On the cardinality of Hausdorff spaces and Pol-Šapirovskii technique

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Abstract. In this paper we make use of the Pol-Šapirovskii technique to prove three cardinal inequalities. The first two results are due to Fedeli [2] and the third theorem of this paper is a common generalization to: (a) (Arhangel'skii [1]) If X is a T_1 space such that (i) $L(X)t(X) \leq \kappa$, (ii) $\psi(X) \leq 2^\kappa$, and (iii) for all $A \in [X]^{\leq 2^\kappa}$, $|\overline{A}| \leq 2^\kappa$, then $|X| \leq 2^\kappa$; and (b) (Fedeli [2]) If X is a T_2-space then $|X| \leq 2^{\mathrm{aql}(X)t(X)\psi_c(X)}$.

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In [2], Fedeli proved, using the language of elementary submodels, two cardinal inequalities which state (1) "if $X \in \mathcal{T}_2$, then $|X| \leq 2^{\mathrm{ac}(X)H\psi(X)}$" and (2) "if $X \in \mathcal{T}_2$, then $|X| \leq 2^{\mathrm{lc}(X)\pi\chi(X)\psi_c(X)}$". Each of these inequalities improve the well known Hajnal-Juhász's inequality: "for $X \in \mathcal{T}_2$, $|X| \leq 2^{c(X)\chi(X)}$". In the first part of this paper we give a proof of the inequalities (1) and (2) without using elementary submodels. Our proof makes use of the Pol-Šapirovskii technique. This technique provides a unified approach to the difficult inequalities in the theory of cardinal functions. The reader is referred to [4] and [3] for a detailed discussion like for additional inequalities in cardinal functions which can be proved using the Pol-Šapirovskii technique.

We refer the reader to [3], [2] and [5] for definitions and terminology not explicitly given. Let L, c, χ, ψ, ψ_c, $\pi\chi$, t, denote the following standard cardinal functions: Lindelöf degree, celularity, character, pseudocharacter, closed pseudocharacter, π-character and tightness, respectively.

Let X be a Hausdorff space. The Hausdorff pseudocharacter, denoted $H\psi(X)$, is the smallest infinite cardinal κ such that for every $x \in X$ there is a collection $\mathcal{U}_x$ of open neighborhoods of x with $|\mathcal{U}_x| \leq \kappa$ and such that $(*)$ if $x \neq y$, there exist $U \in \mathcal{U}_x$ and $V \in \mathcal{U}_y$ with $U \cap V = \emptyset$. If $\mathcal{U}_x$ is a collection of open neighborhoods of x which satisfies $(*)$, we say that $\mathcal{U}_x$ is a H-pseudobase of x.

Definition 1. Let X be a topological space:

(a) $\mathrm{ac}(X)$ is the smallest infinite cardinal κ such that there is a subset S of X with $|S| \leq 2^\kappa$ and for every open collection $\mathcal{U}$ in X, there is a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$, with $\bigcup\mathcal{U} \subseteq \overline{S \cup \bigcup\{\overline{V} : V \in \mathcal{V}\}}$.

(b) $\mathrm{lc}(X)$ is the smallest infinite cardinal κ such that there is a closed subset F of X with $|F| \leq 2^\kappa$ and for every open collection $\mathcal{U}$ in X, there is a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$, with $\bigcup\mathcal{U} \subseteq F \cup \overline{\bigcup\{\overline{V} : V \in \mathcal{V}\}}$.

(c) $\mathrm{aql}(X)$ is the smallest infinite cardinal κ such that there is a subset S of X such that $|S| \leq 2^\kappa$ and for every open cover $\mathcal{U}$ of X there is a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ with $X = S \cup (\bigcup \mathcal{V})$.

Clearly $\mathrm{ac}(X) \leq \mathrm{lc}(X) \leq c(X)$, and $\mathrm{aql}(X) \leq L(X)$ for every topological space.

Theorem 2. *If X is a T_2-space then $|X| \leq 2^{\mathrm{ac}(X)H\psi(X)}$.*

Proof: Let $\kappa = \mathrm{ac}(X)H\psi(X)$, and let S be a subset of X with $|S| \leq 2^\kappa$ and witnessing that $\mathrm{ac}(X) \leq \kappa$. For each $x \in X$, let $\mathcal{B}_x$ an H-pseudobase of x in X, with $|\mathcal{B}_x| \leq \kappa$.

