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A remark on a theorem of Solecki

P. Holický, L. Zajíček, M. Zelený

Abstract. S. Solecki proved that if $\mathcal{F}$ is a system of closed subsets of a complete separable metric space X, then each Suslin set $S \subset X$ which cannot be covered by countably many members of $\mathcal{F}$ contains a $\boldsymbol{G}_\delta$ set which cannot be covered by countably many members of $\mathcal{F}$. We show that the assumption of separability of X cannot be removed from this theorem. On the other hand it can be removed under an extra assumption that the σ-ideal generated by $\mathcal{F}$ is locally determined. Using Solecki's arguments, our result can be used to reprove a Hurewicz type theorem due to Michalewski and Pol, and a nonseparable version of Feng's theorem due to Chaber and Pol.

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1. Introduction

According to [S] we introduce the following notions. Let X be a metric space and $\mathcal{F}$ be a system of closed subsets of X. Then $\mathcal{F}_{\mathrm{ext}}$ stands for the σ-ideal generated by $\mathcal{F}$ (i.e., the system of sets which can be covered by countably many elements of $\mathcal{F}$). A set $A \subset X$ is called *$\mathcal{F}$-approximable* if either $A \in \mathcal{F}_{\mathrm{ext}}$ or there is a $\boldsymbol{G}_\delta$ set $G \subset A$ with $G \notin \mathcal{F}_{\mathrm{ext}}$. The set A is *absolutely approximable* if it is $\mathcal{F}$-approximable for any system $\mathcal{F}$ of closed subsets of X.

S. Solecki generalizing a result of Gy. Petruska ([P]) proved in [S] the following theorem.

Solecki's Theorem. *If X is a Polish space, then each Suslin subset of X is absolutely approximable.*

The first result of our paper says that Suslin subsets of a metric space X need not be absolutely approximable if X is only supposed to be complete. Namely, we prove the following theorem.

Theorem 1.1. *Let X be a topologically complete metric space containing a subspace of density (at least) continuum without isolated points. Then there exists an $\boldsymbol{F}_{\sigma\delta}$ subset of X which is not absolutely approximable.*

In Section 3 we investigate $\mathcal{F}$-approximability of Suslin sets in topologically complete metric spaces. We prove a "non-separable" version of Solecki's theorem

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saying that, under some conditions on $\mathcal{F}$, Suslin sets are $\mathcal{F}$-approximable. It is not difficult to obtain this result since Solecki's proof essentially works. Our main motivation for this slight extension comes from the study of σ-ideals of σ-lower porous sets, which were used also in non-separable Banach spaces. Within this context the next notion turns out to be useful. A σ-ideal $\mathcal{I}$ is $\boldsymbol{F}_\sigma$ *supported* if each set $A \in \mathcal{I}$ is contained in an $\boldsymbol{F}_\sigma$ set belonging to $\mathcal{I}$.

We need also the following definition to formulate our result.

Definition 1.2. Let $\mathcal{A}$ be a system of subsets of a metric space X. We say that $\mathcal{A}$ is *locally determined* if $A \subset X$ belongs to $\mathcal{A}$ whenever for each $x \in A$ there exists a, not necessarily open, neighbourhood U of x such that $U \cap A \in \mathcal{A}$.

Theorem 1.3. *Let $\mathcal{I}$ be a locally determined $\boldsymbol{F}_\sigma$ supported σ-ideal of subsets of a topologically complete metric space X. Then each Suslin set $S \notin \mathcal{I}$ contains a $\boldsymbol{G}_\delta$ set which does not belong to $\mathcal{I}$.*

In other words this theorem says that if $\mathcal{F}$ is a family of closed subsets of a topologically complete metric space X such that the σ-ideal $\mathcal{F}_{\text{ext}}$ is locally determined, then Suslin subsets of X are $\mathcal{F}$-approximable. Since each σ-ideal in a separable metric space is clearly locally determined, it is easy to see that, in Polish spaces, our Theorem 1.3 is just a reformulation of Solecki's theorem.

Using Theorem 1.3, it is not difficult to show (see [ZZ]) that each non-σ-lower porous Suslin subset of a topologically complete metric space contains even a closed subset which is not σ-lower porous. Similar results on σ-cone supported sets in separable Banach spaces and ball small sets in any Hilbert space (see [Z] for the definitions) can be also found in [ZZ].

In the closing remarks we will discuss, besides some reformulations of Theorem 1.3, results which can be obtained using Theorem 1.3. Solecki in [S] shows that his theorem can be used to infer easily a strong version of Hurewicz's theorem in Polish spaces which is due to Kechris, Louveau, and Woodin. The same argument shows that Theorem 1.3 gives the corresponding result in topologically complete metric spaces. Nevertheless, one can find a more straightforward proof of this result in [MP]. Solecki derives also Feng's theorem ([F]) from his result. The same approach can be applied in the nonseparable case as well to reprove a nonseparable version of Feng's result, which was firstly obtained in [CP]. Moreover, we show that each Suslin subset S of a topologically complete metric space can be covered by closures of countably many $\boldsymbol{G}_\delta$ sets contained in S.

In the sequel we employ the following notation. The symbols ω and $\mathbb{N}$ stand for the set of nonnegative integers and for the set of positive integers, respectively. The symbols $\omega^{<\omega}$ and ω^ω denote the set of all finite sequences of nonnegative integers (including the empty sequence) and the set of all infinite sequences of nonnegative integers, respectively. The concatenation of $s \in \omega^{<\omega}$ and a sequence (n), $n \in \omega$, is denoted by $s^\wedge n$. The length of $s \in \omega^{<\omega}$ is denoted by $|s|$. If

$\nu = (\nu(0), \nu(1), \dots) \in \omega^\omega$ and $n \in \omega$, then $\nu \,|\, n$ is a restriction of ν to the first n coordinates, i.e., $\nu \,|\, n = (\nu(0), \nu(1), \dots, \nu(n-1))$.

