Fixed point theorems for n-periodic mappings in Banach spaces

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Abstract. Using modified Halpern iterations, by elementary method, we extend and improve results obtained by W.A. Kirk (Proc. Amer. Math. Soc. **29** (1971), 294) and others, which have recently been presented in Chapter 11 of *Handbook of Metric Fixed Point Theory* (2001).

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1. Introduction

Mappings which are defined on metric spaces and which do not increase distances between pairs of points and their images are called *nonexpansive*. In general, to assure the fixed point property for nonexpansive mappings some assumptions concerning the geometry of the spaces are added (see [8]). Another way is to put some additional restrictions on the mapping itself.

Let C be a nonempty subset of a Banach space E and $T\colon C \to C$. Recall that a mapping T is said to be n-periodic if $T^n = I$ (for $n = 2$, T is called *involution*).

The first fixed point theorem for involutions are due to K. Goebel and E. Złotkiewicz [1], [3]. It is shown that if $T\colon C \to C$, where C is a closed and convex subset of a Banach space, is a k-lipschitzian involution with $k < 2$, then T has at least one fixed point. The proof of this fact is a straightforward verification that starting from any $x \in C$, the sequence of iterates $\{F^n x\}$ for $F = \frac{1}{2}(I + T)$ always converges to a fixed point of T.

Moreover, if the space E satisfies $\varepsilon_0(E) < 1$, the same is true for k-lipschitzian involutions where k satisfies

$$\left(\frac{k}{2}\right)\left(1 - \delta_E\left(\frac{2}{k}\right)\right) < 1.$$

W.A. Kirk [7] extended this result for all Banach spaces by proving that the same is true if T is n-periodic and such that $\|T^i x - T^i y\| \le k\|x - y\|$ for $x, y \in C$, $i = 1, 2, \ldots, n-1$, where

$$\frac{1}{n^2}\Big[(n-1)(n-2)k^2 + 2(n-1)k\Big] < 1. \tag{1}$$

It follows from (1) that for $n = 3$, $k < 1.3452$; for $n = 4$, $k < 1.2078$; for $n = 5$, $k < 1.1280$; for $n = 6$, $k < 1.1147$.

If T is k-lipschitzian with $k > 1$, then $\|T^i x - T^i y\| \le k^{n-1}\|x - y\|$ for $x, y \in C$, $i = 1, 2, \ldots, n-1$. Thus a k-lipschitzian mapping satisfying $T^n = I$ has fixed points if

$$\frac{1}{n^2}\left[(n-1)(n-2)k^{2(n-1)} + 2(n-1)k^{n-1}\right] < 1. \tag{2}$$

It follows from (2) that for $n = 3$, $k < 1.1598$; for $n = 4$, $k < 1.0649$; for $n = 5$, $k < 1.0351$; for $n = 6$, $k < 1.0219$.

In 1973, J. Linhart [9] showed that a k-lipschitzian mappings $T\colon C \to C$ for which $T^n = I$ $(n > 1)$ has a fixed point if

$$\frac{1}{n}\sum_{j=n-1}^{2n-3} k^j < 1. \tag{3}$$

It follows from (3) that for $n = 3$, $k < 1.1745$; for $n = 4$, $k < 1.0741$; for $n = 5$, $k < 1.0412$; for $n = 6$, $k < 1.0262$.

In the present paper, studying a new iteration process, we extend Kirk and Linhart's results for n-periodic mappings.

2. Lipschitzian mappings

Recall that $T\colon C \to C$ is called *k-lipschitzian* if for all $x, y \in C$,

$$\|Tx - Ty\| \le k\|x - y\|.$$

We will start with the following lemma :

Lemma 1 ([4]). *Let C be a nonempty closed subset of a Banach space E and $T\colon C \to C$ be k-lipschitzian. Let $A, B \in \mathbb{R}$, $0 \le A < 1$ and $0 < B$. If for arbitrary $x \in C$ there exists $u \in C$ such that*

$$\|Tu - u\| \le A\|Tx - x\|$$

and

$$\|u - x\| \le B\|Tx - x\|,$$

then T has a fixed point in C.

