On left distributive left idempotent groupoids

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Abstract. We study the groupoids satisfying both the left distributivity and the left idempotency laws. We show that they possess a canonical congruence admitting an idempotent groupoid as factor. This congruence gives a construction of left idempotent left distributive groupoids from left distributive idempotent groupoids and right constant groupoids.

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The left self-distributivity identity

$$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z) \tag{LD}$$

is often studied together with the idempotency identity

$$x \cdot x = x \tag{I}$$

giving left distributive idempotent (LDI) groupoids. However, some structures, for instance the so-called LD-quasigroups [1] (left distributive left quasigroups equipped with another left distributive operation) satisfy, together with left distributivity, a weaker version of idempotency only, called left idempotency:

$$(x \cdot x) \cdot y = x \cdot y. \tag{LI}$$

The first results about left idempotent left distributive groupoids (LDLI) appeared in Kepka [4] where these groupoids were called pseudoidempotent left distributive groupoids. However, the first systematic study of these groupoids seems to have appeared as late as in [2].

In this paper, we study left distributive left idempotent (LDLI) groupoids and show that there exists a canonical congruence that, in fact, is the smallest idempotent congruence. Classes of that congruence are right constant groupoids, *i.e.*, groupoids satisfying the identity

$$x \cdot z = y \cdot z. \tag{RC}$$

This enables us to construct LDLI groupoids starting with an LDI groupoid and a family of right constant groupoids.

Kepka [3] found a decomposition similar to the current one for left symmetric left distributive (LSLD) groupoids. These groupoids form a subvariety of LDLI groupoids given by the identity

$$x \cdot xy = y \tag{LS}$$

and our decomposition is a generalization of the decomposition described for LSLD groupoids.

The smallest idempotent congruence

We begin with technical notes: if not specified differently, each groupoid mentioned here is equipped with the binary operation $(\cdot)$. The expression abc stands for $a \cdot (b \cdot c)$ and similarly a^k means $a \cdot a^{k-1}$.

Lemma 1. *Let G be an LI groupoid and let a be in G. Then we have, for all a, b in G,*

$$a^k b = ab \quad \textit{and} \quad (a^k)^l = a^{k+l-1}.$$

Proof: First of all we prove $a^k b = ab$, for all a, b in G. It is evident for $k = 1$ and for $k > 1$ we have

$$a^k b = (a \cdot a^{k-1})b = (a^{k-1} \cdot a^{k-1})b = a^{k-1}b = ab.$$

Now we prove the other result by induction on l. Since it is true for $l = 1$, we continue with $l > 1$:

$$(a^k)^l = (a^k) \cdot (a^k)^{l-1} = a \cdot a^{k+l-2} = a^{k+l-1},$$

and that is what we wanted to prove. □

Definition 2 ([5])**.** Let G be an LI groupoid. We define ip_G to be the smallest equivalence relation on G satisfying $(a, a^2) \in \mathrm{ip}_G$.

Lemma 3. *Let G be an LI groupoid. Then, for all a, b in G, the following conditions are equivalent:*

(i) $(a, b) \in \mathrm{ip}_G$;
(ii) *there exist positive integers k, l satisfying $a^k = b^l$.*

Proof: (i) ⇒ (ii): The relation $(a, b) \in \mathrm{ip}_G$ means that there exists a sequence $a = a_0, a_1, \dots, a_n = b$, such that we have $a_i = a_{i-1}^2$ or $a_i^2 = a_{i-1}$, for each $1 \leqslant i \leqslant n$. Using induction on n, we show that there exist positive integers k, l satisfying $a^k = b^l$. The claim is evident for $n = 0$. Let us suppose $n \geqslant 1$. The induction hypothesis tells us that there exist k', l' satisfying $a^{k'} = a_{n-1}^{l'}$. We have two possibilities now:

- for $b^2 = a_{n-1}$ we have $b^{l'+1} = (b^2)^{l'} = a_{n-1}^{l'} = a^{k'}$;
- for $b = a_{n-1}^2$, we have $b^{l'} = (a_{n-1}^2)^{l'} = (a_{n-1}^{l'})^2 = (a^{k'})^2 = a^{k'+1}$.

(ii) $\Rightarrow$ (i): Evident. □

Example 4. The relation ip_G is not a congruence in general, for instance

$\cdot$	0	1	2
0	1	0	2
1	1	0	2
2	1	2	2

is a simple LI groupoid with ip_G non-trivial. However, the relation ip_G is a congruence on any LDLI groupoid:

Proposition 5. *For each LDLI groupoid G, the relation ip_G is a congruence and, for any a, b, c in G with $(a, b) \in \mathrm{ip}_G$, we have $ac = bc$.*

PROOF: Consider $(a, b) \in \mathrm{ip}_G$ in G. Then there exist k, l satisfying $a^k = b^l$. Now, for all c in G, we have

$$a \cdot c = a^k \cdot c = b^l \cdot c = b \cdot c,$$
$$(c \cdot a)^k = c \cdot a^k = c \cdot b^l = (c \cdot b)^l.$$

This implies that ip_G is a congruence. □

Note 6. Kepka and Němec [5] proved Proposition 5 for a left cancellative LDLI groupoid. They also proved that, in the case of left cancellative LD groupoids, the LI identity is equivalent to the identity

$$xx \cdot x = xx.$$

This result is not true for non-cancellative ones, as we can see on the following example, which is LD, satisfies the cited identity but it is not LI ($(1 \cdot 1) \cdot 0 \neq 1 \cdot 0$):

$\cdot$	0	1	2
0	1	2	2
1	1	2	2
2	2	2	2

It is easy to see that, for any LDLI groupoid G, the factor $G/\,\mathrm{ip}_G$ is LDI and that the equivalence classes are right constant groupoids. Moreover, two ip_G congruent elements satisfy $a^k = b^l$ for some k and l.

