Properties of one-point completions of a noncompact metrizable space

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Abstract. If a metrizable space X is dense in a metrizable space Y, then Y is called a *metric extension* of X. If T_1 and T_2 are metric extensions of X and there is a continuous map of T_2 into T_1 keeping X pointwise fixed, we write $T_1 \leq T_2$. If X is noncompact and metrizable, then $(\mathcal{M}(X), \leq)$ denotes the set of metric extensions of X, where T_1 and T_2 are identified if $T_1 \leq T_2$ and $T_2 \leq T_1$, i.e., if there is a homeomorphism of T_1 onto T_2 keeping X pointwise fixed. $(\mathcal{M}(X), \leq)$ is a large complicated poset studied extensively by V. Bel'nov [*The structure of the set of metric extensions of a noncompact metrizable space*, Trans. Moscow Math. Soc. **32** (1975), 1–30]. We study the poset $(\mathcal{E}(X), \leq)$ of one-point metric extensions of a locally compact metrizable space X. Each such extension is a (Cauchy) completion of X with respect to a compatible metric. This poset resembles the lattice of compactifications of a locally compact space if X is also separable. For Tychonoff X, let $X^* = \beta X \setminus X$, and let $\mathcal{Z}(X)$ be the poset of zerosets of X partially ordered by set inclusion.
Theorem *If X and Y are locally compact separable metrizable spaces, then $(\mathcal{E}(X), \leq)$ and $(\mathcal{E}(Y), \leq)$ are order-isomorphic iff $\mathcal{Z}(X^*)$ and $\mathcal{Z}(Y^*)$ are order-isomorphic, and iff X^* and Y^* are homeomorphic.* We construct an order preserving bijection $\lambda : \mathcal{E}(X) \to \mathcal{Z}(X^*)$ such that a one-point completion in $\mathcal{E}(X)$ is locally compact iff its image under λ is clopen. We extend some results to the nonseparable case, but leave problems open. In a concluding section, we show how to construct one-point completions geometrically in some explicit cases.

Keywords: metrizable, metric extensions and completions, completely metrizable, one-point metric extensions, extension traces, zerosets, clopen sets, Stone-Čech compactification, $\beta X \setminus X$, hedgehog

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1. Introduction

If X is a dense subspace of a Tychonoff space Y, then Y is called an *extension* of X. Two extensions T_1 and T_2 of X are said to be *equivalent* if there is a homeomorphism of T_1 onto T_2 that keeps X pointwise fixed. Clearly "equivalence" is an equivalence relation on the set of (Tychonoff) extensions of X, and the set of equivalence classes thus generated will be denoted by $\mathcal{E}xt(X)$. Such equivalence classes will be identified with individual members when this causes no confusion.

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Keeping this identification in mind, if T_1 and T_2 are in $\mathcal{E}xt(X)$ and there is a continuous map of T_2 into T_1 that keeps X pointwise fixed, we will write $T_1 \leq T_2$. It is not difficult to see that $(\mathcal{E}xt(X), \leq)$ is a partially ordered set (or *poset*). A detailed discussion of this poset may be found in Section 4.1 of [PW87].

There have been extensive studies of the order structure of various subsets of this poset, especially when the subset consists of compactifications. The work in this paper was motivated initially by V.K. Bel'nov's study of the poset $(\mathcal{M}(X), \leq)$ of all metric extensions of a noncompact metrizable space X. (In this case, Bel'nov called the mappings used to define the partial order $\leq$ *admissible*; see [B74] and especially [B75].) A few of Bel'nov's results follow.

1.1. If X is a locally compact noncompact metric space, then any two members of $\mathcal{M}(X)$ have a common lower bound, but a finite number need not have a greatest lower bound.

1.2. If X is a noncompact metric space, then any countable family in $\mathcal{M}(X)$ has a supremum.

1.3. If X is a metric space that is not locally compact, there are two members of $\mathcal{M}(X)$ that have no common lower bound.

Bel'nov's study of $(\mathcal{M}(X), \leq)$ was much more extensive. The authors have been unable to find any other discussions of the properties of this poset in the research literature.

Others who have studied subsets of the poset ($\mathcal{E}xt(X), \leq)$ have focussed on the relationship between their order structure and the topology of spaces related to X. One of the earliest and most beautiful papers of this sort was written by K. Magill [Ma68]. Let X denote a locally compact Tychonoff space, βX its Stone-Čech compactification, $X^* = \beta X \setminus X$, and let $\mathcal{K}(X)$ denote the set of compact members of $\mathcal{E}xt(X)$. (See 1.5.) In [Ma68] Magill shows that:

1.4. If X and Y are locally compact, then $(\mathcal{K}(X), \leq)$ and $(\mathcal{K}(Y), \leq)$ are lattices and are order-isomorphic if and only if X^* and Y^* are homeomorphic.

Similar results appear, for example, in [Ra73], [MRW72], [MRW74], and [W74], among other places.

The purpose of this paper is to add to the body of such results by studying the poset $(\mathcal{E}(X), \leq)$, where $\mathcal{E}(X)$ denotes the family of one-point metric extensions of a locally compact metrizable space X. Thus our subject matter is close to that of Bel'nov. Our results, however, are similar in form to that of Magill cited above. It comes as a surprise that the poset $(\mathcal{E}(X), \leq)$ has so rich a structure and conveys so much information.

In Section 4, we show that there is a one-one order reversing mapping λ from the poset $(\mathcal{E}(X), \leq)$ into the lattice $\mathcal{Z}(X^*)$ of zerosets of X^* (partially ordered by set inclusion). If X is also separable, then λ maps $\mathcal{E}(X)$ onto $\mathcal{Z}(X^*) \setminus \emptyset$ and hence is an anti-isomorphism. It follows that if X and Y are locally compact separable metrizable spaces, then $(\mathcal{E}(X), \leq)$ and $(\mathcal{E}(Y), \leq)$ are order-isomorphic if and only if $\mathcal{Z}(X^*)$ and $\mathcal{Z}(Y^*)$ are order-isomorphic and if and only if X^* and Y^* are are homeomorphic. Furthermore, the clopen subsets of X^* are precisely the images under λ of the (equivalence classes of) locally compact members of $\mathcal{E}(X)$.

In the final sections of the paper, we present some partial results in the case when the spaces considered are locally compact but not separable, and indicate how some of our results could be described geometrically.

We now review briefly some of the notation and terminology used below and some known facts from the theory of metric spaces and the Stone-Čech compactification. If X is a metric space with metric d, $x \in X$ and $\epsilon > 0$, then $S_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ is called an *open ball of radius ϵ centered at x*. A metrizable space (X, τ) is called *completely metrizable* if there is a metric d on X such that the topology induced on X by d is τ (in which case d is said to be *compatible with τ*) and (X, d) is complete; that is, every d-cauchy sequence converges. Two metrics on X are called *equivalent* if they induce the same topology on X. Topological terminology and theorems used herein come mostly from [E89] and [PW87], and the ordering used on extensions from 4.1 of [PW87], [B74], and [B75]. Additional information on metric extensions may be found in [FGO93] and [V87]. We close this section with:

1.5 The Stone-Čech compactification and related topics. It is well-known that every Tychonoff space X has a dense embedding into a compact space βX such that if Y is any compactification of X, then there is a continuous map of βX onto Y keeping X pointwise fixed. (For this and other background material on βX, see [GJ76], especially Chapter 6, and Chapter 4 of [PW87].) Let $C(X)$ denote the algebra of continuous real-valued functions on X, and $C^*(X)$ its subalgebra of bounded elements. If $f \in C(X)$, then $Z(f) = \{x \in X : f(x) = 0\}$ is called the *zeroset* of f, and the family of zerosets of X is denoted by $\mathcal{Z}(X)$. It is well-known that this latter family is closed under finite union and countable intersection. Also, $\mathcal{Z}(X)$ is the family of all closed subsets of X if X is metrizable. Use will be made in what follows of the following properties of βX.

(i) If $\{Z_1, \ldots, Z_n\}$ is a finite collection of zerosets of X, then:

$$\bigcap_{i=1}^{n} [\mathrm{cl}_{\beta X} Z_i] = \mathrm{cl}_{\beta X}[\bigcap_{i=1}^{n} Z_i].$$

(ii) If X is locally compact, then $\beta X \backslash X$ is compact and $\{\mathrm{cl}_{\beta X} Z \backslash X : Z \in \mathcal{Z}(X)\}$ is a base for the closed subsets of $\beta X \setminus X$.