Theorem 1. *Let C be a nonempty closed convex subset of a Banach space E and $T\colon C \to C$ be a k-lipschitzian mapping $(k > 1)$ such that $T^n = I$ $(n > 1)$. If*

$$k < 2 \quad \text{for} \quad n = 2$$

and

$$k < k_0(n) = \sup_{\alpha \in (0,1)} \Big\{ s > 1 \colon \alpha^2 s^n + (1-\alpha)^n s^{n-1} + \alpha^2 \sum_{j=2}^{n-1} (1-\alpha)^{j-1} s^j \frac{1 - s^{n-j}}{1-s} - 1 = 0 \Big\} \quad \text{for} \quad n > 2,$$

then T has a fixed point in C.

Proof: Let $n = 2$ $(T^2 = I)$. If we put $z = \alpha x_0 + (1-\alpha)Tx_0$ for arbitrary $x_0 \in C$, we get

$$\begin{aligned}
\|z - Tz\| &= \|\alpha x_0 + (1-\alpha)Tx_0 - Tz\| \\
&= \|\alpha(x_0 - Tz) + (1-\alpha)(Tx_0 - Tz)\| \\
&\le \alpha k \|Tx_0 - z\| + (1-\alpha)k\|x_0 - z\| \\
&= \alpha k \|Tx_0 - \alpha x_0 - (1-\alpha)Tx_0\| \\
&\quad + (1-\alpha)k\|x_0 - \alpha x_0 - (1-\alpha)Tx_0\| \\
&= (2\alpha^2 - 2\alpha + 1)k\|x_0 - Tx_0\|.
\end{aligned}$$

If $k < 2$, then $h_\alpha(k) = (2\alpha^2 - 2\alpha + 1)k < 1$ for $\alpha \in (0,1)$. Since

$$\|z - x_0\| = \|\alpha x_0 + (1-\alpha)Tx_0 - x_0\| \le (1-\alpha)\|Tx_0 - x_0\|,$$

Lemma 1 implies the existence of fixed points of T in C.

Now, let $n > 2$. We consider a sequence generated by Halpern's iteration procedure [5] as follows: let x be an arbitrary point in C, i.e.

$$x_0 = x \in C,$$

and

$$\begin{aligned}
x_1 &= \alpha x_0 + (1-\alpha)Tx_0, \\
x_2 &= \alpha x_0 + (1-\alpha)Tx_1, \\
&\cdots\cdots\cdots\cdots\cdots \\
x_{n-2} &= \alpha x_0 + (1-\alpha)Tx_{n-3}, \\
x_{n-1} &= \alpha x_0 + (1-\alpha)Tx_{n-2},
\end{aligned}$$

where $\alpha \in (0,1)$. Put $z = x_{n-1}$; then

$$\begin{aligned}\|z - Tz\| &= \|\alpha x_0 + (1-\alpha)Tx_{n-2} - Tz\| \\ &= \|\alpha(T^n x_0 - Tz) + (1-\alpha)(Tx_{n-2} - Tz)\| \\ &\le \alpha k\|T^{n-1}x_0 - z\| + (1-\alpha)k\|x_{n-2} - z\|.\end{aligned} \tag{4}$$

Now, we have evaluation

$$\begin{aligned}\|T^{n-1}x_0 - z\| &= \|T^{n-1}x_0 - \alpha x_0 - (1-\alpha)Tx_{n-2}\| \\ &= \|\alpha(T^{n-1}x_0 - T^n x_0) + (1-\alpha)(T^{n-1}x_0 - Tx_{n-2})\| \\ &\le \alpha k^{n-1}\|x_0 - Tx_0\| + (1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\|,\end{aligned} \tag{5}$$

