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Duality theory of spaces of vector-valued continuous functions

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Abstract. Let X be a completely regular Hausdorff space, E a real normed space, and let $C_b(X,E)$ be the space of all bounded continuous E-valued functions on X. We develop the general duality theory of the space $C_b(X,E)$ endowed with locally solid topologies; in particular with the strict topologies $\beta_z(X,E)$ for $z=\sigma,\tau,t$. As an application, we consider criteria for relative weak-star compactness in the spaces of vector measures $M_z(X,E')$ for $z=\sigma,\tau,t$. It is shown that if a subset H of $M_z(X,E')$ is relatively $\sigma(M_z(X,E'),C_b(X,E))$-compact, then the set $\operatorname{conv}(S(H))$ is still relatively $\sigma(M_z(X,E'),C_b(X,E))$-compact ($S(H)=$ the solid hull of H in $M_z(X,E')$). A Mackey-Arens type theorem for locally convex-solid topologies on $C_b(X,E)$ is obtained.

Keywords: vector-valued continuous functions, strict topologies, locally solid topologies, weak-star compactness, vector measures

Classification: 46E10, 46E15, 46E40, 46G10

1. Introduction and preliminaries

Let X be a completely regular Hausdorff space and let $(E,\|\cdot\|_E)$ be a real normed space. Let B_E and S_E stand for the closed unit ball and the unit sphere in E, and let E' stand for the topological dual of $(E,\|\cdot\|_E)$. Let $C_b(X,E)$ be the space of all bounded continuous functions $f:X\to E$. We will write $C_b(X)$ instead of $C_b(X,\mathbb{R})$, where $\mathbb{R}$ is the field of real numbers. For a function $f\in C_b(X,E)$ we will write $\|f\|(x)=\|f(x)\|_E$ for $x\in X$. Then $\|f\|\in C_b(X)$ and the space $C_b(X,E)$ can be equipped with the norm $\|f\|_\infty=\sup_{x\in X}\|f\|(x)=\|\|f\|\|_\infty$, where $\|u\|_\infty=\sup_{x\in X}|u(x)|$ for $u\in C_b(X)$.

It turns out that the notion of solidness in the Riesz space (= vector lattice) $C_b(X)$ can be lifted in a natural way to $C_b(X,E)$ (see [NR]). Recall that a subset H of $C_b(X,E)$ is said to be *solid* whenever $\|f_1\|\le\|f_2\|$ (i.e., $\|f_1(x)\|_E\le\|f_2(x)\|_E$ for all $x\in X$) and $f_1\in C_b(X,E)$, $f_2\in H$ imply $f_1\in H$. A linear topology τ on $C_b(X,E)$ is said to be *locally solid* if it has a local base at 0 consisting of solid sets. A linear topology τ on $C_b(X,E)$ that is at the same time locally convex and locally solid will be called a *locally convex-solid topology*.

In [NR] we examine the general properties of locally solid topologies on the space $C_b(X,E)$. In particular, we consider the mutual relationship between locally solid topologies on $C_b(X,E)$ and $C_b(X)$. It is well known that the so-called strict topologies $\beta_z(X,E)$ on $C_b(X,E)$ ($z=t,\tau,\sigma,g,p$) are locally convex-solid topologies (see [Kh, Theorem 8.1], [KhO$_2$, Theorem 6], [KhV$_1$, Theorem 5]).

For a linear topological space (L,ξ), by $(L,\xi)'$ (or L'_ξ) we will denote its topological dual. We will write $C_b(X,E)'$ and $C_b(X)'$ instead of $(C_b(X,E),\|\cdot\|_\infty)'$ and $(C_b(X),\|\cdot\|_\infty)'$ respectively. By $\sigma(L,M)$ and $\tau(L,M)$ we will denote the weak topology and the Mackey topology with respect to a dual pair $\langle L,M\rangle$. For terminology concerning locally solid Riesz spaces we refer to [AB$_1$], [AB$_2$].

In the present paper, we develop the duality theory of the space $C_b(X,E)$ endowed with locally solid topologies (in particular, the strict topologies $\beta_z(X,E)$, where $z=\sigma,\tau,t$).

In Section 2 we examine the topological dual of $C_b(X,E)$ endowed with a locally solid topology τ. We obtain that $(C_b(X,E),\tau)'$ is an ideal of $C_b(X,E)'$. We consider a mutual relationship between topological duals of the spaces $C_b(X)$ and $C_b(X,E)$, which allows us to examine in a unified manner continuous linear functionals on $C_b(X,E)$ by means of continuous linear functionals on $C_b(X)$.

In Section 3 we consider criteria for relative weak-star compactness in spaces of vector measures $M_z(X,E')$ for $z=\sigma,\tau,t$. In particular, we show that if a subset H of $M_z(X,E')$ is relatively $\sigma(M_z(X,E'),C_b(X,E))$-compact, then $\operatorname{conv}(S(H))$ is still relatively $\sigma(M_z(X,E'),C_b(X,E))$-compact (here $S(H)$ stand for the solid hull of H in $M_z(X,E')$; see Definition 3.1 below).

Section 4 deals with the absolute weak and the absolute Mackey topologies on $C_b(X,E)$. A Mackey-Arens type theorem for locally convex-solid topologies on $C_b(X,E)$ is obtained.

Now we recall some properties of locally solid topologies on $C_b(X,E)$ as set out in [NR]. A seminorm ρ on $C_b(X,E)$ is said to be *solid* whenever $\rho(f_1)\le\rho(f_2)$ if $f_1,f_2\in C_b(X,E)$ and $\|f_1\|\le\|f_2\|$.

Note that a solid seminorm on the vector lattice $C_b(X)$ is usually called a Riesz seminorm (see [AB$_1$]).

Theorem 1.1 (see [NR, Theorem 2.2]). *For a locally convex topology τ on $C_b(X,E)$ the following statements are equivalent:*

(i) *τ is generated by some family of solid seminorms;*
(ii) *τ is a locally convex-solid topology.*

From Theorem 1.1 it follows that any locally convex-solid topology τ on $C_b(X,E)$ admits a local base at 0 formed by sets which are simultaneously absolutely convex and solid.

Recall that the algebraic tensor product $C_b(X)\otimes E$ is the subspace of $C_b(X,E)$ spanned by the functions of the form $u\otimes e$, $(u\otimes e)(x)=u(x)e$, where $u\in C_b(X)$ and $e\in E$.

Now we briefly explain the general relationship between locally convex-solid topologies on $C_b(X)$ and $C_b(X,E)$ (see [NR]). Given a Riesz seminorm p on

$C_b(X)$ let us set

$$p^{\vee}(f) := p(\|f\|) \quad \text{for all} \quad f \in C_b(X,E).$$

It is seen that $p^{\vee}$ is a solid seminorm on $C_b(X,E)$. From now on let $e_0 \in S_E$ be fixed. Given a solid seminorm ρ on $C_b(X,E)$ one can define a Riesz seminorm $\rho^{\wedge}$ on $C_b(X)$ by:

$$\rho^{\wedge}(u) := \rho(u \otimes e_0) \quad \text{for all} \quad u \in C_b(X).$$

One can easily show:

Lemma 1.2 (see [NR, Lemma 3.1]). (i) *If ρ is a solid seminorm on $C_b(X,E)$, then $(\rho^{\wedge})^{\vee}(f) = \rho(f)$ for all $f \in C_b(X,E)$.*

(ii) *If p is a Riesz seminorm on $C_b(X)$, then $(p^{\vee})^{\wedge}(u) = p(u)$ for all $u \in C_b(X)$.*

Let τ be a locally convex-solid topology on $C_b(X,E)$ and let $\{\rho_\alpha : \alpha \in \mathcal{A}\}$ be a family of solid seminorms on $C_b(X,E)$ that generates τ. By $\tau^{\wedge}$ we will denote the locally convex-solid topology on $C_b(X)$ generated by the family $\{\rho_\alpha^{\wedge} : \alpha \in \mathcal{A}\}$.

Next, let ξ be a locally convex-solid topology on $C_b(X)$ and let $\{p_\alpha : \alpha \in \mathcal{A}\}$ be a family of solid seminorms on $C_b(X)$ that generates ξ. By $\xi^{\vee}$ we will denote the locally convex-solid topology on $C_b(X,E)$ generated by the family $\{p_\alpha^{\vee} : \alpha \in \mathcal{A}\}$.

As an immediate consequence of Lemma 1.2 we have:

Theorem 1.3 (see [NR, Theorem 3.2]). *For a locally convex-solid topology τ on $C_b(X,E)$ (resp. ξ on $C_b(X)$) we have:*

$$\left(\tau^{\wedge}\right)^{\vee} = \tau \quad \left(\textit{resp.}\ \left(\xi^{\vee}\right)^{\wedge} = \xi\right).$$

The strict topologies $\beta_z(X,E)$ on $C_b(X,E)$, where $z = t,\tau,\sigma,g,p$ have been examined in [F], [KhC], [Kh], [KhO$_1$], [KhO$_2$], [KhO$_3$], [KhV$_1$], [KhV$_2$]. In this paper we will consider the strict topologies $\beta_z(X,E)$, where $z = t,\tau,\sigma$. We will write $\beta_z(X)$ instead of $\beta_z(X,\mathbb{R})$.