2. When does a metrizable space have a one-point completion?

2.1 Theorem. *Suppose $Y \in \mathcal{M}(X)$ is a metric extension of X such that $K = (Y \setminus X)$ is compact. If Y is completely metrizable, then so is X.*

Proof: Since X is the complement of K in Y, X is open and a fortiori a G_δ in Y. Since Y is completely metrizable, so is X. □

The converse of 2.1 also holds.

2.2 Theorem. *If X is completely metrizable, and $Y \in \mathcal{M}(X)$ is a metric extension of X such that $(Y \setminus X)$ is compact, then Y is topologically complete.*

Proof: If Z is a completion of Y, then X is dense in Z and is completely metrizable. By 4.3.23 in [E89], X is a G_δ in Z; say $X = \bigcap_{n<\omega} V_n$ where each V_n is open in Z, and $V_{n+1} \subset V_n$. Now $(Y \setminus X)$ is compact and thus closed in the metric space Z; so $(Y \setminus X) = \bigcap_{n<\omega} W_n$, where each W_n is open in Z and $W_{n+1} \subset W_n$. Hence

$$Y = X \cup (Y \setminus X) = (\bigcap_{n<\omega} V_n) \cup (\bigcap_{j<\omega} W_j) = \bigcap\{(V_n \cup W_j) : n < \omega \text{ and } j < \omega\}$$

because the V_ns and W_js are descending chains. Thus Y is a dense G_δ in the completely metrizable space Z and hence is completely metrizable space by 4.3.23 in [E89]. □

2.3 Theorem. *A metrizable space has a completion with a one-point remainder if and only if it has one with a compact remainder.*

Proof: If Y is a metric extension of X such that $Y \setminus X$ is compact, and T is the quotient space obtained by collapsing $Y \setminus X$ to a point and fixing X pointwise, then because metrizability is preserved under perfect maps by 4.4.15 of [E89], it is routine to verify that T is a one-point metric extension of X. □

3. Extension traces and regular sequences of open sets

3.1 Definitions. Let $\mathcal{U} = (U_n)_{n<\omega}$ be a countable family of distinct nonempty open subsets of a metrizable space X. Consider the following conditions:

(i) $\operatorname{cl} U_{n+1} \subset U_n$ for all $n < \omega$, and
(ii) $\bigcap_{n<\omega} U_n = \emptyset$.

If $\mathcal{U}$ satisfies (i), it is called a *regular sequence of open sets* of X.
If $\mathcal{U}$ satisfies both (i) and (ii), it is called an *extension trace on* X.

The motivation for these definitions comes from Lemmas 3.2, 3.3, and 4.3.

3.2 Lemma. *If $Y = X \cup \{p\}$ is a one-point metric extension of a metrizable space X and d is a compatible metric on Y, then $\{X \cap S_d(p, \frac{1}{n}) : n < \omega\}$ is an extension trace on X.*

What is more interesting is the converse, which is a restatement of Theorem 2 in [A71].

3.3 Theorem (Alexander)**.** *Suppose (X,τ) is metrizable, $p \notin X$, $Y = X \cup \{p\}$, and let $\mathcal{U}$ denote an extension trace on X. Define a family ς on Y as follows:*

$$\varsigma = \tau \cup \{S \subset Y : p \in S, \text{ there is a } U \in \mathcal{U} \text{ such that } U \subset S \cap X, \text{ and } S \cap X \in \tau\}.$$

Then:

(a) *(Y,ς) is a regular topological space containing (X,τ) as a dense subspace, and*

(b) *(Y,ς) is metrizable and hence is a one-point metric extension of X.*

If we are given an extension trace $\mathcal{U}$ on a metrizable space X, we will denote by $Y_{\mathcal{U}}$ the one-point metric extension (Y,ς) described above. When we do this, the unique point in $Y \setminus X$ will be denoted by $p(\mathcal{U})$. Thus $Y_{\mathcal{U}} = X \cup \{p(\mathcal{U})\}$, and $\{\{p(\mathcal{U})\} \cup U : U \in \mathcal{U}\}$ is a neighborhood base at $p(\mathcal{U})$ in $Y_{\mathcal{U}}$.

Thus we see that every extension trace on a metric space X generates a one-point metric extension of X, and every one-point metric extension of a metric space generates an extension trace on X.

3.4 Definition. Let $\mathcal{U} = (U_n)_{n<\omega}$ and $\mathcal{V} = (V_n)_{n<\omega}$ denote two regular sequences of open sets on X. We say that $\mathcal{U}$ *is finer than* $\mathcal{V}$ and denote it by $\mathcal{U} \leq \mathcal{V}$ if for each $n < \omega$, there is a $k_n < \omega$ such that $U_{k_n} \subset V_n$.

3.5 Theorem. *If $\mathcal{U}, \mathcal{V}$, are two extension traces on X, and $Y_{\mathcal{U}}$, and $Y_{\mathcal{V}}$ are defined as above, then the following are equivalent.*

(a) *$Y_{\mathcal{U}} \geq Y_{\mathcal{V}}$.*

(b) *$\mathcal{U}$ is finer than $\mathcal{V}$.*

(c) *For each $n < \omega$, there is a $k_n < \omega$ such that $\mathrm{cl}_X\, U_{k_n} \setminus V_n$ is empty or compact.*

Proof: (a) implies (b). Define $f : Y_{\mathcal{U}} \to Y_{\mathcal{V}}$ by letting $f(x) = x$ if $x \in X$, and $f(p(\mathcal{U})) = p(\mathcal{V})$. Clearly (a) holds if and only if f is continuous, and clearly this latter holds at each point of X. If (a) holds, then f is continuous at $p(\mathcal{U})$. Thus, because for each $n < \omega$, $\{p(\mathcal{V})\} \cup V_n$ is a neighborhood of $p(\mathcal{V})$ in $Y_{\mathcal{V}}$, there is a neighborhood S of $p(\mathcal{U})$ in $Y_{\mathcal{U}}$ such that $S \subset f^{\leftarrow}[p(\mathcal{V}) \cup V_n] = \{p(\mathcal{U})\} \cup V_n$. But there will be some $k_n < \omega$ such that $\{p(\mathcal{U})\} \cup U_{k_n} \subset S$. Thus, $U_{k_n} \subset V_n$ and (b) holds.

(b) implies (a). Suppose conversely that (b) holds. Then, for each $n < \omega$, there is a $k_n < \omega$ such that $U_{k_n} \subset V_n$. Therefore $\{p(\mathcal{U})\} \cup U_{k_n}$ is a neighborhood of $p(\mathcal{U})$ in $Y_{\mathcal{U}}$ that is mapped into the basic neighborhood $\{p(\mathcal{V})\} \cup V_n$ of $p(\mathcal{V})$ in $Y_{\mathcal{V}}$ by f. Thus f is continuous at $p(\mathcal{U})$ and hence is continuous. So (a) holds.

(b) implies (c). By (b), given $n < \omega$, there is a $j_n < \omega$ such that $U_{j_n} \subset V_n$. So $\mathrm{cl}_X\, U_{j_n+1} \setminus V_n = \emptyset$ and (c) follows.

(c) implies (b). Given $n < \omega$, by (c) there is a $j_n < \omega$ such that $\mathrm{cl}_X\, U_{j_n} \setminus V_n$ is compact. Now $\bigcap_{k<\omega} \mathrm{cl}_X\, U_k = \emptyset$ because $(U_k)_{k<\omega}$ is an extension trace, so

$\{(\mathrm{cl}_X U_{j_n} \setminus V_n) \cap \mathrm{cl}_X U_k : k < \omega\}$ is a collection of closed subsets of a compact space $(\mathrm{cl}_X U_{j_n} \setminus V_n)$ with empty intersection. Hence there is a finite subset G of ω such that $\bigcap_{k \in G}(\mathrm{cl}_X U_{j_n} \setminus V_n) \cap \mathrm{cl}_X U_k = \emptyset$. If $m = \max(G \cup \{j_n\})$, then $\mathrm{cl}_X U_m \setminus Vn = \emptyset$, so $U_m \subset V_n$ and m is the desired k_n. □

3.6 Definition. If $\mathcal{U} = (U_n)_{n<\omega}$ and $\mathcal{V} = (V_n)_{n<\omega}$ are two regular sequences of open sets on X and each is finer than the other, we say that $\mathcal{U}$ and $\mathcal{V}$ are *equivalent* and write $\mathcal{U} \approxeq \mathcal{V}$.