If X is a metric space, then $\boldsymbol{G}_\delta(X)$ and $\boldsymbol{F}_\sigma(X)$ denote the system of all $\boldsymbol{G}_\delta$ subsets of X and the system of all $\boldsymbol{F}_\sigma$ subsets of X, respectively. The symbol $\mathcal{P}(A)$ stands for the set of all subsets of A and the Cantor space with the usual topology is denoted by 2^ω.

2. The assumption of separability cannot be removed from Solecki's theorem

The key step for us was the easy observation formulated in Lemma 2.2. It led us to Lemmas 2.3 and 2.4 which are sufficient for an easy construction of a counterexample.

Definition 2.1. Let X be a metric space and $A \subset X$. We say that $T : \boldsymbol{G}_\delta(X) \cap \mathcal{P}(A) \to \boldsymbol{F}_\sigma(X)$ is an *A-cover operation* if $H \subset T(H)$ for each $H \in \boldsymbol{G}_\delta(X) \cap \mathcal{P}(A)$.

Lemma 2.2. *Let X be a metric space and $A \subset X$. Then the following are equivalent.*

(i) *The set A is absolutely approximable.*
(ii) *If T is an A-cover operation, then there exists a sequence of sets $H_n \in \boldsymbol{G}_\delta(X) \cap \mathcal{P}(A)$ $(n \in \mathbb{N})$ such that $A \subset \bigcup_{n\in\mathbb{N}} T(H_n)$.*

Proof: Let (i) hold and T be an A-cover operation. For each $H \in \boldsymbol{G}_\delta(X) \cap \mathcal{P}(A)$ choose closed sets F_i^H $(i \in \mathbb{N})$ such that $T(H) = \bigcup_{i=1}^\infty F_i^H$ and put $\mathcal{F} := \{F_i^H : H \in \boldsymbol{G}_\delta(X) \cap \mathcal{P}(A), i \in \mathbb{N}\}$. Since A is $\mathcal{F}$-approximable, (ii) easily follows.

Suppose that (ii) holds and $\mathcal{F}$ is a system of closed sets such that each $H \in \boldsymbol{G}_\delta(X) \cap \mathcal{P}(A)$ is in $\mathcal{F}_{\text{ext}}$. Then for $H \in \boldsymbol{G}_\delta(X) \cap \mathcal{P}(A)$ we can choose sets $F_i^H \in \mathcal{F}$ for which $H \subset \bigcup_{i=1}^\infty F_i^H$ and put $T(H) = \bigcup_{i=1}^\infty F_i^H$. Then T is an A-cover operation and thus (ii) implies that A is covered by countably many members of $\mathcal{F}$. $\square$

The construction of the set from the next lemma can be found as early as in [B] (cf. [L, pp. 82–86]).

Lemma 2.3. *There exist an $\boldsymbol{F}_{\sigma\delta}$ set $A \subset 2^\omega$ and a sequence $\{T_n\}_{n=1}^\infty$ of A-cover operations such that the set A is covered by no set of the form $\bigcup_{n=1}^\infty T_n(H_n)$, where $H_n \in \boldsymbol{G}_\delta(2^\omega) \cap \mathcal{P}(A)$ for every $n \in \mathbb{N}$.*

Proof: Fix a metric on 2^ω giving the product topology. We define a system $\{F(s) : s \in \omega^{<\omega}\}$ of nonempty perfect subsets of 2^ω such that for every $s \in \omega^{<\omega}$ we have

(i) $F(\emptyset) = 2^\omega$,
(ii) $F(s^\wedge n)$ is a nowhere dense set in $F(s)$ whenever $n \in \omega$,
(iii) $\lim_{n\to\infty} \operatorname{diam} F(s^\wedge n) = 0$,

(iv) $\bigcup\{F(s^\wedge n) : n \in \omega\}$ is dense in $F(s)$,

(v) $F(s^\wedge n) \cap F(s^\wedge m) = \emptyset$ whenever $n, m \in \omega$, $n \neq m$.

Put $F(\emptyset) = 2^\omega$. If a set $F(s)$, $s \in \omega^{<\omega}$, is defined, then we take an open basis $\{V_n : n \in \omega\}$ of the space $F(s)$ containing only nonempty sets. It is easy to find pairwise disjoint nonempty perfect sets $F(s^\wedge n)$, $n \in \omega$, such that $F(s^\wedge n) \subset V_n$, $\operatorname{diam} F(s^\wedge n) < 1/(n+1)$ and $F(s^\wedge n)$ is nowhere dense in $F(s)$. This finishes the construction of the desired system.

Observe that conditions (iii) and (iv) imply that the following condition **(C)** is satisfied.

(C) For every $s \in \omega^{<\omega}$ and every $D \subset F(s)$ not dense in $F(s)$ there exists $k \in \omega$ such that $F(s^\wedge k) \cap D = \emptyset$.