Now we recall the concept of a strict topology on $C_b(X,E)$. Let βX stand for the Stone-Čech compactification of X. For $v \in C_b(X)$, $\overline{v}$ denotes its unique continuous extension to βX. For a compact subset Q of $\beta X \setminus X$ let $C_Q(X) = \{v \in C_b(X) : \overline{v} \mid Q \equiv 0\}$. Let $\beta_Q(X,E)$ be the locally convex topology on $C_b(X,E)$ defined by the family of solid seminorms $\{\varrho_v : v \in C_Q(X)\}$, where $\varrho_v(f) = \sup_{x\in X} |v(x)|\, \|f\|(x)$ for $f \in C_b(X,E)$.

Now let $\mathcal{C}$ be some family of compact subsets of $\beta X \setminus X$. The *strict topology* $\beta_{\mathcal{C}}(X,E)$ on $C_b(X,E)$ determined by $\mathcal{C}$ is the greatest lower bound (in the class of locally convex topologies) of the topologies $\beta_Q(X,E)$, as Q runs over $\mathcal{C}$ (see [NR] for more details). In particular, it is known that $\beta_{\mathcal{C}}(X,E)$ is locally solid (see [NR, Theorem 4.1]).

The strict topologies $\beta_\tau(X,E)$ and $\beta_\sigma(X,E)$ on $C_b(X,E)$ are obtained by choosing the family $\mathcal{C}_\tau$ of all compact subsets of $\beta X \setminus X$ and the family $\mathcal{C}_\sigma$ of all zero subsets of $\beta X \setminus X$ as $\mathcal{C}$, resp. In view of [NR, Corollary 4.4] for $z = \tau, \sigma$ we have

$$\beta_z(X)^\vee = \beta_z(X,E) \quad \text{and} \quad \beta_z(X,E)^\wedge = \beta_z(X).$$

The strict topology $\beta_t(X,E)$ on $C_b(X,E)$ is generated by the family $\{\varrho_v : v \in C_0(X)\}$, where $C_0(X)$ denotes the space of scalar-valued continuous functions on X, vanishing at infinity. It is easy to show that

$$\beta_t(X)^\vee = \beta_t(X,E) \quad \text{and} \quad \beta_t(X,E)^\wedge = \beta_t(X).$$

2. Topological dual of $C_b(X,E)$ with locally solid topologies

For a linear functional Φ on $C_b(X,E)$ let us put

$$|\Phi|(f) = \sup\big\{|\Phi(h)| : h \in C_b(X,E),\ \|h\| \le \|f\|\big\}.$$

The next theorem gives a characterization of the space $C_b(X,E)'$.

Theorem 2.1. *We have*

$$C_b(X,E)' = \big\{\Phi \in C_b(X,E)^\# : |\Phi|(f) < \infty \ \ \textit{for all} \ \ f \in C_b(X,E)\big\},$$

where $C_b(X,E)^\#$ denotes the algebraic dual of $C_b(X,E)$.

PROOF: Indeed, by the way of contradiction, assume that for some $\Phi_0 \in C_b(X,E)'$ we have $|\Phi_0|(f_0) = \infty$ for some $f_0 \in C_b(X,E)$. Hence there exists a sequence (h_n) in $C_b(X,E)$ such that $\|h_n\| \le \|f_0\|$ and $|\Phi_0(h_n)| \ge n$ for all $n \in \mathbb{N}$. Since $\|n^{-1}h_n\|_\infty \to 0$, we get $n^{-1}\Phi_0(h_n) \to 0$, which is in contradiction with $|\Phi_0(h_n)| \ge n$.

Next, assume by the way of contradiction that there exists a linear functional Φ_0 on $C_b(X,E)$ such that $|\Phi_0|(f) < \infty$ for all $f \in C_b(X,E)$ and $\Phi_0 \notin C_b(X,E)'$. Then there exists a sequence (f_n) in $C_b(X,E)$ such that $\|f_n\|_\infty = 1$ and $|\Phi_0(f_n)| > n^3$ for all $n \in \mathbb{N}$. Since $\sum_{n=1}^\infty \frac{1}{n^2}\|\|f_n\|\|_\infty < \infty$ and the space $(C_b(X), \|\cdot\|_\infty)$ is complete, there exists $u_0 \in C_b(X)^+$ such that $\sum_{n=1}^\infty \frac{1}{n^2}\|f_n\| = u_0$. Let $f_0 = u_0 \otimes e_0$ for some fixed $e_0 \in S_E$. Then $\frac{1}{n^2}\|f_n\| \le \|f_0\| = u_0$. Hence for all $n \in \mathbb{N}$, $n < |\Phi_0(f_n/n^2)| \le |\Phi_0|(f_n/n^2) \le |\Phi_0|(f_0) < \infty$, which is impossible. Thus the proof is complete. $\square$

Now we consider the concept of solidness in $C_b(X,E)'$.

Definition 2.1. For $\Phi_1, \Phi_2 \in C_b(X,E)'$ we will write $|\Phi_1| \le |\Phi_2|$ whenever $|\Phi_1|(f) \le |\Phi_2|(f)$ for all $f \in C_b(X,E)$. A subset A of $C_b(X,E)'$ is said to be *solid* whenever $|\Phi_1| \le |\Phi_2|$ with $\Phi_1 \in C_b(X,E)'$ and $\Phi_2 \in A$ implies $\Phi_1 \in A$. A linear subspace I of $C_b(X,E)'$ will be called an *ideal* whenever I is solid.

Since the intersection of any family of solid subsets of $C_b(X,E)'$ is solid, every subset A of $C_b(X,E)'$ is contained in the smallest (with respect to the inclusion) solid set called the *solid hull* of A and denoted by $S(A)$. Note that

$$S(A)=\big\{\Phi\in C_b(X,E)':|\Phi|\le|\Psi|\quad\text{for some}\quad \Psi\in A\big\}.$$

Lemma 2.2. *Let $\Phi\in C_b(X,E)'$. Then for $f\in C_b(X,E)$,*

$$(*)\qquad |\Phi|(f)=\sup\big\{|\Psi(f)|:\Psi\in C_b(X,E)',\ |\Psi|\le|\Phi|\big\}.$$

Moreover, if A is a subset of $C_b(X,E)'$ then for $f\in C_b(X,E)$ we have

$$(**)\qquad \begin{aligned}\sup\big\{|\Phi|(f):\Phi\in A\big\}&=\sup\big\{|\Psi(f)|:\Psi\in S(A)\big\}\\&=\sup\big\{|\Psi(f)|:\Psi\in\operatorname{conv}\big(S(A)\big)\big\}.\end{aligned}$$

Proof: Note first that $|\Phi|$ is a seminorm on $C_b(X,E)$. To see that $|\Phi|(f_1+f_2)\le|\Phi|(f_1)+|\Phi|(f_2)$ holds for $f_1,f_2\in C_b(X,E)$ with $f_1,f_2\ne 0$, assume that $h\in C_b(X,E)$ and $\|h\|\le\|f_1+f_2\|$. Then for $h_i=(\|f_i\|/(\|f_1\|+\|f_2\|))h$ for $i=1,2$ we have $h=h_1+h_2$ and $\|h_i\|\le\|f_i\|$ for $i=1,2$. Thus $|\Phi(h)|\le|\Phi(h_1)|+|\Phi(h_2)|\le|\Phi|(h_1)+|\Phi|(h_2)\le|\Phi|(f_1)+|\Phi|(f_2)$. Hence $|\Phi|(f_1+f_2)\le|\Phi|(f_1)+|\Phi|(f_2)$, as desired. Moreover, one can easily show that $|\Phi|(\lambda f)=|\lambda|\,|\Phi|(f)$ for all $\lambda\in\mathbb{R}$.

For a fixed $f_0\in C_b(X,E)$ we define a functional Ψ_0 on the linear subspace $L_{f_0}=\{\lambda f_0:\lambda\in\mathbb{R}\}$ of $C_b(X,E)$ by putting $\Psi_0(\lambda f_0)=\lambda|\Phi|(f_0)$ for $\lambda\in\mathbb{R}$. It is clear that Ψ_0 is a linear functional on L_{f_0} and $|\Psi_0(\lambda f_0)|=|\Phi|(\lambda f_0)$ for $\lambda\in\mathbb{R}$. Then by the Hahn-Banach extension theorem there exists a linear functional Ψ on $C_b(X,E)$ such that $\Psi(f)\le|\Phi|(f)$ for all $f\in C_b(X,E)$ and $\Psi(\lambda f_0)=\Psi_0(\lambda f_0)$ for all $\lambda\in\mathbb{R}$. Since Ψ is linear and $|\Phi|(f)=|\Phi|(-f)$ we get $|\Psi(f)|\le|\Phi|(f)$ for all $f\in C_b(X,E)$. To see that $|\Psi|\le|\Phi|$ let $f\in C_b(X,E)$ and take $h\in C_b(X,E)$ with $\|h\|\le\|f\|$. Then $|\Psi(h)|\le|\Phi|(h)\le|\Phi|(f)$, so $|\Psi|(f)\le|\Phi|(f)$. Thus $|\Psi|\le|\Phi|$. Moreover, $\Psi(f_0)=\Psi_0(f_0)=|\Phi|(f_0)$, so

$$|\Phi|(f_0)=\sup\big\{|\Psi(f_0)|:\Psi\in C_b(X,E)',\quad |\Psi|\le|\Phi|\big\}.$$

Thus $(*)$ is shown. As a consequence of $(*)$ we easily obtain that $(**)$ holds. □

We now introduce the concept of a *solid dual system*. Let I be an ideal of $C_b(X,E)'$ separating the points of $C_b(X,E)$. Then the pair $\langle C_b(X,E),I\rangle$, under its natural duality

$$\langle f,\Phi\rangle=\Phi(f)\quad\text{for}\quad f\in C_b(X,E),\quad \Phi\in I$$

will be referred to as a *solid dual system*.