Construct a sequence $\{A_\alpha : 0 \leq \alpha < \kappa^+\}$ of sets in X and a sequence $\{\mathcal{V}_\alpha : 0 < \alpha < \kappa^+\}$ of open collections in X such that

(1) $|A_\alpha| \leq 2^\kappa$; $0 \leq \alpha < \kappa^+$;
(2) $\mathcal{V}_\alpha = \bigcup\left\{\mathcal{B}_x : x \in \bigcup_{\beta<\alpha} A_\beta\right\}$; $0 < \alpha < \kappa^+$;
(3) if $\mathcal{C} = \{C_\gamma : \gamma \in \lambda\}$ is a collection $(\lambda \leq \kappa)$ of closed sets in X such that each C_γ has the form $\overline{\bigcup\{\overline{V} : V \in \mathcal{U}_\gamma\}}$, where $\mathcal{U}_\gamma \in [\mathcal{V}_\alpha]^{\leq\kappa}$, and if $X - (S \cup \bigcup\mathcal{C}) \neq \emptyset$, then $A_\alpha - (S \cup \bigcup\mathcal{C}) \neq \emptyset$.

The construction is by transfinite induction. Let $0 < \alpha < \kappa^+$, and assume that A_β and $\mathcal{V}_\beta$ have been constructed for each $\beta < \alpha$. Note that $\mathcal{V}_\alpha$ is defined by (2). For each collection $\mathcal{C} = \{C_\gamma : \gamma \in \lambda\}$ with $\lambda \leq \kappa$ of closed sets in X such that each C_γ has the form $\overline{\bigcup\{\overline{V} : V \in \mathcal{U}_\gamma\}}$, where $\mathcal{U}_\gamma \in [\mathcal{V}_\alpha]^{\leq\kappa}$, and such that $X \neq S \cup \bigcup\{C_\gamma : \gamma \in \lambda\}$, choose one point in $X - \left(S \cup \bigcup\{C_\gamma : \gamma \in \lambda\}\right)$. Let A_α be the set of points chosen in this way. To show that $|A_\alpha| \leq 2^\kappa$, let $F = \bigcup_{\beta<\alpha} A_\beta$; then $\mathcal{V}_\alpha = \bigcup_{x\in F} \mathcal{B}_x$, hence $|\mathcal{V}_\alpha| \leq \sum_{x\in F} |\mathcal{B}_x| \leq \kappa \cdot |F| \leq \kappa \cdot \sum_{\beta\in\alpha} |A_\beta| = \kappa \cdot |\alpha| \cdot 2^\kappa = 2^\kappa$. Since $|A_\alpha| \leq \left|[[\mathcal{V}_\alpha]^\kappa]^\kappa\right| \leq (2^\kappa)^\kappa = 2^\kappa$, we have $|A_\alpha| \leq 2^\kappa$. This completes the construction.

Now let $A = \bigcup_{\alpha<\kappa^+} A_\alpha$ and let $\mathcal{U} = \bigcup\{\mathcal{V}_\alpha : \alpha \in \kappa^+\}$; clearly, $|A| \leq 2^\kappa$.

The proof is complete if $X = (S \cup A)$. Suppose not, and let $p \in X - (S \cup A)$. Let $\mathcal{B} = \{B_\gamma : \gamma \in \lambda\}$ be a family of open neighbourhoodsof p in X, such that $\bigcap\{\overline{B}_\gamma : \gamma \in \lambda\} = \{p\}$ with $\lambda \leq \kappa$. For each $\gamma \in \lambda$, let $V_\gamma = X - \overline{B}_\gamma$ and let $\mathcal{W}_\gamma = \{V \in \mathcal{U} : V \subseteq V_\lambda\}$. Since $\mathrm{ac}(X) \leq \kappa$, for each $\gamma \in \lambda$ there exists $\mathcal{U}_\gamma \in [\mathcal{W}_\gamma]^{\leq\kappa}$ such that $\bigcup\mathcal{W}_\gamma \subseteq S \cup \overline{\bigcup\{\overline{V} : V \in \mathcal{U}_\gamma\}}$. Note that for each $\gamma \in \lambda$, $p \notin S \cup \overline{\bigcup\{\overline{V} : V \in \mathcal{U}_\gamma\}}$. Finally, let $C_\gamma = \overline{\bigcup\{\overline{V} : V \in \mathcal{U}_\gamma\}}$ for each $\gamma \in \lambda$. Since $\mathcal{U}_\gamma \subseteq \mathcal{U}$ and $|\mathcal{U}_\gamma| \leq \kappa$, for all $\gamma \in \lambda$, by the regularity of κ^+ there is an $\alpha \in \kappa^+$ such that $\mathcal{C} = \{C_\gamma : \gamma \in \lambda\}$ is a collection of $\leq \kappa$ closed sets in X, such that each C_γ has the form $\overline{\bigcup\left\{\overline{V} : V \in \mathcal{U}_\gamma\right\}}$, where $\mathcal{U}_\gamma \in [\mathcal{V}_\alpha]^{\leq\kappa}$. Moreover $X - \left(S \cup \bigcup\{C_\gamma : \gamma \in \lambda\}\right) \neq \emptyset$, therefore, by (3), $A_\alpha - \left(S \cup \bigcup\{C_\gamma : \gamma \in \lambda\}\right) \neq \emptyset$. Since $A_\alpha \subseteq A \subseteq S \cup \bigcup\{C_\gamma : \gamma \in \lambda\}$, we reach a contradiction. Thus $X = S \cup A$ and $|X| = |S \cup A| \leq 2^\kappa$. $\square$