We put

$$A = \bigcap_{k=1}^{\infty} \bigcup \{F(s) : s \in \omega^{<\omega}, |s| = k\} \qquad \text{and}$$

$$T_n(H) = \bigcup \{\overline{F(s) \cap H} : s \in \omega^{<\omega}, |s| = n\}, \qquad n \in \mathbb{N},\ H \in \boldsymbol{G}_\delta(2^\omega) \cap \mathcal{P}(A).$$

Since $A \subset \bigcup\{F(s) : s \in \omega^{<\omega}, |s| = n\}$, we have $H \subset T_n(H)$ and so T_n is clearly an A-cover operation.

Now suppose to the contrary that there are $H_n \in \boldsymbol{G}_\delta(2^\omega) \cap \mathcal{P}(A)$, $n \in \mathbb{N}$, such that $A \subset \bigcup_{n=1}^\infty T_n(H_n)$. We will construct a sequence $\{s^n\}_{n=0}^\infty$ of elements of $\omega^{<\omega}$ such that $|s^n| = n$, s^n extends s^{n-1} and $F(s^n) \cap T_n(H_n) = \emptyset$ for every $n \in \mathbb{N}$. Put $s^0 = \emptyset$. Suppose that s^{n-1}, $n \in \mathbb{N}$, was constructed. The set $H_n \cap F(s^{n-1})$ is a subset of $\bigcup_{j=0}^\infty F((s^{n-1})^\wedge j)$, which is meager in $F(s^{n-1})$ by (ii). Since $H_n \cap F(s^{n-1})$ is a $\boldsymbol{G}_\delta$ set, we see that $H_n \cap F(s^{n-1})$ is not dense in $F(s^{n-1})$. Using condition **(C)** we find $k \in \omega$ such that $F((s^{n-1})^\wedge k) \cap (H_n \cap F(s^{n-1})) = F((s^{n-1})^\wedge k) \cap H_n = \emptyset$. Using condition (v) we obtain $F((s^{n-1})^\wedge k) \cap T_n(H_n) = \emptyset$. Put $s^n = (s^{n-1})^\wedge k$. This finishes the construction of the sequence $\{s^n\}_{n=0}^\infty$. The sequence $\{F(s^n)\}_{n=0}^\infty$ is a decreasing sequence of nonempty compact sets and therefore there exists $x \in \bigcap_{n=0}^\infty F(s^n)$. Clearly we have $x \in A \setminus \bigcup_{n=1}^\infty T_n(H_n)$, a contradiction. □

We shall need the following simple set-theoretical lemma. We do not know whether it holds also for $\aleph_1$ (i.e., if we write $\aleph_1$ and $2^{\aleph_1}$ instead of $\mathfrak{c}$ and $2^{\mathfrak{c}}$).

Lemma 2.4. *Let* $\operatorname{card} P = \mathfrak{c}$. *Then there exists a system* $\mathcal{M}$ *of functions* $f : P \to \mathbb{N}$ *such that* $\operatorname{card} \mathcal{M} = 2^{\mathfrak{c}}$ *and, for each one-to-one sequence* $f_1, f_2, \dots$ *of functions from* $\mathcal{M}$*, there exists* $p \in P$ *such that the mapping* $i \mapsto f_i(p)$ *is one-to-one.*

Proof: Choose a set Q with $\operatorname{card} Q = \mathfrak{c}$. We shall define a one-to-one mapping ψ from $2^Q = \{0,1\}^Q$ to $\mathbb{N}^P$ such that $\mathcal{M} := \psi(2^Q)$ will have the desired properties.

For each infinite countable set $C \subset Q$ denote by $\mathcal{V}_C$ the system of all infinite countable sets of functions belonging to $\{0,1\}^C$ and by $\mathcal{V}$ the union of all $\mathcal{V}_C$. For each $T \in \mathcal{V}$ choose a bijection $\beta_T : T \to \mathbb{N}$. Since clearly $\operatorname{card} \mathcal{V} = \mathfrak{c}$, we can choose a bijection $\eta : \mathcal{V} \to P$.

Now, for $g \in 2^Q$, define $g^* = \psi(g) : P \to \mathbb{N}$ as the (clearly uniquely determined) function with the following properties:

(i) $g^*(p) = \beta_T(\varphi)$ if there exist $T \in \mathcal{V}$ and $\varphi \in T$ such that g extends φ and $p = \eta(T)$;

(ii) $g^*(p) = 1$ if such T, φ do not exist.

Now consider a one-to-one sequence $g_1, g_2, \dots$ of elements of 2^Q. Then there clearly exists an infinite countable set $C \subset Q$ such that the sequence of restrictions $g_1\restriction_C, g_2\restriction_C, \dots$ is one-to-one. Thus $T := \{g_1\restriction_C, g_2\restriction_C, \dots\} \in \mathcal{V}$ and property (i) immediately implies that the mapping $i \mapsto \psi(g_i)(p)$ is one-to-one for $p := \eta(T)$.

This just proved property of ψ clearly implies that ψ is one-to-one and subsequently also that $\mathcal{M} := \psi(2^Q)$ has the desired properties. □

Lemma 2.5. *Let X be a metric space containing a discrete family of cardinality $\mathfrak{c}$ of homeomorphic copies of 2^ω. Then there exists an $\boldsymbol{F}_{\sigma\delta}$ set $A \subset X$ which is not absolutely approximable.*

Proof: Let $(C_p)_{p\in P}$ be a discrete system in X such that $\operatorname{card} P = \mathfrak{c}$ and every C_p is homeomorphic to 2^ω. In each C_p find an $\boldsymbol{F}_{\sigma\delta}$ set A^p and A^p-cover operations T_n^p ($n \in \mathbb{N}$) by Lemma 2.3. Clearly $A := \bigcup_{p\in P} A^p$ is an $\boldsymbol{F}_{\sigma\delta}$ set. Find a system $\mathcal{M} \subset \mathbb{N}^P$ by Lemma 2.4, choose a bijection $\beta : \boldsymbol{G}_\delta(X) \cap \mathcal{P}(A) \to \mathcal{M}$ and for every $H \in \boldsymbol{G}_\delta(X) \cap \mathcal{P}(A)$ put $T(H) := \bigcup_{p\in P} T^p_{\beta(H)(p)}(H \cap C_p)$. It is clear that T is an A-cover operation in X.