3.7 Example. Let $X = \mathbb{R}$, $\mathcal{U} = \{(2n, \infty) : n < \omega\}$, and $\mathcal{V} = \{(2n+1, \infty) : n < \omega\}$. Then $\mathcal{U}$ and $\mathcal{V}$ have no set in common, but since for all $n < \omega$

$$(2n+2, \infty) \subset (2n+1, \infty) \subset (2n, \infty),$$

each is finer than the other. Thus, $\mathcal{U} \approxeq \mathcal{V}$. ($\mathbb{R} \cup p(\mathcal{U}) = \mathbb{R} \cup \{+\infty\} \approxeq (0, 1]$ in this case.)

3.8 Definition. Let $\mathbb{E}(X)$ denote the set of equivalence classes $[\mathcal{U}]$ of extension traces on X and partially order it by letting

$$[\mathcal{U}] \leq [\mathcal{V}] \text{ if } \mathcal{V} \text{ is finer than } \mathcal{U}.$$

It is easy to verify that $(\mathbb{E}(X), \leq)$ is a partially ordered set.

The last result of this section is a restatement of Theorem 3.5.

3.9 Theorem. *The poset* $(\mathcal{E}(X), \leq)$ *of* (*equivalence classes of*) *one-point metric extensions of* X *is order-isomorphic to the poset* $(\mathbb{E}(X), \leq)$ *of* (*equivalence classes of*) *extension traces of* X.

4. The partially ordered set of (equivalence classes of) one-point metric extensions of locally compact metric spaces

In this section, we will produce a one-one mapping λ from the poset $(\mathcal{E}(X), \leq)$ of one-point metrizable extensions of a locally compact metric space X into the lattice $(\mathcal{Z}(\beta X \setminus X), \subset)$ of zerosets of $\beta X \setminus X$ under set inclusion. We will show that λ is an order anti-isomorphism onto its image and that the latter is closed under finite unions and intersections. In Sections 5 and 6 respectively, we will consider the cases when X is separable and nonseparable.

Notational conventions: If $A \subset X$, we let $A^* = \mathrm{cl}_{\beta X} A \setminus X$, so $X^* = \beta X \setminus X$. For any space Y, $C(X, Y)$ denotes the set of all continuous functions from X to Y.

4.1 Lemma. *If $\mathcal{U} = (U_n)_{n<\omega}$ is a regular sequence of open sets on a locally compact metrizable space X, then*

$$\bigcap_{n<\omega} (\mathrm{cl}_X\, U_n)^* \in \mathcal{Z}(X^*).$$

Proof: For all $n < \omega$, $X \setminus U_n$ and $\mathrm{cl}\, U_{n+1}$ are disjoint closed subspaces of the (normal) metrizable space X, so by Tietze's extension theorem, there are $f_n \in C(X, [0,1])$ such that $f_n[\ \mathrm{cl}_X\, U_{n+1}] = \{0\}$ and $f_n[X \setminus U_n] = \{1\}$. If F_n is the continuous extension of f_n to $C(\beta X, [0,1])$ and $F = \sum 2^{-n} F_n$, then $F \in C(\beta X, [0,1])$ and $Z(F) \setminus X \in \mathcal{Z}(X^*)$. We will show next that:

$$Z(F) \setminus X = \bigcap_{n<\omega} (\mathrm{cl}_X\, U_n)^*.$$

To see this, assume first that $x \in \bigcap_{n<\omega} (\mathrm{cl}_X\, U_n)^*$. Clearly $x \notin X$. Because $x \in \mathrm{cl}_{\beta X}(\mathrm{cl}_X\, U_{n+1})$ for all $n < \omega$, and since F_n is a closed continuous map,

$$F_n(x) \in F_n[\mathrm{cl}_{\beta X}(\mathrm{cl}_X\, U_{n+1})] = \mathrm{cl}_{[0,1]}\, F_n[\mathrm{cl}_X\, U_{n+1}] = \mathrm{cl}_{[0,1]}\, f_n[\mathrm{cl}_X\, U_{n+1}] = \{0\}$$

since $f_n = F_n | X$. Hence $F(x) = \{0\}$. Thus $\bigcap_{n<\omega} (\mathrm{cl}_X\, U_n)^* \subset Z(F) \setminus X$. Now suppose that $x \notin \bigcap_{n<\omega} (\mathrm{cl}_X\, U_n)^*$. If $x \in X$, then $x \notin Z(f) \setminus X$. If $x \notin X$, then $x \notin \mathrm{cl}_{\beta X}(\mathrm{cl}_X\, U_k)$ for some $k < \omega$. Now $\mathrm{cl}_X\, U_k \cup (X \setminus U_k) = X$, so

$$\mathrm{cl}_{\beta X}(\mathrm{cl}_X\, U_k) \cup \mathrm{cl}_{\beta X}(X \setminus U_k) = \beta X.$$

Hence $x \in \mathrm{cl}_{\beta X}(X \setminus U_k)$. Therefore

$$F_k(x) \in F_k[\mathrm{cl}_{\beta X}(X \setminus U_k] = \mathrm{cl}_{[0,1]}\, f_k[X \setminus U_k] = \{1\}.$$

Thus $F(x) \geq 2^{-k} > 0$ and $x \notin Z[F]$. This completes the proof of the lemma. $\square$

Lemma 4.1 shows how to associate each regular sequence of open sets of X with a zeroset of X^*. The next lemma shows the converse.

4.2 Lemma. *If X is a locally compact metric space, then whenever $Z \in \mathcal{Z}(X^*)$, there is a regular sequence of open sets $(U_n)_{n<\omega}$ on X for which $Z = \bigcap_{n<\omega} (\mathrm{cl}_X\, U_n)^*$.*

Proof: Because X is locally compact, X^* is compact and hence C^*-embedded in βX. So there is an $f \in C(\beta X, [0,1])$ such that $Z = Z(f) \setminus X$. For each $n < \omega$, let $U_n = X \cap f^{\leftarrow}[(0, \frac{1}{n+1})]$. Clearly

$$\mathrm{cl}_X\, U_{n+1} = \mathrm{cl}_X(X \cap f^{\leftarrow}[(0, \tfrac{1}{n+2})]) \subset X \cap f^{\leftarrow}[(0, \tfrac{1}{n+1})] \subset U_n,$$

so $(U_n)_{n<\omega}$ is a regular sequence of open sets. We will show next that $\bigcap_{n<\omega}(\mathrm{cl}_X U_n)^* = Z$.

If $x \in Z = Z(f) \setminus X$, and V is a neighborhood of x in βX, then since $f(x) = 0$, if $n < \omega$, then $V \cap f^{\leftarrow}[[0, \frac{1}{n})]$ is a βX-neighborhood of x that meets the dense subspace X of βX. So

$$\emptyset \neq X \cap V \cap f^{\leftarrow}[(0, \tfrac{1}{n})] = V \cap U_n.$$

Thus $x \in \mathrm{cl}_{\beta X} U_n$, and since n is arbitrary, $Z \subset \bigcap_{n<\omega}(\mathrm{cl}_X U_n)^*$.

Suppose conversely that $x \in \bigcap_{n<\omega}(\mathrm{cl}_X U_n)^*$. Thus $x \notin X$, so to show that $x \in Z$, it suffices to show that $f(x) = 0$. But for each $n < \omega$, we have $x \in \mathrm{cl}_{\beta X} U_n$, and so $f(x) \in f[\mathrm{cl}_{\beta X} U_n] = \mathrm{cl}_{\mathbb{R}} f[U_n] \subset [0, \frac{1}{n})$. Thus $f(x) = 0$ and we are done. □

Combining Lemmas 4.1 and 4.2 yields:

4.3 Corollary. *If we let $\mu[(U_n)_{n<\omega}] = \bigcap_{n<\omega}(\mathrm{cl}_X U_n)^*$, then μ is a well-defined mapping from the set of regular sequences of open sets on X onto $\mathcal{Z}(X^*)$.*

Next we show that if $\mathcal{U}$ and $\mathcal{V}$ are extension traces on X for which the corresponding one-point metric extensions $Y_{\mathcal{U}}$, and $Y_{\mathcal{V}}$ are equivalent, then $\mu(\mathcal{U}) = \mu(\mathcal{V})$.

