $y \in Y_{(p)}$ and $z \in Z_{(q)}$. Consider for $u \in Y \times Z$ the u-components of the three terms. If one of them is nontrivial, we must have $\dim(u, Y \times Z) = p + q - 1$ and the images y', z' of u must be in the closures of y, z, respectively. Dimension reasons show $y' = y$ or $z' = z$. Now the claim follows from one of the two definitions of the cross product and Proposition 4.6.2.

(14.5) Compatibility. The cross product is compatible with the four basic types of maps f_*, g^*, $\{a\}$ and ∂ acting on one of the two factors. This follows from the compatibility with flat pull-back and Definition 1.3. We omit a detailed formulation.

We conclude with a consideration of the intersection pairing for cycles on a smooth variety. Let X be smooth over k and let τ be a coordination of TX. We define

$$I_X\colon C^*(X; N) \times C^*(X; M) \to C^*(X; M),$$

$$I_X(\rho, \mu) = \big(r(\tau) \circ J(X \times X, X)\big)(\rho \times \mu).$$

By (14.4) this is a pairing of complexes. Let

$$\smile\colon A^*(X; N) \times A^*(X; M) \to A^*(X; M)$$

be the induced pairing.

The next theorem follows from the preceding remarks and in particular from Theorem 12.1. It holds accordingly on chain level up to homotopy.

(14.6) THEOREM. *If $M = N$ is a cycle module with ring structure over $B = \operatorname{Spec} k$, the pairing $\smile$ turns $A^*(X;M)$ into an anti-commutative associative ring. If $N = K_*$, the pairing $\smile$ turns $A^*(X;M)$ into a module over $A^*(X;K_*)$.* □

We have defined in particular a ring structure on the classical Chow groups

$$\mathrm{CH}^*(X) = \coprod_p A^p(X;K_*,p)$$

of a smooth variety. This ring structure coincides with the classical one. This may be deduced from the remark at the beginning of Section 11 and (Fulton 1984, Chaps. 5, 6, 8).

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Fakultät für Mathematik
Universität Regensburg
D-93040 Regensburg
Germany
markus.rost@rphs1.physik.uni-regensburg.de