To show that A is not absolutely approximable, suppose on the contrary that condition (ii) of Lemma 2.2 is satisfied. Then there exists a sequence of sets $H_n \in \boldsymbol{G}_\delta(X) \cap \mathcal{P}(A)$ ($n \in \mathbb{N}$) such that $A \subset \bigcup_{n\in\mathbb{N}} T(H_n)$. We can suppose that the sequence $\{H_n\}_{n=1}^\infty$ and therefore also the sequence $\{\beta(H_n)\}_{n=1}^\infty$ is one-to-one. Consequently, (by the definition of $\mathcal{M}$) there exists $p \in P$ such that the sequence $\{\beta(H_n)(p)\}_{n=1}^\infty$ is one-to-one as well. But this contradicts the choice of T_n^p since clearly $A^p \subset \bigcup_{n\in\mathbb{N}} T^p_{\beta(H_n)(p)}(H_n \cap C_p)$. □

Proof of Theorem 1.1: Let Y be a subspace of X of density (at least) continuum without isolated points. Let D_n be a maximal $\frac{1}{n}$-discrete set in Y, $n \in \mathbb{N}$. Then the set $D := \bigcup_{n=1}^\infty D_n$ is dense in Y, hence cardinality of D is at least $\mathfrak{c}$. The König inequality ([K, Lemma 10.40]) gives that there exists $n_0 \in \mathbb{N}$ such that cardinality of D_{n_0} is at least $\mathfrak{c}$. Thus there is a discrete family $\mathcal{B}$ of cardinality $\mathfrak{c}$ consisting of closed balls in $\overline{Y}$, centered at points from D_{n_0}. Using the Perfect Set Theorem (cf. [Ku, 36, V]) we find a homeomorphic copy of 2^ω inside of each $B \in \mathcal{B}$. Applying Lemma 2.5 the proof is complete. □

Proposition 2.6. *Suppose that (CH) holds. Let X be a topologically complete metric space. Then the following are equivalent.*

(i) *Every Suslin subset of X is absolutely approximable.*
(ii) *There exist a σ-discrete set $D \subset X$ and a separable set $S \subset X$ such that $X = D \cup S$.*

PROOF: Let us assume that (ii) is satisfied and A is a Suslin subset of X which is not in $\mathcal{F}_{\mathrm{ext}}$ for some family $\mathcal{F}$ of closed sets. Put $P = \overline{S}$ and $I = X \setminus P$. Since every subset of I is σ-discrete, it is $\boldsymbol{F}_\sigma$. Thus every subset of the open set I is a $\boldsymbol{G}_\delta$ set in X. If $A \cap I \notin \mathcal{F}_{\mathrm{ext}}$, then A is $\mathcal{F}$-approximable because $A \cap I$ is $\boldsymbol{G}_\delta$. In the other case, $A \cap P \notin \mathcal{F}_{\mathrm{ext}}$ and we may use Solecki's theorem to $A \cap P$ in P with the family $\{F \cap P : F \in \mathcal{F}\}$.

It is well-known that X can be decomposed to a σ-discrete set and its perfect complement Y. (Just define Y as the set of all $x \in X$ such that every neighbourhood of x is not σ-discrete and use the existence of a σ-discrete base of X.) If (ii) does not hold, Y is nonseparable. Thus the density of Y is at least continuum by the continuum hypothesis and Theorem 1.1 may be applied to see that (i) does not hold. $\square$

Remark 2.7. If Lemma 2.4 held with $\mathfrak{c}$ replaced by $\aleph_1$ in its statement, then Lemma 2.5 would be true with the same replacement. Consequently, in Theorem 1.1 it would be sufficient to assume that X is a topologically complete metric space containing a nonseparable subspace without isolated points and Proposition 2.6 would be true even without (CH).

3. A nonseparable version of Solecki's theorem

In the proof of Theorem 1.3 we need the notion of the kernel A^* of a set A with respect to a σ-ideal.

Definition 3.1. Let $\mathcal{I}$ be a σ-ideal of subsets of a metric space X and $A \subset X$. Then we define (the *kernel*) A^* as the set of all $x \in A$ such that $A \cap U \notin \mathcal{I}$ for every neighbourhood U of x.

Lemma 3.2. *Let $\mathcal{I}$ be a locally determined σ-ideal of subsets of a metric space X and $A \subset X$.*

(a) *We have $A \setminus A^* \in \mathcal{I}$.*
(b) *If $U \subset X$ is open, then $A \cap U \notin \mathcal{I}$ if and only if $U \cap A^* \neq \emptyset$.*
(c) *If $A = \bigcup_{n\in\omega} A_n$, then $\overline{A^*} = \overline{\bigcup_{n\in\omega} A_n^*}$.*

PROOF: The statement (a) immediately follows from the fact that $\mathcal{I}$ is locally determined and (b) is an easy consequence of (a).

The inclusion $\overline{\bigcup_{n\in\omega} A_n^*} \subset \overline{A^*}$ is obvious by the monotonicity of the operation *.