Theorem 3. *If X is a T_2-space then $|X| \leq 2^{\mathrm{lc}(X)\pi\chi(X)\psi_c(X)}$.*

Proof: Let $\kappa = \mathrm{lc}(X)\pi\chi(X)\psi_c(X)$, and let F be a closed set in X with $|F| \leq 2^\kappa$ and witnessing that $\mathrm{lc}(X) \leq \kappa$. For each $x \in X$, let $\mathcal{V}_x$ a π-base local of x in X such that $|\mathcal{B}_x| \leq \kappa$.

Construct a sequence $\left\{A_\alpha : \alpha \in \kappa^+\right\}$ of sets in X and a sequence $\left\{\mathcal{B}_\alpha : \alpha \in \kappa^+\right\}$ of open collections in X such that:

(1) $\alpha \in \kappa^+$, $|A_\alpha| \leq 2^\kappa$; $0 \leq \alpha \leq \kappa^+$;
(2) $\mathcal{V}_\alpha = \bigcup\left\{\mathcal{B}_x : x \in \bigcup_{\beta<\alpha} A_\beta\right\}$; $0 < \alpha < \kappa^+$;
(3) if $\mathcal{C} = \{C_\gamma : \gamma \in \lambda\}$, with $\lambda \leq \kappa$, is a collection of closed sets in X, where each C_γ has the form $\overline{\bigcup\left\{\overline{V} : V \in \mathcal{U}_\gamma\right\}}$, where $\mathcal{U}_\gamma \in [\mathcal{V}_\alpha]^{\leq\kappa}$ and $X - (F \cup \bigcup\mathcal{C}) \neq \emptyset$, then $A_\alpha - (F \cup \bigcup\mathcal{C}) \neq \emptyset$.

The construction is by transfinite induction. Let $0 < \alpha < \kappa^+$, and assume that A_β and $\mathcal{V}_\beta$ have been constructed for each $\beta < \alpha$. Note that $\mathcal{V}_\alpha$ is defined by (2). Let $P_\alpha = \bigcup_{\beta<\alpha} A_\beta$; we have $\mathcal{V}_\alpha = \bigcup\{\mathcal{B}_x : x \in P_\alpha\}$. Now, for each collection $\mathcal{C} = \{C_\gamma : \gamma \in \lambda\}$, $\lambda \leq \kappa$, of closed sets in X such that each C_γ has the form $\overline{\bigcup\left\{\overline{V} : V \in \mathcal{U}_\gamma\right\}}$, where $\mathcal{U}_\gamma \in [\mathcal{V}_\alpha]^{\leq\kappa}$ and $X \neq F \cup \bigcup\{C_\gamma : \gamma \in \lambda\}$, choose one point in $X - \left(F \cup \bigcup\{C_\gamma : \gamma \in \lambda\}\right)$. Let A_α be the set of points chosen in this way. Observe that $|A_\alpha| \leq \left|[[\mathcal{V}_\alpha]^{\leq\kappa}]^{\leq\kappa}\right| \leq 2^\kappa$. This completes the construction.