where

$$\begin{aligned}&(1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\| \\ &\quad = (1-\alpha)k\|T^{n-2}x_0 - \alpha x_0 - (1-\alpha)Tx_{n-3}\| \\ &\quad = (1-\alpha)k\|\alpha(T^{n-2}x_0 - x_0) + (1-\alpha)(T^{n-2}x_0 - Tx_{n-3})\| \\ &\quad \le (1-\alpha)\alpha k\|T^{n-2}x_0 - x_0\| + (1-\alpha)^2k^2\|T^{n-3}x_0 - x_{n-3}\| \\ &\quad = (1-\alpha)\alpha k\|T^{n-2}x_0 - x_0\| \\ &\qquad + (1-\alpha)^2k^2\|T^{n-3}x_0 - \alpha x_0 - (1-\alpha)Tx_{n-4}\| \\ &\quad = (1-\alpha)\alpha k\|T^{n-2}x_0 - x_0\| \\ &\qquad + (1-\alpha)^2k^2\|\alpha(T^{n-3}x_0 - x_0) + (1-\alpha)(T^{n-3}x_0 - Tx_{n-4})\| \\ &\quad \le (1-\alpha)\alpha k\|T^{n-2}x_0 - x_0\| \\ &\qquad + (1-\alpha)^2k^2\left\{\alpha\|T^{n-3}x_0 - x_0\| + (1-\alpha)k\|T^{n-4}x_0 - x_{n-4}\|\right\} \\ &\quad \le \alpha(1-\alpha)k\|T^{n-2}x_0 - x_0\| \\ &\qquad + \alpha(1-\alpha)^2k^2\|T^{n-3}x_0 - x_0\| + (1-\alpha)^3k^3\|T^{n-4}x_0 - x_{n-4}\| \\ &\quad \le \alpha(1-\alpha)k\|T^{n-2}x_0 - x_0\| + \alpha(1-\alpha)^2k^2\|T^{n-3}x_0 - x_0\| \\ &\qquad + \alpha(1-\alpha)^3k^3\|T^{n-4}x_0 - x_0\| + \dots \\ &\qquad\qquad + \alpha(1-\alpha)^{n-2}k^{n-2}\|Tx_0 - x_0\|.\end{aligned}$$

Finally, using only the triangle inequality and the fact that T is k-lipschitzian we get

$$(1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\| \le \alpha\sum_{j=2}^{n-1}(1-\alpha)^{j-1}k^{j-1}\frac{1-k^{n-j}}{1-k}\|Tx_0 - x_0\|, \tag{6}$$

and consequently from (5) and (6) we obtain

$$\|T^{n-1}x_0 - z\| \leq \left\{\alpha k^{n-1} + \alpha \sum_{j=2}^{n-1} (1-\alpha)^{j-1} k^{j-1} \frac{1-k^{n-j}}{1-k}\right\} \|Tx_0 - x_0\|. \tag{7}$$

For the next expression we have the following evaluation

$$\begin{aligned} \|x_{n-2} - z\| &= \|\alpha x_0 + (1-\alpha)Tx_{n-3} - \alpha x_0 - (1-\alpha)Tx_{n-2}\| \\ &= \|(1-\alpha)(Tx_{n-3} - Tx_{n-2})\| \leq (1-\alpha)k\|x_{n-3} - x_{n-2}\| \leq \dots \\ &\leq (1-\alpha)^{n-2}k^{n-2}\|x_0 - x_1\| = (1-\alpha)^{n-1}k^{n-2}\|x_0 - Tx_0\|. \end{aligned} \tag{8}$$

Combining (4) with (7) and (8) yields

$$\begin{aligned} &\|z - Tz\| \\ &\leq \left\{\alpha^2 k^n + \alpha^2 \sum_{j=2}^{n-1} (1-\alpha)^{j-1} k^j \frac{1-k^{n-j}}{1-k} + (1-\alpha)^n k^{n-1}\right\} \|Tx_0 - x_0\|. \end{aligned} \tag{9}$$

Moreover, we have

$$\begin{aligned} \|z - x_0\| &= \|\alpha x_0 + (1-\alpha)Tx_{n-2} - x_0\| \leq (1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| \\ &= (1-\alpha)k\|\alpha x_0 + (1-\alpha)Tx_{n-3} - T^{n-1}x_0\| \\ &= (1-\alpha)k\|\alpha(T^n x_0 - T^{n-1}x_0)\| + (1-\alpha)(Tx_{n-3} - T^{n-1}x_0)\| \\ &\leq \alpha(1-\alpha)k^n\|Tx_0 - x_0\| + (1-\alpha)^2k^2\|x_{n-3} - T^{n-2}x_0\|. \end{aligned} \tag{10}$$