Let $x \notin \overline{\bigcup_{n\in\omega} A^*_n}$. Then there is an open neighbourhood U of x with $U \cap A^*_n = \emptyset$, $n \in \omega$. So $U \cap A = \bigcup_{n\in\omega}(U \cap A_n) \in \mathcal{I}$ by (a) and consequently, $U \cap A^* = \emptyset$. Thus $x \notin \overline{A^*}$, which proves $\overline{A^*} \subset \overline{\bigcup_{n\in\omega} A^*_n}$. □

For the proof of Theorem 1.3 we shall need also the following easy lemma, whose "separable version" is implicitly used in [S].

Lemma 3.3. *Let X be a metric space, $U \subset X$ be open, $P \subset U$, E be nowhere dense in P, and $\varepsilon > 0$ be arbitrary. Then there is a family $\mathcal{U}$ of open subsets of U such that*

- (a) $\mathrm{diam}(W) < \varepsilon$ *for* $W \in \mathcal{U}$;
- (b) $W \cap P \neq \emptyset$ *for* $W \in \mathcal{U}$;
- (c) $\mathcal{U}$ *is disjoint;*
- (d) $\overline{W} \subset U$ *for* $W \in \mathcal{U}$;
- (e) *if $H \cap W \neq \emptyset$ for all $W \in \mathcal{U}$, then $E \subset \overline{H}$; we abbreviate this property of the pair $\mathcal{U}$ and E by $\overline{\mathcal{U}} \supset E$.*

Proof: For every $k \in \mathbb{N}$ we find a maximal $\frac{1}{k}$-discrete set D_k in $(B_{1/k}(E) \cap P) \setminus \overline{E}$, where $B_{1/k}(E)$ stands for the open $\frac{1}{k}$-neighbourhood of E. The set $D = \bigcup_{k\in\mathbb{N}} D_k$ is discrete in $U \setminus \overline{E}$ and we may assign to each $d \in D$ an open neighbourhood $W(d)$ of d such that $W(d) \cap W(d') = \emptyset$ if $d, d' \in D$ are distinct, $\overline{W(d)} \subset U$ and $\mathrm{diam}(W(d)) < \varepsilon$ for $d \in D$, and $\mathrm{diam}(W(d)) < 1/k$ if $d \in D \cap B_{1/k}(E)$. Put $\mathcal{U} = \{W(d) : d \in D\}$.

Take $x \in E$ and its open δ-neighbourhood $B(x, \delta)$. Choose $k \in \mathbb{N}$ with $1/k < \delta/3$. There is a $y \in B(x, \delta/3) \cap B_{1/k}(E) \cap (P \setminus \overline{E})$ as E is nowhere dense in P. Since $D_k \subset D$ is a maximal $\frac{1}{k}$-discrete set in $(B_{1/k}(E) \cap P) \setminus \overline{E}$, we can find a $d \in D \cap B(x, 2\delta/3) \cap (B_{1/k}(E) \cap P)$. Consequently, $\mathrm{diam}(W(d)) < 1/k$ and we have that $W(d) \subset B(x, \delta)$, which proves that (e) is fulfilled. □

Proof of Theorem 1.3: In what follows we work with a fixed complete metric giving the topology on X. Suppose to the contrary that $S \notin \mathcal{I}$ and every $\boldsymbol{G}_\delta$ set G in X which is contained in S belongs to $\mathcal{I}$.

The first part of Solecki's proof of [S, Theorem 1] (before the construction of Φ and U_τ) can be read as the proof of the following claim. For the readers who are not familiar with [S] we supply a short proof of the claim.

Claim. There is a regular Suslin scheme consisting of closed sets A_s, $s \in \omega^{<\omega}$, such that

- (i) $A_\emptyset \neq \emptyset$;
- (ii) $\bigcup_{\nu\in\omega^\omega} \bigcap_{n\in\omega} A_{\nu\,|\,n} \subset S$;
- (iii) if $U \cap A_s \neq \emptyset$ for some $s \in \omega^{<\omega}$ and open U in X, then there is a set $E \subset U \cap A_s$ nowhere dense in $U \cap A_s$ and not in $\mathcal{I}$;
- (iv) $A_s = \overline{\bigcup_{n\in\omega} A_{s^\wedge n}}$ for $s \in \omega^{<\omega}$.

Proof: Since $S \subset X$ is Suslin, it can be represented by a regular Suslin scheme consisting of closed sets, which means that

$$S = \bigcup_{\nu \in \omega^\omega} \bigcap_{n \in \omega} F_{\nu\,|\,n},$$

where every set $F_{\nu\,|\,n}$ is closed and $F_{\nu\,|\,(n+1)} \subset F_{\nu\,|\,n}$ for $\nu \in \omega^\omega$ and $n \in \omega$.

Put

$$S_s = \bigcup_{\nu \in \omega^\omega} \bigcap_{n \in \omega} F_{s^\wedge(\nu\,|\,n)}$$

for every $s \in \omega^{<\omega}$. Clearly $S = S_\emptyset$ and $S_s = \bigcup_{n \in \omega} S_{s^\wedge n}$ for $s \in \omega^{<\omega}$.

As in [S] put $A_s = \overline{S_s^*}$. By Lemma 3.2(b) condition (i) is satisfied, by part (b) of the same lemma we have (iv). The inclusion (ii) follows since $A_s = \overline{S_s^*} \subset F_s$ by the definition of the sets S_s.