Let $A = \bigcup\left\{A_\alpha : \alpha \in \kappa^+\right\}$ and let $\mathcal{U} = \bigcup\left\{\mathcal{V}_\alpha : \alpha \in \kappa^+\right\}$. It is clear that $|A| \leq 2^\kappa$. The proof is complete if $X = F \cup A$. Assume, on the contrary, that $p \in X - (F \cup A)$, and consider $\mathcal{V} = \{B_\gamma : \gamma \in \lambda\}$, where $\lambda \leq \kappa$, a family of neighbourhoods of p in X such that $\bigcap\{\overline{B}_\gamma : \gamma \in \lambda\} = \{p\}$. For each $\gamma \in \lambda$, let $V_\gamma = X - \overline{B}_\gamma$ and let $\mathcal{W}_\gamma = \{V \subseteq V_\lambda : V \in \mathcal{U}\}$. Since $\mathrm{lc}(X) \leq \kappa$ for each $\gamma \in \lambda$, there exists $\mathcal{U}_\gamma \in \left[\mathcal{W}_\gamma\right]^{\leq\kappa}$ such that $\bigcup\mathcal{W}_\gamma \subseteq F \cup \overline{\{\bigcup\overline{V} : V \in \mathcal{U}_\gamma\}}$. Observe that, for each $\gamma \in \lambda$, $p \notin F \cup \overline{\bigcup\{\overline{V} : V \in \mathcal{U}_\gamma\}}$. Let $\mathcal{W} = \bigcup\{\mathcal{W}_\gamma : \gamma \in \lambda\}$. Finally, for each $\gamma \in \lambda$, let $C_\gamma = \overline{\bigcup\{\overline{V} : V \in \mathcal{U}_\gamma\}}$. Since $\mathcal{U}_\gamma \subseteq \mathcal{U}$ and $\left|\mathcal{U}_\gamma\right| \leq \kappa$ for all $\gamma \in \lambda$, then by the regularity of κ^+ there exists $\alpha \in \kappa^+$ such that $\mathcal{C} = \{C_\gamma : \gamma \in \lambda\}$ is a collection of $\leq \kappa$ closed sets in X and each C_γ has the form $\overline{\bigcup\left\{\overline{V} : V \in \mathcal{U}_\gamma\right\}}$, where $\mathcal{U}_\gamma \in [\bigcup\{\mathcal{V}_x : x \in A_\alpha\}]^{\leq\kappa}$. Moreover, $X - \left(F \cup \bigcup\{C_\gamma : \gamma \in \lambda\}\right) \neq \emptyset$, hence by (3), $A_\alpha - \left(F \cup \bigcup\{C_\gamma : \gamma \in \lambda\}\right) \neq \emptyset$. Since $A_\alpha \subseteq A \subseteq \overline{\bigcup\mathcal{W}} \subseteq F \cup \bigcup\{C_\gamma : \gamma \in \lambda\}$, we reach a contradiction. Thus $X = F \cup A$; therefore $|X| \leq 2^\kappa$. $\square$

Now we turn to the second part of this paper. Another well known cardinal inequality is due to Arhangel'skii [3]: "For $X \in \mathcal{T}_2$, $|X| \leq 2^{L(X)t(X)\psi(X)}$". Fedeli

[2] proved, making use of elementary submodels, that: if X is a T_2-space then $|X| \leq 2^{\mathrm{aql}(X)t(X)\psi_c(X)}$. This result generalizes the Arhangel'skii's inequality. On the other hand, in [1], Arhangel'skii proved that: (a) "If X is a T_1 space such that (i) $L(X)t(X) \leq \kappa$, (ii) $\psi(X) \leq 2^\kappa$, and (iii) for all $A \in [X]^{\leq 2^\kappa}$, $|\overline{A}| \leq 2^\kappa$, then $|X| \leq 2^\kappa$". From this result one easily obtains the Arhangel'skii's inequality mentioned above.

Since $\mathrm{aql}(X) \leq L(X)$ for every topological space X, it is natural to ask if L can be replace by aql in the inequality (a). The next theorem gives an affirmative answer to this question. Our proof makes use of the Pol-Šapirovskii technique.

Theorem 4. *Let X be a T_1-space such that* (i) $\mathrm{aql}(X)t(X) \leq \kappa$, (ii) $\psi(X) \leq 2^\kappa$, *and* (iii) *if $A \in [X]^{\leq 2^\kappa}$ then $|\overline{A}| \leq 2^\kappa$. Then $|X| \leq 2^\kappa$.*

Proof: Let S be an element of $[X]^{\leq 2^\kappa}$ witnessing that $\mathrm{aql}(X) \leq \kappa$. For each $x \in X$, let $\mathcal{B}_x$ an pseudobase of x in X such that $|\mathcal{B}_x| \leq \kappa$.