4.4 Theorem. *If $\mathcal{U} = (U_n)_{n<\omega}$ and $\mathcal{V} = (V_n)_{n<\omega}$ are two equivalent extension traces on X, then $\bigcap_{n<\omega}(\mathrm{cl}_X U_n)^* = \bigcap_{n<\omega}(\mathrm{cl}_X V_n)^*$.*

Proof: By 3.8 and 3.5, we know that $Y_{\mathcal{U}} \geq Y_{\mathcal{V}}$ and $Y_{\mathcal{U}} \leq Y_{\mathcal{V}}$. By the former, if $n < \omega$, there is a $k_n < \omega$ such that $U_{k_n} \subset V_n$. Then:

$$\bigcap_{n<\omega}(\mathrm{cl}_X U_n)^* \subset \bigcap_{n<\omega}(\mathrm{cl}_X U_{k_n})^* \subset \bigcap_{n<\omega}(\mathrm{cl}_X V_n)^*,$$

and as $Y_{\mathcal{U}} \geq Y_{\mathcal{V}}$, the opposite inclusions hold as well. The result follows. □

4.5 Definition. If $\mathcal{U} = (U_n)_{n<\omega}$ is an extension trace on a locally compact metric space X, let $\lambda(Y_{\mathcal{U}}) = \bigcap_{n<\omega}(\mathrm{cl}_X U_n)^*$.

4.6 Theorem. *If X is a locally compact metric space, then λ is a well-defined mapping from $\mathcal{E}(X)$ into $\mathcal{Z}(X^*)$.*

Proof: If $Y_{\mathcal{U}}$ and $Y_{\mathcal{V}}$ are equivalent one-point metric extensions, then $\lambda(Y_{\mathcal{U}}) = \lambda(Y_{\mathcal{V}})$ by 4.4, so λ is defined unambiguously. By 4.1, $\lambda(Y_{\mathcal{U}}) \in \mathcal{Z}(X^*)$. □

4.7 Theorem. *Under the hypotheses given above, λ is one-one.*

Proof: If $Y_{\mathcal{U}}$ and $Y_{\mathcal{V}}$ are not equivalent, we may assume that $Y_{\mathcal{U}} \not\geq Y_{\mathcal{V}}$. By using the equivalence of (a) and (c) in Theorem 3.5 and the negation of (c); we obtain

There is an $n_o < \omega$ such that for all $j < \omega$, $\mathrm{cl}_X U_j \setminus V_{n_o}$ is not compact.

Suppose the family $\{(\mathrm{cl}_X U_j)^* \cap (X \setminus V_{n_o})^* : j < \omega\}$ of compact sets has empty intersection. Then there would be a finite subset G of ω such that $\bigcap_{j\in G}(\mathrm{cl}_X U_j)^* \cap (X \setminus V_{n_o})^* = \emptyset$, and if $k = \max G$, this implies that

$$(\mathrm{cl}_X U_k)^* \cap (X \setminus V_{n_o})^* = \emptyset = [(\mathrm{cl}_X U_k) \cap (X \setminus V_{n_o})]^*$$

by properties of the Stone-Čech compactification noted in 1.6(i). This implies that $\mathrm{cl}_{\beta X}[(\mathrm{cl}_X U_k) \cap (X \setminus V_{n_o})] \subset X$; that is $(\mathrm{cl}_X U_k) \cap (X \setminus V_{n_o})$ is compact. This contradiction yields that $\bigcap_{j<\omega}(\mathrm{cl}_X U_j)^* \cap (X \setminus V_{n_o})^* \neq \emptyset$.

But $\mathrm{cl}_X V_{n_o+1} \subset V_{n_o}$, so $\mathrm{cl}_X V_{n_o+1} \cap (X \setminus V_{n_o}) = \emptyset$, whence

$$\mathrm{cl}_{\beta X}(\mathrm{cl}_X V_{n_o+1}) \cap \mathrm{cl}_{\beta X}(X \setminus V_{n_o}) = \emptyset = (\mathrm{cl}_X V_{n_o+1})^* \cap (X \setminus V_{n_o})^*.$$

Therefore $\bigcap(\mathrm{cl}_X V_n)^* \cap (X \setminus V_0)^* = \emptyset$, which combined with the above tells us that $\bigcap_{n<\omega}(\mathrm{cl}_X U_n)^* \neq \bigcap_{n<\omega}(\mathrm{cl}_X V_n)^*$, i.e., that $\lambda(Y_{\mathcal{U}}) \neq \lambda(Y_{\mathcal{V}})$. Thus λ is one-one as claimed. □

4.8 Lemma. *If $Y_{\mathcal{V}} \geq Y_{\mathcal{U}}$, then $\lambda(Y_{\mathcal{U}}) \supset \lambda(Y_{\mathcal{V}})$.*

Proof: By Theorem 2.5, if $Y_{\mathcal{V}} \geq Y_{\mathcal{U}}$, then for all $n < \omega$, there is a $k_n < \omega$ such that $V_{k_n} \subset U_n$. So $\bigcap_{n<\omega}(\mathrm{cl}_X V_n)^* \subset \bigcap_{n<\omega}(\mathrm{cl}_X V_{k_n})^* \subset \bigcap_{n<\omega}(\mathrm{cl}_X U_n)^*$; i.e., $\lambda(Y_{\mathcal{U}}) \supset \lambda(Y_{\mathcal{V}})$. □

4.9 Lemma. *If $\mathcal{U}$ and $\mathcal{V}$ are extension traces and $\lambda(Y_{\mathcal{U}}) \subset \lambda(Y_{\mathcal{V}})$, then $Y_{\mathcal{U}} \geq Y_V$.*

Proof: If $Y_{\mathcal{U}} \not\geq Y_V$, then arguing exactly as in the proof of Lemma 4.7, we conclude that there is a $V_{n_0} \in \mathcal{V}$ such that $\bigcap_{j<\omega}(\mathrm{cl}_X U_j)^* \cap (X \setminus V_{n_o})^* \neq \emptyset$, while

$$\bigcap_{n<\omega}(\mathrm{cl}_X V_n)^* \subset (\mathrm{cl}_X V_{n_0})^* \subset X^* \setminus (X \setminus V_{n_o})^*,$$

so $\bigcap_{n<\omega}(\mathrm{cl}_X U_n)^* \not\subseteq \bigcap_{n<\omega}(\mathrm{cl}_X V_n)^*$, i.e., $\lambda(Y_{\mathcal{U}}) \not\subseteq \lambda(Y_{\mathcal{V}})$. □

Combining 4.7,4.8. and 4.9, we obtain:

4.10 Theorem. *If X is a locally compact metric space, then $\lambda : \mathcal{E}(X) \to \mathcal{Z}(X^*)$ is an order anti-isomorphism onto its image.*

5. The case when X is separable

The results obtained in Section 4 apply to one-point metric extensions of any locally compact metric space X. We now consider what additional information can be obtained if X is separable. Recall from 3.8C [E89] that a locally compact separable metric space is σ-compact. Moreover, as noted also in 3.8C in [E89], we have

5.1 Proposition. *A locally compact σ-compact noncompact Tychonoff space may be written in the form $X = \bigcup_{n<w} C_n$, where for each $n < \omega$, C_n is a regular open set, $\mathrm{cl}_X\, C_n \subset C_{n+1}$, and $\mathrm{cl}_X\, C_n$ is compact.*

Combining this with 4.2 yields:

5.2 Theorem. *If X is a locally compact separable metric space and $Z \in \mathcal{Z}(X^*)$, then there is an extension trace $\mathcal{V} = (V_n)_{n<\omega}$ on X such that $\lambda(Y_{\mathcal{V}}) = Z$.*

Proof: By 4.2, there is a regular sequence of open sets $(U_n)_{n<\omega}$ on X for which $Z = \bigcap_{n<\omega}(\mathrm{cl}_X\, U_n)^*$. Using the notation in 5.1, let $V_n = U_n \setminus \mathrm{cl}_X\, C_n$. Then

$$\begin{aligned}&\mathrm{cl}_X\, V_{n+1} \subset \mathrm{cl}_X\, U_{n+1} \cap \mathrm{cl}_X(X \setminus \mathrm{cl}_X\, C_{n+1})\\ &\subset U_n \cap (X \setminus \mathrm{int}_X\, \mathrm{cl}_X\, C_{n+1}) = U_n \cap (X \setminus C_{n+1}) \subset U_n \cap (X \setminus \mathrm{cl}_X\, C_n) = V_n\end{aligned}$$

since C_{n+1} is a regular open set. So, $(\mathcal{V}_n)_{n<\omega}$ is a regular sequence of open sets. Finally

$$\bigcap_{n<\omega} V_n = \bigcap_{n<\omega} (U_n \setminus \mathrm{cl}_X\, C_n) = \emptyset$$

since $\bigcup_{n<\omega} C_n = X$. Thus $\mathcal{V} = (\mathcal{V}_n)_{n<\omega}$ is an extension trace on X.