Let $U \cap A_s \neq \emptyset$ for some open $U \subset X$ and $s \in \omega^{<\omega}$. As $\tilde{S} := S \cap (U \cap A_s)$ is a Suslin subset of $U \cap A_s$, the set $\tilde{S}$ has the Baire property in $U \cap A_s$. Thus there are a $\boldsymbol{G}_\delta$ subset G of $U \cap A_s$, and so $\boldsymbol{G}_\delta$ in X, and a set M that is of the first category in $U \cap A_s$ such that $\tilde{S} = G \cup M$.

Now $G \subset \tilde{S} \subset S$ belongs to $\mathcal{I}$ by our assumptions on S. Since $\tilde{S} \supset U \cap S_s^*$ and $U \cap S_s^* \notin \mathcal{I}$ by Lemma 3.2(a) and (b), we have that $\tilde{S} \notin \mathcal{I}$. As $\mathcal{I}$ is a σ-ideal, there is a subset E of M that is nowhere dense in $U \cap A_s$ with $E \notin \mathcal{I}$. $\square$

Also the second part of our proof is essentially contained in [S] but it is (necessarily) formally different.

Put $\mathcal{U}_0 = \{X\}$ and $s_0(X) = \emptyset$, and construct inductively in $i \in \mathbb{N}$, using the preceding two lemmas, a set $E_i(U) \subset U$ for $U \in \mathcal{U}_{i-1}$, a family $\mathcal{U}_i$, and a mapping $s_i : \mathcal{U}_i \to \omega^{<\omega}$ associating to each $U \in \mathcal{U}_i$ a sequence of length i such that

(1) $\mathcal{U}_i$, $i \in \mathbb{N}$, are families of pairwise disjoint open subsets of X of diameter at most $1/i$;
(2) $U \cap A_{s_i(U)} \neq \emptyset$ for $U \in \mathcal{U}_i$;
(3) for $W \in \mathcal{U}_i$ there is $U \in \mathcal{U}_{i-1}$ with $\overline{W} \subset U$ and $s_i(W)$ prolongs $s_{i-1}(U)$;
(4) $E_i(U) \subset U \cap A_{s_{i-1}(U)}$ is nowhere dense in $U \cap A_{s_{i-1}(U)}$ and $E_i(U) \notin \mathcal{I}$ for $U \in \mathcal{U}_{i-1}$;
(5) $\overline{\mathcal{U}_i(U)} \supset E_i(U)$, where $\mathcal{U}_i(U) = \{W \in \mathcal{U}_i : W \subset U\}$ for $U \in \mathcal{U}_{i-1}$.

Let $\mathcal{U}_i$ and s_i be already defined for $i < n$ and some $n \in \mathbb{N}$. Note that by Claim (i) for $n = 1$ and by (2) for $n > 1$ we may use Claim (iii) for every $U \in \mathcal{U}_{n-1}$ and $A_{s_{n-1}(U)}$ to find $E_n(U)$ which is nowhere dense in $U \cap A_{s_{n-1}(U)}$ and not in $\mathcal{I}$.

Using Lemma 3.3 with $P := U \cap A_{s_{n-1}(U)}$, $E := E_n(U)$, and $\varepsilon := \frac{1}{n}$, we find a family $\mathcal{U} = \mathcal{U}_n(U)$ with the corresponding properties and put $\mathcal{U}_n = \bigcup\{\mathcal{U}_n(U) : U \in \mathcal{U}_{n-1}\}$. Let us consider a $W \in \mathcal{U}_n$. Choose the (clearly uniquely determined) $U \in \mathcal{U}_{n-1}$ with $W \in \mathcal{U}_n(U)$. By the definition of $\mathcal{U}_n(U)$ and Lemma 3.3(b), we see that $W \cap A_{s_{n-1}(U)} \neq \emptyset$. Using Claim (iv), we now choose $k \in \omega$ such that $A_{s_{n-1}(U)^\wedge k} \cap W \neq \emptyset$ and put $s_n(W) = s_{n-1}(U)^\wedge k$. This finishes our inductive construction.

Define $G = \bigcap_{i=0}^{\infty} \bigcup \mathcal{U}_i$. The set G is a $\boldsymbol{G}_\delta$ subset of X by (1).

To prove $G \subset S$ consider $x \in G$. Then for every $i \in \omega$ there exists a uniquely determined $U_i \in \mathcal{U}_i$ with $x \in U_i$. According to (3) we find $\mu \in \omega^\omega$ such that $s_i(U_i) = \mu \,|\, i$ for $i \in \omega$. Then $x \in \bigcap_{i \in \omega} A_{\mu \,|\, i}$. Indeed, otherwise there exists $k \in \omega$ with $x \notin A_{\mu \,|\, k}$. Using (1) and the fact that the set $A_{\mu \,|\, k}$ is closed, we find $j \in \omega$ with $U_j \cap A_{\mu \,|\, k} = \emptyset$. Putting $p = \max\{j, k\}$ we obtain $U_p \cap A_{s_p(U_p)} = U_p \cap A_{\mu \,|\, p} \subset U_j \cap A_{\mu \,|\, k} = \emptyset$. This contradicts (2). Thus $G \subset \bigcup_{\nu \in \omega^\omega} \bigcap_{n \in \omega} A_{\nu \,|\, n} \subset S$.

By (1), (2), (3), and the completeness of X, we have $U \cap G \neq \emptyset$ for $U \in \mathcal{U}_i$, $i \in \omega$.