For a subset A of $C_b(X,E)$ and a subset B of I let us set

$$\begin{aligned}A^0&=\big\{\Phi\in I:|\langle f,\Phi\rangle|\le 1\quad\text{for all}\quad f\in A\big\},\\ {}^0B&=\big\{f\in C_b(X,E):|\langle f,\Phi\rangle|\le 1\quad\text{for all}\quad \Phi\in B\big\}.\end{aligned}$$

By making use of Lemma 2.2 we can get the following result.

Theorem 2.3. *Let $\langle C_b(X,E), I\rangle$ be a solid dual system.*

(i) *If a subset A of $C_b(X,E)$ is solid, then A^0 is a solid subset of I.*

(ii) *If a subset B of I is solid, then ${}^0B$ is a solid subset of $C_b(X,E)$.*

Proof: (i) Let $|\Phi_1| \le |\Phi_2|$ with $\Phi_1 \in I$ and $\Phi_2 \in A^0$. Assume that $f \in A$ and let $h \in C_b(X,E)$ with $\|h\| \le \|f\|$. Then $h \in A$, because A is solid, so $|\Phi_2(h)| \le 1$. Hence $|\Phi_2|(f) \le 1$. Thus $|\Phi_1(f)| \le |\Phi_1|(f) \le 1$, so $\Phi_1 \in A^0$. This means that A^0 is a solid subset of I.

(ii) Let $\|f_1\| \le \|f_2\|$ with $f_1 \in C_b(X,E)$ and $f_2 \in {}^0B$. To see that $f_1 \in {}^0B$ assume that $\Phi \in B$. Since B is a solid subset of I, by Lemma 2.2 the identity $|\Phi|(f_2) = \sup\{|\Psi(f_2)| : \Psi \in B,\ |\Psi| \le |\Phi|\}$ holds. Thus for every $\Psi \in B$ with $|\Psi| \le |\Phi|$ we have $|\Psi(f_2)| \le 1$, so $|\Phi|(f_2) \le 1$. Since $|\Phi(f_1)| \le |\Phi|(f_1) \le |\Phi|(f_2) \le 1$, we get $f_1 \in {}^0B$, as desired. □

Theorem 2.4. *Let τ be a locally solid topology on $C_b(X,E)$. Then $(C_b(X,E),\tau)'$ is an ideal of $C_b(X,E)'$.*

Proof: To show that $(C_b(X,E),\tau)' \subset C_b(X,E)'$, by the way of contradiction assume that for some $\Phi_0 \in (C_b(X,E),\tau)'$ we have $\Phi_0 \notin C_b(X,E)'$, so in view of Theorem 2.1 we get $|\Phi_0|(f_0) = \infty$ for some $f_0 \in C_b(X,E)$. Hence there exists a sequence (h_n) in $C_b(X,E)$ such that $\|h_n\| \le \|f_0\|$ and $|\Phi_0(h_n)| \ge n$ for $n \in \mathbb{N}$. Since $n^{-1}f_0 \to 0$ for τ, and τ is locally solid, we get $n^{-1}h_n \to 0$ for τ. Hence $\Phi_0(n^{-1}h_n) \to 0$, which is in contradiction with $|\Phi_0(h_n)| \ge n$.

To see that $(C_b(X,E),\tau)'$ is an ideal of $C_b(X,E)'$ assume that $|\Phi_1| \le |\Phi_2|$ with $\Phi_1 \in C_b(X,E)'$ and $\Phi_2 \in (C_b(X,E),\tau)'$. Let $f_\alpha \xrightarrow{\tau} 0$ and $\varepsilon > 0$ be given. Then there exists a net (h_α) in $C_b(X,E)$ such that $\|h_\alpha\| \le \|f_\alpha\|$ for each α and $|\Phi_2|(f_\alpha) \le |\Phi_2(h_\alpha)| + \varepsilon$. Clearly $h_\alpha \xrightarrow{\tau} 0$, because τ is locally solid, so $\Phi_2(h_\alpha) \to 0$. Since $|\Phi_1(f_\alpha)| \le |\Phi_1|(f_\alpha) \le |\Phi_2|(f_\alpha) \le |\Phi_2(f_\alpha)| + \varepsilon$, we get $\Phi_1(f_\alpha) \to 0$, so $\Phi_1 \in (C_b(X,E),\tau)'$, as desired. □

Theorem 2.5. *For a Hausdorff locally convex topology τ on $C_b(X,E)$ the following statements are equivalent:*

(i) *τ is locally solid;*

(ii) *$(C_b(X,E),\tau)'$ is an ideal of $C_b(X,E)'$ and for every τ-equicontinuous subset A of $(C_b(X,E),\tau)'$ its solid hull $S(A)$ is also τ-equicontinuous.*

Proof: (i) $\Longrightarrow$ (ii) By Theorem 2.4 $(C_b(X,E),\tau)'$ is an ideal of $C_b(X,E)'$, and thus we have the solid dual system $\langle C_b(X,E), (C_b(X,E),\tau)'\rangle$. Assume that a subset A of $(C_b(X,E),\tau)'$ is equicontinuous. Hence $A \subset V^0$ for some solid τ-neighbourhood V of zero. Hence $S(A) \subset S(V^0) = V^0$ (see Theorem 2.3). This means that $S(A)$ is a τ-equicontinuous subset of $(C_b(X,E),\tau)'$.

(ii) $\Longrightarrow$ (i) Let $\mathcal{B}_\tau$ be a local base at zero for τ consisting of absolutely convex, τ-closed sets. Assume that V is τ-neighbourhood of zero. Then there exists $U \in \mathcal{B}_\tau$

such that $U \subset V$. Moreover, the polar set U^0 is a τ-equicontinuous subset of $(C_b(X,E),\tau)'$. By our assumption $S(U^0)$ is also τ-equicontinuous. Hence there exists $W \in \mathcal{B}_\tau$ such that $W \subset {}^0S(U^0)$. Since the set ${}^0S(U^0)$ is solid in $C_b(X,E)$, $S(W) \subset {}^0S(U^0) \subset {}^0(U^0) = \overline{\text{abs conv}\, U}^{\tau} = U \subset V$. This shows that τ is locally solid, as desired. $\square$

For each $\Phi \in C_b(X,E)'$ let

$$\varphi_\Phi(u) = \sup\left\{\left|\Phi(h)\right| : h \in C_b(X,E),\ \ \|h\| \le u\right\} \quad \text{for} \quad u \in C_b(X)^+.$$

One can easily show that $\varphi_\Phi : C_b(X)^+ \to \mathbb{R}^+$ is an additive and positively homogeneous mapping (see [KhO$_1$, Lemma 1]), so φ_Φ has a unique positive extension to a linear mapping from $C_b(X)$ to $\mathbb{R}$ (denoted by φ_Φ again) and given by

$$\varphi_\Phi(u) = \varphi_\Phi(u^+) - \varphi_\Phi(u^-) \quad \text{for all} \quad u \in C_b(X)$$

(see [AB, Lemma 3.1]). Hence $\varphi_\Phi = |\varphi_\Phi|$ holds on $C_b(X)^+$. Since $C_b(X)' = C_b(X)^\sim$ (the order dual of $C_b(X)$) (see [AB$_2$, Corollary 12.5]), we get $\varphi_\Phi \in C_b(X)'$. Moreover, we have:

$$\varphi_\Phi\big(\|f\|\big) = |\Phi|(f) \quad \text{for} \quad f \in C_b(X,E)$$

and

$$\varphi_\Phi(u) = |\Phi|(u \otimes e_0) \quad \text{for} \quad u \in C_b(X)^+.$$

The following lemma will be useful.

Lemma 2.6. (i) *Assume that L is an ideal of $C_b(X)'$. Then the set*

$$C_b(X,E)'_L := \left\{\Phi \in C_b(X,E)' : \varphi_\Phi \in L\right\}$$

is an ideal of $C_b(X,E)'$.

(ii) *Assume that I is an ideal of $C_b(X,E)'$. Then the set*

$$C_b(X)'_I := \left\{\varphi \in C_b(X)' : |\varphi| \le \varphi_\Phi \quad \textit{for some} \quad \Phi \in I\right\}$$

is an ideal of $C_b(X)'$ and $C_b(X,E)'_{C_b(X)'_I} = I$.