Because $V_n \subset U_n$, it follows that $\bigcap_{n<\omega}(\mathrm{cl}_X\, V_n)^* \subset \bigcap_{n<\omega}(\mathrm{cl}_X\, U_n)^*$. Conversely, it is clear that $U_n \subset V_n \cup (\mathrm{cl}_X\, U_n)^*$, so $(\mathrm{cl}_X\, U_n)^* \subset (\mathrm{cl}_X\, V_n)^* \cup (\mathrm{cl}_X\, C_n)^*$. But $(\mathrm{cl}_X\, C_n)^* = \emptyset$ since $\mathrm{cl}_X\, C_n$ is compact, so $(\mathrm{cl}_X\, U_n)^* \subset (\mathrm{cl}_X\, V_n)^*$. Thus $\bigcap_{n<\omega}(\mathrm{cl}_X\, U_n)^* \subset \bigcap_{n<\omega}(\mathrm{cl}_X\, V_n)^*$, and each is equal to Z. □

5.3 Theorem. *If X is a locally compact separable metric space, then the map $\lambda : \mathcal{E}(X) \to \mathcal{Z}(X^*)$ defined in 4.5 is an order reversing bijection onto $\mathcal{Z}(X^*) \setminus \{\emptyset\}$.*

Proof: By 4.10, it suffices to show that λ maps $\mathcal{E}(X)$ onto $\mathcal{Z}(X^*) \setminus \emptyset$. But if $Z \in \mathcal{Z}(X^*) \setminus \{\emptyset\}$, by 5.2 there is an extension trace $\mathcal{V} = (V_n)_{n<\omega}$ on X such that $Z = \bigcap_{n<\omega}(\mathrm{cl}_X\, V_n)^*$, and so $Y_{\mathcal{V}} \in \mathcal{E}(X)$ and $\lambda(Y_{\mathcal{V}}) = Z$. □

The theorem that follows is similar to results proved by K. Magill in [Ma68].

5.4 Theorem. *If X and Y are locally compact separable metrizable spaces, then the following are equivalent.*

(a) *$(\mathcal{E}(X), \leq)$ and $(\mathcal{E}(Y), \leq)$ are order-isomorphic.*
(b) *$\mathcal{Z}(X^*)$ and $\mathcal{Z}(Y^*)$ are order-isomorphic.*
(c) *X^* and Y^* are homeomorphic.*

Proof: The equivalence of (a) and (b) follows from 5.3. Because X^* and Y^* are compact, their topology is determined by the order structure of their lattices of zerosets. Hence (b) and (c) are equivalent. □

5.5 Theorem. *If X is a locally compact separable metric space, and $Y = X\cup\{p\}$ is a one-point metric locally compact extension of X, then its image $\lambda(Y)$ is clopen in X^*.*

Proof: If d is a compatible metric on Y, then since X is locally compact, there is an $n_0 < \omega$ such that $\mathrm{cl}_Y\, S_d(p, \frac{1}{n_0})$ is compact. For all $n < \omega$, let $U_n = X \cap S_d(p, \frac{1}{n_0+n})$, and observe that $\mathrm{cl}_Y\, U_n$ is compact. Then $\mathcal{U} = (U_n)_{n<\omega}$ is an extension trace on X, and clearly the one-point extension $Y_{\mathcal{U}}$ associated with $\mathcal{U}$ is equivalent to Y.

If $n < \omega$, then because $\mathrm{cl}_Y\, U_n = \{p\}\cup\mathrm{cl}_X\, U_n$ and $S_d(p, \frac{1}{n_0+n+1}) = U_{n+1}\cup\{p\}$, we see that:

$$\mathrm{cl}_X\, U_n \setminus U_{n+1} = \mathrm{cl}_Y\, U_n \setminus S_d(p, \tfrac{1}{n_0+n+1})$$

which is compact because it is a closed subspace of the compact set $\mathrm{cl}_Y\, U_n$. Thus $\mathrm{cl}_{\beta X}(\mathrm{cl}_X\, U_n \setminus U_{n+1}) \subset X$, so $(\mathrm{cl}_X\, U_n \setminus U_{n+1})^* = \emptyset$. Now, $\mathrm{cl}_X\, U_n = \mathrm{cl}_X\, U_{n+1} \cup (\mathrm{cl}_X\, U_n \setminus U_{n+1})$, so

$$(\mathrm{cl}_X\, U_n)^* = (\mathrm{cl}_X\, U_{n+1})^* \cup (\mathrm{cl}_X\, U_n \setminus U_{n+1})^* = (\mathrm{cl}_X\, U_{n+1})^*.$$

Because this holds for all $n < \omega$, we see that

$$\lambda(Y) = \lambda(Y_{\mathcal{U}}) = \bigcap_{n<\omega} (\mathrm{cl}_X\, U_n)^* = (\mathrm{cl}_X\, U_1)^* \in \mathcal{Z}(X^*).$$

But by Lemma 2.1 of [Mi82], $(\mathrm{cl}_X\, U_1)^*$ is a P-set of X^*. That is, any G_δ of X^* that contains $(\mathrm{cl}_X\, U_1)^*$ is open in X^*. So, since $(\mathrm{cl}_X\, U_1)^*$ is a zeroset of X^*, it is clopen in X^*. $\square$

5.6 Theorem. *If X is a locally compact separable metric space, and $Y = X\cup\{p\}$ is a one-point metric extension of X such that p has no compact neighborhood, then $\lambda(Y)$ is not clopen.*

Proof: We know that $Y = Y_{\mathcal{U}}$ for some extension trace $\mathcal{U}$ on X. Suppose to the contrary $\lambda(Y_{\mathcal{U}}) = \bigcap_{n<\omega}(\mathrm{cl}_X\, U_n)^*$ is clopen in X^*, where $\mathcal{U} = (U_n)_{n<\omega}$. Then $X^* \setminus \bigcap_{n<\omega}(\mathrm{cl}_X\, U_n)^*$ is also clopen in X^* since X is locally compact. So there is a zeroset A of X such that A^* equals the latter. Thus $\bigcap_{n<\omega}(\mathrm{cl}_X\, U_n)^* \cap A^* = \emptyset$. Because X^* is compact, there is a finite subset $G \subset \omega$ such that $\bigcap_{n\in G}(\mathrm{cl}_X\, U_n)^* \cap A^* = \emptyset$. If $m = \max G$, then by 1.6(i) $(\mathrm{cl}_X\, U_m)^* \cap A^* = \emptyset$. Since $\bigcap_{n<\omega}(\mathrm{cl}_X\, U_n)^* \subset (\mathrm{cl}_X\, U_m)^*$, by the above we obtain $\bigcap_{n<\omega}(\mathrm{cl}_X\, U_n)^* = (\mathrm{cl}_X\, U_m)^*$, and hence $(\mathrm{cl}_X\, U_n)^* = (\mathrm{cl}_X\, U_m)^*$ if $n \geq m$. If follows from Lemma 2.4 of [W71] that the inclusion of the first of these remainders in the second implies $\mathrm{cl}_X(\mathrm{cl}_X\, U_m \setminus \mathrm{cl}_X\, U_n)$ is pseudocompact — and hence compact because X is metrizable. Since $\mathcal{U}$ is an extension trace on X, $(\mathrm{cl}_X\, U_m) \cup \{p\}$ is a closed neighborhood of p in Y.

If $\mathcal{C}$ is an open cover of of $\{p\} \cup (\mathrm{cl}_X\, U_m)$, then there is a $k < \omega$ and $C_0 \in \mathcal{C}$ such that $k \geq m$ and $\{p\} \cup U_k \subset C_0$. It follows from the above that $\mathrm{cl}_X(\mathrm{cl}_X\, U_m \setminus \mathrm{cl}_X\, U_k)$ is compact. Hence, $\mathrm{cl}_X\, U_m \setminus \mathrm{int}_X(\mathrm{cl}_X\, U_k)$ is compact. Because we could have chosen the members of $\mathcal{U}$ to be regular open sets, we may assume that $\mathrm{cl}_X\, U_m \setminus U_k$ is compact. Since it is covered by $\mathcal{C}$, there are finitely many $C_1, \ldots, C_s \in \mathcal{C}$ such that $(\mathrm{cl}_X\, U_m \setminus U_k) \subset \bigcup_{i=1}^s C_i$. Thus, $\{p\} \cup (\mathrm{cl}_X\, U_m) \subset \bigcup_{i=0}^s C_i$, and so $\mathcal{C}$ has a finite subcover.

It follows that $\{p\} \cup (\mathrm{cl}_X\, U_m)$ is compact and $Y_{\mathcal{U}}$ is locally compact, so the theorem is proved. □

For any space X, its Boolean algebra of clopen sets is denoted by $\mathcal{B}(X)$.