Observe that

$$\begin{aligned} &(1-\alpha)^2k^2\|x_{n-3} - T^{n-2}x_0\| \\ &\qquad = (1-\alpha)^2k^2\|\alpha(x_0 - T^{n-2}x_0) + (1-\alpha)(Tx_{n-4} - T^{n-2}x_0)\| \\ &\qquad \leq (1-\alpha)^2\alpha k^2\|x_0 - T^{n-2}x_0\| + (1-\alpha)^3k^3\|x_{n-4} - T^{n-3}x_0\| \leq \dots \\ &\qquad \leq \alpha(1-\alpha)^2k^2\|x_0 - T^{n-2}x_0\| + \alpha(1-\alpha)^3k^3\|x_0 - T^{n-3}x_0\| + \dots \\ &\qquad\quad + \alpha(1-\alpha)^{n-2}k^{n-2}\|x_0 - T^2x_0\| + \alpha(1-\alpha)^{n-1}k^{n-1}\|x_0 - Tx_0\|. \end{aligned}$$

Now, using only the triangle inequality and the fact that T is k-lipschitzian, we have

$$(1-\alpha)^2k^2\|x_{n-3} - T^{n-2}x_0\| \leq \alpha \sum_{j=2}^{n-1} (1-\alpha)^j k^j \frac{1-k^{n-j}}{1-k} \|x_0 - Tx_0\|,$$

which together with (10) gives

$$\|z - x_0\| \le \left\{ \alpha(1-\alpha)k^n + \alpha \sum_{j=2}^{n-1} (1-\alpha)^j k^j \frac{1-k^{n-j}}{1-k} \right\} \|x_0 - Tx_0\|. \tag{11}$$

Since $h_\alpha(k) = \alpha^2 k^n + \alpha^2 \sum_{j=2}^{n-1} (1-\alpha)^{j-1} k^j \frac{1-k^{n-j}}{1-k} + (1-\alpha)^n k^{n-1} < 1$ for all $\alpha \in (0,1)$ and $k < k_0(n)$, by inequality (9) and (11), Lemma 1 implies the existence of fixed points of T in C. □

Remark 1. It follows from Theorem 1 that $k_0(3) \ge 1.3821$ (this evaluation is obtained for $\alpha = 0.345$); $k_0(4) \ge 1.2524$ (for $\alpha = 0.283$); $k_0(5) \ge 1.1777$ (for $\alpha = 0.224$); $k_0(6) \ge 1.1329$ (for $\alpha = 0.185$). All these evaluations are much better than those obtained by W.A. Kirk [7] and J. Linhart [9].

Remark 2. From the above and Lemma 1 it follows that the sequence $\{z_p\}$ generated by the following iteration process

$$\begin{aligned}
x_0 &= x \in C, \\
x_1 &= \alpha x_0 + (1-\alpha)Tx_0, \\
x_2 &= \alpha x_0 + (1-\alpha)Tx_1, \\
&\dots\dots\dots\dots\dots\dots \\
x_{n-2} &= \alpha x_0 + (1-\alpha)Tx_{n-3}, \\
z_1 &= x_{n-1}(x_0) = \alpha x_0 + (1-\alpha)Tx_{n-2}, \quad n > 2
\end{aligned}$$

i. e.,

$$\begin{aligned}
z_1 &= x_{n-1}(x_0), \quad x_0 \in C, \\
z_2 &= x_{n-1}(z_1),
\end{aligned}$$

and

$$z_{p+1} = x_{n-1}(z_p), \quad \text{for} \quad p = 2, 3, \dots,$$

converges strongly to a fixed point of T.

3. Uniformly lipschitzian mappings

Recall that a mapping $T\colon C \to C$ is called *uniformly k-lipschitzian* if for all $n \in \mathbb{N}$ and $x, y \in C$,

$$\|T^n x - T^n y\| \le k\|x - y\|.$$

The class of uniformly lipschitzian mappings was introduced to the fixed point theory by K. Goebel and W.A. Kirk in 1973 (see [2]). This class forms a natural extension of the family of nonexpansive mappings.