Proof: (i) We first show that $C_b(X,E)'_L$ is a linear subspace of $C_b(X,E)'$. Assume that $\Phi_1, \Phi_2 \in C_b(X,E)'_L$, i.e., $\varphi_{\Phi_1}, \varphi_{\Phi_2} \in L$. It is easy to show that $\varphi_{\Phi_1+\Phi_2}(u) \le (\varphi_{\Phi_1} + \varphi_{\Phi_2})(u)$ for $u \in C_b(X)^+$, so $\varphi_{\Phi_1+\Phi_2} \in L$, i.e., $\Phi_1 + \Phi_2 \in C_b(X,E)'_L$. Next, let $\Phi \in C_b(X,E)'_L$ and $\lambda \in \mathbb{R}$. Then $\varphi_\Phi \in L$ and since $\varphi_{\lambda\Phi} = \lambda\varphi_\Phi$, we get $\lambda\Phi \in C_b(X,E)'_L$.

To show that $C_b(X,E)'_L$ is solid in $C_b(X,E)'$, assume that $|\Phi_1| \le |\Phi_2|$ with $\Phi_1 \in C_b(X,E)'$ and $\Phi_2 \in C_b(X,E)'_L$, i.e., $\varphi_{\Phi_2} \in L$. Then for $u \in C_b(X)^+$ we have $\varphi_{\Phi_1}(u) = |\Phi_1|(u \otimes e_0) \le |\Phi_2|(u \otimes e_0) = \varphi_{\Phi_2}(u)$. Hence $\varphi_{\Phi_1} \in L$, because L is an ideal of $C_b(X)'$. Thus $\Phi_1 \in C_b(X,E)'_L$, as desired.

(ii) To prove that $C_b(X)'_I$ is an ideal of $C_b(X)'$ assume that $|\varphi_1| \le |\varphi_2|$, where $\varphi_1 \in C_b(X)'$ and $\varphi_2 \in C_b(X)'_I$. Then $|\varphi_2| \le \varphi_\Phi$ for some $\Phi \in I$, so $|\varphi_1| \le \varphi_\Phi$, and this means that $\varphi_1 \in C_b(X)'_I$.

To show that $I \subset C_b(X,E)'_{C_b(X)'_I}$, assume that $\Phi \in I$. Then $\varphi_\Phi \in C_b(X)'_I$, so $\Phi \in C_b(X,E)'_{C_b(X)'_I}$.

Now, we assume that $\Phi \in C_b(X,E)'_{C_b(X)'_I}$, i.e., $\Phi \in C_b(X,E)'$ and $\varphi_\Phi \in C_b(X)'_I$. It follows that there exists $\Phi_0 \in I$ such that $\varphi_\Phi \le \varphi_{\Phi_0}$. Hence for every $f \in C_b(X,E)$ we have $|\Phi|(f) = \varphi_\Phi(\|f\|) \le \varphi_{\Phi_0}(\|f\|) = |\Phi_0|(f)$. Thus $\Phi \in I$, because I is an ideal of $C_b(X,E)'$. □

Let A be a subset of $C_b(X,E)'_\tau$. Then $S(A) \subset C_b(X,E)'_\tau$ as $C_b(X,E)'_\tau$ is solid (by Theorem 2.4). Hence

$$S(A) = \big\{\Phi \in C_b(X,E)'_\tau : \quad |\Phi| \le |\Psi| \quad \text{for some} \quad \Psi \in A\big\}.$$

In view of Lemma 2.2 for a subset A of $C_b(X,E)'$ and $f \in C_b(X,E)$ we have:

$$(+) \qquad \begin{aligned} \sup\big\{|\Phi|(f) : \Phi \in A\big\} &= \sup\big\{\varphi_\Phi(\|f\|) : \Phi \in A\big\} \\ &= \sup\big\{|\Psi(f)| : \Psi \in S(A)\big\}. \end{aligned}$$

Theorem 2.7. *Let τ be a locally convex-solid Hausdorff topology on $C_b(X,E)$. Then for a subset A of $C_b(X,E)'$ the following statements are equivalent:*

(i) *A is τ-equicontinuous;*
(ii) *$\operatorname{conv}(S(A))$ is τ-equicontinuous;*
(iii) *$S(A)$ is τ-equicontinuous;*
(iv) *the subset $\{\varphi_\Phi : \Phi \in A\}$ of $C_b(X)'$ is $\tau^\wedge$-equicontinuous.*

Proof: (i) $\Longrightarrow$ (ii) In view of Theorem 2.4 we have a solid dual system $\langle C_b(X,E), C_b(X,E)'_\tau\rangle$. Let A be τ-equicontinuous. Then by Theorem 1.1 there is a convex solid τ-neighbourhood V of zero such that $A \subset V^0$. Hence $\operatorname{conv}(S(A)) \subset \operatorname{conv}(S(V^0)) = V^0$ (see Theorem 2.3), and this means that $\operatorname{conv}(S(A))$ is still τ-equicontinuous.

(ii) $\Longrightarrow$ (iii) It is obvious.

(iii) $\Longrightarrow$ (iv) Assume that the subset $S(A)$ of $C_b(X,E)'$ is τ-equicontinuous. Let $\{\rho_\alpha : \alpha \in \mathcal{A}\}$ be a family of solid seminorms on $C_b(X,E)$ that generates τ. Given $\varepsilon > 0$ there exist $\alpha_1, \ldots, \alpha_n \in \mathcal{A}$ and $\eta > 0$ such that $\sup\{|\Psi(f)| : \Psi \in S(A)\} \le \varepsilon$

whenever $\rho_{\alpha_i}(f) \leq \eta$ for $i = 1, 2, \ldots, n$. To show that $\{\varphi_\Phi : \Phi \in A\}$ is $\tau^\wedge$-equicontinuous, it is enough to show that $\sup\{|\varphi_\Phi(u)| : \Phi \in A\} \leq \varepsilon$ whenever $\rho^\wedge_{\alpha_i}(u) \leq \eta$ for $i = 1, 2, \ldots, n$. Indeed, let $u \in C_b(X)$ and $\rho^\wedge_{\alpha_i}(u) \leq \eta$ for $i = 1, 2, \ldots, n$. Then $\rho_{\alpha_i}(u \otimes e_0) \leq \eta$ $(i = 1, 2, \ldots, n)$, so $\sup\{|\Psi(u \otimes e_0)| : \Psi \in S(A)\} \leq \varepsilon$. Hence, in view of (+) we obtain that $\sup\{\varphi_\Phi(|u|) : \Phi \in A\} \leq \varepsilon$, because $\|u \otimes e_0\| = |u|$. But $|\varphi_\Phi(u)| \leq \varphi_\Phi(|u|)$, and the proof is complete.

(iv) $\Longrightarrow$ (i) Assume that the set $\{\varphi_\Phi : \Phi \in A\}$ is $\tau^\wedge$-equicontinuous. Let $\{\rho_\alpha : \alpha \in \mathcal{A}\}$ be a family of solid seminorms on $C_b(X, E)$ that generates τ. Given $\varepsilon > 0$ there exist $\alpha_1, \ldots, \alpha_n \in \mathcal{A}$ and $\eta > 0$ such that $\sup\{|\varphi_\Phi(u)| : \Phi \in A\} \leq \varepsilon$ whenever $u \in C_b(X)$ and $\rho^\wedge_{\alpha_i}(u) \leq \eta$ for $i = 1, 2, \ldots, n$. Let $f \in C_b(X, E)$ with $\rho_{\alpha_i}(f) \leq \eta$ for $i = 1, 2, \ldots, n$. Since $\rho^\wedge_{\alpha_i}(\|f\|) = \rho_{\alpha_i}(\|f\| \otimes e_0) = \rho_{\alpha_i}(f)$ $(i = 1, 2, \ldots, n)$, $\sup\{|\varphi_\Phi(\|f\|)| : \Phi \in A\} \leq \varepsilon$. But $|\Phi(f)| \leq |\Phi|(f) = \varphi_\Phi(\|f\|)$, so $\sup\{|\Phi(f)| : \Phi \in A\} \leq \varepsilon$. This means that A is τ-equicontinuous. □

Corollary 2.8. *Let τ be a locally convex-solid topology on $C_b(X, E)$. Then for $\Phi \in C_b(X, E)'$ the following statements are equivalent:*

(i) *Φ is τ-continuous;*
(ii) *φ_Φ is $\tau^\wedge$-continuous.*

Corollary 2.9. *Let ξ be a locally convex-solid topology on $C_b(X)$. Then for $\Phi \in C_b(X, E)'$ the following statements are equivalent:*

(i) *Φ is $\xi^\vee$-continuous;*
(ii) *φ_Φ is ξ-continuous.*

Remark. For the equivalence (i) $\Longleftrightarrow$ (iv) of Theorem 2.7 for the strict topologies $\beta_z(X, E)$ $(z = \sigma, \tau, t, \infty, g)$ see [KhO3, Lemma 2].