It follows that if X is a locally compact separable metric space for which X^* is zero-dimensional (i.e., has an open base of clopen sets), and we denote by $\mathcal{E}_K(X)$ the poset of (equivalence classes) of locally compact one-point extensions of X, then the topology of X^* determines and is determined by the order structure of $\mathcal{E}_K(X)$. More precisely:

5.7 Theorem. *If X and Y are locally compact separable metrizable spaces whose Stone-Čech remainders are zero-dimensional, then the following are equivalent.*

(a) *The posets $\mathcal{E}_K(X)$ and $\mathcal{E}_K(Y)$ are order-isomorphic.*
(b) *The Boolean algebras $\mathcal{B}(X^*)$ and $\mathcal{B}(Y^*)$ are isomorphic.*
(c) *X^* and Y^* are homeomorphic.*

Proof: This follows immediately from 5.4 and 5.5. □

Theorem 5.7 has some consequences whose validity depend on which set-theoretic assumptions are made. For missing definitions or details in what follows, see [DH99]. A *Parovičenko space* is a compact zero-dimensional space of weight ω_1 with no isolated points in which every nonempty G_δ has a nonempty interior. It is known that every Parovičenko space is homeomorphic with ω^* if and only if the continuum hypothesis (CH) holds. Hence if CH holds and if X is a free union of countably many copies of the Cantor set, then X^* is homeomorphic to ω^*. On the other hand, if the Open Coloring Axiom (which implies the negation of CH) holds then it is not. So, by 5.7, whether or not $(\mathcal{E}_K(X), \leq)$ and $(\mathcal{E}_K(\omega), \leq)$ are order-isomorphic depends on which model of set theory is being used.

6. The case when X is not separable

The result that follows is an easily seen consequence of a theorem of Alexandroff concerning locally separable spaces. See 4.4F in [E89].

6.1 Theorem. *Every locally compact nonseparable metric space is a free union $\oplus\{X_i : i \in I\}$ of uncountably many locally compact separable noncompact metric spaces X_i.*

The subspace of βX introduced next plays a vital role in the study of one-point metric extensions of locally compact nonseparable metrizable spaces.

6.2 Definition. For any nonseparable Tychonoff space X, let

$$\sigma X = \{x \in \beta X : x \text{ is in the closure in } \beta X \text{ of a } \sigma\text{-compact subspace of } X\}.$$

The proof of the following lemma is an exercise.

6.3 Lemma. *If $X = \oplus\{X_i : i \in I\}$ is a free union of uncountably many locally compact separable metric spaces X_i, then:*

$$\sigma X = \bigcup\{\mathrm{cl}_{\beta X}(\bigcup X_i : i \in J) : \ J \text{ a countable subset of } I\}.$$

Moreover, σX is an open subspace of βX, and $\sigma(\sigma X) = \sigma X$.

We know by 5.3 that when X is separable, then λ is an order isomorphism from $\mathcal{E}(X)$ onto $\mathcal{Z}(X^*) \setminus \{\emptyset\}$. But λ will not map $\mathcal{E}(X)$ onto $\mathcal{Z}(X^*) \setminus \{\emptyset\}$ when X is not separable. Next, we investigate those zerosets Z that belong to $\lambda[\mathcal{E}(X)]$.

6.4 Lemma. *Suppose X is a locally compact nonseparable metrizable space. We will denote $\beta X \setminus \sigma X$ by $c\sigma(X)$. If $S \in \mathcal{Z}(X^*)$ and $\emptyset \neq \mathrm{int}_{c\sigma(X)}(S \setminus \sigma X)$, then there is a $T \in \mathcal{Z}(X)$ such that*

$$\emptyset \neq T^* \setminus \sigma X \subset S \setminus \sigma X.$$

Proof: Suppose $x \in \mathrm{int}_{c\sigma(X)}(S \setminus \sigma X)$. By 1.5(ii), there is an $H \in \mathcal{Z}(X)$ such that $x \in c\sigma(X) \setminus H^* \subset S \setminus \sigma X$. Now $\{x\} = \bigcap\{Z^* : x \in Z^* \text{ and } Z \in \mathcal{Z}(X)\}$. Because $x \notin H^*$, $\bigcap\{Z^* : x \in Z^* \text{ and } Z \in \mathcal{Z}(X^*)\} \cap H^* = \emptyset$. Since X^* is compact, there is a finite subset $\{Z_1, \dots, Z_n\}$ of $\mathcal{Z}(X)$ such that $x \in \bigcap_{i=1}^n Z_i^*$ and $\bigcap_{i=1}^n Z_i^* \cap H^* = \emptyset$. Let $T = \bigcap_{i=1}^n Z_i$. Then

$$x \in T^* = (\bigcap_{i=1}^n Z_i)^* = \bigcap_{i=1}^n Z_i^* \subset X^* \setminus H^*.$$

Because $x \in T^* \setminus \sigma X$, it follows that $\emptyset \neq T^* \setminus \sigma X \subset X^* \setminus H^* \subset Z \setminus \sigma X$. □

6.5 Lemma. *If X is a locally compact metrizable space and $(U_n)_{n<\omega}$ is an extension trace on X and $Z = \bigcap_{n<\omega}(\mathrm{cl}_X U_n)^*$, then there does not exist $S \in \mathcal{Z}(X)$ such that $\emptyset \neq S^* \setminus \sigma X \subset Z \setminus \sigma X$.*

Proof: Suppose the contrary; then there exists such an S with $S^* \setminus \sigma X \neq \emptyset$. Now $\bigcap_{n<\omega} U_n = \emptyset$, since $(U_n)_{n<\omega}$ is an extension trace. Hence

$$S = S \setminus \bigcap_{n<\omega} U_n = \bigcup_{n<\omega} (S \setminus U_n).$$

Using the notation of 6.1, suppose that for each $n < \omega$, there is a countable subset $J_n \subset I$ such that

$$S \setminus U_n \subset \bigcup_{j \in J_n} X_j.$$

Let $J = \bigcup_{n<\omega} J_n$. Then J is countable and $S \subset \bigcup_{j\in J} X_j$, which is σ-compact. Then $\mathrm{cl}_{\beta X} S \subset \sigma X$, so $S^* \setminus \sigma X = \emptyset$. This contradiction shows that there is a $k < \omega$ such that $\{i \in I : (S \setminus U_k) \cap X_i \neq \emptyset\}$ is an uncountable set L. For each $i \in L$, choose $y_i \in (S \setminus U_k) \cap X_i$. Then $(y_i)_{i\in L}$ is an uncountable closed discrete subset D of X contained in $S \setminus U_k$. Clearly $D^* \setminus \sigma X \neq \emptyset$, so $(S \setminus U_k)^* \setminus \sigma X \neq \emptyset$. Since $(X \setminus U_k) \cap \mathrm{cl}_X U_{k+1} = \emptyset$ and $(U_j)_{j<\omega}$ is an extension trace on X, it follows that $\mathrm{cl}_{\beta X}(X \setminus U_k) \cap \mathrm{cl}_X U_{k+1} = \emptyset$. But $(X \setminus U_k)^* \supset (S \setminus U_k)^*$ and $Z \subset (\mathrm{cl}_X U_{k+1})^*$, so we see that $[(S \setminus U_k)^* \setminus \sigma X] \cap Z = \emptyset$. But $\emptyset \neq [(S \setminus U_k)^* \setminus \sigma X] \subset S^* \setminus \sigma X$, so $(S^* \setminus \sigma X) \setminus (Z \setminus \sigma X) \neq \emptyset$, in contradiction to assumption. The lemma follows. □

6.6 Lemma. *If X is a locally compact metrizable space and $Z \in \lambda[\mathcal{E}(X)]$, then $\mathrm{int}_{c\sigma(X)}(Z \setminus \sigma X) = \emptyset$.*

Proof: If $Z \in \lambda[\mathcal{E}(X)]$, then there is an extension trace $\mathcal{U} = (U_n)_{n<\omega}$ on X for which $Z = \lambda(Y_{\mathcal{U}}) = \bigcap_{n<\omega}(\mathrm{cl}_X U_n)^*$. By 6.5, there does not exist $T \in \mathcal{Z}(X)$ such that $\emptyset \neq T^* \setminus \sigma X \subset Z \setminus \sigma X$. By 6.4, it follows that $\mathrm{int}_{c\sigma(X)}(Z \setminus \sigma X) = \emptyset$. □

6.7 Theorem. *If X is a locally compact metrizable space and $Z \in \mathcal{Z}(X^*)$, then the following are equivalent.*

(a) $Z \in \lambda[\mathcal{E}(X)]$.
(b) *There does not exist $S \in \mathcal{Z}(X)$ such that $\emptyset \neq S^* \setminus \sigma X \subset Z \setminus \sigma X$.*
(c) $\mathrm{cl}_{\beta X}[\bigcap_{n<\omega}(\mathrm{cl}_X U_n)] \subset \sigma X$, *where $(U_n)_{n<\omega}$ is a regular sequence of open sets for which $Z = \bigcap_{n<\omega}(\mathrm{cl}_X U_n)^*$* (see 4.2).
(d) $\bigcap_{n<\omega}(\mathrm{cl}_X U_n)$ *is σ-compact* (*where $(U_n)_{n<\omega}$ is as in* (c)).