Decomposition of LDLI groupoids

The result of Proposition 5 leads us to introduce the following definition:

Definition 7. A set A is a *connected monounary algebra* if it is equipped with a unary operation α satisfying, for all a, b in A, the relation $\alpha^k(a) = \alpha^l(b)$ for some k, l.

Every right constant groupoid G is equipped with a natural operation $o_G : a \mapsto a^2$ that describes the multiplication on G entirely. On the other hand, we can build, on every monounary algebra, a structure of left idempotent right constant groupoid. We say that a right constant groupoid is *connected* if its corresponding monounary algebra is connected. If G is an LDLI groupoid, all congruence classes of ip_G are connected right constant groupoids, according to Proposition 5. This permits us to find a decomposition of the groupoid G.

Proposition 8. (i) *Let H be an LDI groupoid and let A_a, with $a \in H$, be a pairwise disjoint sets. Let $f_{a,b}$ be a mapping from A_b to A_{ab}, for every a, b in H. Let us define the groupoid $B(H, f)$ to be the set $\bigcup_{a\in H} A_a$ with the operation $*$ defined by $x * y = f_{a,b}(y)$, for x in A_a and y in A_b. Then the groupoid $B(H, f)$ is LI. Moreover, the mappings $f_{a,b}$ satisfy the identity*

$$f_{a,bc} \circ f_{b,c} = f_{ab,ac} \circ f_{a,c} \tag{ld}$$

for all a, b and c in H if and only if the groupoid $B(H, f)$ is LD.

(ii) *Let G be an LDLI groupoid. Then G is equal to $B(G/\mathrm{ip}_G, f)$, where $f_{\bar{a},\bar{b}}(c) = ac$ and $\bar{a}$ stands for the class of ip_G containing a.*

PROOF: (i) Let us take arbitrary a, b, c from H, x from A_a, y from A_b and z from A_c. The element $x * x = f_{a,a}(x)$ belongs also to A_a because H is idempotent. Hence we have $(x * x) * y = f_{a,b}(y) = x * y$. For the left distributivity, since we have

$$\begin{aligned} x * (y * z) = x * f_{b,c}(z) = f_{a,bc}(f_{b,c}(z)) = f_{ab,ac}(f_{a,c}(z)), \\ (x * y) * (x * z) = f_{a,b}(y) * f_{a,c}(z) = f_{ab,ac}(f_{a,c}(z)), \end{aligned}$$

the groupoid $B(H, f)$ is LD if and only if Condition (ld) is satisfied.

(ii) We remark first that the definition of $f_{\bar{a},\bar{b}}$ depends neither on the choice of a, by Proposition 5, nor on the choice of b. The construction yields an LI groupoid and we want to show that the groupoid $B(G/\mathrm{ip}_G, f)$ is equal to $(G, \cdot)$. Let us choose arbitrarily a, b in G, c in $\bar{a}$ and d in $\bar{b}$. Then we have

$$c * d = f_{\bar{a},\bar{b}}(d) = a \cdot d = c \cdot d,$$

which completes the proof. □

Note 9. For all a in G, we have the equality $f_{\bar{a},\bar{a}} = o_G$ on the equivalence class $\bar{a}$. And when considering any a, b in G, the mapping $f_{\bar{a},\bar{b}}$ has to be a homomorphism:

$$f_{\bar{a},\bar{b}}(o_G(d)) = f_{\bar{a},\bar{b}}(f_{\bar{b},\bar{b}}(d)) = f_{\overline{ab},\overline{ab}}(f_{\bar{a},\bar{b}}(d)) = o_G(f_{\bar{a},\bar{b}}(d))$$

holds for any d in $\bar{b}$.

In the sequel, each element of the groupoid $B(H, f)$ is denoted by the pair (a, x) with a in H and x in A_a.

Example 10. Let H be an LDI groupoid and let A be a connected right constant groupoid. Let us take, for each a in H, a disjoint copy of A, denoted A_a. We define the mapping $f_{a,b}$ by $d \mapsto o_{H_b}(d)$, d in A_b. Then the groupoid $B(H, f)$ is isomorphic to the product $H \times A$.

We apply the congruence ip_G to get a classification of all nonidempotent simple LDLI groupoids. Although this classification follows directly from the results about simple LD groupoids presented in [5], we show it here because it uses a different approach.

Definition 11 ([5])**.** The groupoid $\mathrm{Cyc}_r(n)$, with $n \geq 1$, is the set $\{0, 1, \dots, n-1\}$ with the operation $i \cdot j = j - 1$, for $j > 0$, and $i \cdot 0 = n - 1$.
The groupoid $\mathrm{Path}_r(n)$, with $n \geq 1$, is the set $\{0, 1, \dots, n-1\}$ with the operation $i \cdot j = j - 1$, for $j > 0$, and $i \cdot 0 = 0$.

Proposition 12 (Stanovský [6])**.** *The only simple right constant groupoids are, up to isomorphism, the two-element idempotent right constant groupoid,* $\mathrm{Path}_r(2)$ *and* $\mathrm{Cyc}_r(p)$*, for* p *prime.*

Proposition 13. *The only simple nonidempotent LDLI groupoids are, up to isomorphism,* $\mathrm{Path}_r(2)$*, and* $\mathrm{Cyc}_r(p)$*, for* p *prime.*

PROOF: The congruence ip_G on an LDLI groupoid G is not trivial, unless G is idempotent or G is a connected right constant groupoid. The only nonidempotent simple right constant groupoids are, according to Proposition 12, the groupoids $\mathrm{Path}_r(2)$, and $\mathrm{Cyc}_r(p)$. □

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