Corollary 2.10. (i) *Let ξ be a locally convex-solid topology on $C_b(X)$. Then*

$$\big(C_b(X), \xi\big)' = \Big\{\varphi \in C_b(X)' : |\varphi| \leq \varphi_\Phi \ \text{ for some } \ \Phi \in \big(C_b(X, E), \xi^\vee\big)'\Big\}.$$

(ii) *Let τ be a locally convex-solid topology on $C_b(X, E)$. Then*

$$\big(C_b(X), \tau^\wedge\big)' = \Big\{\varphi \in C_b(X)' : |\varphi| \leq \varphi_\Phi \ \text{ for some } \ \Phi \in \big(C_b(X, E), \tau\big)'\Big\}.$$

Proof: (i) Let $\varphi \in (C_b(X), \xi)'$. Define a linear functional Φ_0 on the subspace $C_b(X)(e_0)$ $(= \{u \otimes e_0 : u \in C_b(X)\})$ of $C_b(X, E)$ by putting $\Phi_0(u \otimes e_0) = \varphi(u)$ for $u \in C_b(X)$. Let $\{p_\alpha : \alpha \in \mathcal{A}\}$ be a family of Riesz seminorms generating ξ. Since $\varphi \in (C_b(X), \xi)'$, there exist $c > 0$ and $\alpha_1, \ldots, \alpha_n \in \mathcal{A}$ such that for $u \in C_b(X)$

$$\big|\Phi_0(u \otimes e_0)\big| = \big|\varphi(u)\big| \leq c \max_{1 \leq i \leq n} p_{\alpha_i}(u) = c \max_{1 \leq i \leq n} p^\vee_{\alpha_i}(u \otimes e_0).$$

This means that $\Phi_0 \in (C_b(X)(e_0), \xi^\vee \mid C_b(X)(e_0))'$, so by the Hahn-Banach extension theorem there is $\Phi \in (C_b(X,E), \xi^\vee)'$ such that $\Phi(u \otimes e_0) = \varphi(u)$ for all $u \in C_b(X)$. We shall now show that $|\varphi| \leq \varphi_\Phi$, i.e., $|\varphi|(u) \leq \varphi_\Phi(u)$ for all $u \in C_b(X)^+$. Indeed, let $u \in C_b(X)^+$ be given and let $v \in C_b(X)$ with $|v| \leq u$. Then we have $|\varphi(v)| = |\Phi(v \otimes e_0)| \leq \varphi_\Phi(u)$, so $|\varphi| \leq \varphi_\Phi$, as desired.

Next, assume that $\varphi \in C_b(X)'$ with $|\varphi| \leq \varphi_\Phi$ for some $\Phi \in (C_b(X,E), \xi^\vee)'$. In view of Corollary 2.9, $\varphi_\Phi \in (C_b(X), \xi)'$ and since $(C_b(X), \xi)'$ is an ideal of $C_b(X)'$, we conclude that $\varphi \in (C_b(X), \xi)'$.

(ii) It follows from (i), because $(\tau^\wedge)^\vee = \tau$. □

It is well known that if L is a σ-Dedekind complete vector-lattice and if H is a relatively $\sigma(L_n^\sim, L)$-compact subset of $L_n^\sim$ (resp. a relatively $\sigma(L_c^\sim, L)$-compact subset of $L_c^\sim$), then the set $\operatorname{conv}(S(H))$ is still relatively $\sigma(L_n^\sim, L)$-compact (resp. relatively $\sigma(L_c^\sim, L)$-compact) (see [AB, Corollary 20.12, Corollary 20.10]) (here $L_n^\sim$ and $L_c^\sim$ stand for the order continuous dual and the σ-order continuous dual of L resp.).

Now, we shall show that this property holds in $(C_b(X,E)'_{\beta_z}, \sigma(C_b(X,E)'_{\beta_z}, C_b(X,E)))$ for $z = \sigma, \tau, t$.

Recall that a completely regular Hausdorff space X is called a *P-space* if every G_δ set in X is open (see [GJ, p. 63]).

The following result will be of importance.

Theorem 2.11. *Let H be a norm-bounded and $\sigma(C_b(X,E)'_{\beta_z}, C_b(X,E))$-compact subset of $C_b(X,E)'_{\beta_z}$, where $z = \sigma$ (resp. $z = \tau$ and X is a paracompact space; resp. $z = \tau$ and X is a P-space). Then H is $\beta_z(X,E)$-equicontinuous.*

Proof: See [KhO$_1$, Theorem 5] for $z = \sigma$; [Kh, Theorem 6.1] for $z = \tau$ and [KhC, Lemma 3] for $z = t$. □

Now we are ready to state our main result.

Theorem 2.12. *Let H be a norm bounded subset of $C_b(X,E)'_{\beta_z}$, where $z = \sigma$ (resp. $z = \tau$ and X is a paracompact space; resp. $z = t$ and X is a P-space). Then the following statements are equivalent:*

(i) *H is relatively countably $\sigma(C_b(X,E)'_{\beta_z}, C_b(X,E))$-compact;*
(ii) *H is $\beta_z(X,E)$-equicontinuous;*
(iii) *$\operatorname{conv}(S(H))$ is relatively $\sigma(C_b(X,E)'_{\beta_z}, C_b(X,E))$-compact;*
(iv) *$S(H)$ is relatively $\sigma(C_b(X,E)'_{\beta_z}, C_b(X,E))$-compact;*
(v) *H is relatively $\sigma(C_b(X,E)'_{\beta_z}, C_b(X,E))$-compact.*

Proof: (i) $\Longrightarrow$ (ii) See Theorem 2.11.

(ii) $\Longrightarrow$ (iii) In view of Theorem 2.7 the set $\operatorname{conv}(S(H))$ is $\beta_z(X,E)$-equicontinuous, i.e., there is a neighbourhood of 0 for $\beta_z(X,E)$ such that $\operatorname{conv}(S(H)) \subset V^0$

(= the polar set with respect to the dual pair $\langle\, C_b(X,E), C_b(X,E)'_{\beta_z}\rangle$). Then by the Banach-Alaoglu's theorem the set V^0 is $\sigma(C_b(X,E)'_{\beta_z}, C_b(X,E))$-compact, so the set conv$\,(S(H))$ is relatively $\sigma(C_b(X,E)'_{\beta_z}, C_b(X,E))$-compact.

(iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (i) It is obvious. □

3. Weak-star compactness in some spaces of vector measures

In this section we consider criteria for relative weak-star compactness in some spaces of vector measures $M_z(X,E')$ for $z=\sigma,\tau,t$. In particular, by making use of Theorem 2.11 we show that if a subset H of $M_z(X,E')$ is relatively $\sigma(M_z(X,E'), C_b(X,E))$-compact, then the set conv$\,(S(H))$ is still relatively $\sigma(M_z(X,E'), C_b(X,E))$-compact (here $S(H)$ stand for the solid hull of H is $M_z(X,E')$). We start by recalling some notions and results concerning the topological measure theory (see [V], [S], [Wh]).

Let $B(X)$ be the algebra of subsets of X generated by the zero sets. Let $M(X)$ be the space of all bounded finitely additive regular (with respect to the zero sets) measures on $B(X)$. The spaces of all σ-additive, τ-additive and tight members of $M(X)$ will be denoted by $M_\sigma(X)$, $M_\tau(X)$ and $M_t(X)$ respectively (see [V], [Wh]). It is well known that $M_z(X)$ for $z=\sigma,\tau,t$ are ideals of $M(X)$ (see [Wh, Theorem 7.2]).

Theorem 3.1 (A.D. Alexandroff; [Wh, Theorem 5.1]). *For a linear functional $\varphi : C_b(X)\to\mathbb{R}$ the following statements are equivalent.*

(i) $\varphi\in C_b(X)'$.

(ii) *There exists a unique $\mu\in M(X)$ such that*

$$\varphi(u)=\varphi_\mu(u)=\int_X u\,\mathrm{d}\mu\quad \textit{for all}\quad u\in C_b(X).$$

Moreover, $\mu\ge 0$ if and only if $\varphi_\mu(u)\ge 0$ for all $u\in C_b(X)^+$.

By $M(X,E')$ we denote the set of all finitely additive measures $m : B(X)\to E'$ with the following properties:

(i) For every $e\in E$, the function $m_e : B(X)\to\mathbb{R}$ defined by $m_e(A)=m(A)(e)$, belongs to $M(X)$.

(ii) $|m|(X)<\infty$, where for $A\in B(X)$

$$|m|(A)=\sup\Big\{\Big|\sum_{i=1}^n m(B_i)(e_i)\Big| : \bigcup_{i=1}^n B_i=A,\ B_i\in B(X),\ B_i\cap B_j=\emptyset \\ \text{for } i\ne j,\ e_i\in B_E,\ n\in\mathbb{N}\Big\}.$$

For $z = \sigma, \tau, t$ let

$$M_z(X, E') = \{m \in M(X, E') : m_e \in M_z(X) \text{ for every } e \in E\}.$$

It is well known that $|m| \in M(X)$ (resp. $|m| \in M_z(X)$ for $z = \sigma, \tau, t$) whenever $m \in M(X, E')$ (resp. $m \in M_z(X, E')$ for $z = \sigma, \tau, t$) (see [F, Proposition 3.9]).