Construct an increasing sequence $\left\{A_\alpha : \alpha \in \kappa^+\right\}$ of closed sets in X and a sequence $\left\{\mathcal{V}_\alpha : \alpha \in \kappa^+\right\}$ of open collections in X such that

(1) $|A_\alpha| \leq 2^\kappa$, $0 \leq \alpha < \kappa^+$;
(2) $\mathcal{V}_\alpha = \bigcup\{\mathcal{B}_x : x \in A_\alpha\}$;
(3) if $\mathcal{U} \subseteq \bigcup\left\{\mathcal{B}_x : x \in \mathrm{cl}_X\left(\bigcup_{\beta<\alpha} A_\beta\right)\right\}$ with $|\mathcal{U}| \leq \kappa$ and $X - (S \cup \bigcup\mathcal{U}) \neq \emptyset$, then $A_\alpha - (S \cup \bigcup\mathcal{U}) \neq \emptyset$.

The construction is by transfinite induction. Let $0 < \alpha < \kappa^+$ and assume that A_β and $\mathcal{V}_\beta$ have been constructed for each $\beta \in \alpha$. Note that $\mathcal{V}_\alpha$ is defined by (2). Let $P_\alpha = \mathrm{cl}_X\left(\bigcup_{\beta<\alpha} A_\beta\right)$ and let $\mathcal{C}_\alpha = \bigcup\{\mathcal{B}_x : x \in P_\alpha\}$. Since $\left|\bigcup_{\beta<\alpha} A_\beta\right| \leq 2^\kappa$, it follows by (iii) that $|P_\alpha| \leq 2^\kappa$, hence, $|\mathcal{C}_\alpha| \leq 2^\kappa$. For each $\mathcal{U} \subseteq \mathcal{C}_\alpha$ with $|\mathcal{U}| \leq \kappa$ and $X - (S \cup \bigcup\mathcal{U}) \neq \emptyset$, choose one point in $X - (S \cup \bigcup\mathcal{U})$. Let L_α be the set of points chosen in this way. Clearly $|L_\alpha| \leq 2^\kappa$. Let $A_\alpha = \overline{P_\alpha \cup L_\alpha}$. This completes the construction.

Let $A = \bigcup\left\{A_\alpha : \alpha \in \kappa^+\right\}$ and note that A is closed in X; moreover, clearly $|A| \leq 2^\kappa$. Let $\mathcal{V} = \bigcup\left\{\mathcal{V}_\alpha : \alpha \in \kappa^+\right\}$. The proof is complete if $X = S \cup A$. Suppose not, let $p \in X - (S \cup A)$ and for each $x \in A$, choose $V_x \in \mathcal{B}_x$ such that $p \notin V_x$. Then $\{V_x : x \in A\}$ together with $\{X - A\}$ cover X; hence, there exists $B \subseteq [A]^{\leq\kappa}$ such that $X = S \cup (\bigcup\{V_x : x \in B\}) \cup (X - A)$. Let $U = \bigcup\{V_x : x \in B\}$. Since $|B| \leq \kappa$, by the regularity of κ^+ there exists $\alpha \in \kappa^+$ such that $\{V_x : x \in B\} \subseteq \bigcup\left\{\mathcal{B}_x : x \in \mathrm{cl}_X\left(\bigcup_{\beta<\alpha} A_\beta\right)\right\}$, that is U is the union of $\leq \kappa$ elements of $\bigcup\left\{\mathcal{B}_x : x \in \mathrm{cl}_X\left(\bigcup_{\beta<\alpha} A_\beta\right)\right\}$ and $X - (S \cup U) \neq \emptyset$. Hence by (3), $A_\alpha - (S \cup U) \neq \emptyset$. Since $A_\alpha \subseteq A \subseteq S \cup U$, we reach a contradiction. Thus $X = S \cup A$. □

Now we have the inequality (a), as a consequence of our theorem.

Corollary 5 (Arhangel'skii)**.** *Let X be a T_1-space such that:* (i) $L(X)t(X) \leq \kappa$, (ii) $\psi(X) \leq 2^\kappa$, *and* (iii) *for all* $A \in [X]^{\leq 2^\kappa}$, $|\overline{A}| \leq 2^\kappa$. *Then* $|X| \leq 2^\kappa$.

Another consequence of Theorem 5 is the next theorem due to Fedeli.

Corollary 6. *If X is a T_2-space then* $|X| \leq 2^{\mathrm{aql}(X)\psi_c(X)t(X)}$.

Proof: Let $\kappa = \mathrm{aql}(X)\psi_c(X)t(X)$. It is enough to note that for all $A \in [X]^{\leq 2^\kappa}$, $|\overline{A}| \leq 2^\kappa$. □

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