Proof: (a) implies (b). If $Z \in \lambda[\mathcal{E}(X)]$, then there is an extension trace $(U_n)_{n<\omega}$ on X such that $Z = \bigcap_{n<\omega}(\mathrm{cl}_X U_n)^*$. This implication is now just a restatement of 6.5.

(b) implies (c). If (c) fails, then $\bigcap_{n<\omega}(\mathrm{cl}_X U_n)^* \setminus \sigma X \neq \emptyset$. But then (b) fails (with $\bigcap_{n<\omega}(\mathrm{cl}_X U_n)$ playing the role of S), because clearly

$$\Big(\bigcap_{n<\omega} \mathrm{cl}_X U_n\Big)^* \setminus \sigma X \subset \Big[\bigcap_{n<\omega} (\mathrm{cl}_X U_n)^*\Big] \setminus \sigma X = Z \setminus \sigma X.$$

(c) implies (a). By hypothesis (using the notation of 6.1):

$$\mathrm{cl}_{\beta X}\Big[\bigcap_{n<\omega} \mathrm{cl}_X U_n\Big] \subset \bigcup\{\mathrm{cl}_{\beta X}\Big(\bigcup_{i\in J} X_i\Big) : J \subset I \ \text{ is countable}\}.$$

Now $\bigcup_{i\in J} X_i$ is clopen in X, so $\mathrm{cl}_{\beta X}(\bigcup_{i\in J} X_i)$ is clopen in βX for each countable $J \subset I$. So by compactness, we can find a finite family $(J_i)_{i=1}^k$ of countable subsets of I such that

$$\mathrm{cl}_{\beta X}[\bigcap_{n<\omega} \mathrm{cl}_X U_n] \subset \bigcup\{\mathrm{cl}_{\beta X}(\bigcup_{j\in J_i} X_j) : 1 \le i \le k\} = \mathrm{cl}_{\beta X}(\bigcup_{i\in J_0} X_i).$$

where $J_0 = \bigcup_{i=1}^k J_i$. Because $\bigcup_{i\in J_0} X_i$ is clopen in X, it follows that $\bigcap_{n<\omega} \mathrm{cl}_X U_n \subset \bigcup_{i\in J_0} X_i = T$. Because J_0 is countable, T is a locally compact σ-compact metric space.

Arguing as in the proof of 5.2 we see that there is a regular sequence of open sets $(C_j)_{j<\omega}$ in T such that $\mathrm{cl}_X C_{j+1} \subset C_j$ and $T = \bigcup_{j<\omega} C_j$. Let $V_n = U_n \setminus \mathrm{cl}_X C_n$ for each $n < \omega$. Arguing again as in the proof of 5.2, we see that $(V_n)_{n<\omega}$ is an extension trace $\mathcal{V}$ and $Z = \bigcap_{n<\omega}(\mathrm{cl}_X V_n)^*$. Then $Z = \lambda(Y_{\mathcal{V}})$ and so $Z \in \lambda[\mathcal{E}(X)]$. Thus (a) holds.

Finally note that (c) holds if and only if $\bigcap_{n<\omega}(\mathrm{cl}_X U_n)$ is contained in the union of countably many of the X_i, which is easily seen to be equivalent to (d). □

If the converse of Lemma 6.6 were valid, the following theorem would hold for any locally compact nonseparable metric space.

6.8 Theorem. *If D is an uncountable discrete space and $Z \in \mathcal{Z}(D^*)$, then the following are equivalent:*

(a) $\mathrm{int}_{c\sigma(X)}(Z \setminus \sigma D) = \emptyset$;
(b) $Z \in \lambda[\mathcal{E}(D)]$.

Proof: (b) implies (a) is a special case of 6.6.

(a) implies (b). Since $Z \in \mathcal{Z}(D^*)$ is nonempty, there is a regular sequence of open sets $(A_n)_{n<\omega}$ such that $Z = \bigcap_{n<\omega} A_n^*$. If $\bigcap_{n<\omega} A_n$ is uncountable, then by 6.3 $(\bigcap_{n<\omega} A_n)^* \setminus \sigma D \neq \emptyset$ and this set is thus a nonempty open subset of D^* contained in $Z \setminus \sigma D$, contrary to our hypothesis. Thus $\bigcap_{n<\omega} A_n$ is countable, and so $\mathrm{cl}_{\beta D}(\bigcap_{n<\omega} A_n) \subset \sigma D$. It now follows from 6.7 that $Z \subset \lambda[\mathcal{E}(D)]$. □

Next we show that the set $Z \setminus \sigma X$ can be chosen to be nonempty.

6.9 Theorem. *If X is a nonseparable locally compact metrizable space, then there is a $Z \in \lambda[\mathcal{E}(X)]$ such that $Z \setminus \sigma X \neq \emptyset$.*

Proof: Using the notation of 6.1, let $\{I_k : k < \omega\}$ partition I into countably many uncountable subsets, let $J_n = \bigcup\{I_k : k \geq n\}$ for each $n < \omega$, and let $A_n = \bigcup_{i\in J_n} X_i$. Clearly $\bigcap_{n<\omega} J_n = \emptyset$, and it follows that $(A_n)_{n<\omega}$ is a decreasing sequence of clopen subsets of X for which $\bigcap_{n<\omega} A_n = \emptyset$. Thus $\mathcal{A} = (A_n)_{n<\omega}$ is an extension trace on X, and so $Z = \bigcap_{n<\omega} A_n^* \in \lambda[\mathcal{E}(X)]$ by 4.2. We will show now that $Z \setminus \sigma X \neq \emptyset$. For otherwise, $(\bigcap_{n<\omega} A_n^*) \cap (\beta X \setminus \sigma X) = \emptyset$. Since both sets intersected are compact, it follows that there is a finite subset G of ω such that $(\bigcap_{n\in G} A_n^*) \cap (\beta X \setminus \sigma X) = \emptyset$. If $m = \max G$, then $A_m^* \subset \sigma X$. But since J_m is uncountable, A_m meets uncountably many of the X_i, so $A_m^* \setminus \sigma X \neq \emptyset$. This contradiction shows that $Z \setminus \sigma X \neq \emptyset$. □

We conclude this section with a:

6.10 Question. To what extent can Theorem 6.8 be generalized? In particular, can we replace D by any locally compact nonseparable metric space?

7. Finding one-point metric extensions geometrically

In this section, we provide some examples of ways of creating examples of one-point completions of some locally compact metrizable spaces in a geometric way more easy to visualize than the methods employing the Stone-Čech compactification that were used above.

A valuable tool for this purpose will be presented next. Suppose $\mathfrak{M}$ is an infinite cardinal which we identify with its initial ordinal. (That is, $\mathfrak{M}$ is the cardinality of a well-ordered set.)

For each $\alpha < \mathfrak{M}$, let $[0,1]_\alpha$ denote a copy of the closed interval $[0,1]$ with its usual (Euclidean) metric. Let $H(\mathfrak{M})$ denote the set obtained from $\bigcup_{\alpha<\mathfrak{M}}[0,1]_\alpha$ by collapsing each of the left hand endpoints to a point that will be denoted by $\mathcal{O}$, and these intervals are called *spines*. We define a metric d on $H(\mathfrak{M})$ by letting for $x \in [0,1]_\alpha$ and $y \in [0,1]_\beta$

$$d(x,y) = |x-y| \text{ if } \alpha = \beta, \text{ and } d(x,y) = x + y \text{ otherwise.}$$

The resulting metric space is an example of what is called a *hedgehog with* $\mathfrak{M}$ *spines* and is known to be complete. See 4.15 and 4.3B in [E89].

It will be shown next how to obtain a one-point metric extension of an infinite discrete space D from an extension trace $\mathcal{A}$ by injecting D into a hedgehog H and taking the closure of this image in H. It will be shown also how to tell from properties of the extension trace when the resulting one-point extension is locally compact.