Now we are ready to define the notion of solidness in $M(X, E')$.

Definition 3.1. For $m_1, m_2 \in M(X, E')$ we will write $|m_1| \le |m_2|$ whenever $|m_1|(B) \le |m_2|(B)$ for every $B \in B(X)$. A subset H of $M(X, E')$ is said to be *solid* whenever $|m_1| \le |m_2|$ with $m_1 \in M(X, E')$ and $m_2 \in H$ imply $m_1 \in H$. A linear subspace I of $M(X, E')$ will be called an *ideal* of $M(X, E')$ whenever I is a solid subset of $M(X, E')$.

Proposition 3.2. *$M_z(X, E')$ $(z = \sigma, \tau, t)$ is an ideal of $M(X, E')$.*

Proof: Let $|m_1| \le |m_2|$, where $m_1 \in M(X, E')$ and $m_2 \in M_z(X, E')$. Then $|m_1| \in M(X)$ and $|m_2| \in M_z(X)$, and since $M_z(X)$ is an ideal of $M(X)$ we conclude that $|m_1| \in M_z(X)$. For each $e \in E$ we have $|(m_1)_e|(B) \le \|e\|_E\, |m_1|(B)$ for $B \in B(X)$, so $(m_1)_e \in M_z(X)$, i.e., $m_1 \in M_z(X, E')$. $\square$

Since the intersection of any family of solid subsets of $M(X, E')$ is solid, every subset H of $M(X, E')$ is contained in the smallest (with respect to inclusion) solid set called the *solid hull* of H and denoted by $S(H)$. Note that

$$S(H) = \{m \in M(X, E') : |m| \le |m'| \quad \text{for some} \quad m' \in H\}.$$

Now we recall some results concerning a characterization of the topological duals of $(C_b(X, E), \beta_z(X, E))$ in terms of the spaces $M_z(X, E')$ ($z = \sigma, \tau, t$).

Theorem 3.3. *Assume that $\beta_z(X, E)$ is the strict topology on $C_b(X, E)$, where $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_\sigma(X, E))$ (resp. $z = \tau$; resp. $z = t$). Then for a linear functional Φ on $C_b(X, E)$ the following statements are equivalent.*

(i) *Φ is $\beta_z(X, E)$-continuous.*

(ii) *There exists a unique $m \in M_z(X, E')$ such that*

$$\Phi(f) = \Phi_m(f) = \int_X f \, \mathrm{d}m \quad \text{for every} \quad f \in C_b(X, E).$$

(iii) *The functional φ_Φ is $\beta_z(X)$-continuous.*

Moreover, $\|\Phi_m\| = |m|(X)$ for $m \in M_z(X, E')$.

Proof: (i) $\Longleftrightarrow$ (ii) See [Kh, Theorem 5.3] for $z = \sigma$; [Kh, Corollary 3.9] for $z = \tau$; [F_1, Theorem 3.13] for $z = t$.

(ii) $\Longleftrightarrow$ (iii) It follows from Corollary 2.8, because $\beta_z(X, E)^\wedge = \beta_z(X)$. $\square$

Lemma 3.4. *Assume that $m \in M_z(X,E')$, where $z=\sigma$ and $C_b(X)\otimes E$ is dense in $(C_b(X,E),\beta_\sigma(X,E))$ (resp. $z=\tau$; resp. $z=t$). Then*

$$\varphi_{\Phi_m}(u)=\int_X u\,\mathrm{d}|m|=\varphi_{|m|}(u)\quad \textit{for all}\quad u\in C_b(X).$$

PROOF: Let $u\in C_b(X)^+$ and $m\in M_z(X,E')$. Then for $h\in C_b(X,E)$ with $\|h\|\le u$ by [F$_2$, Lemma 3.11] we have

$$|\Phi_m(h)|=\Big|\int_X h\,\mathrm{d}m\Big|\le\int_X\|h\|\,\mathrm{d}|m|\le\int_X u\,\mathrm{d}|m|=\varphi_{|m|}(u).$$

Hence

$$\varphi_{\Phi_m}(u)=|\Phi_m|(u\otimes e_0)=\sup\big\{|\Phi_m(h)|:h\in C_b(X,E),\ \|h\|\le u\big\}\le\varphi_{|m|}(u).$$

On the other hand, in view of [Kh, Theorem 2.1] we have

$$\varphi_{|m|}(u)=\int_X u\,\mathrm{d}|m|=\sup\big\{|\Phi_m(g)|:g\in C_b(X)\otimes E,\quad \|g\|\le u\big\},$$

so $\varphi_{|m|}(u)\le\varphi_{\Phi_m}(u)$. Thus $\varphi_{|m|}(u)=\varphi_{\Phi_m}(u)$ for all $u\in C_b(X)$. □

Lemma 3.5. *Assume that $m_1,m_2\in M_z(X,E')$, where $z=\sigma$ and $C_b(X)\otimes E$ is dense in $(C_b(X,E),\beta_\sigma(X,E))$ (resp. $z=\tau$; resp. $z=t$). Then the following statements are equivalent:*

(i) $|m_1|\le|m_2|$, *i.e.*, $|m_1|(B)\le|m_2|(B)$ *for every* $B\in B(X)$;
(ii) $\varphi_{|m_1|}(u)\le\varphi_{|m_2|}(u)$ *for every* $u\in C_b(X)^+$;
(iii) $|\Phi_{m_1}|(f)\le|\Phi_{m_2}|(f)$ *for every* $f\in C_b(X,E)$.

PROOF: (i) $\Longleftrightarrow$ (ii) It easily follows from Theorem 3.1.

(ii) $\Longrightarrow$ (iii) In view of Lemma 3.4 we get

$$\begin{aligned}|\Phi_{m_1}|(f)&=\varphi_{\Phi_{m_1}}\big(\|f\|\big)=\varphi_{|m_1|}\big(\|f\|\big)\\&\le\varphi_{|m_2|}\big(\|f\|\big)=\varphi_{\Phi_{m_2}}\big(\|f\|\big)=\big|\Phi_{m_2}\big|(f).\end{aligned}$$

(iii) $\Longrightarrow$ (ii) By Lemma 3.3 for $u\in C_b(X)^+$ and $e_0\in S_E$ we have

$$\begin{aligned}\varphi_{|m_1|}(u)&=\varphi_{\Phi_{m_1}}(u)=\big|\Phi_{m_1}\big|(u\otimes e_0)\\&\le\big|\Phi_{m_2}\big|(u\otimes e_0)=\varphi_{\Phi_{m_2}}(u)=\varphi_{|m_2|}(u).\end{aligned}$$

□

Lemma 3.6. *Assume that $H \subset M_z(X,E')$, where $z=\sigma$ and $C_b(X)\otimes E$ is dense in $(C_b(X,E),\beta_\sigma(X,E))$ (resp. $z=\tau$; resp. $z=t$), and let $\Phi_H=\{\Phi_m : m\in H\}$. Then* $\operatorname{conv}(S(\Phi_H))=\Phi_{\operatorname{conv}(S(H))}$.

PROOF: Assume that $\Phi\in\operatorname{conv}(S(\Phi_H))$. Then $\Phi=\sum_{i=1}^n\alpha_i\Phi_{m_i}=\Phi_{\sum_{i=1}^n\alpha_i m_i}$, where $m_i\in M_z(X,E')$ and $\alpha_i\ge 0$ for $i=1,2,\dots,n$ with $\sum_{i=1}^n\alpha_i=1$, and $|\Phi_{m_i}|\le|\Phi_{m_i'}|$ for some $m_i'\in H$ and $i=1,2,\dots,n$. In view of Lemma 3.5 $|m_i|\le|m_i'|$, i.e., $m_i\in S(H)$ for $i=1,2,\dots,n$ and $\sum_{i=1}^n\alpha_i m_i\in\operatorname{conv}(S(H))$. This means that $\Phi\in\Phi_{\operatorname{conv}(S(H))}$.

Assume that $\Phi\in\Phi_{\operatorname{conv}(S(H))}$. Then $\Phi=\Phi_{\sum_{i=1}^n\alpha_i m_i}=\sum_{i=1}^n\alpha_i\Phi_{m_i}$, where $m_i\in M_z(X,E')$ and $\alpha_i\ge 0$ for $i=1,2,\dots,n$ with $\sum_{i=1}^n\alpha_i=1$, and $|m_i|\le|m_i'|$ for some $m_i'\in H$ and $i=1,2,\dots,n$. By Lemma 3.5 $|\Phi_{m_i}|\le|\Phi_{m_i'}|$ for $i=1,2,\dots,n$, so $\Phi\in\operatorname{conv}(S(\Phi_H))$. □

Corollary 3.7. *Assume that $m_0\in M_z(X,E')$, where $z=\sigma$ and $C_b(X)\otimes E$ is dense in $(C_b(X,E),\beta_\sigma(X,E))$ (resp. $z=\tau$; resp. $z=t$) and let $e\in S_E$. Then for every $u\in C_b(X)^+$ we have:*

$$\int_X u\,\mathrm{d}|m_0|=\sup\Big\{\Big|\int_X u\,\mathrm{d}m_e\Big| : m\in M_z(X,E'),\ |m|\le|m_0|\Big\}.$$

PROOF: Let $m_0\in M_z(X,E')$ and $e\in S_E$. Assume that $\Phi\in C_b(X,E)'$ and $|\Phi|\le|\Phi_{m_0}|$. Since $\Phi_{m_0}\in C_b(X,E)'_{\beta_z}$ (see Theorem 3.3), by making use of Theorem 2.4 we get $\Phi\in C_b(X,E)'_{\beta_z}$. Hence in view of Theorem 3.3 and Lemma 3.5 we see that $\Phi=\Phi_m$ for some $m\in M_z(X,E')$ with $|m|\le|m_0|$.