By our assumptions $G \in \mathcal{I}$. Thus there is a countable family $\mathcal{F} \subset \mathcal{I}$ of closed sets with $G \subset \bigcup \mathcal{F}$. As G is a Baire space, there is an open set V in X and an element F of $\mathcal{F}$ such that $\emptyset \neq V \cap G \subset F \in \mathcal{I}$. Using (1) we find $k \in \omega$ and $U \in \mathcal{U}_k$ such that $U \subset V$. By (5), $\overline{\mathcal{U}_{k+1}(U)} \supset E_{k+1}(U)$, and as stated above $W \cap G \neq \emptyset$ for $W \in \mathcal{U}_{k+1}(U)$. Thus we get that $E_{k+1}(U) \subset U \cap \overline{G}$ and so $E_{k+1}(U) \subset V \cap \overline{G} \subset \overline{V \cap G} \subset F \in \mathcal{I}$. This is a contradiction with $E_{k+1}(U) \notin \mathcal{I}$ stated in (4). □

The next simple proposition shows how one can generate σ-ideals satisfying the assumptions of Theorem 1.3 starting with a given family $\mathcal{F}$ of closed sets. To formulate it we need the following definitions.

Definition 3.4. Let X be a metric space.

(i) We say that a *system $\mathcal{F}$ of closed subsets of X is hereditary* if $F_1 \in \mathcal{F}$ whenever F_1 is closed and there is an $F_2 \in \mathcal{F}$ with $F_1 \subset F_2$.

(ii) Let $\mathcal{F}$ be a system of closed subsets of X. Then the symbol $\mathcal{F}_{hd}$ stands for the system of unions of discrete families of closed subsets of elements of $\mathcal{F}$. (Note that, in other words, it is the smallest hereditary system of closed sets which contains $\mathcal{F}$ and which is closed with respect to the operation of discrete unions.)

(iii) Let $\mathcal{A}$ be a system of subsets of X. We say that $\mathcal{A}$ is *weakly locally determined* if $A \subset X$ belongs to $\mathcal{A}$ whenever for each $x \in X$ there exists a, not necessarily open, neighbourhood U of x such that $U \cap A \in \mathcal{A}$.

Proposition 3.5. *Let X be a metric space.*

(i) *It is equivalent to say about a σ-ideal $\mathcal{I}$ of subsets of X that $\mathcal{I}$ is locally determined, or that $\mathcal{I}$ is weakly locally determined, or that $\mathcal{I}$ is closed with respect to discrete unions.*

(ii) *Let $\mathcal{F}$ be an arbitrary system of closed subsets of X. Then the smallest locally determined σ-ideal $\mathcal{I}$ containing $\mathcal{F}$ is equal to $(\mathcal{F}_{hd})_{\mathrm{ext}}$. In particular, $\mathcal{I}$ is $\boldsymbol{F}_\sigma$ supported.*

(iii) *Let $\mathcal{F}$ be a hereditary weakly locally determined system of closed subsets of X. Then $\mathcal{F}_{\text{ext}}$ is a (clearly $\boldsymbol{F}_\sigma$ supported) locally determined σ-ideal.*

Proof: We prove (i) first. If $\mathcal{I}$ is locally determined, it is clearly weakly locally determined, and it is also immediate to verify that every weakly locally determined σ-ideal is closed with respect to discrete unions. Now suppose that $\mathcal{I}$ is closed with respect to discrete unions and consider $A \subset X$ such that for each $x \in A$ there exists its neighbourhood U with $U \cap A \in \mathcal{I}$. Choose a σ-discrete open base $\mathcal{B}$ in X (cf. [E]). Let $\mathcal{B} = \bigcup_{n=1}^\infty \mathcal{B}_n$, where $\mathcal{B}_n$ are discrete systems. The equality $A = \bigcup_{n=1}^\infty \bigcup \{B \cap A : \ B \in \mathcal{B}_n, B \cap A \in \mathcal{I}\}$ implies $A \in \mathcal{I}$. Thus $\mathcal{I}$ is locally determined.

By (i) each locally determined σ-ideal $\mathcal{I}$ containing $\mathcal{F}$ contains $(\mathcal{F}_{hd})_{\text{ext}}$. Using (i) again, we see that to prove (ii) it is sufficient to show that $(\mathcal{F}_{hd})_{\text{ext}}$ is closed with respect to unions of discrete subfamilies. To this end, let $\{E_a : a \in A\} \subset (\mathcal{F}_{hd})_{\text{ext}}$ be a discrete family. Then there are sets $E_{an} \in \mathcal{F}_{hd}$ such that $E_a \subset \bigcup\{E_{an} : n \in \mathbb{N}\}$. The families $\{E_{an} \cap \overline{E_a} : a \in A\}$, $n \in \mathbb{N}$, form discrete subfamilies of $\mathcal{F}_{hd}$ and therefore the sets $\bigcup\{E_{an} \cap \overline{E_a} : a \in A\}$, $n \in \mathbb{N}$, belong to $\mathcal{F}_{hd}$. Since $\bigcup\{E_a : a \in A\} \subset \bigcup_{n \in \mathbb{N}} \bigcup\{E_{an} \cap \overline{E_a} : a \in A\}$, we get that $\bigcup\{E_a : a \in A\} \in (\mathcal{F}_{hd})_{\text{ext}}$.

To prove (iii) observe that $\mathcal{F} = \mathcal{F}_{hd}$ if $\mathcal{F}$ is as in (iii). Indeed, if $\{E_a : a \in A\}$ is a discrete family of closed sets such that each E_a is contained in some element of $\mathcal{F}$, then the set $\bigcup\{E_a : a \in A\}$ is in $\mathcal{F}$ as $\mathcal{F}$ is hereditary and weakly locally determined. Using (ii), we get that $\mathcal{F}_{\text{ext}}$ is locally determined. □

The next corollary immediately follows by Theorem 1.3 and by the above proposition.