Theorem 2. *Let C be a nonempty closed convex subset of a Banach space E and $T\colon C\to C$ be a mapping such that*

(a) $\|T^i x - T^i y\| \le k\|x-y\|$ *for* $x,y\in C$ *and* $i=1,2,\dots,n-1$,

(b) $T^n = I \quad (n\ge 3)$.

If $k<k_0$, where

$$k < k_0(n) = \sup_{\alpha\in(0,1)}\left\{ s>1\colon \alpha^2 s^2 + (1-\alpha)^n s^{n-1} + \alpha^2 \sum_{j=1}^{n-2} (1-\alpha)^j s^{j+1}[1+(n-j-2)s] - 1 = 0 \right\},$$

then T has a fixed point in C.

Proof: Let $n\ge 3$. As in the proof of Theorem 1 we consider the following iteration procedure:

$$\begin{aligned} x_0 &\in C,\\ x_1 &= \alpha x_0 + (1-\alpha)Tx_0,\\ x_2 &= \alpha x_0 + (1-\alpha)Tx_1,\\ &\dots\dots\dots\dots\dots\dots\\ x_{n-2} &= \alpha x_0 + (1-\alpha)Tx_{n-3},\\ x_{n-1} &= \alpha x_0 + (1-\alpha)Tx_{n-2}, \end{aligned}$$

for $\alpha\in(0,1)$. Let $z=x_{n-1}$. Then from (4) and

$$\begin{aligned} \|T^{n-1}x_0 - z\| &= \|\alpha(T^{n-1}x_0 - T^n x_0) + (1-\alpha)(T^{n-1}x_0 - Tx_{n-2})\|\\ &\le \alpha k\|x_0 - Tx_0\| + (1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\| \end{aligned}$$

where for expression

$$\begin{aligned} &(1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\|\\ &\quad= (1-\alpha)k\|T^{n-2}x_0 - \alpha x_0 - (1-\alpha)Tx_{n-3}\|\\ &\quad= (1-\alpha)k\|\alpha(T^{n-2}x_0 - x_0) + (1-\alpha)(T^{n-2}x_0 - Tx_{n-3})\|\\ &\quad\le (1-\alpha)\alpha k\|T^{n-2}x_0 - x_0\| + (1-\alpha)^2k^2\|T^{n-3}x_0 - x_{n-3}\| \le \dots\\ &\quad\le \alpha(1-\alpha)k\|T^{n-2}x_0 - x_0\| + \alpha(1-\alpha)^2k^2\|T^{n-3}x_0 - x_0\|\\ &\qquad + \alpha(1-\alpha)^3k^3\|T^{n-4}x_0 - x_0\| + \dots\\ &\qquad\qquad + \alpha(1-\alpha)^{n-2}k^{n-2}\|Tx_0 - x_0\|, \end{aligned}$$

using only the triangle inequality and the fact that T is uniformly k-lipschitzian now we get evaluation

$$\begin{aligned}(1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\| &\le \alpha(1-\alpha)k[1+(n-3)k]\|Tx_0 - x_0\| \\ &\quad + \alpha(1-\alpha)^2k^2[1+(n-4)k]\|Tx_0 - x_0\| + \dots \\ &\quad + \alpha(1-\alpha)^{n-2}k^{n-2}\|Tx_0 - x_0\| \\ &= \alpha\sum_{j=1}^{n-2}(1-\alpha)^j k^j[1+(n-j-2)k]\|Tx_0 - x_0\|\end{aligned}$$

which together with (8) gives

$$\begin{aligned}\|z - Tz\| \le \Big\{&\alpha^2k^2 + \alpha^2\sum_{j=1}^{n-2}(1-\alpha)^j k^{j+1}[1+(n-j-2)k] \\ &+ (1-\alpha)^n k^{n-1}\Big\}\|x_0 - Tx_0\|.\end{aligned} \tag{12}$$