Suppose $\mathcal{A} = (A_n)_{n<\omega}$ is an extension trace on D. We will assume that D is well-ordered of cardinality $\mathfrak{M}$, and that $A_0 = D$. Note that $D = \bigcup_{n<\omega}(A_n \setminus A_{n+1})$ is the union of pairwise disjoint sets since $\bigcap_{n<\omega}(A_n) = \emptyset$. So, for each $n < \omega$, we may write $(A_n \setminus A_{n+1}) = \{a(n,\alpha) : \alpha < \mathfrak{M}_n\}$ where $\mathfrak{M}_n = \mathrm{card}(A_n \setminus A_{n+1})$ is the cardinality of $(A_n \setminus A_{n+1})$. Next we define a function $f : D \to H(\mathfrak{M})$ by letting $f[a(n,\alpha)]$ be the point on $[0,1]_\alpha$ at distance $\frac{1}{n+1}$ from $\mathcal{O}$. It is clear that the map f is one-one, and is continuous because D is a discrete space. By the definition of extension trace and the completeness of hedgehogs, we have:

7.1 Proposition. $\mathrm{cl}_H f[D]$ *is (homeomorphic with) the one-point completion of the discrete space D determined by the extension trace $\mathcal{A}$; that is* $\mathrm{cl}_H f[D] = f[D] \cup \{p\}$, *where $p = p(\mathcal{A})$ is its unique nonisolated point.*

We will call $f[D] \cup \{p(\mathcal{A})\}$ a one-point completion of D.

7.2 Proposition. *The completion $f[D] \cup \{p(\mathcal{A})\}$ (where $\mathcal{A} = (A_n)_{n<\omega}$) is locally compact if and only if the sets $A_n \setminus A_{n+1}$ are finite for all but finitely many $n < \omega$.*

Proof: Suppose $m < \omega$ is fixed and $\mathrm{card}(A_m \setminus A_{m+1}) = \mathfrak{M}_m$. Then, by the definition of distance in the hedgehog $H(\mathfrak{M})$ and the function f, there are $\mathfrak{M}_m$ points in $f[A_m \setminus A_{m+1}]$ at distance $\frac{2}{m+1}$ from each other; namely $\{d(f[(m,\alpha)], (f[(m,\beta)]) : \alpha \neq \beta < \mathfrak{M}_m\}$. So, if $\{n < \omega : A_n \setminus A_{n+1}$ is not finite$\}$ is infinite, then every neighborhood of $p(\mathcal{A})$ in $f[D] \cup \{p(\mathcal{A})\}$ contains an infinite closed discrete set, and we may conclude that $f[D] \cup \{p(\mathcal{A})\}$ is not locally compact.

Next assume that the sets $A_n \setminus A_{n+1}$ are eventually finite; that is, there is an $m < \omega$ such that if $n \geq m$, then $A_n \setminus A_{n+1}$ is finite. Before proving the converse implication, we introduce a definition and a lemma. If $a \in D$, let $\varphi(a) = s + 1$ if $a \in A_s \setminus A_{s+1}$. The proof of the following lemma is an exercise since D is the union of the pairwise disjoint sets $A_n \setminus A_{n+1}$.

7.3 Lemma. *If (a_k) is a sequence of distinct elements of A_m, then it has a subsequence $(a_{k(i)})$ such that $\varphi(a_{k(i)})$ diverges to ∞.*

We will show that any sequence (x_k) of distinct elements from the set $f[A_m]$ converges to $\mathcal{O}$. Writing $a_k = f^{\leftarrow}(x_k)$, we see that the sequence (a_k) satisfies the hypothesis of Lemma 7.3, so there is a subsequence $(x_{k(i)})$ of (x_k) defined by $x_{k(i)} = f(a_{k(i)})$. Note that the distance from p to $x_{k(i)}$ is $\frac{1}{\varphi(a_{k(i)})}$ on any spine of the hedgehog, we conclude that the subsequence $x_{k(i)}$ converges to p. This completes the proof of Proposition 7.2. □

Whether use of the axiom of choice has been avoided in the above depends on whether we are willing to assume that the set D given us at the beginning of the construction comes to us with a well-ordering. For if D is well-ordered, so are each or the sets $A_n \setminus A_{n+1}$, and the resulting geometric description of the one-point completions is much easier to visualize than the ones constructed by using the Stone-Čech compactification.

A well-known way of creating a one-point completion of the half-line $[0, \infty)$ that is not locally compact is obtained by embedding $[0, \infty)$ as $S = \{(x, \sin(\frac{1}{x})) : 0 < x \leq 1\}$ in $\mathbb{R}^2$, taking its closure therein, and choosing one point, say $(0, 0)$ on the vertical axis to obtain $S \cup \{(0, 0)\}$. This latter is a completion because it is a G_δ in the complete metric space $S \cup \{(0, y) : 0 \leq |y| \leq 1\}$. While this looks constructive at first glance, the theorem that guarantees that a G_δ in the complete metric space has a compatible metric with respect to which the G_δ is a Cauchy completion of S is not constructive.

We will give a brief outline of how one may use the metric on the separable Hilbert space ℓ_2 of square summable sequences of real numbers to obtain one-point completions of $\mathbb{R}$ that are not locally compact.

Let $\{e_n : 1 \leq n < \infty\}$ denote the usual basis of unit vectors in ℓ_2. If $x, y, z \in \ell_2$, let $[x, y] = \{tx + (1-t)y : 0 \leq t \leq 1\}$ denote the line segment joining x and y, and let $[x, y, z] = [x, y] \cup [y, z]$. Consider the subset T of ℓ_2 obtained by attaching $\{te_1 : t \geq 1\}$ to $[e_1, \frac{1}{2}e_2, e_2]$, to $\cdots [e_n, \frac{1}{n+1}e_{n+1}, e_{n+1}], \cdots$ for $n = 1, 2, \ldots$. It is not difficult to subdivide $\mathbb{R}$ into successive intervals each meeting the next in exactly one point, and use them to construct an order preserving homeomorphism of $\mathbb{R}$ onto T. Then the Cauchy completion of T with respect to the metric of ℓ_2 will be $T \cup \{0\}$. We leave it to the reader to verify that $T \cup \{0\}$ is not locally compact.

Constructing one-point metric completions geometrically with methods that apply in more generality would appear to be a formidable task.

References

[A71] Alexander C.C., *Metrically isolated sets*, Amer. Math. Monthly **78** (1971), 892–895.

[B74] Bel'nov V.K., *Some theorems on metric extensions*, Trans. Moscow Math. Soc. **30** (1974), 119–138.

[B75] Bel'nov V.K., *The structure of set of metric extensions of a noncompact metrizable space*, Trans. Moscow Math. Soc. **32** (1975), 1–30.

[DH99] Dow A., Hart K.P., *ω^* has (almost) no continuous images*, Israel J. Math. **109** (1999), 29–39.

[E89] Engelking R., *General Topology*, Heldermann Verlag, Berlin, 1989.

[FGO93] Fitzpatrick B., Gruenhage G., Ott J., *Topological completions of metrizable spaces*, Proc. Amer. Math. Soc. **117** (1993), 259–267.

[GJ76] Gillman L., Jerison M., *Rings of Continuous Functions Springer, New York, 1976.*

[Ma68] Magill K., *The lattice of compactifications of locally compact space*, Proc. London Math. Soc. **18** (1968), 231–244.

[Mi82] van Mill J., *Weak P-points in Čech-Stone compactifications*, Trans. Amer. Math. Soc. **273** (1982), 657–678.

[MRW72] Mack J., Rayburn M.C., Woods R.G., *Local topological properties and one-point extensions*, Canadian J. Math. **24** (1972), 338–348.

[MRW74] ibid, *Lattices of topological extensions*, Trans. Amer. Math. Soc. **189** (1974), 163–174.

[PW87] Porter J., Woods R.G., *Extension and Absolutes of Hausdorff Spaces*, Springer, New York, 1987.

[Ra73] Rayburn M.C., *On Hausdorff compactifications*, Pacific J. Math **44** (1973), 707–71.

[V87] Villani A., *Spaces with locally compact completions are compact*, Amer. Math. Monthly **94** (1987), 863–865.

[W71] Woods R.G., *Co-absolutes of remainders of $\beta X \setminus X$*, Pacific J. Math. **37** (1971), 545–560.

[W74] Woods R.G., *Zero-dimensional compactifications of locally compact spaces*, Canadian J. Math. **26** (1974), 920–930.

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