Corollary 3.6. *Let X be a topologically complete metric space.*

(i) *Let $\mathcal{F}$ be a hereditary weakly locally determined system of closed subsets of X. Then each Suslin subset of X is $\mathcal{F}$-approximable.*
(ii) *Let $\mathcal{F}$ be an arbitrary system of closed subsets of X. Then each Suslin subset of X is $\mathcal{F}_{hd}$-approximable.*

Remark 3.7. Note that, if $\mathcal{I}$ is a locally determined σ-ideal of subsets of a metric space X and $A \subset X$, then $A^* = A^{**}$ (Lemma 3.2(a) and (b)) and A^* is closed in A. In particular, if $A \notin \mathcal{I}$ is a $\boldsymbol{G}_\delta$ subset of X, then $G = A^*$ is a $\boldsymbol{G}_\delta$ set not in $\mathcal{I}$ with $G = G^*$.

Using this observation and Theorem 1.3, we can find in each Suslin set $S \notin \mathcal{I}$ a $\boldsymbol{G}_\delta$ set $G \subset S$ such that $G \notin \mathcal{I}$ and $G = G^*$.

Due to Proposition 3.5 we get the same conclusion considering as $\mathcal{I}$ the σ-ideals $\mathcal{F}_{\text{ext}}$ (or $(\mathcal{F}_{hd})_{\text{ext}}$), where $\mathcal{F}$ is as in Corollary 3.6(i) (or in Corollary 3.6(ii)).

The next Proposition 3.8 can be interesting on its own right.

Proposition 3.8. *Let X be a topologically complete metric space and $S \subset X$ be a Suslin set. Then there exist $H_n \in \boldsymbol{G}_\delta(X) \cap \mathcal{P}(S)$, $n \in \mathbb{N}$, such that $S \subset \bigcup_{n=1}^\infty \overline{H_n}$.*

Proof: Put $\mathcal{F} = \{\overline{H} : H \in \boldsymbol{G}_\delta(X) \cap \mathcal{P}(S)\}$. Since each $\boldsymbol{G}_\delta$ set H contained in S is covered by an element of $\mathcal{F}$, Corollary 3.6(ii) implies $S \in (\mathcal{F}_{hd})_{\text{ext}}$. Thus it is sufficient to prove that each element of $\mathcal{F}_{hd}$ is covered by an element of $\mathcal{F}$.

Let $\mathcal{M}$ be a discrete family of closed sets such that each element of $\mathcal{M}$ is contained in some element of $\mathcal{F}$. Then for every $M \in \mathcal{M}$ there exists $H(M) \in \boldsymbol{G}_\delta(X) \cap \mathcal{P}(S)$ with $M \subset \overline{H(M)}$. For every $M \in \mathcal{M}$ we may find an open set $O(M) \subset X$ containing M such that $\{O(M)\}_{M \in \mathcal{M}}$ is discrete (as every metric space is collectionwise normal, see e.g. [E, Theorem 5.1.18]). Then the set $H := \bigcup\{H(M) \cap O(M) : M \in \mathcal{M}\}$ is a $\boldsymbol{G}_\delta$ set, $\bigcup \mathcal{M} \subset \overline{H}$, and $H \subset S$. It completes the proof. $\square$

Remark 3.9. Using the same approach as in [S], a nonseparable version of a Hurewicz type theorem from [KLW] can be obtained.

Let S be a Suslin subset of a completely metrizable space X and let $T \subset X$ be such that $S \cap T = \emptyset$. Then either S can be separated from T by an $\boldsymbol{F}_\sigma$ set, or there is a homeomorphic copy $C \subset S \cup T$ of the Cantor set such that $C \cap T$ is countable and dense in C.

The system $\mathcal{F}$ consisting of all closed sets which do not intersect T is weakly locally determined and thus our Corollary 3.6(i) (with Remark 3.7) may be applied in the same way as Solecki's theorem was used in [S]. However, a more straightforward proof of the assertion can be found in [MP].

Remark 3.10. In the same way as in [S] we might prove a nonseparable version of a theorem of Feng.

*Let S be a Suslin subset of a completely metrizable space X and $R \subset X \times X$ be a closed symmetric relation. Then either S can be covered by a σ-discrete family of closed R-homogeneous sets, or, there is a homeomorphic copy $P \subset S$ of the Cantor set which is R-independent. (A set A is R-*homogeneous*, if $(x, y) \in R$ whenever $x, y \in A$, $x \neq y$, and R-*independent *if $(x, y) \notin R$ whenever $x, y \in A$, $x \neq y$.)*

The elegant proof of Solecki of Feng's theorem in Polish spaces can be repeated if we use our Corollary 3.6(ii) (with Remark 3.7) to $(\mathcal{F}_{hd})_{\text{ext}}$, where $\mathcal{F} = \{F \subset X : F \text{ is closed and } F \times F \subset R\}$. However, a more general version of Feng's theorem can be found in [CP, Corollary 3.1].

Finally note that, if the statement of Theorem 1.3 holds in a space X for S from a class $\mathcal{S}^*$ (possibly under some extra set theoretical axioms, see e.g. [S] for the case of Polish X and a class $\mathcal{S}^*$ including co-Suslin sets), then the results mentioned in Remarks 3.9 and 3.10 hold also for $S \in \mathcal{S}^*$.

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Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail: holicky@karlin.mff.cuni.cz
zajicek@karlin.mff.cuni.cz
zeleny@karlin.mff.cuni.cz

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