Analogously as in (10) we obtain

$$\|z - x_0\| \le \alpha(1-\alpha)k^2\|Tx_0 - x_0\| + (1-\alpha)^2k^2\|x_{n-3} - T^{n-2}x_0\|, \tag{13}$$

where for expression $(1-\alpha)^2k^2\|x_{n-3}-T^{n-2}x_0\|$ using only the triangle inequality and the fact that T is uniformly k-lipschitzian now we get evaluation

$$(1-\alpha)^2k^2\|x_{n-3} - T^{n-2}x_0\| \le \alpha\sum_{j=2}^{n-1}(1-\alpha)^j k^j[1+(n-j-1)]\|x_0 - Tx_0\|,$$

which together with (13) gives

$$\begin{aligned}\|z - x_0\| \le \Big\{&\alpha\sum_{j=2}^{n-1}(1-\alpha)^j k^j[1+(n-j-1)] \\ &+ \alpha(1-\alpha)k^2\Big\}\|x_0 - Tx_0\|.\end{aligned} \tag{14}$$

Since $H_\alpha(k) = \alpha^2k^2 + (1-\alpha)^n k^{n-1} + \alpha^2\sum_{j=1}^{n-2}(1-\alpha)^j k^{j+1}[1+(n-j-2)k] < 1$ for all $\alpha \in (0,1)$ and $k < k_0(n)$, by inequalities (12) and (14), Lemma 1 implies the existence of fixed points of T in C. $\square$

Remark 3. It follows from Theorem 2 that $k_0(3) \ge 1.4558$ (this evaluation is obtained for $\alpha = 0.393$); $k_0(4) \ge 1.2917$ (for $\alpha = 0.322$); $k_0(5) \ge 1.2001$ (for $\alpha = 0.255$); $k_0(6) \ge 1.1482$ (for $\alpha = 0.206$). All these evaluations are better than those obtained by W.A. Kirk [7].

4. Nonexpansive iterate

In this section, using Theorems 1 and 2, we extend the result of W.A. Kirk ([7, Theorem 1]). We obtain in all Banach spaces the conditions sufficient to guarantee existence of fixed points for mappings T which are n-periodic, i.e. $T^n = I$.

Theorem 3. *Let E be a reflexive Banach space which has a strictly convex norm and suppose C is a nonempty bounded closed and convex subset of E with normal structure. Suppose the mapping $T\colon C \to C$ has one of the following properties:*

(A) *T^n is nonexpansive $(n > 2)$ and there is a constant $k < k_0(n)$ (see Theorem 1) such that $\|Tx - Ty\| \leq k\|x - y\|$ for all $x, y \in C$,*

or

(B) *T^n is nonexpansive $(n > 2)$ and there is a constant $k < k_0(n)$ (see Theorem 2) such that $\|T^i x - T^i y\| \leq k\|x - y\|$ for all $x, y \in C$ and $i = 1, 2, \ldots, n-1$.*

Then T has a fixed point in C.

Proof: By the result of Browder–Göhde–Kirk [8] the set

$$C^* = \{x \in C \colon T^n x = x\} \neq \emptyset.$$

Because of strict convexity of E, C^* must be convex. Clearly C^* is closed, $T\colon C^* \to C^*$ and T^n is the identity on C^*. Thus, the assumptions for Theorems 1 and 2, respectively, are satisfied for T on C^*, and in consequence T has a fixed point in C. $\square$

Remark 4. For k-lipschitzian involutions $(T^2 = I)$ the assumption of strict convexity is not necessary. In this case we have (see Corollary 1 in [6]):

Let C be a weakly compact convex subset of a Banach space, and suppose C has normal structure. Suppose $T : C \to C$ is k-lipschitzian mapping where k satisfies the condition

$$\frac{k}{2}\left(1 - \delta\left(\frac{2}{k}\right)\right) < 1,$$

for which T^2 is nonexpansive. Then T has a fixed point.

Problems.

1. We do not know whether the new constants are close to optimal or even whether optimal constants exist.
2. Is the assumption of strict convexity necessary for Theorem 3?
3. In which situation it is possibility to replace the condition "T^n is nonexpansive" by the condition "T^n is asymptotically nonexpansive" in the above theorems?

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