Moreover, it is easy to observe that for every $m\in M(X,E')$ and $u\in C_b(X)$ we have:

$$\int_X (u\otimes e)\,\mathrm{d}m=\int_X u\,\mathrm{d}m_e.$$

Thus in view of Lemma 3.4, Lemma 2.2 and Lemma 3.5 we get:

$$\begin{aligned}\int_X u\,\mathrm{d}|m_0| &= \varphi_{\Phi_{m_0}}(u)=|\Phi_{m_0}|(u\otimes e)\\ &=\sup\big\{|\Phi(u\otimes e)| : \Phi\in C_b(X,E)',\ |\Phi|\le|\Phi_{m_0}|\big\}\\ &=\sup\big\{|\Phi_m(u\otimes e)| : m\in M_z(X,E'),\ |m|\le|m_0|\big\}\\ &=\sup\Big\{\Big|\int_X (u\otimes e)\,\mathrm{d}m\Big| : m\in M_z(X,E'),\ |m|\le|m_0|\Big\}\\ &=\sup\Big\{\Big|\int_X u\,\mathrm{d}m_e\Big| : m\in M_z(X,E'),\ |m|\le|m_0|\Big\}.\end{aligned}$$
□

To state our main result we recall some definitions (see [Wh, Definition 11.13, Definition 11.23, Theorem 10.3]).

A subset A of $M_\sigma(X)$ (resp. $M_\tau(X)$) is said to be *uniformly σ-additive* (resp. *uniformly τ-additive*) if whenever $u_n(x) \downarrow 0$ for every $x \in X$, $u_n \in C_b(X)^+$ (resp. $u_\alpha \downarrow 0$ for every $x \in X$, $u_\alpha \in C_b(X)^+$), then $\sup\{|\int_X u_n \,\mathrm{d}\mu| : \mu \in A\} \underset{n}{\longrightarrow} 0$ (resp. $\sup\{|\int_X u_\alpha \,\mathrm{d}\mu| : \mu \in A\} \underset{\alpha}{\longrightarrow} 0$).

A subset A of $M_t(X)$ is said to be *uniformly tight* if given $\varepsilon > 0$ there exists a compact subset K of X such that $\sup\{|\mu|(X \setminus K) : \mu \in A\} \le \varepsilon$.

Now we are in position to prove our desired result.

Theorem 3.8. *For a subset H of $M_z(X, E')$, where $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_\sigma(X, E))$ (resp. $z = \tau$ and X is paracompact; resp. $z = t$ and X is a P-space) the following statements are equivalent.*

(i) *H is relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact.*
(ii) $\operatorname{conv}(S(H))$ *is relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact.*
(iii) *The set $\{|m| : m \in H\}$ in $M_z(X)^+$ is uniformly σ-additive for $z = \sigma$, (resp. uniformly τ-additive for $z = \tau$; resp. uniformly tight for $z = t$).*

Proof: (i) $\Longrightarrow$ (ii) It is seen that H is relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact if and only if Φ_H is relatively $\sigma(C_b(X, E)'_{\beta_z}, C_b(X, E))$-compact. Hence by Theorem 2.12 and Lemma 3.6 the set $\Phi_{\operatorname{conv}(S(H))}$ is still relatively $\sigma(C_b(X, E)'_{\beta_z}, C_b(X, E))$-compact. This means that $\operatorname{conv}(S(H))$ is relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact.

(ii) $\Longrightarrow$ (i) It is obvious.

(i) $\Longleftrightarrow$ (iii) In view of Theorem 2.12 H is relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact if and only if Φ_H is $\beta_z(X, E)$-equicontinuous; hence in view of Theorem 2.7 and Lemma 3.4 the subset $\{\varphi_{|m|} : m \in H\}$ of $(C_b(X), \beta_z(X))'$ is $\beta_z(X)$-equicontinuous. It is known that the subset $\{\varphi_{|m|} : m \in H\}$ of $(C_b(X), \beta_z(X))'$ is $\beta_z(X)$-equicontinuous if and only if the set $\{|m| : m \in H\}$ in $M_z(X)^+$ is uniformly σ-additive for $z = \sigma$ (see [Wh, Theorem 11.14]) (resp. uniformly τ-additive for $z = \tau$ (see [Wh, Theorem 11.24]); resp. uniformly tight for $z = t$ (see [Wh, Theorem 10.7])). □

4. A Mackey-Arens type theorem for locally convex-solid topologies on $C_b(X, E)$

Let I be an ideal of $C_b(X, E)'$ separating points of $C_b(X, E)$. For each $\Phi \in I$ let us put

$$\rho_\Phi(f) = |\Phi|(f) \quad \text{for} \quad f \in C_b(X, E).$$

One can show that ρ_Φ is a solid seminorm on $C_b(X, E)$ (see the proof of Lemma 2.2). We define the *absolute weak topology* $|\sigma|(C_b(X, E), I)$ on $C_b(X, E)$ as the locally convex-solid topology generated by the family $\{\rho_\Phi : \Phi \in I\}$. In view of Lemma 2.2 we have

$$\rho_\Phi(f) = |\Phi|(f) = \sup\bigl\{|\Psi(f)| : \Psi \in I, \quad |\Psi| \leq |\Phi|\bigr\}.$$

This means that $|\sigma|(C_b(X,E), I)$ is the topology of uniform convergence on sets of the form $\{\Psi \in I : |\Psi| \leq |\Phi|\} = S(\{\Phi\})$, where $\Phi \in I$.

Assume that L is an ideal of $C_b(X)'$ separating the points of $C_b(X)$. For each $\varphi \in L$ the function $p_\varphi(u) = |\varphi|(|u|)$ for $u \in C_b(X)$ defines a Riesz semi-norm on $C_b(X)$. The family $\{p_\varphi : \varphi \in I\}$ defines a locally convex-solid topology $|\sigma|(C_b(X), L)$ on $C_b(X)$, called the *absolute weak topology* generated by L (see [AB]).

Recall that $|\sigma|(C_b(X), L)^\vee$ is the locally convex-solid topology on $C_b(X,E)$ generated by the family $\{p_\varphi^\vee : \varphi \in L\}$, where $p_\varphi^\vee(f) = p_\varphi(\|f\|)$ for $f \in C_b(X,E)$.

We shall need the following result.

Lemma 4.1. *Let I be an ideal of $C_b(X,E)'$ separating the points of $C_b(X,E)$. Then*

$$|\sigma|\bigl(C_b(X,E), I\bigr) = |\sigma|\bigl(C_b(X), C_b(X)'_I\bigr)^\vee$$

where $C_b(X)'_I = \{\varphi \in C_b(X)' : |\varphi| \leq \varphi_\Phi$ for some $\Phi \in I\}$.

Proof: Let $\varphi \in C_b(X)'$, i.e., $|\varphi| \leq \varphi_\Phi$ for some $\Phi \in I$. Then for $f \in C_b(X,E)$ we have

$$p_\varphi^\vee(f) = p_\varphi\bigl(\|f\|\bigr) = |\varphi|\bigl(\|f\|\bigr) \leq \varphi_\Phi\bigl(\|f\|\bigr) = |\Phi|(f) = \rho_\Phi(f).$$

This means that $|\sigma|(C_b(X), C_b(X)'_I)^\vee \subset |\sigma|(C_b(X,E), I)$.

Next, let $\Phi \in I$. Then for $f \in C_b(X,E)$ we have

$$\rho_\Phi(f) = |\Phi|(f) = \varphi_\Phi\bigl(\|f\|\bigr) = p_{\varphi_\Phi}\bigl(\|f\|\bigr) = p_{\varphi_\Phi}^\vee(f).$$

This shows that $|\sigma|(C_b(X,E), I) \subset |\sigma|(C_b(X), C_b(X)'_I)^\vee$, and the proof is complete. $\square$

Now we are ready to state the main result of this section.

Theorem 4.2. *Let I be an ideal of $C_b(X,E)'$ separating the points of $C_b(X,E)$. Then*

$$\bigl(C_b(X,E), |\sigma|\bigl(C_b(X,E), I\bigr)\bigr)' = I.$$

Proof: To see that $(C_b(X,E), |\sigma|(C_b(X,E), I))' \subset I$ assume that $\Phi \in (C_b(X,E), |\sigma|(C_b(X,E), I))'$. In view of Lemma 2.6 we have to show that $\Phi \in C_b(X,E)'_{C_b(X)'_I}$, that is $\Phi \in C_b(X,E)'$ and $\varphi_\Phi \in C_b(X)'_I$. In fact, we know that $(C_b(X), |\sigma|(C_b(X), C_b(X)'_I))' = C_b(X)'_I$ (see [AB$_1$, Theorem 6.6]). Assume that $u_\alpha \to 0$ for $|\sigma|(C_b(X), C_b(X)'_I)$. It is enough to show that $\varphi_\Phi(u_\alpha) \to 0$. Indeed, $u_\alpha \otimes e_0 \to 0$ for $|\sigma|(C_b(X), C_b(X)'_I)^\vee$, because for each $\varphi \in C_b(X)'_I$, $p^\vee_\varphi(u_\alpha \otimes e_0) = p_\varphi(u_\alpha)$. Hence by Theorem 4.1 $u_\alpha \otimes e_0 \to 0$ for $|\sigma|(C_b(X,E), I)$. Since $|\varphi_\Phi(u_\alpha)| \leq \varphi_\Phi(|u_\alpha|) = |\Phi|(u_\alpha \otimes e_0) = \rho_\Phi(u_\alpha \otimes e_0)$, we obtain that $\varphi_\Phi(u_\alpha) \to 0$.

Now let $\Phi \in I$. Then for $f \in C_b(X,E)$, $|\Phi(f)| \leq |\Phi|(f) = \rho_\Phi(f)$, so Φ is $|\sigma|(C_b(X,E), I)$-continuous, i.e., $\Phi \in (C_b(X,E), |\sigma|(C_b(X,E), I))'$, as desired.

□

As an application of Theorem 4.2 we have:

Corollary 4.3. *Let I be an ideal of $C_b(X,E)'$ separating the points of $C_b(X,E)$. Then for a subset H of $C_b(X,E)$ the following statements are equivalent:*

(i) *H is bounded for $\sigma(C_b(X,E), I)$;*
(ii) *$S(H)$ is bounded for $\sigma(C_b(X,E), I)$.*

PROOF: (i) $\Longrightarrow$ (ii) By Theorem 4.2 and the Mackey theorem (see [Wi, Theorem 8.4.1]) H is bounded for $|\sigma|(C_b(X,E), I)$. Since the topology $|\sigma|(C_b(X,E), I)$ is locally solid, $S(H)$ is bounded for $|\sigma|(C_b(X,E), I)$. Hence $S(H)$ is bounded for $\sigma(C_b(X,E), I)$.

(ii) $\Longrightarrow$ (i) It is obvious. □

Lemma 4.4. *Let $I_z = \{\Phi_m : m \in M_z(X,E')\}$, where $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X,E), \beta_\sigma(X,E))$ (resp. $z = \tau$; resp. $z = t$). Then*

$$C_b(X)'_{I_z} = \{\varphi_\mu : \mu \in M_z(X)\}.$$

PROOF: Assume that $\varphi \in C_b(X)'_I$, i.e., $\varphi \in C_b(X)'$ and $|\varphi| \leq \varphi_{\Phi_m}$ for some $m \in M_z(X,E')$. Then $\varphi = \varphi_\mu$ for some $\mu \in M(X)$, and $|\varphi_\mu| = \varphi_{|\mu|} \leq \varphi_{\Phi_m} = \varphi_{|m|}$ (see Lemma 3.4). It follows that $|\mu| \leq |m|$, where $|m| \in M_\sigma(X)^+$. Since $M_z(X)$ is an ideal of $M(X)$, we get $\mu \in M_z(X)$.

Conversely, assume that $\mu \in M_z(X)$ and $e_0 \in S_E$ and let $e^* \in E'$ be such that $e^*(e_0) = 1$ and $\|e^*\|_{E'} = 1$. Let us set $m(B) = \mu(B)e^*$ for all $B \in B(X)$. Then $m : B(X) \to E'$ is finitely additive, and for each $e \in E$ we have $m_e(B) = m(B)(e) = (e^*(e)\mu)(B)$ for all $B \in B(X)$. Hence $m_e \in M_z(X)$ for each $e \in E$. It is easy to show that $|m|(B) = |\mu|(B)$ for all $B \in B(X)$, so $|m| \in M_z(X)$. Hence $m \in M_z(X,E')$, and $|\varphi_\mu| = \varphi_{|\mu|} = \varphi_{|m|} = \varphi_{\Phi_m}$, so $\varphi_\mu \in C_b(X)'_{I_z}$, as desired.

□

As an application of Lemma 4.1 and Lemma 4.4 we get:

Corollary 4.5. *For $z = \sigma$ and $C_b(X) \otimes E$ dense in $(C_b(E), \beta_\sigma(X,E))$ (resp. $z = \tau$; resp. $z = t$) we have:*

$$|\sigma|\big(C_b(X,E), M_z(X,E')\big) = |\sigma|\big(C_b(X), M_z(X)\big)^{\vee}$$

and

$$|\sigma|\big(C_b(X,E), M_z(X,E')\big)^{\wedge} = |\sigma|\big(C_b(X), M_z(X)\big).$$

We now define the *absolute Mackey topology* $|\tau|(C_b(X,E), I)$ on $C_b(X,E)$ as the topology on uniform convergence on the family of all solid absolutely convex $\sigma(I, C_b(X,E))$-compact subsets of I. In view of Theorem 2.3 $|\tau|(C_b(X,E), I)$ is a locally convex-solid topology.

The following theorem strengthens the classical Mackey-Arens theorem for the class of locally convex-solid topologies on $C_b(X,E)$.

Theorem 4.6. *Let τ be a locally convex-solid topology on $C_b(X,E)$ and let $(C_b(X,E),\tau)' = I_\tau$. Then*

$$|\sigma|(C_b(X,E), I_\tau) \subset \tau \subset |\tau|(C_b(X,E), I_\tau).$$

Proof: To show that $|\sigma|(C_b(X,E), I_\tau) \subset \tau$, assume that (f_α) is a sequence in $C_b(X,E)$ and $f_\alpha \xrightarrow{\tau} 0$. Let $\Phi \in I_\tau$ and $\varepsilon > 0$ be given. Then there exists a net (h_α) in $C_b(X,E)$ such that $\|h_\alpha\| \le \|f_\alpha\|$ and $\rho_\Phi(f_\alpha) = |\Phi|(f_\alpha) \le |\Phi(h_\alpha)| + \varepsilon$. Since τ is locally solid, $h_\alpha \xrightarrow{\tau} 0$. Hence $h_\alpha \to 0$ for $\sigma(C_b(X,E), I_\tau)$, so $\Phi(h_\alpha) \to 0$, because $\sigma(C_b(X,E), I_\tau) \subset \tau$. Thus $\rho_\Phi(f_\alpha) \to 0$, and this means that $f_\alpha \to 0$ for $|\sigma|(C_b(X,E), I_\tau)$.

Now we show that $\tau \subset |\tau|(C_b(X,E), I_\tau)$. Indeed, let $\mathcal{B}_\tau$ be a local base at zero for τ consisting of solid absolutely convex and τ-closed sets and let $V \in \mathcal{B}_\tau$. Then by Theorem 2.3 and the Banach-Alaoglu's theorem, V^0 is a solid absolutely convex and $\sigma(I_\tau, C_b(X,E))$-compact subset of I_τ. Hence

$${}^0(V^0) = \overline{\text{abs conv } V}^{\sigma} = \overline{\text{abs conv } V}^{\tau} = V,$$

so τ is the topology of uniform convergence on the family $\{V^0 : V \in \mathcal{B}_\tau\}$. It follows that $\tau \subset |\tau|(C_b(X,E), I_\tau)$. □

Corollary 4.7. *Let $I_z = \{\Phi_m : m \in M_z(X,E')\}$, where $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X,E), \beta_z(X,E))$ (resp. $z = \tau$ and X is paracompact; resp. $z = t$ and X is a P-space). Then*

$$\beta_z(X,E) = |\tau|\big(C_b(X,E), M_z(X,E')\big) = \tau\big(C_b(X,E), M_z(X,E')\big),$$

and for a locally convex-solid topology τ on $C_b(X,E)$ with $C_b(X,E)'_\tau = I_z$ we have:

$$|\sigma|\big(C_b(X,E), M_z(X,E')\big) \subset \tau \subset \beta_z(X,E).$$

Proof: It is known that under our assumptions $\beta_z(X,E)$ is a Mackey topology (see [KhO$_1$, Corollary 6] for $z=\sigma$, [Kh, Theorem 6.2] for $z=\tau$ and [Kh, Theorem 5] for $z=t$). Hence $\tau(C_b(X,E),M_z(X,E'))=\beta_z(X,E)$. On the other hand, since $\beta_z(X,E)$ is a locally convex-solid topology and $(C_b(X,E),\beta_z(X,E))'=I_z$, by Corollary 4.6 we get $\beta_z(X,E)\subset|\tau|(C_b(X,E),M_z(X,E'))$. $\square$

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